

Constructions for large spatial point-line (n_k) configurations

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Abstract

Highly symmetric figures, such as regular polytopes, can serve as a scaffolding on which spatial (n_k) point-line configurations can be built. We give several constructions using this method in dimension 3 and 4. We also explore possible constructions of point-line configurations obtained as Cartesian products of smaller ones. Using suitable powers of well-chosen configurations, we obtain infinite series of (n_k) configurations for which both n and k are arbitrarily large. We also combine the method of polytopal scaffolding and the method of powers to construct further examples. Finally, we formulate an incidence statement concerning a (100_4) configuration in 3-space derived from the product of two complete pentalaterals; it is posed as a conjecture.

Keywords: Spatial configuration, Platonic solid, regular 4-polytope, product of configurations, incidence statement.

Math. Subj. Class.: 51A20, 51A45, 51E30, 52B15

1 Introduction

By a (p_q, n_k) configuration we mean a set consisting of p points and n lines such that k of the points lie on each line and q of the lines pass through each point [7, 16, 19, 21]. If, in particular, $p = n$, then $q = k$; in this case the notation (n_k) is used, and we speak of a *balanced configuration* [16]. We consider configurations embedded either in Euclidean or projective spaces.

In the last decades, there has been a revival of interest in point-line configurations; the developments and results are summarized in the quite recent research monograph by Branko Grünbaum [16]. This book deals predominantly with planar configurations. However, as the author notices in the Postscript, a “...seemingly safe guess is that there will be interest in higher-dimensional analogues of the material described in this book”.

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In fact, the theory was and has not been restricted to planar configurations. Research in higher dimensions go back to Cayley, Cremona, Veronese and others [4, 7, 10, 16]. One of the most well known spatial configurations is Reye’s $(12_4, 16_3)$ configuration in projective 3-space [19, 25]. Its construction is based on the ordinary cube; the cube serves, so to say, as a “scaffolding”: once the configuration has been built, the underlying cube is deleted. Moreover, the configuration inherits its symmetry (at least, in this case, in a combinatorial sense).

We apply the same building principle in several of our constructions. For an underlying polytope, we choose a regular polytope in dimension 3 or 4. In all but one of these cases, the geometric symmetry group of the configuration will be the same as that of the underlying polytope. We note that the term *geometric symmetry group*, or briefly, *symmetry group*, is meant in the usual sense; i.e., it denotes the group of isometries of the ambient (Euclidean) space that map the configuration onto itself. The combinatorial counterpart of this notion is the group of *automorphisms*, i.e., the group of incidence-preserving permutations of points and lines (both among themselves); Coxeter simply calls it the *group of the configuration* [7]. (The distinction between these two types of groups will play some role in Section 5.)

In Section 4 we apply the notion of a product, which can be considered as the Cartesian product of configurations. Using this tool, we construct two infinite series and a finite class of (n_k) configurations which are powers of smaller configurations. In this way very high n and k values can be attained (and so can the dimension of the space spanned by these configurations). We think this may be interesting for future research; for, as it is emphasized just recently, little is known in general on the existence of such large configurations [3].

We summarize our results in the following theorem.

Theorem 1.1. *There exist (n_k) configurations which form the following classes:*

1. *infinite series of type*

$$(18(t + 1)_3), \quad (36(t + 1)_3) \quad \text{and} \quad (90(t + 1)_3), \quad (t = 1, 2, \dots);$$

they have the symmetry group of a regular tetrahedron, cube and dodecahedron, respectively, and each spans the Euclidean space \mathbb{E}^3 ;

2. *infinite series of type*

$$\left(\left(\binom{2k+1}{2}^k \right)_{2k} \right), \quad (k = 2, 3, \dots);$$

they span the projective space \mathbb{P}^{2k} ;

3. *infinite series of type*

$$\left(((2k)^{2k-2}(2k+1)k)_{2k} \right), \quad (k = 2, 3, \dots);$$

they span the projective space \mathbb{P}^{2k} ;

4. *finite class of types*

$$(240_3), \quad (768_3) \quad \text{and} \quad (28\,800_3),$$

with the symmetry group of a regular 4-simplex, a regular 4-cube, or a regular 120-cell, respectively; each spans the Euclidean space \mathbb{E}^4 ;

5. finite class of types

$$\begin{aligned} (14\ 400_4) &\subset \mathbb{E}^6 \\ (5\ 832\ 200_6) &\subset \mathbb{E}^9 \\ (3\ 317\ 760\ 000_8) &\subset \mathbb{E}^{12} \\ \left((2.43 \cdot 10^{10})_{10} \right) &\subset \mathbb{E}^{15} \\ \left((2^{18} \cdot 3^{12} \cdot 5^6)_{12} \right) &\subset \mathbb{E}^{18}, \end{aligned}$$

each full-dimensional in the given Euclidean space;

6. sporadic examples of type

$$(180_3), \quad (60_4), \quad (540_4) \quad \text{and} \quad (780_4),$$

with the symmetry group of a regular dodecahedron, all of them spanning the Euclidean space \mathbb{E}^3 .

In Sections 2–4 below, we construct all these configurations, and thus their existence is proved; the formula numbers of them at the location where they are actually constructed are as follows: 1: (3.8); 2: (4.1); 3: (4.2); 4: (3.14); 5: (4.5); 6: (3.10), (2.1), (2.2) and (2.3).

In each case, an essential part of the constructive proof is to exclude unintended incidences (i.e., incidences that do not belong to the given configuration, cf. [16], Section 2.6). We emphasize that this possibility has been checked in each case. However, in all but one case this part of the proof was omitted, to save space (the one exception is the case of type (2.2), where the problem has been indicated in Remark 2.2).

In stating our results, we avoid using the term “ d -dimensional” for a configuration that we construct in some space of dimension d . The reason is that the *dimension* of a configuration \mathcal{C} is defined as the largest dimension of the space that is spanned by \mathcal{C} (see p. 24 and Section 5.6 in [16]). Thus, we can only state that each configuration in our theorem above *spans* the given space (sometimes we also say, equivalently, that the configuration is *full-dimensional* in the given space). Investigating the actual dimension of our configurations is beyond the scope of this paper.

We think that the symmetry group is an essential property of our configurations realized in some Euclidean space. Although we did not investigate, we believe that several of these configurations have the maximal symmetry which is possible in the given space. (We note that e.g. for convex polytopes a question like this is far from trivial [12, 17].) It would also be interesting to know that for $d' < d$, whether there is a configuration in \mathbb{E}^d whose symmetry group is a subgroup of the orthogonal group of $\mathbb{E}^{d'}$ and which is not symmetrically realizable in $\mathbb{E}^{d'}$. We consider these and other related questions as a possible subject of future study.

Finally, in Section 5 we present an incidence conjecture. It is suggested by one of the new configurations that we found. This is a (100_4) configuration in projective 3-space consisting of four quintuples of complete pentalaterals (thus there are altogether 400 incidences, 100 for each quintuple). Informally, the conjecture states that the incidences belonging to three quintuples of the complete pentalaterals imply the remaining 100 incidences.

