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On 12-regular nut graphs*

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Abstract

A nut graph is a simple graph whose adjacency matrix is singular with 1-dimensional kernel such that the corresponding eigenvector has no zero entries. In 2020, Fowler *et al.* characterised for each $d \in \{3, 4, \ldots, 11\}$ all values n such that there exists a d-regular nut graph of order n. In the present paper, we resolve the first open case $d = 12$, i.e. we show that there exists a 12-regular nut graph of order n if and only if $n \geq 16$. We also present a result by which there are infinitely many circulant nut graphs of degree $d \equiv 0 \pmod 4$ and no circulant nut graphs of degree $d \equiv 2 \pmod 4$. The former result partially resolves a question by Fowler *et al.* on existence of vertex-transitive nut graphs of order n and degree d. We conclude the paper with problems, conjectures and ideas for further work.

Keywords: Nut graph, adjacency matrix, singular matrix, core graph, Fowler construction, regular graph.

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1 Introduction

Let G be a simple graph with the vertex set $V(G) = \{0, 1, \ldots, n-1\}$. Its adjacency matrix $\mathbb{A}$ is a symmetric $n \times n$ matrix with entries $a_{i,j}$, where $0 \leq i, j \leq n-1$, such that $a_{i,j} = 1$ if $\{i, j\}$ is an edge of G and $a_{i,j} = 0$ otherwise. A graph G is a *nut graph* if $\mathbb{A}$ has eigenvalue 0 and no eigenvector has zero entries. As a consequence, the eigenspace corresponding to the eigenvalue 0 is 1-dimensional. Observe that if the eigenspace corresponding to 0 has dimension greater than one, then there exists an eigenvector containing entry 0 that is different from $\mathbf{0} = (0, 0, \ldots, 0)^T$. For an introductory treatment of spectral graph theory, which links graphs to linear algebra, see e.g. [3, 4, 7].

Nut graphs have been studied in [6, 9, 11, 12, 16, 17, 18, 19, 20, 22], see also the webpage `https://hog.grinvin.org/Nuts` within the House of Graphs [2, 5]. Recently, this concept was extended to signed graphs [1]. Nut graphs have chemical applications, see e.g. [9, 8, 21]. However, in the present paper we consider 12-regular graphs, so our motivation is purely mathematical.

In [22], Gutman and Sciriha showed that the smallest non-trivial nut graph has order 7. In [10], Fowler *et al.* determined all nut graphs on up to 10 vertices and all chemical nut graphs on up to 16 vertices. The smallest order for which a regular nut graph exists is 8; see also [9]. In [9], Fowler *et al.* presented the following question.

If there exists a d-regular graph of order n, then we say that the order n is *admissible* regarding the degree d. Obviously, if d is even then every $n \geq d + 1$ is admissible. If d is odd then every even $n \geq d + 1$ is admissible.

Question 1.1. Is it true that for each degree $d \geq 3$ there are only finitely many admissible orders n such that there does not exist a d-regular nut graph of order n?

In the attempt to answer Question 1.1, the 'Fowler Construction' played an important role; see also [11]. This construction implies the following theorem.

Theorem 1.2. *Let G be a nut graph on n vertices and let u be a vertex of G of degree d. Then there exists a nut graph of order $n + 2d$ that is obtained from G by adding $2d$ new vertices and rearranging the edges in a certain way. In the newly obtained nut graph the degrees of the new vertices are d and the degrees of the original vertices are not changed.*

Obviously, if G is a d-regular graph of order n, then the new graph is d-regular of order $n + 2d$. Hence, to answer Question 1.1 positively for specific degree d, it suffices to find d-regular graphs for $2d$ consecutive orders. In [11] ($d = 3, 4$) and [9] ($5 \leq d \leq 11$), the authors found all pairs (d, n), such that $d \leq 11$ and there exists a d-regular nut graph of order n. In the present paper, we extend this result to $d = 12$. We prove the following statement.

Theorem 1.3. *There exists a 12-regular nut graph of order n if and only if $n \geq 16$.*

To prove the 'positive part' of Theorem 1.3, it suffices to find 12-regular nut graphs of orders $n \in \{16, 17, \ldots, 39\}$. We present these graphs in the following section. For odd orders there is not much to say; we did a computer search and thus we provide a list of graphs that we found. However, for even orders we can say more.

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A graph G is called *vertex-transitive* if all vertices are equivalent under the action of the automorphism group $\mathrm{Aut}(G)$. In other words, for each pair of vertices $u, v \in V(G)$ there exist an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(u) = v$. In [9], the following necessary condition for a vertex-transitive nut graph was given.

Theorem 1.4. *Let G be a vertex-transitive nut graph of degree d on n vertices. Then n and d satisfy the following conditions. Either*

(1) $d \equiv 0 \pmod 4$, $n \equiv 0 \pmod 2$ *and* $n \geq d + 4$, *or*

(2) $d \equiv 0 \pmod 2$, $n \equiv 0 \pmod 4$ *and* $n \geq d + 6$.

The existence of vertex-transitive nut graphs is interesting in its own right, see [9, Question 9]. For our research it is important that, by Theorem 1.4, there may exist vertex-transitive 12-regular graphs of even orders $n \geq 16$. We found such graphs among circulant graphs.

2 Results

We start with the 'negative part' of Theorem 1.3. There is only one 12-regular graph of order 13, namely the complete graph K_{13}, and it is not a nut graph. The unique 12-regular graph of order 14 is obtained by removing a matching from K_{14}, and again, this graph is not a nut graph. Finally, there are 17 graphs of order 15 which are 12-regular. They are obtained by removing a 2-factor from K_{15}. Using the SageMath software [23], we analysed all such graphs and concluded that none of them is a nut graph.

Now we turn our attention to the 'positive part' of Theorem 1.3. We start with more general results for even orders. The following lemma is in fact implied in the text preceding Proposition 1 in [11]. We present it here in a slightly more general setting together with its short proof.

Lemma 2.1. *Let G be a d-regular graph on n vertices such that its adjacency matrix $\mathbb{A}$ is singular. Then for every eigenvector $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})^T$ corresponding to eigenvalue* 0*, we have*

$$\sum_{i=0}^{n-1} c_i = 0.$$

Proof. In every d-regular graph, the eigenvector $\mathbf{1} = (1, 1, \ldots, 1)$ corresponds to the eigenvalue d. Since eigenspaces are mutually orthogonal, we have $\mathbf{c} \cdot \mathbf{1} = 0$. □

Let $V = \{0, 1, \ldots, n-1\}$ and let $1 \leq a_1 < a_2 < \cdots < a_t \leq \frac{n}{2}$. By $C(n, \{a_1, a_2, \ldots, a_t\})$ we denote a graph on the vertex set V in which two vertices $i, j \in V$ are adjacent if and only if $|i - j| = a_k$, where $1 \leq k \leq t$. The graph $C(n, \{a_1, a_2, \ldots, a_t\})$ is called a *circulant graph* and it is regular. Its degree is $2t - 1$ if $a_t = \frac{n}{2}$ and $2t$ otherwise. In fact, circulant graphs are vertex-transitive since $\varphi\colon i \to i + 1$ is an automorphism of $C(n, \{a_1, a_2, \ldots, a_t\})$ (the addition is modulo n).

Circulant graphs are easy to describe and easy to handle. Therefore, it would be nice if there were many nut graphs among them. We prove one positive and one negative result about circulant graphs. We start with the following lemma.

Lemma 2.2. *Let $G = C(n, \{a_1, a_2, \ldots, a_t\})$ be a circulant nut graph, and let $\mathbb{A}$ be its adjacency matrix. Then $(1, -1, 1, -1, \ldots)^T$ is an eigenvector corresponding to eigenvalue* 0.

Proof. We use the well-known fact that if $\mathbf{b}$ and $\mathbf{c}$ are eigenvectors corresponding to eigenvalue λ, then $\mathbf{b}+\mathbf{c}$ is also an eigenvector corresponding to eigenvalue λ.

Let $\mathbf{b}=(b_0,b_1,\dots,b_{n-1})^T$ be an eigenvector corresponding to 0. Denote $b_0=p$ and $b_1=q$. Since $\varphi\colon i\to 2-i$ is an automorphism of G (the addition being modulo n), there is an eigenvector $\mathbf{c}=(c_0,c_1,\dots,c_{n-1})^T$ such that $c_{2-i}=-b_i$, $0\le i\le n-1$. Then $c_1=-b_1=-q$ and $c_2=-b_0=-p$. Since $b_1+c_1=0$ and $\mathbf{b}+\mathbf{c}$ is an eigenvector, we must have $\mathbf{b}+\mathbf{c}=\mathbf{0}$ because G is a nut graph. Hence, $b_2+c_2=0$ and therefore $b_2=p$. Now repeating the process we get $\mathbf{b}=(p,q,p,q,\dots)$. Observe that n is even by Theorem 1.4. Thus, by Lemma 2.1, we have $q=-p$ and so $(1,-1,1,-1,\dots)$ is an eigenvector corresponding to eigenvalue 0. □

Our negative result covers all circulant graphs of degree $d\equiv 2\pmod 4$.

Theorem 2.3. *There is no circulant nut graph of degree d if $d\equiv 2\pmod 4$.*

Proof. Let $d\equiv 2\pmod 4$. Denote $t=\frac{d}{2}$. Observe that t is an odd number. By way of contradiction, assume that $G=C(n,\{a_1,a_2,\dots,a_t\})$ is a circulant nut graph. Then n is even by Theorem 1.4. Let $\mathbb{A}=(\mathbf{a}_0,\mathbf{a}_1,\dots,\mathbf{a}_{n-1})^T$ be the adjacency matrix of G. By Lemma 2.2, $\mathbf{c}=(1,-1,1,-1,\dots)^T$ is an eigenvector corresponding to eigenvalue 0, so that $\mathbb{A}\mathbf{c}=\mathbf{0}$, and in particular $\mathbf{a}_0\mathbf{c}=0$. However,

$$\mathbf{a}_0\mathbf{c}=c_{a_1}+c_{a_2}+\cdots+c_{a_t}+c_{n-a_1}+c_{n-a_2}+\cdots+c_{n-a_t}.$$

Since $c_{a_i}=c_{n-a_i}$ for every i, $1\le i\le t$ (observe that the difference between indices a_i and $n-a_i$ is even), we have $\mathbf{a}_0\mathbf{c}=2(c_{a_1}+c_{a_2}+\cdots+c_{a_t})$, which implies that $c_{a_1}+c_{a_2}+\cdots+c_{a_t}=0$. However, the sum of an odd number of odd numbers is odd, a contradiction. □

Now we prove the positive result. Notice that this result also partially resolves Question 9 from [9] about the existence of vertex-transitive nut graphs of order n and degree d.

Theorem 2.4. *Let $d\equiv 0\pmod 4$ and let n be even. Then $C(n,\{1,2,\dots,\frac{d}{2}\})$ is a nut graph if and only if $\frac{d}{2}+1$ is coprime to n and $\frac{d}{4}$ is coprime to $\frac{n}{2}$.*

Proof. Let $t=\frac{d}{2}$. Then t is even and the graph is $G=C(n,\{1,2,\dots,t\})$.

Let $\mathbb{A}$ be the adjacency matrix of G. By Lemma 2.2, $\mathbf{b}=(1,-1,1,-1,\dots)^T$ is an eigenvector of $\mathbb{A}$ corresponding to eigenvalue 0. Thus $\mathbb{A}\mathbf{b}=\mathbf{0}$. Our aim is to show that if $t+1$ is coprime to n and $\frac{t}{2}$ is coprime to $\frac{n}{2}$, then $\mathbb{A}\mathbf{c}=\mathbf{0}$ if and only if $\mathbf{c}$ is a multiple of $\mathbf{b}$.

So let $\mathbb{A}\mathbf{c}=\mathbf{0}$, where $\mathbf{c}=(c_0,c_1,\dots,c_{n-1})^T$. Let $\mathbb{A}=(\mathbf{a}_0,\mathbf{a}_1,\dots,\mathbf{a}_{n-1})^T$. Then

$$\begin{aligned}\mathbf{a}_t\mathbf{c}&=c_0+c_1+\cdots+c_{t-1}+c_{t+1}+c_{t+2}+\cdots+c_{2t}=0,\\ \mathbf{a}_{t+1}\mathbf{c}&=c_1+c_2+\cdots+c_t+c_{t+2}+c_{t+3}+\cdots+c_{2t+1}=0.\end{aligned}$$

Subtracting the two equations we get

$$\mathbf{a}_t\mathbf{c}-\mathbf{a}_{t+1}\mathbf{c}=c_0-c_t+c_{t+1}-c_{2t+1}=0,$$

and analogously

$$\mathbf{a}_{2t+1}\mathbf{c}-\mathbf{a}_{2t+2}\mathbf{c}=c_{t+1}-c_{2t+1}+c_{2t+2}-c_{3t+2}=0.$$

This gives

$$c_0 - c_t = c_{2t+2} - c_{3t+2},$$

and analogously

$$\begin{aligned} c_{2t+2} - c_{3t+2} &= c_{4t+4} - c_{5t+4}, \\ c_{4t+4} - c_{5t+4} &= c_{6t+6} - c_{7t+5}, \qquad \text{etc.} \end{aligned}$$

So if the odd number $t+1$ is coprime to the even number n, we get

$$c_0 - c_t = c_{2(t+1)} - c_{t+2(t+1)} = \cdots = c_2 - c_{t+2},$$

which gives

$$c_2 - c_0 = c_{t+2} - c_t,$$

and analogously we get

$$\begin{aligned} c_{t+2} - c_t &= c_{2t+2} - c_{2t}, \\ c_{2t+2} - c_{2t} &= c_{3t+2} - c_{3t}, \qquad \text{etc.} \end{aligned}$$

Here, t and n are both even. But if $\frac{t}{2}$ is coprime to $\frac{n}{2}$ then

$$c_2 - c_0 = c_{t+2} - c_t = \cdots = c_4 - c_2.$$

Hence,

$$c_2 - c_0 = c_4 - c_2 = c_6 - c_4 = \cdots$$

Now, if $c_2 > c_0$ then $c_0 < c_2 < c_4 < \cdots < c_0$, a contradiction. Analogously, if $c_2 < c_0$ then $c_0 > c_2 > c_4 > \cdots > c_0$, a contradiction. So $c_0 = c_2 = \cdots = c_{n-2}$ and analogously $c_1 = c_3 = \cdots = c_{n-1}$. Hence if $c_0 = p$, then $\mathbf{c} = (p, -p, p, -p, \dots)$ by Lemma 2.1, and the eigenspace corresponding to eigenvalue 0 is 1-dimensional.

Now suppose that $t+1$ is not coprime to n. Set $\mathbf{b} = \mathbf{0}$. We will change some entries of $\mathbf{b}$. Since $t+1$ is odd, there is an even k such that $(t+1)k \equiv 0 \pmod{n}$ and $1 \leq k < n$. Set

$$b_0 = 1, \quad b_{t+1} = -1, \quad b_{2(t+1)} = 1, \quad b_{3(t+1)} = -1, \quad \dots,$$

where the indices are modulo n. We have changed k entries of $\mathbf{b}$ and since k is even, the last changed entry has value -1. Thus some entries of $\mathbf{b}$ remained 0's and nevertheless $\mathbb{A}\mathbf{b} = \mathbf{0}$, since if j-th entry of $\mathbf{a}_i$ is 1, then either $(j + (t+1))$-th or $(j - (t+1))$-th (modulo n) entry of $\mathbf{a}_i$ is also 1 (while the other is 0). Hence, G is not a nut graph in this case.

Finally, suppose that $\frac{t}{2}$ is not coprime to $\frac{n}{2}$. Then there exists a number k such that $k \mid \frac{t}{2}$, $k \mid \frac{n}{2}$ and $k > 1$. Again, set $\mathbf{b} = \mathbf{0}$. We will change some entries of $\mathbf{b}$. Set

$$b_0 = b_2 = b_4 = \cdots = b_{2(k-2)} = 1 \qquad \text{and} \qquad b_{2(k-1)} = -(k-1),$$

and repeat this pattern for all even indices of $\mathbf{b}$. Since $k \mid \frac{n}{2}$, this pattern is repeated exactly $\frac{n}{2k}$ times. And since every $\mathbf{a}_i$ contains two disjoint sets of t consecutive 1's, we have $\mathbb{A}\mathbf{b} = \mathbf{0}$. But half of the entries of $\mathbf{b}$ are 0's and therefore G is not a nut graph. $\square$

Observe that the only requirement for n in Theorem 2.4 is that n is even and $n > d$. However, if $n = d + 2$ then $\frac{d}{2} + 1$ is not coprime to n, and so $n \geq d + 4$. Hence, by Theorem 2.4, for $d = 12$ the following circulant graphs are nut graphs:

$$C(16, \{1, 2, 3, 4, 5, 6\}), \quad C(20, \{1, 2, 3, 4, 5, 6\}), \quad C(22, \{1, 2, 3, 4, 5, 6\}),$$
$$C(26, \{1, 2, 3, 4, 5, 6\}), \quad C(32, \{1, 2, 3, 4, 5, 6\}), \quad C(34, \{1, 2, 3, 4, 5, 6\}), \text{ and}$$
$$C(38, \{1, 2, 3, 4, 5, 6\}).$$

Using the computer [23] we found the following graphs that are nut graphs:

$$C(18, \{1, 2, 3, 4, 5, 8\}), \quad C(24, \{1, 2, 3, 4, 5, 8\}), \quad C(28, \{1, 2, 3, 4, 5, 10\}),$$
$$C(30, \{1, 2, 3, 4, 5, 8\}), \text{ and} \quad C(36, \{1, 2, 3, 4, 5, 8\}).$$

3 Concluding remarks and further work

From the very beginning of our work on this paper, the nut circulant graphs were continuously present, which fact motivates us explicitly to pose here the following problem.

Problem 3.1. Find which circulant graphs are nut graphs.

By the arguments in this paper, any circulant nut graph must satisfy the conditions of Theorem 1.4(1), i.e. the order n is even, the degree d is divisible by 4, and $n \geq d + 4$. We believe that for any such pairs n and d, there exists a circulant nut graph.

Conjecture 3.2. *For every d, where $d \equiv 0 \pmod 4$, and for every even n, $n \geq d + 4$, there exists a circulant nut graph $C(n, \{a_1, a_2, \ldots, a_{d/2}\})$ of degree d.*

And, as a very particular case of the above conjecture, by restricting to 12-regular graphs, we also propose.

Conjecture 3.3. *For every even n, $n \geq 16$, there exists a circulant nut graph $C(n, \{a_1, a_2, \ldots, a_6\})$ of degree 12.*

By Theorem 1.4, if n is odd then there is no vertex-transitive nut graph of order n and degree 12. In this case all graphs were found by a computer search. If G is a regular graph that contains edges u_1v_1 and u_2v_2 but does not contain edges u_1v_2, u_2v_1, then *rewiring* (i.e. removing edges u_1v_1, u_2v_2 and adding edges u_1v_2, u_2v_1; it is also known as a *Ryser switch* [15]) yields another regular graph. Our approach was to start with a 'random' 12-regular graph of odd order and perform a sequence of rewirings. In this way all graphs in the Appendix were obtained. For instance, the graph on 21 vertices, whose kernel eigenvector contains only values 1 and -2, was obtained from $C(21, \{1, 2, 3, 4, 5, 6\})$ by removing the edges $(0, 16)$ and $(2, 7)$ and adding the edges $(0, 7)$ and $(2, 16)$.

Note that kernel eigenvectors of all graphs in the Appendix on $n = 3k$ vertices (for $k = 7, 9, 11, 13$) contain only values 1 and -2. All those graphs have a special structure. Let $V = V_1 \cup V_{-2}$ be the partition of vertices with respect to the kernel eigenvector entry. In each case, the graph induced by V_{-2} is isomorphic to a graph that can be obtained from $C(k, \{1, 2\})$ by at most one rewiring, while the graph induced by V_1 is isomorphic to a graph that can be obtained from $C(2k, \{1, 2, 3, 4\})$ by at most one rewiring. Moreover, let $\mathrm{BiC}(n, S)$ be the graph with the vertex set $V = \{v_0, \ldots, v_{n-1}, u_0, \ldots, u_{n-1}\}$ and the edge set $E = \{v_i u_{(i+s) \bmod n} : 0 \leq i < n, s \in S\}$. This graph is a special kind of bicirculant (see [13, 14] and references cited therein). The set V_1 can be partitioned into two subsets $V_1 = V_1' \cup V_1''$, $|V_1'| = |V_1''|$, such that the graph induced by edges from V_{-2} to V_1' is isomorphic to a graph that can be obtained from $\mathrm{BiC}(k, \{0, 1, 2, 3\})$ by at most one rewiring. Similarly, the graph induced by edges from V_{-2} to V_1'' is also isomorphic to a graph that can be obtained from $\mathrm{BiC}(k, \{0, 1, 2, 3\})$ by at most one rewiring.

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References

[1] N. Bašić, P. W. Fowler, T. Pisanski and I. Sciriha, On singular signed graphs with nullspace spanned by a full vector: Signed nut graphs, 2020, `arXiv:2009.09018 [math.CO]`.

[2] G. Brinkmann, K. Coolsaet, J. Goedgebeur and H. Mélot, House of Graphs: A database of interesting graphs, *Discrete Appl. Math.* **161** (2013), 311–314, doi:10.1016/j.dam.2012.07.018.

[3] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Universitext, Springer, New York, 2012, doi:10.1007/978-1-4614-1939-6.

[4] F. R. K. Chung, *Spectral Graph Theory*, volume 92 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, 1997.

[5] K. Coolsaet, P. W. Fowler and J. Goedgebeur, Nut graphs, homepage of Nutgen, `http://caagt.ugent.be/nutgen/`.

[6] K. Coolsaet, P. W. Fowler and J. Goedgebeur, Generation and properties of nut graphs, *MATCH Commun. Math. Comput. Chem.* **80** (2018), 423–444.

[7] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs: Theory and Applications*, Johann Ambrosius Barth, Heidelberg, 3rd edition, 1995.

[8] P. W. Fowler, N. Bašić and T. Pisanski, Charting the space of chemical nut graphs, *MATCH Commun. Math. Comput. Chem.* **86** (2021), 519–538.

[9] P. W. Fowler, J. B. Gauci, J. Goedgebeur, T. Pisanski and I. Sciriha, Existence of regular nut graphs for degree at most 11, *Discuss. Math. Graph Theory* **40** (2020), 533–557, doi:10.7151/dmgt.2283.

[10] P. W. Fowler, B. T. Pickup, T. Z. Todorova, M. Borg and I. Sciriha, Omni-conducting and omni-insulating molecules, *J. Chem. Phys.* **140** (2014), Article No. 054115, doi:10.1063/1.4863559.

[11] J. B. Gauci, T. Pisanski and I. Sciriha, Existence of regular nut graphs and the Fowler construction, *Appl. Anal. Discrete Math.* (2021), doi:10.2298/aadm190517028g.

[12] I. Gutman and I. Sciriha, Graphs with maximum singularity, *Graph Theory Notes N. Y.* **30** (1996), 17–20.

[13] I. Kovács, B. Kuzman, A. Malnič and S. Wilson, Characterization of edge-transitive 4-valent bicirculants, *J. Graph Theory* **69** (2012), 441–463, doi:10.1002/jgt.20594.

[14] A. Malnič, D. Marušič, P. Šparl and B. Frelih, Symmetry structure of bicirculants, *Discrete Math.* **307** (2007), 409–414, doi:10.1016/j.disc.2005.09.033.

[15] H. J. Ryser, Combinatorial properties of matrices of zeros and ones, *Canadian J. Math.* **9** (1957), 371–377, doi:10.4153/cjm-1957-044-3.

[16] I. Sciriha, On the construction of graphs of nullity one, *Discrete Math.* **181** (1998), 193–211, doi:10.1016/s0012-365x(97)00036-8.

[17] I. Sciriha, A characterization of singular graphs, *Electron. J. Linear Algebra* **16** (2007), 451–462, doi:10.13001/1081-3810.1215.

[18] I. Sciriha, Coalesced and embedded nut graphs in singular graphs, *Ars Math. Contemp.* **1** (2008), 20–31, doi:10.26493/1855-3974.20.7cc.

[19] I. Sciriha, Graphs with a common eigenvalue deck, *Linear Algebra Appl.* **430** (2009), 78–85, doi:10.1016/j.laa.2008.06.033.

[20] I. Sciriha, Maximal core size in singular graphs, *Ars Math. Contemp.* **2** (2009), 217–229, doi:10.26493/1855-3974.115.891.

[21] I. Sciriha and P. W. Fowler, Nonbonding orbitals in fullerenes: Nuts and cores in singular polyhedral graphs, *J. Chem. Inf. Model.* **47** (2007), 1763–1775, doi:10.1021/ci700097j.

[22] I. Sciriha and I. Gutman, Nut graphs: Maximally extending cores, *Util. Math.* **54** (1998), 257–272.

[23] The Sage Developers, SageMath, the Sage Mathematics Software System (Version 9.2), 2020, http://www.sagemath.org/.

Appendix A 12-regular nut graphs of odd orders

Here, we list one 12-regular nut graph of odd order n for each $n \in \{17, 19, \dots, 39\}$. Each graph is given in the adjacency-lists (of neighbours of each vertex) representation, formatted as a Python dictionary. We also give the corresponding kernel eigenvector **c** as a list of integer entries.

Order $n = 17$.

{0: [1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 15, 16], 1: [0, 2, 3, 4, 6, 7, 8, 9, 10, 11, 15, 16], 2: [0, 1, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15], 3: [0, 1, 4, 6, 7, 8, 9, 11, 12, 14, 15, 16], 4: [0, 1, 2, 3, 5, 6, 8, 9, 10, 11, 13, 16], 5: [0, 2, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15], 6: [1, 2, 3, 4, 5, 7, 8, 9, 12, 13, 14, 15], 7: [1, 2, 3, 5, 6, 8, 10, 11, 12, 13, 14, 16], 8: [0, 1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 14], 9: [0, 1, 2, 3, 4, 5, 6, 10, 12, 13, 14, 16], 10: [0, 1, 2, 4, 5, 7, 8, 9, 12, 14, 15, 16], 11: [0, 1, 2, 3, 4, 7, 8, 12, 13, 14, 15, 16], 12: [0, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 16], 13: [2, 4, 5, 6, 7, 8, 9, 11, 12, 14, 15, 16], 14: [3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16], 15: [0, 1, 2, 3, 5, 6, 10, 11, 12, 13, 14, 16], 16: [0, 1, 3, 4, 7, 9, 10, 11, 12, 13, 14, 15]}

$\mathbf{c} = [3, -3, -2, 2, 1, 2, -1, -2, 3, -1, -1, 1, 1, -1, 1, -1, -2]$

Order $n = 19$.

{0: [1, 2, 5, 7, 9, 10, 11, 12, 13, 14, 16, 18], 1: [0, 3, 5, 6, 7, 10, 12, 13, 14, 15, 17, 18], 2: [0, 4, 6, 7, 8, 9, 10, 11, 12, 16, 17, 18], 3: [1, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18], 4: [2, 5, 6, 7, 8, 11, 12, 13, 14, 15, 17, 18], 5: [0, 1, 4, 7, 8, 9, 11, 12, 13, 14, 15, 17], 6: [1, 2, 3, 4, 7, 8, 9, 10, 14, 15, 16, 17], 7: [0, 1, 2, 3, 4, 5, 6, 8, 9, 11, 15, 16], 8: [2, 3, 4, 5, 6, 7, 9, 11, 14, 15, 17, 18], 9: [0, 2, 5, 6, 7, 8, 10, 11, 12, 13, 16, 17], 10: [0, 1, 2, 3, 6, 9, 11, 12, 13, 14, 16, 18], 11: [0, 2, 3, 4, 5, 7, 8, 9, 10, 16, 17, 18], 12: [0, 1, 2, 3, 4, 5, 9, 10, 13, 14, 15, 16], 13: [0, 1, 3, 4, 5, 9, 10, 12, 14, 15, 16, 17], 14: [0, 1, 3, 4, 5, 6, 8, 10, 12, 13, 15, 18], 15: [1, 4, 5, 6, 7, 8, 12, 13, 14, 16, 17, 18], 16: [0, 2, 3, 6, 7, 9, 10, 11, 12, 13, 15, 18], 17: [1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 15, 18], 18: [0, 1, 2, 3, 4, 8, 10, 11, 14, 15, 16, 17]}

$\mathbf{c} = [5, 10, 6, -10, -3, -1, 4, -1, -5, 1, 1, -5, -4, -3, -4, 2, -4, 7, 4]$

Order $n = 21$.

{0: [1, 2, 3, 4, 5, 6, 7, 15, 17, 18, 19, 20], 1: [0, 2, 3, 4, 5, 6, 7, 16, 17, 18, 19, 20], 2: [0, 1, 3, 4, 5, 6, 8, 16, 17, 18, 19, 20], 3: [0, 1, 2, 4, 5, 6, 7, 8, 9, 18, 19, 20], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 19, 20], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 20], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [0, 1, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13], 8: [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15], 10: [4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16], 11: [5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19], 14: [8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20], 15: [0, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20], 16: [1, 2, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20], 17: [0, 1, 2, 11, 12, 13, 14, 15, 16, 18, 19, 20], 18: [0, 1, 2, 3, 12, 13, 14, 15, 16, 17, 19, 20], 19: [0, 1, 2, 3, 4, 13, 14, 15, 16, 17, 18, 20], 20: [0, 1, 2, 3, 4, 5, 14, 15, 16, 17, 18, 19]}

$\mathbf{c} = [1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1]$

Order $n = 23$.

{0: [1, 2, 4, 6, 7, 8, 10, 11, 13, 19, 20, 21], 1: [0, 4, 5, 6, 7, 9, 11, 13, 16, 17, 20, 22], 2: [0, 3, 4, 6, 8, 11, 12, 13, 16, 19, 20, 21], 3: [2, 4, 5, 8, 9, 10, 12, 13, 14, 16, 17, 18], 4: [0, 1, 2, 3, 6, 7, 8, 14, 15, 16, 21, 22], 5: [1, 3, 7, 10, 11, 12, 14, 15, 17, 18, 19, 20], 6: [0, 1, 2, 4, 11, 12, 14, 17, 18, 19, 20, 22], 7: [0, 1, 4, 5, 10, 11, 12, 16, 18, 19, 21, 22], 8: [0, 2, 3, 4, 9, 10, 12, 13, 15, 16, 21, 22], 9: [1, 3, 8, 10, 11, 13, 14, 15, 18, 19, 21, 22], 10: [0, 3, 5, 7, 8, 9, 11, 13, 17, 18, 19, 21], 11: [0, 1, 2, 5, 6, 7, 9, 10, 13, 14, 15, 20], 12: [2, 3, 5, 6, 7, 8, 13, 14, 17, 19, 20, 22], 13: [0, 1, 2, 3, 8, 9, 10, 11, 12, 14, 15, 19], 14: [3, 4, 5, 6, 9, 11, 12, 13, 15, 16, 17, 20], 15: [4, 5, 8, 9, 11, 13, 14, 17, 18, 20, 21, 22], 16: [1, 2, 3, 4, 7, 8, 14, 17, 18, 20, 21, 22], 17: [1, 3, 5, 6, 10, 12, 14, 15, 16, 19, 20, 21], 18: [3, 5, 6, 7, 9, 10, 15, 16, 19, 20, 21, 22], 19: [0, 2, 5, 6, 7, 9, 10, 12, 13, 17, 18, 22], 20: [0, 1, 2, 5, 6, 11, 12, 14, 15, 16, 17, 18], 21: [0, 2, 4, 7, 8, 9, 10, 15, 16, 17, 18, 22], 22: [1, 4, 6, 7, 8, 9, 12, 15, 16, 18, 19, 21]}

$\mathbf{c} = [6, -24, -7, 13, 39, 1, 27, 4, -18, -4, 10, 3, -14, -14, 28, 1, -22, -2, 3, 6, -28, 2, -10]$

Order $n = 25$.
{0: [3, 4, 5, 7, 9, 10, 12, 13, 17, 19, 22, 23], 1: [2, 3, 5, 11, 12, 15, 16, 18, 19, 20, 21, 23], 2: [1, 3, 4, 5, 10, 13, 14, 17, 20, 21, 23, 24], 3: [0, 1, 2, 5, 8, 10, 14, 16, 20, 21, 23, 24], 4: [0, 2, 6, 8, 9, 10, 11, 13, 18, 21, 23, 24], 5: [0, 1, 2, 3, 10, 13, 14, 17, 18, 19, 20, 24], 6: [4, 8, 9, 10, 11, 12, 14, 17, 19, 20, 21, 22], 7: [0, 8, 9, 11, 12, 15, 16, 18, 19, 22, 23, 24], 8: [3, 4, 6, 7, 9, 10, 11, 13, 17, 18, 22, 23], 9: [0, 4, 6, 7, 8, 10, 11, 12, 14, 15, 18, 21], 10: [0, 2, 3, 4, 5, 6, 8, 9, 15, 16, 17, 18], 11: [1, 4, 6, 7, 8, 9, 12, 13, 14, 17, 19, 20], 12: [0, 1, 6, 7, 9, 11, 13, 14, 15, 18, 21, 22], 13: [0, 2, 4, 5, 8, 11, 12, 16, 20, 21, 22, 23], 14: [2, 3, 5, 6, 9, 11, 12, 15, 16, 17, 19, 22], 15: [1, 7, 9, 10, 12, 14, 16, 17, 19, 20, 22, 24], 16: [1, 3, 7, 10, 13, 14, 15, 17, 18, 19, 20, 24], 17: [0, 2, 5, 6, 8, 10, 11, 14, 15, 16, 21, 23], 18: [1, 4, 5, 7, 8, 9, 10, 12, 16, 21, 22, 24], 19: [0, 1, 5, 6, 7, 11, 14, 15, 16, 21, 22, 24], 20: [1, 2, 3, 5, 6, 11, 13, 15, 16, 22, 23, 24], 21: [1, 2, 3, 4, 6, 9, 12, 13, 17, 18, 19, 23], 22: [0, 6, 7, 8, 12, 13, 14, 15, 18, 19, 20, 24], 23: [0, 1, 2, 3, 4, 7, 8, 13, 17, 20, 21, 24], 24: [2, 3, 4, 5, 7, 15, 16, 18, 19, 20, 22, 23]}

$\mathbf{c} = [29, 20, -31, 7, 5, -13, 32, -19, -12, 1, 31, -12, -8, -6, -49, 17, 3, -17, -21, 20, 33, 7, 1, -2, -16]$

Order $n = 27$.
{0: [2, 3, 4, 5, 6, 7, 21, 22, 23, 24, 25, 26], 1: [2, 3, 4, 5, 6, 7, 8, 22, 23, 24, 25, 26], 2: [0, 1, 3, 4, 5, 6, 7, 8, 23, 24, 25, 26], 3: [0, 1, 2, 4, 5, 6, 7, 8, 9, 24, 25, 26], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 25, 26], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 26], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [0, 1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13], 8: [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15], 10: [4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16], 11: [5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19], 14: [8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20], 15: [9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21], 16: [10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22], 17: [11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23], 18: [12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24], 19: [13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25], 20: [14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26], 21: [0, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26], 22: [0, 1, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26], 23: [0, 1, 2, 17, 18, 19, 20, 21, 22, 24, 25, 26], 24: [0, 1, 2, 3, 18, 19, 20, 21, 22, 23, 25, 26], 25: [0, 1, 2, 3, 4, 19, 20, 21, 22, 23, 24, 26], 26: [0, 1, 2, 3, 4, 5, 20, 21, 22, 23, 24, 25]}

$\mathbf{c} = [1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1]$

Order $n = 29$.
{0: [1, 3, 5, 6, 9, 10, 11, 13, 14, 19, 26, 28], 1: [0, 2, 4, 5, 11, 16, 17, 18, 19, 21, 26, 27], 2: [1, 3, 8, 9, 10, 11, 13, 24, 25, 26, 27, 28], 3: [0, 2, 12, 13, 17, 20, 21, 23, 24, 25, 26, 27], 4: [1, 5, 6, 9, 11, 15,

16, 17, 20, 22, 23, 28], 5: [0, 1, 4, 7, 12, 15, 16, 19, 20, 22, 24, 25], 6: [0, 4, 7, 8, 9, 11, 15, 17, 18, 19, 21, 22], 7: [5, 6, 8, 11, 12, 13, 15, 16, 18, 20, 22, 24], 8: [2, 6, 7, 10, 12, 15, 19, 20, 21, 24, 26, 27], 9: [0, 2, 4, 6, 12, 14, 15, 20, 22, 23, 24, 27], 10: [0, 2, 8, 13, 16, 17, 18, 20, 21, 23, 25, 26], 11: [0, 1, 2, 4, 6, 7, 12, 16, 17, 19, 20, 23], 12: [3, 5, 7, 8, 9, 11, 14, 15, 18, 19, 21, 25], 13: [0, 2, 3, 7, 10, 14, 15, 21, 23, 25, 27, 28], 14: [0, 9, 12, 13, 15, 18, 22, 23, 24, 26, 27, 28], 15: [4, 5, 6, 7, 8, 9, 12, 13, 14, 18, 22, 27], 16: [1, 4, 5, 7, 10, 11, 18, 20, 21, 25, 27, 28], 17: [1, 3, 4, 6, 10, 11, 18, 19, 22, 24, 27, 28], 18: [1, 6, 7, 10, 12, 14, 15, 16, 17, 19, 23, 24], 19: [0, 1, 5, 6, 8, 11, 12, 17, 18, 23, 26, 27], 20: [3, 4, 5, 7, 8, 9, 10, 11, 16, 25, 26, 28], 21: [1, 3, 6, 8, 10, 12, 13, 16, 22, 23, 25, 26], 22: [4, 5, 6, 7, 9, 14, 15, 17, 21, 24, 25, 27], 23: [3, 4, 9, 10, 11, 13, 14, 18, 19, 21, 24, 28], 24: [2, 3, 5, 7, 8, 9, 14, 17, 18, 22, 23, 28], 25: [2, 3, 5, 10, 12, 13, 16, 20, 21, 22, 26, 28], 26: [0, 1, 2, 3, 8, 10, 14, 19, 20, 21, 25, 28], 27: [1, 2, 3, 8, 9, 13, 14, 15, 16, 17, 19, 22], 28: [0, 2, 4, 13, 14, 16, 17, 20, 23, 24, 25, 26]}

$\mathbf{c} = [1, 1, 37, -13, -20, -42, 21, -5, -36, 25, 5, 30, 41, -25, 21, -6, 6, 17, 34, -34, -14, -13, 7, -51, -16, 39, 5, -21, 6]$

Order $n = 31$.
{0: [5, 10, 12, 13, 17, 18, 21, 22, 24, 26, 27, 29], 1: [3, 6, 7, 8, 10, 14, 17, 20, 23, 25, 27, 30], 2: [4, 7, 9, 10, 18, 21, 22, 23, 24, 25, 27, 28], 3: [1, 4, 5, 11, 13, 16, 17, 18, 19, 24, 25, 29], 4: [2, 3, 5, 11, 12, 13, 18, 21, 25, 26, 28, 29], 5: [0, 3, 4, 6, 7, 9, 11, 14, 17, 25, 27, 29], 6: [1, 5, 8, 9, 11, 13, 18, 20, 22, 26, 29, 30], 7: [1, 2, 5, 9, 10, 12, 20, 24, 25, 26, 27, 30], 8: [1, 6, 9, 14, 15, 17, 18, 20, 21, 22, 23, 30], 9: [2, 5, 6, 7, 8, 12, 14, 15, 19, 24, 27, 28], 10: [0, 1, 2, 7, 12, 13, 15, 18, 19, 21, 24, 28], 11: [3, 4, 5, 6, 12, 15, 17, 20, 22, 23, 29, 30], 12: [0, 4, 7, 9, 10, 11, 14, 16, 18, 21, 27, 30], 13: [0, 3, 4, 6, 10, 16, 20, 23, 24, 25, 26, 27], 14: [1, 5, 8, 9, 12, 15, 17, 18, 19, 20, 22, 23], 15: [8, 9, 10, 11, 14, 17, 19, 20, 21, 27, 28, 30], 16: [3, 12, 13, 18, 19, 21, 22, 23, 24, 26, 28, 29], 17: [0, 1, 3, 5, 8, 11, 14, 15, 20, 22, 23, 29], 18: [0, 2, 3, 4, 6, 8, 10, 12, 14, 16, 24, 25], 19: [3, 9, 10, 14, 15, 16, 20, 21, 22, 23, 26, 28], 20: [1, 6, 7, 8, 11, 13, 14, 15, 17, 19, 24, 25], 21: [0, 2, 4, 8, 10, 12, 15, 16, 19, 25, 27, 29], 22: [0, 2, 6, 8, 11, 14, 16, 17, 19, 23, 28, 30], 23: [1, 2, 8, 11, 13, 14, 16, 17, 19, 22, 26, 28], 24: [0, 2, 3, 7, 9, 10, 13, 16, 18, 20, 28, 30], 25: [1, 2, 3, 4, 5, 7, 13, 18, 20, 21, 26, 29], 26: [0, 4, 6, 7, 13, 16, 19, 23, 25, 27, 29, 30], 27: [0, 1, 2, 5, 7, 9, 12, 13, 15, 21, 26, 30], 28: [2, 4, 9, 10, 15, 16, 19, 22, 23, 24, 29, 30], 29: [0, 3, 4, 5, 6, 11, 16, 17, 21, 25, 26, 28], 30: [1, 6, 7, 8, 11, 12, 15, 22, 24, 26, 27, 28]}

$\mathbf{c} = [1, 91, -39, 14, 39, 33, 75, -48, -37, 2, 146, -14, -13, 23, 20, 6, -84, -32, 27, 38, -93, -66, -43, 21, -79, -43, 18, -15, 59, 1, -8]$

Order $n = 33$.
{0: [1, 2, 3, 4, 5, 6, 27, 28, 29, 30, 31, 32], 1: [0, 2, 3, 4, 5, 6, 7, 11, 28, 29, 31, 32], 2: [0, 1, 3, 4, 5, 6, 7, 8, 29, 30, 31, 32], 3: [0, 1, 2, 4, 5, 6, 7, 8, 9, 30, 31, 32], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 31, 32], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 32], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13], 8: [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15], 10: [4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 30], 11: [1, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19], 14: [8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20], 15: [9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21], 16: [10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22], 17: [11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23], 18: [12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24], 19: [13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25], 20: [14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26], 21: [15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27], 22: [16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28], 23: [17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29], 24: [18, 19, 20, 21, 22, 23,

25, 26, 27, 28, 29, 30], 25: [19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31], 26: [20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32], 27: [0, 21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32], 28: [0, 1, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32], 29: [0, 1, 2, 23, 24, 25, 26, 27, 28, 30, 31, 32], 30: [0, 2, 3, 10, 24, 25, 26, 27, 28, 29, 31, 32], 31: [0, 1, 2, 3, 4, 25, 26, 27, 28, 29, 30, 32], 32: [0, 1, 2, 3, 4, 5, 26, 27, 28, 29, 30, 31]}

$\mathbf{c} = [1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1]$

Order $n = 35$.
{0: [1, 2, 3, 4, 5, 6, 29, 30, 31, 32, 33, 34], 1: [0, 2, 3, 4, 5, 6, 7, 30, 31, 32, 33, 34], 2: [0, 1, 3, 4, 5, 6, 7, 8, 15, 31, 32, 33], 3: [0, 1, 2, 4, 5, 6, 8, 9, 15, 32, 33, 34], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 31, 33, 34], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 34], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 21], 8: [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15], 10: [5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 25], 11: [5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 16, 17, 18, 19, 34], 14: [8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20], 15: [2, 3, 9, 10, 11, 12, 14, 16, 17, 18, 19, 20], 16: [10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22], 17: [11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23], 18: [12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24], 19: [13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25], 20: [14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26], 21: [7, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27], 22: [16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28], 23: [17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29], 24: [18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30], 25: [10, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30], 26: [20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32], 27: [21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33], 28: [22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34], 29: [0, 23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34], 30: [0, 1, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34], 31: [0, 1, 2, 4, 26, 27, 28, 29, 30, 32, 33, 34], 32: [0, 1, 2, 3, 26, 27, 28, 29, 30, 31, 33, 34], 33: [0, 1, 2, 3, 4, 27, 28, 29, 30, 31, 32, 34], 34: [0, 1, 3, 4, 5, 13, 28, 29, 30, 31, 32, 33]}

$\mathbf{c} = [1, -1, -1, -3, 3, 2, -1, -1, 1, 1, -2, 2, -2, -1, 3, -1, -1, 2, -2, -2, 5, -1, -1, 1, -2, -2, 6, -3, -1, 1, -1, 5, -1, -4, 1]$

Order $n = 37$.
{0: [1, 2, 3, 4, 5, 6, 31, 32, 33, 34, 35, 36], 1: [0, 2, 3, 4, 5, 6, 7, 18, 22, 32, 33, 35], 2: [0, 1, 3, 4, 5, 6, 7, 8, 33, 34, 35, 36], 3: [0, 1, 2, 4, 5, 6, 7, 8, 9, 34, 35, 36], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 35, 36], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 36], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13], 8: [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 15, 32], 10: [4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16], 11: [5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 19, 36], 14: [8, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 35], 15: [9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21], 16: [10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22], 17: [11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23], 18: [1, 12, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24], 19: [13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25], 20: [14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26], 21: [15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27], 22: [1, 16, 17, 18, 19, 20, 21, 23, 24, 26, 27, 28], 23: [17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29], 24: [18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30], 25: [19, 20, 21, 23, 24, 26, 27, 28, 29, 30, 31, 34], 26: [20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32], 27: [21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33], 28: [22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34], 29: [23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34, 35], 30: [24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36], 31: [0, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36], 32: [0, 1, 9, 26, 27, 28, 29, 30, 31, 33, 34, 36], 33: [0, 1, 2, 27, 28, 29, 30, 31, 32, 34, 35, 36], 34: [0, 2, 3, 25, 28, 29, 30, 31, 32, 33, 35, 36], 35: [0, 1, 2, 3, 4, 14, 29, 30, 31, 33, 34, 36], 36: [0, 2, 3, 4, 5, 13, 30, 31, 32, 33, 34, 35]}

$\mathbf{c} = [2, -3, -4, 5, 1, -1, -1, -4, 5, 2, -5, 1, 1, -1, 6, -5, -4, 7, -1, -5, 4, -5, 3, 6, -5, -5, 8, -3, 1, 1, -4, 3, 4, -7, -1, 3, 1]$

Order $n = 39$.

{0: [1, 2, 3, 4, 5, 6, 15, 33, 34, 36, 37, 38], 1: [0, 2, 3, 4, 5, 6, 7, 34, 35, 36, 37, 38], 2: [0, 1, 3, 4, 5, 6, 7, 8, 35, 36, 37, 38], 3: [0, 1, 2, 4, 5, 6, 7, 8, 9, 36, 37, 38], 4: [0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 37, 38], 5: [0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 38], 6: [0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12], 7: [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13], 8: [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14], 9: [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15], 10: [4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16], 11: [5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 17, 35], 12: [6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18], 13: [7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19], 14: [8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20], 15: [0, 9, 10, 12, 13, 14, 16, 17, 18, 19, 20, 21], 16: [10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22], 17: [11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23], 18: [12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24], 19: [13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25], 20: [14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26], 21: [15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27], 22: [16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28], 23: [17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29], 24: [18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30], 25: [19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31], 26: [20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32], 27: [21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33], 28: [22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34], 29: [23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34, 35], 30: [24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36], 31: [25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37], 32: [26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37, 38], 33: [0, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38], 34: [0, 1, 28, 29, 30, 31, 32, 33, 35, 36, 37, 38], 35: [1, 2, 11, 29, 30, 31, 32, 33, 34, 36, 37, 38], 36: [0, 1, 2, 3, 30, 31, 32, 33, 34, 35, 37, 38], 37: [0, 1, 2, 3, 4, 31, 32, 33, 34, 35, 36, 38], 38: [0, 1, 2, 3, 4, 5, 32, 33, 34, 35, 36, 37]}

$\mathbf{c} = [1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1, 1, -2, 1]$

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Preferential attachment processes approaching the Rado multigraph

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Abstract

We consider a preferential attachment process in which a multigraph is built one node at a time. The number of edges added at stage t, emanating from the new node, is given by some prescribed function $f(t)$, generalising a model considered by Kleinberg and Kleinberg in 2005 where f was presumed constant. We show that if $f(t)$ is asymptotically bounded above and below by linear functions in t, then with probability 1 the infinite limit of the process will be isomorphic to the *Rado multigraph*. This structure is the natural multigraph analogue of the Rado graph, which we introduce here.

Keywords: Preferential attachment, random graphs, multigraphs, Rado graph.

Math. Subj. Class.: 05C80

1 Introduction

In recent decades, there has been much interest in modelling and analysing the many networks which appear in the real world, in contexts such as the world wide web or online social networks. This work has drawn heavily on the mathematical study of random graphs, a subject with its origins in the 1959 work of Erdős and Rényi, [15]. They principally studied the graphs which emerge from the following process: begin with a collection of nodes, and independently connect every pair with an edge, with some fixed probability p.

Erdős-Rényi random graph theory has two distinct facets. First, researchers have analysed the finite graphs which arise. Here, questions of interest include the emergence of a giant component and the degree distribution of the nodes, and analyses are typically highly sensitive to the value of p. In [3], Bollobás provides a comprehensive discussion of such matters.

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The second angle of approach is to consider this process on a countably infinite set of nodes. In this case, a remarkable theorem of Erdős and Rényi guarantees that, irrespective of the value of $p \in (0, 1)$, the resulting graph will with probability 1 be isomorphic to the *Rado graph*. This famous graph is axiomatised by the following schema: given any finite disjoint sets of nodes U and V, there exists a node v connected to each node in V and none in U. This graph exhibits many properties which logicians and combinatorists enjoy. To start with, it is *universal* in that it isomorphically embeds every finite and countably infinite graph. It is also *countably categorical*, meaning that any two countable models of the above axioms will be isomorphic. The graph is *1-transitive* in that for any any two nodes v_1, v_2 there is an automorphism α where $\alpha(v_1) = v_2$. It is *ultrahomogeneous*: any isomorphism between finite induced subgraphs extends to an automorphism of the whole structure. (Analogues of these facts are proved for a new structure, the Rado multigraph, in Proposition 2.2 below.) The Rado graph continues to attract the attention of today's permutation group-theorists; it is known that its automorphism group is simple (in the group-theoretic sense), and satisfies the *strong small-index property*. In [5], Cameron provides a recent survey of such matters. Beyond this, the Rado graph satisfies several subtler properties, notably *rank-1 supersimplicity* and *1-basedness*, which make it a central object of study for today's model theorists. Wagner provides an authoritative account in [17].

In more recent years, however, network science has grown beyond the Erdős-Rényi approach, to embrace alternative methods for modelling real-world networks. The most prominent of these is perhaps the *preferential attachment* (PA) mechanism introduced by Barabási and Albert in [2]. Another notable class of models derive from the *web-copying* mechanism introduced by Adler and Mitzenmacher in [1].

In PA models, a new node is introduced at each time step, and then connected to each pre-existing node with a probability depending on the current degree of that node, according to a *rich-get-richer* paradigm. PA processes can exhibit several properties observed in real-world networks (but absent in Erdős-Rényi graphs), notably *scale-freeness* meaning that the proportion of nodes of degree k is asymptotically proportional to $k^{-\gamma}$ for some fixed γ and all k.

What can we say of the infinite limits of these processes? Results of Bonato and Janssen [4] have made significant progress for web-copying models. Less work has been done in the case of PA processes. The work of Oliveira and Spencer [14] studying the *Growing Network* model of Krapivsky and Redner [11] and of Drinea, Enachescu, and Mitzenmacher [7] is a notable exception. Of greatest relevance to the current paper, however, is the work of Kleinberg and Kleinberg [10]. There the following PA process is considered: at each time-step, a single node and a constant number C of edges are added. The new edges all emanate from the new node, with their end-points independently chosen among the pre-existing nodes, with probabilities proportional to their degrees. The resulting structures are analysed as *directed multigraphs*: all edges are directed, two or more may share the same start and end-points.

Kleinberg and Kleinberg prove that if $C = 1$ or $C = 2$, then there exists an infinite structure, to which, with probability 1, the infinite limit of the process will be isomorphic. However, the analogous result fails for $C \geq 3$: given two instantiations of the process, there is a positive probability that their infinite limits will fail to be isomorphic.

In this paper we extend the results and methods of [10], by considering a process which adds $f(t)$ many edges at stage t for some function $f\colon \mathbf{N} \to \mathbf{N}$. Again the start-point of every edge is the new node, and the end-points are chosen independently with probability proportional to the nodes' degrees. It follows easily from the results of [10] that whenever f is non-constant, or constant with value ≥ 3, there is a positive probability that the infinite limiting structures of two instantiations will be non-isomorphic as directed multigraphs. However, by forgetting the directions of edges, and looking for isomorphisms as multigraphs, we are able to recover a new categoricity result. In Theorem 3.2 we rigorously establish a sufficient criterion for the resulting structure to be isomorphic to the *Rado multigraph* with probability 1. (This structure is the natural multigraph analogue of the Rado graph, and is defined in Definition 2.1 below.) Our criterion is that f is asymptotically bounded above and below by positive non-constant linear functions of t.

In [9], the author uses similar machinery to analyse a Preferential Attachment process in which parallel edges are not permitted, and the new node $t+1$ is connected to each pre-existing node u independently with probability $\frac{d_u(t)}{t}$. Thus the number of new edges is not prescribed, but is itself a random variable. It is shown in [9] that, so long as the initial graph is neither edgeless nor complete, with probability 1 the infinite limit of the process will be the Rado graph augmented with a finite number of either universal or isolated nodes.

We describe the structure of the remainder of the paper:

- In Section 2 we introduce the infinite *Rado Multigraph*.
- In Section 3 we introduce MPA_f, our main variant of the preferential attachment process, as well as a secondary variant GPA_f. We describe suitable hypotheses on the function f, and we prove some initial results. We state our main result, Theorem 3.2, which asserts that under appropriate conditions MPA_f approaches the Rado multigraph.
- In Section 4 we develop the theory of martingales for the process MPA_f, our main probabilistic tool.
- In Section 5, we complete the proof of Theorem 3.2.
- In Section 6, we close with some discussion of possible further directions of study.

