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Quasi m-Cayley circulants

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Abstract

A graph Γ is called a *quasi m-Cayley graph on a group G* if there exists a vertex $\infty \in V(\Gamma)$ and a subgroup G of the vertex stabilizer $\text{Aut}(\Gamma)_\infty$ of the vertex ∞ in the full automorphism group $\text{Aut}(\Gamma)$ of Γ, such that G acts semiregularly on $V(\Gamma)\setminus\{\infty\}$ with m orbits. If the vertex ∞ is adjacent to only one orbit of G on $V(\Gamma)\setminus\{\infty\}$, then Γ is called a *strongly quasi m-Cayley graph on G*. In this paper complete classifications of quasi 2-Cayley, quasi 3-Cayley and strongly quasi 4-Cayley connected circulants are given.

Keywords: Arc-transitive, circulant, quasi m-Cayley graph.

Math. Subj. Class.: 05C15, 05C10

1 Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected, and groups are finite. Given a graph Γ we let $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\text{Aut}(\Gamma)$ be the set of its vertices, edges, arcs and the automorphism group of Γ, respectively. A graph Γ is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* if its automorphism group acts transitively on $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$, respectively.

Let G be a finite group with identity element 1, and let $S \subset G\setminus\{1\}$ be such that $S^{-1} = S$. We define the *Cayley graph* $Cay(G, S)$ on the group G with respect to the connection set S to be the graph with vertex set G, in which two vertices $x, y \in G$ are adjacent if and only if $x^{-1}y \in S$. A *circulant* of order n is a Cayley graph on a cyclic group of order n.

In this paper we consider quasi-semiregular actions on graphs, a natural generalization of semiregular actions on graphs, which have been an active topic of research in the last decades (see, for example, [1, 2, 3, 4, 5, 8, 9, 11]). Following [7] we say that a group G acts

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quasi-semiregularly on a set X if there exists an element ∞ in X such that G fixes ∞, and the stabilizer G_x of any element $x \in X\backslash\{\infty\}$ is trivial. The element ∞ is called the point at infinity. A graph Γ is called *quasi m-Cayley on G* if the group G acts quasi-semiregularly on $V(\Gamma)$ with m orbits on $V(\Gamma)\backslash\{\infty\}$. If G is cyclic and $m = 1$ (respectively, $m = 2$, $m = 3$ and $m = 4$) then Γ is said to be *quasi circulant* (respectively, *quasi bicirculant*, *quasi tricirculant* and *quasi tetracirculant*). In addition, if the point at infinity ∞ is adjacent with only one orbit of G_∞ then we say that Γ is a *strongly quasi m-Cayley graph* on G.

Quasi m-Cayley graphs were first defined in 2011 by Kutnar, Malnič, Martinez and Marušič [7], who showed which strongly quasi $m-$Cayley graphs are strongly regular graphs.

In this paper, we consider which circulants are also quasi m-Cayley graphs. Our main results are stated in the following three theorems.

Theorem 1.1. Let Γ be a quasi 2-Cayley graph of order n which is also a connected circulant. Then either Γ is isomorphic to the complete graph K_n, or $n \equiv 1 \pmod 4$ is a prime and Γ is isomorphic to the Paley graph $P(n)$. Moreover, Γ is a quasi bicirculant.

Theorem 1.2. Let Γ be a connected circulant. Then Γ is also a quasi 3-Cayley graph if and only if either $\Gamma = K_n$, or replacing Γ with its complement if necessary, $\Gamma \cong Cay(\mathbb{Z}_n, S)$, where S is the set of all non-zero cubes in $\mathbb{Z}_n$, and n is a prime such that $n \equiv 1 \pmod 3$. Moreover, Γ is a quasi tricirculant.

Theorem 1.3. Let Γ be a connected circulant. Then Γ is a strongly quasi 4-Cayley graph on a group G if and only if $\Gamma \cong C_9$ or $\Gamma \cong Cay(\mathbb{Z}_n, S)$, where S is the set of all fourth powers in $\mathbb{Z}_n\backslash\{0\}$, and n is a prime such that $n \equiv 1 \pmod 4$. Moreover, Γ is a quasi tetracirculant.