2 First examples of spatial configurations

Before presenting our more detailed constructions, we remark that there are spatial point-line configurations which need little or no construction. They are there in the zoo of geometric figures which have been known for a long time; one just has to realize them. The first configuration presented here is just such an example. It is a nice

$$(60_4) \tag{2.1}$$

configuration, provided by a polyhedron called the *great icosidodecahedron*. The lines of the configuration are spanned by the edges, and half of its points are the vertices of this polyhedron.

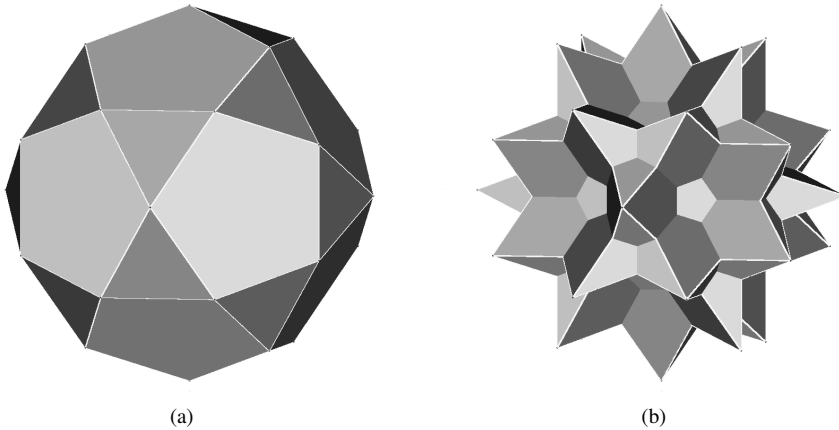


Figure 1: The icosidodecahedron (a), and the great icosidodecahedron (b).

The great icosidodecahedron is one of the 53 non-regular non-convex uniform polyhedra [8, 18, 22] (we note that a polyhedron is called *uniform* if its faces are regular polygons and its symmetry group is transitive on its vertices). It also occurs in [6], given by its Coxeter's symbol $\left\{ \begin{smallmatrix} 5/2 \\ 3 \end{smallmatrix} \right\}$; this indicates that it has triangles and pentagrams (i.e. $\{5/2\}$ *star-polygons*, see [6]) as faces. The number of these faces is 20 and 12, respectively. Its name refers to its close relationship with its convex hull, the icosidodecahedron (one of the Archimedean solids). In particular, its 30 vertices coincide with those of the icosidodecahedron.

The mutual position of its two pentagram faces in non-parallel planes is of two kinds (just like that of the faces of the regular dodecahedron): the angle between them is either $\arctan 2$, or $\pi - \arctan 2$. The angle between the planes of two such faces sharing a common vertex is the acute angle. In addition to its vertices, the edges of a pentagram have five other intersection points (these points can be called “internal vertices” if, instead of a pentagram, we speak of a—complete—pentalateral, cf. Section 5 below). These “internal vertices” of the pentagrams do not belong to the vertex set of the polyhedron (and, in strict sense, not even to that of the pentagram). But two pentagrams in planes with obtuse angle between them share such an “internal vertex”. Taking into account these latter points as

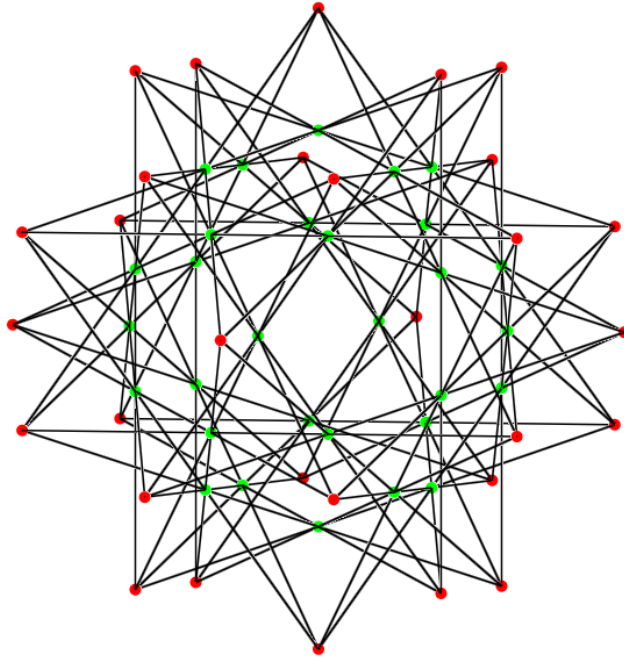


Figure 2: The (60_4) configuration associated with the great icosidodecahedron.

well, we have a system consisting of $12 \times 5 = 60$ edges, and $30+30$ points, the latter all tetravalent. Replacing the edges by the lines that are spanned by them, we obtain directly the configuration (2.1). The symmetry properties of the underlying polyhedron imply that this configuration has two orbits of points and a single orbit of lines. This relatively high degree of symmetry makes it particularly interesting.

The same configuration is also provided by two other types of polyhedra in the same natural way. Namely, both the *great icosihemidodecahedron* $\left\{ \begin{smallmatrix} 10/3 \\ 3 \end{smallmatrix} \right\}$ and the *great dodecahemidodecahedron* $\left\{ \begin{smallmatrix} 10/3 \\ 5/2 \end{smallmatrix} \right\}$ has a system of vertices and edges coinciding with that of the great icosidodecahedron; and, the other 30 points of the configuration are provided likewise. With the details omitted, we just remark that all three types of these polyhedra can be derived from the regular dodecahedron.

We note that a much more simple example can be obtained from the ordinary cube, in the following way. Let the points be the 8 vertices of the cube, the 6 centres of the faces of the cube and the centre of the cube. As lines, choose the 12 diagonals of the faces of the cube, plus the 3 lines between the centres of two opposite faces of the cube. Thus we obtain a (15_3) configuration.

Our next two examples require some more steps of construction. We start from two planar configurations. The first is a (25_4) configuration, due to Jürgen Bokowski ([15], Figure 4; see also [16], Figure 3.3.13). The other is closely related to this and is due to Branko Grünbaum ([15], Figure 9). These configurations are shown in our Figure 3.

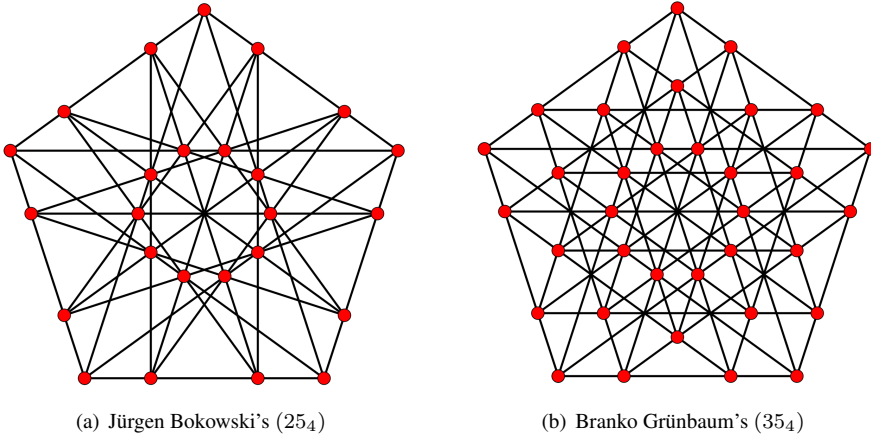


Figure 3: Two planar configurations.