2 The Rado multigraph

We begin by defining the infinite structure which our finitary processes will be shown to approach. So far as we are aware, this structure has not previously appeared in the literature. However the reader familiar with the Rado graph will find little of surprise. (For clarity, we work with the convention that $0 \in \mathbf{N}$.)

Definition 2.1. A *finitary loopless multigraph* is a structure (V, E) where V is a set of vertices, and E is a finitary multiset of unordered pairs from V. That is to say every element of E is of the form $e = \{v_i, v_j\}$ (written $v_i v_j$) where $v_i, v_j \in V$ are distinct, E is itself unordered, and each e has a multiplicity $m_e \in \mathbf{N}$ describing the number of occurrences of e within E. (If e does not occur within E we consider it to have multiplicity 0.)

The Rado Multigraph is a finitary loopless multigraph where V is countably infinite and which additionally satisfies the following axiom:

• For any $n \geq 1$, any $m_1, \ldots, m_n \in \mathbf{N}$, and any distinct $u_1, \ldots, u_n \in V$ there exists $v \in V$ such that the multiplicity of vu_i is exactly m_i.

We now show that the Rado Multigraph is unique up to isomorphism. We also take the opportunity to observe that several other familiar properties of the Rado graph hold in our context although we shall not use them directly:

Proposition 2.2. *Let $\mathcal{M}$ and $\mathcal{M}'$ be structures satisfying* Definition 2.1 *of the Rado Multigraph. Then the following hold:*

1. $\aleph_0$-categoricity: $\mathcal{M} \cong \mathcal{M}'$.

2. 1-transitivity: *Given vertices v_1, v_2 in $\mathcal{M}$ there exists some $\alpha \in \mathrm{Aut}(\mathcal{M})$ where $\alpha(v_1) = v_2$.*

3. Ultrahomogeneity: *If A, B are finite substructures of $\mathcal{M}$ and $\gamma : A \cong B$ is a multigraph-isomorphism, then there exists $\alpha \in \mathrm{Aut}(\mathcal{M})$ where $\alpha \restriction_A = \gamma$.*

   (Note: here we treat A and B as induced substructures*: for any vertices $u, v \in A$ the multiplicity of uv within A equals that within $\mathcal{M}$).*

4. Universality: *Any finite or countably infinite finitary loopless multigraph can be isomorphically embedded in $\mathcal{M}$.*

Proof. We concentrate on proving statement 1. (Statements 2-4 follow from minor alterations to our argument. We leave the reader to fill in the details.) We proceed by a standard back-and-forth argument. First we list the elements of $\mathcal{M}$ as $a_1, a_2, a_3, \ldots$ and similarly $b_1, b_2, b_3, \ldots$ for $\mathcal{M}'$. Now we argue inductively. Suppose i is even, and suppose $(a'_1, \ldots, a'_i) \cong (b'_1, \ldots, b'_i)$ have been chosen. Let $a'_{i+1} = a_j$ where j is minimum such that $a_j \notin \{a'_1, \ldots, a'_i\}$.

Let $(m_1, \ldots, m_i)$ be the vector counting the edges between a'_{i+1} and $(a'_1, \ldots, a'_i)$. Notice that each $m_j \in \mathbf{N}$ by the assumption of finitariness. Then by hypothesis there exists b'_{i+1} joined to $(b'_1, \ldots, b'_i)$ in a fashion described by $(m_1, \ldots, m_i)$. Hence $(a'_1, \ldots, a'_{i+1}) \cong (b'_1, \ldots, b'_{i+1})$.

Odd steps are identical, exchanging the roles of $\mathcal{M}$ and $\mathcal{M}'$. Thus we build an isomorphism $\mathcal{M} \cong \mathcal{M}'$. □

Our concern in the current work is on PA processes. However, we remark in passing that the Rado multigraph also arises from the following process in the style of Erdős-Rényi. We shall not use this result and leave the proof as an easy adaptation of the corresponding classical result about the Rado graph.

Proposition 2.3. *Let $(p_j)_{j \geq 1}$ be any sequence lying entirely in $(0, 1)$. Let V be a countably infinite set. Let $\mathcal{M}$ be multigraph arising from the following random process.*

- *At step 0, the structure has no edges.*

- *At step $j \geq 1$, consider every pair of distinct $v_1, v_2 \in V$ where $v_i v_j$ currently has multiplicity $j - 1$, and connect v_1, v_2 with a new j^{th} edge with probability p_j, independently of the behaviour of all other vertices.*

Then with probability 1*, $\mathcal{M}$ is isomorphic to the Rado multigraph.*

3 Preferential attachment with prescribed edge growth

In this section we shall describe two variants of the preferential attachment process, establish some of their basic properties, and formally state our main result. The process of principal interest will be MPA_f which builds a directed multigraph. We will also mention a natural variant GPA_f, which builds a directed graph. Each proceeds by adding, at each time step, a single node along with a prescribed number of directed edges emanating from it. The number of these edges is determined by some fixed function $f : \mathbf{N} \to \mathbf{N}$. (In fact the directions of the edges will play no role in the theory: we shall analyse the resulting structures as undirected (multi)graphs. However in the interim it will be convenient to refer to the 'start-' and 'end-points' of each edge, so we preserve directedness for the time being.) We shall work over some initial directed (multi)graph G' containing no isolated nodes (i.e. nodes of degree 0). However our results will be independent of the choice of G', so the reader may choose to focus on the case where G' is trivial.

Definition 3.1 (The process MPA_f)**.** Let $G' = (V', E')$ be a finite directed multigraph containing no isolated nodes (so E' is a multiset of ordered pairs from V'). Suppose that G' contains $|E'| = e'$ edges and $|V'| = v'$ nodes. We will assume $V' = \{1, \ldots, v'\}$.

Suppose that the function $f : \mathbf{N} \to \mathbf{N}$ satisfies:

- $f(0) = e'$.
- $f(t) = 0$ whenever $1 \leq t \leq v' - 1$.
- $f(t) \geq 1$ for all $t \geq v'$.

At each time-step $t \geq 1$, we shall construct a multigraph $G(t)$ with vertex set $V(t)$ and edge multiset $E(t)$.

First we impose $G(1) = \ldots = G(v') = G'$.

Whenever $t \geq v'$, we will have $V(t) := \{1, \ldots, t\}$ and

$$E(t+1) = E(t) \cup \mathcal{E}(t+1)$$

where $|\mathcal{E}(t+1)| = f(t)$.

The start-point of each edge in $\mathcal{E}(t+1)$ is the new node $t+1$. The end-points are chosen independently from $V(t)$, with probabilities directly proportional to their degrees in $G(t)$.

Notice that, the degrees used to calculate the probabilities are taken from $G(t)$, which is to say the model does not notice any incremental updating of degrees between $G(t)$ and $G(t+1)$. One can imagine the endpoints of the $f(t)$ many new edges being selected simultaneously, and independently of each other.

Notice too that our assumption that $f(t) \neq 0$ for $t \geq v'$ (along with our assumption on G') serves to ensure that there are never any isolated nodes.

We may now state our main result. (Recall the asymptotic notation $g_1 = \Theta(g_2)$ for functions g_1, g_2 as meaning that there exist $c_2 \geq c_1 > 0$ so that for all large enough t we have $c_1 \cdot g_2(t) \leq g_1(t) \leq c_2 \cdot g_2(t)$.)

Theorem 3.2. *Suppose that G' is a finite directed multigraph containing no isolated nodes, that f satisfies the requirements from* Definition 3.1*, and also that $f(t) = \Theta(t)$. Then, with probability 1, the infinite limit of $\mathrm{MPA}_f(G')$ is isomorphic, as an undirected multigraph, to the Rado multigraph.*

Before we commence the proof of this theorem, we remark that we expect that a similar result to apply to a graph variant of the process, which we briefly introduce:

Definition 3.3 (The process GPA$_f$)**.** Let $G' = (V', E')$ be a finite directed graph containing no isolated nodes. Suppose that G' contains $|E'| = e'$ edges and $|V'| = v'$ nodes. We will assume $V' = \{1, \dots, v'\}$.

Suppose that the function $f\colon \mathbf{N} \to \mathbf{N}$ satisfies:

- $f(0) = e'$.
- $f(t) = 0$ whenever $1 \leq t \leq v' - 1$.
- $1 \leq f(t) \leq t$ for all $t \geq v'$.

At each time-step $t \geq 1$, we shall construct a graph $G(t)$ with vertex set $V(t)$ and edge set $E(t)$.

First we impose $G(1) = \dots = G(v') = G'$.

Whenever $t \geq v'$, we will have $V(t) := \{1, \dots, t\}$ and

$$E(t+1) = E(t) \cup \mathcal{E}(t+1)$$

where $|\mathcal{E}(t+1)| = f(t)$.

The start-point of each edge in $\mathcal{E}(t+1)$ is the new node $t+1$. The end-points of the edges are selected sequentially from $V(t)$, without replacement, with the choice at each step made from the remaining unselected elements of $V(t)$ with probabilities directly proportional to their degrees in $G(t)$.

Conjecture 3.4. *Suppose that $G' = (V', E')$ be a finite directed graph containing no isolated nodes, that f satisfies the conditions in* Definition 3.3*, and further that there are constants $0 < c_1 \leq c_2 < 1$ where $c_1 \cdot t \leq f(t) \leq c_2 \cdot t$ for all large enough t. Then, with probability 1, the infinite limit of GPA$_f(G')$ is isomorphic as an undirected graph to the Rado graph.*

Our arguments will be independent of G', and thus we shall largely suppress mention of it. Let us now consider the distribution of edges at stage $t+1$. First notice that $|E(t)| = F(t) := \sum_{i=0}^{t-1} f(i)$. Hence in MPA$_f$, at stage $t+1$ given any pre-existing node $u \leq t$, the probability that any given edge in $\mathcal{E}(t+1)$ has its end-point at u is exactly $\frac{d_u(t)}{2F(t)}$, where $d_u(t)$ is the degree of u in $G(t)$. Thus the expected number of edges in $\mathcal{E}(t+1)$ with endpoint at u is $\frac{f(t) \cdot d_u(t)}{2F(t)}$.

In GPA$_f$ this probability distribution is more complicated, and the expected number of edges u receives at stage $t+1$ depends in a more detailed way upon $G(t)$. This is the primary obstacle to extending the current work to a proof of Conjecture 3.4.

Our standing assumption will be that we are working in MPA$_f$. We shall leave the case of GPA$_f$ open, but make some remarks about it as we proceed.

Our assumption in Theorem 3.2 is that $f(t) = \Theta(t)$. However we shall be able to develop much of the theory under the following weaker hypotheses:

Assumption 3.5.

$$\sum_{s=0}^{\infty} \frac{f(s)}{F(s)} = \infty. \tag{3.1}$$

$$\sum_{s=0}^{\infty} \left(\frac{f(s)}{F(s)} \right)^2 < \infty. \tag{3.2}$$

We briefly discuss this. Assumption 3.5 easily follows in full, for instance, if $f(t) = \Theta(t^{\alpha})$ for some $\alpha \geq 0$.

However part (2) fails in general for polynomially bounded functions, an example being:

$$f(t) = \begin{cases} t & \text{when } t = 2^n \text{ for } n \in \mathbf{N} \\ 1 & \text{otherwise.} \end{cases}$$

On the other hand, both parts do hold for some exponential functions, such as $f(t) = \lfloor \frac{1}{4} t^{-\frac{3}{4}} e^{\frac{1}{4}t} \rfloor$.

In all cases, it will be useful to extend the domain of f to $\mathbf{R}^{\geq 0}$. We choose to do this as a step function, via $f(t) := f(\lfloor t \rfloor)$. (Of course there may be more natural ways to achieve the same thing, however this choice will be convenient, as the fourth point in the following Lemma makes clear.) We now gather together some observations about the extended function f. These follow immediately from our previous conditions.

Corollary 3.6. *The following hold:*

- $f(t) = e'$ *for* $0 \leq t < 1$.
- $f(t) = 0$ *whenever* $1 \leq t < v'$.
- $f(t) \geq 1$ *for all* $t \geq v'$.
- f *is Lebesgue-measurable with antiderivative* $\int_0^t f(s)ds =: F(t)$. *(This notation is consistent with the previous interpretation of* F *since the two functions coincide at integer points.)*
- F *is monotonic increasing everywhere and strictly so for* $t \geq v'$.

Under our additional hypothesis we can say a little more:

Lemma 3.7. *Suppose that* Assumption 3.5(2) *holds. Then for any* $\beta \geq 1$, *there exists* $K_\beta > 0$ *so that for any* $t \geq m \geq 0$*:*

$$\left| \int_m^t \frac{f(s)}{F(s)^\beta} ds - \sum_{s=m}^{t} \frac{f(s)}{F(s)^\beta} \right| < K_\beta.$$

Proof. Let M be such that whenever $s \geq M$ then $f(s) < F(s)$. Such a value must exist by Assumption 3.5(2).

It is enough to prove the result this for all $m \geq M$, since one can then add

$$\max \left\{ \int_0^M \frac{f(s)}{F(s)^\beta} ds, \sum_{s=0}^{M} \frac{f(s)}{F(s)^\beta} \right\}$$

to K_β to obtain the result for all m. Thus we shall assume $m \geq M$.

Firstly, it is immediate by consideration of $F \restriction_{[s,s+1]}$ that

$$\sum_{s=m}^{t} \frac{f(s)}{F(s+1)^{\beta}} < \int_{m}^{t} \frac{f(s)}{F(s)^{\beta}} ds < \sum_{s=m}^{t} \frac{f(s)}{F(s)^{\beta}}.$$

Next we shall appeal to Newton's generalised binomial theorem, that whenever $a, b, \beta \in \mathbf{C}$ with $0 < |b| < |a|$, then $(a+b)^{\beta} = \sum_{j=0}^{\infty} C(\beta, j) a^{\beta-j} b^{j}$, where $C(\beta, j)$ are the generalised binomial coefficients.

When $a = 1$, the series has radius of convergence 1 in b. We shall also use the fact that the series remains convergent for $|b| = 1$, so long as Re$(\beta) > 0$, which of course holds in the context of this Lemma. (See [6] p.17, for example.) Now,

$$\begin{aligned}
\sum_{s=m}^{t} \frac{f(s)}{F(s)^{\beta}} - \sum_{s=m}^{t} \frac{f(s)}{F(s+1)^{\beta}} &= \sum_{s=m}^{t} \frac{f(s)}{F(s)^{\beta}} - \frac{f(s)}{(F(s)+f(s))^{\beta}} \\
&< \sum_{s=m}^{t} \frac{f(s)\,(F(s)+f(s))^{\beta} - f(s)F(s)^{\beta}}{F(s)^{2\beta}} \\
&= \sum_{s=m}^{t} \frac{f(s)\left(\sum_{j=1}^{\infty} C(\beta,j) F(s)^{\beta-j} f(s)^{j}\right)}{F(s)^{2\beta}} \\
&< \sum_{s=m}^{t} \frac{\sum_{j=1}^{\infty} C(\beta,j) F(s)^{\beta-1} f(s)^{2}}{F(s)^{2\beta}} \\
&< \sum_{s=m}^{t} \frac{2^{\beta} f(s)^{2}}{F(s)^{1+\beta}} \leq 2^{\beta} \sum_{s=m}^{t} \frac{f(s)^{2}}{F(s)^{2}} < 2^{\beta} \cdot K := K_{\beta}
\end{aligned}$$

where K is the finite bound provided in Assumption 3.5(2).

□

The next two results hold in GPA$_f$ as well as MPA$_f$:

Lemma 3.8. *Suppose that* Assumption 3.5(1) *holds. Then for any node u, any stage t_0, and any state of the graph $G(t_0)$, the probability that v never receives another edge is* 0.

Proof. Suppose that $d_u(t_0) = N \geq 1$. The probability that u never receives a further edge is therefore given by (or in GPA$_f$ is bounded above by)

$$\prod_{t=t_0}^{\infty} \left(1 - \frac{N}{2F(t)}\right)^{f(t)}.$$

We shall show that this is 0. It is clearly enough to do so in the case $N = 1$. Taking logarithms, it is therefore enough to show that

$$\sum_{t=t_0}^{\infty} f(t) \ln\left(1 + \frac{1}{2F(t)-1}\right)$$

diverges to ∞. Now as for small enough x, we know $\ln(1+x) > \frac{1}{2}x$. Thus for large enough t,

$$\ln\left(1 + \frac{1}{2F(t)-1}\right) > \frac{1}{4F(t)}.$$

Thus the result follows from Assumption 3.5(1). $\square$

Corollary 3.9. *Suppose that* Assumption 3.5(1) *holds. Then for any node u, given any state of the graph $G(t_0)$, with probability* 1 *it will be true that $d(t) \to \infty$ as $t \to \infty$.*

Proof. This follows automatically from Lemma 3.8 by the countable additivity of the probability measure. $\square$

4 Martingale theory

In this section, we apply some machinery from the theory of martingales to the process MPA_f, generalising the theory developed in [10]. We shall assume throughout that we are working in MPA_f, and begin with the following easy result, which does not transfer immediately to GPA_f.

Remark 4.1. Given any node u, define $U_u(t+1) := d_u(t+1) - d_u(t)$ and $\mu_u(t) := \mathbf{E}\left(U_u(t+1)\middle|\middle|d_u(t)\right)$. Then

$$\frac{\mu_u(t)}{d_u(t)} = \frac{f(t)}{2F(t)}.$$

In particular, if $f(t) = \Theta\left(t^\alpha\right)$ where $\alpha \geq 0$ then $\mu_u(t) = \Theta\left(\frac{d_u(t)}{t}\right)$.

The next two results are the key to our analysis, and generalise Proposition 3.1 of [10]

Proposition 4.2. *Suppose that* Assumption 3.5(2) *holds. For any node u, define*

$$A(t) = A_u(t) := \prod_{j=1}^{t-1}\left(1 + \frac{f(j)}{2F(j)}\right)$$

and $X(t) := X_u(t) = \frac{d_u(t)}{A_u(t)}$. Then

(i) *$X(t)$ is a martingale.*

(ii) *Thus, for any node u, with probability 1, there exists $x_u \geq 0$ such that*

$$\lim_{t\to\infty} \frac{d_u(t)}{A(t)} = x_u.$$

(iii) $A(t) = \Theta\left(F(t)^{\frac{1}{2}}\right)$

Proof. Employing Remark 4.1, the first part is straightforward:

$$\begin{aligned}\mathbf{E}(X(t+1)||X(t)) &= \frac{1}{A(t+1)}\mathbf{E}(d(t+1)||d(t))\\ &= \frac{1}{A(t+1)}\left(d(t)+\mu(t)\right)\\ &= \frac{1}{A(t+1)}d(t)\left(1+\frac{\mu(t)}{d(t)}\right)\\ &= \frac{1}{A(t)}d(t) = X(t).\end{aligned}$$

Part (ii) follows from (i) via Doob's convergence theorem, which gives us that $X(t) \to X$ for some random variable X.

Hence all that remains to understand the behaviour of $A(t)$ for large t, to establish (iii). By taking logarithms and employing the standard bounds $x - \frac{1}{2}x^2 < \ln(1+x) < x$, we see:

$$\frac{1}{2}\sum_{s=1}^{t-1}\frac{f(s)}{F(s)} - \frac{1}{8}\sum_{s=1}^{t-1}\frac{f(s)^2}{F(s)^2} < \ln A(t) < \frac{1}{2}\sum_{s=1}^{t-1}\frac{f(s)}{F(s)}.$$

Therefore by Assumption 3.5 and Lemma 3.7, it follows that

$$\frac{1}{2}\int_{s=1}^{t-1}\frac{f(s)}{F(s)}ds - K < \ln A(t) < \frac{1}{2}\int_{s=1}^{t-1}\frac{f(s)}{F(s)}ds + K'$$

for some constants K and K' from which the result follows. □

We need a little more information about the distribution of the x_u provided by the preceding result:

Proposition 4.3. *Suppose that f satisfies* Assumption 3.5 *in full. Given any time t_0, state $G_0(t_0)$, and node u,* $\mathbf{P}\left(x_u > 0\right) = 1$.

Proof. Our proof closely follows that of Proposition 3.1 of [10].

We take u as fixed and shall suppress mention of it, writing $X(n)$ for $X_u(n)$, etc., throughout.

Given any $n > m > 0$ define $\tilde{X}_m(n) := \left(X(n) - X(m)\right)^2$. Then for fixed m, it is an elementary fact that the sequence $\tilde{X}_m(n)$ forms a submartingale. We now proceed via a sequence of claims.

Claim 1

$$\mathbf{E}\left(\tilde{X}_m(n)\big|\big|X(m)\right) = \sum_{t=m}^{n-1}\mathbf{E}\Big(X(t+1)^2\big|\big|X(m)\Big) - \mathbf{E}\Big(X(t)^2\big|\big|X(m)\Big).$$

Proof of Claim 1.

$$\begin{aligned}\mathbf{E}\Big(\tilde{X}_m(n)\big|\big|X(m)\Big) &= \mathbf{E}\Big(X(n)^2 - 2X(n)X(m) + X(m)^2\big|\big|X(m)\Big)\\ &= \mathbf{E}\Big(X(n)^2\big|\big|X(m)\Big) - 2X(m)\mathbf{E}\Big(X(n)\big|\big|X(m)\Big) + X(m)^2\\ &= \mathbf{E}\Big(X(n)^2\big|\big|X(m)\Big) - X(m)^2.\end{aligned}$$

Unpacking the sum in the statement of the claim gives the same result. □

Claim 2 There exists $K > 0$ such that for all large enough m and all $n > m$

$$\mathbf{E}(\tilde{X}_m(n)||X(m)) < X(m) \cdot \frac{K}{F(m)^{\frac{1}{2}}}.$$

Proof of Claim 2

Proof of Claim 2. Recall $U(t+1) := d(t+1) - d(t)$. Now $U(t+1)$ is binomially distributed via $b\left(f(t), \frac{d(t)}{2F(t)}\right)$ meaning, as already observed, that $\mathbf{E}(U(t+1)||d(t) = d) = \mu(t) = \frac{d \cdot f(t)}{2F(t)}$ and also $\mathbf{Var}(U(t+1)||d(t) = d) = \frac{d \cdot f(t)}{2F(t)}\left(1 - \frac{d}{2F(t)}\right)$. Thus, writing f and F for $f(t)$ and $F(t)$ respectively,

$$\begin{aligned}\mathbf{E}\Big(U(t+1)^2||d(t) = d\Big) &= \left(\frac{df}{2F}\right)^2 + \frac{df}{2F}\left(\frac{2F-d}{2F}\right) \\ &< \frac{fd}{2F} + \frac{f^2d^2}{4F^2}.\end{aligned}$$

At the same time,

$$\begin{aligned}&\mathbf{E}\Big(d(t+1)^2||d(t) = d\Big) \\ &= \mathbf{E}\Big(\left(U(t+1) + d\right)^2||d(t) = d\Big) \\ &= \mathbf{E}(U(t+1)^2||d(t) = d) + 2d\mathbf{E}(U(t+1)||d(t) = d) + d^2 \\ &< \frac{fd}{2F} + \frac{f^2d^2}{4F^2} + 2d \cdot \frac{df}{2F} + d^2 \\ &= \frac{fd}{2F} + \left(1 + \frac{f}{2F}\right)^2 d^2.\end{aligned}$$

Recall the definition of the martingale $X(t) := \frac{d(t)}{A(t)}$. Thus

$$\begin{aligned}\mathbf{E}\Big(X(t+1)^2\,\Big|\Big|d(t) = d\Big) &= \left(\frac{1}{A(t+1)^2}\right) \cdot \mathbf{E}\Big(d(t+1)^2\,\Big|\Big|\,d(t) = d\Big) \\ &< \frac{1}{A(t+1)^2}\left(\frac{fd}{2F} + \left(1 + \frac{f}{2F}\right)^2 d^2\right) \\ &= \frac{fA(t)}{2F \cdot A(t+1)^2}X(t) + \left(1 + \frac{f}{2F}\right)^2\left(\frac{A(t)}{A(t+1)}\right)^2 X(t)^2 \\ &< \frac{f}{2F \cdot A(t)}X(t) + X(t)^2.\end{aligned}$$

Hence, by the law of total expectation,

$$\mathbf{E}(X(t+1)^2||X(m)) - \mathbf{E}(X(t)^2||X(m)) < \frac{f}{2F \cdot A(t)}X(m).$$

Summing this up over successive terms (and appealing to Claim 1, Proposition 4.2(ii) and Lemma 3.7) we get

$$\begin{aligned}
\mathbf{E}(\tilde{X}_m(n)||X(m)) &< X(m) \cdot \sum_{t=m}^{n-1} \frac{f}{2FA(t)} \\
&< X(m) \cdot \sum_{t=m}^{n-1} \frac{f}{2FA(t)} \\
&= O\left(X(m) \cdot \sum_{t=m}^{n-1} \frac{f}{F^{\frac{3}{2}}} \right) \\
&= O\left(X(m) \cdot \int_{t=m}^{n-1} \frac{f(t)}{F(t)^{\frac{3}{2}}} dt \right) \\
&< X(m) \cdot \frac{K}{F(m)^{\frac{1}{2}}} \text{ for some } K > 0. \qquad \square
\end{aligned}$$

Proof of Proposition 4.3, *continued:* We may now prove the proposition. We proceed by defining a sequence of times: $n_0 = t_0$. Let n_{i+1} be the least n (if any exists) such that $X(n) < \frac{1}{2} X(n_i)$. Otherwise $n_{i+1} = \infty$.

The trick is to apply the Kolmogorov-Doob inequality (see for instance [10]) to $\tilde{X}_{n_i}(n)$:

$$\begin{aligned}
\mathbf{P}(n_{i+1} < \infty || n_i < \infty) &= \mathbf{P}\left(\min_{n \geq n_i} X(n) < \frac{1}{2} X(n_i) \Big|\Big| X(n_i) \right) \\
&\leq \mathbf{P}\left(\max_{n \geq n_i} \tilde{X}_{n_i}(n) > \frac{1}{4} X(n_i)^2 \Big|\Big| X(n_i) \right) \\
&= \lim_{N \to \infty} \mathbf{P}\left(\max_{n: N \geq n \geq n_i} \tilde{X}_{n_i}(n) > \frac{1}{4} X(n_i)^2 \Big|\Big| X(n_i) \right) \\
&\leq \frac{4}{X(n_i)^2} \cdot \lim_{N \to \infty} \mathbf{E}(\tilde{X}_{n_i}(N) || X(n_i)) \\
&= O\left(\frac{4}{X(n_i)^2} \cdot \frac{1}{F(n_i)^{\frac{1}{2}}} \cdot X(n_i) \right) \\
&= O\left(\frac{1}{d(n_i)} \right).
\end{aligned}$$

It follows from Corollary 3.9 that $\mathbf{P}(n_{i+1} < \infty || n_i < \infty) \to 0$ as $i \to \infty$, from which the result follows. $\square$

We record one more result regarding the martingale $X(t)$:

Corollary 4.4. *Suppose that* Assumption 3.5(2) *holds. Then the martingale $X(t)$ is bounded in $\mathcal{L}_2$, that is to say* $\sup_t \mathbf{E}\left(X(t)^2\right) < \infty$.

Proof. By a standard result (see for example Theorem 12.1 of [18]), it is sufficient to show that $\sum_{j=0}^{\infty} \mathbf{E}\left(|X_{j+1} - X_j|^2\right) < \infty$.

Notice that by Remark 4.1

$$|X(t+1) - X(t)| = \frac{d(t+1) - \left(1 + \frac{\mu(t)}{d(t)}\right) d(t)}{A(t+1)}$$
$$= \frac{U(t+1) - \mu(t)}{A(t+1)}$$

Hence

$$\mathbf{E}\left(|X(t+1) - X(t)|^2\right) = \frac{\mathbf{Var}(U(t+1))}{A(t+1)^2}$$
$$= O\left(\frac{d(t)f(t)}{2F(t)}\left(1 - \frac{d(t)}{2F(t)}\right) \cdot \frac{1}{A(t+1)^2}\right)$$
$$= O\left(\frac{d(t)}{A(t+1)} \cdot \frac{f(t)}{F(t)A(t+1)}\right)$$
$$= O\left(X(t) \cdot \frac{f(t)}{F(t)A(t+1)}\right)$$
$$= O\left(\frac{f(t)}{F(t)A(t+1)}\right)$$
$$= O\left(\frac{f(t)}{F(t)^{\frac{3}{2}}}\right).$$

Thus

$$\sum_{j=0}^{t} \mathbf{E}\left(|X_{j+1} - X_j|^2\right) = O\left(\int_{j=0}^{t} \frac{f(j)}{F(j)^{\frac{3}{2}}} dj\right)$$
$$= O\left(K - F(t)^{-\frac{1}{2}}\right) = O(K). \qquad \square$$

5 Proof of main result

Definition 5.1. A *witness request* W is a set of pairs of the form $W = \{(u_i, m_i) \mid 1 \leq i \leq n\}$ where $(u_1, \ldots, u_n)$ is a sequence of nodes and $(m_1, \ldots, m_n)$ an accompanying sequence of non-negative integers.

A *witness* for W is a node connected to each u_i with multiplicity m_i.

We write the event $W[t]$ to mean that W is satisfied by some witness by time t.

Observe from the structure of the process that $W[t] \Rightarrow W[t']$ for all $t' \geq t$. The following is the major step towards our goal:

Proposition 5.2. *Suppose that $f(t) = \Theta(t)$, and that $G(t_0)$ is a state of the graph at time t_0. Let W be a witness request. Let $\varepsilon > 0$. Then there exist $t_1 > t_0$ such that $\mathbf{P}\Big(W[t_1] \,||\, G(t_0)\Big) > 1 - \varepsilon$.*

Proof. We consider only stages from $t_0 + 1$ onwards, and everything that occurs is conditioned upon $G(t_0)$, which we shall therefore suppress.

Suppose $W = \{(u_i, m_i) \mid 1 \leq i \leq n\}$. We shall write $m = \sum_{i=1}^{n} m_i$, and, abusing notation, $U_i = U_{u_i}(t+1)$, meaning the number of new edges which u_i gains at the $t+1$st stage, taking the dependency on t as given when the intended value is obvious. Similarly we write d_i for $d_{u_i}(t)$. (We shall not consider d_j for any j other than the u_i, so this will not cause confusion.)

We shall employ vector notation, writing $\mathbf{U}(t+1) := \mathbf{U} = (U_1, \ldots, U_n)$ and $\mathbf{m} := (m_1, \ldots, m_n)$. Thus our focus is the event $\mathbf{U} = \mathbf{m}$. Let us first compute the probability of this event in terms of the d_i. The relevant distribution is multinomial $M(f, p_i, \ldots, p_n, q)$ where $p_i = \frac{d_i}{2F}$ and $q = 1 - \sum p_i$ (again omitting the dependencies on t). Therefore

$$\begin{aligned}\mathbf{P}\left(\mathbf{U} = \mathbf{m} \,||\, d_1, \ldots d_n\right) &= \frac{f!}{m_1! \cdot \ldots m_n! \cdot (f-m)!} \cdot q^{f-m} \cdot \prod_i {p_i}^{m_i} \\ &= \Theta\left(\left(1 - \frac{\sum_i d_i}{2F}\right)^{f-m} \cdot \left(\frac{f}{2F}\right)^m \cdot \prod_i {d_i}^{m_i}\right)\end{aligned}$$

noticing that $\frac{f!}{(f-m)!} \sim f^m$.

Now we employ our assumption that $f(t) = \Theta(t)$ from which it also follows that $\frac{1}{2F} = \Theta\left(\frac{1}{t^2}\right)$ and $\frac{f}{2F} = \Theta\left(\frac{1}{t}\right)$. Thus there exist constants $c_1, c_2, C_0, N > 0$ depending only on $G_0(t_0)$ such that for all $t \geq N$,

$$\mathbf{P}\left(\mathbf{U} = \mathbf{m} \,||\, d_1, \ldots d_n\right) \geq C_0 \cdot \left(1 - \frac{\sum_i d_i}{c_1 t^2}\right)^{c_2 t - m} \cdot t^{-m} \cdot \prod_i d_i^{m_i}. \tag{5.1}$$

Our aim is to bound this probability below, away from 0 over a long enough range of t. We write $X_i = \frac{d_i}{A_i}$ for the martingale supplied by Proposition 4.2, with $x_i := x_{u_i} > 0$ for its limit supplied by Proposition 4.2 and Lemma 4.3. We will not attempt to condition on the actual values x_i, but only on the fact that these values are not extreme (**NE**).

First, choose $\kappa_1, \kappa_2 > 0$ such that

$$\kappa_1 t < A(t) < \kappa_2 t$$

for all large enough t. This is guaranteed to occur by Proposition 4.2(iii) since $F(t)^{\frac{1}{2}} = \Theta(t)$. We increase N if necessary to ensure that this holds. Notice that since $A(t)$ is entirely predictable in advance, the value of N remains dependent only on $G_0(t_0)$.

Now, for any $y_2 > y_1 > 0$, define the following event:

$$\mathbf{NE}(y_1, y_2) : \quad \left(\bigwedge_{i=1}^{n} y_1 < x_i < y_2\right).$$

We shall apply this in the following case: given $\delta > 0$ choose $y_2(\delta) > y_1(\delta) > 0$ so that $\mathbf{P}(\neg\mathbf{NE}(y_1, y_2)) < \delta$. (We shall specify δ later, and will only need to consider one such value. Thus we shall consider δ fixed for the purposes of what follows.)

By Corollary 4.4, $X_i(t) \to x_i$ in expectation, and thus in probability. More precisely, for any $\eta > 0$, we may increase $N > 0$ by some quantity depending only on η so that for all $t \geq N$ and all $i \leq n$

$$\mathbf{E}\Big(|X_i(t) - x_i|\Big) < \left(\frac{\eta}{n}\right)^2.$$

Thus, by Markov's inequality

$$\mathbf{P}\left(|X_i(t) - x_i| > \frac{\eta}{n}\right) < \frac{\eta}{n}.$$

Hence defining the event that all the $X_i(t)$ are close (**Cl**) to their respective x_i

$$\mathbf{Cl}(t,\eta) := \bigwedge_{i=1}^{n} \left(|X_i(t) - x_i| < \frac{\eta}{n} \right)$$

we have for all $t \geq N$

$$\mathbf{P}\left(\mathbf{Cl}(t,\eta)\right) > 1 - \eta. \tag{5.2}$$

Again, we shall pick a value of η later. Notice also that

$$\mathbf{P}\left(\mathbf{Cl}(t,\eta)\right) < \mathbf{P}\left(\mathbf{Cl}(t,\eta) \,\Big|\Big|\, \mathbf{NE}(y_1,y_2)\right) + \delta.$$

So

$$\mathbf{P}\left(\mathbf{Cl}(t,\eta) \,\Big|\Big|\, \mathbf{NE}(y_1,y_2)\right) > 1 - \eta - \delta. \tag{5.3}$$

Next, we define a bound for $d_i(t)$. Given $\delta, \eta > 0$ as before, let $b_1(\eta) = b_1(\delta,\eta) := \kappa_1 \cdot \left(y_1 - \frac{\eta}{n}\right)$ and $b_2(\eta) = b_2(\delta,\eta) := \kappa_2 \cdot \left(y_2 + \frac{\eta}{n}\right)$, insisting that η is small enough that $b_1 > 0$. Then we define the event

$$\mathbf{Bo}(t,b_1,b_2) := \bigwedge_{i=1}^{n} \left(b_1 \cdot t < d_i(t) < b_2 \cdot t\right).$$

Observe now that for $t \geq N$

$$(\mathbf{NE}(y_1(\delta),y_2(\delta)) \,\&\, \mathbf{Cl}(t,\eta)) \Rightarrow \mathbf{Bo}(t,b_1(\eta),b_2(\eta)). \tag{5.4}$$

Hence

$$\mathbf{P}\left(\mathbf{Bo}(t,b_1,b_2) \Big|\Big| \mathbf{NE}(y_1(\delta),y_2(\delta))\right) \geq 1 - \eta - \delta.$$

Thus we obtain the unconditional bound:

$$\mathbf{P}\left(\mathbf{Bo}(t,b_1,b_2)\right) \geq (1-\eta-\delta)(1-\delta). \tag{5.5}$$

Now we use the bound obtained in (5.1) and see that whenever $b_1 \leq b_1' < b_2' \leq b_2$

$$\begin{aligned}
\mathbf{P}\left(\mathbf{U}=\mathbf{m} \,\Big|\Big|\, \mathbf{Bo}(t,b_1',b_2')\right) &> C_0 \cdot \left(1 - \frac{n \cdot b_2 \cdot t}{c_1 t^2}\right)^{c_2 t - m} \cdot t^{-m} \cdot (b_1 \cdot t)^m \\
&= C_0 \cdot b_1^m \cdot \left(1 - \frac{n b_2}{c_1 t}\right)^{c_2 t - m} \\
&= C_0 \cdot b_1^m \cdot \left(1 - \frac{n b_2 c_2}{c_1} \cdot \frac{1}{c_2 t}\right)^{c_2 t} \cdot \left(1 - \frac{n b_2}{c_1 t}\right)^{-m} \\
&\to C_0 \cdot b_1^m \cdot e^{-\frac{n b_2 c_2}{c_1}} := C_3 > 0.
\end{aligned}$$

Hence, by letting $C_4 = C_4(\delta,\eta) := \frac{1}{2} C_3$ and increasing N again if necessary (and again by some predictable amount), we have for all $t \geq N$

$$\mathbf{P}\left(\mathbf{U}=\mathbf{m} \,\Big|\Big|\, \mathbf{Bo}(t,b_1',b_2')\right) > C_4. \tag{5.6}$$

Now for any $\zeta > 0$, we may let $M = M(\zeta,\delta,\eta)$ be large enough that $(1-C_4)^M < \zeta$. The goal therefore is to locate M places where $\mathbf{Bo}(t,b_1(\eta),b_2(\eta))$ holds, and argue that the probability that all of them fail to produce an instance of $\mathbf{U} = \mathbf{m}$ is bounded above by ζ.

Notice that bound (5.6) holds independently for all $t \geq N$: the arguments are unaffected by previous values of $\mathbf{U}$ so long as $\mathbf{Bo}(t, b_1', b_2')$ holds. However, the same is not true for bound (5.5). By conditioning on whether or not $\mathbf{U}(t') = \mathbf{m}$ holds, we risk affecting $\mathbf{P}\left(\mathbf{Bo}(t, b_1(\eta), b_2(\eta))\right)$ for $t > t'$.

To navigate this obstacle, we shall locate a range $[t_2, t_2 + M)$ within which the bound $\mathbf{Bo}(t, b_1(\eta), b_2(\eta))$ is guaranteed to hold, barring a certain extreme event $\neg\mathbf{Sh}$ defined below, which will have a probability bounded above by θ for arbitrarily small θ.

We wish t_2 to satisfy the tighter bound $\mathbf{Bo}(t_2, b_1(\frac{\eta}{2}), b_2(\frac{\eta}{2}))$. Notice that appropriate adaptations of (5.2), (5.4), and (5.5) above guarantee that for large enough t_2,

$$\mathbf{P}\left(\neg\mathbf{Bo}\left(t_2, b_1\left(\tfrac{\eta}{2}\right), b_2\left(\tfrac{\eta}{2}\right)\right)\right) < \tfrac{\eta}{2} + 2\delta. \tag{5.7}$$

However, as already indicated, $\mathbf{Bo}\left(t_2, b_1\left(\frac{\eta}{2}\right), b_2\left(\frac{\eta}{2}\right)\right)$ on its own is not quite enough to guarantee $\mathbf{Bo}(t_2 + j, b_1(\eta), b_2(\eta))$ for $j \leq M$. So let us describe the extra ingredient we require. Notice that $U_i(t)$ is a binomial distribution with a long right tail, since the number of trials $f(t)$ is of the order of t, and the probability of success per trial is $\frac{d_i(t)}{2F(t)}$ which is of order $\frac{1}{t}$. We shall show that we may ignore the extremity of this tail, thus allowing us to impose a tighter upper bound than $f(t)$ on $U_i(t)$ for all $t \in [t_2, t_2 + M)$.

In Theorem 1.1 from [3], we find a useful bound for the right-tail of a binomial distribution $U \sim b(f, p)$: if $u > 1$ and $1 \leq S := \lceil ufp \rceil \leq f - 1$ then

$$\mathbf{P}(U \geq S) < \left(\frac{u}{u-1}\right) \cdot \mathbf{P}(U = S).$$

Let us apply this in the case $S = \lceil t^\alpha \rceil$ for some fixed $\alpha \in \left(\frac{1}{2}, 1\right)$. (Its exact value does not matter.) Then

$$u = u(t) = \frac{t^\alpha}{pf} = \frac{t^\alpha 2F(t)}{d(t)f(t)}.$$

Assembling the bounds $c_1 t \leq f(t) \leq c_2 t$ and $c_1 t^2 \leq 2F(t) \leq c_2 t^2$ and $\mathbf{Bo}(t, b_1', b_2')$ where $b_1 \leq b_1' < b_2' \leq b_2$ and employing the standard bound for the binomial coefficient $\binom{f}{S} \leq \left(\frac{f \cdot e}{S}\right)^S$, we find

$$\begin{aligned}
&\mathbf{P}\Big(U_i \geq t^\alpha \;\Big|\Big|\; \mathbf{Bo}(t, b_1', b_2')\Big) \\
&\quad < \left(\frac{\frac{c_2}{c_1 b_1} t^\alpha}{\frac{c_1}{c_2 b_2} t^\alpha - 1}\right) \cdot \left((ec_2 + 1) \cdot t^{1-\alpha}\right)^{t^\alpha} \cdot \left(\frac{b_2}{c_1 t}\right)^{t^\alpha} \cdot \left(1 - \frac{b_1}{c_2 t}\right)^{t - \lceil t^\alpha \rceil} \\
&\quad < \left(\frac{B}{t^\alpha}\right)^{t^\alpha}.
\end{aligned}$$

for some $B > 0$. Notice again that this holds independently of the specific values of b_1' and b_2', so long as $b_1 \leq b_1' < b_2' \leq b_2$. Now we define a new event, that the tails are short (**sh**):

$$\mathbf{sh}(t) := \bigwedge_{i=1}^{n} U_i(t+1) < t^\alpha.$$

After increasing B to allow for the non-independence of the n different U_i we now see that:

$$\mathbf{P}\left(\neg\mathbf{sh}(t)\,\middle\|\,\mathbf{Bo}(t, b_1(\eta), b_2(\eta))\right) < n \cdot \left(\frac{B}{t^\alpha}\right)^{t^\alpha}. \tag{5.8}$$

Putting these events together, define

$$\mathbf{Sh}(t_2) := \forall t \in [t_2, t_2 + M)\ \ \mathbf{sh}(t).$$

To obtain a similar bound for $\mathbf{P}\left(\neg\mathbf{Sh}(t_2)||\mathbf{Bo}\left(t_2, b_1\left(\frac{\eta}{2}\right), b_2\left(\frac{\eta}{2}\right)\right)\right)$ we first show that for $j \leq M$

$$\left(\mathbf{Bo}\left(t_2, b_1\left(\tfrac{\eta}{2}\right), b_2\left(\tfrac{\eta}{2}\right)\right)\ \&\ \bigwedge_{i=0}^{j-1}\mathbf{sh}(t_2 + j - 1)\right) \tag{5.9}$$
$$\Rightarrow \mathbf{Bo}(t_2 + j, b_1(\eta), b_2(\eta)).$$

Suppose that $\mathbf{Bo}\left(t_2, b_1\left(\frac{\eta}{2}\right), b_2\left(\frac{\eta}{2}\right)\right)$ holds. We address the lower bound first, for which we do not require the hypothesis on **sh**. Instead, for all $j \leq M$, clearly

$$d(t_2 + j) \geq d(t_2) \geq b_1\left(\tfrac{\eta}{2}\right) \cdot t_2 = \kappa_1 \cdot \left(y_1 - \frac{\eta}{2n}\right) \cdot t_2.$$

If additionally $t_2 \geq \frac{2nMy_1}{\eta}$, then the final term above exceeds

$$\kappa_1 \cdot \left(y_1 - \frac{\eta}{n}\right) \cdot (t_2 + M) \geq b_1(\eta) \cdot (t_2 + j).$$

Now we obtain the corresponding upper bound. By our assumption on **sh**,

$$\begin{aligned} d(t_2 + j) &\leq d(t_2) + M \cdot (t_2 + M)^\alpha \\ &\leq \kappa_2 \cdot \left(y_2 + \frac{\eta}{2n}\right) \cdot t_2 + M \cdot (t_2 + M)^\alpha \\ &\leq \kappa_2 \cdot \left(y_2 + \frac{\eta}{n}\right) \cdot t_2 \end{aligned}$$

if $t_2 \geq \max\left\{M, \left(\frac{4Mn}{\kappa_2\eta}\right)^{\frac{1}{1-\alpha}}\right\}$, which completes the proof of Implication (5.9).

Implication (5.9) allows us to take the M-fold sum of (5.8), finding

$$\mathbf{P}\left(\neg\mathbf{Sh}(t_2)\,\middle\|\,\mathbf{Bo}\left(t_2, b_1\left(\tfrac{\eta}{2}\right), b_2\left(\tfrac{\eta}{2}\right)\right)\right) < \sum_{t=t_2}^{t_2+M} n\left(\frac{B}{t^\alpha}\right)^{t^\alpha} \to 0$$

as $t_2 \to \infty$. Thus for any $\theta > 0$ for all large enough t_2 we have

$$\mathbf{P}\left(\neg\mathbf{Sh}(t_2)\,\middle\|\,\mathbf{Bo}\left(t_2, b_1\left(\tfrac{\eta}{2}\right), b_2\left(\tfrac{\eta}{2}\right)\right)\right) < \theta. \tag{5.10}$$

Finally, we may complete the argument, setting $\delta = \frac{\varepsilon}{8}$ and $\theta = \zeta = \frac{\varepsilon}{4}$ and $\eta = \frac{\varepsilon}{2}$ and $t_1 := t_2 + M$. For large enough t, we may update bound (5.6) to get

$$\mathbf{P}\left((\neg\mathbf{U}(t_2 + 1) = \mathbf{m})\ \&\ \mathbf{sh}(t_2)\,\middle\|\,\mathbf{Bo}\left(t_2, b_1\left(\tfrac{\eta}{2}\right), b_2\left(\tfrac{\eta}{2}\right)\right)\right) < 1 - C_4.$$

Similarly,

$$\mathbf{P}\Big(\big(\neg\mathbf{U}(t_2+j+1)=\mathbf{m}\big)\ \&\ \mathbf{sh}(t_2+j) \\ \Big|\Big| \bigwedge_{i=0}^{j-1} \mathbf{sh}(t_2+i)\ \&\ \mathbf{Bo}\left(t_2, b_1\left(\tfrac{\eta}{2}\right), b_2\left(\tfrac{\eta}{2}\right)\right)\Big) \\ < 1-C_4.$$

As observed earlier, these bounds hold independently of the previous values of $\mathbf{U}$, meaning that

$$\mathbf{P}\Big(\big(\neg\mathbf{U}(t_2+j+1)=\mathbf{m}\big)\ \&\ \mathbf{sh}(t_2+j) \\ \Big|\Big| \bigwedge_{i=0}^{j} \neg\mathbf{U}(t_2+i)\ \&\ \bigwedge_{i=0}^{j-1} \mathbf{sh}(t_2+i)\ \&\ \mathbf{Bo}\left(t_2, b_1\left(\tfrac{\eta}{2}\right), b_2\left(\tfrac{\eta}{2}\right)\right)\Big) \\ < 1-C_4.$$

Taking the product of these bounds, and denoting the failure of our desired result by $\mathbf{Fa}(t_2) := \forall t \in [t_2, t_2+M)\ \ (\mathbf{U}(t+1) \neq \mathbf{m})$, we see that

$$\mathbf{P}\Big(\mathbf{Fa}(t_2)\ \&\ \mathbf{Sh}(t_2)\ \Big|\Big|\ \mathbf{Bo}(t_2, b_1\left(\tfrac{\eta}{2}\right), b_2\left(\tfrac{\eta}{2}\right)\Big) < \zeta$$

and so by bounds (5.7) and (5.10)

$$\mathbf{P}\Big(\mathbf{Fa}(t_2)\Big) < \zeta + \theta + \tfrac{\eta}{2} + 2\delta = \varepsilon. \qquad \square$$

We may now complete the proof of our main result, which we first restate for the reader's convenience:

Theorem 5.3. *Suppose that G' is a finite directed multigraph containing no isolated nodes, that f satisfies the requirements from* Definition 3.1, *and also that $f(t) = \Theta(t)$. Then, with probability 1, the infinite limit of MPA$_f(G')$ is isomorphic, as an undirected multigraph, to the Rado multigraph.*

Proof. First notice that there are countably many witness requests. Thus we may organise them into a list $(W_j : j \geq 1)$.

Let $\varepsilon > 0$. Again everything that occurs is conditioned upon $G_0(t_0)$. We shall show that the probability of all witness requests eventually being satisfied exceeds $1-\varepsilon$. Suppose inductively that we have found time t_j so that so that $\mathbf{P}(\bigwedge_{i=1}^{j} W_i[t_j]) > 1 - \left(1 - \frac{1}{2^j}\right)\varepsilon$.

Let $\mathcal{G} = \mathcal{G}_j$ be the set of all states $G = G(t_j)$ of the graph at time t_j consistent with $G_0(t_0)$ and with $\bigwedge_{i=1}^{j} W_i[t_j]$. Notice that $\mathcal{G}$ is a finite set, that $\mathbf{P}\left(G(t_j)\ ||\ G_0(t_0)\right) > 0$ for each $G \in \mathcal{G}$, and by assumption that $\sum_{G\in\mathcal{G}} \mathbf{P}\left(G(t_j)\ ||\ G(t_0)\right) > 1 - \left(1 - \frac{1}{2^j}\right)\varepsilon$.

Consider now W_{j+1} and let $\varepsilon' < \frac{1}{2^{j+1}}\varepsilon$. Now given each $G^{(k)} \in \mathcal{G}$, by Proposition 5.2 there exist $t^{(k)} \geq t_j$ such that

$$\mathbf{P}\left(W_{j+1}\left[t^{(k)}\right]\ ||\ G^{(k)}(t_j)\right) > 1 - \varepsilon'.$$

Let $t_{j+1} := \max\{t^{(k)} \mid G^{(k)} \in \mathcal{G}\}$. Then

$$\begin{aligned}
&\mathbf{P}\left(\bigwedge_{i=1}^{j+1} W_i[t_{j+1}] \mid\mid G_0(t_0)\right) \\
&\geq \sum_k \mathbf{P}\left(\bigwedge_{i=1}^{j+1} W_i[t_{j+1}] \mid\mid G^{(k)}(t_j)\right) \cdot \mathbf{P}\left(G^{(k)}(t_j) \mid\mid G_0(t_0)\right) \\
&= \sum_k \mathbf{P}\left(W_{j+1}[t_{j+1}] \mid\mid G^{(k)}(t_j)\right) \cdot \mathbf{P}\left(G^{(k)}(t_j) \mid\mid G_0(t_0)\right) \\
&> \sum_k (1-\varepsilon') \cdot \mathbf{P}\left(G^{(k)}(t_j) \mid\mid G_0(t_0)\right) \\
&> (1-\varepsilon')\left(1 - \left(1 - \frac{1}{2^j}\right)\varepsilon\right) > 1 - \left(1 - \frac{1}{2^{j+1}}\right)\varepsilon. \qquad \square
\end{aligned}$$

6 Future work

We close this paper with a short discussion of possible future directions of study. As noted earlier, one goal is to translate the current work into the domain of graphs (rather than multigraphs) by proving Conjecture 3.4, which we restate:

Conjecture 6.1. *Suppose that $G' = (V', E')$ be a finite directed graph containing no isolated nodes, that f satisfies the conditions in* Definition 3.3*, and further that there are constants $0 < c_1 \leq c_2 < 1$ where $c_1 \cdot t \leq f(t) \leq c_2 \cdot t$ for all large enough t. Then, with probability 1, the infinite limit of GPA$_f(G')$ is isomorphic as an undirected graph to the Rado graph.*

A second avenue to investigate is the limit of an MPA$_f$ process when f is strictly between the constant case (considered by Kleinberg and Kleinberg in [10]) and the linear growth rate analysed here. A natural starting point would be the case $f(t) = \Theta\left(\sqrt{t}\right)$. One might hope somewhere within this regime to identify a connection to (a multigraph analogue of) the theory of Shelah-Spencer sparse random graphs as elucidated in [16]. The author recently established a connection between Shelah-Spencer graphs and the limits of finitary random processes in [8], albeit in a context rather simpler than preferential attachment.

Thirdly, a central role in the contemporary study of graph limits is played by the theory of graphons, as developed, for instance, by Lovász in [13]. Thus it is natural to seek to connect the current work to that body of knowledge. Although graphons as originally conceived do not allow for multi-edges, in [12] a theory of convergence of sequences of multigraphs is developed within the broader setting of *decorated graphs* and *Banach space valued graphons*, so this is an initial point of contact to consider. (I am grateful to the anonymous reviewer for bringing this to my attention.)

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References

[1] M. Adler and M. Mitzenmacher, Towards compressing web graphs, in: *Proceedings DCC 2001. Data Compression Conference*, volume 1, 2001 pp. 203–212, doi:10.1109/dcc.2001.917151.

[2] A.-L. Barabási and R. Albert, Emergence of scaling in random networks, *Science* **286** (1999), 509–512, doi:10.1126/science.286.5439.509.

[3] B. Bollobás, *Random graphs (2nd edition)*, number 73 in Cambridge Studies in Advanced Mathematics, Cambridge university press, 2001, doi:doi.org/10.1017/cbo9780511814068.

[4] A. Bonato and J. Janssen, Infinite limits and adjacency properties of a generalized copying model, *Internet Math.* **4** (2007), 199–223, doi:10.1080/15427951.2007.10129292.

[5] P. J. Cameron, The random graph, in: *The mathematics of Paul Erdős, II*, Springer, Berlin, volume 14 of *Algorithms Comb.*, pp. 333–351, 1997, doi:10.1007/978-3-642-60406-5_32.

[6] G. F. Carrier, M. Krook and C. E. Pearson, *Functions of a Complex Variable: Theory and Technique*, SIAM, 2005, http://bookstore.siam.org/cl49/.

