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BOUNDS ON THE GENERALIZED AND THE JOINT SPECTRAL RADIUS OF HADAMARD PRODUCTS OF BOUNDED SETS OF POSITIVE OPERATORS ON SEQUENCE SPACES

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BOUNDS ON THE GENERALIZED AND THE JOINT SPECTRAL RADIUS OF HADAMARD PRODUCTS OF BOUNDED SETS OF POSITIVE OPERATORS ON SEQUENCE SPACES

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Abstract. Recently, K.M.R. Audenaert (2010), R.A. Horn and F. Zhang (2010), Z. Huang (2011) and A.R. Schep (2011) proved inequalities between the spectral radius of Hadamard products of finite and infinite non-negative matrices that define operators on sequence spaces and the spectral radius of their ordinary matrix product. We extend these results to the generalized and the joint spectral radius of bounded sets of such operators. Moreover, we prove new inequalities even in the case of the usual spectral radius of non-negative matrices. We also obtain related results in max algebra.

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1. Introduction

In [29], X. Zhan conjectured that for non-negative $n \times n$ matrices A and B the spectral radius $\rho(A \circ B)$ of the Hadamard product satisfies

$$\rho(A \circ B) \leq \rho(AB),$$

where AB denotes teh usual matrix product of A and B. This conjecture was confirmed by K.M.R. Audenaert in [2] by proving

$$\rho(A \circ B) \leq \rho^{\frac{1}{2}}((A \circ A)(B \circ B)) \leq \rho(AB). \tag{1}$$

These inequalities were established via a trace description of the spectral radius. Using the fact that the Hadamard product is a principal submatrix of the Kronecker product, R.A. Horn and F. Zhang proved in [15] the inequalities

$$\rho(A \circ B) \leq \rho^{\frac{1}{2}}(AB \circ BA) \leq \rho(AB) \tag{2}$$

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and also the right-hand side inequality in (1). Applying the techniques of [15], Huang proved that

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_m) \le \rho(A_1 A_2 \cdots A_m) \tag{3}$$

for $n \times n$ non-negative matrices $A_1, A_2, \cdots, A_m$ (see [16]). In [22] and [23], A.R. Schep extended inequalities (1) and (2) to non-negative matrices that define bounded operators on sequence spaces (in particular on l^p spaces, $1 \le p < \infty$). In the proofs certain results on the Hadamard product from [8] were used. In [22] it was claimed in Theorem 2.7 that

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}((A \circ A)(B \circ B)) \le \rho^{\frac{1}{2}}(AB \circ BA) \le \rho(AB). \tag{4}$$

However, the proof of Theorem 2.7 actually demonstrates that

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}((A \circ A)(B \circ B)) \le \rho^{\frac{1}{2}}(AB \circ AB) \le \rho(AB). \tag{5}$$

It turns out that $\rho(AB \circ BA)$ and $\rho(AB \circ AB)$ may in fact be different and that (4) is false in general. This typing error was corrected in [23].

In this paper we generalize the mentioned results to the setting of the generalized and the joint spectral radius of bounded sets of non-negative matrices that define bounded operators on Banach sequence spaces. Moreover, we also prove new inequalities even in the case of the usual spectral radius of non-negative matrices. In particular, we prove that

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}((A \circ A)(B \circ B)) \le \rho(AB \circ AB)^{\frac{1}{4}} \rho(BA \circ BA)^{\frac{1}{4}} \le \rho(AB)$$

and

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}(AB \circ BA) \le \rho(AB \circ AB)^{\frac{1}{4}} \rho(BA \circ BA)^{\frac{1}{4}} \le \rho(AB)$$

(see Corollary 3.9).

In the last section we also obtain related results in max algebra, which is an attractive setting for describing certain conventionally non-linear problems in a linear fashion.

2. Preliminaries

Throughout the paper, let R denote the set $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$ or the set $\mathbb{N}$ of all natural numbers. Let $S(R)$ be the vector lattice of all complex sequences $(x_i)_{i \in R}$. A Banach space $L \subseteq S(R)$ is called a *Banach sequence space* if $x \in S(R)$, $y \in L$ and $|x| \le |y|$ imply that $x \in L$ and $\|x\|_L \le \|y\|_L$. Note that in the literature such a space L is usually called a Banach function space over a measure space (R, μ), where μ denotes the counting measure on R. The cone of non-negative elements in L is denoted by L_+.

Similarly as in [9] and [21] let us denote by $\mathcal{L}$ the collection of all Banach sequence spaces L satisfying the property that $e_i = \chi_{\{i\}} \in L$ and $\|e_i\|_L = 1$ for all $i \in R$. Standard examples of spaces from $\mathcal{L}$ are Euclidean spaces, the well known l^p spaces ($1 \le p \le \infty$) and the space c_0 of all null convergent sequences, equipped with the usual norms. The set $\mathcal{L}$ also contains all cartesian products $L = X \times Y$ for $X, Y \in \mathcal{L}$, equipped with the norm $\|(x,y)\|_L = \max\{\|x\|_X, \|y\|_Y\}$.

A matrix $A = [a_{ij}]_{i,j\in R}$ is called *non-negative* if $a_{ij} \geq 0$ for all $i, j \in R$. Given matrices A and B, we write $A \leq B$ if the matrix $B - A$ is non-negative.

By an *operator* on a Banach sequence space L we always mean a linear operator on L. We say that a non-negative matrix A defines an operator on L if $Ax \in L$ for all $x \in L$, where $(Ax)_i = \sum_{j\in R} a_{ij}x_j$. Then $Ax \in L_+$ for all $x \in L_+$ and so A defines a *positive* operator on L. Recall that this operator is always bounded, i.e., its operator norm

$$\|A\| = \sup\{\|Ax\| : x \in L_+, \|x\| \leq 1\}$$

is finite. Also, its spectral radius $\rho(A)$ is always contained in the spectrum. For the theory of Banach function spaces, Banach lattices and positive operators we refer the reader to the books [28], [17] and [1].

Given non-negative matrices $A = [a_{ij}]_{i,j\in R}$ and $B = [b_{ij}]_{i,j\in R}$, let $A \circ B = [a_{ij}b_{ij}]_{i,j\in R}$ be the *Hadamard (or Schur) product* of A and B and let $A^{(t)} = [a_{ij}^t]_{i,j\in R}$ be the *Hadamard (or Schur) power* of A for $t \geq 0$. Here we use the convention $0^0 = 1$.