Put 12 copies of the (25_4) configuration onto the faces of a regular dodecahedron so that the vertices of the pentagonal “frame” of the configuration coincide with the midpoints of edges of the dodecahedron. Then delete the edges of the dodecahedron and all the external edges of the 12 pentagonal frames. Thus we obtained a system, which is not a configuration; however, for each of its lines there are precisely four points incident with it. A system like this deserves to be the subject of a new definition.

Definition 2.1. A set consisting of p points and n lines is called a *semiconfiguration* if either of the following conditions hold:

- (1) each point is incident with precisely q lines; or
- (2) each line is incident with precisely k points.

The type of a semiconfiguration is denoted by (p_q, n_*) or (p_*, n_k) , respectively.

If one wants to specify which version is actually used, one may call it a P -semiconfiguration or an L -semiconfiguration; thus, the abbreviation refers to the fact that the incidences are uniformly distributed among the points or the lines, respectively. Clearly, a system is a configuration if and only if it is both P -semiconfiguration and L -semiconfiguration.

Using this notion, we see that the system we obtained in the present step is an L -semiconfiguration of type $(270_*, 240_4)$. It contains a class of 120 trivalent points, a class of 120 tetravalent points, and a class of 30 tetravalent points. These classes are distinguished by the position of their points; in fact, they are transitivity classes with respect to the symmetry group of the dodecahedron. To obtain a balanced configuration, take a second, concentric and homothetic copy of this semiconfiguration; thus we have an “outer shell” and an “inner shell” of points. Finally, connect the trivalent points of these shells by radial lines. Thus we have 60 new lines, all incident with four points. At the same time, the 2×120 trivalent points turned into tetravalent. Hence we obtained a configuration of type

$$(540_4). \tag{2.2}$$

It has 6 orbits of points and 7 orbits of lines. It is shown in Figure 4. Although this figure, due to the large number of its elements, is necessarily somewhat crowded, the two shells can be distinguished; the radial lines are indicated by orange colour.

Remark 2.2. The relative size of the outer and the inner shell must be chosen carefully in order to avoid unintended incidences. Clearly, there are infinitely many choices.

Using the (35_4) configuration, and proceeding analogously, we obtain a

$$(780_4) \tag{2.3}$$

configuration. It has 8 orbits of points and 13 orbits of lines.

We note that one may find several analogous cases on the basis of our examples (2.2) and (2.3). For example, a geometrically different but completely analogous (540_4) configuration can be obtained by starting from another planar (25_4) configuration that is shown in [2], Figure 4a. Furthermore, starting from the same planar configurations \mathcal{P} with pentagonal symmetry as above, one can also obtain geometrically different examples if one chooses other points of \mathcal{P} to tack onto the edges of the dodecahedron's face, and delete the appropriate lines (the radial lines will also be different in these cases).

The full icosahedral symmetry can also be reduced so as to obtain a chiral configuration in this construction as well. One just has to replace the starting planar configuration by a suitable chiral one. What is more, even movable spatial examples can be obtained in

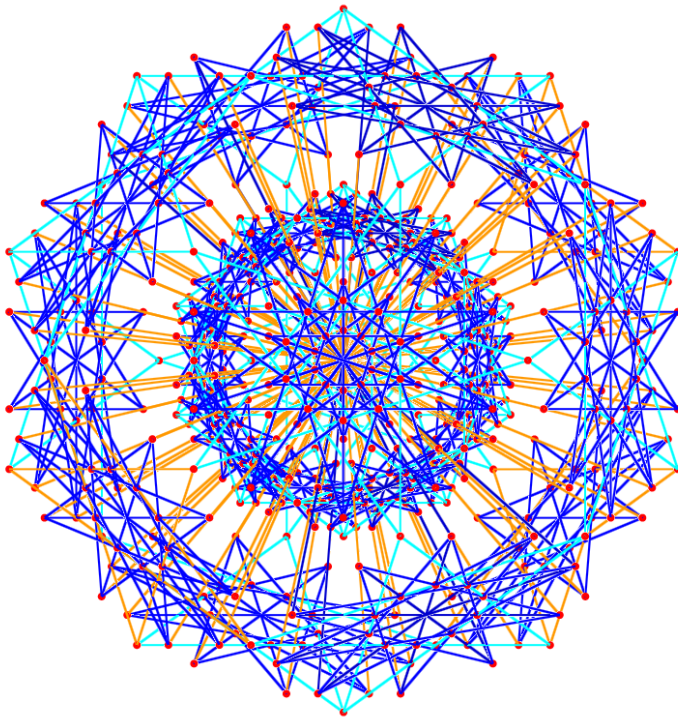


Figure 4: A (540_4) configuration.

this way, too; see a beautiful construction in [1] for movable planar (n_4) configurations (the example with 10-fold rotational symmetry given there in Figure 4 may be a possible candidate to this purpose). We do not pursue this idea here.

3 Classes of configurations based on regular polytopes

First we construct three infinite series of balanced configurations, so that we make use of the structure and symmetry properties of Platonic solids. (We note that these series have already been mentioned in [13], p. 327).

In what follows, TP denotes a Platonic solid whose 1-skeleton is a trivalent graph, i.e. TP is a tetrahedron, cube, or dodecahedron. It is well known that the Petrie polygon of these polytopes is a (regular, skew) quadrangle, hexagon, or decagon, respectively. (We recall that the Petrie polygon of a regular 3-polytope is a skew polygon such that any two consecutive edges, but no three, belong to a face of the polytope [6].) Given a Petrie polygon, consider for each of its vertices the third edge emanating from it but not belonging to the Petrie polygon. Take a point on each of these edges such that it subdivides, but not bisects, the edge in an arbitrary but fixed ratio; moreover, it is closer to the endpoint of the edge belonging to the Petrie polygon than to the other endpoint. Connect these points by straight line segments in the cyclic order induced by the Petrie polygon. What is obtained is again a regular skew polygon, clearly having the same number of edges as the Petrie polygon we started from. We shall call it a *P-polygon*. The vertices of a *P-polygon*, together with the lines spanned by their edges, form a (p_2) configuration, where p is 4, 6 or 10 according as TP is the tetrahedron, the cube or the dodecahedron, respectively. Clearly, the number of Petrie polygons and *P-polygons* is the same in a given TP, that is 3, 4 or 6, respectively. It follows that taking the disjoint union of all these *P-polygons* (more precisely, the corresponding (p_2) configurations), one obtains a (non-connected) configuration whose type is

$$(12_2), (24_2) \text{ or } (60_2). \tag{3.1}$$

We call this configuration a *P-system* (see Figure 5).

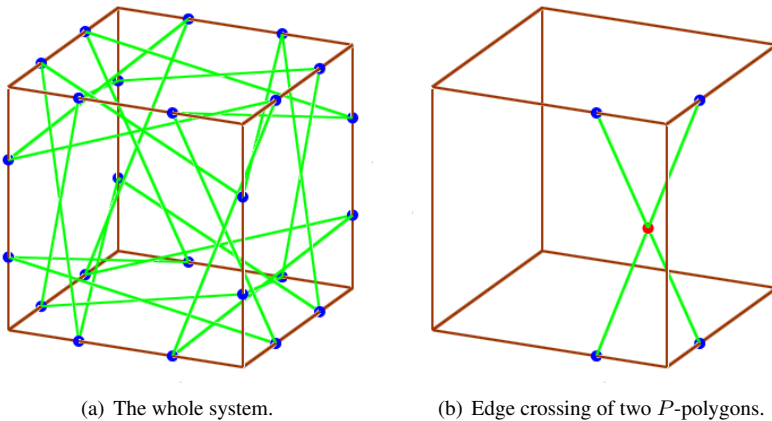


Figure 5: The *P-system* (24_2) in the cube.