The paper is organized as follows. In Section 2 we recall the classification of connected arc-transitive circulants. In Section 3 we prove Theorem 1.1 and in Section 4 we prove Theorems 1.2 and 1.3.

2 Arc-transitive circulants

We begin this section with the following lemma:

Lemma 2.1. Let Γ be a connected vertex-transitive strongly quasi $m-$Cayley graph. Then Γ is arc-transitive.

Proof. Since Γ is vertex transitive, it is sufficient to prove that there exists a vertex v such that the stabilizer $\mathrm{Aut}(\Gamma)_v$ acts transitively on the neighborhood of v. It is obvious that if we choose the point at infinity for v, this condition is satisfied. □

The previous lemma implies that we can somehow restrict our study to the connected arc-transitive circulants, therefore it is important to understand the structure of such graphs.

To state the classification of connected arc-transitive circulants, which has been obtained independently by Kovács [6] and Li [10], we need to recall certain graph products and the concept of normal Cayley graphs.

The *wreath (lexicographic) product* $\Sigma[\Gamma]$ of a graph Γ by a graph Σ is the graph with vertex set $V(\Sigma) \times V(\Gamma)$ such that $\{(u_1, u_2), (v_1, v_2)\}$ is an edge if and only if either $\{u_1, v_1\} \in E(\Sigma)$, or $u_1 = v_1$ and $\{u_2, v_2\} \in E(\Gamma)$. For a positive integer b and a graph

Σ, denote by $b\Sigma$ the graph consisting of b vertex-disjoint copies of the graph Σ. The graph $\Sigma[\overline{K_b}]-b\Sigma$ is called the *deleted wreath (deleted lexicographic) product* of Σ and $\overline{K_b}$, where $\overline{K_b} = bK_1$.

Let $\Gamma = Cay(G,S)$ be a Cayley graph on a group G. Denote by $\text{Aut}(G,S)$ the set of all automorphisms of G which fix S setwise, that is,

$$\text{Aut}(G,S) = \{\sigma \in \text{Aut}(G) | S^\sigma = S\}.$$

It is easy to check that $\text{Aut}(G,S)$ is a subgroup of $\text{Aut}(\Gamma)$ and that it is contained in the stabilizer of the identity element $1 \in G$. It follows from the definition of Cayley graph that the left regular representation G_L of G induces a regular subgroup of $\text{Aut}(\Gamma)$. Following Xu [12], $\Gamma = Cay(G,S)$ is called a *normal Cayley graph* if G_L is normal in $\text{Aut}(\Gamma)$, that is, if $\text{Aut}(G,S)$ coincides with the vertex stabilizer $1 \in G$. Moreover, if Γ is a normal Cayley graph, then $\text{Aut}(\Gamma) = G_L \rtimes \text{Aut}(G,S)$.

Proposition 2.1. [6, 10] Let Γ be a connected arc-transitive circulant of order n. Then one of the following holds:

(i) $\Gamma \cong K_n$;

(ii) $\Gamma = \Sigma[\overline{K_d}]$, where $n = md$, $m, d > 1$ and Σ is a connected arc-transitive circulant of order m;

(iii) $\Gamma = \Sigma[\overline{K_d}] - d\Sigma$, where $n = md$, $d > 3$, $gcd(d,m) = 1$ and Σ is a connected arc-transitive circulant of order m;

(iv) Γ is a normal circulant.

In Section 3 and 4 two lemmas (that show that arc-transitive circulants described in Proposition 2.1(ii) and (iii) are not strongly quasi k-Cayley graphs) will be needed.

Lemma 2.2. Let Γ be an arc-transitive circulant, described in Proposition 2.1(ii). Then Γ is not a strongly quasi k-Cayley graph for any $k \in \mathbb{N}$.