Let Σ be a bounded set of bounded operators on L. For $m \geq 1$, let

$$\Sigma^m = \{A_1A_2\cdots A_m : A_i \in \Sigma\}.$$

The generalized spectral radius of Σ is defined by

$$\rho(\Sigma) = \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} \rho(A)]^{1/m} \tag{6}$$

and is equal to

$$\rho(\Sigma) = \sup_{m\in\mathbb{N}} [\sup_{A\in\Sigma^m} \rho(A)]^{1/m}.$$

The joint spectral radius of Σ is defined by

$$\hat{\rho}(\Sigma) = \lim_{m\to\infty} [\sup_{A\in\Sigma^m} \|A\|]^{1/m}. \tag{7}$$

It is well known that $\rho(\Sigma) = \hat{\rho}(\Sigma)$ for a precompact set Σ of compact operators on L (see e.g. [25], [26]), in particular for a bounded set of complex $n \times n$ matrices (see e.g. [5], [10], [24], [7]). This equality is called the Berger-Wang formula or also the generalized spectral radius theorem (for a new elegant proof in the finite dimensional case see [7]). However, in general $\rho(\Sigma)$ and $\hat{\rho}(\Sigma)$ may differ even in the case of a bounded set Σ of compact positive operators on L as the following example from [24] shows. Let $\Sigma = \{A_1, A_2, \ldots\}$ be a bounded set of compact operators on $L = l^2$ defined by $A_ke_k = e_{k+1}, (k \in \mathbb{N})$ and $A_ke_j = 0$ for $j \neq k$. Then $(A_{i_1}A_{i_2}\cdots A_{i_k})^2 = 0$ for arbitrary $k \in \mathbb{N}$ and any subset $\{i_1, i_2, \ldots, \imath_k\} \subset \mathbb{N}$. Thus $\rho(\Sigma) = 0$. Since

$$A_mA_{m-1}\cdots A_1e_1 = e_{m+1}, \quad m \in \mathbb{N},$$

$$A_mA_{m-1}\cdots A_1e_j = 0, \quad j \neq 1,$$

we have $\hat{\rho}(\Sigma) \geq \limsup_{m\to\infty} \|A_m \cdots A_1\|^{1/m} = 1$ and so $\rho(\Sigma) \neq \hat{\rho}(\Sigma)$.

In [13], the reader can find an example of two positive non-compact weighted shifts A and B on $L = l^2$ such that $\rho(\{A, B\}) = 0 < \hat{\rho}(\{A, B\})$.

The theory of the generalized and the joint spectral radius has many important applications for instance to discrete and differential inclusions, wavelets, invariant subspace theory (see e.g. [5], [7], [27], [25], [26] and the references cited there). In particular, $\hat{\rho}(\Sigma)$ plays a central role in determining stability in convergence properties of discrete and differential inclusions. In this theory the quantity $\log \hat{\rho}(\Sigma)$ is known as the maximal Lyapunov exponent (see e.g. [27]).

We will frequently use the following well known fact that

$$\rho(\Psi\Sigma) = \rho(\Sigma\Psi) \text{ and } \hat{\rho}(\Psi\Sigma) = \hat{\rho}(\Sigma\Psi),$$

where $\Psi\Sigma = \{AB : A \in \Psi, B \in \Sigma\}$.

3. Results on the generalized and joint spectral radius

Before proving our main results, let us first state some results that we will need in our proofs. The following result was proved in [8, Theorem 3.3] and [19, Theorem 5.1 and Remark 5.2] using only basic analythic methods and elementary facts.

Theorem 3.1. *Given $L \in \mathcal{L}$, let $\{A_{ij}\}_{i=1,j=1}^{k,m}$ be non-negative matrices that define operators on L. If α_1, α_2,..., α_m are positive numbers such that $\sum_{i=1}^m \alpha_i \geq 1$, then the matrix $\left(A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}\right) \ldots \left(A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)}\right)$ also defines an operator on L and satisfies the inequalities*

(8)
$$\left(A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}\right) \ldots \left(A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)}\right) \leq (A_{11} \cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m} \cdots A_{km})^{(\alpha_m)},$$

(9)
$$\left\|\left(A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}\right) \ldots \left(A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)}\right)\right\| \leq \|A_{11} \cdots A_{k1}\|^{\alpha_1} \cdots \|A_{1m} \cdots A_{km}\|^{\alpha_m}$$

and

(10)
$$\rho\left(\left(A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}\right) \ldots \left(A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)}\right)\right) \leq \rho\left(A_{11} \cdots A_{k1}\right)^{\alpha_1} \cdots \rho\left(A_{1m} \cdots A_{km}\right)^{\alpha_m}.$$

The following special case of Theorem 3.1 was considered in the finite dimensional case by several authours using different methods (for references see e.g. [9], [8], [19]).

Corollary 3.2. *Given $L \in \mathcal{L}$, let $A_1, \ldots, A_m$ be non-negative matrices that define operators on L and α_1, α_2,..., α_m positive numbers such that $\sum_{i=1}^m \alpha_i \geq 1$. Then we have*

$$\|A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}\| \leq \|A_1\|^{\alpha_1} \|A_2\|^{\alpha_2} \cdots \|A_m\|^{\alpha_m}$$

and

$$\rho(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}) \leq \rho(A_1)^{\alpha_1}\, \rho(A_2)^{\alpha_2} \cdots \rho(A_m)^{\alpha_m}. \tag{11}$$

We will also need the following result.

Proposition 3.3. *Given $L \in \mathcal{L}$, let A, B, C and D be non-negative matrices that define operators on L. Then the following inequalities hold*

$$(A \circ B)(C \circ D) \le (A^{(2)} D^{(2)})^{(\frac{1}{2})} \circ (B^{(2)} C^{(2)})^{(\frac{1}{2})}, \tag{12}$$

$$(A \circ B)(C \circ D) \le AC \circ BD, \tag{13}$$

$$(A \circ B)(C \circ D) \le AD \circ BC. \tag{14}$$

Proof. Using $A = (A^{(2)})^{(\frac{1}{2})}$ (and similarly for B, C and D) and applying (8) we obtain

$$(A \circ B)(C \circ D) = (A \circ B)(D \circ C) \le (A^{(2)} D^{(2)})^{(\frac{1}{2})} \circ (B^{(2)} C^{(2)})^{(\frac{1}{2})},$$

which proves (12) (for a simple direct proof see [22, Proposition 2.3]).