We emphasize that when constructing the P -system, the same ratio is used in the definition of each P -polygon. Thus it follows from the construction that the P -system inherits the (geometric) symmetry of TP, i.e. it has the same symmetry group. Moreover, note that both its points and its lines form a single transitivity class.

Looking at this configuration more closely, we find that it can be extended to form a non-balanced but connected configuration. For, observe that each edge e of TP belongs to precisely two distinct Petrie polygons. Moreover, these Petrie polygons are mirror images of each other with respect to the mirror plane of TP containing the edge e . It follows that the edges of the corresponding P -polygons cross each other in the vicinity of e in a point lying in that mirror plane. Furthermore, this point also lies in the mirror plane that is perpendicular to the former plane and bisects e . This amounts to saying that this point is on the line connecting the centre of TP with the midpoint of e . (For an example of such a crossing point in the cube, see Figure 5b, where it is indicated by red colour).

The number of these crossing points equals the number of the edges of TP. Hence, adding them to the P -system, we obtain a new configuration, which we shall call a P -configuration. Its type is

$$(18_2, 12_3), \quad (36_2, 24_3) \quad \text{or} \quad (90_2, 60_3). \quad (3.2)$$

The special position of the crossing points provides the possibility of a further extension of this configuration, as follows. Shrink the 1-skeleton of the original TP until the midpoints of its edges coincide with the crossing points, and add it to the configuration. Then, remove the vertices of this skeleton, and replace each of its edges by the line spanned by it. The new structure that is obtained is a subfiguration.

A *subfiguration* (p_q, n_k) is defined as a set of points and lines with incidences as in the definition of configurations, but with the difference that each of the p points is incident with at most q of the n lines, and each line is incident with at most k of the points [16]. If we want to emphasize that the number of the missing incidences is s (in comparison to a (p_q, n_k) configuration), we say that it is an $\#s$ -subfiguration. (We note that in the converse case the notion of a *superfiguration* has also been introduced, in a similar way, in [16]).

Thus, the subfiguration that is obtained is of type

$$(18_3)^-, \quad (36_3)^- \quad \text{or} \quad (90_3)^-, \quad (3.3)$$

with s equal to 24, 48 or 120, respectively. (Here the superscript refers to the missing incidences; we use this notation to avoid confusion with a configuration of type (n_k) .) One half of the missing incidences of this subfiguration belong to the one subsystem, and the other half of them belongs to the other subsystem, of which it is composed. For example, in the cubic case there are 24 missing incidences because the points of the P -system are of valency two, instead of three; and there are 24 other missing incidences, since all the 8 vertices of the cube skeleton (which are of valency three, and have been removed) are missing. We shall call these two kinds of defective points type A and type B, respectively. Likewise, the corresponding subsystems will be referred to as type A and B, respectively.

Due to the equal number of the defective points of the two types in the two subsystems, this subfiguration can serve as a repetitive unit; hence we shall call it an R -unit. The repetition is meant in the following way. Take a copy of an R -unit R_1 , and shrink it so as to obtain a homothetic copy R_2 , such that the points of type A of R_2 fit onto the corresponding lines in the subsystem of type B of R_1 ; then take the union $R_1 \sqcup R_2$. Using Figure 5a,

in the cubic case this can simply be conceived as if the brown lines belonged to R_1 and the green lines belonged to R_2 .

Observe that the new figure obtained in this way is again an $\#s$ -subfiguration such that s remains the same; for, half of the defects both in R_1 and R_2 have been repaired, but the other half in both of them remained. At the same time, both the number of points and lines have been doubled.

The operation that we applied here is not simply a disjoint union; for, new incidences occurred, and (in our particular case) the result is a connected structure. Thus we think it is appropriate to fix these properties in a separate definition. As that will refer not only to configurations, first we give a common name for all the four related types of structures used in this paper: we shall call such a structure an *X-figuration*, where “X-” may mean either “con”, “semicon”, “sub” or “super”.

Definition 3.1. By the *incidence sum*¹ of X-figurations \mathcal{F}_1 and \mathcal{F}_2 we mean the X-figuration \mathcal{F} which is the disjoint union of \mathcal{F}_1 and \mathcal{F}_2 , together with a specified set $I \subseteq \mathcal{P}_1 \times \mathcal{L}_2 \cup \mathcal{P}_2 \times \mathcal{L}_1$ of incident point-line pairs, where \mathcal{P}_i denotes the point set and \mathcal{L}_i denotes the line set of \mathcal{F}_i , for $i = 1, 2$. We denote it by $\mathcal{F}_1 \oplus_I \mathcal{F}_2$.

Note that \mathcal{F}_1 and \mathcal{F}_2 may form distinct incidence sums depending on the set I ; we do not consider here such cases; on the other hand, if the set I is fixed and is clear from the context (as in our present case), it can be omitted from the operation symbol.

Accordingly, in the present step of our construction we obtained the subfiguration of the form $R_1 \oplus R_2$. Furthermore, it is clearly seen that the process by which we obtained $R_1 \oplus R_2$ from R_1 can be repeated arbitrary many times. Thus, let λ be a shrinking factor defined by the equality $R_2 = \lambda R_1$, and set $R = R_1$. Then, starting with R , we obtain after $t - 1$ steps the subfiguration

$$\bigoplus_{i=1}^t \lambda^{i-1} R, \quad (3.4)$$

which is still an $\#s$ -subfiguration with $s = 24, 48$ or 120 , and is of type

$$((18t)_3)^-, \quad ((36t)_3)^- \quad \text{or} \quad ((90t)_3)^-, \quad (3.5)$$

respectively.

Finally, we have to extend this subfiguration, so as to obtain a configuration. First we construct a unit which, when added, closes the structure “outside”. This construction also is analogous for each of the three types of TP; we explain it in the case of the tetrahedron. Start from the 1-skeleton of a regular tetrahedron, and take the midpoints of its edges. Add these points to the structure, so that one obtains a spatial graph with 10 vertices and 12 edges, such that four vertices are trivalent and six vertices are bivalent. For each bivalent vertex, take a line connecting it to the centre of the tetrahedron, and shift the vertex along this line outwards, each to the same extent; simultaneously, the edges incident to these vertices are stretched, and remain straight line segments. Although any ratio would serve our purpose, we note that if the distance of these shifted vertices from the centre is twice that of the original, then the angle between any two adjacent edges is $\arccos(-1/3)$. This is the famous “tetrahedral bond angle” in organic chemistry, and the figure that we obtained is precisely the carbon skeleton of a hydrocarbon molecule called *adamantane*², well known

¹The present (improved) version of this definition was proposed by Tomaž Pisanski.