Proof. We have $\Gamma = \Sigma[\overline{K_d}]$, where $n = md$, $m, d > 1$ and Σ is a connected arc-transitive circulant of order m. Suppose that Γ is a strongly quasi k-Cayley graph on a group G. Then $val(\Gamma) = (n-1)/k = (md-1)/k$. On the other hand, since $\Gamma = \Sigma[\overline{K_d}]$, we have $val(\Gamma) = val(\Sigma) \cdot d$. These two facts combined together imply that $d(m - k \cdot val(\Sigma)) = 1$, and so $d = 1$, a contradiction. □

Lemma 2.3. Let Γ be an arc-transitive circulant, described in Proposition 2.1(iii). Then Γ is not a strongly quasi k-Cayley graph for any $k \in \mathbb{N}$.

Proof. We have $\Gamma = \Sigma[\overline{K_d}] - d\Sigma$, where $n = md$, $d > 3$, $gcd(d,m) = 1$, and Σ is an arc-transitive circulant of order m. Suppose that Γ is also a strongly quasi k-Cayley graph on a group G. By [10, Theorem 1.1] the m copies of the graph $\overline{K_d}$ form an imprimitivity block system $\mathcal{B}$ for $\text{Aut}(\Gamma)$. Clearly the block $B \in \mathcal{B}$ containing the point at infinity, that is, the trivial orbit of G, is fixed by G. This implies that $|G|$ divides $d-1$. On the other hand, since the valency of Γ is $|G|$, we have $|G| \geq d-1$. Combining these results we obtain $|G| = d-1$. Thus, connectedness of Γ implies that $m = 2$. However, then there are $2d-1$ vertices in Γ different from the point at infinity, and they cannot be divided into k orbits of size $d-1$ for any natural number k. Therefore, there are no strongly quasi k-Cayley graphs amongst the graphs from Proposition 2.1(iii) for any natural number $k \geq 1$. □

Lemma 2.4. Let Γ be an arc-transitive circulant, described in Proposition 2.1(iv). If Γ is also a strongly quasi m-Cayley graph on a group G, then the order of Γ has at most $m+1$ divisors.

Proof. Let $\Gamma = Cay(\mathbb{Z}_n, S)$ be a normal circulant. Let $A = \text{Aut}(\Gamma)$. Since Γ is a normal Cayley graph, $A \cong \mathbb{Z}_n \rtimes \text{Aut}(\mathbb{Z}_n, S)$. We may, without loss of generality, assume that the point at infinity corresponds to the vertex $0 \in \mathbb{Z}_n$, and so $G \leq \text{Aut}(\mathbb{Z}_n, S) \leq \text{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$. Therefore, $G \lesssim \mathbb{Z}_n^*$. Since G has m orbits on $\mathbb{Z}_n \backslash \{0\}$, then $\text{Aut}(\mathbb{Z}_n)$ has at most m orbits on $\mathbb{Z}_n \backslash \{0\}$, and at most $m+1$ orbits on $\mathbb{Z}_n$. Elements in the same orbit of $\text{Aut}(\mathbb{Z}_n)$ are clearly of the same order in $\mathbb{Z}_n$. There exist an element in $\mathbb{Z}_n$ of order d, if and only if d divides n. Therefore the number of divisors of n, denoted by $\tau(n)$, is not greater than $m+1$, i.e. $\tau(n) \leq m+1$. □

3 Quasi 2-Cayley graphs

In this section the connected circulants are considered. In particular, connected circulants that are also quasi 2-Cayley graphs are classified (see Theorem 1.1). If a graph Γ of order n is a quasi 2-Cayley graph on a group G, which is not a strongly quasi 2-Cayley graph, then it is isomorphic to the complete graph K_n. Namely, in such a graph, the point at infinity ∞ is adjacent to both nontrivial orbits of G, and thus it is adjacent to all the vertices different from ∞. Consequently, we can conclude that Γ has valency $|V(\Gamma)| - 1$, and so Γ is a complete graph. In order to classify all connected circulants that are also quasi 2-Cayley graphs it therefore suffices to characterize strongly quasi 2-Cayley graphs that are also connected circulants, we do this in Theorem 3.1.

Theorem 3.1. Let Γ be a connected circulant. Then Γ is also a strongly quasi 2-Cayley graph if and only if Γ is isomorphic to the Paley graph $P(p)$, where p is a prime such that $p \equiv 1 \pmod 4$. Moreover, Γ is a quasi bicirculant.