The inequality (13) is a special case of (8). We include a simple proof for completeness. The (i,j)th entry of the matrix $(A \circ B)(C \circ D)$ equals $\sum_{k \in R} a_{ik} b_{ik} c_{kj} d_{kj}$ and we have

$$\sum_{k \in R} a_{ik} b_{ik} c_{kj} d_{kj} = \sum_{k \in R} (a_{ik} c_{kj})(b_{ik} d_{kj}) \le \sum_{k \in R} a_{ik} c_{kj} \sum_{k \in R} b_{ik} d_{kj},$$

which proves (13).

By (13) we have

$$(A \circ B)(C \circ D) = (A \circ B)(D \circ C) \le AD \circ BC,$$

which proves (14). □

Let Ψ and Σ be sets of non-negative matrices and $\alpha > 0$. Then $\Psi \circ \Sigma$ and $\Psi^{(\alpha)}$ denote respectively the *Hadamard (Schur) product* of Ψ and Σ and the *Hadamard (Schur) power* of Ψ, e.g.,

$$\Psi \circ \Sigma = \{A \circ B : A \in \Psi, B \in \Sigma\} \text{ and } \Psi^{(\alpha)} = \{A^{(\alpha)} : A \in \Psi\}.$$

The following result on the generalized and the joint spectral radius was stated in ([19, Corollary 5.3]) only in the case of bounded sets of $n \times n$ non-negative matrices, however the same proof works in our more general setting by using Theorem 3.1.

Theorem 3.4. *Given $L \in \mathcal{L}$, let $\Psi_1, \ldots \Psi_m$ be bounded sets of non-negative matrices that define operators on L and let $\alpha_1, \ldots \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i \ge 1$. Then we have*

$$\rho(\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)}) \le \rho(\Psi_1)^{\alpha_1} \cdots \rho(\Psi_m)^{\alpha_m} \tag{15}$$

and

$$\hat{\rho}(\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)}) \le \hat{\rho}(\Psi_1)^{\alpha_1} \cdots \hat{\rho}(\Psi_m)^{\alpha_m}. \tag{16}$$

We are now ready to prove the following result for the generalized and joint spectral radius, which generalizes (5).

Theorem 3.5. *Given $L \in \mathcal{L}$, let Ψ and Σ be bounded sets of non-negative matrices that define operators on L. Then we have*

$$\rho(\Psi \circ \Sigma) \le \rho(\Psi^{(2)}\Sigma^{(2)})^{\frac{1}{2}} \le \rho((\Psi \circ \Psi)(\Sigma \circ \Sigma))^{\frac{1}{2}} \le \rho(\Psi\Sigma \circ \Psi\Sigma)^{\frac{1}{2}} \le \rho(\Psi\Sigma) \tag{17}$$

and

$$\hat{\rho}(\Psi \circ \Sigma) \le \hat{\rho}(\Psi^{(2)}\Sigma^{(2)})^{\frac{1}{2}} \le \hat{\rho}((\Psi \circ \Psi)(\Sigma \circ \Sigma))^{\frac{1}{2}} \le \hat{\rho}(\Psi\Sigma \circ \Psi\Sigma)^{\frac{1}{2}} \le \hat{\rho}(\Psi\Sigma). \tag{18}$$

Proof. To prove the first inequality in (17), choose $A \in (\Psi \circ \Sigma)^{2m}$. Then there exist $A_i \in \Psi$ and $B_i \in \Sigma$ for $i = 1, \ldots, 2m$, such that

$$A = (A_1 \circ B_1)(A_2 \circ B_2) \cdots (A_{2m-1} \circ B_{2m-1})(A_{2m} \circ B_{2m}).$$

By (12) and (8) we have

$$\begin{aligned} A &\le ((A_1^{(2)}B_2^{(2)})^{(\frac{1}{2})} \circ (B_1^{(2)}A_2^{(2)})^{(\frac{1}{2})}) \cdots ((A_{2m-1}^{(2)}B_{2m}^{(2)})^{(\frac{1}{2})} \circ (B_{2m-1}^{(2)}A_{2m}^{(2)})^{(\frac{1}{2})}) \\ &\le B^{(\frac{1}{2})} \circ C^{(\frac{1}{2})}, \end{aligned}$$

where

$$B = A_1^{(2)}B_2^{(2)} \cdots A_{2m-1}^{(2)}B_{2m}^{(2)} \in (\Psi^{(2)}\Sigma^{(2)})^m$$

and

$$C = B_1^{(2)}A_2^{(2)} \cdots B_{2m-1}^{(2)}A_{2m}^{(2)} \in (\Sigma^{(2)}\Psi^{(2)})^m.$$

By Corollary 3.2 we obtain $\rho(A) \le \rho(B)^{\frac{1}{2}}\rho(C)^{\frac{1}{2}}$. This implies

$$\rho(\Psi \circ \Sigma)^2 \le \rho(\Psi^{(2)}\Sigma^{(2)})^{1/2}\rho(\Sigma^{(2)}\Psi^{(2)})^{1/2}.$$

Therefore

$$\rho(\Psi \circ \Sigma)^2 \le \rho(\Psi^{(2)}\Sigma^{(2)}),$$

since $\rho(\Psi^{(2)}\Sigma^{(2)}) = \rho(\Sigma^{(2)}\Psi^{(2)})$. This proves the first inequality in (17).

The second inequality in (17) is trivial, since $\Psi^{(2)}\Sigma^{(2)} \subset (\Psi \circ \Psi)(\Sigma \circ \Sigma)$.

For the proof of the third inequality in (17) take $A \in ((\Psi \circ \Psi)(\Sigma \circ \Sigma))^m$. Then there exist $A_i, B_i \in \Psi$ and $C_i, D_i \in \Sigma$ for $i = 1, \ldots, m$ such that

$$A = (A_1 \circ B_1)(C_1 \circ D_1) \cdots (A_m \circ B_m)(C_m \circ D_m).$$

By (13) we have

$$A \le (A_1C_1 \circ B_1D_1) \ldots (A_mC_m \circ B_mD_m) \in (\Psi\Sigma \circ \Psi\Sigma)^m,$$

which implies $\rho((\Psi \circ \Psi)(\Sigma \circ \Sigma)) \le \rho(\Psi\Sigma \circ \Psi\Sigma)$. The last inequality in (17) follows from Theorem 3.4. This completes the proof of (17) and the inequalities (18) are proved simply by replacing the spectral radius with the operator norm $\|\cdot\|$ in the proof above. $\square$

If we interchange the roles of Ψ and Σ in Theorem 3.5, we obtain the following result.