[7] E. Drinea, M. Enachescu and M. D. Mitzenmacher, Variations on random graph models for the web, 2001, https://www.eecs.harvard.edu/~michaelm/ListByYear.html.

[8] R. Elwes, Evolving Shelah-Spencer graphs, *Math. Log. Q.* **67** (2020), 6–17, doi:10.1002/malq.201900053.

[9] R. Elwes, A preferential attachment process approaching the Rado graph, *Proc. Edinb. Math. Soc. (2)* **63** (2020), 443–455, doi:10.1017/s0013091519000336.

[10] R. D. Kleinberg and J. M. Kleinberg, Isomorphism and embedding problems for infinite limits of scale-free graphs., in: *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, 2005 pp. 277–286, doi:10.5555/1070432.1070472.

[11] P. L. Krapivsky and S. Redner, Organization of growing random networks, *Phys. Rev. E* **63** (2001), 066123, doi:10.1103/physreve.63.066123.

[12] D. Kunszenti-Kovács, L. Lovász and B. Szegedy, Multigraph limits, unbounded kernels, and Banach space decorated graphs, 2014, arXiv:1406.7846 [math.CO].

[13] L. Lovász, *Large Networks and Graph Limits*, volume 60 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, 2012, doi:10.1090/coll/060.

[14] R. Oliveira and J. Spencer, Connectivity transitions in networks with super-linear preferential attachment, *Internet Math.* **2** (2005), 121–163, doi:10.1080/15427951.2005.10129101.

[15] E. Paul and R. Alfréd, On random graphs I, *Publ. Math. Debrecen* **6** (1959), 290–297, https://users.renyi.hu/~p_erdos/Erdos.html.

[16] J. Spencer, *The strange logic of random graphs*, volume 22, Springer Science & Business Media, 2001, doi:10.1007/978-3-662-04538-1.

[17] F. O. Wagner, *Simple Theories*, volume 503, Springer Netherlands, 2000, doi:10.1007/978-94-017-3002-0.

[18] D. Williams, *Probability with Martingales*, Cambridge university press, 1991, doi:10.1017/cbo9780511813658.

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Bracing frameworks consisting of parallelograms*

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Abstract

A rectangle in the plane can be continuously deformed preserving its edge lengths, but adding a diagonal brace prevents such a deformation. Bolker and Crapo characterized combinatorially which choices of braces make a grid of squares infinitesimally rigid using a *bracing graph*: a bipartite graph whose vertices are the columns and rows of the grid, and a row and column are adjacent if and only if they meet at a braced square. Duarte and Francis generalized the notion of the bracing graph to rhombic carpets, proved that the connectivity of the bracing graph implies rigidity and stated the other implication without proof. Nagy Kem gives the equivalence in the infinitesimal setting. We consider continuous deformations of braced frameworks consisting of a graph from a more general class and its placement in the plane such that every 4-cycle forms a parallelogram. We show that rigidity of such a braced framework is equivalent to the non-existence of a special edge coloring, which is in turn equivalent to the corresponding bracing graph being connected.

Keywords: Flexibility, rigidity, bracing, rhombic tiling.

Math. Subj. Class.: 52C25, 51K99, 70B99

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1 Introduction

A planar framework is a graph together with a placement of its vertices in the plane. If there is a non-trivial flex (a deformation of the placement preserving the distances between adjacent vertices that is not induced by a rigid motion), then the framework is said to be flexible, otherwise rigid. Bolker and Crapo [4] studied infinitesimal flexibility of a framework corresponding to a grid of squares with some squares being braced by adding diagonals, see Figure 1. They construct a bipartite graph by taking the columns and rows of the grid to be the two parts of the vertex set; a column and row are connected if and only if their common square is braced. They showed that a braced grid is infinitesimally rigid, i.e., has no non-trivial first order flex, if and only if the bipartite graph is connected.

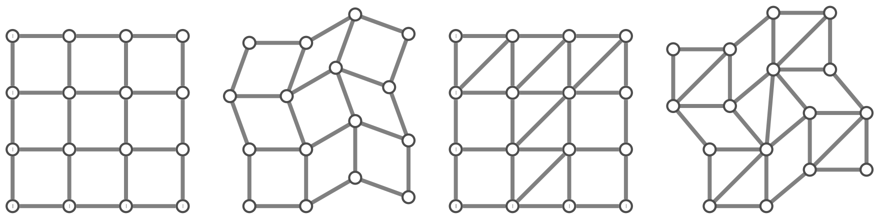


Figure 1: Grid frameworks can be deformed in a way that preserves the edge lengths. By bracing it (i.e. by adding diagonal edges) we can reduce the number of degrees of freedom. The left braced grid is rigid whereas the right one allows a flex.

Generalizations to rectangular grids with holes [14, 24] or placing longer diagonals [13] than those for a single 4-cycle have been studied as well as bracing by cables [20]. Grids with rectilinear boundary are discussed in [17]. Extensions to cubic grids have been studied [3, 21]. The papers [9, 28] describe the number of randomly added braces for which the transition from flexible to rigid occurs. A related problem to the rigidity of a grid is the rigidity of one- and multi-story building [19, 23], also with cables [5, 22, 25]. Simple forms of bracing grids are also known to be suitable as a puzzle, for science communication and for student's exercises (see for instance [26]).

In this work, we focus on parallelograms instead of squares, and we allow a richer combinatorial structure than grids. Flexibility of rhombic/parallelogramic tilings is studied by physicists due to its relation with quasicrystals [29]. The bracing of rhombic carpets, which are 1-skeleta of finite simply connected pieces of rhombic tilings, was investigated by Wester [27]. Duarte and Francis [8] formalized the notions necessary to study the flexibility of rhombic carpets: a natural step from columns and rows of a grid towards a rhombic carpet is to take *ribbons*. These are sequences of rhombi such that every two consecutive ones share an edge and all these edges are parallel. Following the idea of Bolker and Crapo, Duarte and Francis construct a *bracing graph* whose vertices are the ribbons and two ribbons are adjacent if they have a common rhombus that is braced. They prove that if the constructed graph is connected, then the braced rhombic carpet is rigid. Further, they state the other implication without proof. We thank Eliana Duarte for pointing out this statement to us and sharing some hints about a possible proof [7]. Nagy Kem [18] translates the infinitesimal rigidity of a braced rhombic carpet to the rigidity of an auxiliary framework which in turn corresponds to the connectivity of the bracing graph.

We formulate the problem of the flexibility of braced structures in terms of frameworks. In particular, we define ribbons as equivalence classes on edges of the underlying graph using its 4-cycles. We consider a special class of graphs, which we call *ribbon-cutting* graphs. A connected graph is ribbon-cutting if every ribbon is an edge cut, i.e., removing the edges of the ribbon makes the graph disconnected. Regarding the placement, we ask all 4-cycles to form parallelograms, see Figure 2. Notice that the frameworks we consider — we call them P-frameworks — form a proper superset of the frameworks corresponding to rhombic carpets and rectangular grids (without holes). The question we address is analogous to the one by Bolker, Crapo, Duarte and others, namely, characterization of choices of braces of parallelograms yielding flexible/rigid P-frameworks. Contrary to Bolker, Crapo and Nagy Kem, we consider continuous flexes, not infinitesimal ones. Furthermore, we use a recently established method of special edge colorings to prove our results.

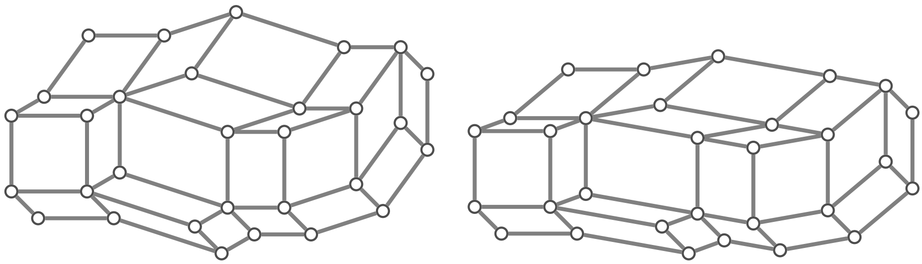


Figure 2: Carpet frameworks can be deformed in a way that preserves the edge lengths.

The notion of *NAC-colorings* was developed in our previous paper [11]. A NAC-coloring is a surjective edge coloring of a graph by red and blue such that for every cycle of the graph, either all edges have the same color or there are at least two edges of each color. We proved that a graph has a flexible framework if and only if it has a NAC-coloring. It appears that the techniques used to prove the theorem fit nicely to the context of bracing P-frameworks if we restrict ourselves to certain NAC-colorings: a NAC-coloring is called *cartesian* if there are no two vertices connected by a red path and blue path simultaneously. The non-existence of a cartesian NAC-coloring serves as a bridge in the proof that a braced P-framework is rigid if and only if the corresponding bracing graph (defined analogously to [8]) is connected. Our results can be summarized as follows.

Theorem 1.1. *For a braced P-framework* (G, ρ)*, the following statements are equivalent:*

(i) (G, ρ) *is rigid,*

(ii) *G has no cartesian NAC-coloring, and*

(iii) *the bracing graph of G is connected.*

In particular, the minimum number of braces making a framework rigid is one less than the number of ribbons of its underlying graph.

Theorem 1.1 extends the result by Duarte and Francis [8], which is the implication (iii) $\implies$ (i) for rhombic carpets. These form a proper subclass of P-frameworks used in this paper since all rhombic carpets are planar embeddings in graph theoretical sense, contrary to P-frameworks.

We implemented the concepts introduced in the current paper by extending our SAGEMATH package FLEXRILOG [12]. We encourage the reader to experiment with the Jupyter notebook available at `https://jan.legersky.cz/bracingFrameworks`.

The paper is organized as follows: Section 2 recalls the notions from rigidity theory and NAC-colorings. We define ribbons, parallelogram placements and P-frameworks in Section 3 and prove some basic results. We also formalize bracing and the notion of bracing graph in our context. Section 4 provides the proofs yielding Theorem 1.1. We put additional material to the appendix, for instance, we show that rhombic carpets are P-frameworks in Appendix B.

2 Preliminaries

In this section we recall basic notions and definitions commonly used in rigidity theory. The ideas are based on previous work using special edge colorings to find flexes of graphs. We introduce these colorings here and describe what we mean by flexibility.

Definition 2.1. Let $G = (V_G, E_G)$ be a connected graph. A map $\rho : V_G \to \mathbb{R}^2$ such that $\rho(u) \neq \rho(v)$ for all edges $uv \in E_G$ is a *placement*. The pair (G, ρ) is called a *framework*.

Definition 2.2. Two frameworks (G, ρ) and (G, ρ') are *equivalent* if

$$\|\rho(u) - \rho(v)\| = \|\rho'(u) - \rho'(v)\|$$

for all $uv \in E_G$. Two placements ρ and ρ' are *congruent* if there exists a Euclidean isometry M of $\mathbb{R}^2$ such that $M\rho'(v) = \rho(v)$ for all $v \in V_G$.

Definition 2.3. A *flex* of the framework (G, ρ) is a continuous path $t \mapsto \rho_t$, $t \in [0, 1]$, in the space of placements of G such that $\rho_0 = \rho$ and each (G, ρ_t) is equivalent to (G, ρ). The flex is called trivial if ρ_t is congruent to ρ for all $t \in [0, 1]$.

We define a framework to be *(proper) flexible* if there is a non-trivial flex in $\mathbb{R}^2$ (with injective placements). Otherwise it is called *rigid*.

In a previous paper [11], we introduced a special edge coloring, which is called NAC-coloring, in order to classify the graphs that have flexible frameworks.

Definition 2.4. Let G be a graph. A coloring of edges $\delta \colon E_G \to \{\text{blue}, \text{red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle in G, either all edges have the same color, or there are at least 2 edges in each color (see Figure 3). The NAC-coloring δ gives subgraphs

$$G^\delta_{\text{red}} = (V_G, \{e \in E_G \colon \delta(e) = \text{red}\}) \text{ and } G^\delta_{\text{blue}} = (V_G, \{e \in E_G \colon \delta(e) = \text{blue}\})\,.$$

We remark that flexibility in [11] is defined in the following sense: an edge labeling by positive real numbers (interpreted as lengths for edges) is called *flexible* if there are infinitely many non-congruent placements inducing it. Notice that if a framework has a flex, then the induced edge labeling (corresponding to the lengths) is flexible. On the other hand, assuming we have a flexible edge labeling, then placements inducing the edge lengths corresponding to the labeling form an algebraic variety containing an algebraic curve of placements. Considering a nonsingular point in this curve, a local parametrization of the curve around this point gives a flex. Therefore, we can state the result as follows.

Theorem 2.5 ([11])**.** *A connected non-trivial graph allows a flexible framework if and only if it has a NAC-coloring.*

Two non-adjacent vertices u and v overlap in the flex constructed in the proof of the theorem in [11] if and only if there is a red path from u to v and a blue path from u to v. In order to avoid overlapping vertices, we focus on a special type of NAC-colorings.

Definition 2.6. A NAC-coloring δ of a graph G is called *cartesian* if no two distinct vertices are connected by a red and blue path simultaneously.

We chose the name *cartesian* due to its connection with cartesian products of graphs, see Appendix A.

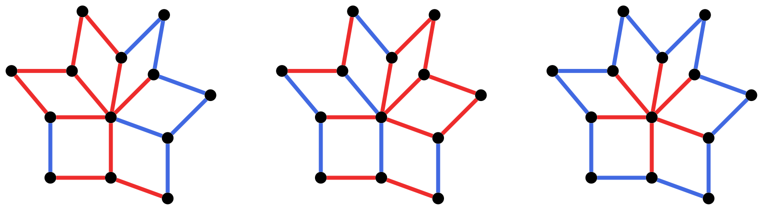


Figure 3: A coloring that is not NAC-coloring (left), cartesian NAC-coloring (middle) and non-cartesian NAC-coloring (right).

Remark 2.7. A NAC-coloring δ of a graph G is cartesian if and only if for every connected component R of G^{δ}_{red} and B of G^{δ}_{blue}, the intersection of the vertex sets of R and B contains at most one vertex.

Notice that in a cartesian NAC-coloring, a 4-cycle subgraph is monochromatic, or the opposite edges have the same color.

3 Ribbons and parallelogram placements

In this section we describe bracings of graphs (Section 3.2). We mainly consider a class of graphs (Section 3.1) which essentially consists of four-cycles which we want to place in the plane, forming parallelograms. Having these 4-cycles in mind we start by defining an equivalence relation on the edges. The equivalence classes, called *ribbons*, generalize the notion of rows and columns in a rectangular grid. Ribbons are a concept that is also used in other places under various names (stripes, worms, de Bruijn lines) and for different purpose (see for instance [2, 6, 10, 29]).

Definition 3.1. Let G be a graph. Consider the relation on the set of edges, where two edges are in relation if they are opposite edges of a 4-cycle subgraph of G. An equivalence class of the reflexive-transitive closure of the relation is called a *ribbon*. Figure 4 shows all ribbons for some small graphs. A ribbon r is *simple* if the subgraph induced by r does not contain any 4-cycle (see Figure 5a for an example of a non-simple ribbon).

In the case of rectangular grids, there is a natural way how to order the edges in a ribbon, i.e., a row or column. In our context, there is no natural order of the edges in a ribbon as Figure 5b indicates.

From now on, given a walk W, the notation $(u, v) \in W$ means that the edge uv belongs to W and u precedes v in W. Similarly, for a ribbon r, the notation $(u, v) \in r \cap W$ means $(u, v) \in W$ and $uv \in r$. If $(u, v) \in r \cap W$ is used to iterate in a sum, the edges of W

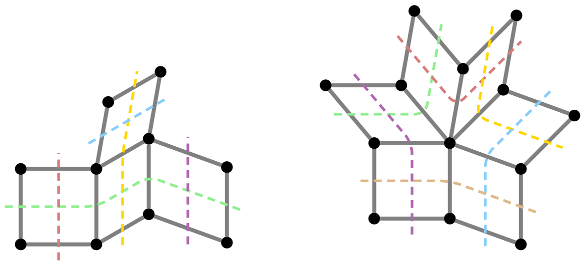
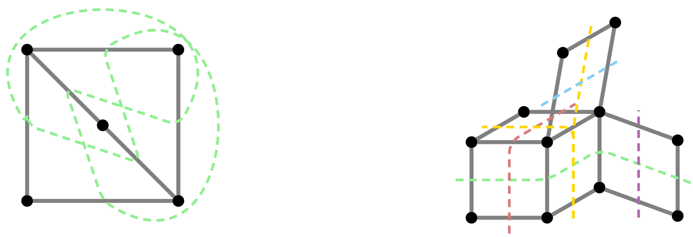


Figure 4: The ribbons of the graphs are indicated by dashed lines. All edges intersecting the line belong to the same ribbon.



(a) The only ribbon is not simple. (b) The yellow ribbon has three "ends".

Figure 5: Special cases that might happen for ribbons.

must be considered as a multiset: the summand corresponding to (u, v) is included as many times as uv occurs in W with u preceding v in W. Similarly for the cardinality of $r \cap W$:

$$|r \cap W| = \sum_{(u,v) \in r \cap W} 1 \,.$$

Recall that a set of edges r is an *edge cut* of a connected graph G if the graph $G \setminus r = (V_G, E_G \setminus r)$ is disconnected.

Lemma 3.2. *Let G be a connected graph with a simple ribbon r, which is an edge cut. Then $G \setminus r$ has exactly two connected components. If $w, w' \in V_G$ and W is a walk from w to w', then $|r \cap W|$ is odd if and only if r separates w and w', i.e., w and w' are in the different connected components of $G \setminus r$. In particular, if W is a closed walk in G, then $|r \cap W|$ is even.*

Proof. Let uv be an edge of r. For every edge $u'v' \in r$, there exists a sequence of edges $u_1v_1, \ldots, u_kv_k$ such that $u = u_1, v = v_1, u_k = u', v_k = v'$ and $(u_i, v_i, v_{i+1}, u_{i+1})$ is a 4-cycle in G. Hence, there are walks $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ in G. An edge u_iu_{i+1} is in r if and only if v_iv_{i+1} is in r. But if $u_iu_{i+1}, v_iv_{i+1} \in r$, then $(u_i, v_i, v_{i+1}, u_{i+1})$ would be a 4-cycle in the subgraph induced by r, which is not possible since r is simple. Hence, no edge of the two walks is in r. This shows that every vertex of an edge in r is either connected to u, or v in $G \setminus r$, thus, $G \setminus r$ has two connected components.

If W is a walk from w to w', then $|W \cap r|$ is even if and only if w and w' are in the same connected component of $G \setminus r$. $\square$

We want to consider graphs that somehow consist of parallelograms. For interpreting this idea we need to look at frameworks rather than graphs.

Definition 3.3. Let G be a connected graph. A placement $\rho : V_G \to \mathbb{R}^2$ for G such that ρ is injective and each 4-cycle in G forms a parallelogram[1] in ρ is called a *parallelogram placement*.

Remark 3.4. Let ρ be a parallelogram placement of a connected graph G. Edges of a ribbon of G are parallel line segments of the same length in ρ.

Remark 3.5. By Remark 3.4, if there was a 4-cycle induced by a ribbon, then two opposite vertices of the 4-cycle would coincide in a parallelogram placement, which contradicts injectivity of the placement. Hence, if a graph allows a parallelogram placement, then all its ribbons are simple.

The following properties of parallelogram placements are needed later on.

Lemma 3.6. *Let G be a connected graph with a parallelogram placement ρ and ribbon r which is an edge cut. If the vertex set of r is $V_1 \cup V_2$, where all vertices of V_i belong to the same connected component of $G \setminus r$, then $\rho(V_2)$ is a translation of $\rho(V_1)$. In particular, the vector $\rho(u_2) - \rho(u_1)$ is the same for all edges $u_1u_2 \in r$, $u_i \in V_i$.*

Proof. The ribbon r is simple by Remark 3.5. Lemma 3.2 gives the partition $V_1 \cup V_2$ with $|V_1| = |V_2|$. The vector $\rho(u_2) - \rho(u_1)$ is the same for all $u_1u_2 \in r$, $u_i \in V_i$, by Remark 3.4. □

Lemma 3.7. *Let G be a connected graph with a parallelogram placement ρ. Let r be a ribbon of G which is an edge cut and W be a walk in G. If $|r \cap W|$ is even, then*

$$\sum_{(w_1,w_2) \in r \cap W} (\rho(w_2) - \rho(w_1)) = 0\,.$$

Proof. Let $W = (u_0, u_1, \ldots, u_m)$ be a walk. All ribbons are simple by Remark 3.5. Let $V_1 \cup V_2$ be as in Lemma 3.6. Let the edges of W that are in r be $u_{j_1}u_{j_1+1}, \ldots, u_{j_k}u_{j_k+1}$ with $j_1 < j_2 < \cdots < j_k$, k is even by assumption. We have that $u_{j_1}, u_{j_2+1}, u_{j_3}, \ldots, u_{j_k+1} \in V_1$ and $u_{j_1+1}, u_{j_2}, u_{j_3+1}, \ldots, u_{j_k} \in V_2$. By Lemma 3.6,

$$\sum_{i=1}^{k} (\rho(u_{j_i+1}) - \rho(u_{j_i})) = 0\,. \qquad \square$$

3.1 Frameworks and graphs consisting of parallelograms

For proving the statements in this paper, we require frameworks with a parallelogram placement where all ribbons of the underlying graph are edge cuts. This yields a so called P-framework. We describe two different subclasses of P-frameworks: an illustrating approach is to start from a set of connected parallelograms with additional properties and form a framework. This will be a carpet framework. The second one is a recursive construction. Furthermore, we describe the relations between these classes. For streamlining the paper, we put some results to Appendix B.

Definition 3.8. A graph G is called *ribbon-cutting graph* if it is connected and every ribbon is an edge cut. If ρ is a parallelogram placement of G, we call the framework (G, ρ) a *P-framework*.

[1] Here we consider only non-degenerate parallelograms, namely, not all vertices are collinear.

A rectangular lattice graph (grid graph) with its natural placement is a P-framework as well as the frameworks in Figure 2 and the graphs in Figure 4 and 5b with the placements given by their layouts.

There are ribbon-cutting graphs without any parallelogram placement. Figure 6 shows such a graph, for which the non-existence of a parallelogram placement follows from failing one of the necessary conditions given by Theorem 3.9. On the other hand, the graph in Figure 7 is not ribbon-cutting but has a parallelogram placement.

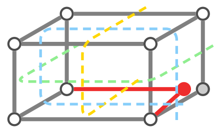


Figure 6: The graph of the framework is ribbon-cutting, but it has no parallelogram placement: if the red vertex and edges were placed forming a parallelogram, the two filled vertices would coincide. Theorem 3.9 also shows this, since these two vertices are not separated by any ribbon.

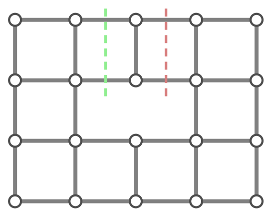


Figure 7: A parallelogram placement of a graph with ribbons that are not edge cuts.

Theorem 3.9. *If (G, ρ) is a P-framework, then there are no odd cycles in G, i.e., the graph is bipartite, and every two vertices are separated by a ribbon.*

Proof. Remark 3.5 guarantees that all ribbons of G are simple. Every ribbon intersects a cycle in an even number of edges by Lemma 3.2, hence, the cycle is even.

Let u and v be two distinct vertices. Let $W = (u = u_0, u_1, \dots, u_m = v)$ be a walk. Let R be the set of ribbons which contain at least one edge of W. Since u and v are distinct and ρ is injective, we have

$$0 \neq \rho(u_m) - \rho(u_0) = \sum_{i=1}^{m} (\rho(u_i) - \rho(u_{i-1})) = \sum_{r \in R} \sum_{(w_1, w_2) \in r \cap W} (\rho(w_2) - \rho(w_1)) \,.$$

All ribbons r such that $|r \cap W|$ is even have a zero contribution by Lemma 3.7. Hence, there must be a ribbon r' such that $|r' \cap W|$ is odd. The ribbon r' separates u and v by Lemma 3.2. □

The following definition is a slight generalization of rhombic carpets used in [8] where we allow parallelograms instead of rhombi.

Definition 3.10. Let S be a finite set of arbitrary parallelograms in $\mathbb{R}^2$ (including interiors) such that:

- if a point belongs to two parallelograms, then it is either a vertex of both, or an interior point of an edge of both (in particular, if a point belongs to more than two parallelograms, then it is a vertex),

- the boundary of the union $\bigcup S$ is a simple polygon.

The framework obtained by taking the 1-skeleton of S together with the vertex positions is called a *carpet framework* (see Figure 2 for an example).

Every carpet framework is a P-framework, see Corrolary B.6. In order to prove this we introduce a class of graphs $\mathcal{G}_{\text{rec}}$, which contains the underlying graphs of carpet frameworks. The definition is done recursively by adding vertices in a way that a parallelogram placement can be extended. In order to streamline the paper, these discussions can be seen in Appendix B.

3.2 Bracings

A general P-framework is flexible with many degrees of freedom. By adding edges to the graph we can reduce this number. In particular we are interested in adding diagonal edges of 4-cycles. This process is called the bracing of the graph or framework.

Definition 3.11. A *braced ribbon-cutting graph* is a graph $G = (V_G, E_c \cup E_d)$ where E_c and E_d are two non-empty disjoint sets such that the graph (V_G, E_c) is a ribbon-cutting graph and the edges in E_d correspond to diagonals of some 4-cycles of (V_G, E_c). These diagonals are also called *braces*. If r is a ribbon of (V_G, E_c), then

$$r \cup \{u_1u_3 \in E_d \colon \exists \text{ 4-cycle } (u_1, u_2, u_3, u_4) \text{ of } (V_G, E_c) \text{ s.t. } u_1u_2, u_3u_4 \in r\}$$

is a *ribbon* of the braced ribbon-cutting graph G.

The framework (G, ρ) is called *braced P-framework* if G is a braced ribbon-cutting graph and ρ is a parallelogram placement for (V_G, E_c). Figure 8 shows an example.

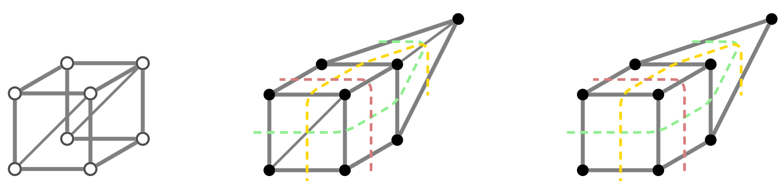


Figure 8: An example of a braced P-framework (left) with the underlying ribbon-cutting graph (right) and its bracing (middle).

Remark 3.12. A ribbon of a braced ribbon-cutting graph $(V, E_c \cup E_d)$ is an edge cut since the corresponding ribbon of (V, E_c) is an edge cut.

We construct a new graph, which encodes the relations between the ribbons, i.e., we ask whether they share 4-cycles. A subgraph of this graph indicates whether some of the shared 4-cycles is braced.

Definition 3.13. Let G be a braced ribbon-cutting graph. The *ribbon graph* Γ of G is the graph with the set of vertices being the set of ribbons of G and two ribbons r_1, r_2 are adjacent if and only if there is a 4-cycle (u_1, u_2, u_3, u_4) in the underlying unbraced graph of G such that $u_1u_2, u_3u_4 \in r_1$ and $u_1u_4, u_2u_3 \in r_2$. The subgraph (V_Γ, E_b) of Γ, where

$$E_b = \{r_1r_2 \in E_\Gamma : r_1 \cap r_2 \text{ is a non-empty subset of braces of } G\},$$

is called the *bracing (sub)graph*. See Figure 9 for an example of these definitions.

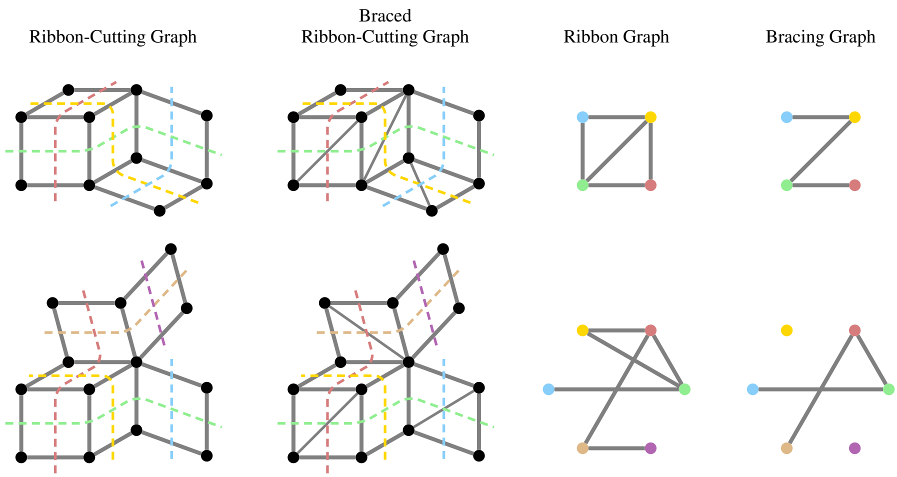


Figure 9: Two ribbon-cutting graphs with an example of a bracing as well as the corresponding ribbon graph and bracing graph. The vertices in the ribbon graph and the bracing graph are colored in correspondence with the indicated ribbons.

We remark that the bracing subgraph according to the definition in [8] does not contain the ribbons which have no brace. In our definition these ribbons are isolated vertices.

There are no loops in ribbon and bracing graphs if all ribbons of the underlying unbraced ribbon-cutting graph are simple. An edge in a bracing graph does not determine uniquely a braced 4-cycle (see the yellow and green ribbon in Figure 8).

Now we have all definitions to recall the main theorem of [8]. In the next section we extend this theorem to P-frameworks and also prove the other direction.

Theorem 3.14 ([8])**.** *Let (G, ρ) be a braced carpet framework. If the bracing graph of G is connected, then (G, ρ) is rigid.*

4 Flexibility of braced P-frameworks

In this section we determine when a bracing makes the framework rigid and in which cases it remains flexible. We use cartesian NAC-colorings for that. The theory is therefore based on [11]. Indeed, we show that a P-framework is flexible if and only if it has a cartesian NAC-coloring. This finally leads to a proof of the main theorem.

Cartesian NAC-colorings of a subclass of ribbon-cutting graphs can be characterized using ribbons.

Lemma 4.1. *Let G be a braced ribbon-cutting graph such that every two vertices are separated by a ribbon. A NAC-coloring of G is cartesian if and only if each ribbon of G is monochromatic.*

Proof. Let δ be a NAC-coloring. If δ is cartesian, then all 4-cycles are either monochromatic or opposite edges have the same color. Since the edges of a braced 4-cycle have the same color, ribbons are monochromatic. On the other hand, if the ribbons are monochromatic, then two vertices cannot be connected by a blue and red path simultaneously since they are separated by a ribbon. □

Theorem 4.2. *If a braced P-framework (G, ρ) is flexible, then G has a cartesian NAC-coloring.*

Proof. A NAC-coloring for G can be constructed as in the proof of [11, Theorem 3.1]. The zero set of the following system of equations for coordinates (x_u, y_u) for $u \in V_G$ describes all placements of G inducing the same edge lengths as ρ:

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \|\rho(u) - \rho(v)\|^2 \quad \text{for all } uv \in E_G\,. \tag{4.1}$$

In order to remove rigid motions, we fix the position of an edge $\bar{u}\bar{v}$ by setting

$$x_{\bar{u}} = 0\,, \quad y_{\bar{u}} = 0\,, \quad x_{\bar{v}} = \|\rho(\bar{u}) - \rho(\bar{v})\|\,, \quad y_{\bar{v}} = 0\,. \tag{4.2}$$

We also impose that each 4-cycle (u_1, u_2, u_3, u_4) in G is a parallelogram:

$$\begin{aligned} x_{u_2} - x_{u_1} &= x_{u_3} - x_{u_4}\,, \quad & x_{u_4} - x_{u_1} &= x_{u_3} - x_{u_2}\,, \\ y_{u_2} - y_{u_1} &= y_{u_3} - y_{u_4}\,, \quad & y_{u_4} - y_{u_1} &= y_{u_3} - y_{u_2}\,. \end{aligned} \tag{4.3}$$

The existence of a flex of (G, ρ) implies that there are infinitely many placements in the zero set of the system consisting of Equations 4.1, 4.2 and 4.3. Hence, there is an irreducible algebraic curve C in the zero set. For every $u, v \in V_G$ such that $uv \in E_G$, we define $W_{u,v}$ in the complex function field of C by

$$W_{u,v} = (x_v - x_u) + i(y_v - y_u)\,.$$

There exists a valuation ν of the function field of C yielding a NAC-coloring δ of G by taking $\delta(uv) =$ red if $\nu(W_{u,v}) > 0$ and $\delta(uv) =$ blue otherwise, see [11, Theorem 3.1] for the details. Since $W_{u_1,u_2} = W_{u_4,u_3}$ for the opposite edges u_1u_2 and u_3u_4 of a 4-cycle (u_1, u_2, u_3, u_4), we have that $\delta(u_1u_2) = \delta(u_3u_4)$. Therefore, ribbons are monochromatic since a 4-cycle with a diagonal is monochromatic. The NAC-coloring δ is cartesian by Lemma 4.1 since every two vertices are separated by a ribbon by Theorem 3.9 applied to the underlying unbraced P-framework and Remark 3.12. □

Lemma 4.3. *Let (G, ρ) be a P-framework. Let $u, v \in V_G$ and W, W' be walks from u to v in G. If G has a cartesian NAC-coloring δ and $c \in \{red, blue\}$, then*

$$\sum_{\substack{(w_1,w_2)\in W \\ \delta(w_1w_2)=c}} (\rho(w_2) - \rho(w_1)) = \sum_{\substack{(w_1,w_2)\in W' \\ \delta(w_1w_2)=c}} (\rho(w_2) - \rho(w_1))\,.$$

Proof. Let $\widehat{W}$ be the walk obtained by concatenating W and the inverse of W'. We consider the sum

$$\sum_{\substack{(w_1,w_2)\in\widehat{W}\\ \delta(w_1w_2)=c}} (\rho(w_2)-\rho(w_1))\,.$$

Since each ribbon r is monochromatic in a cartesian NAC-coloring and $\widehat{W}$ is closed, the number of edges in $r\cap\widehat{W}$ included in the sum is even by Lemma 3.2 (the ribbons are simple by Remark 3.5). Hence, the sum is zero by Lemma 3.7. □

Using the lemma we show the reverse direction of Theorem 4.2. The proof is constructive, i.e., it provides a flex. In order to do so, we adapt the "zigzag" grid construction from [11]. A "zigzag" grid is determined by a column of points, whose copies are translated to other positions. Such a grid can flex so that the distances among all vertices in the same column/row remain constant. The idea of the construction of a flexible framework for a given graph with a NAC-coloring is to place the vertices of the graph to the "zigzag" grid using the NAC-coloring so that every blue, resp. red, component is in one column, resp. row of the grid, see Figure 10. Since there are no diagonals, the framework is flexible.

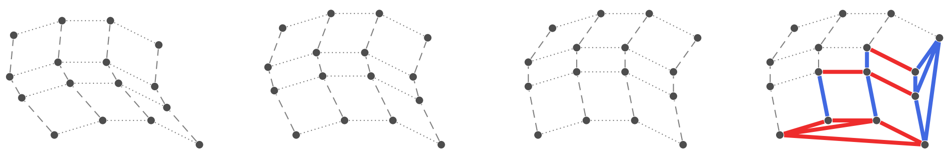


Figure 10: A flex of a "zigzag" grid and an example of a braced ribbon-cutting graph with a NAC-coloring placed to the grid so that there are no diagonals.

In case of P-frameworks (G,ρ), the grid can be chosen so that the flex starts at ρ if the graph allows a cartesian NAC-coloring, see Figure 11. We formalize these observations in the following theorem.

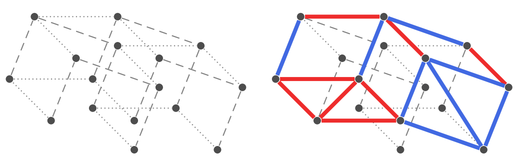


Figure 11: Any parallelogram placement of a braced ribbon cutting graph with a cartesian NAC-coloring can be obtained via some "zigzag" grid.

Theorem 4.4. *If a braced P-framework has a cartesian NAC-coloring, then it is flexible.*

Proof. Let (G',ρ) be a braced P-framework and δ' be a cartesian NAC-coloring of G'. We can assume that $\rho(\bar{u})=(0,0)$ for a fixed vertex $\bar{u}\in V_{G'}$. Let G be the graph G' with braces removed and δ be the NAC-coloring of G obtained by restricting δ'.

Let $R_1,\dots,R_m$, resp. $B_1,\dots,B_n$, be the vertex sets of the connected components of G^{δ}_{red}, resp. G^{δ}_{blue}. We define a map $\rho_{\text{red}}:\{R_1,\dots,R_m\}\to\mathbb{R}^2$ as follows: for R_i, let W

be any walk in G from $\bar{u}$ to a vertex of G in R_i and

$$\rho_{\text{red}}(R_i) = \sum_{\substack{(w_1,w_2)\in W \\ \delta(w_1w_2)=\text{blue}}} (\rho(w_2) - \rho(w_1)) \,.$$

Lemma 4.3 guarantees that it is well-defined, namely, the sum is independent of the choice of W and the vertex in R_i. We define $\rho_{\text{blue}} : \{B_1, \ldots, B_n\} \to \mathbb{R}^2$ analogously by swapping red and blue.

For $t \in [0, 2\pi]$ and $v \in V_G = V_{G'}$, where $v \in R_i \cap B_j$, let

$$\rho_t(v) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \cdot \rho_{\text{red}}(R_i) + \rho_{\text{blue}}(B_j) \,.$$

If W is a walk in G from $\bar{u}$ to v, then

$$\begin{aligned} \rho_0(v) &= \rho_{\text{red}}(R_i) + \rho_{\text{blue}}(B_j) \\ &= \sum_{\substack{(w_1,w_2)\in W \\ \delta(w_1w_2)=\text{blue}}} (\rho(w_2) - \rho(w_1)) + \sum_{\substack{(w_1,w_2)\in W \\ \delta(w_1w_2)=\text{red}}} (\rho(w_2) - \rho(w_1)) \\ &= \sum_{(w_1,w_2)\in W} (\rho(w_2) - \rho(w_1)) = \rho(v) - \rho(\bar{u}) = \rho(v) \,. \end{aligned}$$

We follow the argument from [11] that the lengths of the edges in $E_{G'}$ are constant along ρ_t. Notice that the vertex sets of G^{δ}_{red}, resp. G^{δ}_{blue}, and $G'^{\delta'}_{\text{red}}$, resp. $G'^{\delta'}_{\text{blue}}$, are the same since each brace has the same color as the 4-cycle it braces. Let uv be an edge in $E_{G'}$ with $u \in R_i \cap B_j$ and $v \in R_k \cap B_\ell$. If uv is red, then $i = k$ and hence

$$\|\rho_t(v) - \rho_t(u)\| = \|\rho_{\text{blue}}(B_\ell) - \rho_{\text{blue}}(B_j)\| \,.$$

On the other hand, if uv is blue, then $j = \ell$ and

$$\begin{aligned} \|\rho_t(v) - \rho_t(u)\| &= \left\| \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \cdot (\rho_{\text{red}}(R_k) - \rho_{\text{red}}(R_i)) \right\| \\ &= \|\rho_{\text{red}}(R_k) - \rho_{\text{red}}(R_i)\| \,. \end{aligned}$$

Since $\rho_0 = \rho$ and ρ is injective, none of the edges has length zero. Therefore, ρ_t is a flex of (G', ρ). □

Finally, we connect the results of flexibility and NAC-colorings with the connectivity of the bracing graph.

Theorem 4.5. *Let G be a braced ribbon-cutting graph such that every two vertices are separated by a ribbon. The bracing graph of G is connected if and only if G does not have a cartesian NAC-coloring.*

Proof. Let B be the bracing graph of G. In a cartesian NAC-coloring of G, ribbons are monochromatic by Lemma 4.1. Hence, if two ribbons are adjacent in B, then the union of their edges is monochromatic. Therefore, if B is connected, all edges of G must have the same color, namely, no cartesian NAC-coloring exists.

For the opposite implication, assume B is not connected. We color the edges of the ribbons of one connected component by red and the rest by blue. To show that this surjective edge coloring is a NAC-coloring, consider a cycle C. Let uv be an edge of C and r be the ribbon containing uv. Since r separates G, r contains another edge $u'v'$ of C. Since ribbons are monochromatic, either all edges of C have the same color or there are two edges of each color. The obtained NAC-coloring is cartesian by Lemma 4.1. □

This forms the last part of the proof of Theorem 1.1.

Proof of Theorem 1.1. Let (G, ρ) be a braced P-framework. Every two vertices are separated by a ribbon by Theorem 3.9 and Remark 3.12. Hence, (G, ρ) is rigid if and only if G has no cartesian NAC-coloring (Theorems 4.2 and 4.4) if and only if the bracing graph of G is connected (Theorem 4.5).

Each edge of the bracing graph corresponds to at least one brace. The minimum number of braces making the framework rigid follows from the fact that the number of edges of a spanning tree of the bracing graph is one less than the number of vertices, i.e., ribbons. The result is also illustrated in Figure 12. □

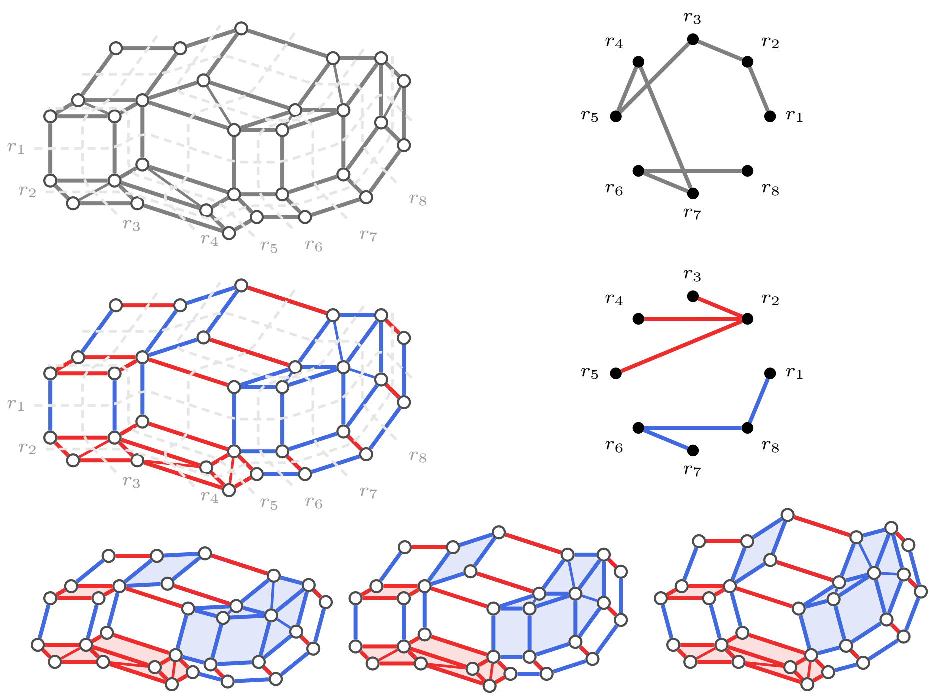


Figure 12: Two bracings of a P-framework where the first one is rigid as visible by the connectivity of the bracing graph. The second bracing yields a flexible framework since the bracing graph is not connected. We show three instances of the flex that is possible with the bracing and the unique resulting cartesian NAC-coloring thereof (shaded parallelograms preserves their shapes).

A consequence of Theorem 1.1 is that rigidity of a braced P-framework is a combinatorial property, not a geometric one.

Corollary 4.6. *If G is a braced ribbon-cutting graph admitting a parallelogram placement, then either*

(i) (G, ρ) *is rigid for all parallelogram placements ρ of G, or*

(ii) (G, ρ) *is flexible for all parallelogram placements ρ of G.*

Conclusion

We have applied the theory of NAC-colorings to P-frameworks generalizing previous results in the area of bracing grids. In fact, we have shown that a P-framework is rigid if and only if it has no cartesian NAC-coloring if and only if the bracing graph is connected. Notice that a consequence of this statement is that a braced rectangular grid/rhombic carpet is rigid if and only if it is infinitesimally rigid. This is not the case for grids with holes as there are instances which are rigid but not infinitesimally rigid (an example can be obtained by bracing all squares besides those with the indicated ribbons in Figure 7).

Similarly as in rectangular grids, there are plenty of interesting questions for further generalizations such as graphs with holes, different types of diagonals or higher dimensions. For P-frameworks these questions are subject to further research.

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References

[1] A. Biswas, Flexibility and movability in Cayley graphs, *Ann. Comb.* (2022), doi:10.1007/s00026-022-00569-4.

[2] O. Bodini, T. Fernique, M. Rao and E. Rémila, Distances on rhombus tilings, *Theoret. Comput. Sci.* **412** (2011), 4787–4794, doi:10.1016/j.tcs.2011.04.015.

[3] E. D. Bolker, Bracing Rectangular Frameworks. II, *SIAM J. Appl. Math.* **36** (1979), 491–508, doi:10.1137/0136037.

[4] E. D. Bolker and H. Crapo, Bracing Rectangular Frameworks. I, *SIAM J. Appl. Math.* **36** (1979), 473–490, doi:10.1137/0136036.

[5] N. Chakravarty, G. Holman, S. McGuinness and A. Recski, One-story buildings as tensegrity frameworks, *Structural Topology* **12** (1986), 11–18, `https://hdl.handle.net/2099/1029`.

[6] N. Destainville, M. Widom, R. Mosseri and F. Bailly, Random tilings of high symmetry: I. Mean-field theory, *J. Stat. Phys.* **120** (2005), 799–835, doi:10.1007/s10955-005-6989-y.

[7] E. M. Duarte Gélvez, E-mail conversation, 2020.

[8] E. M. Duarte Gélvez and G. K. Francis, Stability of quasicrystal frameworks in 2D and 3D, in: *Proceedings of the First Conference Transformables 2013*, School of Architecture Seville, Spain, 2013 .

[9] W. G. Ellenbroek and X. Mao, Rigidity percolation on the square lattice, *EPL (Europhysics Letters)* **96** (2011), 54002, doi:10.1209/0295-5075/96/54002.

[10] D. Frettlöh and E. Harriss, Parallelogram Tilings, Worms, and Finite Orientations, *Discrete. Comput. Geom.* **49** (2013), 531–539, doi:10.1007/s00454-012-9478-5.

[11] G. Grasegger, J. Legerský and J. Schicho, Graphs with Flexible Labelings, *Discrete Comput. Geom.* **62** (2019), 461–480, doi:10.1007/s00454-018-0026-9.

[12] G. Grasegger and J. Legerský, FlexRiLoG—A SageMath Package for Motions of Graphs, in: A. M. Bigatti, J. Carette, J. H. Davenport, M. Joswig and T. de Wolff (eds.), *Mathematical Software – ICMS 2020*, Springer, volume 12097 of *Lecture Notes in Computer Science*, 2020 pp. 442–450, doi:10.1007/978-3-030-52200-1_44.

[13] Z. Gáspár, N. Radics and A. Recski, Square grids with long "diagonals", *Optim. Methods Softw.* **10** (1998), 217–231, doi:10.1080/10556789808805712.

[14] Z. Gáspár, N. Radics and A. Recski, Rigidity of square grids with holes, *Comput. Assist. Methods Eng. Sci.* **6** (1999), 329–335.

[15] R. Hammack, E-mail conversation, 2019.

[16] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs, Second Edition*, CRC Press, Inc., USA, 2nd edition, 2011, doi:10.1201/b10959.

[17] Y. Ito, Y. Kobayashi, Y. Higashikawa, N. Katoh, S. H. Poon and M. Saumell, Optimally bracing grid frameworks with holes, *Theor. Comput. Sci.* **607** (2015), 337–350, doi:10.1016/j.tcs.2015.07.062.

[18] G. Nagy Kem, Bracing rhombic structure by one-dimensional tensegrities, *Meccanica* **52** (2017), 1283–1293, doi:10.1007/s11012-016-0477-0.

[19] N. Radics, Rigidity of multi-story buildings, *Discrete Appl. Math.* **115** (2001), 187–198, doi:10.1016/s0166-218x(01)00224-4.

[20] N. Radics and A. Recski, Applications of combinatorics to statics — rigidity of grids, *Discrete Appl. Math.* **123** (2002), 473–485, doi:10.1016/s0166-218x(01)00350-x.

[21] A. Recski, Bracing cubic grids — a necessary condition, *Discrete Math.* **73** (1988), 199–206, doi:10.1016/0012-365x(88)90148-3.

[22] A. Recski, One-story buildings as tensegrity frameworks II, *Structural Topology* **17** (1991), 43–52, https://hdl.handle.net/2099/1065.

[23] A. Recski, Applications of combinatorics to statics — a second survey, *Discrete Math.* **108** (1992), 183–188, doi:10.1016/0012-365x(92)90673-4.

[24] A. Recski, Maps of matroids with applications, *Discrete Math.* **303** (2005), 175–185, doi:10.1016/j.disc.2004.12.023.

[25] A. Recski and W. Schwärzler, One-story buildings as tensegrity frameworks III, *Discrete Appl. Math.* **39** (1992), 137–146, doi:10.1016/0166-218x(92)90164-6.

[26] B. Servatius, Graphs, digraphs, and the rigidity of grids, *UMAP Journal* **16** (1995), 37–63.

[27] T. Wester, The Structural Morphology of Penrose and Quasicrystal Patterns Part I, *Adaptables2006, TU/e, International Conference On Adaptable Building Structures* (2006), 2–6.

[28] L. Zhang, D. Z. Rocklin, B. G. G. Chen and X. Mao, Rigidity percolation by next-nearest-neighbor bonds on generic and regular isostatic lattices, *Phys. Rev. E* **91** (2015), 1–18, doi:10.1103/physreve.91.032124.

[29] D. Zhou, L. Zhang and X. Mao, Topological Boundary Floppy Modes in Quasicrystals, *Phys. Rev. X* **9** (2019), 21054, doi:10.1103/physrevx.9.021054.

Appendix

A Cartesian products of graphs and NAC-colorings

In this section, we show a connection of cartesian NAC-colorings with cartesian products of graphs. Recall that the cartesian product of graphs G and H is given by

$$G \,\square\, H = (V_G \times V_H, \{(u,u')(v,v') \colon (u = v \wedge u'v' \in E_H) \vee (u' = v' \wedge uv \in E_G)\}) .$$

By coloring edges coming from G by red and the rest blue, the following holds.

Theorem A.1 ([1, 15])**.** *The cartesian product of any two nontrivial graphs G and H has a cartesian NAC-coloring.*

We remark that the statement of Theorem A.1 has been pointed out by [15] independently of [1]. In [1] a cartesian NAC-coloring is called *good* since applying the grid construction described in [11] yields a proper flexible framework, whereas for a non-cartesian NAC-coloring there are overlapping vertices. Our naming is motivated by the fact that the converse statement can be proved using ideas from [16] about embeddings of graphs into cartesian products.

Theorem A.2. *If a graph G has a cartesian NAC-coloring, then there are graphs Q_1, Q_2 with at least two vertices each and an injective graph morphism $h : G \to Q_1 \,\square\, Q_2$ such that each vertex in $V_{Q_1} \cup V_{Q_2}$ occurs as a coordinate of a vertex in $h(G)$. In particular, G can be viewed as a subgraph of $Q_1 \,\square\, Q_2$.*

Proof. Let δ be a cartesian NAC-coloring of G. Let $R_1, \dots, R_m$, resp. $B_1, \dots, B_n$, be the vertex sets of the connected components of G^δ_{red}, resp. G^δ_{blue}. Since δ is surjective and no blue edge can connect vertices of the same red component [11, Lemma 2.4], $m \geq 2$ and $n \geq 2$. Let $\pi_{\text{red}} : V_G \to \{R_1, \dots, R_m\}$ and $\pi_{\text{blue}} : V_G \to \{B_1, \dots, B_m\}$ map a vertex to the vertex set of its red, resp. blue, component, namely, $\pi_{\text{red}}(v) = R_i$ and $\pi_{\text{blue}}(v) = B_j$ if $v \in R_i \cap B_j$. We define the following quotient graphs

$$\begin{aligned} Q_1 &= (\{R_1, \dots, R_m\}, \{\pi_{\text{red}}(u)\pi_{\text{red}}(v) \colon uv \in E_G \text{ and } \delta(uv) = \text{blue}\}) \;, \\ Q_2 &= (\{B_1, \dots, B_n\}, \{\pi_{\text{blue}}(u)\pi_{\text{blue}}(v) \colon uv \in E_G \text{ and } \delta(uv) = \text{red}\}) \;. \end{aligned}$$

Let Q be the cartesian product of Q_1 and Q_2, and $h : V_G \to V_Q$ be the graph morphism given by

$$h(v) = (\pi_{\text{red}}(v), \pi_{\text{blue}}(v)) \,.$$

We check that it is indeed a morphism: if uv is an edge of G, w.l.o.g. red, then $\pi_{\text{red}}(u) = \pi_{\text{red}}(v)$ and $\pi_{\text{blue}}(u) \neq \pi_{\text{blue}}(v)$ from the properties of NAC-colorings. Thus, $h(u)h(v) = (\pi_{\text{red}}(u), \pi_{\text{blue}}(u))(\pi_{\text{red}}(u), \pi_{\text{blue}}(v))$ which is an edge of Q. The morphism h is injective by Remark 2.7 since δ is cartesian. Each vertex in $V_{Q_1} \cup V_{Q_2}$ occurs as a coordinate of a vertex in $h(G)$, since π_{red} and π_{blue} are surjective. $\square$

B Carpet frameworks are P-frameworks

As indicated in Section 3.1, we prove here that a carpet framework is P-framework. For this, we define the following class of graphs.

Definition B.1. We define the class of graphs $\mathcal{G}_{\text{rec}}$ recursively. The 4-cycle graph is in $\mathcal{G}_{\text{rec}}$. There are two types of construction (see also Figure 13):

ADD4-CYCLE: If $G \in \mathcal{G}_{\text{rec}}$ with $uv \in E_G$, then the graph $(V_G \cup \{w_1, w_2\}, E_G \cup \{uw_1, w_1w_2, w_2v\})$ is in $\mathcal{G}_{\text{rec}}$, where $w_1, w_2 \notin V_G$.