Proof. Let Γ be the Paley graph $P(p)$, where p is a prime, such that $p \equiv 1 \pmod 4$. It is well known that the Paley graphs are connected arc-transitive circulants, and, as was observed in [7], they are also strongly quasi 2-Cayley graphs.

Conversely, let Γ be a connected circulant $Cay(\mathbb{Z}_n, S)$ of order n not isomorphic to the complete graph K_n, which is also a strongly quasi 2-Cayley graph on a group G. Then $|G| = (n-1)/2$ and Γ is of valency $(n-1)/2$. Lemma 2.1 tells us that Γ is an arc-transitive graph, and moreover Proposition 2.1, Lemma 2.2 and Lemma 2.3 combined together imply that Γ is a normal circulant. The theorem now follows from the three claims below.

CLAIM 1: n is an odd prime.

It is obvious that n must be odd, since 2 divides $n-1$. By Lemma 2.4 we have that $\tau(n) \leq 3$. Thus we have the following two possibilities for n:

- $n = p$, where p is a prime;
- $n = p^2$, where p is a prime.

Suppose that the latter case hold. Let $A = \text{Aut}(\Gamma)$. Since Γ is a normal Cayley graph, we have $A \cong \mathbb{Z}_n \rtimes \text{Aut}(\mathbb{Z}_n, S)$. We may, without loss of generality, assume that the point at infinity corresponds to the vertex $0 \in \mathbb{Z}_n$, and so $G \leq \text{Aut}(\mathbb{Z}_n, S) \leq \text{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$. Therefore, $\mathbb{Z}_n^*$ contains a subgroup G of order $(n-1)/2$. Since $|\mathbb{Z}_n^*| \leq n-1$ and $|G|$

divides $|\mathbb{Z}_n^*|$ we obtain that $|\mathbb{Z}_n^*| = n-1$ or $(n-1)/2$. Since, by assumption, n is not a prime, we have $|\mathbb{Z}_n^*| = (n-1)/2$. This gives in the following equation

$$\frac{p^2-1}{2} = p(p-1)$$

which has the unique solution $p = 1$, a contradiction.

CLAIM 2: $n \equiv 1 \pmod 4$.

Since $S = -S$, and no element in $\mathbb{Z}_n$ can be its own inverse, we have that the number of elements in S is even, and since $|S| = \frac{n-1}{2}$, we have $n \equiv 1 \pmod 4$.

CLAIM 3: Γ is isomorphic to the Paley graph $P(n)$.

By Claim 1, n is a prime. Therefore the group $\mathbb{Z}_n^*$ is cyclic, and thus since G is a subgroup of $\mathbb{Z}_n^*$, G is cyclic as well. By [6, Remark 2], we have $\mathrm{Aut}(\Gamma) = \{g \mapsto g^\sigma + h \mid \sigma \in K, h \in \mathbb{Z}_n\}$, for a suitable group $K < \mathrm{Aut}(\mathbb{Z}_n)$, and S is the orbit under K of a generating element of $\mathbb{Z}_n$, that is, $S = Orb_K(g)$ for some generating element g of $\mathbb{Z}_n$. Now we have that $\mathrm{Aut}(\Gamma)_0 = \{g \mapsto g^\sigma + h \mid \sigma \in K, h \in \mathbb{Z}_n : 0^\sigma + h = 0\} = \{g \mapsto g^\sigma \mid \sigma \in K\} \cong K$. So we see that $G \lesssim K$. Since $S = Orb_K(g) \lesssim Orb_G(g)$, and $|S| = |Orb_G(g)|$ we have that $S \cong Orb_G(g)$, which gives us that $S \cong G$ (taking $g = 1$). Now, since G is the index 2 subgroup of the cyclic group $\mathbb{Z}_n^*$, G is of the form $G = \langle x^2 \rangle$ where x generates $\mathbb{Z}_n^*$. Therefore G consists of all squares in $\mathbb{Z}_n^*$ and $S \cong G$, implying that Γ is isomorphic to the Paley graph $P(n)$ as claimed.