Corollary 3.6. *Given $L \in \mathcal{L}$, let Ψ and Σ be bounded sets of non-negative matrices that define operators on L. Then we have*

$$\rho((\Psi \circ \Psi)(\Sigma \circ \Sigma))^{\frac{1}{2}} \leq \rho(\Sigma\Psi \circ \Sigma\Psi)^{\frac{1}{2}} \leq \rho(\Psi\Sigma)$$

and

$$\hat{\rho}((\Psi \circ \Psi)(\Sigma \circ \Sigma))^{1/2} \leq \hat{\rho}(\Sigma\Psi \circ \Sigma\Psi)^{1/2} \leq \hat{\rho}(\Psi\Sigma).$$

The following result generalizes and sharpens the inequalities (2).

Theorem 3.7. *Given $L \in \mathcal{L}$, let Ψ and Σ be bounded sets of non-negative matrices that define operators on L. Then we have*

(19)
$$\rho(\Psi \circ \Sigma) \leq \rho(\Psi\Sigma \circ \Sigma\Psi)^{\frac{1}{2}} \leq \rho((\Psi\Sigma)^{(2)})^{\frac{1}{4}}\rho((\Sigma\Psi)^{(2)})^{\frac{1}{4}} \leq \rho(\Psi\Sigma \circ \Psi\Sigma)^{\frac{1}{4}}\rho(\Sigma\Psi \circ \Sigma\Psi)^{\frac{1}{4}} \leq \rho(\Psi\Sigma)$$

and

(20)
$$\hat{\rho}(\Psi \circ \Sigma) \leq \hat{\rho}(\Psi\Sigma \circ \Sigma\Psi)^{\frac{1}{2}} \leq \hat{\rho}((\Psi\Sigma)^{(2)})^{\frac{1}{4}}\hat{\rho}((\Sigma\Psi)^{(2)})^{\frac{1}{4}} \leq \hat{\rho}(\Psi\Sigma \circ \Psi\Sigma)^{\frac{1}{4}}\hat{\rho}(\Sigma\Psi \circ \Sigma\Psi)^{\frac{1}{4}} \leq \hat{\rho}(\Psi\Sigma).$$

Proof. For the proof of the first inequality in (19), choose $A \in (\Psi \circ \Sigma)^{2m}$. Then there exist $A_i \in \Psi$ and $B_i \in \Sigma$ for $i = 1, \dots, 2m$, such that

$$A = (A_1 \circ B_1)(A_2 \circ B_2) \cdots (A_{2m-1} \circ B_{2m-1})(A_{2m} \circ B_{2m}).$$

By (14) we obtain

$$A \leq (A_1B_2 \circ B_1A_2) \cdots (A_{2m-1}B_{2m} \circ B_{2m-1}A_{2m}) \in (\Psi\Sigma \circ \Sigma\Psi)^m$$

and thus

$$\rho(A) \leq \rho((A_1B_2 \circ B_1A_2) \cdots (A_{2m-1}B_{2m} \circ B_{2m-1}A_{2m})).$$

This implies the first inequality in (19).

Also $A_iB_{i+1} \circ B_iA_{i+1} = ((A_iB_{i+1})^{(2)} \circ (B_iA_{i+1})^{(2)})^{(\frac{1}{2})}$ for $i = 1, \dots, 2m-1$ and thus we have by (8)

$$(A_1B_2 \circ B_1A_2) \cdots (A_{2m-1}B_{2m} \circ B_{2m-1}A_{2m}) \leq$$
$$((A_1B_2)^{(2)} \cdots (A_{2m-1}B_{2m})^{(2)})^{(\frac{1}{2})} \circ ((B_1A_2)^{(2)} \cdots (B_{2m-1}A_{2m})^{(2)})^{(\frac{1}{2})}$$

By (11) we obtain

$$\rho((A_1B_2 \circ B_1A_2) \cdots (A_{2m-1}B_{2m} \circ B_{2m-1}A_{2m})) \leq \rho(B)^{\frac{1}{2}}\rho(C)^{\frac{1}{2}},$$

where

$$B = (A_1B_2)^{(2)} \cdots (A_{2m-1}B_{2m})^{(2)} \in ((\Psi\Sigma)^{(2)})^m$$

and

$$C = (B_1A_2)^{(2)} \cdots (B_{2m-1}A_{2m})^{(2)} \in ((\Sigma\Psi)^{(2)})^m.$$

This implies

$$\rho(\Psi\Sigma \circ \Sigma\Psi) \leq \rho((\Psi\Sigma)^{(2)})^{\frac{1}{2}}\rho((\Sigma\Psi)^{(2)})^{\frac{1}{2}},$$

which proves the second inequality in (19).

The third inequality in (19)is trivial, since $(\Psi\Sigma)^{(2)} \subset \Psi\Sigma \circ \Psi\Sigma$.

The fourth inequality in (19) follows from Theorem 3.4 since

$$\rho(\Psi\Sigma \circ \Psi\Sigma)^{\frac{1}{4}} \rho(\Sigma\Psi \circ \Sigma\Psi)^{\frac{1}{4}} \le \rho(\Psi\Sigma)^{\frac{1}{2}} \rho(\Sigma\Psi)^{\frac{1}{2}} = \rho(\Psi\Sigma).$$

The inequalites (20) are proved by replacing the spectral radius with the operator norm $\|\cdot\|$ in the proof above. □

The following result complements Theorems 3.5 and 3.7.