CLOSE4-CYCLE: If $G \in \mathcal{G}_{\text{rec}}$ with $uv, vw \in E_G$ and the vertex v is separated from any vertex in $V_G \setminus \{u, v, w\}$ by a ribbon which does not contain uv or vw, then the graph $(V_G \cup \{w'\}, E_G \cup \{uw', w'w\})$, where $w' \notin V_G$, is in $\mathcal{G}_{\text{rec}}$.
Note that the separation assumption is needed for avoiding situations as described in Figure 6.

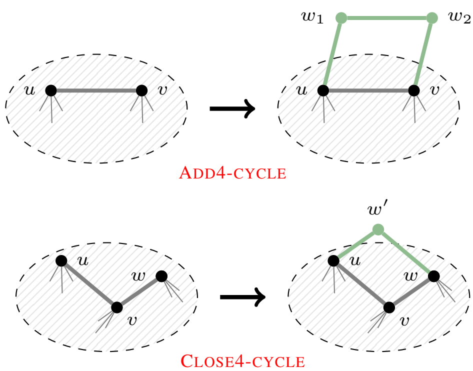

Figure 13: Two recursive $\mathcal{G}_{\text{rec}}$ constructions.

Figure 5b gives an example of a graph in $\mathcal{G}_{\text{rec}}$ that is not the underlying graph of a carpet framework. It is easy to use the construction to show that the class has the ribbon-cutting property.

Proposition B.2. *Every graph in $\mathcal{G}_{rec}$ is ribbon-cutting.*

Proof. By structural induction: the 4-cycle graph is ribbon-cutting. ADD4-CYCLE preserves the property since $\{uw_1, w_2v\}$ is a new ribbon and w_1w_2 belongs to the ribbon of uv. CLOSE4-CYCLE does so as well: the edges uw' and $w'w$ belong to the ribbons of vw and uv respectively. If any ribbon of the extended graph were not an edge cut, than it would not be an edge cut in the original graph. Notice that the separation assumption is not needed for this. $\square$

Recall that for a P-framework (G, ρ), any ribbon r is simple by Remark 3.5 and $G \setminus r$ has two connected components by Lemma 3.2. This allows us to translate the vertices of one of the components by a constant vector.

Remark B.3. Let (G, ρ) be a P-framework and r be a ribbon of G. Let V_1 and V_2 be the vertex sets of the two connected components of $G \setminus r$. For every vector $t \in \mathbb{R}^2 \setminus \{\rho(u_1) - \rho(u_2) \colon u_1 \in V_1, u_2 \in V_2\}$, the placement ρ' of G given by $\rho'(v) = \rho(v) + t$ if $v \in V_2$ and $\rho'(v) = \rho(v)$ otherwise is a parallelogram placement.

We are going to show the relation between P-frameworks, carpet frameworks and the graphs in $\mathcal{G}_{\text{rec}}$. Namely, the underlying graphs of carpet frameworks are in $\mathcal{G}_{\text{rec}}$, which is in turn a subset of the underlying graphs of P-frameworks. For this we need an equivalent condition to the separation assumption in CLOSE4-CYCLE.

Lemma B.4. *For a P-framework (G, ρ) and $uv, vw \in E_G$, the following are equivalent:*

(i) *The vertex v is separated from any vertex in $V_G \setminus \{u, v, w\}$ by a ribbon which does not contain uv or vw.*

(ii) *There exists a parallelogram placement ρ' of the graph $G' = (V_G \cup \{w'\}, E_G \cup \{uw', w'w\})$, where $w' \notin V_G$.*

Proof. (i) $\implies$ (ii) If we want to extend ρ to a parallelogram placement of G', the position $\rho(w')$ of the new vertex w' is uniquely determined by the requirement that $(\rho(u), \rho(v), \rho(w), \rho(w'))$ is a parallelogram. We can assume that $\rho(u), \rho(v), \rho(w)$ are not collinear, hence, $\rho(v) \neq \rho(w')$. If it is not so, we replace ρ by a parallelogram placement obtained by Remark B.3 for the ribbon of uv and a non-zero translation.

If $\rho : V_{G'} \to \mathbb{R}^2$ is injective, we are done. Otherwise, $\rho(w') = \rho(u')$ for a unique vertex $u' \in V_G \setminus \{u, v, w\}$. By assumption, there is a ribbon r separating v from u such that $uv, vw \notin r$. Thus, u, v, w are in the same connected component of $G \setminus r$, whereas u' is in the other one. Using Remark B.3, there is a parallelogram placement ρ' of G such that $\rho(w') \neq \rho'(u')$. Moreover, the translation vector can be chosen so that the whole image $\rho'(V_G)$ avoids $\rho(w')$. Therefore, ρ' uniquely extends to a parallelogram placement of G' by setting $\rho'(w') = \rho(w')$.

$\neg$(i) $\implies$ $\neg$(ii) Assume that $u' \in V_G \setminus \{u, v, w\}$ is a vertex such that it is separated from v only by the ribbon of uv or vw. Let $W = (v = u_0, u_1, \ldots, u_m = u')$ be a walk from v to u'. Let R be the set of ribbons which contains at least one edge of W. All ribbons are simple by Remark 3.5. By the assumption and Lemma 3.2, $|r \cap W|$ is even for every ribbon r avoiding uv and vw. For any parallelogram placement ρ of G, we have

$$
\begin{aligned}
\rho(u') - \rho(v) &= \sum_{i=1}^{m} (\rho(u_i) - \rho(u_{i-1})) = \sum_{r \in R} \sum_{(w_1, w_2) \in r \cap W} (\rho(w_2) - \rho(w_1)) \\
&\overset{3.7}{=} \sum_{\substack{r \in R \\ uv \in r \vee vw \in r}} \sum_{(w_1, w_2) \in r \cap W} (\rho(w_2) - \rho(w_1)) \\
&\overset{3.6}{=} \alpha(\rho(w) - \rho(v)) + \beta(\rho(u) - \rho(v)) \,,
\end{aligned}
$$

where $\alpha, \beta \in \{0, 1\}$. Actually, $\alpha = \beta = 1$, otherwise $\rho(u') = \rho(w)$ or $\rho(u') = \rho(u)$, which violates injectivity. Hence, $\rho(u') = \rho(w) + \rho(u) - \rho(v)$. Assume for contradiction that there is a parallelogram placement ρ' of G'. Since $\rho'|_{V_G}$ is a parallelogram placement of G, we have by the previous $\rho'(u') = \rho'(w) + \rho'(u) - \rho'(v)$. But this is a contradiction since $\rho'(w') = \rho'(w) + \rho'(u) - \rho'(v)$ as well and $w' \neq u'$. $\square$

Corollary B.5. *There exists a P-framework (G, ρ) for every $G \in \mathcal{G}_{rec}$.*

Proof. We proceed by structural induction. The 4-cycle can be placed as a parallelogram. For a graph G' constructed using ADD4-CYCLE from G, a parallelogram placement of G can be extended to a parallelogram placement of G' by placing the two new vertices to

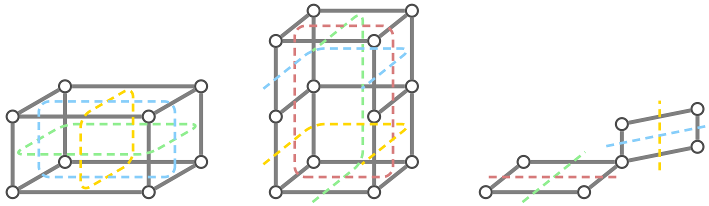

Figure 14: P-frameworks whose underlying graphs are not in $\mathcal{G}_{\text{rec}}$.

form a parallelogram so that the placement is injective. If G' is constructed from G by CLOSE4-CYCLE, then there exists a parallelogram placement of G' by Lemma B.4. □

Note that there are P-frameworks whose underlying graphs are not in $\mathcal{G}_{\text{rec}}$, see Figure 14.

Corollary B.6. *If (G, ρ) is a carpet framework, then $G \in \mathcal{G}_{rec}$. In particular, (G, ρ) is a P-framework.*

Proof. By the definition of carpet framework, ρ is a parallelogram placement. Once we show that $G \in \mathcal{G}_{\text{rec}}$, the fact that (G, ρ) is a P-framework follows from Proposition B.2.

We proceed by induction on the number of parallelograms yielding a carpet framework. Let S be the set of parallelograms in $\mathbb{R}^2$ giving a carpet framework (G', ρ') according to Definition 3.10. If $|S| = 1$, then (G', ρ') is the 4-cycle with a parallelogram placement, hence, $G' \in \mathcal{G}_{\text{rec}}$. Suppose that $|S| \geq 2$. The boundary of $\bigcup S$ is a simple polygon M with k edges. We divide the parallelograms having an edge in the polygon M into the following categories (see Figure 15):

- K_1 — parallelograms with one edge in M,
- K_2 — parallelograms with two incident edges in M such that the vertex that is not in these two edges is not in M,
- K_2' — parallelograms with two incident edges in M that are not in K_2,
- K_2'' — parallelograms with two opposite edges in M,
- K_3 — parallelograms with three edges in M.

Clearly, $k = |K_1| + 2|K_2| + 2|K_2'| + 2|K_2''| + 3|K_3|$. The sum of the interior angles of the simple polygon M equals $(k - 2)\pi$. Considering contributions to the sum for parallelograms in the categories above (see Figure 15), we have

$$\begin{aligned} |K_1|\pi + |K_2|\pi + 2|K_2'|\pi + 2|K_2''|\pi + 2|K_3|\pi &\leq (k-2)\pi \\ \iff \qquad 2 &\leq |K_2| + |K_3|\,. \end{aligned}$$

For a parallelogram s in $K_2 \cup K_3$, $S \setminus s$ satisfies the assumptions of Definition 3.10. Thus, we have a carpet framework (G, ρ) and G is in $\mathcal{G}_{\text{rec}}$ by induction assumption. If $s \in K_3$, then G can be extended to G' by ADD4-CYCLE. If $s \in K_2$, then G can be extended to G' by CLOSE4-CYCLE, since the separation assumption is satisfied by Lemma B.4 and the placement ρ'. □

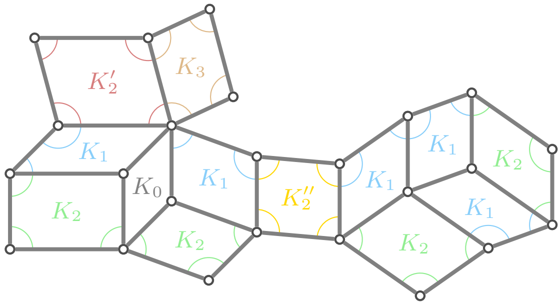

Figure 15: An example of dividing parallelograms into categories according to their intersection with the boundary. The angles whose contribution to the sum of the interior angles is considered are indicated. Note that the parallelogram labeled K_0 belongs to S but is not part of the boundary.

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Constructing integer-magic graphs via the combinatorial nullstellensatz*

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Abstract

Let A be a nontrivial additive abelian group and $A^* = A \setminus \{0\}$. A graph is A-magic if there exists an edge labeling f using elements of A^* which induces a constant vertex labeling of the graph. Such a labeling f is called an A-magic labeling and the constant value of the induced vertex labeling is called an A-magic value. In this paper, we use the Combinatorial Nullstellensatz to construct nontrivial classes of $\mathbb{Z}_p$-magic graphs, prime $p \geq 3$. For these graphs, some lower bounds on the number of distinct $\mathbb{Z}_p$-magic labelings are also established.

Keywords: integer-magic graph, integer-magic labeling, Combinatorial Nullstellensatz

Math. Subj. Class.: 05C78

1 Introduction

Let $G = (V, E)$ be a graph, where G might be disconnected and/or a multigraph. For any nontrivial additive abelian group A, let $A^* = A \setminus \{0\}$. A mapping $f : E(G) \to A^*$ is called an *edge labeling* of G. Any such edge labeling induces a *vertex labeling* $f^+ : V(G) \to A$, where the label at a vertex is the sum of the edge labels incident to that vertex. Here, a loop label is counted only once. An edge labeling f whose induced mapping f^+ on $V(G)$

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is a constant is called an *A-magic labeling* of G. In this case, the constant is called the *A-magic value* of f and G is called an *A-magic* graph. If G has a $\mathbb{Z}_k$-magic labeling (for some $k \geq 2$), then G is an *integer-magic* graph. The *integer-magic spectrum* of a graph G is the set $\mathrm{IM}(G) = \{k \geq 2 : G \text{ is } \mathbb{Z}_k\text{-magic}\}$. Generally speaking, it is quite difficult to determine the integer-magic spectrum of a graph. Note that the integer-magic spectrum of a graph is not to be confused with the set of achievable magic values.

The concept of an A-magic graph was first introduced in [12]. Since then, A-magic graph labelings have been studied in [15, 20, 22, 37, 39, 41] and $\mathbb{Z}_k$-magic graphs were investigated in [11, 13, 14, 16, 17, 18, 19, 21, 24, 25, 30, 31, 32, 33, 42, 38, 40]. $\mathbb{Z}$-magic graphs were considered by Stanley [43, 44], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous Diophantine equations. They were also considered in [2, 34].

Labelings form a large and important area of study in graph theory. First formally introduced by Rosa [29] in the 1960s, graph labelings have captivated the interest of many mathematicians in the ensuing decades. In addition to the intrinsic beauty of the subject matter, graph labelings have applications (discussed in papers by Bloom and Golomb [4, 3]) in graph factorization problems, X-ray crystallography, radar pulse code design, and addressing systems in communication networks. The interested reader is directed to Gallian's [6] dynamic survey, which contains 2900+ references to research papers and books on the topic of graph labelings.

2 Preliminaries

All graph-theoretic terms (which are not explicitly defined) are standard ones and can be found in [7]. Throughout this paper, we consider general graphs which might be disconnected and/or a multigraph. We first note a few important facts which are known about $\mathbb{Z}_k$-magic labelings. Lemmas 2.2 and 2.4 are found in [20], whereas Lemma 2.1 is a slight generalization of a lemma found in [20].

Lemma 2.1. *For a graph G, let $i(v)$ denote the number of edges, multiedges and loops incident to $v \in V(G)$. Then, G is $\mathbb{Z}_2$-magic $\iff$ $i(v)$ are of the same parity, for all $v \in V(G)$.*

Lemma 2.2. *If G is $\mathbb{Z}_k$-magic and $k|n$, then G is $\mathbb{Z}_n$-magic.*

Remark 2.3. The converse of Lemma 2.2 is not true, in general. For example, it was shown in [13] that $\mathrm{IM}(K_4 - \{uv\}) = \{4, 6, 8, \dots\}$. In particular, $K_4 - \{uv\}$ is $\mathbb{Z}_6$-magic. However, $K_4 - \{uv\}$ is not $\mathbb{Z}_3$-magic.

Lemma 2.4. *Let p be prime. If G is $\mathbb{Z}_p$-magic for some magic value $t \neq 0$, then G is $\mathbb{Z}_p$-magic with magic value t' for any nonzero $t' \in \mathbb{Z}_p$.*

Proof. Let $b = t't^{-1}$. Multiply all of the edge labels by b. Since $\mathbb{Z}_p$ is a field, this gives edge labels which are non-zero. Hence, we have the desired $\mathbb{Z}_p$-magic labeling. □

Lemmas 2.1 and 2.2 allow us to focus on primes $p \geq 3$. Because of Lemma 2.4, it suffices to look at $\mathbb{Z}_p$-magic labelings with magic values equal to 0 and 1.

3 The Combinatorial Nullstellensatz

In [1], Alon proved the following result and successfully applied it to problems in additive number theory and graph theory.

Theorem 3.1 (Combinatorial Nullstellensatz)**.** *Let $f = f(x_1, \dots, x_m)$ be a polynomial of degree d over a field $\mathbb{F}$. Suppose that the coefficient of the monomial $x_1^{t_1} \cdots x_m^{t_m}$ in f is nonzero and $t_1 + \cdots + t_m = d$. If $S_1, \dots, S_m$ are subsets of $\mathbb{F}$ with $|S_i| \geq t_i + 1$, then there exists an $\underline{x}' = (x_1', x_2', \dots, x_m') \in S_1 \times \cdots \times S_m$ for which $f(\underline{x}') \neq 0$.*

Example 3.2. Let $f(x_1, x_2, x_3, x_4) = x_1^4 x_2 x_3 - 2x_1^5 + x_1^2 x_2^2 x_3^2 + x_4^2 \in \mathbb{Z}_3[x_1, x_2, x_3, x_4]$. We will apply Theorem 3.1 on the term $x_1^2 x_2^2 x_3^2$ in f. Note that $\deg(f) = 6 = \deg(x_1^2 x_2^2 x_3^2)$. We choose $S_1 = \{0, 1, 2\}$, $S_2 = \{0, 1, 2\}$, $S_3 = \{0, 1, 2\}$ and $S_4 = \{2\}$. Then, Theorem 3.1 implies that there exist $s_i \in S_i$, where $1 \leq i \leq 4$, such that $f(s_1, s_2, s_3, s_4) \neq 0$. Note that the Combinatorial Nullstellensatz cannot be applied to any of the other monomial terms in f.

After its discovery, the Combinatorial Nullstellensatz would soon become a powerful tool in extremal combinatorics [10]. With regards to graph labeling and coloring problems, it has been used to prove theorems on anti-magic labelings, neighbor sum distinguishing total colorings, and list colorings [27, 36, 8]. For a recent research monograph on the Combinatorial Nullstellensatz and graph coloring problems, the reader is directed to [45].

4 The Hartke polynomials

Let $G = (V, E)$, where $|V(G)| = n$, $|E(G)| = m$, and the edges, multiedges and loops of G are identified with variables $x_1, x_2, \dots, x_m$. As mentioned previously, we will focus on $\mathbb{Z}_p$-magic labelings (prime $p \geq 3$) and magic values equal to 0 and 1. For fixed prime $p \geq 3$ and $t \in \{0, 1\}$, define the polynomials f_t in $\mathbb{Z}_p[x_1, \dots, x_m]$ as

$$f_t(\underline{x}) = f_t(x_1, \dots, x_m) = \prod_{v \in V(G)} \left[1 - \left(t - \sum_{v \in x_j} x_j \right)^{p-1} \right], \tag{4.1}$$

where the addition and multiplication are taken modulo p. The given factorization of (4.1) and its factors are called the *canonical factorization* and *canonical factors* of f_t, respectively. The f_t are called *Hartke polynomials* and were introduced in [23]. Note that each Hartke polynomial describes a unique graph G up to isomorphism.

In this section, we recall the basic properties of f_t (see [23]) and give additional analysis of these polynomials. This is used in conjunction with Theorem 3.1 in subsequent sections of this paper, where we construct $\mathbb{Z}_p$-magic graphs.

Remark 4.1. Note that $\deg(f_t(\underline{x})) = (p-1) \cdot |V(G)|$. This follows from:

1. There are $|V(G)|$ canonical factors of $f_t(\underline{x})$.
2. Each of the canonical factors is of degree $p - 1$.
3. Theorem in [9]: Let R be a commutative ring with unity and $g, h \in R[x_1, x_2, \dots, x_m]$. If R has no zero divisors, then $\deg(gh) = \deg(g) + \deg(h)$.

Observations 4.2. Let $\underline{x}'$ be an m-tuple in $\mathbb{Z}_p^* \times \mathbb{Z}_p^* \times \cdots \times \mathbb{Z}_p^*$. Then, we note the following:

1. $f_t(\underline{x})$ is defined for all connected multigraphs G.
2. The range of $f_t(\underline{x})$ is $\{0, 1\}$. This follows from the fact that each canonical factor of f_t takes on a value of 0 or 1, due to Fermat's Little Theorem [9]: If p is prime, then $a^p = a$ for all $a \in \mathbb{Z}_p$.
3. $f_0(\underline{x}') = 1 \Rightarrow \underline{x}'$ is a $\mathbb{Z}_p$-magic labeling of G with magic value 0.
4. $f_1(\underline{x}') = 1 \Rightarrow \underline{x}'$ is a $\mathbb{Z}_p$-magic labeling of G with magic value 1.
5. $f_0(\underline{x}') = 0$ and $f_1(\underline{x}') = 0 \Rightarrow \underline{x}'$ is not a $\mathbb{Z}_p$-magic labeling of G with magic value 0 or 1.
6. $f_0(\underline{x}') = 1 \Rightarrow f_1(\underline{x}') = 0$. If $f_0(\underline{x}') = 1$, then $\underline{x}'$ is a $\mathbb{Z}_p$-magic labeling of G with magic value 0. Thus, $\underline{x}'$ is not a $\mathbb{Z}_p$-magic labeling of G with magic value 1.
7. $f_1(\underline{x}') = 1 \Rightarrow f_0(\underline{x}') = 0$. This is the contrapositive of Observation 6.

Two techniques are often used to establish results in graph labeling problems. Either a construction of a desired labeling is obtained through ingenuity, or one shows the non-existence of the labeling (via proof by contradiction). In practice, these methods can be time-consuming and difficult to use.

In [23], the Combinatorial Nullstellensatz and Hartke polynomials were used to prove that certain graphs were $\mathbb{Z}_p$-magic (prime $p \geq 3$), without having to construct an actual $\mathbb{Z}_p$-magic labeling. As far as the authors know, it was the first time that a nonconstructive method was used to analyze integer-magic graph labelings. The focus of this paper is to use the Combinatorial Nullstellensatz to construct $\mathbb{Z}_p$-magic graphs, for prime $p \geq 3$. In particular, we construct *Hartke* $\mathbb{Z}_p$-magic graphs.

Definition 4.3. Let $p \geq 3$ be prime. A graph G is called *Hartke* $\mathbb{Z}_p$-magic if Theorem 3.1 can be used on a Hartke polynomial f_t of G to prove that G is $\mathbb{Z}_p$-magic. In this case, a nonvanishing monomial term M of degree $(p-1) \cdot |V(G)|$ in the expansion of f_t (where Theorem 3.1 is applied in such a manner) is called a *Hartke term*.

Example 4.4. Let $p = 3$ and G_7 be the graph illustrated in Figure 1. Note that G_7 is the graph F4 in [28]. Using Mathematica 12, we see that $\deg(f_0(\underline{x})) = 16$ and that $f_0(\underline{x})$ contains the monomial term $14336 x_5 x_6 \cdots x_{20} \equiv 2 x_5 x_6 \cdots x_{20} \pmod 3$. Let $S_i = \{1, 2\}$ for $i = 5, 6, \ldots, 20$, and $S_i = \{1\}$ for $i = 1, 2, 3$ and 4. So by Theorem 3.1, we have that $f_0(\underline{x}') \neq 0$, for some $\underline{x}' \in S_1 \times S_2 \times \cdots \times S_{20}$. Thus, $f_0(\underline{x}') = 1$ and we conclude that G_7 is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0.

Proposition 4.5. *Let $p \geq 3$ be prime and G be a graph with Hartke polynomial f_t. Then, G is Hartke $\mathbb{Z}_p$-magic with magic value $t \iff f_t$ has a nonvanishing monomial term M of degree $(p-1) \cdot |V(G)|$, where all the exponents t_i satisfy* $0 \leq t_i \leq p-2$.

Proof. $(\Longrightarrow)$. Suppose that G is a Hartke $\mathbb{Z}_p$-magic graph with Hartke polynomial f_t. Then, there exists a Hartke term M of degree $(p-1) \cdot |V(G)|$ in f_t. Since G is a Hartke $\mathbb{Z}_p$-magic graph, there exist nonempty subsets $S_1, S_2, \ldots, S_m$ of $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$ corresponding to the m variables in f_t, which satisfy the hypothesis of Theorem 3.1 (when applied to M). In particular, all of the exponents t_i of M satisfy $0 \leq t_i \leq p-2$.

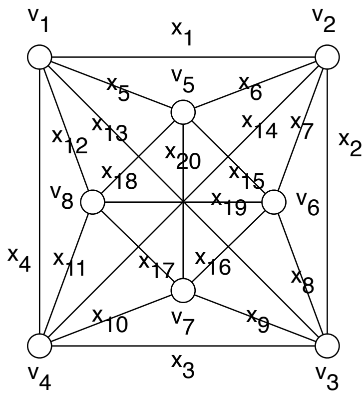

Figure 1: G_7 has a $\mathbb{Z}_3$-magic labeling with magic value 0.

($\Longleftarrow$). Suppose that f_t has a nonvanishing monomial term M of degree $(p-1)\cdot|V(G)|$, where all of the exponents t_i satisfy $0 \le t_i \le p-2$. For each exponent t_i (associated with variable x_i) appearing in M, choose a nonempty subset S_i of $\mathbb{Z}_p^* = \{1, 2, \dots, p-1\}$ where $|S_i| \ge t_i + 1$. Thus by Theorem 3.1, G has a $\mathbb{Z}_p$-magic labeling. In particular, G is Hartke $\mathbb{Z}_p$-magic with magic value t. □

Proposition 4.6. *Let $p \ge 3$ be prime. If G is a Hartke $\mathbb{Z}_p$-magic graph, then $|E(G)| \ge \frac{p-1}{p-2} \cdot |V(G)|$.*

Proof. We prove the contrapositive. If $|E(G)| < \frac{p-1}{p-2} \cdot |V(G)|$, then a straightforward counting argument shows that every nonvanishing monomial of degree $(p-1)\cdot|V(G)|$ in f_t has an exponent $t_i \ge p-1$. Thus by Proposition 4.5, G is not a Hartke $\mathbb{Z}_p$-magic graph. □

Remark 4.7. The converse of Proposition 4.6 is not true, in general. For example, let $p = 3$ and consider the graph G comprised of P_3 with K_6 attached at an end-vertex. Then, G has 8 vertices and 17 edges. Thus, the inequality $|E(G)| \ge \frac{p-1}{p-2} \cdot |V(G)|$ is satisfied. However, G is not $\mathbb{Z}_3$-magic since P_3 is not $\mathbb{Z}_3$-magic; hence, G is not Hartke $\mathbb{Z}_3$-magic.

Theorem 4.8. *Suppose $p \ge 3$ is prime and G is a graph. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ denote the sets of nonvanishing monomial terms of degree $(p-1)\cdot|V(G)|$, of $f_0(\underline{x})$ and $f_1(\underline{x})$, respectively. Then, $\mathcal{M}_0 = \mathcal{M}_1$.*

Proof. Let $M \in \mathcal{M}_0$. For each vertex $v \in V(G)$, let b_v denote the sum inside the corresponding canonical factor in Equation (4.1). Observe that every term in the expansion of $(0 - b_v)^{p-1} = b_v^{p-1}$ is of degree $p-1$. Thus in the expansion of $f_0(\underline{x})$, M arises from the product of terms $\Pi_{v\in V(G)} b_v^{p-1}$. More specifically, M is equal to a product consisting of one term from each b_v^{p-1}.

Now, let us examine $f_1(\underline{x})$ carefully. First, we note that every term in the expansion of $(1 - b_v)^{p-1}$ is of the form $\_1^k b_v^{p-1-k}$, where $0 \le k \le p-1$. In the expansion of the $f_1(\underline{x})$, every nonvanishing monomial term of degree $(p-1)\cdot|V(G)|$ will arise from a product of terms, one from each of the b_v^{p-1}. Monomial terms of $f_1(\underline{x})$ which do not arise in this manner have degree at most $(p-1)\cdot(|V(G)|-1)+(p-2) = (p-1)\cdot|V(G)|-1$. In particular, $M \in \mathcal{M}_1$.

This argument is reversible. Thus, the claim is established. □

Corollary 4.9. *Let $p \geq 3$ be prime. Then, G is a Hartke $\mathbb{Z}_p$-magic graph with magic value $0 \iff G$ is a Hartke $\mathbb{Z}_p$-magic graph with magic value 1.*

Proof. This follows immediately from Theorems 3.1 and 4.8. □

Example 4.10. This example illustrates the proof of Corollary 4.9. Let $p = 3$. Consider the graph G_7 (from Example 4.4) illustrated in Figure 1. In that example, the monomial term $14336x_5x_6\cdots x_{20} \equiv 2x_5x_6\cdots x_{20} \pmod 3$ of degree $8\cdot 2 = 16$ was found in $f_0(\underline{x})$. This was then used to show that G_7 is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0. By Theorem 4.8, $f_1(\underline{x})$ must also contain this particular Hartke term. This is easily verified by using Mathematica 12. Hence, we conclude that G_7 is a Hartke $\mathbb{Z}_3$-magic graph with magic value 1.

5 Constructing $\mathbb{Z}_p$-magic graphs

Definition 5.1. Let $p \geq 2$ be prime, $t \in \{0, 1\}$ and G have a $\mathbb{Z}_p$-magic labeling with magic value t. Then, G is called an *edge-stable* $\mathbb{Z}_p$-magic graph if the addition of any number of simple edges, multiedges and/or loops to G results in a $\mathbb{Z}_p$-magic graph with magic value t.

Example 5.2. The 1-vertex loop graph is an edge-stable $\mathbb{Z}_p$-magic graph. In [13], it was shown that $\mathrm{IM}(K_4 - \{e\}) = \{4, 6, 8, \dots\}$. Thus, C_4 is $\mathbb{Z}_p$-magic but not edge-stable, for all primes p.

Theorem 5.3. *Let $p \geq 3$ be prime. Adding simple edges, multiedges and/or loops to a Hartke $\mathbb{Z}_p$-magic graph results in a new Hartke $\mathbb{Z}_p$-magic graph. In particular, every Hartke $\mathbb{Z}_p$-magic graph is edge-stable.*

Proof. Suppose that G is a Hartke $\mathbb{Z}_p$-magic graph with Hartke polynomial f_t. Let G^* (with Hartke polynomial f_t^*) be obtained by adding simple edges, multiedges and/or loops to G. First, note that $\deg(f_t) = \deg(f_t^*)$. Since G is Hartke $\mathbb{Z}_p$-magic, there exists a Hartke term M in f_t. By Proposition 4.5, all of the exponents t_i of M satisfy $0 \leq t_i \leq p-2$. Furthermore, M also appears in the expansion of f_t^*. By Proposition 4.5, G^* is a Hartke $\mathbb{Z}_p$-magic graph with magic value t. □

Example 5.4. Let $p = 5$ and G_5 be the first graph illustrated in Figure 2. Then, $f_1(\underline{x}) \in \mathbb{Z}_5[x_1, x_2, \dots, x_8]$, where

$$\begin{aligned} f_1(\underline{x}) = {}& [1-(1-(x_1+x_3))^4]\cdot[1-(1-(x_1+x_2+x_6+x_7))^4]\cdot \\ & [1-(1-(x_2+x_8))^4]\cdot[1-(1-(x_3+x_4))^4]\cdot \\ & [1-(1-(x_4+x_5+x_7+x_8))^4]\cdot[1-(1-(x_5+x_6))^4]. \end{aligned}$$

Using Mathematica 12, we see that $\deg(f_1(\underline{x})) = 24$ and that $f_1(\underline{x})$ contains the monomial term $1069056x_1^3x_2^3\cdots x_8^3 \equiv x_1^3x_2^3\cdots x_8^3 \pmod 5$. Let $S_i = \{1, 2, 3, 4\}$, for $i = 1, 2, \dots, 8$. By Theorem 3.1, we have that $f_1(\underline{x}') \neq 0$, for some $\underline{x}' \in S_1 \times S_2 \times \cdots \times S_8$. Thus, $f_1(\underline{x}') = 1$ and we conclude that G_5 is a Hartke $\mathbb{Z}_5$-magic graph with magic value 1.

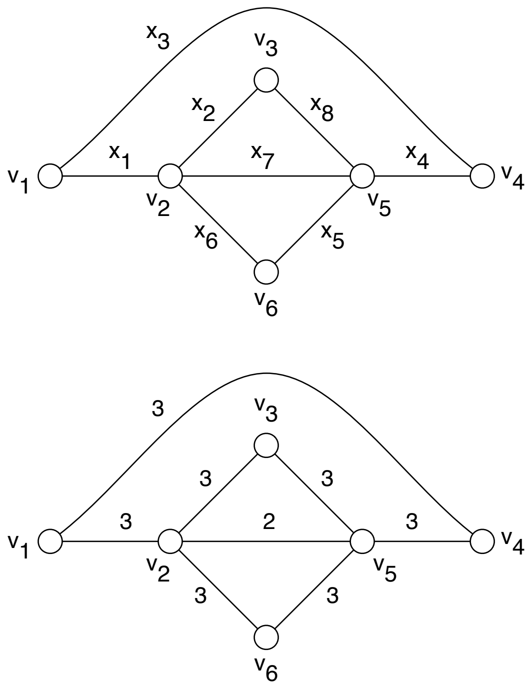

Figure 2: A $\mathbb{Z}_5$-magic labeling of G_5 with magic value 1.

With some considerable effort (by hand), one can obtain a $\mathbb{Z}_5$-magic labeling of G_5 with magic value 1, as illustrated in the second graph of Figure 2.

Suppose we add a loop or an additional edge to G_5. Then, there exist $\mathbb{Z}_5$-magic labelings with magic value 1, for these new graphs. Figures 3, 4 and 5 illustrate Theorem 5.3.

Remark 5.5. Theorem 5.3 says that any graph which contains a Hartke $\mathbb{Z}_p$-magic graph as a spanning subgraph is also a Hartke $\mathbb{Z}_p$-magic graph. Note that the converse of Theorem 5.3 is not necessarily true. The 1-vertex loop graph in Example 5.2 illustrates an edge-stable $\mathbb{Z}_p$-magic graph which is not Hartke $\mathbb{Z}_p$-magic.

We now establish some lower bounds for the number of $\mathbb{Z}_p$-magic labelings with magic value t for a given graph. Symmetry is ignored when counting these labelings. For example, if one labeling can be obtained by "rotating" another labeling, then these two labelings are counted separately.

Theorem 5.6. *Let $p \geq 3$ be prime and G be a Hartke $\mathbb{Z}_p$-magic graph. Suppose that G^* is obtained by adding z simple edges, multiedges and/or loops to G. Then, the number of different (ignoring symmetry) $\mathbb{Z}_p$-magic labelings of G^* (with magic value t) is greater than or equal to $(p-1)^z$.*

Proof. Let G, G^*, f_t, f_t^* and M be defined as in the proof of Theorem 5.3. There, we saw that M is also a term in the expansion of f_t^*. The variables in f_t^* corresponding to the

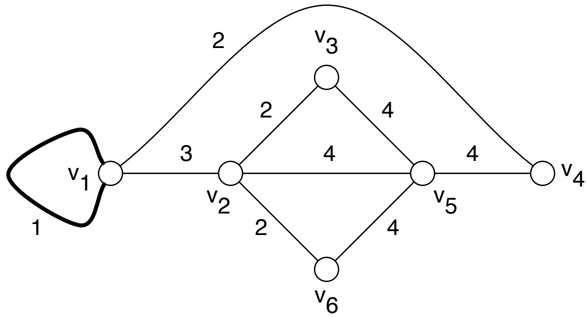

Figure 3: A $\mathbb{Z}_5$-magic labeling (magic value 1) of G_5 with a loop (labeled 1) at v_1.

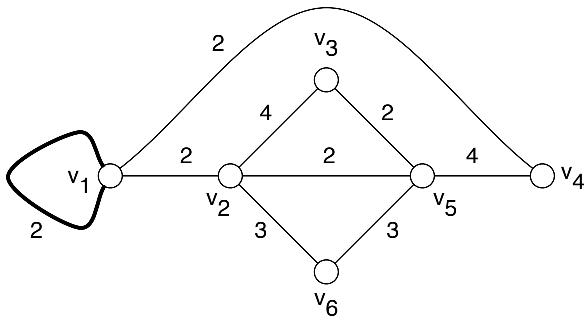

Figure 4: A $\mathbb{Z}_5$-magic labeling (magic value 1) of G_5 with a loop (labeled 2) at v_1.

additional z simple edges, multiedges, and/or loops do not appear in M. Hence we can apply Theorem 3.1 to M in f_t^*, where each of the z new variables can take on any of the $p-1$ non-zero elements from $\mathbb{Z}_p$. Thus for each $t \in \{0, 1\}$, there are at least $(p-1)^z$ different $\mathbb{Z}_p$-magic labelings of G^* with magic value t. $\square$

Example 5.7. In Example 4.4, we saw that graph G_7 (Figure 1) is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0. By Theorem 4.8 and Corollary 4.9, G_7 is also a Hartke $\mathbb{Z}_3$-magic graph with magic value 1. Let G_7^* denote the graph obtained by adding loop x_{21} and multiple edge x_{22} to G_7, as illustrated in Figure 6. Since the monomial term $14336x_5x_6 \cdots x_{20} \equiv 2x_5x_6 \cdots x_{20} \pmod 3$ is a Hartke term in the f_t of G_7^*, there exist $\mathbb{Z}_3$-magic labelings of G_7^* (with magic value t), with x_{21} and x_{22} having labels 1 or 2. Thus, there are at least $(3-1)^2 = 4$ different $\mathbb{Z}_3$-magic labelings of G_7^* with magic value t. For this particular example, even more can be said. The Hartke term $2x_5x_6 \cdots x_{20} \pmod 3$ does not involve x_1, x_2, x_3 and x_4. Each of these particular edges can be labeled with 1 or 2. Hence for each $t \in \{0, 1\}$, there are at least $(3-1)^6 = 64$ different $\mathbb{Z}_3$-magic labelings of G_7^* with magic value t.

Definition 5.8. Let $p \geq 3$ be prime, G be a Hartke $\mathbb{Z}_p$-magic graph, and M be a Hartke term of f_t. Then, the *excess set* of M (denoted by $\mathcal{E}_M$) is the set of variables that have

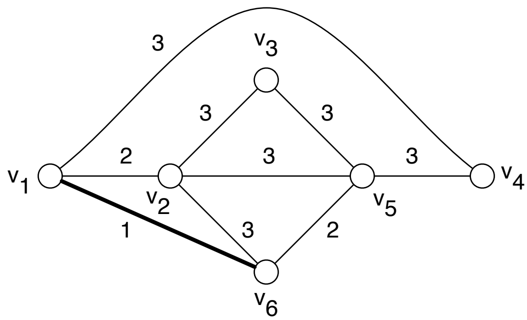

Figure 5: A $\mathbb{Z}_5$-magic labeling (magic value 1) of G_5 with edge v_1v_6 (labeled 1).

exponent zero in M.

Theorem 5.9. *Let $p \geq 3$ be prime, G be a Hartke $\mathbb{Z}_p$-magic graph, and M be a Hartke term of f_t. Then for each $t \in \{0,1\}$, G has at least $(p-1)^{|\mathcal{E}_M|}$ different $\mathbb{Z}_p$-magic labelings with magic value t. Furthermore if $G - E$ is connected, where E is any subset of edges corresponding to variables in $\mathcal{E}_M$, then $G - E$ has a $\mathbb{Z}_p$-magic labeling with magic value t.*

Proof. Suppose that $p \geq 3$ is prime, G is a Hartke $\mathbb{Z}_p$-magic graph, and M is a Hartke term of f_t. The variables in $\mathcal{E}_M$ do not appear in M. Thus, we can apply Theorem 3.1 to M in f_t, where each variable in $\mathcal{E}_M$ can take on any of the $p-1$ non-zero elements from $\mathbb{Z}_p$. Thus for each $t \in \{0,1\}$, there are at least $(p-1)^{|\mathcal{E}_M|}$ different $\mathbb{Z}_p$-magic labelings of G with magic value t. Finally, M will still be a Hartke term of the Hartke polynomials of $G - E$, where E is any subset of edges (corresponding to variables in $\mathcal{E}_M$). Hence, Theorem 3.1 can be applied to the Hartke polynomials of $G - E$ and we conclude that $G - E$ has a $\mathbb{Z}_p$-magic labeling with magic value t. □

Example 5.10. Consider the graph G_7 in Example 4.4. Then $G_7 - E$, where E is any subset of edges (corresponding to x_1, x_2, x_3, x_4), has a $\mathbb{Z}_3$-magic labeling with magic value t. This is because its associated Hartke polynomial contains the same Hartke term $14336x_5x_6\cdots x_{20} \equiv 2x_5x_6\cdots x_{20} \pmod 3$.

Corollary 5.11. *Let $p \geq 3$ be prime and G be a Hartke $\mathbb{Z}_p$-magic graph. Suppose that $|E(G)| \geq (p-1)\cdot|V(G)|$. Then, G has at least $(p-1)^{[|E(G)|-(p-1)\cdot|V(G)|]}$ different $\mathbb{Z}_p$-magic labelings with magic value t, for each $t \in \{0,1\}$.*

Proof. Suppose that $p \geq 3$ is prime, G is a Hartke $\mathbb{Z}_p$-magic graph, and $|E(G)| \geq (p-1)\cdot|V(G)|$. Let M be a Hartke term of f_t. Note that when $|E(G)| = \frac{p-1}{p-2}\cdot|V(G)|$, the corollary follows from Theorem 5.9. When $|E(G)| > \frac{p-1}{p-2}\cdot|V(G)|$, M includes at most $(p-1)\cdot|V(G)|$ distinct variables. Since $|E(G)| \geq (p-1)\cdot|V(G)|$, we have that $|\mathcal{E}_M| \geq |E(G)| - (p-1)\cdot|V(G)|$. By Theorem 5.9, the result follows. □

Theorem 5.12. *Let $p \geq 3$ be prime. Then, the disjoint union of Hartke $\mathbb{Z}_p$-magic graphs is a Hartke $\mathbb{Z}_p$-magic graph.*

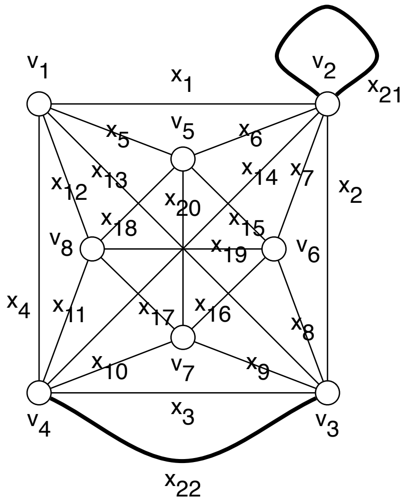

Figure 6: For each $t \in \{0, 1\}$, G_7^* has at least $(3-1)^6 = 64$ different $\mathbb{Z}_3$-magic labelings with magic value t.

Proof. It suffices to show the claim is true for the disjoint union of two Hartke $\mathbb{Z}_p$-magic graphs H_1 and H_2, with magic value $t \in \{0, 1\}$. Note that the degrees of the Hartke polynomials of H_1, H_2, and the disjoint union of H_1 and H_2 are $(p-1) \cdot |V(H_1)|$, $(p-1) \cdot |V(H_2)|$, and $(p-1) \cdot (|V(H_1)| + |V(H_2)|)$, respectively. Let M_1 and M_2 be Hartke terms in the Hartke polynomials of H_1 and H_2, respectively. Note that $M_1 \cdot M_2$ does not vanish, since the coefficients come from a field. Thus, the degree of $M_1 \cdot M_2$ is $(p-1) \cdot (|V(H_1)| + |V(H_2)|)$ and $M_1 \cdot M_2$ appears in the expansion of the Hartke polynomial of the disjoint union of H_1 and H_2. We also see that $M_1 \cdot M_2$ is a Hartke term, since M_1 and M_2 individually are Hartke terms. Therefore, the disjoint union of Hartke $\mathbb{Z}_p$-magic graphs is a Hartke $\mathbb{Z}_p$-magic graph. $\square$

Definition 5.13. A *weak join* of graphs $H_1, H_2, \ldots, H_r$ is defined to be a connected graph with vertex set $\bigcup_{i=1}^{r} V(H_i)$ and edge set $Z \cup (\bigcup_{i=1}^{r} E(H_i))$, where Z is a set of simple and/or multiedges of the form uv with $u \in V(H_i)$ and $v \in V(H_j)$, where $i \neq j$.

Example 5.14. Figure 7 illustrates a weak join of C_6, W_6 (of order six) and P_2.

Theorem 5.15. *Let $p \geq 3$ be prime. Then, a weak join of Hartke $\mathbb{Z}_p$-magic graphs is a Hartke $\mathbb{Z}_p$-magic graph.*

Proof. Let $H_1, H_2, \ldots, H_r$ be Hartke $\mathbb{Z}_p$-magic graphs. By Theorem 5.12, the disjoint union $\bigcup_{i=1}^{r} H_i$ is a Hartke $\mathbb{Z}_p$-magic graph. Since a weak join of $H_1, H_2, \ldots, H_r$ is formed by adding simple edges and/or multiedges between the H_i in $\bigcup_{i=1}^{r} H_i$, the claim is established by Theorem 5.3. $\square$

Example 5.16. Let $p = 3$ and G_3 be the top half of the graph in Figure 8. Then,

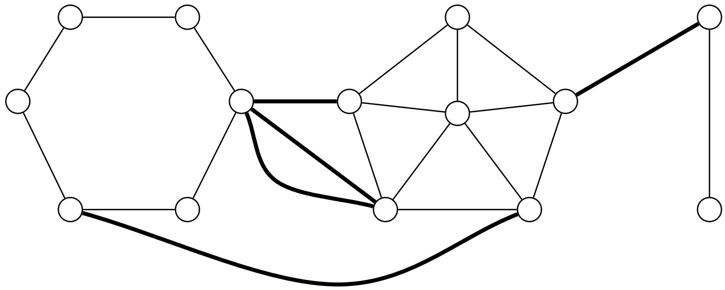

Figure 7: A weak join of C_6, the wheel graph W_6 and P_2.

$f_0(\underline{x}) \in \mathbb{Z}_3[x_1, x_2, \ldots, x_{15}]$, where

$$\begin{aligned} f_0(\underline{x}) = {} & [1-(0-(x_1+x_7+x_8+x_{14}+x_{15}))^2]\cdot[1-(0-(x_1+x_2+x_{11}+x_{12}))^2]\cdot \\ & [1-(0-(x_2+x_3+x_8+x_9))^2]\cdot[1-(0-(x_3+x_4+x_{12}+x_{13}))^2]\cdot \\ & [1-(0-(x_4+x_5+x_9+x_{10}+x_{15}))^2]\cdot[1-(0-(x_5+x_6+x_{13}+x_{14}))^2]\cdot \\ & [1-(0-(x_6+x_7+x_{10}+x_{11}))^2]. \end{aligned}$$

Using Mathematica 12, we see that $\deg(f_0(\underline{x})) = 14$ and that $f_0(\underline{x})$ contains the monomial term $-6400x_1x_2\cdots x_{11}x_{12}x_{14}x_{15} \equiv 2x_1x_2\cdots x_{11}x_{12}x_{14}x_{15} \pmod 3$. Let $S_i = \{1,2\}$, for $i = 1, 2, \ldots, 12, 14, 15$ and $S_{13} = \{1\}$. So by Theorem 3.1, we have that $f_0(\underline{x}') \neq 0$, for some $\underline{x}' \in S_1 \times S_2 \times \cdots \times S_{15}$. Thus, $f_0(\underline{x}') = 1$ and G_3 is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0.

Now, let G_4 (graph G1121 from [28]) be the bottom half of the graph in Figure 8. Then, $f_0(\underline{y}) \in \mathbb{Z}_3[y_1, y_2, \ldots, y_{14}]$, where

$$\begin{aligned} f_0(\underline{y}) = {} & [1-(0-(y_1+y_5+y_6+y_9+y_{10}+y_{14}))^2]\cdot[1-(0-(y_1+y_2+y_{11}))^2]\cdot \\ & [1-(0-(y_2+y_3+y_7+y_8+y_{10}))^2]\cdot[1-(0-(y_3+y_4+y_{13}+y_{14}))^2]\cdot \\ & [1-(0-(y_4+y_5))^2]\cdot[1-(0-(y_6+y_7+y_{11}+y_{12}))^2]\cdot \\ & [1-(0-(y_8+y_9+y_{12}+y_{13}))^2]. \end{aligned}$$

Using Mathematica 12, we see that $\deg(f_0(\underline{y})) = 14$ and that $f_0(\underline{y})$ contains the monomial term $-4096y_1y_2\cdots y_{13}y_{14} \equiv 2y_1y_2\cdots y_{13}y_{14} \pmod 3$. Let $S_i = \{1,2\}$, for $i = 1, 2, \ldots, 14$. So by Theorem 3.1, we have that $f_0(\underline{y}') \neq 0$, for some $\underline{y}' \in S_1 \times S_2 \times \cdots \times S_{14}$. Thus, $f_0(\underline{y}') = 1$ and G_4 is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0.

The graph G in Figure 8 is a weak join of G_3 and G_4, where z_1, z_2 and z_3 are the additional simple and multiedges used to create the weak join. Let $f_0(\underline{r}) \in \mathbb{Z}_3[r_1, r_2, \ldots, r_{32}]$, where

$$r_i = \begin{cases} x_i & \text{if } 1 \leq i \leq 15; \\ y_{i-15} & \text{if } 16 \leq i \leq 29; \\ z_{i-29} & \text{if } 30 \leq i \leq 32, \end{cases}$$

be a Hartke polynomial of G. Using Mathematica 12, we see that $\deg(f_0(\underline{r})) = 28$ and that $f_0(\underline{r})$ contains the monomial term

$$\begin{aligned}
&(-6400 r_1 r_2 \cdots r_{11} r_{12} r_{14} r_{15}) \cdot (-4096 r_{16} r_{17} \cdots r_{28} r_{29}) \\
&\quad \equiv (2 r_1 r_2 \cdots r_{11} r_{12} r_{14} r_{15}) \cdot (2 r_{16} r_{17} \cdots r_{28} r_{29}) \pmod 3 \\
&\quad \equiv r_1 r_2 \cdots r_{11} r_{12} r_{14} r_{15} r_{16} r_{17} \cdots r_{28} r_{29} \pmod 3.
\end{aligned}$$

Let $S_i = \{1, 2\}$, for $i = 1, 2, \ldots, 12, 14, 15, 16, 17 \ldots 28, 29$ and $S_{13} = S_{30} = S_{31} = S_{32} = \{1\}$. So by Theorem 3.1, we have that $f_0(\underline{r}') \neq 0$, for some $\underline{r}' \in S_1 \times S_2 \times \cdots \times S_{32}$. Thus, $f_0(\underline{r}') = 1$ and G is a Hartke $\mathbb{Z}_3$-magic graph with magic value 0.

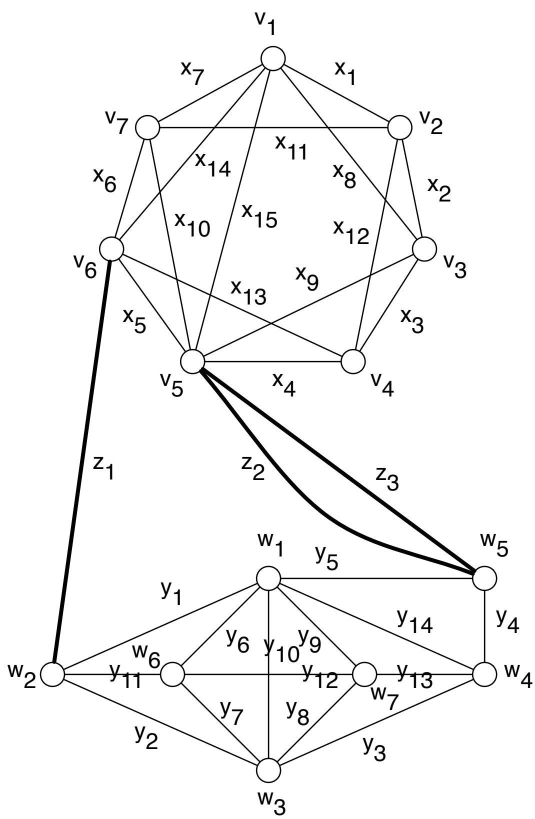

Figure 8: A weak join of Hartke $\mathbb{Z}_3$-magic graphs G_3 and G_4 is a Hartke $\mathbb{Z}_3$-magic graph.

In [22], the $\mathbb{Z}_k$-magic property was analyzed for various classical graph products. There, it was shown that if G and H are connected $\mathbb{Z}_k$-magic graphs, then the Cartesian and lexicographic products of G and H are $\mathbb{Z}_k$-magic, for $k \in \{2, 3, 4, \ldots\}$. However, if instead we strengthen the restriction on G and weaken the restriction on H, then we obtain additional results. To this end, recall the following definitions [5].

Definition 5.17. Let G and H be connected graphs. Then, the *Cartesian* product $G \square H$ is a graph which has vertex set $V(G \square H) = \{(g, h) : g \in V(G) \text{ and } h \in V(H)\}$ and edge set $E(G\Box H)$, where two vertices (g,h) and (g',h') are adjacent if ($g = g'$ and h adj h') or ($h = h'$ and g adj g').

Definition 5.18. Let G and H be connected graphs. Then, the *lexicographic* product $G \circ H$ is a graph which has vertex set $V(G \circ H) = \{(g,h) : g \in V(G) \text{ and } h \in V(H)\}$ and edge set $E(G \circ H)$, where two vertices (g,h) and (g',h') are adjacent if ($g = g'$ and h adj h') or (g adj g').

Definition 5.19. Let G and H be connected graphs. Then, the *strong* product $G \boxtimes H$ is a graph which has vertex set $V(G \boxtimes H) = \{(g,h) : g \in V(G) \text{ and } h \in V(H)\}$ and edge set $E(G \boxtimes H)$, where two vertices (g,h) and (g',h') are adjacent if ($g = g'$ and h adj h') or ($h = h'$ and g adj g') or (h adj h' and g adj g').

Of these three graph products, only the lexicographic product is not commutative.

Example 5.20. Figure 9 illustrates $P_2\Box P_3$, $P_2 \circ P_3$, $P_3 \circ P_2$ and $P_2 \boxtimes P_3$.