It is obvious that G must be cyclic, so the graph Γ is in fact a quasi bicirculant. □

Proof of Theorem 1.1: It follows from Theorem 3.1 and the paragraph preceding it. □

In general, if Γ is a vertex transitive quasi 2-Cayley graph on a group G, not isomorphic to the complete graph, then it is a strongly regular graph of a rank 3 group. Namely, the orbits of G are contained in the orbits of the stabilizer of the $\mathrm{Aut}(\Gamma)_\infty$ and since there are just two nontrivial orbits of G, then there are exactly two nontrivial orbits of the $\mathrm{Aut}(\Gamma)_\infty$ which in fact must coincide with the orbits of G. Therefore $\mathrm{Aut}(\Gamma)$ must be a rank 3-group, and the graphs of the rank 3 groups are strongly regular graphs.

4 Quasi 3-Cayley and 4-Cayley graphs

In this section we will deal with the question which connected circulants are also quasi 3-Cayley graph or strongly quasi 4-Cayley graphs. We first consider the case of strongly quasi 3-Cayley graphs.

Theorem 4.1. Let Γ be a connected circulant. Then Γ is also a strongly quasi 3-Cayley if and only if $\Gamma \cong Cay(\mathbb{Z}_n, S)$ where S is the set of all non-zero cubes in $\mathbb{Z}_n$, and n is a prime such that $n \equiv 1 \pmod 3$. Moreover, Γ is a quasi tricirculant.

Proof. Let $\Gamma = Cay(\mathbb{Z}_p, S)$ where $p \equiv 1 \pmod 3$ is a prime and S is the set of all non-zero cubes in $\mathbb{Z}_p$. Since p is a prime, it is well known that $\mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_p^*$ is a cyclic group of order $p-1$. Let $G = \langle a^3 \rangle$, where a is a generating element of $\mathbb{Z}_p^*$. Then G consists of all non-zero cubes in $\mathbb{Z}_p$, and $|G| = \frac{p-1}{3}$. The action of G on $\mathbb{Z}_p$ defined by $x^g = g \cdot x$ gives G as the subgroup of $\mathrm{Aut}(\Gamma)$. The group G acts quasi-semiregularly on $\mathbb{Z}_p$ with $0 \in \mathbb{Z}_p$

as the point at infinity. Namely, it is easy to check that $G_0 = G$, and that the stabilizer of any element $x \in \mathbb{Z}_p \backslash \{0\}$ is trivial. Since $|G| = \frac{p-1}{3}$, it follows that G has 3 orbits on $\mathbb{Z}_p \backslash \{0\}$, and therefore Γ is a quasi 3-Cayley graph. Since one of the orbits of G is the set S, the point at infinity is adjacent to only one orbit of G, so Γ is in fact a strongly quasi 3-Cayley graph. By the construction Γ is an arc-transitive circulant since $G \leq \mathrm{Aut}(\Gamma)_0$ acts transitively on the set of vertices adjacent to the vertex 0.

Conversely, let Γ be a connected circulant of order n, which is also a strongly quasi 3-Cayley graph on a group G. Then $|G| = \frac{n-1}{3}$. From Lemma 2.1 we have that Γ is arc-transitive, and therefore Proposition 2.1, Lemma 2.2 and Lemma 2.3 combined together imply that Γ is a normal circulant. Therefore, we can assume that $\Gamma = Cay(\mathbb{Z}_n, S)$, and that $G \leq \mathrm{Aut}(\mathbb{Z}_n, S) \leq \mathrm{Aut}(\mathbb{Z}_n)$, implying that $\frac{n-1}{3} | \varphi(n)$, where $\varphi(n)$ is the Euler totient function.

CLAIM 1: n is a prime number.

Let

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_t^{k_t},$$

be a canonic factorization of a positive integer n. From Lemma 2.4, we have $\tau(n) \leq 4$. Now we can calculate

$$\tau(n) = (k_1 + 1)(k_2 + 1) \cdots (k_t + 1).$$

We have the following possibilities for n:

- $n = p$,
- $n = p^2$;
- $n = p^3$;
- $n = pq$,

where p and q are different primes.

If $n = p^2$, then the only solution of $\frac{n-1}{3} | \varphi(n)$ is $p = 2$ and $n = 4$. However, if $n = 4$, the graph Γ is of valency 1, so it is not a connected graph.