Proposition 3.8. *Given $L \in \mathcal{L}$, let Ψ and Σ be bounded sets of non-negative matrices that define operators on L. Then we have*

$$\rho((\Psi \circ \Psi)(\Sigma \circ \Sigma)) \le \rho(\Psi\Sigma \circ \Psi\Sigma)^{\frac{1}{2}} \rho(\Sigma\Psi \circ \Sigma\Psi)^{\frac{1}{2}} \tag{21}$$

and

$$\hat{\rho}((\Psi \circ \Psi)(\Sigma \circ \Sigma)) \le \hat{\rho}(\Psi\Sigma \circ \Psi\Sigma)^{\frac{1}{2}} \hat{\rho}(\Sigma\Psi \circ \Sigma\Psi)^{\frac{1}{2}}. \tag{22}$$

Proof. By Theorem 3.5 we have

$$\rho((\Psi \circ \Psi)(\Sigma \circ \Sigma)) = \rho((\Psi \circ \Psi)(\Sigma \circ \Sigma))^{\frac{1}{2}} \rho((\Sigma \circ \Sigma)(\Psi \circ \Psi))^{\frac{1}{2}} \le \rho(\Psi\Sigma \circ \Psi\Sigma)^{\frac{1}{2}} \rho(\Sigma\Psi \circ \Sigma\Psi)^{\frac{1}{2}},$$

which proves (21). The inequality (22) is proved similarly. □

The following result follows from Theorem 3.5, Theorem 3.7 and Proposition 3.8 by taking $\Psi = \{A\}$ and $\Sigma = \{B\}$.

Corollary 3.9. *Given $L \in \mathcal{L}$, let A and B be non-negative matrices that define operators on L. Then we have*

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}((A \circ A)(B \circ B)) \le \rho(AB \circ AB)^{\frac{1}{4}} \rho(BA \circ BA)^{\frac{1}{4}} \le \rho(AB) \tag{23}$$

and

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}(AB \circ BA) \le \rho(AB \circ AB)^{\frac{1}{4}} \rho(BA \circ BA)^{\frac{1}{4}} \le \rho(AB). \tag{24}$$

The following example shows that $\rho(AB \circ AB)$, $\rho(BA \circ BA)$ and $\rho(AB \circ BA)$ may in fact be different.

Example 3.10. Let $A = \begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 3 & 3 \end{bmatrix}$. Then $AB = \begin{bmatrix} 6 & 6 \\ 9 & 12 \end{bmatrix}$, $BA = \begin{bmatrix} 3 & 3 \\ 9 & 15 \end{bmatrix}$ and so

$$AB \circ AB = \begin{bmatrix} 36 & 36 \\ 81 & 144 \end{bmatrix}, \quad BA \circ BA = \begin{bmatrix} 9 & 9 \\ 81 & 225 \end{bmatrix}, \quad AB \circ BA = \begin{bmatrix} 18 & 18 \\ 81 & 180 \end{bmatrix}.$$

It follows $\rho(AB \circ AB) = 18(5 + 3\sqrt{2}) \doteq 166.368$, $\rho(BA \circ BA) = 9(13 + 3\sqrt{17}) \doteq 228.324$ and $\rho(AB \circ BA) = 9(11 + 3\sqrt{11}) \doteq 188.549$. We see that $\rho(AB \circ AB) < \rho(AB \circ BA) < \rho(BA \circ BA)$ and thus in general neither of the inequalities between $\rho(AB \circ AB)$ and

$\rho(AB \circ BA)$ hold. This fact was independently observed by A. Schep ([23] and private communication).

Remark 3.11. The previous example also shows that the second inequality in (24) may be strict. This is also true for the second inequality in (23), since $\rho((A \circ A)(B \circ B)) = 9(7 + 3\sqrt{5}) \doteq 123.374$.

On the otherhand both of the mentioned inequalities are sharp (take e.g. $A = B = I$ or 0).

The following result follows from Theorem 3.5.

Proposition 3.12. *Given $L \in \mathcal{L}$, let Ψ and Σ be bounded sets of non-negative matrices that define operators on L. Then we have*

$$\rho(\Psi\Sigma \circ \Sigma\Psi) \le \rho(\Psi^2\Sigma^2) \ \textit{and} \ \hat{\rho}(\Psi\Sigma \circ \Sigma\Psi) \le \hat{\rho}(\Psi^2\Sigma^2). \tag{25}$$

Proof. By Theorem 3.5 applied to $\Psi\Sigma$ and $\Sigma\Psi$ we have

$$\rho(\Psi\Sigma \circ \Sigma\Psi) \le \rho(\Psi\Sigma\Sigma\Psi) = \rho(\Psi^2\Sigma^2),$$

which proves the first inequality in (25). The second inequality in (25) is proved similarly. □

Corollary 3.13. *Given $L \in \mathcal{L}$, let A and B be non-negative matrices that define operators on L. Then we have*

$$\rho(AB \circ BA) \le \rho(A^2B^2). \tag{26}$$

The inequality (26) is not weaker than the inequality (24) as the following example from [23] shows.

Example 3.14. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = AB \circ AB = (A \circ A)(B \circ B) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $BA = BA \circ BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, while $AB \circ BA = A^2 = B^2 = 0$. Therefore $\rho(AB \circ BA) = \rho(A^2B^2) = 0$, while $\rho(AB) = \rho((A \circ A)(B \circ B)) = \rho(AB \circ AB) = \rho(BA \circ BA) = 1$.

This example also illustrates that in (26) we can not replace $AB \circ BA$ by $(A \circ A)(B \circ B)$ or $AB \circ AB$.

To conclude this section we prove the following generalization of the inequality (3).

Theorem 3.15. *Given $L \in \mathcal{L}$, let $\Psi_1, \Psi_2, \ldots, \Psi_m$ be bounded sets of non-negative matrices that define operators on L. Then we have*

$$\rho(\Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_m) \le \rho(\Psi_1\Psi_2 \cdots \Psi_m) \tag{27}$$

and

$$\hat{\rho}(\Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_m) \le \hat{\rho}(\Psi_1\Psi_2 \cdots \Psi_m). \tag{28}$$

Proof. Take $A \in (\Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_m)^{mk}$. Then $A = A_1 A_2 \cdots A_k$, where

$$A_i = (A_{i\,1\,1} \circ A_{i\,1\,2} \circ \cdots \circ A_{i\,1\,m})\,(A_{i\,2\,1} \circ A_{i\,2\,2} \circ \cdots \circ A_{i\,2\,m}) \ldots (A_{im1} \circ A_{im2} \circ \cdots \circ A_{imm})$$

for some $A_{i\,j\,1} \in \Psi_1, \ldots, A_{i\,j\,m} \in \Psi_m$ and all $j = 1, \ldots, m$, $i = 1, \ldots, k$. Then

$$A_i = (A_{i\,1\,1} \circ A_{i\,1\,2} \circ \cdots \circ A_{i\,1\,m})\,(A_{i\,2\,2} \circ \cdots \circ A_{i\,2\,m} \circ A_{i\,2\,1}) \ldots (A_{i\,m\,m} \circ A_{i\,m\,1} \circ \cdots \circ A_{i\,m\,m-1})\,.$$