²The name refers to the fact that this is a repetitive unit of the diamond crystal lattice.

to chemists. This skeleton is shown in Figure 6 (by courtesy of H. Ramezani, from [24]). A related figure can be obtained from the 1-skeleton of either kind of TP in an analogous way; so we shall call each of them an *adamantane skeleton*. (Note that they have the symmetry of the type of TP from which it has been derived).

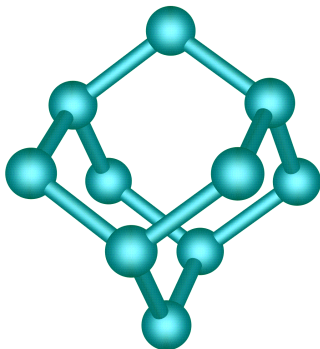


Figure 6: The adamantane skeleton.

We extend the adamantane skeleton in the following way. Take the 1-skeleton of a TP of suitable size and position (and of the corresponding type) such that the midpoints of its edges coincide with the bivalent vertices of the adamantane skeleton. Form the union of these figures, then replace each of its edges by the line spanned by it. We obtain a P -semiconfiguration of type

$$(14_3, 18_*) , \quad (28_3, 36_*) \quad \text{or} \quad (70_3, 90_*) . \quad (3.6)$$

This semiconfiguration will serve as a closure unit so as to close our construction “outside”. In fact, observe that it has defective lines, i.e., lines that are incident with two points, instead of three (the lines corresponding to the original half-edges of TP). The number of these lines is 12, 24 or 60, respectively, and this is the same as the number of the missing incidences. On the other hand, it is half of the number of missing incidences of our subfiguration (3.4) (these latter come from points of type A). Furthermore, by taking a copy of a suitable size of this closure unit, one can form the incidence sum of it with the subfiguration (3.4). The result is a semiconfiguration of type

$$\begin{aligned} & ((18t + 14)_3, (18(t + 1))_*), \quad ((36t + 28)_3, (36(t + 1))_*) \\ & \text{or} \quad ((90t + 70)_3, (90(t + 1))_*). \end{aligned} \quad (3.7)$$

The very last step of our construction is to close our system “inside”. This is very simple, since 4, 8 or 20 points are missing from the smallest R -unit of the subfiguration (3.4) (points of type B , each representing three incidences). These are nothing else than the vertices of a tetrahedron, cube or dodecahedron, respectively. We just add these points to the system, and our construction is ready, resulting in three infinite series of balanced configurations, whose type is

$$(18(t + 1)_3), \quad (36(t + 1)_3) \quad \text{and} \quad (90(t + 1)_3), \quad (t = 1, 2, \dots). \quad (3.8)$$

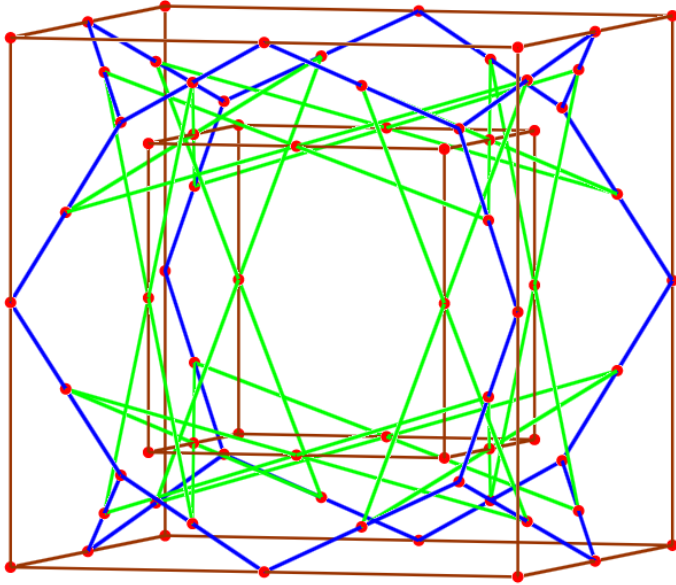


Figure 7: A (72_3) configuration: the cubic case with $t = 1$ of (3.8).

We note that the latter “closure units”, consisting merely of points (but with fixed mutual position), can also be conceived as semiconfigurations. Denoting them by C_I , and those in (3.6) by C_O , we see that our configurations of type (3.8) can be described (and in fact, have been constructed) in the following form:

$$C_O \oplus \left(\bigoplus_{i=1}^t \lambda^{i-1} R \right) \oplus C_I, \tag{3.9}$$

where the middle term is the subfiguration (3.4).

We emphasize that throughout the construction, the original symmetry of the TP we started from is preserved. Thus the (geometric) symmetry group of all of the configurations which we obtained here is equal to that of the corresponding Platonic solid.

Our next construction provides a sporadic example. In this construction we apply some structural elements that have already been constructed above.

Start from the *compound* of five tetrahedra, which can be obtained by inscribing these tetrahedra in a regular dodecahedron [6] (see Figure 8). This is made possible by the property that the set of vertices of the dodecahedron can be partitioned into five quadruples such that within a quadruple, the vertices are at a distance 3 from each other (regarded in the graph of the dodecahedron). The same compound also determines a partition of the set of edges of the dodecahedron into five sextuples in the following way. Consider a compound inscribed in a dodecahedron of a fixed size. Apply a dilation to this compound. It is chosen so that the following condition holds. Let $ABCD$ be any path of length 3 in the graph of the dodecahedron, and let M be the midpoint of the corresponding tetrahedron edge AD in the compound. Then the dilate M' of M coincides with the midpoint of the edge BC of the dodecahedron. Thus, each sextuple of the dodecahedron edges corresponds

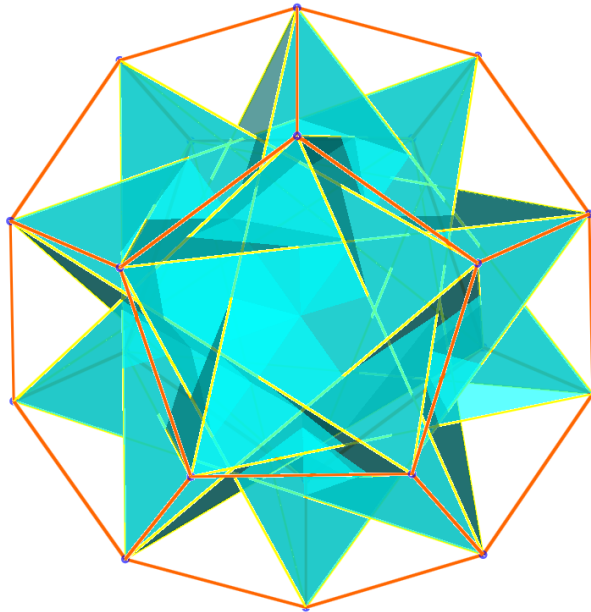


Figure 8: The compound of five tetrahedra inscribed in a dodecahedron.

to the set of edges of a tetrahedron in the compound. A consequence is that one can inscribe a (tetrahedral) adamantane skeleton in each sextuple of the edge-midpoints of the dodecahedron so that the bivalent vertices of the adamantane skeleton coincide with these edge-midpoints. In such an inscribed adamantane skeleton, we inscribe a P -configuration (3.2) constructed previously in this section, in the sense that the points of the latter fit onto the edges of former; moreover, this is performed so that the original local tetrahedral symmetry is preserved. Then we add a tetrahedron skeleton to this structure, so that the midpoints of the tetrahedron edges coincide with the “crossing points” of the P -configuration (cf. Figure 5b). Replace each edge by the corresponding line; thus, taking into account all the points, lines and incidences, we obtained a $(32_*, 30_3)$ semiconfiguration inscribed in a sextuple of edges of the dodecahedron.