If $n = p^3$, then there is no solution of the above equation.

If n is a product of two different primes, then we have $|\mathbb{Z}_n^*| = (n-1)/3$ or $2(n-1)/3$. In the first case $\mathbb{Z}_n^* \cong G$, so $\mathbb{Z}_n^*$ acts semiregularly on $\mathbb{Z}_n \backslash \{0\}$, and it is not difficult to see that this is not the case for $n = pq$. If $|\mathbb{Z}_n^*| = 2(n-1)/3$, then we obtain the following equation

$$(p-1)(q-1) = \frac{2(pq-1)}{3}.$$

The only solutions in natural numbers of the above equation are $(p, q) \in \{(4, 7), (5, 5), (7, 4)\}$, so there are no two different primes p, q satisfying the given equation.

Having in mind all the written above, we conclude that n is a prime.

CLAIM 2: Γ is isomorphic to the Cayley Graph $Cay(\mathbb{Z}_n, S)$, where S is set of all non zero cubes in $\mathbb{Z}_n$, and n is a prime such that $n \equiv 1 \pmod 3$.

Similarly as in the previous section, it can be shown that $G \cong S$. Since G is an index 3 subgroup of $\mathbb{Z}_n^*$, we have $G = \langle x^3 \rangle$, where x is a generating element of $\mathbb{Z}_n^*$. It follows that G consists of all cubes in $\mathbb{Z}_n^*$, so Γ is isomorphic to $Cay(\mathbb{Z}_n, S)$, where S is the set of all non zero cubes in $\mathbb{Z}_n$ and $n \equiv 1 \pmod 3$ is a prime. It is obvious from the mentioned above, that the group G must be cyclic, therefore, Γ is in fact a quasi tricirculant. □

Proof of the Theorem 1.2: Let Γ be a connected circulant of order n, which is also a quasi 3-Cayley on a group G. The point at infinity is adjacent to all three nontrivial orbits of G, if and only if Γ is isomorphic to K_n. If the point at infinity is adjacent to just one nontrivial orbit of G, then Γ is a strongly quasi 3-Cayley graph, therefore, Theorem 4.1 gives us the desired result. If the point at infinity is adjacent to two nontrivial orbits of G, then we consider the complement $\Sigma = \overline{\Gamma}$ of the graph Γ. The graph Σ is a quasi 3-Cayley graph on G, and actually it is a strongly quasi 3-Cayley graph on G. Since Σ is the complement of a circulant it is also a circulant. Suppose that Σ is not connected. Then, since it is vertex-transitive, it is the disjoint union of some isomorphic graphs. The point at infinity is adjacent to one orbit of G, so the connected components of Σ must have at least $1 + \frac{n-1}{3}$ points. Therefore $n = k \cdot n_1$, where k is the number of connected components, and n_1 is the number of points in each of the components. We have noticed that $n_1 \geq 1 + \frac{n-1}{3}$, thus $k \leq 2$. If $k = 1$ then Σ is connected. Suppose that $k = 2$. Then there are two connected components of Γ, say Γ_1 and Γ_2, each containing $n/2$ points. Suppose that $\infty \in \Gamma_1$. Let Δ_1, Δ_2 and Δ_3 be a nontrivial orbits of G, and let the point at infinity be adjacent to Δ_1. Then $\Delta_1 \subset \Gamma_1$. Since Γ_1 and Γ_2 have the same size, it means that at least one of Δ_2 and Δ_3 have points both in Γ_1 and Γ_2. Suppose that $u, v \in \Delta_2$, such that $u \in \Gamma_1$ and $v \in \Gamma_2$. Since u and v are in the same orbit of G then there exist $g \in G$ which maps u to v. However, g fixes ∞, and consequently g fixes Γ_1, a contradiction.

Having in mind all the written above, we see that Σ is a connected circulant, which is also a strongly quasi 3-Cayley graph. Therefore we have the desired result. $\square$

We will continue this section with the proof of Theorem 1.3.