By (8) we have

$$A = A_1 A_2 \cdots A_k \le B_1 \circ B_2 \circ \cdots \circ B_m,$$

where

$$B_1 = \prod_{i=1}^{k} A_{i\,1\,1} A_{i\,2\,2} \cdots A_{i\,m\,m} \in (\Psi_1 \Psi_2 \cdots \Psi_m)^k,$$

$$B_2 = \prod_{i=1}^{k} A_{i\,1\,2} A_{i\,2\,3} \cdots A_{i\,m\,1} \in (\Psi_2 \Psi_3 \cdots \Psi_1)^k,$$

$$\cdots\cdots\cdots$$

$$B_m = \prod_{i=1}^{k} A_{i\,1\,m} A_{i\,2\,1} \cdots A_{i\,m\,m-1} \in (\Psi_m \Psi_1 \cdots \Psi_{m-1})^k.$$

By Corollary 3.2 we have

$$\rho(A) \le \rho(B_1)\rho(B_2) \cdots \rho(B_m),$$

which implies

$$\rho(\Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_m)^m \le \rho(\Psi_1 \Psi_2 \cdots \Psi_m)\rho(\Psi_2 \Psi_3 \cdots \Psi_1) \cdots \rho(\Psi_m \Psi_1 \cdots \Psi_{m-1}) =$$
$$= \rho(\Psi_1 \Psi_2 \cdots \Psi_m)^m.$$

This proves (27) and the inequality (28) is proved similarly. □

4. Related results in max algebra

In this final section we prove some related results in max algebra. The algebraic system max algebra and its isomorphic versions provide an attractive way of describing a class of non-linear problems appearing for instance in manufacturing and transportation scheduling, information technology, discrete event-dynamic systems, combinatorial optimisation, mathematical physics, DNA analysis, ...(see e.g. [4], [6], [3], [21] and the references cited there). Max algebra's usefulness arises from a fact that these non-linear problems become linear when described in the max algebra language. Moreover, recently max algebra techniques were used to solve certain linear algebra problems (see e.g. [11], [12]).

The max algebra consists of the set of non-negative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product ab, where $a, b \ge 0$. The operations between matrices and vectors in the max algebra are defined by analogy with the usual linear algebra. For instance, the product of $n \times n$ non-negative matrices A and B in the max algebra is denoted by

$A \otimes B$ (not the Kronecker product), where $[A\otimes B]_{ij} = \max_{k=1,\ldots,n} a_{ik}b_{kj}$. The notation $A^2_\otimes$ means $A\otimes A$, and $A^k_\otimes$ denotes the k-th max power of A. If $x = [x_i] \in \mathbb{R}^n$ is a non-negative vector, then the notation $A \otimes x$ means $[A \otimes x]_i = \max_{j=1,\ldots,n} a_{ij}x_j$. The usual associative and distributive laws hold in this algebra. The role of the spectral radius in max algebra is played by the maximum circuit geometric mean.

The weighted directed graph $\mathcal{D}(A)$ associated with A has a vertex set $\{1, 2, \ldots , n\}$ and edges (i, j) from a vertex i to a vertex j with weight a_{ij} if and only if $a_{ij} > 0$. A path of length k is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots , (i_k, i_{k+1})$. A circuit of length k is a path with $i_{k+1} = i_1$, where $i_1, i_2, \ldots , i_k$ are distinct. Associated with this circuit is the *circuit geometric mean* known as $(a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1})^{1/k}$. The maximum circuit geometric mean in $\mathcal{D}(A)$ is denoted by $\mu(A)$. Note that circuits (i_1, i_1) of length 1 (loops) are included here and that we also consider empty circuits, i.e., circuits that consist of only one vertex and have length 0. For empty circuits, the associated circuit geometric mean is zero.

There are many different descriptions of the maximum circuit geometric mean $\mu(A)$ (see e.g. [18] and the references cited there). It is known that $\mu(A)$ is the largest max eigenvalue of A. Moreover, if A is irreducible, then $\mu(A)$ is the unique max eigenvalue and every max eigenvector is positive (see e.g. [4, Theorem 2], [6], [3]). Also, the max version of Gelfand formula holds, i.e.,

$$\mu(A) = \lim_{m\to\infty} \|A^m_\otimes\|^{1/m}$$

for an arbitrary vector norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ (see e.g. [18] and the references cited there). Thus $\mu(A^k_\otimes) = \mu(A)^k$ for all $k \in \mathbb{N}$.

Let Ψ be a bounded set of $n \times n$ non-negative matrices. For $m \geq 1$, let

$$\Psi^m_\otimes = \{A_1 \otimes A_2 \otimes \cdots \otimes A_m : A_i \in \Psi\}.$$

The max algebra version of the generalized spectral radius $\mu(\Psi)$ of Ψ, is defined by

$$\mu(\Psi) = \limsup_{m\to\infty} [\sup_{A\in\Psi^m_\otimes} \mu(A)]^{1/m}$$

and is equal to

$$\mu(\Psi) = \sup_{m\in\mathbb{N}} [\sup_{A\in\Psi^m_\otimes} \mu(A)]^{1/m}.$$

Also the max algebra version of the Berger-Wang formula holds, i.e.,

$$\mu(\Psi) = \lim_{m\to\infty} [\sup_{A\in\Psi^m_\otimes} \|A\|]^{1/m}$$

for an arbitrary vector norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ (see e.g. [18]). The quantity $\log \mu(\Psi)$ measures the worst case cycle time of certain discrete event systems and it is sometimes called the worst case Lyapunov exponent (see e.g. [3], [20], [14] and the references cited there).

It is not difficult to see that

$$A^{(t)}_1 \otimes \cdots \otimes A^{(t)}_m = (A_1 \otimes \cdots \otimes A_m)^{(t)} \tag{29}$$

for all $n\times n$ non-negative matrices $A_1, \dots, A_m$ and $t > 0$. This implies that $\mu(\Psi^{(t)}) = \mu(\Psi)^t$ for all $t > 0$ (see also [18]). We also have $\mu(\Psi \otimes \Sigma) = \mu(\Sigma \otimes \Psi)$, where
$\Psi \otimes \Sigma = \{A \otimes B : A \in \Psi, B \in \Sigma\}$.