Corollary 5.21. *Let $p \geq 3$ be prime. Suppose that G is a Hartke $\mathbb{Z}_p$-magic graph and H is a graph. Then, the Cartesian product $G\Box H$ is a Hartke $\mathbb{Z}_p$-magic graph.*

Proof. Let T be a spanning tree of H. $G\Box T$ is obtained by replacing each vertex of T with a copy of G and replacing each edge of T with edges connecting the corresponding vertices of copies of G. Since G is a Hartke $\mathbb{Z}_p$-magic graph, $G\Box T$ is a weak join of Hartke $\mathbb{Z}_p$-magic graphs. By Theorem 5.15, $G\Box T$ is a Hartke $\mathbb{Z}_p$-magic graph. Finally, for each of the edges in H which are not in T, add edges connecting the corresponding vertices of copies of G in $G\Box T$ to obtain $G\Box H$. Since $G\Box T$ is a Hartke $\mathbb{Z}_p$-magic graph, we see that $G\Box H$ is a Hartke $\mathbb{Z}_p$-magic graph, by Theorem 5.3. □

Corollary 5.22. *Let $p \geq 3$ be prime. Suppose that G is a Hartke $\mathbb{Z}_p$-magic graph and H is a graph. Then, $G \circ H$, $H \circ G$ and $G \boxtimes H$ are Hartke $\mathbb{Z}_p$-magic graphs.*

Proof. First, note that $V(G\Box H) = V(G \circ H) = V(H \circ G) = V(G \boxtimes H)$. We also see that the edge sets of $G \circ H$, $H \circ G$, and $G \boxtimes H$ contain the edge set of $G\Box H$. Since $G\Box H$ is Hartke $\mathbb{Z}_p$-magic by Corollary 5.21, these other products are Hartke $\mathbb{Z}_p$-magic, by Theorem 5.3. □

6 Further directions and some open questions

Throughout this paper, we used the Combinatorial Nullstellensatz in the construction of Hartke $\mathbb{Z}_p$-magic graphs, for prime $p \geq 3$. Graphs of this type were found to have an edge-stability property. This was used to further construct non-trivial Hartke $\mathbb{Z}_p$-magic graphs.

It is natural for the reader to wonder if connected simple Hartke $\mathbb{Z}_p$-magic graphs exist, for all orders $n \geq 6$ and prime $p \geq 3$. The authors of this paper believe that this is true.

Conjecture 6.1. *Let $p \geq 3$ be prime. Then, there exists a connected <u>simple</u> Hartke $\mathbb{Z}_p$-magic graph G, for all $|V(G)| \geq 6$.*

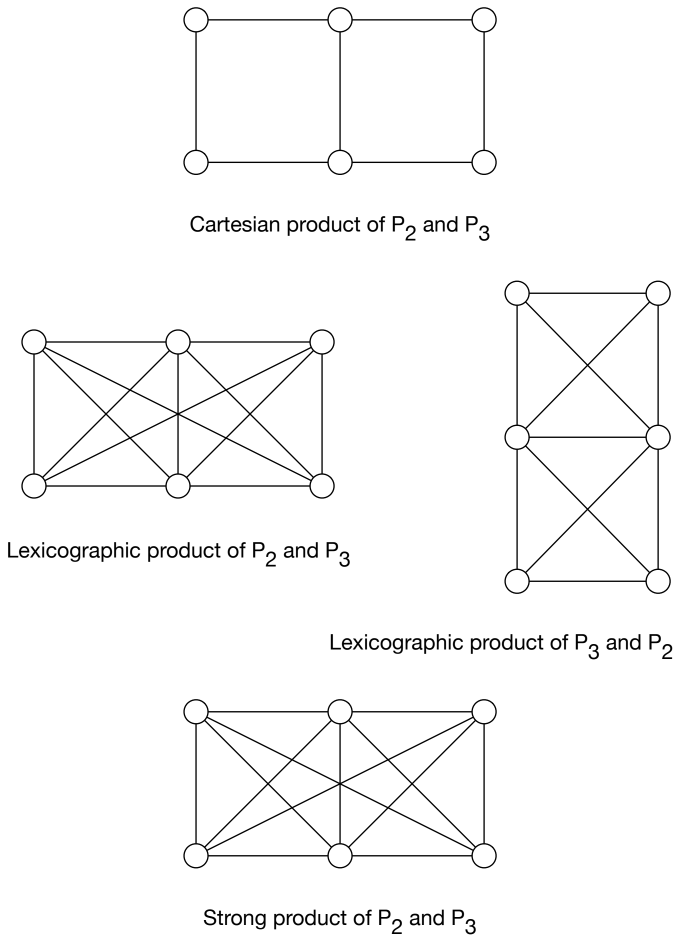

Figure 9: $P_2 \Box P_3$, $P_2 \circ P_3$, $P_3 \circ P_2$ and $P_2 \boxtimes P_3$.

The Combinatorial Nullstellensatz can be generalized in different ways. Theorem 3.1 is true over integral domains. The Generalized Combinatorial Nullstellensatz [35] sharpens Theorem 3.1; instead of analyzing a monomial with degree $= \deg(f)$, it suffices to consider a monomial that does not divide any other monomial term in f. In [26], Michalek remarks that the Combinatorial Nullstellensatz is true over any commutative ring R with unity, as long as $a - b$ is not a zero divisor in R, for any $a, b \in S_i$ $(i = 1, 2, \dots, m)$. Can any of these generalizations of the Combinatorial Nullstellensatz help us in analyzing the $\mathbb{Z}_p$-magic graph labeling problem (prime $p \geq 3$)?

Here are some other questions one might consider.

1. Are there natural classes of Hartke $\mathbb{Z}_p$-magic graphs?
2. Let $p \geq 3$ be prime and G be a connected simple graph satisfying $|E(G)| \geq \frac{p-1}{p-2} \cdot |V(G)|$. What is the probability that G is a Hartke $\mathbb{Z}_p$-magic graph?

3. Are there other types of graph labeling problems where the Combinatorial Nullstellensatz or its various generalizations can be used?

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References

[1] N. Alon, Combinatorial Nullstellensatz, *Comb. Probab. Comput.* **8** (1999), 7–29, doi:10.1017/s0963548398003411, recent trends in combinatorics (Mátraháza, 1995).

[2] C. Berge, Regularisable graphs, *Ann. Discrete Math.* **3** (1978), 11–19, doi:10.1016/s0167-5060(08)70493-x.

[3] G. Bloom and S. Golomb, Applications of numbered undirected graphs, *Proc. of the IEEE* **65** (1977), 562–570, doi:10.1109/proc.1977.10517.

[4] G. S. Bloom and S. W. Golomb, Numbered complete graphs, unusual rulers, and assorted applications, in: *Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976)*, volume 642, 1978 pp. 53–65. Lecture Notes in Math., Vol. 642, doi:10.1007/bfb0070359.

[5] N. Bray and E. W. Weisstein, *Graph Product*, MathWorld–A Wolfram Web Resource, `https://mathworld.wolfram.com/GraphProduct.html`.

[6] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Comb.* **23** (2020), Dynamic Survey 6, 43, doi:10.37236/27.

[7] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1994.

[8] D. Hefetz, Anti-magic graphs via the combinatorial nullstellensatz, *J. Graph Theory* **50** (2005), 263–272, doi:10.1002/jgt.20112.

[9] T. W. Hungerford, *Algebra*, Springer-Verlag, New York, 1974, doi:10.1007/978-1-4612-6101-8.

[10] S. Jukna, *Extremal Combinatorics With Applications in Computer Science*, Springer-Verlag, Berlin Heidelberg, 2011, doi:10.1007/978-3-642-17364-6.

[11] M. C. Kong, S.-M. Lee and H. S. H. Sun, On magic strength of graph, *Ars Comb.* **45** (1997), 193–200.

[12] S. Lee, I. Wen and H. Sun, On group-magic graphs, *J. Comb. Math. Comb. Comput.* **38** (2001), 197–207.

[13] S.-M. Lee, Y.-S. Ho and R. M. Low, On the integer-magic spectra of maximal planar and maximal outerplanar graphs, in: *Proceedings of the Thirty-Fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 168, 2004 pp. 83–90.

[14] S.-M. Lee and F. Saba, On the integer-magic spectra of two-vertex sum of paths, in: *Proceedings of the Thirty-Fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 170, 2004 pp. 3–15.

[15] S.-M. Lee, F. Saba, E. Salehi and H. Sun, On the V_4-magic graphs, in: *Proceedings of the Thirty-third Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2002)*, volume 156, 2002 pp. 59–67.

[16] S.-M. Lee, F. Saba and G. C. Sun, Magic strength of the kth power of paths, *Congr. Numer.* **92** (1993), 177–184, twenty-second Manitoba Conference on Numerical Mathematics and Computing (Winnipeg, MB, 1992).

[17] S.-M. Lee and E. Salehi, Integer-magic spectra of amalgamations of stars and cycles, *Ars Comb.* **67** (2003), 199–212.

[18] S.-M. Lee, E. Salehi and H. Sun, Integer-magic spectra of trees with diameters at most four, *J. Comb. Math. Comb. Comput.* **50** (2004), 3–15.

[19] S.-M. Lee, L. Valdés and Y.-S. Ho, On group-magic trees, double trees and abbreviated double trees, *J. Comb. Math. Comb. Comput.* **46** (2003), 85–95, 15th MCCCC (Las Vegas, NV, 2001).

[20] R. M. Low and S.-M. Lee, On group-magic Eulerian graphs, *J. Comb. Math. Comb. Comput.* **50** (2004), 141–148.

[21] R. M. Low and S.-M. Lee, On the integer-magic spectra of tessellation graphs, *Australas. J. Comb.* **34** (2006), 195–210, https://ajc.maths.uq.edu.au/?page=get_volumes&volume=34.

[22] R. M. Low and S.-M. Lee, On the products of group-magic graphs, *Australas. J. Comb.* **34** (2006), 41–48, https://ajc.maths.uq.edu.au/?page=get_volumes&volume=34.

[23] R. M. Low and D. Roberts, Application of the combinatorial Nullstellensatz to integer-magic graph labelings, *Theory Appl. Graphs* **9** (2022), Art. 3, 14, https://digitalcommons.georgiasouthern.edu/tag/vol9/iss1/3/.

[24] R. M. Low and W. C. Shiu, On the integer-magic spectra of graphs, in: *Proceedings of the Thirty-Ninth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 191, 2008 pp. 193–203.

[25] R. M. Low and L. Sue, Some new results on the integer-magic spectra of tessellation graphs, *Australas. J. Comb.* **38** (2007), 255–266, https://ajc.maths.uq.edu.au/?page=get_volumes&volume=38.

[26] M. Michał ek, A short proof of combinatorial Nullstellensatz, *Am. Math. Monthly* **117** (2010), 821–823, doi:10.4169/000298910x521689.

[27] J. Przybył o, Neighbour sum distinguishing total colourings via the combinatorial Nullstellensatz, *Discrete Appl. Math.* **202** (2016), 163–173, doi:10.1016/j.dam.2015.08.028.

[28] R. C. Read and R. J. Wilson, *An Atlas of Graphs*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1998.

[29] A. Rosa, On certain valuations of the vertices of a graph, in: *Theory of Graphs (Internat. Sympos., Rome, 1966)*, Gordon and Breach, New York; Dunod, Paris, pp. 349–355, 1967.

[30] E. Salehi, Zero-sum magic graphs and their null sets, *Ars Comb.* **82** (2007), 41–53.

[31] E. Salehi, On zero-sum magic graphs and their null sets, *Bull. Inst. Math. Acad. Sin. (N.S.)* **3** (2008), 255–264, https://web.math.sinica.edu.tw/bulletin/archives_articlecontent21.jsp?bid=MjAwODIwMg==.

[32] E. Salehi and P. Bennett, On integer-magic spectra of caterpillars, *J. Comb. Math. Comb. Comput.* **61** (2007), 65–71.

[33] E. Salehi and P. Bennett, Integer-magic spectra of trees of diameter five, *J. Comb. Math. Comb. Comput.* **66** (2008), 105–111.

[34] Ľ. Šándorová and M. Trenkler, On a generalization of magic graphs, in: *Combinatorics (Eger, 1987)*, North-Holland, Amsterdam, volume 52 of *Colloq. Math. Soc. János Bolyai*, pp. 447–452, 1988.

[35] U. Schauz, Algebraically solvable problems: describing polynomials as equivalent to explicit solutions, *Electron. J. Comb.* **15** (2008), Research Paper 10, 35, doi:10.37236/448.

[36] U. Schauz, A paintability version of the combinatorial Nullstellensatz, and list colorings of k-partite k-uniform hypergraphs, *Electron. J. Comb.* **17** (2010), Research Paper 176, 13, doi: 10.37236/448.

[37] W. C. Shiu and R. M. Low, Group magicness of complete N-partite graphs, *J. Comb. Math. Comb. Comput.* **58** (2006), 129–134.

[38] W. C. Shiu and R. M. Low, Integer-magic spectra of sun graphs, *J. Comb. Optim.* **14** (2007), 309–321, doi:10.1007/s10878-007-9052-x.

[39] W. C. Shiu and R. M. Low, Ring-magic labelings of graphs, *Australas. J. Comb.* **41** (2008), 147–158, https://ajc.maths.uq.edu.au/?page=get_volumes&volume=41.

[40] W. C. Shiu and R. M. Low, $\mathbb{Z}_k$-magic labellings of fans and wheels with magic-value zero, *Australas. J. Comb.* **45** (2009), 309–316, https://ajc.maths.uq.edu.au/?page=get_volumes&volume=45.

[41] W. C. Shiu and R. M. Low, Group-magic labelings and null sets of complete n-partite graphs with a deleted edge, *Australas. J. Comb.* **57** (2013), 3–19, https://ajc.maths.uq.edu.au/?page=get_volumes&volume=57.

[42] W. C. Shiu and R. M. Low, The integer-magic spectra and null sets of the Cartesian product of trees, *Australas. J. Comb.* **70** (2018), 157–167, https://ajc.maths.uq.edu.au/?page=get_volumes&volume=70.

[43] R. P. Stanley, Linear homogeneous Diophantine equations and magic labelings of graphs, *Duke Math. J.* **40** (1973), 607–632, doi:10.1215/s0012-7094-73-04054-4.

[44] R. P. Stanley, Magic labelings of graphs, symmetric magic squares, systems of parameters, and Cohen-Macaulay rings, *Duke Math. J.* **43** (1976), 511–531, doi:10.1215/s0012-7094-76-04342-8.

[45] X. Zhu and R. Balakrishnan, *Combinatorial Nullstellensatz with Applications to Graph Colouring*, Chapman and Hall/CRC, 2022, doi:10.1201/9781003138679.

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On generalised Petersen graphs of girth 7 that have cop number 4^*

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Abstract

We show that if $n = 7k/i$ with $i \in \{1, 2, 3\}$ then the cop number of the generalised Petersen graph $GP(n, k)$ is 4, with some small previously-known exceptions. It was previously proved by Ball et al. (2015) that the cop number of any generalised Petersen graph is at most 4. The results in this paper explain all of the known generalised Petersen graphs that actually have cop number 4 but were not previously explained by Morris et al. in a recent preprint, and places them in the context of infinite families. (More precisely, the preprint by Morris et al. explains all known generalised Petersen graphs with cop number 4 and girth 8, while this paper explains those that have girth 7.)

Keywords: Cops and robbers, generalised Petersen graphs, girth, cop number.

Math. Subj. Class.: 05C57, 91A46

1 Introduction

Cops and robbers is a game that can be played on any graph. There are two players: one playing the cops, with one or more pieces and the other playing the robber, with a single piece. Both players have perfect information: they can see the graph and the current locations of all pieces. The two players take turns, with the cop player going first. On

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their first turns, each player chooses a vertex on which to place each of their pieces. On all subsequent turns, they may move any number of their pieces to any neighbouring vertex. The object of the game for the cops is to capture the robber by landing on the same vertex as them with any of the cop pieces. The object for the robber is to avoid being captured forever. The *cop number* of a graph is the minimum number of cops required in order for the cop player to have a winning strategy.

This game first appears in [5] and [6], and has been studied extensively on many families of graphs (see [3] as an excellent reference).

We are interested in studying this game only on the well-known family of generalised Petersen graphs. The *generalised Petersen graph* $GP(n, k)$ is a graph on $2n$ vertices whose vertex set is the union of $A = \{a_0, \ldots, a_{n-1}\}$ and $B = \{b_0, \ldots, b_{n-1}\}$. The edges have one of three possible forms:

- $\{a_i, a_{i+1}\}$;
- $\{a_i, b_i\}$; and
- $\{b_i, b_{i+k}\}$,

where subscripts are calculated modulo n. This family generalises the Petersen graph, which is $GP(5, 2)$. We require $n \geq 5$, and $0 < k < n/2$. This ensures that the graphs are cubic, and avoids some isomorphic copies of graphs.

The girth of a generalised Petersen graph is at most 8, and is well-understood from the parameters n and k (see [2, Theorem 5]). Up to isomorphisms, a generalised Petersen graph has girth 7 if and only if its girth is not 6 or less, and its parameters satisfy one of the following conditions:

- $n = 7k/i$ where $i \in \{1, 2, 3\}$;
- $k = 4$;
- $n = 2k + 3$; or
- $n = 3k \pm 2$.

Generalised Petersen graphs of girth 6 or less are characterised (up to isomorphisms) as those having $k \leq 3$, $n = 2k + 2$, or $n = jk$ where $j \in \{5/2, 3, 4, 5, 6\}$. By carefully checking these with the girth at most 7 conditions for compatibility, we obtain the following precise characterisation. Up to isomorphisms, a generalised Petersen graphs has girth 7 if and only if $k \geq 4$ and its parameters satisfy one of the following conditions:

- $k = 4$ and $n \notin \{10, 12, 16, 20, 24\}$;
- $n = 7k/i$ where $i \in \{1, 2, 3\}$, and if $i = 3$ then $k \neq 6$;
- $n = 2k + 3$ and $k \neq 6$; or
- $n = 3k \pm 2$ and $(n, k) \neq (10, 4)$.

In [1], Ball et al. showed that the cop number of every generalised Petersen graph is at most 4. They also provided a list of all generalised Petersen graphs with $n \leq 40$ that attain this bound. (Minor corrections to this list were included in [4], which also proved that every generalised Petersen graph of girth 8 that appears on this list falls into an infinite class of generalised Petersen graphs of girth 8 that have cop number 4.) Almost all graphs on this list have girth 8, but there are three graphs of girth 7 on the list:

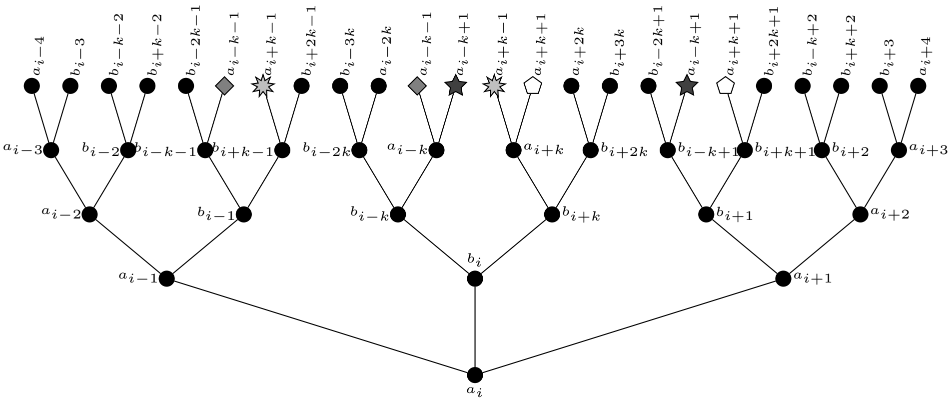

Figure 1: The vertices within distance 4 of a_i, in our families of graphs.

- $GP(28, 8)$;
- $GP(35, 10)$; and
- $GP(35, 15)$.

Notably, all of these graphs have parameters of the form $n = 7k/2$ or $n = 7k/3$. In this paper, we show that with the exception of the generalised Petersen graphs with $n < 42$ that do not appear above, every generalised Petersen graph whose parameters have the form $n = 7k/i$ where $i \in \{1, 2, 3\}$ has cop number 4. Thus, in light of the results of [4], we show that all known generalised Petersen graphs with cop number 4 are included in infinite families that have this property.

For graphs in the family we are studying (generalised Petersen graphs with $n = 7k/i$, $i \in \{1, 2, 3\}$ and $n \geq 42$ or $(n, k) \in \{(28, 8), (35, 10), (35, 15)\}$), the neighbourhood (up to distance 4) of an arbitrary vertex $a_i \in A$ is shown in Figure 1, while that for an arbitrary vertex $b_i \in B$ is shown in Figure 2. Note that at distance 4, some vertices may be the same as others; a pair of nodes that are depicted by the same non-standard shape (e.g. a diamond) represent the same vertex. In our results, we will assume that either $n \geq 42$, or $(n, k) \in \{(28, 8), (35, 10), (35, 15)\}$. Careful calculations (left to the reader) verify that this assumption avoids any additional vertices at distance 4 being the same as each other, so that the graphs are as depicted. This will be of critical importance in our proofs.

2 Main result

We begin by defining what it means for a robber to be trapped.

Definition 2.1. The cops have *trapped* the robber if there is a cop on or adjacent to each v_i ($i \in \{1, 2, 3\}$), where v_1, v_2, and v_3 are the neighbours of the vertex the robber is on (no matter whose turn it is), or if at least one cop is adjacent to the robber on the cops' turn.

Observe that if the cops have trapped the robber, then the robber will be caught on the cops' next move if the robber moves, or within the next two cop moves if the robber passes, and this is the only configuration for which this is true.

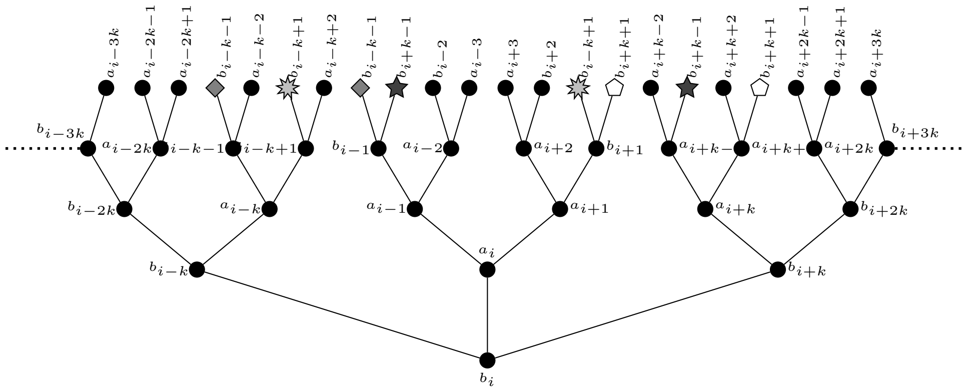

Figure 2: The vertices within distance 4 of b_i, in our families of graphs. The dotted lines represent an edge between b_{i+3k} and b_{i-3k}.

We intend to demonstrate that three cops do not have a winning strategy on the graphs we are studying. Together with the upper bound of [1], this is sufficient to show that the cop number for these graphs is 4. In order to prove that three cops do not have a winning strategy, we will first show that if three cops have not already trapped the robber, the robber always has a legal move whereby the cops cannot trap the robber on the cops' next move. After setting up some notation, we will define three cases that together encompass all possible configurations the cops can ever be in relative to the robber's position, and will explain that these cases are exhaustive. Our first lemma will show that one of these three cases can only arise in our graphs if the robber is already trapped. Our second lemma shows that if either of the other cases applies, the robber has a legal move whereby the cops cannot trap the robber on the cops' next turn. These lemmas set up an inductive argument showing that the robber can always avoid becoming trapped unless they begin the game trapped. In the proof of our main theorem, we show that the robber can always find a starting position that is not trapped, thus completing the proof.

We now introduce the labelling system used throughout the rest of this paper and all the proofs within. We assume throughout that the game is being played with 3 cops, C_1, C_2, and C_3. We will use r to denote the vertex the robber starts on. Label the three vertices adjacent to r with v_1, v_2, and v_3. When we mention a "branch" from v_i ($i \in \{1, 2, 3\}$), we are referring to v_i and all of the other 7 vertices along any paths from the robber's vertex to the vertices at distance 4 from the robber, using only a fixed one of v_i's neighbours (excluding r from our count). All of this is illustrated in Figure 3.

We now outline the three cases included in [4] as the only possible sets of configurations for the robber to be in, relative to the positions of the three cops. They are as follows:

- **Case 1.** For at least one vertex v_i ($i \in \{1, 2, 3\}$), v_i has no cop on one of its branches, and no cop within distance 2 of the robber on the other branch.

- **Case 2.** For every cop C_i ($i \in \{1, 2, 3\}$), C_i is not within distance 2 of the robber.

- **Case 3.** For at least one vertex v_i ($i \in \{1, 2, 3\}$), v_i has at least one cop on its branches who is at distance 2 or less from the robber. If any vertices v_j ($j \in \{1, 2, 3\}$)

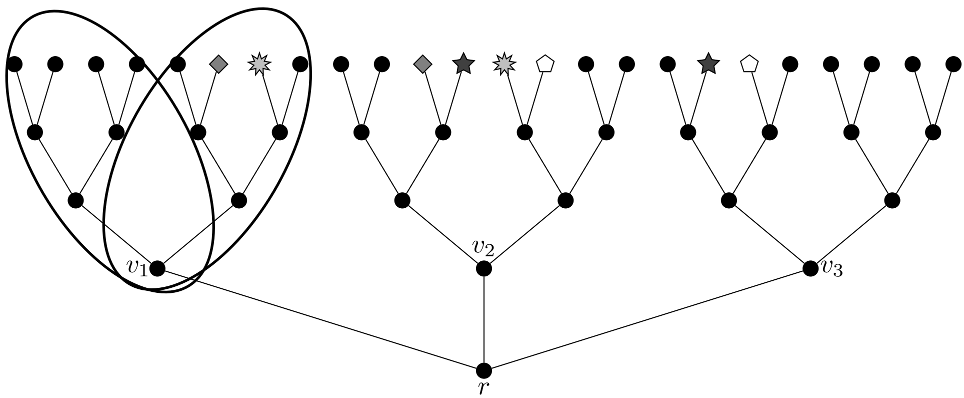

Figure 3: This figure shows our basic notation. In the illustration, r is a vertex in A. The two branches from v_1 are circled.

are not covered by that clause, then for each such v_j, v_j has at least one cop on each of its branches.

Now that we have stated our three cases, we will explain why these cases cover all of the possibilities. Suppose we have a set-up that does not conform to Case 2. This means that there is at least one cop who is at distance 2 or less from the robber. This cop is on v_i ($i \in \{1, 2, 3\}$), or one of its adjacent vertices (other than r). If the situation also does not satisfy the assumptions of Case 3, then there must be at least one vertex (say v_j, where $j \in \{1, 2, 3\}$ and $j \neq i$) that has no cop on one of its branches, and the closest cop through v_j is at distance 3 or more from the robber. This means that Case 1 applies to vertex v_j.

We will now prove our first lemma, which will be used in our main lemma.

Lemma 2.2. *Suppose that $n = 7k/i$ where $i \in \{1, 2, 3\}$, and either $n \geq 42$ or $(n, k) \in \{(28, 8), (35, 10), (35, 15)\}$. If we play cops and robbers with 3 cops on $GP(n, k)$, then the only way for the configuration of the robber with respect to the cops to fall into Case 3 is if the robber is trapped.*

Proof. To prove this lemma, we will use a case analysis on the possible configurations within Case 3, and show why each is impossible unless the robber is trapped. Our proof holds regardless of whether the robber is on a vertex in A or a vertex in B. We assume that both the cop and robber players use their moves optimally.

We now move to our case analysis.

Case A. For exactly one vertex v_i ($i \in \{1, 2, 3\}$), v_i satisfies the conditions of Case 3 by having at least one cop on its branches who is at distance 2 or less from the robber.

In order for our situation to fall under Case A, v_i satisfies the conditions of Case 3 already, but there are two other vertices, v_j and v_k (where $\{i, j, k\} = \{1, 2, 3\}$), that still need to satisfy these conditions. This means that each of v_j and v_k requires both of its branches to be covered by cops. There is no way for two cops to cover all four branches, unless those two cops are directly on v_j and v_k, which does not fall under Case A. This means that it is impossible for a configuration of cops and robber on our graphs to fall under Case A. ∎

Case B. For exactly two vertices v_i and v_j ($i, j \in \{1, 2, 3\}$), v_i and v_j satisfy the conditions of Case 3 by each having at least one cop on its branches who is at distance 2 or less from the robber.

Assuming our scenario meets the parameters of Case B, there is only one vertex v_k (where $\{i, j, k\} = \{1, 2, 3\}$) which does not yet satisfy the conditions of Case 3. This final vertex is required to have at least one cop on each of its branches. Since two of the three cops have already been placed, we only have one cop left. No v_k has a vertex that is a part of both of its branches, other than v_k itself, which cannot have a cop on it because of the conditions of Case B. ■

Case C. For every vertex v_i ($i \in \{1, 2, 3\}$), v_i satisfies the conditions of Case 3 by having at least one cop on its branches who is at distance 2 or less from the robber.

In order for the parameters of Case C to be met, there must be a cop on each of v_1, v_2, and v_3, or the adjacent vertices (excluding r, of course). In this case, the robber is trapped. ■ □

We now prove our main lemma, which will be used inductively in the proof of our theorem.

Lemma 2.3. *Suppose that $n = 7k/i$ where $i \in \{1, 2, 3\}$, and either $n \geq 42$ or $(n, k) \in \{(28, 8), (35, 10), (35, 15)\}$. If we play cops and robbers on $GP(n, k)$ with three cops, then unless the robber starts their turn trapped, they always have a legal move so that they cannot be trapped by the cops on the cops' turn.*

Proof. To prove this lemma, we will use a case analysis of the current positions of the cops compared to that of the robber (on the robber's turn). We will use the techniques from [4] to prove this lemma. We assume that the three cops have not yet trapped the robber. We also assume that both the cop and robber players use their moves optimally. We will go on to prove that unless the robber starts their turn trapped, they always have a legal move so that the robber cannot be trapped at the end of the cops' turn.

In Lemma 2.2, we proved that Case 3 cannot apply to our graphs. This means that we need only prove this lemma under the assumption that the scenario falls into Case 1 or Case 2. These proofs are below.

Case 1. For at least one vertex v_i ($i \in \{1, 2, 3\}$), v_i has no cop on one of its branches, and no cop within distance 2 of the robber on the other branch.

Let us assume that the vertex satisfying the requirements of this case is v_1 (the other cases are similar). Equally, let us assume that the left branch is the one with no cop. This situation is shown in Figure 4. Since there is no cop within distance 2 of the robber on vertex v_1, the robber can go to v_1 safely. No cop can catch the robber on this move, since no cop can be within distance 1 of v_1 at the start of this turn. Since there are no cops on the left branch from v_1, there are no cops within distance 4 of the robber in that direction before the robber's move, so even after the cops' turn there cannot be a cop within distance 2 of the robber in that direction. Therefore, the robber is not trapped at the end of the cops' turn. ■

Case 2. For every cop C_i ($i \in \{1, 2, 3\}$), C_i is not within distance 2 of the robber.

Figure 5 illustrates this case. In this scenario, the robber can move to any of the three vertices adjacent to their position (v_1, v_2, or v_3). Let's have the robber choose v_1. Since no cops were within distance 2 of the robber at the beginning, once the robber has moved,

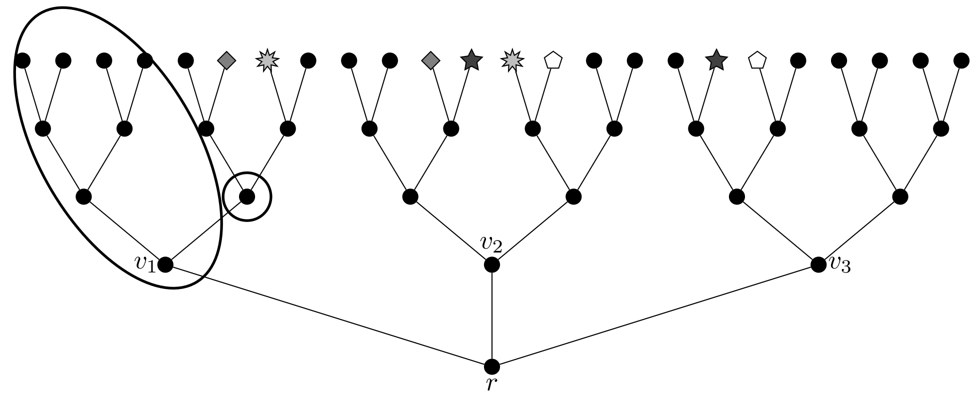

Figure 4: The vertices that we assume do not have cops on them in Case 1 are circled. The illustration shows this when $r \in A$.

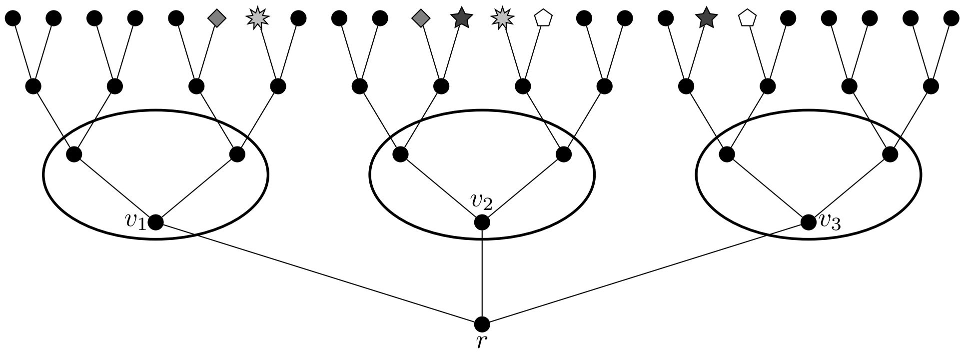

Figure 5: The vertices that cannot have cops on them in Case 2 are circled. The illustration shows this when $r \in A$.

they cannot be within distance 1 of the robber, so cannot catch the robber. Furthermore, since no cop was on vertex v_1, v_2, v_3, or any of their neighbours, after the cops' move no cop is on r or any of its neighbours (and r is a neighbour of the robber's new position). This means that the robber is not trapped. ■

We have now proven that the robber cannot be trapped at the end of the cops' turn if they did not start their turn trapped. □

This allows us to prove our main result.

Theorem 2.4. *Suppose that* $n = 7k/i$ *where* $i \in \{1, 2, 3\}$, *and* $n \geq 42$ *or* $(n, k) \in \{(28, 8), (35, 10), (35, 15)\}$. *Then the cop number of the graph* $GP(n, k)$ *is* 4.

Proof. We begin by showing that there is always a vertex the robber can choose for their first move, so that they do not begin the game trapped.

Let w be an arbitrary vertex with neighbours v_1, v_2, and v_3. Recall that for the robber to be trapped, there must be cops on each of its neighbours or their adjacent vertices, or

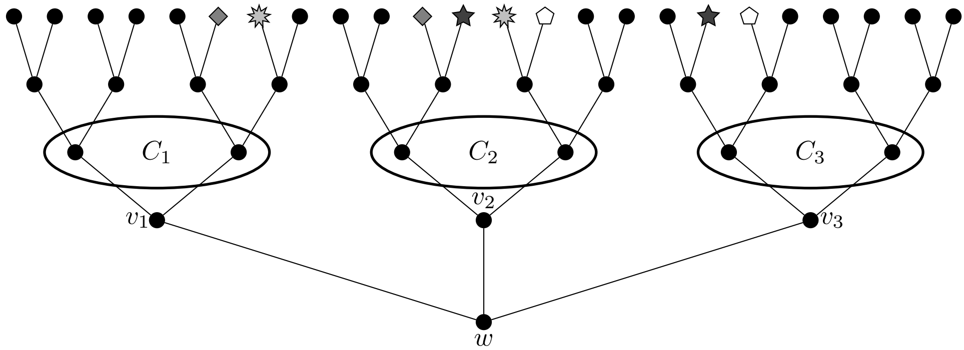

Figure 6: The circles in this figure show the ideal positions for the cops at the beginning of the game, when $w \in A$.

there must be at least one cop directly adjacent to the robber. In order for the cops to be in a position where they could have the robber trapped, a cop should be placed on each of v_1, v_2, and v_3 or their adjacent vertices. Without loss of generality, let us assume that C_i ($i \in \{1, 2, 3\}$) is on v_i or an adjacent vertex.

Suppose momentarily that all three cops choose to go on vertices adjacent to w. Under this scenario, should the robber go on any vertex adjacent to a cop, they will be trapped. This gives us 10 vertices where the robber cannot go without being trapped (including those with cops already on them).

Now, suppose that all three cops choose to go on vertices at distance 2 from w. In this set-up, in addition to the robber being trapped if they are adjacent to a cop, they will also be trapped if they go on w, since there are cops adjacent to v_1, v_2, and v_3. Therefore, in this case, there are 13 different vertices where the robber cannot go without being trapped.

This means that our second set of positions, as shown in Figure 6, are the ideal choices for the cops when there are only three of them, and if they go in the ideal positions, they can still only ensure that the robber will be trapped if placed on one of 13 different vertices. Since $n \geq 28$, our graph has at least 56 vertices, and the robber still has many choices that do not leave them trapped, meaning that the robber will not start trapped.

Combining this with the results of Lemma 2.3, we conclude that the robber never has to become trapped, so $c(G) > 3$.

By [1], if G is a generalised Petersen graph, then $c(G) \leq 4$, so $c(G) = 4$. □

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References

[1] T. Ball, R. W. Bell, J. Guzman, M. Hanson-Colvin and N. Schonsheck, On the cop number of generalized Petersen graphs, *Discrete Math.* **340** (2017), 1381–1388, doi:10.1016/j.disc.2016.10.004.

[2] M. Boben, T. Pisanski and A. Žitnik, I-graphs and the corresponding configurations, *J. Combin. Des.* **13** (2005), 406–424, doi:10.1002/jcd.20054.

[3] A. Bonato and R. J. Nowakowski, *The game of cops and robbers on graphs*, volume 61 of *Student Mathematical Library*, American Mathematical Society, Providence, RI, 2011, doi:10.1090/stml/061.

[4] J. Morris, T. Runte and A. Skelton, Most Generalized Petersen graphs of girth 8 have cop number 4, 2020, arXiv:2009.00693 [math.CO], https://arxiv.org/abs/2009.00693.

[5] R. Nowakowski and P. Winkler, Vertex-to-vertex pursuit in a graph, *Discrete Math.* **43** (1983), 235–239, doi:10.1016/0012-365X(83)90160-7.

[6] A. Quilliot, *Problèmes de jeux, de point fixe, de connectivité et de représentation sur des graphes, des ensembles ordonnés et des hypergraphes*, Ph.D. thesis, Université de Paris VI, 1983.

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Topology of clique complexes of line graphs*

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Abstract

The clique complex of a graph G is a simplicial complex whose simplices are all the cliques of G, and the line graph $L(G)$ of G is a graph whose vertices are the edges of G and the edges of $L(G)$ are incident edges of G. In this article, we determine the homotopy type of the clique complexes of line graphs for several classes of graphs including triangle-free graphs, chordal graphs, complete multipartite graphs, wheel-free graphs, and 4-regular circulant graphs. We also give a closed form formula for the homotopy type of these complexes in several cases.

Keywords: H-free complex, clique complex, line graph, chordal graphs, wheel-free graphs, circulant graphs.

Math. Subj. Class.: 05C69, 55P15

1 Introduction

For a fixed graph H, a subgraph K of a graph G is called H-free if K does not contain any subgraph isomorphic to H. Finding H-free subgraphs of a graph G is a well studied

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notion in extremal graph theory. It is easy to observe that H-freeness is a hereditary property, thereby giving a simplicial complex for any fixed graph G, denoted $\mathcal{F}(G, H)$. These complexes have already been studied for different graphs H, for example see [2, 7, 11, 14, 15, 22] when H is a complete graph, [9, 19, 21] when H is a star graph, [8, 13] when H is r disjoint copies of complete graphs on 2 vertices, denoted rK_2. Replacing H by a class of trees on fixed number of vertices has also gained attention in the last decade, for instance see [6, 18, 20].

In this article, our focus will be on the complex $\mathcal{F}(G, 2K_2)$. In [13], Linusson et. al have studied $\mathcal{F}(G, rK_2)$ for complete graphs and complete bipartite graphs. They showed that $\mathcal{F}(G, rK_2)$ is homotopy equivalent to a wedge of $(3r-4)$-dimensional spheres when G is a complete graph, and $(2r-3)$-dimensional spheres if G is a complete bipartite graph. Further, Holmsen and Lee [8] studied these complexes for general graph G and they showed that $\mathcal{F}(G, rK_2)$ is $(3r-3)$-Leray[1], and it is $(2r-2)$-Leray if G is bipartite. It is worth noting that $\mathcal{F}(G, 2K_2)$ is the clique complex of the line graph of G, denoted $\Delta L(G)$. Clique complexes of graphs have a very rich literature in topological combinatorics, for instance see [1, 10]. Recently, Nikseresht [17] studied some algebraic properties like Cohen-Macaulay, sequentially Cohen-Macaulay, Gorenstein, etc., of clique complexes of line graphs. In this article, we determine the exact homotopy type of $\Delta L(\_)$ for various classes of graphs including triangle-free graphs, chordal graphs, complete multipartite graphs, wheel-free graphs and 4-regular circulant graphs. Moreover, we show that $\Delta L(\_)$ in each of these cases is homotopy equivalent to a wedge of equidimensional spheres, except for the 4-regular circulant graphs for which it is homotopy equivalent to a wedge of circles and 2-spheres.

This article is organized as follows. In Section 2, we recall various definitions and results which are needed. In the next section, we analyze $\Delta L(G)$ as the 2-skeleton of the clique complex of G and then use it to compute the homotopy type of $\Delta L(G)$ for various classes of graphs including triangle free graphs and chordal graphs. In Section 3.1, by realizing ΔL as a functor from the category of graphs to the category of simplicial complexes, we compute the homotopy type of $\Delta L(G)$ when G is a complete multipartite graph. Section 4 is devoted towards the study of ΔL of wheel-free graphs and 4-regular circulant graphs. In the last section, we discuss a few questions that arise naturally from the work done in this article.

2 Preliminaries

A (*simple*) *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is called the set of vertices and $E(G) \subseteq \binom{V(G)}{2}$, the set of (unordered) edges of G. The vertices $v_1, v_2 \in V(G)$ are said to be adjacent, if $(v_1, v_2) \in E(G)$ and this is also denoted by $v_1 \sim v_2$. For $v \in V(G)$, the set of its *neighbours*, $N_G(v)$, in G is $\{x \in V(G) : x \sim v\}$. The *degree* of a vertex $v \in V(G)$ is the cardinality of the set of its neighbours. A vertex v of degree 1 is called a *leaf vertex* and the vertex adjacent to v is called the *parent* of v. An edge adjacent to a leaf vertex is called a *leaf/hanging edge*. A graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a *subgraph* of the graph G. For a nonempty subset $U \subseteq V(G)$, the induced subgraph $G[U]$, is the subgraph of G with vertices $V(G[U]) = U$ and $E(G[U]) = \{(a, b) \in E(G) : a, b \in U\}$. Similarly, for a nonempty subset $F \subseteq E(G)$,

[1]A simplicial complex K is called *d-Leray* (over a field $\mathbb{F}$) if $\tilde{H}_i(\mathsf{L}, \mathbb{F}) = 0$ for all $i \geq d$ and for every induced subcomplex $\mathsf{L} \subseteq \mathsf{K}$.

the induced subgraph $G[F]$, is the subgraph of G with vertices $V(G[F]) = \{v \in V(G) : v \in e$ for some $e \in F\}$ and $E(G[F]) = F$. In this article, the graph $G[V(G) \setminus A]$ will be denoted by $G - A$ for $A \subsetneq V(G)$.

For $n \geq 1$, the *complete graph* on n vertices is a graph where any two distinct vertices are adjacent, and is denoted by K_n. For $n \geq 3$, the *cycle graph* C_n is the graph with $V(C_n) = \{1, \ldots, n\}$ and $E(C_n) = \{(i, i+1) : 1 \leq i \leq n-1\} \cup \{(1, n)\}$. For $r \geq 1$, the *path graph* of length r is a graph with vertex set $V(P_r) = \{0, \ldots, r\}$ and edge set $E(P_r) = \{(i, i+1) : 0 \leq i \leq r-1\}$, and is denoted by P_r. The *line graph*, $L(G)$, of a graph G is the graph with $V(L(G)) = E(G)$ and $E(L(G)) = \{((v_1, v_2), (v_1', v_2')) : |\{v_1, v_2\} \cap \{v_1', v_2'\}| = 1\}$.

An (*abstract*) *simplicial complex* $\mathcal{K}$ is a collection of finite sets such that if $\tau \in \mathcal{K}$ and $\sigma \subset \tau$, then $\sigma \in \mathcal{K}$. The elements of $\mathcal{K}$ are called the *simplices* (or *faces*) of $\mathcal{K}$. If $\sigma \in \mathcal{K}$ and $|\sigma| = k+1$, then σ is said to be *k-dimensional*. The maximal simplices of $\mathcal{K}$ are also called *facets* of $\mathcal{K}$. The set of 0-dimensional simplices of $\mathcal{K}$ is denoted by $V(\mathcal{K})$, and its elements are called *vertices* of $\mathcal{K}$. A *subcomplex* of a simplicial complex $\mathcal{K}$ is a simplicial complex whose simplices are contained in $\mathcal{K}$. For $k \geq 0$, the *k-skeleton* of a simplicial complex $\mathcal{K}$ is a subcomplex consisting of all the simplices of dimension $\leq k$ and it is denoted by $\mathcal{K}^{(k)}$. The *cone* of a simplicial complex $\mathcal{K}$ with apex w, denoted $\mathcal{C}_w(\mathcal{K})$, is a simplicial complex whose facets are $\sigma \cup \{w\}$ for each facet σ of $\mathcal{K}$. In this article, we always assume empty set as a simplex of any simplicial complex and we consider any simplicial complex as a topological space, namely its geometric realization. For the definition of geometric realization, we refer to Kozlov's book [12].

The *clique complex*, $\Delta(G)$, of a graph G is the simplicial complex whose simplices are subsets $\sigma \subseteq V(G)$ such that $G[\sigma]$ is a complete graph.

The clique complex of line graph of a graph G, $\Delta L(G)$, has also been studied by Linusson, Shareshian, and Welker in [13] and Holmsen and Lee in [8]. They denoted these complexes by $NM_2(G)$ and proved the following results.

Theorem 2.1 (Theorem 1.1, [13]). *Let n be a positive integers. Then $NM_2(K_n)$ is homotopy equivalent to a wedge of spheres of dimension* 2.

Theorem 2.2 (Theorem 1.1, [8]). *For a graph G, the complex $NM_2(G)$ is* 3*-Leray.*

It is worth mentioning here that the Lerayness of the clique complex of the complement of a claw-free graph has also been studied in [3] and [16].

2.1 (Homotopy) Pushout

Let X, Y, Z be topological spaces, and $p : X \to Y$ and $q : X \to Z$ be continuous maps. The *pushout* of the diagram $Y \xleftarrow{p} X \xrightarrow{q} Z$ is the space

$$\left(Y \bigsqcup Z\right) / \sim,$$

where $\sim$ denotes the equivalence relation $p(x) \sim q(x)$, for $x \in X$.

The *homotopy pushout* of $Y \xleftarrow{p} X \xrightarrow{q} Z$ is the space $\big(Y \sqcup (X \times [0,1]) \sqcup Z\big) / \sim$, where $\sim$ denotes the equivalence relation $(x, 0) \sim p(x)$, and $(x, 1) \sim q(x)$, for $x \in X$. It can be shown that homotopy pushouts of any two homotopy equivalent diagrams are homotopy equivalent.

Remark 2.3. If spaces are simplicial complexes and maps are subcomplex inclusions, then their homotopy pushout and pushout spaces are equivalent up to homotopy. For elaborate discussion on this, we refer interested reader to [5, Chapter 7].

Remark 2.4. Consider a diagram $Y \xleftarrow{p} X \xrightarrow{q} Z$ and let W be its pushout object. If p, q are null-homotopic, then $W \simeq Y \bigvee Z \bigvee \Sigma(X)$, where $\simeq$ and $\bigvee$ respectively denotes homotopy equivalence and the wedge of topological spaces; and $\Sigma(X)$ denotes the suspension of X.

2.2 Simplicial Collapse

Let $\mathcal{K}$ be a simplicial complex and $\tau, \sigma \in \mathcal{K}$ be such that $\sigma \subsetneq \tau$ and τ is the only maximal simplex in $\mathcal{K}$ that contains σ. A *simplicial collapse* of $\mathcal{K}$ is the simplicial complex $\mathcal{L}$ obtained from $\mathcal{K}$ by removing all those simplices γ of $\mathcal{K}$ such that $\sigma \subseteq \gamma \subseteq \tau$. Here σ is called a *free face* of τ and (σ, τ) is called a *collapsible pair*. We denote this collapse by $\mathcal{K} \searrow \mathcal{L}$. A complex is called *collapsible* if it collapses onto a point by applying a sequence of simplicial collapses. It is a simple observation that, if $\mathcal{K} \searrow \mathcal{L}$ then $\mathcal{K} \simeq \mathcal{L}$ (in fact, $\mathcal{K}$ deformation retracts onto $\mathcal{L}$). Throughout this article, we write $\mathcal{K} \searrow \langle A_1, A_2, \ldots, A_r\rangle$ to mean that $\mathcal{K}$ collapses onto a subcomplex whose faces are $A_1, \ldots, A_r$ and all its subsets.

We now give a result about collapsing which will be used throughout this article. To the best of our knowledge, this result is not present in writing anywhere in literature.

Lemma 2.5. *Let $\mathcal{K}$ be a simplicial complex and σ be a facet of $\mathcal{K}$. Let $A, B, C \subset \sigma$ be such that $A, B \neq \emptyset$ and $\sigma = A \sqcup B \sqcup C$. For each $a \in A$ and $b \in B$, let $\{a, b\}$ be a free face of σ in $\mathcal{K}$.*

1. *If $C \neq \emptyset$, then $\sigma \searrow \langle \sigma \setminus A, \sigma \setminus B\rangle$.*
2. *If $C = \emptyset$, $|A| = 1$ and $|B| \geq 2$, then for each $a \in A$ and $b \in B$, $\sigma \searrow \langle B, \{a, b\}\rangle$.*
3. *If $C = \emptyset$ and $|A|, |B| \geq 2$, then for each $a \in A$ and $b \in B$, $\sigma \searrow \langle A, B, \{a, b\}\rangle$.*

Proof. Let $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_q\}$. Without loss of generality, we can assume that $a_p = a$ and $b_q = b$.

1. We first do collapsing using the vertex a_1 from set A and then use similar arguments for the rest. Since $\{a_1, b_1\}$ is a free face of σ, $\sigma \searrow \langle \sigma \setminus \{b_1\}, \sigma \setminus \{a_1\}\rangle$. Now $(\{a_1, b_2\}, \sigma \setminus \{b_1\})$ is a collapsible pair and therefore $\sigma \setminus \{b_1\} \searrow \langle \sigma \setminus \{b_1, b_2\}, \sigma \setminus \{a_1, b_1\}\rangle$. Hence $\sigma \searrow \langle \sigma \setminus \{b_1, b_2\}, \sigma \setminus \{a_1\}\rangle$. Inductively, assume that for some $1 \leq j < q$, $\sigma \searrow \langle \sigma \setminus \{b_1, \ldots, b_j\}, \sigma \setminus \{a_1\}\rangle$. Now $(\{a_1, b_{j+1}\}, \sigma \setminus \{b_1, \ldots, b_j\})$ is a collapsible pair and therefore $\sigma \searrow \langle \sigma \setminus \{b_1, \ldots, b_j, b_{j+1}\}, \sigma \setminus \{a_1\}\rangle$. Using induction, we get that $\sigma \searrow \langle \sigma \setminus \{b_1, \ldots, b_q\} = \sigma \setminus B, \sigma \setminus \{a_1\}\rangle$.

   If $p = 1$, then this completes the proof, otherwise we proceed as follows:

   By doing similar collapsing using vertices $a_2, a_3, \ldots, a_{p-1}$ from set A, we get that $\sigma \searrow \langle \sigma \setminus B, \sigma \setminus \{a_1, \ldots, a_{p-1}\}\rangle$. Now $(\{a_p, b_1\}, \sigma \setminus \{a_1, \ldots, a_{p-1}\})$ is a collapsible pair and therefore $\sigma \setminus \{a_1, \ldots, a_{p-1}\} \searrow \langle \sigma \setminus \{a_1, \ldots, a_{p-1}, a_p\} = \sigma \setminus A, \sigma \setminus \{a_1, \ldots, a_{p-1}, b_1\}\rangle$. Thus $\sigma \searrow \langle \sigma \setminus B, \sigma \setminus A, \sigma \setminus \{a_1, \ldots, a_{p-1}, b_1\}\rangle$. We now show that $\sigma \setminus \{a_1, \ldots, a_{p-1}, b_1\} \searrow \langle \sigma \setminus A, \sigma \setminus B\rangle$.

Since $C \neq \emptyset$, it is easy to observe that by using collapsible pairs in the following order:

$$(\{a_p, b_2\}, \sigma \setminus \{a_1, \dots, a_{p-1}, b_1\}), \dots, (\{a_p, b_q\}, \sigma \setminus \{a_1, \dots, a_{p-1}, b_1, \dots, b_{q-1}\}),$$

and applying the collapses, we get that $\sigma \setminus \{a_1, \dots, a_{p-1}, b_1\} \searrow \langle \sigma \setminus \{a_1, \dots, a_{p-1}, a_p, b_1, \dots, b_{q-1}\}, \sigma \setminus \{a_1, \dots, a_{p-1}, b_1, \dots, b_{q-1}, b_q\}\rangle$. Since $\sigma \setminus \{a_1, \dots, a_{p-1}, a_p, b_1, \dots, b_{q-1}\} \subseteq \sigma \setminus A$ and $\sigma \setminus \{a_1, \dots, a_{p-1}, b_1, \dots, b_{q-1}, b_q\} \subseteq \sigma \setminus B$, we get that $\sigma \searrow \langle \sigma \setminus A, \sigma \setminus B\rangle$.

2. Here $\sigma = \{a, b_1, \dots, b_q\}$. Using similar arguments as in the first paragraph of previous case and using the collapsible pairs in the following order:

$$(\{a, b_1\}, \sigma), (\{a, b_2\}, \{a, b_2, \dots, b_q\}), \dots, (\{a, b_{q-1}\}, \{a, b_{q-1}, b_q\}),$$

and doing the collapses, we get the desired result.