Proof of Theorem 1.3: Let $\Gamma = C_9$. Then $\Gamma \cong Cay(\mathbb{Z}_9, \{\pm 1\})$. Then the group $G = \{1, -1\} \subset \mathbb{Z}_9^*$ acts quasi semiregularly on $\mathbb{Z}_9$ with 0 as the point at infinity.

Let $\Gamma \cong Cay(\mathbb{Z}_p, S)$, where S is the set of all fourth powers in $\mathbb{Z}_p \backslash \{0\}$, and p is a prime such that $p \equiv 1 \pmod 4$. Define $G = \langle a^4 \rangle$, where a is some generating element of $\mathbb{Z}_p^*$, which is cyclic in this case. We have that G acts quasi-semiregularly on $\mathbb{Z}_p^*$ with 0 as the point at infinity. Since $|G| = \frac{p-1}{4}$, it follows that G has 4 orbits on $\mathbb{Z}_p \backslash \{0\}$, and therefore Γ is a quasi 4-Cayley graph. It is also easy to see that 0 is adjacent to only one orbit of G on $\mathbb{Z}_p \backslash \{0\}$, therefore Γ is a strongly quasi 4-Cayley graph. By the construction, Γ is a connected arc-transitive circulant.

Conversely, let Γ be a connected circulant of order n which is also a strongly quasi 4-Cayley graph on a group G. Then $|G| = (n-1)/4$. Using Lemma 2.1 we have that Γ is arc-transitive, and so Proposition 2.1, Lemma 2.2 and Lemma 2.3 combined together imply that Γ is a normal circulant. Therefore, we can assume that $\Gamma = Cay(\mathbb{Z}_n, S)$, and that $G \leq \mathrm{Aut}(\mathbb{Z}_n, S) \leq \mathrm{Aut}(\mathbb{Z}_n)$, implying that

$$\frac{n-1}{4} | \varphi(n). \tag{4.1}$$

Using Lemma 2.4 we obtain $\tau(n) \leq 5$. So we have the following possibilities:

- $n = p$,
- $n = p^2$,
- $n = p^3$,
- $n = p^4$,
- $n = pq$,

where p and q are different primes.

If $n = p^2$, then the only solution of (4.1) is $n = 9$. In this case, the valency of Γ is $(9-1)/4 = 2$, so $\Gamma \cong C_9$.

In the cases when $n = p^3$, and $n = p^4$ there is no prime satisfying (4.1).

When $n = pq$, we have that $(p-1)(q-1) = \alpha \cdot (pq-1)/4$, where $\alpha \in \{1, 2, 3\}$. If $\alpha = 1$, then we have $\mathbb{Z}_n^* = G$, so $\mathbb{Z}_n^*$ must act semiregularly on $\mathbb{Z}_n \backslash \{0\}$, which is not the case. If $\alpha = 2$, then there are no two different primes satisfying $(p-1)(q-1) = (pq-1)/2$, and finally, when $\alpha = 3$, we have that $n = 5 \cdot 13$ is the only possibility. In this case, Γ is a connected arc-transitive circulant on 65 vertices, which has valency 16. Since G is an index 3 subgroup of $\mathbb{Z}_{65}^* \cong \mathbb{Z}_4 \times \mathbb{Z}_{12}$, then we can calculate $G \cong \{\pm 1, \pm 8, \pm 12, \pm 14, \pm 18, \pm 21, \pm 27, \pm 31\}$, and we can see that G does not act semiregularly on $\mathbb{Z}_{65} \backslash \{0\}$. Namely, the non identity element $21 \in G$ fixes the point $13 \in \mathbb{Z}_{65} \backslash \{0\}$.

Assume now that n is a prime. Similarly as in the proof of Theorem 3.1, we obtain $G \cong S$, and therefore, since G is an index 4 subgroup of $\mathbb{Z}_n^*$, we have $G = \langle x^4 \rangle$, where x is some generating element of $\mathbb{Z}_n^*$. Therefore, $\Gamma \cong Cay(\mathbb{Z}_n, S)$, where S is the set of all fourth powers in $\mathbb{Z}_n \backslash \{0\}$.

From the mentioned above, it is clear that G is a cyclic group, so Γ is in fact a quasi tetracirculant. □

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