For proving the max algebra analogues of the results from the previous section we will need the following result, which was essentially proved in [19]. It follows from [19, Theorem 5.1 and Remark 5.2] and (29).

Theorem 4.1. *Let $\{A_{ij}\}_{i=1,j=1}^{k,m}$ be $n \times n$ non-negative matrices and let α_1, α_2,..., α_m be positive numbers. Then we have*

$$\left(A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}\right) \otimes \ldots \otimes \left(A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)}\right)$$
$$\le (A_{11} \otimes \cdots \otimes A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m} \otimes \cdots \otimes A_{km})^{(\alpha_m)}$$

and

$$\mu\left(\left(A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}\right) \otimes \ldots \otimes \left(A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)}\right)\right)$$
$$\le \mu\left(A_{11} \otimes \cdots \otimes A_{k1}\right)^{\alpha_1} \cdots \mu\left(A_{1m} \otimes \cdots \otimes A_{km}\right)^{\alpha_m}.$$

The following result on the max version of the generalized spectral radius was already stated in [19, Corollary 5.3].

Corollary 4.2. *Let $\Psi_1, \dots \Psi_m$ be bounded sets of $n \times n$ non-negative matrices and let $\alpha_1, \dots \alpha_m$ be positive numbers.*

$$\mu(\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)}) \le \mu(\Psi_1)^{\alpha_1} \cdots \mu(\Psi_m)^{\alpha_m}. \tag{30}$$

In contrast with the linear algebra case, we also have the following result.

Corollary 4.3. *If Ψ is a bounded set of $n \times n$ non-negative matrices, then*

$$\mu(\Psi \circ \Psi) = \mu(\Psi)^2. \tag{31}$$

Proof. Since $\Psi^{(2)} \subset \Psi \circ \Psi$, we have

$$\mu(\Psi^{(2)}) \le \mu(\Psi \circ \Psi) \le \mu(\Psi)^2,$$

where the second inequality follows from (30). Since also $\mu(\Psi^{(2)}) = \mu(\Psi)^2$, the result follows. $\square$

By replacing the sums with $\max_{k=1,\dots,n}$ in the proof of (13) and (14), we obtain the following result.

Proposition 4.4. *Let A, B, C and D be $n \times n$ non-negative matrices. Then we have*

$$(A \circ B) \otimes (C \circ D) \le A \otimes C \circ B \otimes D \ \text{ and } \ (A \circ B) \otimes (C \circ D) \le A \otimes D \circ B \otimes C. \tag{32}$$

Remark 4.5. It is easy to verify that

$$A \otimes C = (A^{(2)} \otimes C^{(2)})^{(\frac{1}{2})} \tag{33}$$

for $n \times n$ non-negative matrices A and C. Thus the max version of (12) is only a restatement of (32).

By applying Theorem 4.1 an Corollary 4.2 we can now prove the following max algebra version of Theorem 3.15. We omit the proof, since it is similar to the proof of Theorem 3.15.

Theorem 4.6. *If $\Psi_1, \Psi_2, \ldots, \Psi_m$ are bounded sets of $n \times n$ non-negative matrices, then*

$$\mu(\Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_m) \leq \mu(\Psi_1 \otimes \Psi_2 \otimes \cdots \otimes \Psi_m)$$

Moreover, applying Proposition 4.4 and Corollary 4.2 we can also prove the following max algebra analogue of Theorem 3.7 and Proposition 3.12. The proof is similar as in the previous section and we omit the details.

Theorem 4.7. *If Ψ and Σ are bounded sets of $n \times n$ non-negative matrices, then we have*

$$\mu(\Psi \circ \Sigma) \leq \mu(\Psi \otimes \Sigma \circ \Sigma \otimes \Psi)^{1/2} \leq \mu(\Psi \otimes \Sigma)$$

and

$$\mu(\Psi \otimes \Sigma \circ \Sigma \otimes \Psi) \leq \mu(\Psi_{\otimes}^2 \otimes \Sigma_{\otimes}^2).$$

For single matrices we obtain the following result.

Corollary 4.8. *Let $A_1, \ldots, A_m, A, B$ be $n \times n$ non-negative matrices. Then the following inequalities hold:*

$$\mu(A_1 \circ \cdots \circ A_m) \leq \mu(A_1 \otimes \cdots \otimes A_m),$$

$$\mu(A \circ B) \leq \mu(A \otimes B \circ B \otimes A)^{1/2} \leq \mu(A \otimes B), \tag{34}$$

$$\mu(A \otimes B \circ B \otimes A) \leq \mu(A_{\otimes}^2 \otimes B_{\otimes}^2). \tag{35}$$

Remark 4.9. Let Ψ and Σ be bounded sets of $n \times n$ non-negative matrices. In contrast with the linear algebra case, we have the following

$$\mu(\Psi^{(2)} \otimes \Sigma^{(2)})^{\frac{1}{2}} = \mu((\Psi \circ \Psi) \otimes (\Sigma \circ \Sigma))^{\frac{1}{2}} = \mu(\Psi \otimes \Sigma \circ \Psi \otimes \Sigma)^{\frac{1}{2}} = \mu(\Sigma \otimes \Psi \circ \Sigma \otimes \Psi)^{\frac{1}{2}} = \mu(\Psi \otimes \Sigma). \tag{36}$$

Indeed, similarly as in the proof Theorem 3.5 (and using (31)) one can prove

$$\mu(\Psi^{(2)} \otimes \Sigma^{(2)})^{\frac{1}{2}} \leq \mu((\Psi \circ \Psi) \otimes (\Sigma \circ \Sigma))^{\frac{1}{2}} \leq \mu(\Psi \otimes \Sigma \circ \Psi \otimes \Sigma)^{\frac{1}{2}} = \mu(\Psi \otimes \Sigma).$$

However, by (33) we have $\Psi \otimes \Sigma = (\Psi^{(2)} \otimes \Sigma^{(2)})^{(\frac{1}{2})}$ and this implies (36).