3. Using similar arguments as in the proof of case 1, we get that $\sigma \searrow \langle \sigma \setminus B, \sigma \setminus \{a_1, \dots, a_{p-1}\}\rangle$. Observe that $\sigma \setminus \{a_1, \dots, a_{p-1}\} = \{a_p, b_1, \dots, b_q\}$. We now use case 2 to collapse $\sigma \setminus \{a_1, \dots, a_{p-1}\}$ and get the result.

$\square$

Induction along with Lemma 2.5 gives the following result.

Corollary 2.6. *Let $\mathcal{K}$ be a simplicial complex and let $\sigma = A_1 \sqcup A_2 \dots \sqcup A_k \sqcup C$ be a facet of $\mathcal{K}$, where $A_j = \{a_1^j, \dots, a_{l_j}^j\}$ for $1 \leq j \leq k$. Let $\{a_s^i, a_t^j\}$ be a free face of σ for each $i \neq j$, $s \in [l_i]$ and $t \in [l_j]$.*

(1) *If $C = \emptyset$, then $\sigma \searrow \langle A_1, \dots, A_k, \{a_{l_1}^1, a_{l_k}^k\}, \{a_{l_2}^2, a_{l_k}^k\}, \dots, \{a_{l_{k-1}}^{k-1}, a_{l_k}^k\}\rangle$.*

(2) *If $C \neq \emptyset$, then $\sigma \searrow \langle A_1 \sqcup C, \dots, A_k \sqcup C\rangle$.*

3 Structural properties of ΔL

In this section, we first analyze $\Delta L(G)$ as the 2-skeleton of the clique complex of G and then we use it to compute the homotopy type of $\Delta L(G)$ for various classes of graphs including triangle free graphs and chordal graphs. Later we realize ΔL as a functor from the category of graphs to the category of simplicial complexes and compute the homotopy type of $\Delta L(G)$, when G is a complete multipartite graph, using the functoriality of ΔL.

The *nerve* of a family of sets $(A_i)_{i \in I}$ is the simplicial complex $\mathbf{N} = \mathbf{N}(\{A_i\})$ defined on the vertex set I so that a finite subset $\sigma \subseteq I$ is in $\mathbf{N}$ precisely when $\bigcap_{i \in \sigma} A_i \neq \emptyset$.

Theorem 3.1 ([4, Theorem 10.6(i)]). *Let $\mathcal{K}$ be a simplicial complex and $(\mathcal{K}_i)_{i \in I}$ be a family of subcomplexes such that $\mathcal{K} = \bigcup_{i \in I} \mathcal{K}_i$. Suppose every nonempty finite interSection $\mathcal{K}_{i_1} \cap \dots \cap \mathcal{K}_{i_t}$ for $i_j \in I, t \in \mathbb{N}$ is contractible, then $\mathcal{K}$ and $\mathbf{N}(\{\mathcal{K}_i\})$ are homotopy equivalent.*

Remark 3.2. Since any non-empty interSection of simplices is again a simplex, Theorem 3.1 implies that the nerve of the facets of a simplicial complex $\mathcal{K}$ is homotopy equivalent to $\mathcal{K}$.

Lemma 3.3. *For any graph G, the complex $\Delta L(G)$ is homotopy equivalent to the 2-skeleton of the clique complex of G, i.e.,*

$$\Delta L(G) \simeq (\Delta(G))^{(2)}.$$

Proof. Let $\mathcal{F}$ be the set of facets of $\Delta L(G)$. It is easy to observe that any $F \in \mathcal{F}$ is either a collection of three edges of G forming a triangle in G or of the form $\sigma_a = \{(a, v) : v \in N_G(a)\}$ for some $a \in V(G)$. From Remark 3.2, $\mathbf{N}(\mathcal{F}) \simeq \Delta L(G)$. It is therefore enough to show that $\mathbf{N}(\mathcal{F}) \simeq (\Delta(G))^{(2)}$.

Let $\mathcal{K}$ be a simplicial complex obtained from $(\Delta(G))^{(2)}$ by adding a barycenter to every 2-simplex of $(\Delta(G))^{(2)}$. Clearly $\mathcal{K} \simeq (\Delta(G))^{(2)}$ and $\mathcal{K} \cong \mathbf{N}(\mathcal{F})$. □

Let $v(G)$ and $e(G)$ denote the number of vertices and edges in a graph G, respectively. For a given graph G, define $u(G) = e(G) - e(T)$, where T is a spanning tree of G. The following is an immediate consequence of Lemma 3.3.

Corollary 3.4. *Let G be a triangle free graph. Then $\Delta L(G)$ is homotopy equivalent to G (considered as a 1-dimensional simplicial complex). In particular*

$$\Delta L(G) \simeq \bigvee_{u(G)} \mathbb{S}^1,$$

A graph G is called *chordal* if it has no induced cycle of length greater than 3, *i.e.*, each cycle of length more than 3 has a chord.

Theorem 3.5. *Let G be a connected chordal graph. Then $\Delta L(G)$ is homotopy equivalent to a wedge of 2-dimensional spheres.*

Proof. It is well known that the 2-skeleton of the clique complex of a chordal graph is simply connected. Also, we know that any 2-dimensional simply connected simplicial complex is homotopy equivalent to a wedge of 2-spheres (cf. [4, (9.19)]). The result therefore follows from Lemma 3.3. □

Let G be a graph. The *cone* over G with apex vertex w, denoted as C_wG, is the graph with $V(C_wG) = V(G) \sqcup \{w\}$ and edge set $E(C_wG) = E(G) \cup \{(w, v) : v \in V(G)\}$.

Lemma 3.6. *Let G be a graph with m triangles. Then $\Delta L(C_wG)$ is homotopy equivalent to a wedge of m spheres of dimension 2.*

Proof. Let $\mathcal{F} = \{F \in (\Delta(C_wG))^{(2)} : |F| = 3 \text{ and } w \notin F\}$. Observe that $|\mathcal{F}| = m$. Clearly $(\Delta(C_wG))^{(2)} \setminus \mathcal{F} \cong \mathcal{C}_w(G)$ (here G is considered as a 1-dimensional simplicial complex) and therefore $(\Delta(C_wG))^{(2)} \setminus \mathcal{F}$ is contractible. Hence, from [12, Proposition 7.8], we have that $(\Delta(C_wG))^{(2)} \simeq (\Delta(C_wG))^{(2)}/((\Delta(C_wG))^{(2)} \setminus \mathcal{F})$. Thus, from Lemma 3.3, $\Delta L(C_wG) \simeq (\Delta(C_wG))^{(2)} \simeq \bigvee_{|\mathcal{F}|} \mathbb{S}^2$. □

The *suspension* ΣG of a graph G is the graph with vertex set $V(\Sigma G) = V(G) \sqcup \{a, b\}$ and $E(\Sigma G) = E(G) \cup \{(a, v) : v \in V(G)\} \cup \{(b, v) : v \in V(G)\}$.

Lemma 3.7. *Let G be a triangle free connected graph on at least two vertices. Then*

$$\Delta L(\Sigma G) \simeq \Sigma(\Delta L(G)).$$

Proof. We know that $\Delta L(\Sigma G) \simeq (\Delta(\Sigma G))^{(2)}$. Since G is triangle free, $(\Delta(\Sigma G))^{(2)} = \Sigma((\Delta(G))^{(2)}) \simeq \Sigma(\Delta L(G))$. □

3.1 Complete multipartite graphs

Let $m_1, m_2, \ldots, m_r$ be positive integers and let A_i be a set of cardinality m_i for $1 \leq i \leq r$. A complete multipartite graph $K_{m_1,\ldots,m_r}$ is a graph on vertex set $A_1 \sqcup \ldots \sqcup A_r$ and two vertices are adjacent if and only if they lie in different A_i's. Using functoriality of ΔL, we study the homotopy type of ΔL of complete multipartite graphs. The following result can be obtained from Lemma 3.3.

Corollary 3.8. *Let H be a connected induced subgraph of connected graphs G_1 and G_2. Then,*

$$\Delta L(G_1 \bigcup_H G_2) \simeq \begin{cases} \Delta L(G_1) \bigsqcup \Delta L(G_2) & \text{if } |\mathrm{V(H)}| = 0, \\ \Delta L(G_1) \bigvee \Delta L(G_2) & \text{if } |\mathrm{V(H)}| = 1, \\ \Delta L(G_1) \bigcup_{\Delta L(H)} \Delta L(G_2) & \text{otherwise.} \end{cases}$$

In particular, for any $1 \leq s \leq m_r - 1$, $\Delta L(K_{m_1,m_2,\ldots,m_r})$ is homotopy equivalent to the pushout of

$$\Delta L(K_{m_1,m_2,\ldots,m_{r-1},s}) \hookleftarrow \Delta L(K_{m_1,m_2,\ldots,m_{r-1}}) \hookrightarrow \Delta L(K_{m_1,m_2,\ldots,m_{r-1},m_r-s}).$$

Proposition 3.9. *For $m, n, r \in \mathbb{N}$, $\Delta L(K_{m,n}) \simeq \bigvee_t \mathbb{S}^1$ and $\Delta L(K_{m,n,r}) \simeq \bigvee_{t(r-1)} \mathbb{S}^2$, where $t = mn - (m + n - 1)$.*

Proof. For the graph $K_{m,n}$, recall $u(K_{m,n}) = e(K_{m,n}) - e(T_{m,n}) = mn - (m + n - 1)$, where $T_{m,n}$ denotes a spanning tree of $K_{m,n}$. Since any bipartite graph is triangle-free, Corollary 3.4 implies that $\Delta L(K_{m,n})$ is homotopy equivalent to $\bigvee_{u(K_{m,n})} \mathbb{S}^1$.

Further, $K_{m,n,1}$ is a cone over the triangle-free graph $K_{m,n}$, and hence $\Delta L(K_{m,n,1})$ is contractible by Lemma 3.6. Now using Remark 2.4 and Corollary 3.8, $\Delta L(K_{m,n,2}) \simeq \{\text{pt}\} \bigvee \{\text{pt}\} \bigvee \left(\bigvee_{u(K_{m,n})} \mathbb{S}^2\right)$. Inductively, we construct $K_{m,n,r}$ as a pushout of $K_{m,n,r-1} \hookleftarrow K_{m,n} \hookrightarrow K_{m,n,1}$. Then repeating the similar arguments, we get that

$$\Delta L(K_{m,n,r}) \simeq \bigvee_{u(K_{m,n})(r-1)} \mathbb{S}^2. \qquad \square$$

Note. The homotopy type of $\Delta L(K_{m,n})$ has also been computed by Linusson et al. in [13, Theorem 1.4] using discrete Morse theory.

Theorem 3.10. *For a complete r-partite graph $K_{m_1,\ldots,m_r}$, $\Delta L(K_{m_1,\ldots,m_r})$ is homotopy equivalent to a wedge of $\mathbb{S}^2$'s for $r > 2$.*

Proof. From Proposition 3.9, $\Delta L(K_{m_1,m_2,m_3})$ is homotopy equivalent to a wedge of 2-dimensional spheres. So let $r > 3$. Since $K_{m_1,m_2,m_3,1}$ is a cone over K_{m_1,m_2,m_3}, Lemma 3.6 gives that $\Delta L(K_{m_1,m_2,m_3,1})$ is homotopy equivalent to a wedge of $\mathbb{S}^2$'s. If $m_4 > 1$, we inductively construct K_{m_1,m_2,m_3,m_4} as a pushout of

$$K_{m_1,m_2,m_3,m_4-1} \hookleftarrow K_{m_1,m_2,m_3} \hookrightarrow K_{m_1,m_2,m_3,1}.$$

Note that after applying ΔL to this diagram and using Proposition 3.9, we get the following pushout diagram

$$\begin{tikzcd}
\bigvee \mathbb{S}^2 \arrow[r, hook] \arrow[d, hook] & (\bigvee \mathbb{S}^2) \vee (\bigvee \mathbb{B}^3) \arrow[d] \\
(\bigvee \mathbb{S}^2) \vee (\bigvee \mathbb{B}^3) \arrow[r] & X \simeq \Delta L(K_{m_1,m_2,m_3,m_4})
\end{tikzcd}$$

Figure 1: $\Delta L(K_{m_1,m_2,m_3,m_4})$ as a pushout

Here $\mathbb{B}^3$ denotes a closed ball of dimension 3. From Figure 1, it is easy to see that $\Delta L(K_{m_1,m_2,m_3,m_4})$ is homotopy equivalent to a wedge of spheres of dimension 2 and 3. However, Theorem 2.2 implies that $\tilde{H}_3(\Delta L(K_{m_1,m_2,m_3,m_4})) = 0$, and hence $\Delta L(K_{m_1,m_2,m_3,m_4}) \simeq \bigvee \mathbb{S}^2$. We now repeat these arguments for $K_{m_1,\ldots,m_r}$ to get the desired result. □

4 Wheel-free graphs and 4-regular circulant graphs

For $n \geq 3$, a wheel graph on $n+1$ vertices, denoted by W_n, is a graph isomorphic to cone over a cycle graph C_n. For a wheel graph, we call apex vertex of the cone as the center of the wheel.

Theorem 4.1. *Let G be a connected wheel-free graph. Then $\Delta L(G)$ is homotopy equivalent to a wedge of circles.*

Proof. In view of Lemma 3.3, we can assume that G does not have any leaf. Observe that any maximal simplex σ in $\Delta L(G)$ is one of the following two types

1. $\sigma = \{e_1, \ldots, e_t\}$ such that $\bigcap_{i\in[t]} e_i = \{x\}$, *i.e.*, every edge shares a common vertex.
2. $\sigma = \{e_1, e_2, e_3\}$ and e_1, e_2, e_3 forms a triangle in G.

We now show that each facet of $\Delta L(G)$ can be collapsed to a 1-dimensional subcomplex. First we collapse facets of type 1.

Case 1: Let $\sigma = \{e_1, \ldots, e_t\}$ be a facet and $\bigcap_{i\in[t]} e_i = \{x\}$. Without loss of generality, we can assume that $t \geq 3$. We partition the set $N_G(x) = A \sqcup B$, where

- $A = \bigcup_{i\in[r_1]}\{a_i\}$, where each a_i is an isolated vertex in the induced subgraph $G[N_G(x)]$.
- $B = \bigsqcup_{j\in[r_2]} B_j$, where each $B_j \subseteq N_G(x)$ such that $G[B_j]$ is a connected component of $G[N_G(x)]$ and has more than 1 vertex.

Since G is wheel-free graph, it is easy to see that $G[B_j]$ is a tree for each $j \in [r_2]$. Define,

$$\begin{aligned} \mathcal{A} &= \{(x, a_i) : i \in [r_1]\} \text{ and} \\ \mathcal{B} &= \bigsqcup_{j\in[r_2]} \{(x, a) : a \in B_j\}, \end{aligned} \tag{4.1}$$

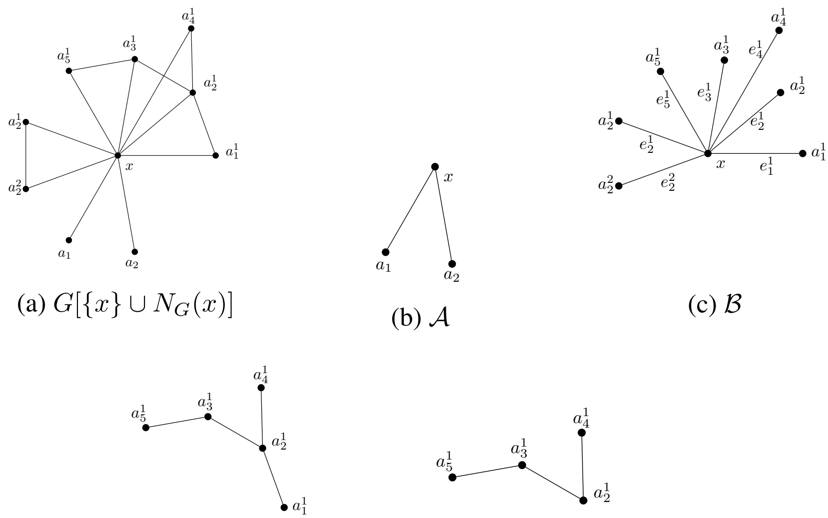
(cf. Figure 2). For each $j \in [r_2]$, let $\mathcal{B}_j = \{(x, a) : a \in B_j\}$. It is easy to see that each $\{e, e'\}$ is a free face of σ whenever

(i) $e \in \mathcal{B}_i$, $e' \in \mathcal{B}_j$ and $i \neq j$, or

(ii) $e \in \mathcal{A}$ and $e' \in \mathcal{B}$, or

(iii) $e, e' \in \mathcal{A}$.

For each $j \in [r_2]$, let $\mathcal{B}_j = \{e_1^j, e_2^j, \ldots, e_{l_j}^j\}$ and $\mathcal{A} = \{e_1, \ldots, e_{r_1}\}$. By Corollary 2.6(1), $\sigma \searrow \langle \mathcal{B}_1, \ldots, \mathcal{B}_{r_2}, \mathcal{A}, \{e_{l_1}^1, e_{r_1}\}, \ldots, \{e_{l_{r_2}}^{r_2}, e_{r_1}\}\rangle$.



(a) $G[\{x\} \cup N_G(x)]$ (b) $\mathcal{A}$ (c) $\mathcal{B}$

(d) $G[B_1]$ (e) $G[B_1 - a_1^1]$

Figure 2

Again using Corollary 2.6(2), $\mathcal{A} \searrow \langle \{e_1, e_{r_1}\}, \ldots, \{e_{r_1-1}, e_{r_1}\}\rangle$. We now show that each $\mathcal{B}_j$ can be collapsed onto a 1-dimensional subcomplex of $\Delta L(G)$. We prove this by induction on the number of elements in $\mathcal{B}_j$. If $l_j = 2$ then the simplex $\mathcal{B}_j$ itself is of dimension 1.

Fix $j \in [r_2]$ and assume that $l_j \geq 3$. For each $1 \leq i \leq l_j$, let $e_i^j = (x, a_i^j)$. Without loss of generality, we can assume that the a_1^j is a leaf vertex in tree $G[B_j]$ and a_2^j is its parent. This implies that for any $i \in \{3, \ldots, l_j\}$, there exist no triangle in G which contains both the edges e_1^j and e_i^j.

Since e_1^j and e_3^j are not part of any triangle, $(\{e_1^j, e_3^j\}, \mathcal{B}_j)$ is a collapsible pair and therefore $\mathcal{B}_j \searrow \langle \mathcal{B}_j \setminus \{e_1^j\}, \mathcal{B}_j \setminus \{e_3^j\}\rangle$. If $l_j = 3$, then $\mathcal{B}_j \setminus \{e_1^j\}$ and $\mathcal{B}_j \setminus \{e_3^j\}$ are 1-dimensional. Assume $l_j \geq 4$.

Now $(\{e_1^j, e_4^j\}, \mathcal{B}_j \setminus \{e_3^j\})$ is a collapsible pair and therefore $\mathcal{B}_j \setminus \{e_3^j\} \searrow \langle \mathcal{B}_j \setminus \{e_3^j, e_4^j\}, \mathcal{B}_j \setminus \{e_3^j, e_1^j\}\rangle$. Since $\mathcal{B}_j \setminus \{e_3^j, e_1^j\} \subseteq \mathcal{B}_j \setminus \{e_1^j\}$, $\mathcal{B}_j \searrow \langle \mathcal{B}_j \setminus \{e_1^j\}, \mathcal{B}_j \setminus \{e_3^j, e_4^j\}\rangle$. It is easy to observe that by using collapsible pairs in the following order:

$$(\{e_1^j, e_5^j\}, \mathcal{B}_j \setminus \{e_3^j, e_4^j\}), \ldots (\{e_1^j, e_{l_j}^j\}, \mathcal{B}_j \setminus \{e_3^j, e_4^j, \ldots, e_{l_j-1}^j\})$$

and applying the collapses, we get that $\mathcal{B}_j \searrow \langle \mathcal{B}_j \setminus \{e_1^j\}, \{e_1^j, e_2^j\}\rangle$. Since a_1^j is a leaf vertex of $G[B_j]$, $G[B_j - \{a_1^j\}]$ is again a tree (cf. Figure 2e). Therefore, from induction, we have that $\mathcal{B}_j \setminus \{e_1^j\}$ collapses onto a 1-dimensional subcomplex of $\Delta L(G)$. Which implies that, $\mathcal{B}_j$ collapses onto a subcomplex of dimension 1.

Case 2: Once we have collapsed all simplices of type 1 then given any simplex $\{e_1, e_2, e_3\}$ of type 2, it is easy to see that $(\{e_1, e_2\}, \{e_1, e_2, e_3\})$ is always a collapsible pair. Thus we can collapse all these simplices to a 1-dimensional subcomplex of $\Delta L(G)$.

Since G is connected, $\Delta L(G)$ is connected. Therefore the result follows from the fact that any connected one dimensional simplicial complex is homotopy equivalent to a wedge of circles. $\square$

It is an easy observation that any graph which is not K_4 and has maximal degree 3 is a wheel-free graph, and hence the previous result implies the following corollary.

Corollary 4.2. *Let G be a connected graph of maximal degree at most 3 and $G \not\cong K_4$. Then $\Delta L(G)$ is homotopy equivalent to a wedge of circles.*

Let $n \geq 3$ be a positive integer and $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $C_n(S)$ is the graph, whose set of vertices $V(C_n(S)) = \{0, 1, \ldots, n-1\}$ and any two vertices x and y adjacent if and only if $x - y \pmod{n} \in S \cup (-S)$, where $-S = \{n - a : a \in S\}$. Circulant graphs are also Cayley graphs of $\mathbb{Z}_n$, the cyclic group on n elements. Since $n \notin S$, $C_n(S)$ is a simple graph, *i.e.*, does not contains any loop. Further, $C_n(S)$ are $|S \cup (-S)|$-regular graphs. We now prove a structural result for 4-regular circulant graphs. This result will enable us to do the computation of the homotopy type of their ΔL.

Proposition 4.3. *Let G be a 4-regular circulant graph. Then each connected component of G is either wheel-free or isomorphic to K_5 or to ΣC_4.*

Proof. Let $\{s, t\}$ be the generating set of G such that $s < t$ and $V(G) = \{0, 1, \ldots, n-1\}$. Symmetry in the circulant graph implies that connected components of G are isomorphic. Suppose G is not wheel-free and say it has a subgraph H isomorphic to W_m, a wheel on $m+1$ vertices. Since G is 4-regular, $3 \leq m \leq 4$. Without loss of generality we can assume that 0 is the center vertex of W_m. Clearly, $N_G(0) = \{s, t, n-s, n-t\}$.
Case 1: $m = 3$.

Since $W_3 \cong K_4$, we see that $|V(W_3) \cap N_G(0)| = 3$. Therefore, either $s \sim t$ or $n - s \sim n - t$ in W_3. In both the cases we get that $t = 2s$. Since $s < t$, $n - t < n - s$. If $N_{W_3}(0) = \{s, t, n-t\}$, then $n - t < n - s$ implies that $n - t = 3s$, thus $n = 3s + 2s = 5s$. Similar analysis for any 3 element subset of $N_G(0)$ implies that $n = 5s$. This implies that $N_G(0) = \{s, 2s, 3s, 4s\}$ and therefore $G[N_G(0) \cup \{0\}]$ is a clique in G. Moreover, 4-regularity of G implies that $G[\{0, s, 2s, 3s, 4s\}]$ is a component.
Case 2: $m = 4$.

Let $a \sim b \sim c \sim d \sim a$ be the outer cycle of W_4. By symmetry, the vertex a is also a center of a wheel with 5 vertices, say W_4'. Let $N_G(a) = \{b, d, 0, x\}$. If $x = c$, then $\{a, b, c, 0\}$ forms a K_4. By Case 1, we get that $\{a, b, c, d, 0\}$ forms a K_5. Therefore, let $x \neq c$, then $x \not\sim 0$ implies that $d \sim x \sim b \sim 0 \sim d$ is the outer cycle of W_4'. Similarly, b is the center of some other wheel with 5 vertices, say W_4''. Since $N_G(b) = \{a, 0, c, x\}$, the outer cycle of W_4'' is given by $a \sim x \sim c \sim 0 \sim a$. Therefore, $N_G(x) = N_G(0) = \{a, b, c, d\}$. Again, the 4-regularity of G implies that $G[\{0, a, b, c, d, x\}]$ is a component and is isomorphic to ΣC_4. $\square$

Corollary 4.4. *Let G be a 4-regular circulant graph. Then each connected component of $\Delta L(G)$ is homotopy equivalent to either a wedge of circles or a wedge of 2-spheres.*

Proof. We first note that for any cycle graph C_r, $\Delta L(C_r) \simeq \mathbb{S}^1$ whenever $r \geq 4$. From Proposition 4.3, each connected component of G is either wheel-free or isomorphic to K_5 or to ΣC_4. Since $\Delta L(\text{wheel-free}) \simeq \bigvee \mathbb{S}^1$ (cf. Theorem 4.1), $\Delta L(K_5) \simeq \bigvee \mathbb{S}^2$ (cf. Theorem 2.1) and $\Delta L(\Sigma C_4) \simeq \Sigma(\Delta L(C_4)) \simeq \Sigma(\mathbb{S}^1) = \mathbb{S}^2$ (cf. Lemma 3.7), the result follows. □

It is to note here that the computations done in this Section gives the exact homotopy type of ΔL for all the 2, 3 and 4-regular circulant graphs.

5 Further directions

For any simplicial complex $\mathcal{K} \simeq \left(\bigvee_m \mathbb{S}^1\right) \vee \left(\bigvee_n \mathbb{S}^2\right)$, Corollary 3.8 shows that $\Delta L(\left(\bigvee_m C_4\right) \vee \left(\bigvee_n K_4\right)) \simeq \mathcal{K}$, where the wedge of graphs is taken along a vertex as 1-dimensional simplicial complexes. It is well known that the clique complex functor Δ is universal (*i.e.*, given any simplicial complex $\mathcal{K}$, there is a graph G such that $\Delta(G) \simeq \mathcal{K}$), whereas the line graph functor L is not (for example, $K_{1,3}$ is not a line graph of any graph). This raises the following natural question.

Question 5.1. Is the functor ΔL universal from the category of graphs to the category of 3-Leray simplicial complexes?

We note here that for all the classes of graphs considered in this article, ΔL has always been a wedge of equidimensional spheres. Therefore, it would be interesting to know the following.

Question 5.2. Can the classes of graphs be classified whose ΔL is a wedge of equidimensional spheres? More specifically, can we classify those graphs whose ΔL is simply connected?

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References

[1] M. Adamaszek, Clique complexes and graph powers, *Israel J. Math.* **196** (2013), 295–319, doi:10.1007/s11856-012-0166-1.

[2] J. A. Barmak, Star clusters in independence complexes of graphs, *Adv. Math.* **241** (2013), 33–57, doi:10.1016/j.aim.2013.03.016.

[3] T. Bıyıkoğlu and Y. Civan, Castelnuovo-Mumford regularity of graphs, *Combinatorica* **38** (2018), 1353–1383, doi:10.1007/s00493-017-3450-1.

[4] A. Björner, Topological methods, in: *Handbook of combinatorics, Vol. 1, 2*, Elsevier Sci. B. V., Amsterdam, pp. 1819–1872, 1995.

[5] G. E. Bredon, *Topology and Geometry*, volume 139 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1997, corrected third printing of the 1993 original.

[6] P. Deshpande, S. Shukla and A. Singh, Distance r-domination number and r-independence complexes of graphs, *Eur. J. Comb.* **102** (2022), 14, doi:10.1016/j.ejc.2022.103508.

[7] A. Dochtermann and A. Engström, Algebraic properties of edge ideals via combinatorial topology, *Electron. J. Comb.* **16** (2009), Research Paper 2, 24, doi:10.37236/68.

[8] A. F. Holmsen and S. Lee, Leray numbers of complexes of graphs with bounded matching number, *J. Comb. Theory, Ser. A* **189** (2022), 44, doi:10.1016/j.jcta.2022.105618.

[9] J. Jonsson, *Simplicial Complexes of Graphs*, volume 1928 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2008, doi:10.1007/978-3-540-75859-4.

[10] M. Kahle, Sharp vanishing thresholds for cohomology of random flag complexes, *Ann. of Math. (2)* **179** (2014), 1085–1107, doi:10.4007/annals.2014.179.3.5.

[11] K. Kawamura, Independence complexes of chordal graphs, *Discrete Math.* **310** (2010), 2204–2211, doi:10.1016/j.disc.2010.04.021.

[12] D. Kozlov, *Combinatorial Algebraic Topology*, volume 21 of *Algorithms and Computation in Mathematics*, Springer, Berlin, 2008, doi:10.1007/978-3-540-71962-5.

[13] S. Linusson, J. Shareshian and V. Welker, Complexes of graphs with bounded matching size, *J. Algebraic Comb.* **27** (2008), 331–349, doi:10.1007/s10801-007-0092-1.

[14] R. Meshulam, Domination numbers and homology, *J. Comb. Theory Ser. A* **102** (2003), 321–330, doi:10.1016/S0097-3165(03)00045-1.

[15] S. Moradi, t-clique ideal and t-independence ideal of a graph, *Comm. Algebra* **46** (2018), 3377–3387, doi:10.1080/00927872.2017.1412445.

[16] E. Nevo, Regularity of edge ideals of C_4-free graphs via the topology of the lcm-lattice, *J. Comb. Theory Ser. A* **118** (2011), 491–501, doi:10.1016/j.jcta.2010.03.008.

[17] A. Nikseresht, Algebraic properties of clique complexes of line graphs, 2020, `arXiv:2007.13082 [math.CO]`.

[18] G. Paolini and M. Salvetti, Weighted sheaves and homology of Artin groups, *Algebr. Geom. Topol.* **18** (2018), 3943–4000, doi:10.2140/agt.2018.18.3943.

[19] A. Singh, Higher matching complexes of complete graphs and complete bipartite graphs, *Discrete Math.* **345** (2022), 10, doi:10.1016/j.disc.2021.112761.

[20] T. Szabó and G. Tardos, Extremal problems for transversals in graphs with bounded degree, *Combinatorica* **26** (2006), 333–351, doi:10.1007/s00493-006-0019-9.

[21] M. L. Wachs, Topology of matching, chessboard, and general bounded degree graph complexes, *Algebra Universalis* **49** (2003), 345–385, doi:10.1007/s00012-003-1825-1.

[22] R. Woodroofe, Vertex decomposable graphs and obstructions to shellability, *Proc. Am. Math. Soc.* **137** (2009), 3235–3246, doi:10.1090/S0002-9939-09-09981-X.

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On algebraic structure of the Reed-Muller codes

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Abstract

It is known that the Reed-Muller codes over a prime field may be described as the radical powers of a modular group algebra. In this paper, we give a new proof of the same result in a quotient of a polynomial ring. Special elements in a prime field are studied. An interpolation polynomial is introduced in order to characterize the coefficients of the Jennings polynomials.

Keywords: Reed-Muller codes, finite field, interpolation polynomial, Jennings basis.

Math. Subj. Class.: 94B05, 16N40, 12E05, 12E20

1 Introduction

Reed-Muller codes are among the oldest known families of codes. They were discovered by I.S. Reed and D.E. Muller in 1954. These codes were initially given as binary codes, but generalizations to q-ary were provided with q a prime power. Reed-Muller codes were studied by many authors (see, e.g. [3, 5, 7, 8, 9, 11, 12]).

These codes form a class of practically important codes. They have found widespread applications. A powerful Reed-Muller code was used by Mariner 9 to send back clear pictures from Mars to Earth in 1972.

A great advantage of the Reed-Muller codes is that they are relatively easy to decode by using majority logic decoding.

One of the interesting properties of the Reed-Muller codes is that there are several ways to describe them. They may be described by using finite geometries [2]. Group algebra approach can be used to characterize the Reed-Muller codes. This approach enables Berman and Charpin to identify the Reed-Muller codes with the radical powers in a suitable modular group algebra for the binary and p-ary cases. This is the famous theorem of Berman for the binary case [4]. The p-ary case was treated by Charpin [6]. Many authors (see, e.g. [1, 9, 10]) have studied this property of Reed-Muller codes.

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In this paper, we utilize the quotient algebra $B = \mathbb{F}_p[X_0, \ldots, X_{m-1}]/\langle X_0^p - 1, \ldots, X_{m-1}^p - 1\rangle$ as the ambient space of the codes. The radical powers of B are linearly generated by the Jennings bases. We give some properties of special elements of the finite field $\mathbb{F}_p$. Then, we obtain the coefficients of the Jennings polynomials by means of an appropriate interpolation function. We can apply this fact to show that the radical powers of B are the Reed-Muller codes of length p^m over $\mathbb{F}_p$.

2 Preliminary results

In this section, we give some properties of the following special elements of the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, \ldots p-1\}$

$$a_{i,d} := \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^d \tag{2.1}$$

where i and d are integers such that $0 \leq i, d \leq p-1$.

Proposition 2.1. *We have*

$$a_{i,d} \; = \; i \sum_{k=0}^{d-1} \binom{d-1}{k} a_{i-1,k}$$

for all integers d *and* i *such that* $1 \leq i, d \leq p-1$.

Proof. We have

$$\sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^d \; = \; \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} j^d \; = \; \sum_{j=1}^{i} (-1)^{i-j} \cdot j \cdot \binom{i}{j} j^{d-1}.$$

Since

$$\binom{i}{j} = \frac{i}{j} \binom{i-1}{j-1},$$

the last expression become

$$i \sum_{j=1}^{i} (-1)^{i-1-(j-1)} \binom{i-1}{j-1} ((j-1)+1)^{d-1}.$$

By using the relation

$$((j-1)+1)^{d-1} = \sum_{k=0}^{d-1} \binom{d-1}{k} (j-1)^k,$$

and introducing $J = j-1$, the last expression become

$$i \sum_{k=0}^{d-1} \binom{d-1}{k} (\sum_{J=0}^{i-1} (-1)^{i-1-J} \binom{i-1}{J} J^k). \qquad \square$$

Proposition 2.2. *Let i be an integer such that $1 \leq i \leq p-1$. We have*

$$a_{i,d} = 0$$

for $d = 0, 1, ..., i-1$.

Proof. By induction on i:

– for $i = 1$: thus $d = 0$, we have, by convention $0^0 = 1$,

$$(-1)^1 \cdot \binom{1}{0} \cdot 0^0 + (-1)^0 \cdot \binom{1}{1} \cdot 1^0 = -1 + 1 = 0$$

– suppose the assertion is true for $i-1$, i.e.

$$\sum_{j=0}^{i-1} (-1)^{i-1-j} \binom{i-1}{j} j^d = 0\ ,\ for\ \ d = 0, 1, ..., i-2$$

and let us prove that it is also true for i.

- For $d = 0$,

$$\sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} = 0.$$

- Let d be such that $1 \leq d \leq i-1$, thus $0 \leq d-1 \leq i-2$; according to Proposition 2.1,

$$\begin{aligned} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^d &= i \sum_{k=0}^{d-1} \binom{d-1}{k} \left(\sum_{j=0}^{i-1} (-1)^{i-1-j} \binom{i-1}{j} j^k \right) \\ &= i \sum_{k=0}^{d-1} \binom{d-1}{k} \cdot 0 = 0. \end{aligned}$$ □

Proposition 2.3. *Let i be an integer such that $1 \leq i \leq p-1$. Thus*

$$a_{i,i} \neq 0.$$

Proof. By induction on i.

– For $i = 1$,

$$(-1)^1 \binom{1}{0} 0^1 + (-1)^0 \binom{1}{1} 1^1 = 0 + 1 \neq 0$$

– Assume that the assertion holds for $i-1$ with $1 \leq i-1 \leq p-2$ (therefore, $2 \leq i \leq p-1$), i.e.

$$\sum_{j=0}^{i-1} (-1)^{i-1-j} \binom{i-1}{j} j^{i-1} \neq 0$$

And we have to show that the assertion is also true for i.

Using Proposition 2.1 and Proposition 2.2, we have

$$\sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^i = i \sum_{k=0}^{i-2} \binom{i-1}{k} \cdot 0 + i \sum_{j=0}^{i-1} (-1)^{i-1-j} \binom{i-1}{j} j^{i-1} \neq 0.$$ □

3 Interpolation polynomial

In this section, we study the following interpolation polynomial.

Definition 3.1. Taking account of (2.1), we define

$$H_i(Y) = (a_{i,0} - a_{i,p-1}) - \sum_{d=0}^{p-2} a_{i,d}\, Y^{p-1-d} \quad \in \mathbb{F}_p[Y]. \tag{3.1}$$

Theorem 3.2. *We have*

$$\deg(H_i(Y)) = p - 1 - i \tag{3.2}$$

where deg *denotes the degree of the polynomial.*

Proof. This is clear by using Proposition 2.2 and Proposition 2.3 in (3.1). □

Recall the following well known Lemmas.

Lemma 3.3. *We have*

$$\binom{p-1}{d} = (-1)^d \mod p$$

for $d = 0, 1, \ldots, p-1$.

Proof. It can be proved by induction on d. □

Lemma 3.4. *For* $a \in \mathbb{F}_p$, *we have*

$$a^{p-1} = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a \neq 0. \end{cases}$$

Proof. It is clear that $0^{p-1} = 0$, because $p \geq 2$.

For the second case, note that $\mathbb{F}_p - \{0\}$ is a multiplicative group of order $p-1$. □

Theorem 3.5. *For an integer* k *such that* $0 \leq k \leq i$, *we have*

$$H_i(k) = (-1)^{i-k}\binom{i}{k}.$$

Proof. By using (3.1), (2.1) and the Lemma 3.3, we have

$$\begin{aligned} H_i(Y) &= (a_{i,0} - a_{i,p-1}) - \sum_{d=0}^{p-2} a_{i,d}\, Y^{p-1-d} \\ &= a_{i,0} - \sum_{d=0}^{p-1} a_{i,d}\, Y^{p-1-d} \\ &= \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} - \sum_{d=0}^{p-1} (\sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^d)\, Y^{p-1-d} \\ &= \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} - \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} [\sum_{d=0}^{p-1} j^d\, Y^{p-1-d}] \\ &= \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} - \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} [\sum_{d=0}^{p-1} (-1)^d (-j)^d\, Y^{p-1-d}] \\ &= \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} - \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} [\sum_{d=0}^{p-1} \binom{p-1}{d} (-j)^d\, Y^{p-1-d}] \\ &= \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} - \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} (Y-j)^{p-1} \\ &= \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} [1 - (Y-j)^{p-1}]. \end{aligned}$$

And by Lemma 3.4, we have the result. □

Remark 3.6. For an integer k such that $i < k \leq p-1$, we have

$$H_i(k) \ = 0.$$

4 Application to Reed-Muller codes

In this section, we give a new proof of the theorem of Berman and Charpin. Recall that $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, \ldots, p-1\}$ is the field of p elements with p a prime number. Let m be a positive integer.

Definition 4.1. A linear code of length p^m over $\mathbb{F}_p$ is a linear subspace of the vector space

$$(\mathbb{F}_p)^{p^m} = \{(c_0, c_1, \ldots, c_{p^m-1}) \mid c_t \in \mathbb{F}_p, \text{ for all } t\}.$$

We consider the quotient algebra

$$B = \mathbb{F}_p[X_0, \ldots, X_{m-1}]/\langle X_0^p - 1, \ldots, X_{m-1}^p - 1\rangle \tag{4.1}$$

where $I = \langle X_0^p - 1, \ldots, X_{m-1}^p - 1\rangle$ is the ideal of the polynomial ring $\mathbb{F}_p[X_0, \ldots, X_{m-1}]$ generated by $X_0^p - 1, \ldots, X_{m-1}^p - 1$.

We denote

$$x_0 = X_0 + I, \ldots, x_{m-1} = X_{m-1} + I. \tag{4.2}$$

Remark 4.2. Note that $x_t^p = 1 = x_t^0$ for $t = 0, \dots, m-1$. Then, the exponent i_t in $x_t^{i_t}$ can be viewed as an integer in $\mathbb{Z}/p\mathbb{Z} = \{0, 1, \dots, p-1\}$.

We have

$$B = \left\{ \sum_{i_0=0}^{p-1} \cdots \sum_{i_{m-1}=0}^{p-1} c_{i_0,\dots,i_{m-1}} x_0^{i_0} \dots x_{m-1}^{i_{m-1}} \mid c_{i_0,\dots,i_{m-1}} \in \mathbb{F}_p \right\}. \tag{4.3}$$

Let us fix an order on the set of monomials

$$\left\{ x_0^{i_0} \dots x_{m-1}^{i_{m-1}} \mid i_t \in \mathbb{Z}/p\mathbb{Z}, \text{ for all } t \right\}.$$

Then, we can consider the isomorphism of vector spaces

$$\begin{array}{cccc} \Phi & : & B & \longrightarrow & (\mathbb{F}_p)^{p^m} \\ \sum_{i_0=0}^{p-1} \cdots \sum_{i_{m-1}=0}^{p-1} c_{i_0,\dots,i_{m-1}} x_0^{i_0} \dots x_{m-1}^{i_{m-1}} & & & \longmapsto & (c_{i_0,\dots,i_{m-1}})_{0 \le i_0,\dots,i_{m-1} \le p-1}. \end{array} \tag{4.4}$$

Therefore, B can be considered as the ambient space for the linear codes of length p^m over $\mathbb{F}_p$, and the polynomial $\sum_{i_0=0}^{p-1} \cdots \sum_{i_{m-1}=0}^{p-1} c_{i_0,\dots,i_{m-1}} x_0^{i_0} \dots x_{m-1}^{i_{m-1}}$ of B can be identified with the vector $(c_{i_0,\dots,i_{m-1}})_{0 \le i_0,\dots,i_{m-1} \le p-1}$ of $(\mathbb{F}_p)^{p^m}$ and vice-versa.

B is a local ring with maximal ideal R which is the radical of B, i.e.

$$R = \operatorname{rad}(B). \tag{4.5}$$

Let d be an integer such that $0 \le d \le m(p-1)$. Consider the powers R^d of R. We have the following sequence of ideals:

$$\{0\} \subset R^{m(p-1)} \subset \cdots \subset R^2 \subset R \subset B.$$

For simplicity, in virtue of (4.2) and Remark 4.2, we use the following notations

$$\begin{aligned} \mathbf{x} &:= (x_0, \dots, x_{m-1}), \\ \mathbf{Y} &:= (Y_0, \dots, Y_{m-1}), \\ \mathbf{i} &:= (i_0, \dots, i_{m-1}) \in (\mathbb{Z}/p\mathbb{Z})^m, \\ \mathbf{j} &\le \mathbf{i} \text{ if } \ j_t \le i_t, \text{ for all } t, \text{ with } \ \mathbf{i}, \mathbf{j} \in (\mathbb{Z}/p\mathbb{Z})^m, \\ \mathbf{x}^{\,\mathbf{i}} &:= x_0^{i_0} \cdot \ldots \cdot x_{m-1}^{i_{m-1}} \text{ and} \\ \lfloor \mathbf{i} \rfloor &:= i_0 + \ldots + i_{m-1}. \end{aligned}$$

Definition 4.3. The Jennings polynomial is defined by

$$J_{\mathbf{i}}(\mathbf{x}) := (x_0 - 1)^{i_0} \cdot \ldots \cdot (x_{m-1} - 1)^{i_{m-1}}$$

with $\mathbf{i} := (i_0, \dots, i_{m-1}) \in (\mathbb{Z}/p\mathbb{Z})^m$.

Remark 4.4. (i) A linear basis of R^d over $\mathbb{F}_p$ called the Jennings basis of R^d is

$$E_d := \{ J_{\mathbf{i}}(\mathbf{x}) \mid \mathbf{i} \in (\mathbb{Z}/p\mathbb{Z})^m, \ \lfloor \mathbf{i} \rfloor \ge d \}.$$

(ii) We have

$$\dim_{\mathbb{F}_p}(R^d) = \text{card}\,\{\mathbf{i} \in (\mathbb{Z}/p\mathbb{Z})^m \mid \ \lfloor \mathbf{i} \rfloor \ \geq d\ \} \tag{4.6}$$

where $\dim_{\mathbb{F}_p}$ denotes the dimension of the vector space over $\mathbb{F}_p$ and card means the number of elements in the set.

By taking account of the relation (3.1), we have the following definition.

Definition 4.5. For $\mathbf{i} := (i_0, \ldots, i_{m-1}) \in (\mathbb{Z}/p\mathbb{Z})^m$, we define the interpolation polynomial

$$H_{\mathbf{i}}(\mathbf{Y}) := H_{i_0}(Y_0) \cdot \ldots \cdot H_{i_{m-1}}(Y_{m-1}) \quad \in \mathbb{F}_p[Y_0, \ldots, Y_{m-1}].$$

Theorem 4.6. *For* $\mathbf{i} \in (\mathbb{Z}/p\mathbb{Z})^m$, *we have*

$$\deg(H_{\mathbf{i}}(\mathbf{Y})) = m(p-1) - \lfloor \mathbf{i} \rfloor. \tag{4.7}$$

Proof. It is obvious by (3.2). □

Theorem 4.7. *For* $\mathbf{i} \in (\mathbb{Z}/p\mathbb{Z})^m$, *we have*

$$J_{\mathbf{i}}(\mathbf{x}) = \sum_{\mathbf{j} \leq \mathbf{i}} H_{\mathbf{i}}(\mathbf{j})\mathbf{x}^{\mathbf{j}}.$$

Proof. By Theorem 3.5, we have

$$\begin{aligned} J_{\mathbf{i}}(\mathbf{x}) &= \prod_{t=0}^{m-1} (x_t - 1)^{i_t} \\ &= \prod_{t=0}^{m-1} (\sum_{j_t=0}^{i_t} (-1)^{i_t - j_t} \binom{i_t}{j_t} x_t^{j_t}) \\ &= \prod_{t=0}^{m-1} (\sum_{j_t=0}^{i_t} H_{i_t}(j_t) x_t^{j_t}) \\ &= \sum_{\mathbf{j} \leq \mathbf{i}} (\prod_{t=0}^{m-1} H_{i_t}(j_t))\mathbf{x}^{\mathbf{j}} \\ &= \sum_{\mathbf{j} \leq \mathbf{i}} H_{\mathbf{i}}(\mathbf{j})\mathbf{x}^{\mathbf{j}}. \end{aligned}$$

□

Remark 4.8. (i) If there is a t such that $j_t > i_t$, then $H_{\mathbf{i}}(\mathbf{j}) = 0$.

(ii) The polynomial $J_{\mathbf{i}}(\mathbf{x})$ can be identified with the vector $(H_{\mathbf{i}}(\mathbf{j}))_{\mathbf{j} \in (\mathbb{Z}/p\mathbb{Z})^m}$.

Recall that $\mathbf{Y} := (Y_0, ..., Y_{m-1})$. Consider the vector space of the reduced polynomials in m variables over $\mathbb{F}_p$

$$P(m,p) := \left\{P(\mathbf{Y}) \in \mathbb{F}_p[Y_0, ..., Y_{m-1}] \mid \deg_{Y_t}(P) \leq p-1, \text{ for all } t\right\}$$

where $\deg_{Y_t}(P)$ is the degree of the polynomial $P(\mathbf{Y})$ with respect to the variable Y_t.

Let ω be an integer such that $0 \leq \omega \leq m(p-1)$. Consider the subspace of $P(m,p)$ defined by

$$P_\omega(m,p) := \{P(\mathbf{Y}) \in P(m,p) \mid \deg(P) \leq \omega\}$$

where $\deg(P)$ is the total degree of the polynomial $P(\mathbf{Y})$.

We have the following isomorphism of vector spaces:

$$\begin{array}{cccc} \Psi : & P(m,p) & \longrightarrow & B \\ & P(\mathbf{Y}) & \longmapsto & \displaystyle\sum_{\mathbf{j}\in(\mathbb{Z}/p\mathbb{Z})^m} P(\mathbf{j})\mathbf{x}^{\mathbf{j}} \end{array} \tag{4.8}$$

Definition 4.9. The Reed-Muller code of length p^m over $\mathbb{F}_p$ and of order ω $(0 \leq \omega \leq m(p-1))$ is the subspace of $(\mathbb{F}_p)^{p^m}$ defined by

$$RM_{\mathbb{F}_p}(m,\omega) := \left\{ (P(\mathbf{j}))_{\mathbf{j}\in(\mathbb{Z}/p\mathbb{Z})^m} \in (\mathbb{F}_p)^{p^m} \mid P(\mathbf{Y}) \in P_\omega(m,p) \right\}. \tag{4.9}$$

Remark 4.10. (i) According to the isomorphisms (4.4) and (4.8), the Reed-Muller code $RM_{\mathbb{F}_p}(m,\omega)$ is isomorphic to $P_\omega(m,p)$.

(ii) We have

$$\dim_{\mathbb{F}_p}(RM_{\mathbb{F}_p}(m,\omega)) = \operatorname{card}\left\{ \prod_{t=0}^{m-1} Y_t^{e_t} \mid 0 \leq e_t \leq p-1, \sum_{t=0}^{m-1} e_t \leq \omega \right\} \tag{4.10}$$

We know give a new proof of the following theorem

Theorem 4.11 (Berman-Charpin)**.** *Let ω be an integer such that $0 \leq \omega \leq m(p-1)$. We have*

$$RM_{\mathbb{F}_p}(m,\omega) = R^{m(p-1)-\omega}$$

where R is defined in (4.5).

Proof. For simplicity, let $d = m(p-1) - \omega$. By (4.7), we have $\deg(H_{\mathbf{i}}(\mathbf{Y})) \leq \omega$, for $\lfloor \mathbf{i} \rfloor \geq d$. And it follows from Remark 4.4(i), Remark 4.8(ii) and (4.9) that

$$R^d \subseteq RM_{\mathbb{F}_p}(m,\omega).$$

It remains to show that $\dim_{\mathbb{F}_p}(RM_{\mathbb{F}_p}(m,\omega)) = \dim_{\mathbb{F}_p}(R^d)$.

By (4.10), we have

$$\dim_{\mathbb{F}_p}(RM_{\mathbb{F}_p}(m,\omega)) = \operatorname{card}\{\, \mathbf{i} \in (\mathbb{Z}/p\mathbb{Z})^m \mid \lfloor \mathbf{i} \rfloor \leq \omega \,\}$$

and by (4.6), we have

$$\dim_{\mathbb{F}_p}(R^d) = \operatorname{card}\{\, \mathbf{i} \in (\mathbb{Z}/p\mathbb{Z})^m \mid \lfloor \mathbf{i} \rfloor \geq d \,\}.$$

It is clear that the map

$$\begin{array}{cccc} \theta & : & (\mathbb{Z}/p\mathbb{Z})^m & \longrightarrow & (\mathbb{Z}/p\mathbb{Z})^m \\ & & \mathbf{i} = (i_0, \ldots, i_{m-1}) & \longmapsto & \theta(\,\mathbf{i}\,) = (p-1-i_0, \ldots, p-1-i_{m-1}) \end{array}$$

is a bijection with $\theta^{-1} = \theta$. And we obtain

$$\lfloor \theta(\,\mathbf{i}\,) \rfloor = \sum_{t=0}^{m-1} p - 1 - i_t = m(p-1) - \lfloor \mathbf{i} \rfloor.$$

It follows that $\lfloor \mathbf{i} \rfloor = m(p-1) - \lfloor \theta(\mathbf{i}) \rfloor$.

Thus, we have the following equivalence

$$\lfloor \mathbf{i} \rfloor \leq \omega \iff \lfloor \theta(\mathbf{i}) \rfloor \geq m(p-1) - \omega.$$

This implies that

$$\operatorname{card} \{ \mathbf{i} \in (\mathbb{Z}/p\mathbb{Z})^m \mid \lfloor \mathbf{i} \rfloor \leq \omega \} = \operatorname{card} \{ \mathbf{i} \in (\mathbb{Z}/p\mathbb{Z})^m \mid \lfloor \mathbf{i} \rfloor \geq m(p-1) - \omega \}. \quad \square$$

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References

[1] E. F. Assmus, Jr., On Berman's characterization of the Reed-Muller codes, *J. Statist. Plann. Inference* **56** (1996), 17–21, doi:10.1016/s0378-3758(96)00004-3.

[2] E. F. Assmus, Jr. and J. D. Key, Polynomial codes and finite geometries, in: *Handbook of Coding Theory, Vol. I, II*, North-Holland, Amsterdam, pp. 1269–1343, 1998, https://documents.pub/document/polynomial-codes-and-finite-geometries-keyjkeychapterappdf-polynomial-codes.html?page=1.

[3] T. Berger and P. Charpin, The automorphism group of generalized Reed-Muller codes, *Discrete Math.* **117** (1993), 1–17, doi:10.1016/0012-365x(93)90321-j.

[4] S. D. Berman, On the theory of group codes, *Cybernetics* **3** (1967), 25–31, doi:10.1007/bf01072842.

[5] I. F. Blake and R. C. Mullin, *The Mathematical Theory of Coding*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, doi:10.1016/c2013-0-10384-4.

[6] P. Charpin, Une generalisation de la construction de berman des codes de reed et muller p-aires, *Commun. Algebra* **16** (1988), 2231–2246, doi:10.1080/00927878808823689.

[7] P. Delsarte, J.-M. Goethals and F. J. MacWilliams, On generalized Reed-Muller codes and their relatives, *Information and Control* **16** (1970), 403–442, doi:10.1016/s0019-9958(70)90214-7.

[8] T. Kasami, S. Lin and W. Peterson, New generalizations of the Reed-Muller codes–I: Primitive codes, *IEEE Trans. Inf. Theory* **14** (1968), 189–199, doi:10.1109/tit.1968.1054127.

[9] E. Kouselo, S. Gonsales, V. T. Markov, K. Martines and A. A. Nechaev, Ideal representations of Reed-Solomon and Reed-Muller codes, *Algebra Logic* **51** (2012), 297–320, 414, 417, doi:10.1007/s10469-012-9183-8.

[10] P. Landrock and O. Manz, Classical codes as ideals in group algebras, *Des. Codes Cryptogr.* **2** (1992), 273–285, doi:10.1007/bf00141972.

[11] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error Correcting Codes*, North Holland, 1977, doi:10.1016/s0924-6509(08)x7030-8.

[12] A. Poli, Codes stables sous le groupe des automorphismes isométriques de $A = \mathbf{F}_p[X_1, \cdots, X_n]/(X_1^p - 1, \cdots, X_n^p - 1)$, *C. R. Acad. Sci. Paris Sér. A-B* **290** (1980), A1029–A1032.