By inscribing altogether five copies of this semiconfiguration into the 1-skeleton of the dodecahedron in the same way (and replacing its 30 edges by lines), a configuration of type

$$(180_3) \tag{3.10}$$

is obtained (it is mentioned in [13], too). The symmetry group of the compound of tetrahedra which we started from is the rotation group T of the tetrahedron; thus this compound is a *chiral* figure, i.e. it has no mirror symmetry. Our new configuration inherited this symmetry group, so it is a chiral configuration. Its set of points decomposes into 6 orbits, while there are 4 orbits of lines. These latter orbits are indicated with different colours in Figure 9. Note, in addition, that the structure of this configuration is closely related to those described above in this section. In fact, the orbits correspond to those of the tetrahedral case of (3.8), with $t = 1$. The only difference is that the outermost tetrahedral orbit is

replaced by a dodecahedral orbit, and the others are multiplied by five.

Our last class constructed here is based on certain regular 4-polytopes. We start from a TP that we used above. Take a P -system inscribed in it (that is, inscribed in the sense that the points of the configuration lie on the edges of TP). Then take a smaller homothetic copy of TP in concentric position, and also inscribe a P -system in this copy. The smaller P -system is chosen so that it is not the homothetic copy of the larger one, but each triple of their vertices in the vicinity of a vertex of the TP (determining it) is relatively at a smaller distance from that vertex, than the triple of vertices of the larger P -system is from the corresponding vertex of the larger TP.

If the two P -systems were homothetic copies of each other (with respect to their common centre), then the lines connecting their corresponding points would meet all in a common point (in fact, in the centre). However, due to our particular choice, these lines meet now in threes, forming altogether 4, 8 or 20 points of intersection (depending on the type of TP). This is a consequence of the threefold rotational symmetries of TP. Thus for each such triples of lines in the vicinity of a given vertex v of TP there is a point of intersection which lies on the axis of rotation connecting v with the centre, and this point also is located in the vicinity of v . We shall not use these points later, they just served to explain the location of the connecting lines. On the contrary, we need the connecting lines in the following, so we shall call them c -lines. Another condition for the c -lines is that they are not perpendicular to the edges of TP (this can also be ensured by a suitable choice of the P -systems).

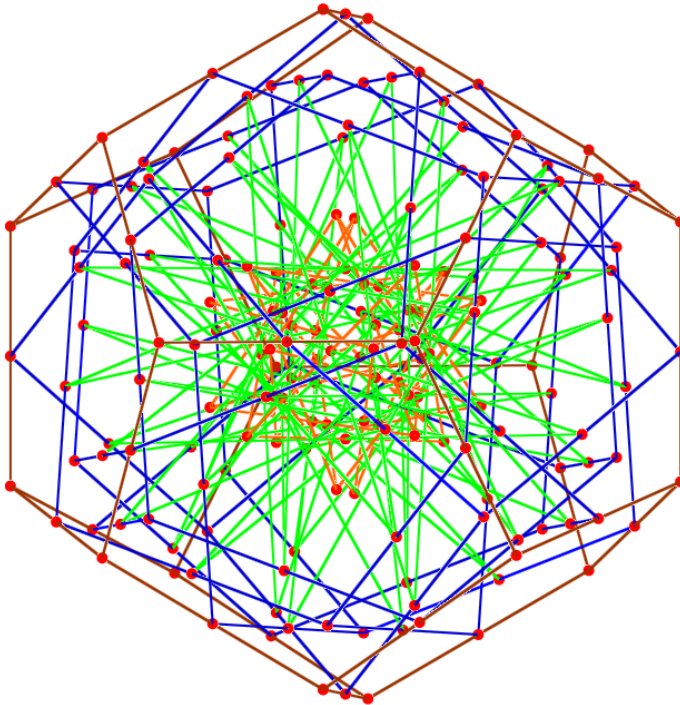


Figure 9: A (180_3) configuration derived from the compound of five tetrahedra.

The two copies of the P -system, together with the c -lines, form a configuration whose type is

$$(24_3, 36_2), \quad (48_3, 72_2) \quad \text{or} \quad (120_3, 180_2) \tag{3.11}$$

(see (3.1) above in this section for the type of the P -system).

By adding the crossing points of the lines of the P -systems, discussed in the first construction of this section, one obtains an s -subfiguration of type

$$(36_3)^-, \quad (72_3)^- \quad \text{or} \quad (180_3)^- \tag{3.12}$$

with $s = 24, 48$ or 120 , respectively (note that the missing incidences are equally distributed between the points and the lines).

Let now P be a regular 4-polytope whose facets are of type TP. Thus P is a regular 4-simplex, a regular 4-cube, or a regular 120-cell. It has 5, 8, or 120 facets, respectively [6]. We put a copy of the subfiguration (3.12) in each of the facets of P , so that each of the points of such a subfiguration is in the interior of the facet, and the whole system preserves the original symmetry of P . This results in a (non-connected) subfiguration of type

$$(180_3)^-, \quad (576_3)^- \quad \text{or} \quad (21\,600_3)^-, \tag{3.13}$$

respectively. We convert it to a connected structure as follows.

Consider a facet F of P , and a c -line connecting two points of the P -systems within F . Clearly, this c -line intersects the edge of F , which is in the vicinity of these points (since it is within the local mirror plane of the facet lying on that edge). There are three facets of P meeting in a common edge; thus the point of intersection of the c -lines is trivalent. There are two such points on each edge of P . Thus, by adding all these points to our structure (3.13), the number of (trivalent) points increases by 20, 64 or 2400, respectively.

Note that with this completion the number of the missing incidences has been halved. The other half is supplied as follows. Take the 1-skeleton of TP, and put a pair of its copies in each facet of P . The size and location of these copies is such that for each of them there is a P -system in (3.13) in which the crossing points coincide with the midpoints of the edges in TP. Finally, replace each of the edges by the line spanned by it. In this way we supplied not only the rest of the missing incidences, but completed the structure by 40, 128 or 4800 points, and by 60, 192 or 7200 lines, respectively. As a result, we obtained three new balanced configurations in \mathbb{E}^4 whose type is

$$(240_3), \quad (768_3) \quad \text{and} \quad (28\,800_3). \tag{3.14}$$

Note that in each step of the construction the original symmetry was preserved, thus the symmetry group of these configurations is equal to that of the regular 4-polytope we started from. In each of these three configurations of type (n_3) the number n is twice the order of the corresponding symmetry group. Furthermore, in all three cases, there are 7 orbits of points and 5 orbits of lines.

4 Cartesian product of point-line configurations

We explore here the following notion.