Of course, we also have

$$\mu(\Psi \otimes \Sigma) = \mu((\Psi \otimes \Sigma)^{(2)})^{\frac{1}{4}} \mu((\Sigma \otimes \Psi)^{(2)})^{\frac{1}{4}} = \mu(\Psi \otimes \Sigma \circ \Psi \otimes \Sigma)^{\frac{1}{4}} \mu(\Sigma \otimes \Psi \circ \Sigma \otimes \Psi)^{\frac{1}{4}}.$$

We conclude the paper with some examples, showing that the inequalities in (34) and (35) may be strict. All these inequalities are also sharp (take e.g. $A = B = 0$ or I).

Examples 4.10. (i) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Then $A \otimes B = A^2_{\otimes} = A$, $B \otimes A = B^2_{\otimes} = B$ and so $A \circ B = A \otimes B \circ B \otimes A = 0$ and $A^2_{\otimes} \otimes B^2_{\otimes} = A \otimes B = A$. Thus $\mu(A \circ B) = \mu(A \otimes B \circ B \otimes A) = 0 < \mu(A \otimes B) = \mu(A^2_{\otimes} \otimes B^2_{\otimes}) = 1$. So (35) and the second inequality in (34) may be strict.

(ii) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $B = I$. Then $A \circ B = I$, $A \otimes B = B \otimes A = A$ and so $A \otimes B \circ B \otimes A = A^{(2)}$. Thus $\mu(A \circ B) = 1 < \mu(A \otimes B \circ B \otimes A)^{\frac{1}{2}} = \mu(A \otimes B) = \sqrt{2}$. So also the first inequality in (34) may be strict.

(iii) Let A and B be from Example 3.14. Then $\mu(A \circ B) = \mu(A \otimes B \circ B \otimes A) = \mu(A^2_{\otimes} \otimes B^2_{\otimes}) = 0 < \mu(A \otimes B) = 1$. Thus (35) is not weaker than the second inequality in (34).

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References

[1] C.D. Aliprantis and O. Burkinshaw, *Positive operators*, Academic Press, Orlando, 1985.

[2] K.M.R. Audenaert, Spectral radius of Hadamard product versus conventional product for non-negative matrices, *Linear Algebra Appl.* 432(1) (2010), 366–368.

[3] F.L. Baccelli, G. Cohen, G.-J. Olsder and J.-P.Quadrat, *Synchronization and Linearity*, John Wiley, Chichester, New York, 1992.

[4] R.B. Bapat, A max version of the Perron-Frobenius theorem, *Linear Algebra Appl.* 275-276, (1998), 3–18.

[5] M.A. Berger and Y. Wang, Bounded semigroups of matrices, *Linear Algebra Appl.* 166 (1992), 21–27.

[6] P. Butkovič, *Max-linear systems: theory and algorithms*, Springer-Verlag, London, 2010.

[7] X. Dai, Extremal and Barabanov semi-norms of a semigroup generated by a bounded family of matrices, J. Math. Anal. Appl. 379 (2011) 827-833.

[8] R. Drnovšek and A. Peperko, Inequalities for the Hadamard weighted geometric mean of positive kernel operators on Banach function spaces, *Positivity* 10 (2006), 613–626.

[9] R. Drnovšek and A. Peperko, On the spectral radius of positive operators on Banach sequence spaces, *Linear Algebra Appl.* 433 (2010) 241-247.

[10] L. Elsner, The generalized spectral radius theorem: An analytic-geometric proof, *Linear Algebra Appl.* 220 (1995), 151–159.

[11] L. Elsner and P. van den Driessche, Bounds for the Perron root using max eigenvalues, *Linear Algebra Appl.* 428 (2008), 2000-2005.

[12] S. Gaubert and M. Sharify, Tropical scaling of polynomial matrices, *Lecture Notes in Control and Information Sciences* 389 (2009), 291–303.

[13] P.S. Guinand, On quasinilpotent semigroup of operators, *Proc. Amer. Math. Soc.* 86 (1982), 485–486.

[14] B.B. Gursoy and O. Mason, $P^1_{\max}$ and $S_{\max}$ properties and asymptotic stability in the max algebra, *Lin. Algebra Appl.* 435 (2011), 1008-1018.
[15] R.A. Horn and F. Zhang, Bounds on the spectral radius of a Hadamard product of nonnegative or positive semidefinite matrices, *Electron. J. Linear Algebra* 20, (2010), 90–94.
[16] Z. Huang, On the spectral radius and the spectral norm of Hadamard products of nonnegative matrices, *Linear Algebra Appl.* 434 (2011), 457–462.
[17] P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Berlin, 1991.
[18] A. Peperko, On the max version of the generalized spectral radius theorem, *Linear Algebra Appl.* 428 (2008), 2312–2318.
[19] A. Peperko, Inequalities for the spectral radius of non-negative functions, *Positivity* 13 (2009), 255-272.
[20] A. Peperko, On the continuity of the generalized spectral radius in max algebra, *Linear Algebra Appl.* 435 (2011), 902–907.
[21] A. Peperko, On the functional inequality for the spectral radius of compact operators, *Linear and Multilinear Algebra* 59 (4) (2011), 357-364
[22] A.R. Schep, Bounds on the spectral radius of Hadamard products of positive operators on l_p-spaces, *Electronic J. Linear Algebra* 22, (2011), 443–447.
[23] A.R. Schep, Corrigendum for "Bounds on the spectral radius of Hadamard products of positive operators on l_p-spaces", preprint 2011.
[24] M.-H. Shih, J.-W. Wu, C.-T. Pang, Asymptotic stability and generalized Gelfand spectral radius formula, *Linear Alg. Appl.* 252 (1997), 61–70.
[25] V.S. Shulman and Yu.V. Turovskii, Joint spectral radius, operator semigroups and a problem of W.Wojtyński, *J. Funct. Anal.* 177 (2000), 383–441.
[26] V. S. Shulman, Yu. V. Turovskii, Application of topological radicals to calculation of joint spectral radii, (2008), arxiv:0805.0209v1 [math.FA].
[27] F. Wirth, The generalized spectral radius and extremal norms, *Linear Algebra Appl.* 342 (2002), 17–40.
[28] A.C. Zaanen, *Riesz spaces II*, North Holland, Amsterdam, 1983.
[29] X. Zhan, Unsolved matrix problems, Talk given at *Advanced Workshop on Trends and Developments in Linear Algebra*, ICTP, Trieste, Italy, July 6-10, 2009.

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