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Deterministic bootstrap percolation on trees

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Abstract

In a graph, k-bootstrap percolation is a process by which an "infection" spreads from an initial set of infected vertices, according to the rule that on each iteration an uninfected vertex with k infected neighbors becomes infected. This process continues until either every vertex is infected or every uninfected vertex has fewer than k infected neighbors. We are particularly interested in the case where every vertex is eventually infected. The cardinality of a smallest set that results in this is the *k-bootstrap percolation number* of the graph. In this paper, we determine the k-bootstrap percolation number for trees of small diameter, spiders, complete N-ary trees, and caterpillars. For these graph families we also consider the smallest number of iterations needed for any smallest set to spread to the entire graph. Finally, we give an upper bound for the k-bootstrap percolation number for general trees which improves upon previous results.

Keywords: Bootstrap Percolation,Trees

Math. Subj. Class.: 05D99,05C05

1 Introduction

Let G be a graph with vertex set V and edge set E. We define the *diameter* of a graph to be the maximum distance between any pair of vertices in V and we define the *periphery* of a graph to be the subgraph induced by all vertices in V whose distance to some other vertex in V is equal to the diameter. Since we are primarily concerned with trees, we note that any vertex on the periphery of a tree is necessarily a leaf (in other words, a vertex of degree one).

We begin with $\mathcal{A}_k^0(G) \subseteq V(G)$, a collection of infected vertices. On the t^{th} iteration we add newly infected vertices to $\mathcal{A}_k^{t-1}(G)$ if they have at least k neighbors in $\mathcal{A}_k^{t-1}(G)$, to

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form $\mathcal{A}_k^t(G)$. This process is repeated until vertices not in $\mathcal{A}_k^t(G)$ have strictly fewer than k neighbors in $\mathcal{A}_k^t(G)$ or all vertices of G are in $\mathcal{A}_k^t(G)$. The above process is called k-*bootstrap percolation*. In particular, for a graph G we are interested in the size of a smallest $\mathcal{A}_k^0(G)$ so that the entire graph is eventually infected. Throughout the paper, we call such a set a *percolating set* and we call the size of this set, denoted $\mathrm{bp}_k(G)$, the k-*bootstrap percolation number* of the graph. Among all smallest percolating sets, there is one that infects all of the vertices of the graph in the minimum number of iterations. We denote this minimum value $t_k(G)$.

We would now like to give a brief (but far from complete) overview of the history of bootstrap percolation. In 1968, Bollabás considered an edge coloring of graphs [6] called "weak saturation," which later came to be called "graph bootstrap percolation" [2]. Bootstrap percolation on vertices was introduced by Chalupa, Leath, and Reich [8]. Their study was motivated by a problem in magnetic systems and considered only on a special class of lattices. In the paper by Chalupa et al. and in most subsequent papers on bootstrap percolation on lattices, the initial set of infected vertices, $\mathcal{A}_k^0(G)$, is chosen at random. Bootstrap percolation with $\mathcal{A}_k^0(G)$ being chosen at random has also been considered in [1, 3, 4, 5, 7, 18, 19, 20]. Alternatively, instead of choosing our initial set randomly, we choose $\mathcal{A}_k^0(G)$ in order to insure that every vertex in a graph is eventually infected. While this deterministic approach seems to be less common historically, it has been considered in [9, 10, 12, 13, 22, 23, 24, 25] and the appendix of [4]. We should also mention that in addition to the standard bootstrap percolation considered in this paper, there are also several variants. For example, two-way bootstrap percolation, which has been considered in [21, 27, 28], and the previously mentioned bootstrap percolation on edges, which has been recently considered in [14].

In this paper, we will be primarily concerned with bootstrap percolation on trees. One primary motivation for considering trees is that they are minimally connected graphs. With this in mind, trees are natural to consider in the context of the extremal values of the k-bootstrap percolation number due to the fact that every graph has a spanning subtree. We should mention that our paper can most naturally be considered an extension of the work of Riedl in [25]. In [25], Riedl finds upper and lower bounds on the k-bootstrap percolation number for all trees[1], and uses these bounds to find the precise k-bootstrap percolation number for certain N-ary trees. In this paper, we begin by obtaining exact values of the k-bootstrap percolation number for various commonly occuring families of trees. It should be noted that we reproduce Riedl's formula for the k-bootstrap percolation number of certain N-ary trees, though we use a different and more concrete method. Furthermore, our main result improves upon the upper bound (See Theorem 5.1) appearing in [25].

In Section 2, we make some basic observations which will be used throughout the remainder of the paper. In Section 3.1, we consider families of trees of small diameter. In Section 3.2, we consider families of spiders. In Section 3.3, we consider complete N-ary trees. This result is also given in [25], since in this case the upper and lower bounds are tight, so that the ceiling of the lower bound is equal to the floor of the upper bound. In Section 4, we consider caterpillars. In Section 5, we present a sharp upper bound for $\mathrm{bp}_k(T)$ and compare this bound to the bounds given in [25]. In Section 6, we use this bound to give $\mathrm{bp}_k(T)$ for the trees on eleven vertices or less that do not fall into one of our families. Finally, we give several open problems for future avenues of research in Section 7.

[1]We note that the main theorem in [25] is stated incorrectly on page 3. However, it is correctly stated in their abstract.

2 Basic observations

In this section we state several fundamental results which will be useful for the remainder of the paper.

We begin with four observations about percolating sets that hold for all graphs.

Observation 2.1. Let G be a graph. (i) All vertices of degree less than k must belong to any percolating set for G. (ii) In k-bootstrap percolation, if u and v are adjacent vertices such that $deg(u) = deg(v) = k$, then at least one of u and v must be in every minimum k-bootstrap set.

Proof. Let v be a vertex of G with degree less than k. If v is not in a percolating set, then v can never have at least k infected neighbors. So, v will never be infected.

Likewise, suppose that u and v are adjacent vertices of degree k. If u is not in a percolating set, then v can never have at least k infected neighbors. Reversing the roles of u and v yields the result. $\square$

Clearly, any percolating set for $k+1$ is also a percolating set for k. Ergo, the next proposition follows immediately from Observation 2.1.

Observation 2.2. If G is a connected graph with maximum degree Δ, then

$$1 = \mathrm{bp}_1(G) \le \mathrm{bp}_2(G) \le \cdots \le \mathrm{bp}_{\Delta+1}(G) = |V(G)|.$$

Observation 2.3. We have $\mathrm{bp}_1(G)$ is the number of connected components in G.

Based on Observation 2.3, we will only consider the case where $k \ge 2$ for the remainder of this paper.

To obtain an upper bound for the k-bootstrap percolation number we consider the k-domination number of G. The *neighborhood* of a vertex x, denoted $N(x)$, is the set of all vertices adjacent to x. If $|N(x)| = 1$, then x is a *leaf*. A *k-domination set* is a set $S \subseteq V(G)$ such that for all $x \in V(G)$, either $x \in S$ or $|N(x) \cap S| \ge k$. If among all k-domination sets, S has the least number of vertices, then S is a *minimum k-domination set*. The cardinality of such a set is the *k-domination number* of G. This number is denoted $\gamma_k(G)$. The k-domination number was introduced by Fink and Jacobson in 1985 [11]. For more information on domination and its variations, please refer to [16, 17]. If $\mathcal{A}_k^0(G)$ is equal to a k-domination set for G, then after a single iteration the entire graph will be infected. From this, the following bound is immediate.

Observation 2.4. For any graph G, $\mathrm{bp}_k(G) \le \gamma_k(G)$.

3 Graph families

In this section, we restrict our attention to certain families of trees for which we can derive the exact k-bootstrap percolation number.

3.1 Trees of small diameter

We give the k-bootstrap percolation number for all trees with diameter less than or equal to five.

We begin with diameter two. A tree of diameter two is a *star*. This graph has a center vertex x adjacent to n leaves, $y_1,\ldots,y_n$. This graph is denoted $K_{1,n}$.

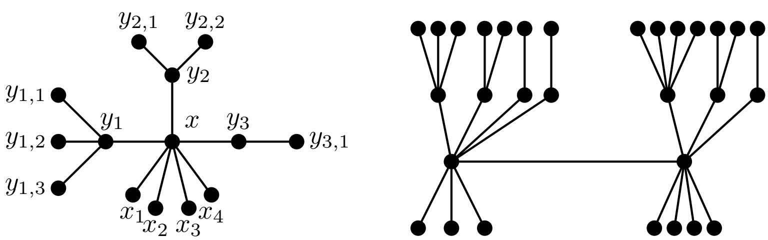

Figure 1: The graphs $K_{1,3}(4;3,2,1)$ and $S_{4,3}(3;3,2,1,1;4;4,2,1)$

Theorem 3.1. *Let $k \geq 2$. For the star $K_{1,n}$, we have the following:*

1. *If $n \leq k-1$, then $\mathrm{bp}_k(K_{1,n}) = n+1$ and $t_k(K_{1,n}) = 0$.*
2. *If $n \geq k$, then $\mathrm{bp}_k(K_{1,n}) = n$ and $t_k(K_{1,n}) = 1$.*

Proof. By Observation 2.1, all vertices in $\{y_1, \ldots, y_n\}$ must be in every percolating set. If $\deg(x) = n \leq k-1$, then x must also be in every percolating set and part 1) follows. If $n \geq k$, then x will get infected after one iteration and part 2) follows. $\square$

It follows from the previous theorem that the bound in Observation 2.4 is sharp. In particular, for the star $K_{1,k}$ we have that $\mathrm{bp}_k(K_{1,k}) = k = \gamma_k(K_{1,k})$.

A tree of diameter three is a *double star*. This graph has two adjacent central vertices x and y. The vertex x is adjacent to r leaves, $x_1,\ldots,x_r$. The vertex y is adjacent to s leaves, $y_1,\ldots,y_s$. This graph is denoted $S_{r,s}$.

Theorem 3.2. *Let $k \geq 2$ and $r \geq s \geq 1$. For the double star $T = S_{r,s}$, we have the following:*

1. *If $r \leq k-2$, then $\mathrm{bp}_k(T) = r+s+2$ and $t_k(T) = 0$.*
2. *If $r \geq k-1$ and $s \leq k-2$, then $\mathrm{bp}_k(T) = r+s+1$ and $t_k(T) = 1$.*
3. *If $r = s = k-1$, then $\mathrm{bp}_k(T) = r+s+1$ and $t_k(T) = 1$.*
4. *If $r \geq k$ and $s = k-1$, then $\mathrm{bp}_k(T) = r+s$ and $t_k(T) = 2$.*
5. *If $s \geq k$, then $\mathrm{bp}_k(T) = r+s$ and $t_k(T) = 1$.*

Proof. By Observation 2.1, all leaves must be initially infected, so $\mathrm{bp}_k(T) \geq r+s$. If $r \leq k-2$, then x and y must both be initially infected, i.e., $\mathrm{bp}_k(T) = r+s+2$ and $t_k(T) = 0$. If $r \geq k-1$ and $s \leq k-2$, then y must be initially infected and x is infected after one iteration, so $\mathrm{bp}_k(T) = r+s+1$ and $t_k(T) = 1$. If $r = s = k-1$ then either x or y must be initially infected, and the other is infected after one iteration, so $\mathrm{bp}_k(T) = r+s+1$ and $t_k(T) = 1$. If $r \geq k$ and $s = k-1$, then x is infected after one iteration and y is infected after two iterations, so $\mathrm{bp}_k(T) = r+s$ and $t_k(T) = 2$. If $s \geq k$, then both x and y are infected after one iteration, so $\mathrm{bp}_k(T) = r+s$ and $t_k(T) = 1$. $\square$

Any tree of diameter four can be obtained by appending leaves to the existing vertices of $K_{1,n}$, where $n \geq 2$. Suppose that we append c leaves to x, namely $x_1, \ldots, x_c$ and $a_i \geq 1$ leaves to y_i, namely $y_{i,1}, \ldots, y_{i,a_i}$ for $i = 1, \ldots, n$. The resulting graph will be denoted $K_{1,n}(c; a_1, \ldots, a_n)$. Without loss of generality, assume that $a_1 \geq \cdots \geq a_n \geq 1$. An example is shown in Figure 1.

There exist non-negative integers p and q such that the following holds:

- $a_i \geq k \geq 2$ if and only if $i \leq p$.
- $a_i = k - 1$ if and only if $p + 1 \leq i \leq n - q$.
- $a_i \leq k - 2$ if and only if $i \geq n - q + 1$.

Thus p is the number of y_i with at least k leaves and q is the number of y_i with at most $k - 2$ leaves.

Theorem 3.3. *Let $k \geq 2$. For $T = K_{1,n}(c; a_1, \ldots, a_n)$, we have the following:*

(1) *If $p + q + c \geq k$, then $\mathrm{bp}_k(T) = c + a_1 + \cdots + a_n + q$ and*

$$t_k(T) = \begin{cases} 1 & \textit{if } q + c \geq k \textit{ and } n = p + q \\ 2 & \textit{if } q + c \geq k \textit{ and } n \geq p + q + 1 \\ 2 & \textit{if } q + c \leq k - 1 \textit{ and } n = p + q \\ 3 & \textit{if } q + c \leq k - 1 \textit{ and } n \geq p + q + 1. \end{cases}$$

(2) *If $p + q + c \leq k - 1$, then $\mathrm{bp}_k(T) = c + a_1 + \cdots + a_n + q + 1$ and*

$$t_k(T) = \begin{cases} 0 & \textit{if } n = q \\ 1 & \textit{if } p \geq 1 \textit{ or } n \geq p + q + 1. \end{cases}$$

Proof. By Observation 2.1, for all i, j, and ℓ, $y_{i,j}$ and x_ℓ must be in every percolating set. Further note that $\deg(y_i) = a_i + 1$. Thus, y_{n-q+1},..., y_n must also be in the initial set. It follows that $\mathrm{bp}_k(T) \geq c + a_1 + \cdots + a_n + q$. Likewise, $y_1,\ldots,y_p$ will be infected after one iteration since they have at least k neighbors in the initial set. Similarly, $y_{p+1},\ldots,y_{n-q}$ will be infected in the iteration after x is infected.

Suppose that $p + q + c \geq k$. If $q + c \geq k$ and $n = p + q$, then x will be infected in one iteration. Since, $n = p + q$, then $\{y_{p+1}, \ldots, y_{n-q}\} = \emptyset$. Thus, the entire graph is infected and $t_k(T) = 1$.

If $q + c \geq k$ and $n \geq p + q + 1$, then x is infected in one iteration and $y_{p+1},\ldots,y_{n-q}$ are infected in two iterations. Hence, $t_k(T) = 2$.

If $q + c \leq k - 1$ and $n = p + q$, then $\{y_{p+1}, \ldots, y_{n-q}\} = \emptyset$. Since $p + q + c \geq k$ but $q + c \leq k - 1$, x gets infected one iteration after $y_1,\ldots,y_p$ are infected. Thus every vertex is infected after two steps and $t_k(T) = 2$.

Similarly, if $q + c \leq k - 1$ and $n \geq p + q + 1$, then $\{y_{p+1}, \ldots, y_{n-q}\} \neq \emptyset$. Since $p + q + c \geq k$ but $q + c \leq k - 1$, x is infected one iteration after $y_1,\ldots,y_p$ are infected. The vertices $y_{p+1},\ldots,y_{n-q}$ are infected one iteration later. Thus every vertex is infected after three steps and $t_k(T) = 3$. This proves part (1).

Now, suppose that $p + q + c \leq k - 1$. As before, y_1,...,y_p, y_{n-q+1},..., y_n, and x_1,.., x_c are either in the initial set, or (in the case of y_1,...,y_p) infected after one step. Hence, x has at most $k-1$ neighbors that will eventually be infected. Thus x must be in the initial set. It follows $\text{bp}_k(T) \geq c + a_1 + \cdots + a_n + q + 1$. Thus if $n = q$, then $p = 0$ and every vertex must be in the initial set. Hence $t_k(T) = 0$.

If $p \geq 1$ or $n \geq p + q + 1$, then $y_1, \ldots, y_{n-q}$ are infected after one iteration. Thus $t_k(T) = 1$. This proves part (2). □

Any tree of diameter five can be obtained by appending leaves to the existing vertices of the double star. We append c_1 leaves to x, namely $w_1, ..., w_{c_1}$. We append c_2 leaves to y, namely $z_1, ..., z_{c_2}$. Similarly, we append a_i leaves to x_i, namely $x_{i,1}, ..., x_{i,a_i}$, and b_j leaves to y_j, namely $y_{j,1}, ..., y_{j,b_j}$. A diameter five tree with these parameters is denoted $S_{r,s}(c_1; a_1, ..., a_r; c_2; b_1, ..., b_s)$ (see Figure 1). Without loss of generality, assume that $a_1 \geq ... \geq a_r \geq 1$ and $b_1 \geq .... \geq b_s \geq 1$.

For convenience of notation, define $X_i = \{x_{i,1}, ..., x_{i,a_i}\}$ and $Y_j = \{y_{j,1}, ..., y_{j,b_j}\}$ for $i = 1, ..., r$ and $j = 1, ..., s$. Note that the set of leaves is

$$L = \{w_1, ..., w_{c_1}, z_1, ..., z_{c_2}\} \cup X_1 \cup \cdots \cup X_r \cup Y_1 \cup \cdots \cup Y_s.$$

and that

$$|L| = c_1 + c_2 + \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j.$$

Given $k \geq 2$, there exist non-negative integers p_1, p_2, q_1, q_2 such that the following holds:

- $a_i \geq k$ if and only if $i \leq p_1$.
- $b_j \geq k$ if and only if $j \leq p_2$.
- $a_i \leq k-2$ if and only if $i \geq r - q_1 + 1$.
- $b_j \leq k-2$ if and only if $j \geq s - q_2 + 1$.

Because our result follows in a very similar manner to the proof of Theorem 3.3, we omit the details of the proof and only provide the initial sets. In each case, it is straightforward to verify that the set in question is a minimum percolating set. While we have omitted the time parameter, this can easily be calculated from these sets.

Theorem 3.4. *For a given k, the k-bootstrap percolation number of $T = S_{r,s}(c_1; a_1, ..., a_r; c_2; b_1, ..., b_s)$ is as follows:*

(i) *If $p_1 + q_1 + c_1 \leq k-2$ and $p_2 + q_2 + c_2 \leq k-2$, then $\text{bp}_k(T) = |L| + q_1 + q_2 + 2$.*

(ii) *If $p_1 + q_1 + c_1 = p_2 + q_2 + c_2 = k-1$ or at most one of $p_1 + q_1 + c_1$ or $p_2 + q_2 + c_2$ is less than or equal to $k-2$, then $\text{bp}_k(T) = |L| + q_1 + q_2 + 1$.*

(iii) *If $p_1 + q_1 + c_1 \geq k-1$ and $p_2 + q_2 + c_2 \geq k-1$, with at most one of $p_1 + q_1 + c_1$ or $p_2 + q_2 + c_2$ equaling $k-1$, then $\text{bp}_k(T) = |L| + q_1 + q_2$.*

Proof. (i) Take the set $L \cup \{x_{r-q_1+1}, ..., x_r\} \cup \{y_{s-q_2+1}, ..., y_s\} \cup \{x, y\}$.

(ii) If $p_1 + q_1 + c_1 \leq k-2$, then take the set

$$L \cup \{x_{r-q_1+1}, ..., x_r\} \cup \{y_{s-q_2+1}, ..., y_s\} \cup \{x\}.$$

If $p_2 + q_2 + c_2 \leq k - 2$, then take the set

$$L \cup \{x_{r-q_1+1}, ..., x_r\} \cup \{y_{s-q_2+1}, ..., y_s\} \cup \{y\}.$$

If $p_1 + q_1 + c_1 = p_2 + q_2 + c_2 = k - 1$, then we can take

$$L \cup \{x_{r-q_1+1}, ..., x_r\} \cup \{y_{s-q_2+1}, ..., y_s\}$$

along with either x or y. While either x or y may be chosen, we give the following procedure for choosing x and y which minimizes the number of iterations needed to completely infect the graph:

Suppose that $r - p_1 - q_1 = 0$. Except for y, every neighbor of x is either in the initial set or (in the case of x_1,...x_{p_1}) infected after one step. Thus, by including y in the initial set, we guarantee that every vertex is infected after one step (if $p_1 = 0$) or two steps (if $p_1 \geq 1$). As either x or y will not be in the initial set, this gives us the minimum number of iterations. Using an analogous argument, if $s - p_2 - q_2 = 0$, then we include x in the initial set.

Suppose that $r - p_1 - q_1 \geq 1$ and $s - p_2 - q_2 \geq 1$. Note that this means that $\{x_{p_1+1}, ..., x_{r-q_1}\}$ and $\{y_{p_2+1}, ..., y_{s-q_2}\}$ are non-empty sets. These sets are infected one step after their corresponding center vertex. If $p_1 = 0$, then choosing y guarantees that x is infected on the first step and every vertex is infected in two. Note that choosing x in the case where $r - p_1 - q_1 \geq 1$, $s - p_2 - q_2 \geq 1$, $p_1 = 0$, and $p_2 \geq 1$ will result in $\{y_{p_2+1}, ..., y_{s-q_2}\}$ becoming infected after three iterations. Using an analogous argument, if $r - p_1 - q_1 \geq 1$, $s - p_2 - q_2 \geq 1$, and $p_2 = 0$, then we choose x for our initial set.

Suppose that $r - p_1 - q_1 \geq 1$, $s - p_2 - q_2 \geq 1$, $p_1 \geq 1$, and $p_2 \geq 1$. By choosing x to be in our initial set, x_1,...,x_{r-q_1}, y_1,...,y_{p_2} are infected after one step, y is infected on the second iteration, and y_{p_2+1},...,y_{s-q_2} are infected in three steps. By reversing the roles of x and y, we see that we do no better by choosing y to be in the initial set.

(iii) Choose $L \cup \{x_{r-q_1+1}, ..., x_r\} \cup \{y_{s-q_2+1}, ..., y_s\}$ as our initial set. □

3.2 Spiders

In this section, we consider bootstrap percolation on spiders, which are also commonly referred to as asters or starlike trees. Let $x_1, \ldots, x_e, y_1, \ldots, y_o$ be positive integers with x_i even for $1 \leq i \leq e$ and y_j odd for $1 \leq j \leq o$. We construct a *spider*, denoted by $S = S(x_1, \ldots, x_e, y_1, \ldots, y_o)$, as follows. First, S has a single vertex of degree larger than 2, which we denote by c. We then add an edge from c to a single leaf from each of the paths P_{x_i}, P_{y_j} for $1 \leq i \leq e$, $1 \leq j \leq o$. Note that for $k \geq 3$, the k-bootstrap percolation number is straightforward to determine using Observation 2.1. However, we include it for completeness.

Proposition 3.5. *Suppose $k \geq 3$ and let S be as above. Then,*

$$\mathrm{bp}_k(S) = \begin{cases} \sum_{i=1}^{e} x_i + \sum_{j=1}^{o} y_j + 1 & \textit{if } e + o \leq k - 1 \\ \sum_{i=1}^{e} x_i + \sum_{j=1}^{o} y_j & \textit{if } e + o \geq k, \end{cases}$$

and

$$t_k(S) = \begin{cases} 0 & \textit{if } e + o \leq k - 1 \\ 1 & \textit{if } e + o \geq k. \end{cases}$$

We now proceed to determine the 2-bootstrap percolation number for S, which is somewhat more involved than the previous result.

Theorem 3.6. *Let $S = S(x_1, \ldots, x_e, y_1, \ldots, y_o)$. Then,*

$$\mathrm{bp}_2(S) = \begin{cases} \sum_{i=1}^{e} \frac{x_i}{2} + 1 & o = 0 \\ \sum_{i=1}^{e} \frac{x_i}{2} + \frac{y_1+1}{2} + 1 & o = 1 \\ \sum_{i=1}^{e} \frac{x_i}{2} + \sum_{j=1}^{o} \frac{y_j+1}{2} & o \geq 2, \end{cases}$$

and

$$t_2(S) = \begin{cases} 2 & o \geq 2, e \geq 1 \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. First, we must initially infect all leaves of S. Second, for each path attached to the center vertex, we initially infect every other vertex starting from the leaf. Note that this will completely infect each odd length path after one step.

Case 1: $o \geq 2$.

In this case, the initially infected vertices given above also infect the center vertex c after one step. Note, if $e = 0$, then we are finished.

Now, suppose that $e \geq 1$. Then, after one step each even length path will have one infected end point and the other end point will be attached to the infected center vertex. Thus, the entire graph is infected after two steps. Therefore, $\sum_{i=1}^{e} \frac{x_i}{2} + \sum_{j=1}^{o} \frac{y_j+1}{2}$ vertices are initially infected. If we initially infect fewer than this number vertices, then there will be either a leaf or a vertex of degree two which is never infected. This yields the result in this case.

Case 2: $o \leq 1$.

In this case, the initially infected vertices will not infect the center vertex c. Thus, we must initially infect one additional vertex, and we see that initially infecting c will ensure that all vertices of the even length paths are infected after one step. Then $\sum_{i=1}^{e} \frac{x_i}{2} + \sum_{j=1}^{o} \frac{y_j+1}{2} + 1$ are initially infected and this is the smallest possible percolating set. This yields the result in this case. □

3.3 N-ary trees

In this section we consider N-ary trees and give a formula for the k-bootstrap percolation number of a complete N-ary tree.

For $N \geq 1$, we say that a tree T is an *N-ary tree* of height h if T is a rooted tree in which each vertex has no more than N children and no child can be further than distance h from the root. Note that when $N = 1$, a N-ary tree of height h is simply a path on $h + 1$ vertices. For example, the path P_2 is a 1-ary tree of height $h = 1$. With this in mind we begin with the following result.

Theorem 3.7. *For a path on n vertices, P_n, we have*

$$\mathrm{bp}_k(P_n) = \begin{cases} \lceil \frac{n+1}{2} \rceil & \textit{if } k = 2 \\ n & \textit{if } k \geq 3, \end{cases}$$

and

$$t_k(P_n) = \begin{cases} 1 & \textit{if } k = 2 \textit{ and } n \geq 3 \\ 0 & \textit{if } k \geq 3 \textit{ or } n = 1 \textit{ or } n = 2. \end{cases}$$

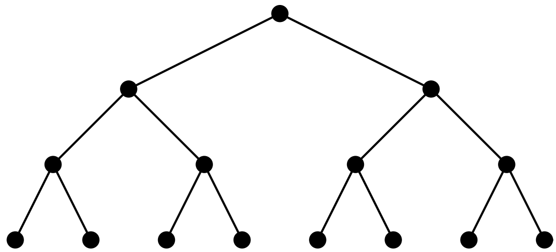

Figure 2: The complete binary tree, $T_{2,3}$

Proof. First, suppose that $k \geq 3$. Then, the degree of each vertex of P_n is less than k. By Observation 2.1, a percolating set for P_n contains every vertex of P_n. So $\mathrm{bp}_k(P_n) = n$.

Second, suppose that $k = 2$. Note, if $\mathcal{A}_2^0(P_n) \subseteq V(P_n)$ is of size $\lceil \frac{n+1}{2} \rceil - 1$, then there must be a vertex in $V(P_n) \backslash \mathcal{A}_2^0(P_n)$ which does not have two neighbors in $\mathcal{A}_2^0(P_n)$. Such a vertex would never be infected. Thus, a percolating set has cardinality at least $\lceil \frac{n+1}{2} \rceil$.

Now, label the vertices of P_n by $\{v_1, \ldots, v_n\}$. Define the following subset of $V(P_n)$,

$$\mathcal{A}_2^0(P_n) = \begin{cases} \{v_1, v_3, \ldots, v_{n-3}, v_{n-1}, v_n\} & \text{if } n \equiv 0 \pmod 2 \\ \{v_1, v_3, \ldots, v_{n-2}, v_n\} & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Note, $|\mathcal{A}_2^0(P_n)| = \lceil \frac{n+1}{2} \rceil$. Furthermore, it is clear that after a single iteration, every vertex of P_n will be infected. Thus, $\mathrm{bp}_k(P_n) = \lceil \frac{n+1}{2} \rceil$ and $t_k(P_n) = 1$ if $n \geq 3$. $\square$

We now consider the case where $N > 1$. An N-ary tree is *complete* if every vertex has either 0 or N children and all leaves are distance h from the root. Note, for fixed N and h, there is one complete N-ary tree of height h. We denote this graph $T_{N,h}$. A complete 2-ary tree (also called a binary tree) of height three is given in Figure 2.

Furthermore, we have that the number of vertices of $T_{N,h}$ is

$$|V(T_{N,h})| = \sum_{i=0}^{h} N^i = \frac{N^{h+1} - 1}{N - 1},$$

and the number of leaves is N^h. For convenience, we will denote the root vertex of $T_{N,h}$ by v_0, the set of children of v_0 by S_1, and so on until we have that set of all leaves of $T_{N,h}$ is denoted S_h. We now present the main result of this section.

Theorem 3.8. *Let $k, N \geq 2$. Then,*

$$\mathrm{bp}_k(T_{N,h}) = \begin{cases} N^h & \textit{if } k \leq N \\ \frac{N^{h+2}-1}{N^2-1} & \textit{if } k = N+1, h \equiv 0 \pmod 2 \\ \frac{N^{h+2}-1}{N^2-1} + \frac{N}{N+1} & \textit{if } k = N+1, h \equiv 1 \pmod 2 \\ \frac{N^{h+1}-1}{N-1} & \textit{if } k \geq N+2, \end{cases}$$

and

$$t_k(T_{N,h}) = \begin{cases} h & \textit{if } k \leq N \\ 1 & \textit{if } k = N+1 \\ 0 & \textit{if } k \geq N+2. \end{cases}$$

Proof. First, suppose that $k \leq N$. By Observation 2.1, each leaf must be in $\mathcal{A}_k^0(T_{N,h})$, i.e., $S_h \subseteq \mathcal{A}_k^0(T_{N,h})$, so $\mathrm{bp}_k(T_{N,h}) \geq N^h$. Furthermore, since each non-leaf has N children, we see that after one iteration all of the vertices in S_{h-1} will be infected, after a second iteration all of the vertices in S_{h-2} will be infected, and the process repeats h times until the entire tree is infected. Hence, we have only the leaves in $\mathcal{A}_k^0(T_{N,h})$, so $\mathrm{bp}_k(T_{N,h}) = N^h$ and $t_k(T_{N,h}) = h$.

Second, suppose that $k \geq N+2$. Then, every vertex of $T_{N,h}$ has degree strictly less than k, hence every vertex of $T_{N,h}$ must be in $\mathcal{A}_k^0(T_{N,h})$. Since $|V(T_{N,h})| = \frac{N^{h+1}-1}{N-1}$, the result follows.

Finally, suppose that $k = N+1$. Since every vertex of degree strictly less than k must be in $\mathcal{A}_k^0(T_{N,h})$, we have $S_h \cup \{v_0\} \subseteq \mathcal{A}_k^0(T_{N,h})$. We begin by proving that

$$\mathrm{bp}_k(T_{N,h}) \leq \begin{cases} \frac{N^{h+2}-1}{N^2-1} & \text{if } h \equiv 0 \pmod 2 \\ \frac{N^{h+2}-1}{N^2-1} + \frac{N}{N+1} & \text{if } h \equiv 1 \pmod 2. \end{cases}$$

Suppose that h is even. Let $\mathcal{A}_k^0(T_{N,h}) = S_h \cup S_{h-2} \cup \cdots \cup S_2 \cup \{v_0\}$. Then, after a single iteration we have that every vertex in $T_{N,h}$ is infected. Thus,

$$\mathrm{bp}_k(T_{N,h}) \leq |\mathcal{A}_k^0(T_{N,h})| = \sum_{i=0}^{\frac{h}{2}} |S_{2i}| = \sum_{i=0}^{\frac{h}{2}} N^{2i} = \frac{N^{h+2}-1}{N^2-1}.$$

Suppose that h is odd. Let $\mathcal{A}_k^0(T_{N,h}) = S_h \cup S_{h-2} \cup \cdots \cup S_3 \cup S_1 \cup \{v_0\}$. Then, after a single iteration we have that every vertex in $T_{N,h}$ is infected. Thus,

$$\mathrm{bp}_k(T_{N,h}) \leq 1 + \sum_{i=0}^{\frac{h-1}{2}} |S_{2i+1}| = 1 + \sum_{i=0}^{\frac{h-1}{2}} N^{2i+1} = \frac{N^{h+2}-1}{N^2-1} + \frac{N}{N+1}.$$

We want to show that the set $\mathcal{A}_k^0(T_{N,h})$ above has the smallest possible size. Every edge can be used at most once to infect a neighboring vertex, and at least $N+1$ edges must be used to infect one vertex. The number of edges in $T(N,h)$ is $\frac{N^{h+1}-1}{N-1} - 1 = \frac{N^{h+1}-N}{N-1}$, therefore at most $\lfloor \frac{N^{h+1}-N}{N-1}\frac{1}{N+1} \rfloor = \lfloor \frac{N^{h+1}-N}{N^2-1} \rfloor$ new vertices can be infected. Therefore the cardinality of the percolating set must be at least $\frac{N^{h+1}-1}{N-1} - \lfloor \frac{N^{h+1}-N}{N^2-1} \rfloor$. If h is even, then $\frac{N^{h+1}-N}{N^2-1} = N\sum_{i=0}^{\frac{h}{2}-1} N^{2i}$ is an integer, and this lower bound on the size of the k-bootstrap set equals the upper bound above. If h is odd, then $\frac{N^{h+1}-N}{N^2-1} = N\sum_{i=0}^{\frac{h-3}{2}} N^{2i+1} + \frac{N}{N+1}$ is not an integer. In this case, taking the floor reduces the total by $\frac{N}{N+1}$, again giving a lower bound which equals the upper bound above.

Therefore, the given example of a percolating set is minimum. □

Recall that in Observation 2.4, we showed that $\mathrm{bp}_k(G) \leq \gamma_k(G)$. To see that $\gamma_k(G) - \mathrm{bp}_k(G)$ can be made arbitrarily large, consider $T_{k,h}$, where h is sufficiently large. As

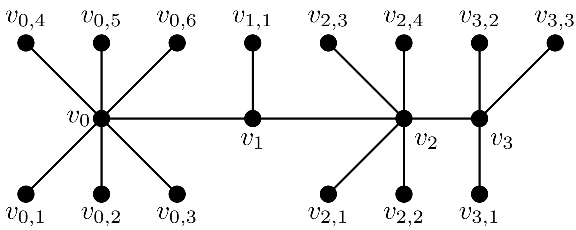

Figure 3: The caterpillar $P_4(6, 1, 4, 3)$

shown in Theorem 3.8, $\mathrm{bp}_k(T_{k,h}) = k^h$. However,

$$\gamma_k(T_{k,h}) = \sum_{i=0}^{\lfloor h/2 \rfloor} k^{h-2i} = \begin{cases} \frac{k^{h+2}-1}{k^2-1} & h \equiv 0 \pmod 2 \\ \frac{k^{h+2}-1}{k^2-1} + \frac{k}{k+1} - 1 & h \equiv 1 \pmod 2. \end{cases}$$

To see this, note that a k-domination set of minimum size must contain the leaves of the tree and every vertex that is of even distance from its closest leaf. Hence as h increases, this difference becomes arbitrarily large. It is interesting to note that $\gamma_k(T_{k,h}) = \mathrm{bp}_{k+1}(T_{k,h})$ when h is even and $\gamma_k(T_{k,h}) + 1 = \mathrm{bp}_{k+1}(T_{k,h})$ when h is odd. This shows that the difference $\mathrm{bp}_{k+1}(G) - \mathrm{bp}_k(G)$ in Observation 2.2 can be made arbitrarily large.

4 Caterpillars

In this section, we give a closed formula for the k-bootstrap percolation number of a caterpillar. A caterpillar is obtained from the path on r vertices by appending leaves to the existing vertices of the path. The vertices of the original path, which are called the *spine* of the caterpillar, are labeled v_1,...,v_r in the natural way, and we call r the *spine length*. We append x_i leaves to v_i for $1 \leq i \leq r$. The caterpillar with parameters r, x_1,...,x_r will be denoted $P_r(x_1, \ldots, x_r)$ (see Figure 3). Without loss of generality, we will assume that for $i \in \{1, r\}$, $x_i \geq 1$.

For the caterpillar $C = P_r(x_1, \ldots, x_r)$, our initial percolating set must contain every vertex of degree less than k by Observation 2.1. Thus for $k \geq 2$, this set must contain every leaf. Further, it must contain all v_i such that $x_i \leq k - 3$ for $1 \leq i \leq r$. Likewise, if $x_1 \leq k - 2$, then v_1 is in the set. Similarly, if $x_r \leq k - 2$, then v_r is in the set. Note that if $x_i \geq k$, then v_i will not be included in our percolating set as these vertices will be infected after one step.

The above discussion tells us nothing about the following vertices:

- v_1 if $x_1 = k - 1$.
- v_r if $x_r = k - 1$.
- v_i if $x_i \in \{k - 2, k - 1\}$ and $2 \leq i \leq r - 1$.

We call these vertices *sensitive*. We partition the sensitive vertices into two sets, S_1 and S_2, as follows. We let S_1 consist of all v_i satisfying $x_i = k - 1$ and $1 \leq i \leq r$. We let S_2 consist of all v_i satisfying $v_i = k - 2$ and $2 \leq i \leq r - 1$.

Consider the subgraph induced by $S_1 \cup S_2$. Label the connected components of this subgraph $L_1,\ldots,L_m$. We call these connected components *sensitive strings*. By definition, two sensitive strings are separated by at least one vertex whose inclusion in the initial set is decided according to the above discussion. For this reason, we may consider each sensitive string individually. Our goal for each sensitive string is to determine the minimum number of vertices to include in our initial set so that the entire string is eventually infected. We denote this number $w(L_i)$ for $1 \leq i \leq m$.

Lemma 4.1. *Let $k \geq 2$, let $C = P_r(x_1,\ldots,x_r)$ be a caterpillar, $L_1,\ldots,L_m$ be the sensitive strings in C, and $w(L_i)$ for $1 \leq i \leq m$ be as above. Then, for $1 \leq i \leq m$ we have*

1. *If $v_1 \notin V(L_i)$ and $v_r \notin V(L_i)$, then*
$$w(L_i) = \left\lfloor \frac{|V(L_i) \cap S_2|}{2} \right\rfloor.$$

2. *If $v_1 \in V(L_i)$ or $v_r \in V(L_i)$ but $\{v_1, v_r\} \not\subseteq V(L_i)$, then*
$$w(L_i) = \left\lfloor \frac{1 + |V(L_i) \cap S_2|}{2} \right\rfloor.$$

3. *If $v_1 \in V(L_i)$ and $v_r \in V(L_i)$, then*
$$w(L_i) = \left\lfloor \frac{2 + |V(L_i) \cap S_2|}{2} \right\rfloor.$$

Proof. Let $1 \leq i \leq m$ be fixed and let $S = V(L_i) \cap S_2 = \{s_1, ..., s_t\}$ be a sequence. To prove part 1), we choose for $A_k^0(C)$ every other s_j beginning with s_2. It is necessary to initially infect every other vertex in S because if two vertices in S are not initially infected, they must have an infected vertex between them by Observation 2.1. We choose to begin with s_2 because S is flanked by vertices that are either initially infected or will eventually become infected. Now, consider a connected component of the subgraph induced by $V(L_i) \cap S_1$. In Case 1, where L_i contains no endpoint of the spine, if this component of S_1 lies to the left of s_1 or to the right of s_t it will be eventually infected by the vertex to its left (right). If, on the other hand, it lies between two connected components of S then either the s_j to its left or the s_j to its right will have an even subscript and eventually infect all of its vertices[2]. This establishes part 1).

To prove part 2), we assume without loss of generality that $v_1 \in V(L_i)$ and $v_r \notin V(L_i)$. Note that L_i is adjacent on the right to a vertex w which is either in our initial set or will be infected eventually. Hence, our result will follow in a similar manner to the proof of part 1). However, the appropriate set of vertices to include from S is now

$$S' = \begin{cases} \{s_1, s_3, \ldots, s_{t-1}\} & \text{if } t \equiv 0 \pmod 2 \\ \{s_1, s_2, s_4 \ldots, s_{t-1}\} & \text{if } t \equiv 1 \pmod 2. \end{cases}$$

[2]If the size of S is even, then we could just as well have initially infected the s_j with odd subscripts. If the size of S is odd, then initially infecting the s_i with even subscripts is necessary for the number of initially infected vertices to be minimum.

Note that this gives us the desired result of

$$w(L_i) = \left\lfloor \frac{1 + |V(L_i) \cap S_2|}{2} \right\rfloor.$$

As for part 3), note that if $v_1 \in V(L_i)$ and $v_r \in V(L_i)$, then the entire spine is a sensitive string. Hence, our result will follow in a similar manner to the proof of part 1). However, the appropriate set of vertices to include from $V(L_i) \cap S_2$ is now

$$S' = \begin{cases} \{s_1, s_3, \ldots, s_{t-1}, s_t\} & \text{if } t \equiv 0 \pmod 2 \\ \{s_1, s_3, \ldots, s_t\} & \text{if } t \equiv 1 \pmod 2. \end{cases}$$

Note that this gives us the desired result of

$$w(L_i) = \left\lfloor \frac{2 + |V(L_i) \cap S_2|}{2} \right\rfloor.$$

The final remaining case is when the entire spine of the caterpillar is in S_1, so that $m = 1$ and S is empty. In this case, we choose the middle vertex (or one of the two middle vertices) in S_1 to include in the initially infected set. The formula in part 3) gives the correct weight of the spine as 1. □

Combining Observation 2.1 and Lemma 4.1, we obtain the main result of this section. Note that a caterpillar with spine length one is a star. The k-bootstrap percolation number of such a caterpillar was given in Theorem 3.1. For this reason, we assume that $r \geq 2$. For convenience of exposition, we let $d_{\leq \ell}(C)$ denote the number of vertices in C of degree less than or equal to ℓ.

Theorem 4.2. *Let $k \geq 2$ and let $C = P_r(x_1, \ldots, x_r)$ with $r \geq 2$. Let $L_1, \ldots, L_m$ be the sensitive strings in C. For each L_i we let $w(L_i)$ be as above. The k-bootstrap percolation number of the caterpillar is*

$$\mathrm{bp}_k(C) = \sum_{i=1}^{m} w(L_i) + d_{\leq k-1}(C).$$

Proof. This is a straightforward combination of Observation 2.1 and Lemma 4.1. □

Note that a double star is a caterpillar with a spine of length two, so this gives an alternate proof of Theorem 3.2. Moreover, we can use the above result to show that the bound from Observation 2.4 is sharp. Consider the caterpillar $P_n(t, \ldots, t)$, where $k \geq 4$ and $t \leq k - 3$. Every vertex has degree less than k. Hence, every vertex must be in a k-domination set and in a percolating set. We also mention that we omit the time parameter in this setting due to the length and tedium of the required calculation as well as the complexity of the resulting formula.

5 An upper bound

In this section we present a sharp upper bound for the k-bootstrap percolation number of a tree. We then compare this bound to other known bounds.

Before stating the theorem, we define the following notation. Recall that $d_{\leq k}(T)$ is the number of vertices in T of degree less than or equal to k. Similarly, we let $d_k(T)$ be the number of degree k vertices in T, and we let $d_{\geq k}(T)$ be the number of vertices in T of degree greater than or equal to k. Furthermore, for a vertex $s \in V(T)$, we set $\ell(s)$ to be the number of leaves adjacent to s.

Theorem 5.1. *Let T be a tree and $k \geq 2$. Then,* $\mathrm{bp}_k(T) \leq d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$.

Proof. We proceed by induction on n, the number of vertices of the tree T.

Up to isomorphism, there is only one tree on two vertices and one tree on three vertices. The result is easily verified in both cases.

Suppose for induction that the result is true for all trees with at most n vertices.

Let T be a tree with $n+1$ vertices and choose a leaf $v \in V(T)$ on the periphery of T with unique neighbor s. As shown in Theorem 3.1, this result holds for stars. For this reason, we will assume that T is not a star. Note, since v is on the periphery and T is not a star, we have that $\deg(s) = \ell(s) + 1$. We now consider several cases.

Case 1: $\ell(s) \leq k-2$ or $\ell(s) \geq k+1$.

In this case, we remove the leaf v from T and denote the resulting tree T'. Note, $d_{\leq k-1}(T') = d_{\leq k-1}(T) - 1$ and $d_k(T') = d_k(T)$. By the induction hypothesis, T' has a percolating set, denoted S, of cardinality at most

$$d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor = d_{\leq k-1}(T) - 1 + \left\lfloor \frac{d_k(T)}{2} \right\rfloor.$$

Thus, $S \cup \{v\}$ eventually infects all of T and has cardinality at most $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$.

Case 2: $\ell(s) = k-1$ and $d_k(T)$ is even.

In this case, we remove the leaf v from T and denote the resulting tree T'. Note, $d_{\leq k-1}(T') = d_{\leq k-1}(T)$ and $d_k(T') = d_k(T) - 1$ since the degree of s in T is k and has been decreased by one in T'. By the induction hypothesis, T' has a percolating set, denoted S, of cardinality at most

$$\begin{aligned} d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor &= d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T) - 1}{2} \right\rfloor \\ &= d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor - 1, \end{aligned}$$

where we have used that $d_k(T)$ is even in the second equality. Thus, $S \cup \{v\}$ eventually infects all of T and has cardinality at most $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$.

Case 3: $\ell(s) = k-1$ and $d_k(T)$ is odd.

First, we label the leaves of s by $L = \{v = v_1, v_2, \ldots, v_{k-1}\}$. Furthermore, let $t \in V(T)$ be a non-leaf with $st \in E(T)$, which is possible since T is not a star. We now remove s and its $k-1$ adjacent leaves from T to obtain a tree T'.

If $\deg(t) \leq k-1$ or $\deg(t) \geq k+2$ in T, then $d_{\leq k-1}(T') = d_{\leq k-1}(T) - (k-1)$ and $d_k(T') = d_k(T) - 1$. This follows because we have removed $k-1$ leaves, a vertex of degree k, and decreased the degree of t by one. Then, T' has a percolating set, denoted S, of cardinality at most

$$d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor = d_{\leq k-1}(T) - (k-1) + \left\lfloor \frac{d_k(T)-1}{2} \right\rfloor$$
$$= d_{\leq k-1}(T) - (k-1) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor,$$

where we have used that $d_k(T)$ is odd in the second equality. Note, $t \in S$, and hence $S \cup L$ eventually infects all of T and has cardinality at most $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$.

If $\deg(t) = k$, then $d_{\leq k-1}(T') = d_{\leq k-1}(T) - (k-2)$ and $d_k(T') = d_k(T) - 2$. This follows because we have removed $k-1$ leaves, a vertex of degree k, and decreased the degree of t by one. Then, T' has a percolating set, denoted S, of cardinality at most

$$d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor = d_{\leq k-1}(T) - (k-2) + \left\lfloor \frac{d_k(T)-2}{2} \right\rfloor$$
$$= d_{\leq k-1}(T) - (k-1) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor.$$

Note, $t \in S$, and hence $S \cup L$ eventually infects all of T and has cardinality at most $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$.

If $\deg(t) = k+1$, then $d_{\leq k-1}(T') = d_{\leq k-1}(T) - (k-1)$ and $d_k(T') = d_k(T)$. This follows because we have removed $k-1$ leaves, a vertex of degree k, and decreased the degree of t by one. Then, T' has a percolating set, denoted S, of cardinality at most

$$d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor = d_{\leq k-1}(T) - (k-1) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor.$$

Note, as S eventually infects all of T', we have that t will eventually be infected, and hence $S \cup L$ eventually infects all of T and has cardinality at most $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$.

Case 4: $\ell(s) = k$ and $d_k(T)$ is even.

In this case, we remove the leaf v from T and denote the resulting tree T'. Note, $d_{\leq k-1}(T') = d_{\leq k-1}(T) - 1$ and $d_k(T') = d_k(T) + 1$ since the degree of s in T is $k+1$ and has been decreased by one in T'. By the induction hypothesis, T' has a percolating set, denoted S, of cardinality at most

$$d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor = d_{\leq k-1}(T) - 1 + \left\lfloor \frac{d_k(T)+1}{2} \right\rfloor$$
$$= d_{\leq k-1}(T) - 1 + \left\lfloor \frac{d_k(T)}{2} \right\rfloor,$$

where we have used that $d_k(T)$ is even in the second equality. Thus, $S \cup \{v\}$ eventually infects all of T and has cardinality at most $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$.

Case 5: $\ell(s) = k$ and $d_k(T)$ is odd.

In this case, we remove two leaves from T, say v and w, which are both supported by s, and denote the resulting tree T'. Note, $d_{\leq k-1}(T') = d_{\leq k-1}(T) - 1$ and $d_k(T') = d_k(T)$

since the degree of s in T is $k+1$ and has been decreased by two in T'. By the induction hypothesis, T' has a percolating set, denoted S, of cardinality at most

$$d_{\leq k-1}(T') + \left\lfloor \frac{d_k(T')}{2} \right\rfloor = d_{\leq k-1}(T) - 1 + \left\lfloor \frac{d_k(T)}{2} \right\rfloor.$$

Note, $s \in S$ since $\deg(s) = k-1$ in T'. Thus, $(S \cup \{v, w\}) \backslash \{s\}$ eventually infects all of T and has cardinality at most $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$.

In all cases we have shown that T has a percolating set of cardinality at most $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$, and hence $\mathrm{bp}_k(T) \leq d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor$. The proof follows by induction. □

We now make a few observations concerning the above bound.

1. The above bound is sharp for paths when $k = 2$ and for the family of caterpillars of the form $P_n(k-2, k-2, \ldots, k-2, k-2)$, where $k \geq 3$.
2. We can make the difference $d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor - \mathrm{bp}_k(T)$ arbitrarily large using the family of caterpillars $P_n(k, k-2, k, k-2, \ldots, k-2, k)$.
3. For a connected graph G, we can remove edges from G to obtain a spanning tree T of G. Then the inequality $\mathrm{bp}_k(G) \leq \mathrm{bp}_k(T)$ combined with the above upper bound gives an upper bound for $\mathrm{bp}_k(G)$.

We conclude this section by comparing our above result with the bounds obtained by Riedl in [25]. The upper and lower bounds for $\mathrm{bp}_k(T)$ given by Riedl can be found in Proposition 3 (lower bound) and Theorem 4 (upper bound) of [25] and are given by

$$\frac{(k-1)n+1}{k} \leq \mathrm{bp}_k(T) \leq \frac{kn + d_{\leq k-1}(T)}{k+1}, \tag{5.1}$$

where n is the order of the tree and $d_{\leq k-1}(T)$ is defined before the statement of Theorem 5.1. It should be noted that our quantity $\mathrm{bp}_k(T)$ is denoted in [25] as $m(T, k)$. Moreover, the upper bound given in [25] is actually an upper bound for a different, but larger, quantity than $\mathrm{bp}_k(T)$.

We first mention that following the statement of Proposition 3 in [25], Riedl mentions that for $k = 2$ his bound is sharp for odd length paths, and for $k > 2$ his lower bound is sharp for complete k-ary trees and complete $k-1$-ary trees. Note, this is precisely the cases of Theorem 3.8 with $k = N, N+1$.

With regards to the bound in Theorem 5.1, we have that this is equal to the lower bound in Equation 5.1 for paths of odd length when $k = 2$. Moreover, by writing $n = d_{\leq k-1}(T) + d_{\geq k}(T)$, we can rewrite the upper bound in Equation 5.1 as

$$\mathrm{bp}_k(T) \leq d_{\leq k-1}(T) + \frac{k d_{\geq k}(T)}{k+1}.$$

As $d_{\geq k}(T) \geq d_k(T)$ and $k > 1$ we have

$$d_{\leq k-1}(T) + \frac{k d_{\geq k}(T)}{k+1} \geq d_{\leq k-1}(T) + \frac{k d_k(T)}{k+1} \geq d_{\leq k-1}(T) + \left\lfloor \frac{d_k(T)}{2} \right\rfloor,$$

which is precisely the upper bound in Theorem 5.1. Hence, Theorem 5.1 gives an improvement upon the upper bound in [25].

6 Trees of small order

In this section, we use the above results to complete the characterization of trees on eleven vertices or less. Throughout, we denote such a tree by T.

There are 201 non-isomorphic trees on ten vertices or less (see Harary [15] or Steinbach's "Field Guide to Simple Graphs" [26]). All but seven of these can be classified as spiders, caterpillars, or trees of diameter at most five. These seven trees all have degree sequence $[3, 3, 2, 2, 2, 2, 1, 1, 1, 1]$. Riedl's lower bound (Equation 5.1) shows that the 2-bootstrap percolation number satisfies $\mathrm{bp}_2(T) \geq 6$. The bound given in Theorem 5.1 shows that $\mathrm{bp}_2(T) \leq 6$. Therefore, $\mathrm{bp}_2(T) = 6$ and $t_2(T) \leq 2$ in these cases. Except for the four cases in which the two vertices of degree three are adjacent, we have that $\mathrm{bp}_3(T) = 8$ and $t_3(T) = 1$. In the four cases in which the two vertices of degree three are adjacent, we have that $\mathrm{bp}_3(T) = 9$ and $t_3(T) = 1$ by Observation 2.1.

As for the 235 non-isomorphic trees on eleven vertices, all but 42 of these can be classified as spiders, caterpillars, or trees of diameter at most five. Note that Riedl's lower bound guarantees that $\mathrm{bp}_2(T) \geq 6$ and $\mathrm{bp}_3(T) \geq 8$.

Fifteen of these have degree sequence $[3, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1]$. In these cases, the bound given in Theorem 5.1 shows that $\mathrm{bp}_2(T) \leq 6$. Hence $\mathrm{bp}_2(T) = 6$. The trivial lower bound given by Observation 2.1 shows that $\mathrm{bp}_3(T) \geq 9$ while the bound given in Theorem 5.1 shows that $\mathrm{bp}_3(T) \leq 10$. It is straightforward to check that $\mathrm{bp}_3(T) = 9$ for these trees if and only if their two vertices of degree three are not adjacent. Otherwise, we have that $\mathrm{bp}_3(T) = 10$.