Definition 4.1. Let \mathcal{C}_1 be a (p_q, m_k) configuration in an Euclidean space \mathbb{E}_1 and \mathcal{C}_2 be an (r_s, n_k) configuration in an Euclidean space \mathbb{E}_2 . Observe that these two configurations

have the same number k of points on each line. The Cartesian product of \mathcal{C}_1 and \mathcal{C}_2 is the $((pr)_{(q+s)}, (pn+rm)_k)$ configuration $\mathcal{C}_1 \times \mathcal{C}_2$ in $\mathbb{E}_1 \times \mathbb{E}_2$ whose point set is the Cartesian product of the point sets of \mathcal{C}_1 and \mathcal{C}_2 and where there is a line incident to two points (x_1, x_2) and (y_1, y_2) if and only if either $x_1 = y_1$ and there is a line incident to x_2 and y_2 in \mathcal{C}_2 , or $x_2 = y_2$ and there is a line incident to x_1 and y_1 in \mathcal{C}_1 .

We emphasize that the incidence degree of the lines of the two configurations \mathcal{C}_1 and \mathcal{C}_2 have to coincide. Therefore, in terms of abstract algebra, this product is merely a partial operation on the set of configurations (it is not defined for any pair of configurations). This shows that, when applied to configurations, the analogy of this kind of product with the classical Cartesian product of other objects (like polytopes, graphs, etc.) is not complete, in strict sense. On the other hand, one observes that if the incidence degrees differ, then this product can still be defined, and it results in a *semiconfiguration* (see Definition 2.1). Furthermore, the definition of the product can also be extended to semiconfigurations. Thus, the larger set of semiconfigurations will be closed under this product, and the partial operation extends to a total operation. Hence using the term *Cartesian product* is still justified, in this sense.

A consequence of the definition that if both \mathcal{C}_1 and \mathcal{C}_2 is full-dimensional in \mathbb{E}_1 resp., in \mathbb{E}_2 , then $\mathcal{C}_1 \times \mathcal{C}_2$ is also full-dimensional in $\mathbb{E}_1 \times \mathbb{E}_2$. We note, however, that one cannot say that in the product the dimensions of \mathcal{C}_1 and \mathcal{C}_2 are added (see the remark on the *dimension* of a configuration in the Introduction). Thus, we do not think that our definition of product would automatically imply the additivity of dimension of configurations.

We remark that our motivating example is the spatial version of the Gray configuration consisting of 27 points and 27 lines. Actually, it provided the intuitive idea for the definition above, see Figure 10. We note that the (27_3) Gray configuration can in fact be decomposed into the product of three $(3_1, 1_3)$ configurations; however, to visualize the intuitive idea we think the decomposition given in Figure 10 is better. More generally, the $((k^k)_k)$ generalized Gray configuration is the k th power of the $(k_1, 1_k)$ configuration. (For the Gray configuration and the generalized Gray configuration, see [23]).

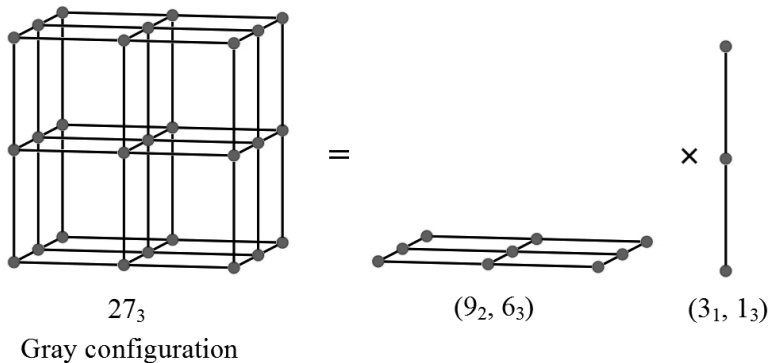


Figure 10: The Gray configuration as a product.

Accordingly, we formulated the definition above in the context of Euclidean geometry. However, an analogous construction also works in projective spaces, which can be described as follows. First recall that a d -dimensional (real) projective space \mathbb{P}^d can be

defined as the set of one-dimensional (linear) subspaces of \mathbb{R}^{d+1} . Given \mathbb{P}^d in this way, let $\{e_1, \dots, e_k, e_{k+1}, \dots, e_{d+1}\}$ be a basis in the corresponding vector space \mathbb{R}^{d+1} . Now we have a projective space \mathbb{P}^k given as the set of one-dimensional subspaces of the vector space spanned by the basis $\{e_1, \dots, e_k, e_{k+1}\}$, and a projective space \mathbb{P}^{d-k} determined analogously by the basis $\{e_{k+1}, \dots, e_{d+1}\}$. In this case we say that \mathbb{P}^d is decomposed to the *direct sum* of the spaces \mathbb{P}^k and \mathbb{P}^{d-k} . More generally, let \mathbb{P}^k and \mathbb{P}^l be two projective spaces. If there are projective isomorphisms $\mathbb{P}^k \cong \overline{\mathbb{P}}^k$ and $\mathbb{P}^l \cong \overline{\mathbb{P}}^l$ such that $\overline{\mathbb{P}}^k$ and $\overline{\mathbb{P}}^l$ form a direct sum decomposition of a space \mathbb{P}^{k+l} , then \mathbb{P}^{k+l} is said to be the direct sum of the spaces \mathbb{P}^k and \mathbb{P}^l . It is not hard to see that this definition determines a unique bijection from the Cartesian product $\mathbb{P}^k \times \mathbb{P}^l$ to \mathbb{P}^{k+l} ; thus the points in \mathbb{P}^{k+l} can uniquely be represented by pairs (P, Q) with $P \in \mathbb{P}^k, Q \in \mathbb{P}^l$.

Now given the configurations \mathcal{C}_1 and \mathcal{C}_2 , embedded in \mathbb{P}_1 and \mathbb{P}_2 , respectively, both full-dimensional, the point set of their product consists of pairs (P_1, P_2) with $P_1 \in \mathbb{P}_1, P_2 \in \mathbb{P}_2$; furthermore, two points (P_1, P_2) and (Q_1, Q_2) are connected by a line in the product if and only if either $P_1 = Q_1$ and P_2 and Q_2 are connected in \mathcal{C}_2 , or $P_2 = Q_2$ and P_1 and Q_1 are connected in \mathcal{C}_1 . Clearly the product configuration is full-dimensional in the direct sum of \mathbb{P}_1 and \mathbb{P}_2 , and its type is determined by the types of \mathcal{C}_1 and \mathcal{C}_2 in the same way as before.

It is clear that the product of a configuration with itself can be repeated, i.e. it can be raised to a power; given a configuration of a suitable type, this may provide a balanced configuration (as we have seen above in the case of generalized Gray configurations). In what follows we give some classes of such examples.

EXAMPLES: CLASS 1.

Consider n lines in the projective plane \mathbb{P}^2 in general position, i.e. such that no more than two of them intersect in one point. Together with all their points of intersection, they form a configuration $((\binom{n}{2})_2, n_{n-1})$, which we call a *complete n -lateral*. (We note that it has already appeared in this context in [21], see p. 85, Satz 21.)