Thirteen of these have degree sequence $[3, 3, 3, 2, 2, 2, 1, 1, 1, 1, 1]$. Again, the bound given in Theorem 5.1 shows that $\mathrm{bp}_2(T) \leq 6$. Combining this with Riedl's lower bound yields $\mathrm{bp}_2(T) = 6$. Further, due to Riedl and Theorem 5.1, we have that $8 \leq \mathrm{bp}_3(T) \leq 9$. Of these, only one has no two vertices of degree three adjacent. Therefore, $\mathrm{bp}_3(T) = 8$ in this case. For the remaining twelve, $\mathrm{bp}_3(T) = 9$ due to Observation 2.1.

The fourteen remaining trees have degree sequence $[4, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1]$. Combining Riedl's bound with Theorem 5.1 yields $6 \leq \mathrm{bp}_2(T) \leq 7$. It is straightforward to check that eight of these have $\mathrm{bp}_2(T) = 6$ and the remaining six have $\mathrm{bp}_2(T) = 7$. Note that the trivial lower bound and Theorem 5.1 guarantee that $\mathrm{bp}_3(T) = 9$. Observation 2.1 shows that $\mathrm{bp}_4(T) = 10$ in all of these cases.

7 Open problems

In this section, we give open problems related to this study as possible avenues for future research.

Suppose that we want every vertex to be infected within t iterations. Among all k-bootstrap sets that will infect the graph within t iterations, choose one with minimum cardinality. What is the cardinality of such a set?

Suppose that we limit the size of the initial set. What is the maximum number of vertices that can be infected? How is this maximum changed if we also limit the number of iterations?

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References

[1] J. Balogh and B. Bollobás, Bootstrap percolation on the hypercube, *Probab. Theory and Related Fields* **134** (2006), 624–648, doi:10.1007/s00440-005-0451-6.

[2] J. Balogh, B. Bollobás, H. Duminil-Copin and R. Morris, The sharp threshold for bootstrap percolation in all dimensions, *Trans. Am. Math. Soc.* **364** (2012), 2667–2701, doi:10.1090/s0002-9947-2011-05552-2.

[3] J. Balogh, Y. Peres and G. Pete, Bootstrap percolation on infinite trees and non-amenable groups, *Comb. Probab. Comput.* **15** (2006), 715–730, doi:10.1017/s0963548306007619.

[4] J. Balogh and G. Pete, Random disease on the square grid, *Random Struct. Algor.* **13** (1998), 409–422, doi:10.1002/(sici)1098-2418(199810/12)13:3/4⟨409::aid-rsa11⟩3.3.co;2-5.

[5] J. Balogh and B. G. Pittel, Bootstrap percolation on the random regular graph, *Random Struct. Algor.* **30** (2007), 257–286, doi:10.1002/rsa.20158.

[6] B. Bollobás, Weakly k-saturated graphs, in: *Beiträge zur Graphentheorie (Kolloquium, Manebach, 1967)*, Teubner, Leipzig, pp. 25–31, 1968.

[7] B. Bollobás, *The Art of Mathematics: Coffee time in Memphis*, Cambridge Univ. Press, 2006, doi:10.1017/CBO9780511816574.

[8] J. Chalupa, P. L. Leath and G. R. Reich, Bootstrap percolation on a Bethe lattice, *J. Phys. C* **12** (1979), L31, doi:10.1088/0022-3719/12/1/008.

[9] A. Coja-Oghlan, U. Feige, M. Krivelevich and D. Reichman, Contagious sets in expanders, in: *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, SIAM, Philadelphia, PA, 2015 pp. 1953–1987, doi:10.1137/1.9781611973730.131.

[10] U. Feige, M. Krivelevich and D. Reichman, Contagious sets in random graphs, *Ann. Appl. Probab.* **27** (2017), 2675–2697, doi:10.1214/16-aap1254.

[11] J. F. Fink and M. S. Jacobson, On n-domination, n-dependence and forbidden subgraphs, in: *Graph theory with applications to algorithms and computer science*, Wiley, New York, pp. 301–311, 1985, doi:10.5555/21936.25447.

[12] D. Freund, M. Poloczek and D. Reichman, Contagious sets in dense graphs, *Eur. J. Comb.* **68** (2018), 66–78, doi:10.1016/j.ejc.2017.07.011.

[13] K. Gunderson, Minimum degree conditions for small percolating sets in bootstrap percolation, *Electron. J. Comb.* **27** (2020), Paper No. 2.37, 22, doi:10.37236/6937.

[14] L. Hambardzumyan, H. Hatami and Y. Qian, Lower bounds for graph bootstrap percolation via properties of polynomials, *J. Comb. Theory Ser. A* **174** (2020), 105253, 12, doi:10.1016/j.jcta.2020.105253.

[15] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969, doi:10.1201/9780429493768.

[16] T. W. Haynes, S. T. Hedetniemi and P. J. Slater (eds.), *Domination in graphs: Advanced topics*, volume 209 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, Inc., New York, 1998, doi:10.1201/9781315141428.

[17] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, volume 208 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, Inc., New York, 1998, doi:10.1201/9781482246582.

[18] A. Holroyd, Sharp metastability threshold for two-dimensional bootstrap percolation, *Probab. Theory Related Fields* **125** (2003), 195–224, doi:10.1007/s00440-002-0239-x.

[19] S. Janson, On percolation in random graphs of given vertex degrees, *Electron. J. Probab.* **14** (2009), 87–118, doi:10.1214/ejp.v14-603.

[20] S. Janson, T. Łuczak, T. Turova and T. Vallier, Bootstrap percolation on the random graph $G_{N,P}$, *Ann. Appl. Probab.* **22** (2012), 1989–2047, doi:10.1214/11-aap822.

[21] C. Jeger and A. Zehmakan, Dynamic monopolies in two-way bootstrap percolation, *Discrete Appl. Math.* **262** (2019), 116–126, doi:10.1016/j.dam.2019.02.011.

[22] R. Morris, Minimal percolating sets in bootstrap percolation, *Electron. J. Comb.* **16** (2009), R2, doi:10.37236/91.

[23] N. Morrison and J. Noel, Extremal bounds for bootstrap percolation in the hypercube, *J. Comb. Theory. A* **156** (2018), 61–84, doi:10.1016/j.jcta.2017.11.018.

[24] D. Reichman, New bounds for contagious sets, *Discrete Math.* **312** (2012), 1812–1814, doi: 10.1016/j.disc.2012.01.016.

[25] E. Riedl, Largest and smallest minimal percolating sets in trees, *Electron. J. Comb.* **19** (2012), P64, doi:10.37236/2152.

[26] P. Steinbach, Field Guide to Simple Graphs, 1990.

[27] A. Zehmakan, Target set in threshold models, *Acta Math. Univ. Comenian. (N.S.)* **88** (2019), 1079–1086, http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/1175.

[28] A. Zehmakan, Tight bounds on the minimum size of a dynamic monopoly, in: *Language and automata theory and applications*, Springer, Cham, volume 11417 of *Lecture Notes in Comput. Sci.*, pp. 381–393, 2019, doi:10.1007/978-3-030-13435-8_2.

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On distance signless Laplacian spectra of power graphs of the integer modulo group*

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Abstract

For a finite group $\mathcal{G}$, the power graph $\mathcal{P}(\mathcal{G})$ is a simple connected graph whose vertex set is the set of elements of $\mathcal{G}$ and two distinct vertices are adjacent if and only if one is a power of the other. In this article, we obtain the distance signless Laplacian spectrum of power graphs of the integer modulo groups $\mathbb{Z}_n$. We characterize the values of n, for which power graphs of $\mathbb{Z}_n$ is distance signless Laplacian integral.

Keywords: Signless Laplacian matrix, distance signless Laplacian matrix, finite groups, power graphs.

Math. Subj. Class.: 05C50, 05C25, 15A18

1 Introduction

A graph $G = G(V(G), E(G))$ consists of the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G)$. The cardinalities of $V(G)$ and $E(G)$ are called the *order* and the *size* of G and are taken as n and m. The set of vertices incident on $v \in V(G)$, denoted by $N(v)$, is the *neighborhood* of v. The *degree* of v, denoted by d_v, is the cardinality of $N(v)$. A graph G is said to be regular if degree of each vertex is same. We assume all our graphs are connected and simple. Our notations are standard and are taken from [16].

The adjacency matrix $A = (a_{ij})$ of G is an $n \times n$ matrix whose (i, j)-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise. Let $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the

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diagonal matrix of vertex degrees $d_i = d_{v_i}$, $i = 1, 2, \dots, n$ of G. The positive semi-definite matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are respectively the Laplacian and the signless Laplacian matrices and their multiset of eigenvalues are respectively the Laplacian spectrum and signless Laplacian spectrum of G. More about these matrices can be seen in [6].

In a graph G, the *distance* between two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is defined as the length of a shortest path between u and v. The *diameter* of G is the maximum distance between any two vertices of G. The *distance matrix* of G, denoted by $\mathcal{D}(G)$, is defined as $\mathcal{D}(G) = (d_{uv})$, where $d_{uv} = d(u, v)$ and $d_{uu} = 0$. A complete survey of the matrix $\mathcal{D}(G)$ is given in [3]. The *transmission* of the vertex v (or transmission degree), denoted by $Tr(v)$, is defined to be the sum of the distances from v to all other vertices in G, that is, $Tr(v) = \sum_{u \in V(\mathcal{G})} d(u, v)$. We observe that transmission of v_i is same as the i^{th} row sum of the matrix $\mathcal{D}(G)$.

Let $Tr(G) = diag(Tr_1, Tr_2, \dots, Tr_n)$ be the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen [2] introduced the distance Laplacian $\mathcal{L}(G) = Tr(G) - \mathcal{D}(G)$ and the distance signless Laplacian $\mathcal{Q}(G) = Tr(G) + \mathcal{D}(G)$ for the distance matrix of a connected graph. These matrices are real symmetric positive and semi-definite. Our interest is in the matrix $\mathcal{Q}(G)$, we denote its eigenvalues by ρ_i's, order them as $\rho_n \leq \rho_{n-1} \leq \cdots \leq \rho_1$, where ρ_1 is known as the distance signless Laplacian spectral radius G. Further information about the matrix $\mathcal{Q}(G)$ can be seen in [2, 4].

Kelarev and Quinn [12] defined the directed power graph of a semigroup S as a directed graph with vertex set S in which two vertices $x, y \in S$ are joined by an arc from x to y if and only if $x \neq y$ and $y^i = x$ for some positive integer i. Chakrabarty et al. [8] defined the undirected power graph $\mathcal{P}(\mathcal{G})$ of a group $\mathcal{G}$ as an undirected graph with vertex set as $\mathcal{G}$ and two vertices $x, y \in G$ are adjacent if and only if $x^i = y$ or $y^j = x$, for $2 \leq i, j \leq n$. Let $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{n-1}\}$ be the cyclic group of integers modulo n. Then $\overline{a}$ and $\overline{b}$ in $\mathcal{P}(\mathbb{Z}_n)$ are adjacent if there exists a positive integer m such that either $\overline{a} = m\overline{b}$ or $\overline{b} = m\overline{a}$. Such graphs have valuable applications and are related to automata theory [11], besides being useful in characterizing finite groups. More on power graphs can be seen in [1, 7, 8, 14]. Laplacian spectrum of power graphs of finite cyclic and dihedral groups have been investigated in [9]. In [9], it is shown that Laplacian spectral radius of power graph of any finite group coincides with the order of group $\mathcal{G}$. Spectral properties of adjacency matrix of $\mathcal{P}(\mathcal{G})$ were investigated in [15]. Other spectral results of power graphs can be seen in [5, 18, 19, 20, 21].

The identity of the group G is denoted by e. The proper power graph of $\mathcal{P}(\mathcal{G})$, denoted by $\mathcal{P}(\mathcal{G}^*) = \mathcal{P}(\mathcal{G} \setminus \{e\})$, is obtained by removing the vertex e. By U_n, we denote the set $\{\overline{a} \in \mathbb{Z}_n \mid 1 \leq \overline{a} < n,\ \gcd(\overline{a}, n) = 1\}$ and $U_n^* = U_n \cup \{\overline{0}\}$. $\mathbb{M}_n(\mathbb{F})$ denotes the set of $n \times n$ matrices with entries from field $\mathbb{F}$. Also, K_n, $K_{1,n-1}$ and P_n respectively denote the complete graph, the star and the path. For other undefined notations and terminology, the readers are referred to [6, 10, 16].

The rest of the paper is organized as follows. In Section 2, we find the distance signless Laplacian spectrum of the power graph $\mathcal{P}(\mathbb{Z}_n)$ in terms of the adjacency eigenvalues and the eigenvalues of the quotient matrix. We end up with some comments for further work.

2 Distance signless Laplacian spectra of power graphs of finite cyclic group $\mathbb{Z}_n$

Let n be a positive integer with canonical decomposition $n = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}$ and let $\tau(n)$ [13] denotes the number of positive factors of n. Then

$$\tau(n) = (n_1 + 1)(n_2 + 1) \dots (n_r + 1). \tag{2.1}$$

The *Euler's totient function* $\phi(n)$ [13] denotes the number of positive integers less or equal to n and relatively prime to n. If n be a positive integer, then

$$\sum_{d|n} \phi(d) = n. \tag{2.2}$$

A divisor d of n is a proper divisor of n, if $1 < d < n$. Let $d_1, d_2, \dots, d_t$ be the distinct proper divisors of n. Let $\mathbb{G}_n$ be a simple graph with vertex set $\{d_1, d_2, \dots, d_t\}$ in which two distinct vertices are adjacent if and only if $d_i | d_j$, for $1 \le i < j \le t$. If n is in canonical decomposition, then by Equation (2.1), the size of $\mathbb{G}_n$ is given by $|V(\mathbb{G}_n)| = \prod_{i=1}^{r}(n_i + 1) - 2$. If $n = pq$, where $p < q$ are primes, then p does not divide q. So $\mathbb{G}_n$ is disconnected when n is a product of two distinct primes.

Let A be $m \times m$ matrix

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} \end{pmatrix},$$

whose rows and columns are partitioned according to a partition $P = \{P_1, P_2, \dots, P_m\}$ of $X = \{1, 2, \dots, n\}$. The quotient matrix Q is the $m \times m$ matrix whose entries are the average row sums of the blocks $A_{i,j}$ of A. The partition P is called equitable if each block $A_{i,j}$ of A has constant row (and column) sum and in such case the matrix Q is known as equitable quotient matrix. In general, the eigenvalues of Q interlace the eigenvalues of A. If the partition is equitable, then each eigenvalue of Q [6] is an eigenvalue of A.

Let $G_i = G_i(V_i, E_i)$ be graphs of order n_i, where $i = 1, \dots, n$. The *joined union* [22] of graphs $G_1, G_2, \dots, G_n$, denoted by $G[G_1, G_2, \dots, G_n]$, is defined as the graph $H(W, F)$ with

$$W = \bigcup_{i=1}^{n} V_i \quad \text{and} \quad F = \bigcup_{i=1}^{n} E_i \cup \bigcup_{\{v_i, v_j\} \in E} V_i \times V_j.$$

In other words, if G has n vertices labelled as $\{1, 2, \dots, n\}$, then v_i in G_i and v_j in G_j are adjacent in the joined union if i and j are adjacent in G. Thus, the usual join of two graphs G_1 and G_2 is a special case of the joined union $K_2[G_1, G_2] = G_1 \triangledown G_2$ where K_2 is the complete graph of order 2.

The following result [17] gives the distance signless Laplacian spectrum of the join of two regular graphs.

Theorem 2.1. *Let G_1 and G_2 be r_1 and r_2 regular graphs of order n_1 and n_2, respectively. Let $\lambda_1 = r_1, \lambda_2, \dots, \lambda_{n_1}$ and $\mu_1 = r_2, \mu_2, \dots, \mu_{n_2}$ be the adjacency eigenvalues of G_1*

and G_2, respectively. Then the distance signless Laplacian spectrum of $G_1 \nabla G_2$ with order $n = n_1 + n_2$ consists of $n_1 - 1$ eigenvalues of the type $2n - n_2 - r_1 - 4 - \lambda_i$, $(2 \leq i \leq n_1)$, and $n_2 - 1$ eigenvalues of the type $2n - n_1 - r_2 - 4 - \mu_j$, $(2 \leq j \leq n_2)$ together with the two eigenvalues of the following matrix

$$\begin{pmatrix} n + 3n_1 - 2r_1 - 4 & n_2 \\ n_1 & n + 3n_2 - 2r_2 - 4 \end{pmatrix}, \tag{2.3}$$

where $2 \leq i \leq n_1$ and $2 \leq j \leq n_2$.

The next result gives the distance signless Laplacian spectrum of the joined union of regular graphs $G_1, \dots, G_n$, in terms of adjacency spectrum of the graphs $G_1, G_2, \dots, G_n$ and the eigenvalues of quotient matrix.

Theorem 2.2 ([17])**.** *Let G be a graph of order n having vertex set $V(G) = \{v_1, \dots, v_n\}$. Let G_i be r_i regular graphs of order n_i having adjacency eigenvalues $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \dots \geq \lambda_{in_i}$, where $i = 1, 2, \dots, n$. The distance signless Laplacian spectrum of the joined union graph $G[G_1, \dots, G_n]$ of order $N = \sum_{i=1}^{n} n_i$ consists of the eigenvalues $2n_i + n_i' - r_i - \lambda_{ik} - 4$ for $i = 1, \dots, n$ and $k = 2, 3, \dots, n_i$, where $n_i' = \sum_{k=1, k \neq i}^{n} n_k d_G(v_i, v_k)$. The remaining n eigenvalues are given by the equitable quotient matrix*

$$Q = \begin{pmatrix} 4n_1 + n_1' - 2r_1 - 4 & n_2 d_G(v_1, v_2) & \dots & n_n d_G(v_1, v_n) \\ n_1 d_G(v_2, v_1) & 4n_2 + n_2' - 2r_2 - 4 & \dots & n_n d_G(v_2, v_n) \\ \vdots & \vdots & \ddots & \vdots \\ n_1 d_G(v_n, v_1) & n_2 d_G(v_n, v_2) & \dots & 4n_n + n_n' - 2r_n - 4 \end{pmatrix}.$$

The following result says that $n - 2$ is always a distance signless Laplacian eigenvalue of the power graph.

Theorem 2.3. *Let $\mathcal{G}$ be a finite group of order $n \geq 3$. Then $n - 2$ is a distance signless Laplacian eigenvalues of $\mathcal{P}(\mathcal{G})$ with multiplicity at least $b - 1$, where b is the number of elements of $\mathcal{G}$ which generate all elements of group $\mathcal{G}$.*

Proof. Let $B(\mathcal{G})$ be the set of vertices of $\mathcal{G}$ consisting of the identity e and those elements of $\mathcal{G}$ which generate all elements of $\mathcal{G}$. Thus, by the definition of power graph, the induced subgraph $\mathcal{P}(B(\mathcal{G}))$ is the complete graph K_b, where b is the cardinality of $B(\mathcal{G})$. To avoid triviality, we assume that $\mathcal{G} \setminus B(\mathcal{G}) \neq \{\}$, so that we obtain, $\mathcal{P}(\mathcal{G}) \cong K_b \nabla \mathcal{P}(\mathcal{G} \setminus B(\mathcal{G}))$. By using Theorem 2.1, we get the distance signless Laplacian eigenvalue

$$2n - n_2 - r_1 - \lambda_{1k} - 4 = 2n - n + b - b + 1 + 1 - 4 = n - 2,$$

with at least multiplicity $b - 1$, since $n - 2$ can also be the eigenvalue of matrix (2.3). $\square$

Taking in particular $\mathcal{G} = \mathbb{Z}_n$, a finite cyclic group of order n, we have the following observation.

Corollary 2.4. *Let $\mathbb{Z}_n$ be a finite cyclic group of order $n \geq 3$. Then $n - 2$ is a distance signless Laplacian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity at least $\phi(n)$, where ϕ is Euler's totient function.*

Proof. Since identity $\overline{0}$ and the invertible elements of the group $\mathbb{Z}_n$, which are $\phi(n)$ in number, generate all the elements of group $\mathbb{Z}_n$, therefore $\mathcal{P}(\mathbb{Z}_n) = K_{\phi(n)+1} \nabla \mathcal{P}(\mathbb{Z}_n \setminus U_n^*)$, where $U_n^* = U_n \cup \{\overline{0}\}$ and $\mathbb{Z}_n \setminus U_n^* \neq \{\}$, since power graph of the empty set is empty. Thus, by Theorem 2.3, $n-2$ is a distance signless Laplacian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity at least $\phi(n)$. □

It will be interesting to characterize the power graphs for which equality holds in Theorem 2.3 and Corollary 2.4. Therefore, we have the following problem.

Problem 2.5. Characterize the power graphs $\mathcal{P}(\mathcal{G})$, where $\mathcal{G}$ is a finite group of order $n \geq 3$ having $n-2$ as the distance signless Laplacian eigenvalue with multiplicity $b-1$, where b is the number of group $\mathcal{G}$ which generate all the elements of $\mathcal{G}$. Also, characterize the power graphs $\mathcal{P}(\mathbb{Z}_n)$ having $n-2$ as the distance signless Laplacian eigenvalue with multiplicity exactly $\phi(n)$.

From Theorem 2.3, we observe that if $\mathcal{P}(\mathcal{G} \setminus B(\mathcal{G}))$ is known, then the distance signless Laplacian spectrum of $\mathcal{P}(\mathcal{G})$ can be completely determined. So, it will be interesting to study the structure of $\mathcal{P}(\mathcal{G} \setminus B(\mathcal{G}))$, and looking for graph parameters related to it.

If $\mathcal{G}$ is a finite group, then $\mathcal{P}(\mathcal{G})$ [8] is complete if and only if $\mathcal{G}$ is cyclic group of prime power order.

The following result [14] shows that the power graph of a cyclic group $\mathbb{Z}_n$ can be written as the joined union of complete graphs.

Theorem 2.6. *If $\mathbb{Z}_n$ is a finite cyclic group, then the power graph has the following form:*

$$\begin{aligned}\mathcal{P}(\mathbb{Z}_n) &= K_{\phi(n)+1} \nabla \mathbb{G}_n[K_{\phi(d_1)}, K_{\phi(d_2)}, \dots, K_{\phi(d_t)}]\\ &= H[K_{\phi(n)+1}, K_{\phi(d_1)}, K_{\phi(d_2)}, \dots, K_{\phi(d_t)}],\end{aligned}$$

where $H = K_{\phi(n)+1} \nabla \mathbb{G}_n$ with the vertex set $\{v_1, \dots, v_{t+1}\}$ and t is the number of proper divisors of n.

We note that if $n = p$, then $\mathbb{G}_n$ is empty graph and we get $\mathcal{P}(\mathbb{Z}_p) = K_p$. By applying Theorem 2.2, we can find the distance signless Laplacian spectrum of $\mathcal{P}(\mathcal{G})$ in terms of adjacency eigenvalues of K_ω and the eigenvalues of the quotient matrix. It is well known that the adjacency eigenvalues of K_ω are $\{\omega, (-1)^{[\omega-1]}\}$. As t is the number of divisors of n, so by Theorem 2.2, $n-t$ out of the n distance signless Laplacian eigenvalues are known to be non negative integers and the remaining t distance signless Laplacian eigenvalues are the eigenvalues of the quotient matrix Q.

In the following result, we obtain the distance signless Laplacian eigenvalues of the power graph of $\mathbb{Z}_n$.

Theorem 2.7. *The distance signless Laplacian spectrum of $\mathcal{P}(\mathbb{Z}_n)$ consists of the eigenvalues*

$$\Big\{(n-2)^{[\phi(n)]}, (\phi(d_1) + n_2' - 2)^{[\phi(d_1)-1]}, (\phi(d_2) + n_3' - 2)^{[\phi(d_2)-1]},\\ \dots, (\phi(d_t) + n_{t+1}')^{[\phi(d_t)-1]}\Big\},$$

where d_i, $1 \leq i \leq t$ are the proper divisors of n and $n_i' = \sum\limits_{k=2, k\neq i}^{t+1} \phi(d_k) d_{\mathcal{P}(\mathbb{Z}_n)}(v_i, v_k)$. The remaining $t+1$ distance signless Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ are the eigenvalues *of the following quotient matrix*

$$Q = \begin{pmatrix} n+\phi(n)-1 & \phi(d_1) & \dots & \phi(d_n) \\ (\phi(n)+1)d(v_2,v_1) & d'_2 & \dots & \phi(d_t)d(v_2,v_{t+1}) \\ \vdots & \vdots & \ddots & \vdots \\ (\phi(n)+1)d(v_{t+1},v_1) & (\phi(d_1))d(v_{t+1},v_2) & \dots & d'_{t+1} \end{pmatrix}, \tag{2.4}$$

where $d'_i = 2n_i + n'_i - 2$, *for* $i = 2, \dots, t+1$ *and* $n_1 = \phi(n)+1$, $n_j = \phi(d_j)$, $j = 2, 3, \dots, n$.

Proof. Let $\mathbb{Z}_n$ be a finite cyclic group of order n. Then it is well known that the identity $\overline{0}$ and $\phi(n)$ elements of $\mathbb{Z}_n$ together generate every other element of $\mathbb{Z}_n$. Thus, by the definition power graph, $\phi(n)+1$ vertices are connected to every vertex of $\mathcal{P}(\mathbb{Z}_n)$. So, by Theorem 2.6, we have

$$\mathcal{P}(\mathbb{Z}_n) = H[K_{\phi(n)+1}, K_{\phi(d_1)}, K_{\phi(d_2)}, \dots, K_{\phi(d_t)}].$$

Clearly, $n_1 = \phi(n)+1$ and $n_i = \phi(d_{i-1})$, for $i = 2, \dots, t+1$. Now, using the fact that $\sum\limits_{1,n \neq d|n} \phi(d) = n - \phi(n) - 1$ and Theorem 2.3, we see that $n-2$ is the distance signless Laplacian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity $\phi(n)$. Again, using Theorem 2.2 and the adjacency eigenvalues of K_ω, we get

$$2n_2 + n'_2 - r_2 - \lambda_{2k} - 4 = 2n_2 + n'_2 - n_2 + 1 + 1 - 4 = n_2 + n'_2 - 2 = \phi(d_1) + n'_2 - 4$$

as the distance signless Laplacian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity $\phi(d_1)-1$. Similarly other distance signless Laplacian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ are $\phi(d_i) + n'_i - 4$ with multiplicities $\phi(d_i)-1$, for $i = 3, 4, \dots, t+1$. In this way, we have obtained $n-t$ distance signless Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ and the remaining distance signless Laplacian eigenvalues are given by matrix (2.4). □

The following are consequences of Theorem 2.7.

Corollary 2.8. *If* $n = p^{m_1}$, *where* p *is a prime and* m_1 *is a non negative integer, then the distance signless Laplacian spectrum of* $\mathcal{P}(\mathbb{Z}_n)$ *is* $\{2n-2, (n-2)^{[n-1]}\}$.

Proof. If $n = p_1^m$, where p is prime and m_1 is a non negative integer, then as shown in [8], $\mathcal{P}(\mathbb{Z}_n) \cong K_n$ and distance signless Laplacian spectrum, of K_n is $\{2n-2, (n-2)^{[n-1]}\}$. □

Corollary 2.9. *If* $n = pq$, *where* p *and* q $(p < q)$ *are primes, then the distance signless Laplacian spectrum of* $\mathcal{P}(\mathbb{Z}_n)$ *is*

$$\left\{(n-2)^{[\phi(n)]}, (n+\phi(q)-2)^{[\phi(p)-1]}, (n+\phi(p)-2)^{[\phi(q)-1]}\right\}$$

and the zeros of the following cubic polynomial

$$\begin{aligned} x^3 - &(1+2p+2q+2pq)x^2 + \left(-40+4p+3p^2+8q+8pq+p^2q+q^2+pq^2+p^2q^2\right)x \\ &- 76+16p-p^2-2p^3+24q+70pq-6p^2q-p^3q-5q^2-16pq^2-13p^2q^2 \\ &+ p^3q^2+pq^3+p^2q^3. \end{aligned}$$

Proof. As p and q are the proper divisor of n, so $\mathbb{G}_n$ is $2K_1$ and $K_1 \nabla 2K_1$ is the path P_3 on three vertices. Thus, by Theorem 2.6, the power graph of $\mathcal{P}(\mathbb{Z}_n)$ is

$$\mathcal{P}(\mathbb{Z}_n) = P_3[K_{p-1}, K_{\phi(pq)+1}, K_{q+1}].$$

Also, $(n_1', n_2', n_3') = (\phi(pq)+1+2\phi(q), \phi(p)+\phi(q), 2\phi(p)+\phi(pq)+1)$. Now, by Corollary 2.4, $n-2$ is the distance signless Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity $\phi(n)$. Again, by using the Theorem 2.7, $n_1 + n_1' - 2 = n + \phi(q) - 2$ and $n + \phi(p) - 2$ are the distance signless Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicities $\phi(p) - 1$ and $\phi(q) - 1$, respectively. The remaining distance signless Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ are given by the following matrix

$$\begin{pmatrix} pq+p+q-4 & pq-p-q+2 & 2(q-1) \\ p-1 & 2pq-p-q & q-1 \\ 2(p-1) & pq-p-q+2 & pq+p+q-4 \end{pmatrix}. \qquad \square$$

Proceeding as in Corollary 2.9, the distance signless Laplacian spectrum of $\mathcal{P}(\mathbb{Z}_n)$ can be discussed, for $n = pqr$, where p, q and r, $(p < q < r)$ are primes.

Corollary 2.10. *If $n = pqr$, where p, q and r $(p < q < r)$ are primes, then the distance signless Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ are*

$$\Big\{(n-2)^{[\phi(n)]}, (n+n_3+n_4+n_7-2)^{[n_2-1]}, (n+n_2+n_4+n_6-2)^{[n_3-1]},$$
$$(n+n_2+n_3+n_5-2)^{[n_4-1]}, (n+n_4+n_6+n_7-2)^{[n_5-1]},$$
$$(n+n_3+n_5+n_7-2)^{[n_6-1]}(n+n_2+n_5+n_6-2)^{[n_7-1]}\Big\},$$

where $n_2 = \phi(p)$, $n_3 = \phi(q)$, $n_4 = \phi(r)$, $n_5 = \phi(pq)$, $n_6 = \phi(pr)$ and $n_7 = \phi(qr)$. The remaining 7 distance signless Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ are the eigenvalues of the following matrix

$$\begin{pmatrix} n+\phi(n)-4 & \phi(p) & \phi(q) & \phi(r) & \phi(pq) & \phi(pr) & \phi(qr) \\ \phi(n)+1 & d_2 & 2\phi(q) & 2\phi(r) & \phi(pq) & \phi(pr) & 2\phi(qr) \\ \phi(n)+1 & 2\phi(p) & d_3 & 2\phi(r) & \phi(pq) & 2\phi(pr) & \phi(qr) \\ \phi(n)+1 & 2\phi(p) & 2\phi(q) & d_4 & 2\phi(pq) & \phi(pr) & \phi(qr) \\ \phi(n)+1 & \phi(p) & \phi(q) & 2\phi(r) & d_5 & 2\phi(pr) & 2\phi(qr) \\ \phi(n)+1 & \phi(p) & 2\phi(q) & \phi(r) & 2\phi(pq) & d_6 & 2\phi(qr) \\ \phi(n)+1 & 2\phi(p) & \phi(q) & \phi(r) & 2\phi(pq) & 2\phi(pr) & d_7 \end{pmatrix},$$

where $d_2 = n+n_2+n_3+n_4+n_7-2$, $d_3 = n+n_2+n_3+n_4+n_6-2$, $d_4 = n+n_2+n_3+n_4+n_5-2$, $d_5 = n+n_4+n_5+n_6+n_7-2$, $d_6 = n+n_3+n_5+n_6+n_7-2$, and $d_7 = n+n_2+n_5+n_6+n_7-2$.

We recall that the proper power graph $\mathcal{P}(\mathcal{G}^*)$ of a group $\mathcal{G}$ is the power graph of $\mathcal{G} \setminus \{e\}$. The proper power graph $\mathcal{P}(\mathbb{Z}_n^*)$ is connected, as $\mathbb{Z}_n$ is a cycle group and there exists at least one element say $\bar{1}$ connected to all the other vertices of $\mathcal{P}(\mathbb{Z}_n^*)$. Thus the distance signless Laplacian matrix makes sense on $\mathcal{P}(\mathbb{Z}_n^*)$. Analogues of Theorems 2.3 and 2.2 can be proved on the proper power graph $\mathcal{P}(\mathbb{Z}_n^*)$. We state them without proofs.

Theorem 2.11. *Let $\mathbb{Z}_n$ be a finite cyclic group of order $n \geq 3$. Then $n-2$ is the distance signless Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n^*)$ with multiplicity at least $\phi(n)-1$.*

Theorem 2.12. *The distance signless Laplacian spectrum of $\mathcal{P}(\mathbb{Z}_n^*)$ is*

$$\Big\{(n-2)^{[\phi(n)-1]},(\phi(d_1)+n_2'-2)^{[\phi(d_1)-1]},(\phi(d_2)+n_3'-2)^{[\phi(d_2)-1]},\\ \dots,(\phi(d_t)+n_{t+1}')^{[\phi(d_t)-1]}\Big\}$$

together with the eigenvalues of the following quotient matrix

$$Q=\begin{pmatrix} n+\phi(n)-2 & \phi(d_1) & \dots & \phi(d_n) \\ \phi(n)d(v_2,v_1) & d_2' & \dots & \phi(d_t)d(v_2,v_{t+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(n)d(v_{t+1},v_1) & (\phi(d_1))d(v_{t+1},v_2) & \dots & d_{t+1}' \end{pmatrix},$$

where $d_i'=2n_i+n_i'-2$, $n_i'=\sum_{k=2,k\neq i}^{t+1}\phi(d_k)d_{\mathcal{P}(\mathbb{Z}_n^*)}(v_i,v_k)$, for $i=2,\dots,t+1$, $n_1=\phi(n)$, $n_j=\phi(d_j)$, $j=2,3,\dots,t$.

The following Lemma can be found in [4].

Lemma 2.13. *Let G be a connected graph on n vertices. If $n-2$ is a distance signless Laplacian eigenvalue with multiplicity μ, then the complement $\overline{G}$ of G contains at least μ components, each of which is bipartite or an isolated vertex.*

Now, we discuss Problem 2.5 for the power group of the group $\mathbb{Z}_n$.

Proposition 2.14. *Let $\mathcal{P}(\mathbb{Z}_n)$ be the power group of order $n\geq 2$. Then $n-2$ is the distance signless Laplacian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity $\phi(n)$ if and only if n is prime power or product of two distinct primes.*

Proof. By Corollaries 2.8 and 2.9, we see that $n-2$ is the distance signless Lapalcian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity exactly $\phi(n)$. Also, complement of $\mathcal{P}(\mathbb{Z}_n)$ are not necessarily bipartite or an isolated vertex when n is other than prime power or product of two distinct primes. Thus, by Lemma 2.13, we see that $\mathcal{P}(\mathbb{Z}_{p^{m_1}})$ and $\mathcal{P}(\mathbb{Z}_{pq})$ are the only two candidates of $\mathcal{P}(\mathbb{Z}_n)$, where $n-2$ is the distance signless Laplacian eigenvalue with multiplicity $\phi(n)$. □

The following Lemma [2] states that upon edge deletion, the distance signless Laplacian eigenvalues increase.

Lemma 2.15. *Let G be a connected graph of order n and size m, where $m\geq n$ and let $G'=G-e$ be a connected graph obtained from G by deleting an edge. Let $\rho_1^L(G)\geq\rho_2^L(G)\geq\cdots\geq\rho_n^L(G)$ and $\rho_1^L(G')\geq\rho_2^L(G')\geq\cdots\geq\rho_n^L(G')$ respectively be the distance Laplacian eigenvalues of G and G'. Then $\rho_i^L(G')\geq\rho_i^L(G)$ holds for all $1\leq i\leq n$.*

Since $\mathcal{P}(\mathbb{Z}_{p^{m_1}})$ is the complete graph, so the following consequence is immediate from the above lemma.

Proposition 2.16. *Let $\mathcal{P}(\mathcal{G})$ be the power graph of order $n=p^{m_1}$, where p is prime and m_1 is positive integer. Then*

$$\rho_i(\mathcal{P}(\mathcal{G}))\geq\rho_i(\mathcal{P}(\mathbb{Z}_{p^{m_1}})),$$

equality holds if and only $\mathcal{G}\cong\mathbb{Z}_{p^{m_1}}$.

From Proposition 2.16, it follows that $\mathcal{P}(\mathbb{Z}_n)$ has minimal spectrum among all power graphs whose order is a prime power. It will be an interesting to characterize the extremal power graph with some given spectral graph invariant.

A matrix $M \in \mathbb{M}_n(\mathbb{R})$ is said to be integral, if its spectrum consists of only integers. Likewise, a graph G is signless Laplacian integral if and only if the matrix $\mathcal{Q}(G)$ is integral. In case of the joined union, $G[G_1.G_2,\ldots,G_n]$ is integral if and only if each of G_i and its associated equitable quotient matrix is integral. Thus for the power graphs, we have the following result.

Proposition 2.17. *The power graph $\mathcal{P}(\mathbb{Z}_n)$ is signless Laplacian integral if and only if n is prime power.*

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References

[1] J. Abawajy, A. Kelarev and M. Chowdhury, Power graphs: a survey, *Electron. J. Graph Theory Appl.* **1** (2013), 125–147, doi:10.5614/ejgta.2013.1.2.6.

[2] M. Aouchiche and P. Hansen, Two Laplacians for the distance matrix of a graph, *Linear Algebra Appl.* **439** (2013), 21–33, doi:10.1016/j.laa.2013.02.030.

[3] M. Aouchiche and P. Hansen, Distance spectra of graphs: a survey, *Linear Algebra Appl.* **458** (2014), 301–386, doi:10.1016/j.laa.2014.06.010.

[4] M. Aouchiche and P. Hansen, On the distance signless Laplacian of a graph, *Linear Multilinear Algebra* **64** (2016), 1113–1123, doi:10.1080/03081087.2015.1073215.

[5] S. Banerjee and A. Adhikari, Signless Laplacian spectrum of power graphs of finite cyclic groups, *AKCE Int. J. Graphs Comb.* **17** (2020), 356–366, doi:10.1016/j.akcej.2019.03.009.

[6] A. E. Brouwer and W. H. Haemers, *Spectra of graphs*, Universitext, Springer, New York, 2012, doi:10.1007/978-1-4614-1939-6.

[7] P. J. Cameron and S. Ghosh, The power graph of a finite group, *Discrete Math.* **311** (2011), 1220–1222, doi:10.1016/j.disc.2010.02.011.

[8] I. Chakrabarty, S. Ghosh and M. K. Sen, Undirected power graphs of semigroups, *Semigroup Forum* **78** (2009), 410–426, doi:10.1007/s00233-008-9132-y.

[9] S. Chattopadhyay and P. Panigrahi, On Laplacian spectrum of power graphs of finite cyclic and dihedral groups, *Linear Multilinear Algebra* **63** (2015), 1345–1355, doi:10.1080/03081087.2014.936435.

[10] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 2nd edition, 2013.

[11] A. Kelarev, *Graph algebras and automata*, volume 257 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, Inc., New York, 2003.

[12] A. V. Kelarev and S. J. Quinn, Directed graphs and combinatorial properties of semigroups, *J. Algebra* **251** (2002), 16–26, doi:10.1006/jabr.2001.9128.

[13] T. Koshy, *Elementary number theory with applications*, Elsevier/Academic Press, 2007.

[14] Z. Mehranian, A. Gholami and A. R. Ashrafi, A note on the power graph of a finite group, *Int. J. Group Theory* **5** (2016), 1–10, doi:10.22108/ijgt.2016.6013.

[15] Z. Mehranian, A. Gholami and A. R. Ashrafi, The spectra of power graphs of certain finite groups, *Linear Multilinear Algebra* **65** (2017), 1003–1010, doi:10.1080/03081087.2016.1221375.

[16] S. Pirzada, *An Introduction to Graph Theory*, Universities Press, Orient BlackSwan, Hyderabad, 2012.

[17] S. Pirzada, B. A. Rather, M. Aijaz and T. A. Chishti, Distance signless laplacian spectrum of graphs and spectrum of zero divisor graphs of $\mathbb{Z}_n$, *Linear Multilinear Algebra* (2020), doi:10.1080/03081087.2020.1838425.

[18] S. Pirzada, B. A. Rather and T. A. Chishti, On distance Laplacian spectrum of zero divisor graphs of the ring $\mathbb{Z}_n$, *Carpathian Math. Publ.* **13** (2021), 48–57, doi:10.15330/cmp.13.1.48-57.

[19] S. Pirzada, B. A. Rather, T. A. Chishti and U. Samee, On normalized Laplacian spectrum of zero divisor graphs of commutative ring $\mathbb{Z}_n$, *Electron. J. Graph Theory Appl.* **9** (2021), 331–345, doi:10.5614/ejgta.2021.9.2.7.

[20] B. A. Rather, S. Pirzada, I. Bhat and T. Chishti, On Randíc spectrum of zero divisor graph of commutative ring $\mathbb{Z}_n$, *Commun. Comb. Optim.* (2021), doi:10.22049/cco.2021.27202.1212.

[21] B. A. Rather, S. Pirzada, T. A. Naikoo and Y. Shang, On Laplacian eigenvalues of the zero-divisor graph associated to the ring of integers modulo n, *Mathematics* **9** (2021), doi:10.3390/math9050482.

[22] D. Stevanović, Large sets of long distance equienergetic graphs, *Ars Math. Contemp.* **2** (2009), 35–40, doi:10.26493/1855-3974.93.e75.

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The dominated chromatic number of middle graphs

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Abstract

A dominated coloring of a graph is a proper vertex coloring such that every color class is dominated with at least one vertex. The minimum number of colors needed for a dominated coloring of a graph G is the dominated chromatic number of G. The middle graph $M(G)$ of a graph G is the graph obtained by subdividing each edge of G exactly once and joining all these newly introduced vertices of adjacent edges of G. For a graph G without isolated vertices, the dominated chromatic number of $M(G)$ is completely determined.

Keywords: Dominated coloring, dominator coloring, dominated chromatic number, middle graph.

Math. Subj. Class.: 05C15, 05C69

1 Introduction

Let $G = (V, E)$ be a finite, undirected and simple graph with the vertex set $V = V(G)$ and edge set $E = E(G)$. The open neighborhood of $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. For a subset S of $V(G)$, the subgraph obtained from G by deleting all vertices in S and all edges incident with S is denoted by $G - S$.

In [6], Hamada and Yoshimura defined the middle graph of a graph. The middle graph $M(G)$ of a graph G is the graph obtained by subdividing each edge of G exactly once and joining all these newly introduced vertices of adjacent edges of G. The precise definition of $M(G)$ is as follows. The vertex set $V(M(G))$ is $V(G) \cup E(G)$. Two vertices $v, w \in V(M(G))$ are adjacent in $M(G)$ if

(i) $v, w \in E(G)$ and v, w are adjacent in G or

(ii) $v \in V(G)$, $w \in E(G)$ and v, w are incident in G.

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Definition 1.1. A dominated coloring of a graph G is a proper (vertex) k-coloring $\{C_1, C_2, \ldots, C_k\}$ where for each $i \in \{1, 2, \ldots, k\}$, there exists a vertex $u \in V(G)$ such that $C_i \subseteq N_G(u)$. We say that u dominates C_i or C_i is dominated by u. The minimum number of colors needed for a dominated coloring of a graph G is the dominated chromatic number of G and is denoted by $\chi_{\mathrm{dom}}(G)$.

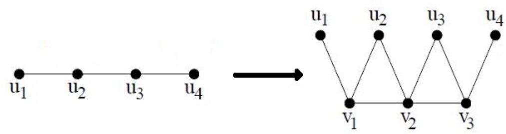

Figure 1: The middle graph $M(P_4)$ of P_4.

Example 1.2. In Figure 1, $\{\{u_1, u_2\}, \{v_1, u_3\}, \{v_2, u_4\}, \{v_3\}\}$ is a dominated coloring of $M(P_4)$. Indeed, the dominated chromatic number of $M(P_4)$ is four.

A dominated coloring of G using $\chi_{\mathrm{dom}}(G)$ colors is called a $\chi_{\mathrm{dom}}(G)$-coloring of G. Merouane et al. introduced the concept of a dominated coloring in [9]. They adopted algorithmic approach for dominated coloring problems and proved that if G is a triangle-free graph, then $\chi_{\mathrm{dom}}(G)$ is equal to its total domination number. In [3], Chen provided an application of dominated coloring in social networks. In [8], Klažar and Tavakoli proposed an application of dominated coloring in genetic networks.

As a concept closely related to dominated colorings, a dominator coloring of a graph is a proper coloring in which every vertex dominates every vertex of at least one color class. This concept was introduced for the first by Gera et al. in [5]. There have been several follow-up studies in [1, 2, 4, 10]. Recently, Klažar and Tavakoli showed that although dominated colorings and dominator colorings appear quite similar, they are strikingly different on corona products (see [8]). In [7], Kazemnejad et al. considered total dominator coloring in middle graphs and gave the total dominator chromatic number of middle graph of several known families of graphs. In contrast, in this note we completely determine the dominated chromatic number of all middle graphs. We shall prove the following:

Theorem 1.3. *Let G be a graph without isolated vertices. Then*

$$\chi_{\mathrm{dom}}(M(G)) = \left\lceil \frac{|V(G)| + |E(G)|}{2} \right\rceil .$$

As an application of Theorem 1.3, for a star $K_{1,n}$ we obtain a $\chi_{\mathrm{dom}}(M(K_{1,n}))$-coloring as follows: Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$, where v is the central vertex. If n is even, then $\{\{v_i, vv_{i+1}\} \mid i = 2k-1, 1 \le k \le \frac{n}{2}\} \cup \{\{v_j, vv_{j-1}\} \mid j = 2k, 1 \le k \le \frac{n}{2}\} \cup \{\{v\}\}$ is a $\chi_{\mathrm{dom}}(M(K_{1,n}))$-coloring. If n is odd, then $\{\{v_i, vv_{i+1}\} \mid i = 2k-1, 1 \le k \le \frac{n-1}{2}\} \cup \{\{v_j, vv_{j-1}\} \mid j = 2k, 1 \le k \le \frac{n-1}{2}\} \cup \{\{v, v_n\}, \{vv_n\}\}$ is a $\chi_{\mathrm{dom}}(M(K_{1,n}))$-coloring.

2 Proof of main theorem

In this section, we prove our main theorem.

Proof of Theorem 1.3. Since $\chi_{\rm dom}(G) = \sum_{i=1}^{t} \chi_{\rm dom}(G_i)$ for a graph G with components $G_1, ..., G_t$, it suffices to consider the case of a connected graph G. We proceed by proving two claims.

Claim 1. $\left\lceil \frac{|V(G)|+|E(G)|}{2} \right\rceil \leq \chi_{\rm dom}(M(G))$.

Let $\{V_1, \ldots, V_k\}$ be a dominated coloring of $M(G)$. Let $v \in V(G)$ belong to V_i, and let $e = vw \in E(G)$ satisfy $N_{M(G)}(e) \supseteq V_i$. Then $|V_i \cap N_{M(G)}[w]| \leq 1$. This implies that $|V_i| \leq 2$.

Let $e \in E(G)$ belong to V_i, and let $x \in V(M(G))$ satisfy $N_{M(G)}(x) \supseteq V_i$. If $x \in V(G)$, then $|V_i| = 1$ since every edge of $N_{M(G)}(x) \setminus \{e\}$ is adjacent to e. If $x \in E(G)$, then $|V_i \cap (N_{M(G)}(x) \setminus \{e\})| \leq 1$. This implies that $|V_i| \leq 2$.

Thus, every dominated coloring of $M(G)$ needs at least $\left\lceil \frac{|V(G)|+|E(G)|}{2} \right\rceil$ colors.

Claim 2. $\left\lceil \frac{|V(G)|+|E(G)|}{2} \right\rceil = \chi_{\rm dom}(M(G))$.

We prove the equality by induction on the order of G. For a graph G of order $n \leq 3$, it is easy to see that $\left\lceil \frac{|V(G)|+|E(G)|}{2} \right\rceil = \chi_{\rm dom}(M(G))$. Let G be a graph of order $n \geq 4$. Suppose that every graph G' of order $n'(< n)$ has $\left\lceil \frac{|V(G')|+|E(G')|}{2} \right\rceil = \chi_{\rm dom}(M(G'))$.

Now we fix a vertex $v \in V(G)$ and consider $G' := G - \{v\}$. Denote the set of edges between v and G' by $E(v, G')$. Let $C = \{V_1, \ldots, V_k\}$ be a $\chi_{\rm dom}(M(G'))$-coloring of $M(G')$, and let $l := |E(v, G')|$ and $E(v, G') = \{vv_1, vv_2, \ldots, vv_l\}$. It follows from the proof of Claim 1 that at most two vertices in $\{v_1, v_2, \ldots, v_l\}$ belong to the same color class. We divide our consideration into two cases.

Case 1: l is odd.

Without loss of generality, we can assume that $v_i \in V_i$ for each $i \in \{1, \ldots, \frac{l+1}{2}\}$. We show that C is extended to a $\chi_{dom}(M(G))$-coloring of $M(G)$ by adding $\frac{l+1}{2}$ color classes.

Set

$$V_i' := (V_i \setminus \{v_i\}) \cup \{vv_i\}$$

for $1 \leq i \leq \frac{l+1}{2}$,

$$V_{k+i} := \{v_i, vv_{\frac{l+1}{2}+i}\}$$

for $1 \leq i \leq \frac{l+1}{2} - 1$,

$$V_{k+\frac{l+1}{2}} = \{v, v_{\frac{l+1}{2}}\}.$$

Then $\{V_i' \mid 1 \leq i \leq \frac{l+1}{2}\} \cup \{V_{\frac{l+1}{2}+1}, \ldots, V_k\} \cup \{V_{k+i} \mid 1 \leq i \leq \frac{l+1}{2} - 1\} \cup \{V_{k+\frac{l+1}{2}}\}$ is a $\chi_{\rm dom}(M(G))$-coloring of $M(G)$.

Case 2: l is even.

Without loss of generality, we can assume that $v_i \in V_i$ for each $i \in \{1, \ldots, \frac{l}{2}\}$. We show that C is extended to a $\chi_{\rm dom}(M(G))$-coloring of $M(G)$ by adding $\frac{l}{2}$ or $\frac{l}{2} + 1$ color classes. We consider two subcases depending on $|V(G')| + |E(G')|$.

Subcase 2.1: $|V(G')| + |E(G')|$ is even. Note that $|V(G')| + |E(G')| = 2k$.

Set

$$V'_i := (V_i \setminus \{v_i\}) \cup \{vv_i\}$$

for $1 \leq i \leq \frac{l}{2}$,

$$V_{k+i} := \{v_i, vv_{\frac{l}{2}+i}\}$$

for $1 \leq i \leq \frac{l}{2}$,

$$V_{k+\frac{l}{2}+1} = \{v\}.$$

Then $\{V'_i \mid 1 \leq i \leq \frac{l}{2}\} \cup \{V_{\frac{l}{2}+1}, \ldots, V_k\} \cup \{V_{k+i} \mid 1 \leq i \leq \frac{l}{2}\} \cup \{V_{k+\frac{l}{2}+1}\}$ is a $\chi_{\text{dom}}(M(G))$-coloring of $M(G)$.

Subcase 2.2: $|V(G')| + |E(G')|$ is odd. Note that there is only a singleton in C, which is denoted by V_k.

Set

$$V'_i := (V_i \setminus \{v_i\}) \cup \{vv_i\}$$

for $1 \leq i \leq \frac{l}{2}$,

$$V_{k+i} := \{v_i, vv_{\frac{l}{2}+i}\}$$

for $1 \leq i \leq \frac{l}{2}$,

$$V'_k = V_k \cup \{v\}.$$

Then $\{V'_i \mid 1 \leq i \leq \frac{l}{2}\} \cup \{V_{\frac{l}{2}+1}, \ldots, V_{k-1}\} \cup \{V'_k\} \cup \{V_{k+i} \mid 1 \leq i \leq \frac{l}{2}\}$ is a $\chi_{\text{dom}}(M(G))$-coloring of $M(G)$. □

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References

[1] M. Chellali and F. Maffray, Dominator colorings in some classs of graphs, *Graphs Comb.* **28** (2012), 97–107, doi:10.1007/s00373-010-1012-z.

[2] Q. Chen, C. Zhao and M. Zhao, Dominator colorings of certain Cartesian products of paths and cycles, *Graphs Comb.* **33** (2017), 73–83, doi:10.1007/s00373-016-1742-7.

[3] Y. H. Chen, The dominated coloring problem and its application, in: B. Murgante, S. Misra, A. M. A. C. Rocha, C. Torre, J. G. Rocha, M. I. Falcão, D. Taniar, B. O. Apduhan and O. Gervasi (eds.), *Computational Science and Its Applications – ICCSA 2014*, Springer International Publishing, Cham, 2014 pp. 132–145, doi:10.1007/978-3-319-09153-2_10.

[4] R. Gera, On dominator colorings in graphs, *Graph theory notes N. Y.* **52** (2007), 25–30, https://faculty.nps.edu/rgera/research/publications.html.

[5] R. Gera, C. Rasmussen and S. Horton, Dominator colorings and safe clique partitions, *Congr. Numerantium* **181** (2006), 19–32, http://hdl.handle.net/10945/25506.

[6] T. Hamada and I. Yoshimura, Traversability and connectivity of the middle graph of a graph, *Discrete Math.* **14** (1976), 247–255, doi:10.1016/0012-365x(76)90037-6.

[7] F. Kazemnejad, B. Pahlavsay, E. Palezzato and M. Torielli, Total dominator coloring number of middle graphs, *Discrete Math. Algorithms Appl.* (2022), doi:10.1142/s1793830922500768, in press.

[8] S. Klavžar and M. Tavakoli, Dominated and dominator colorings over (edge) corona and hierarchical products, *Appl. Math. Comput.* **390** (2021), 7, doi:10.1016/j.amc.2020.125647, id/No 125647.

[9] H. B. Merouane, M. Haddad, M. Chellali and H. Kheddouci, Dominated colorings of graphs, *Graphs Comb.* **31** (2015), 713–727, doi:10.1007/s00373-014-1407-3.

[10] A. Mohammed Abid and T. R. Ramesh Rao, Dominator coloring of Mycielskian graphs, *Australas. J. Comb.* **73** (2019), 274–279, https://ajc.maths.uq.edu.au/?page=get_volumes&volume=73.