Taking $(2k+1)$ -laterals $(k = 2, 3, \dots)$, we have the following infinite series of balanced configurations obtained as powers:

- complete 5-lateral: $(10_2, 5_4)^2 = (100_4) \subset \mathbb{P}^4$
- complete 7-lateral: $(21_2, 7_6)^3 = (9261_6) \subset \mathbb{P}^6$
- complete 9-lateral: $(36_2, 9_8)^4 = (1\,679\,616_8) \subset \mathbb{P}^8$
- complete 11-lateral: $(55_2, 11_{10})^5 = (503\,284\,375_{10}) \subset \mathbb{P}^{10}$
- \vdots

The general element of this series can be given as

$$\left(\left(\binom{2k+1}{2} \right)_2, (2k+1)_{2k} \right)^k = \left(\left(\binom{2k+1}{2} \right)_{2k}^k \right), \tag{4.1}$$

and is full-dimensional in the projective space \mathbb{P}^{2k} .

EXAMPLES: CLASS 2.

Again, let $k = 2, 3, \dots$, and start from the simple configuration $((2k)_1, 1_{2k})$ consisting of $2k$ points and a single projective line. Raise it to the power $2k - 2$ so as to obtain a

configuration of type

$$\left(\left((2k)^{2k-2} \right)_{2k-2}, \left((2k)^{2k-3} (2k-2) \right)_{2k} \right).$$

Then form the product of this configuration with the complete $(2k + 1)$ -lateral. The result is a balanced configuration of type

$$\left(\left((2k)^{2k-2} (2k + 1) k \right)_{2k} \right), \tag{4.2}$$

which spans the projective space \mathbb{P}^{2k} .

We note that in this series the number of points grows faster than in the former one. For comparison, we give the type of the first four members:

$$(160_4), (27\ 216_6), (9\ 437\ 184_8), \left((5.5 \cdot 10^9)_{10} \right).$$

EXAMPLES: CLASS 3.

The method of scaffolding polytopes and raising to powers can also be combined to obtain balanced configurations. Here we construct in this way a finite class of examples in Euclidean space.

We start from the well-known Archimedean solid, the rhombicosidodecahedron [9] (see Figure 11). It can obtained from the regular dodecahedron by truncation [5, 11]; thus it is bounded by 12 pentagons, 20 triangles and 30 squares, originating from the faces, vertices and edges of the dodecahedron, respectively. Its 60 vertices can be given in the following form:

$$(\pm 1, \pm 1, \pm \tau^3)^c, (\pm \tau, \pm 2\tau, \pm \tau^2)^c, (0, \pm(2 + \tau), \pm \tau^2)^c, \tag{4.3}$$

where the superscript denotes that all cyclic permutations of the coordinates are to be taken, and τ denotes the golden mean: $\tau = \frac{1}{2} (1 + \sqrt{5})$.

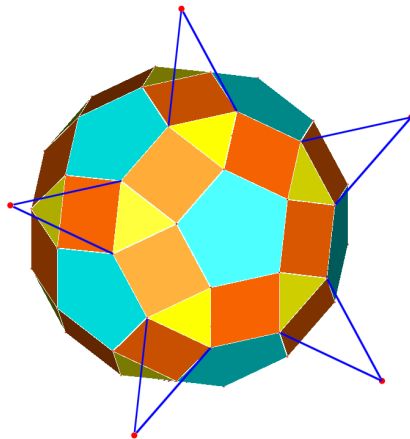


Figure 11: The rhombicosidodecahedron with a regular pentalateral.

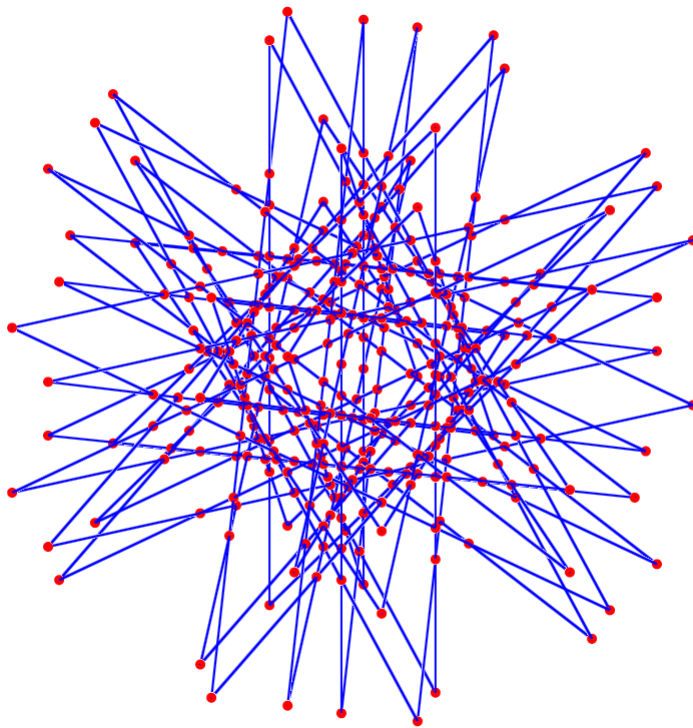


Figure 12: The $(360_2, 60_{12})$ configuration based on the rhombicosidodecahedron.

We use the rhombicosidodecahedron (briefly, RID) as a scaffolding to construct a class of configurations whose types are as follows:

$$(120_2, 60_4); \quad (180_2, 60_6); \quad (240_2, 60_8); \quad (300_2, 60_{10}); \quad (360_2, 60_{12}). \quad (4.4)$$

Observe that in the boundary complex of the RID, the *link* (we use this term following e.g. [26], p. 237) of a pentagonal face forms a regular decagon. Connecting by straight lines the vertices of this decagon that are pairwise at a distance 3 from each other, one obtains a $(10_2, 5_4)$ configuration, which is a regular complete pentalateral (see the definition of a complete n -lateral in our examples of Class 1; now we are in \mathbb{E}^3 , and this figure is regular in Euclidean sense, i.e. its symmetry group is the dihedral group D_5). Figure 11 shows which one of the two possible positions of such a pentalateral is chosen (it can also be seen that five of its points are inside the RID). Clearly there are altogether 12 such regular pentalaterals, and they form a single orbit under the action of the symmetry group of the RID (this group is obviously the full icosahedral group I_h). Hence we obtain a system of 60 lines, which together with the vertices of the pentalaterals form a $(120_2, 60_4)$ configuration (see the first type of (4.4)).

It turns out, however, that there are altogether 360 intersection points of these 60 lines, so that the whole set of points and lines forms a $(360_2, 60_{12})$ configuration (the last type of (4.4)). This can be explained using the symmetry properties of the regular dodecahedron or, equivalently, of the RID. First, recall that a regular dodecahedron has altogether 15

mirror planes. For each edge of the dodecahedron, there are precisely three mirror planes in special position: one lies on it, one is its perpendicular bisector and one is parallel to it. The others intersect it obliquely. The 60 lines are parallel in pairs to the 30 edges of the dodecahedron (and, none of them lie on a mirror plane). Hence, for each of these lines, too, there are precisely 12 mirror planes in oblique position. Because these planes are mirror planes, their intersections with the lines provide points in which precisely two of the 60 lines meet. This is equivalent to the fact that on a given line no two of the 12 intersection points coincide. For, the coincidence means that more than two planes (not perpendicular to each other) meet in such a point, which implies that more than two lines meet in that point. But such multiple intersection does not occur here; this can be visually checked in a model constructed by a dynamic geometry software³. Figure 12 shows a screenshot of this model.

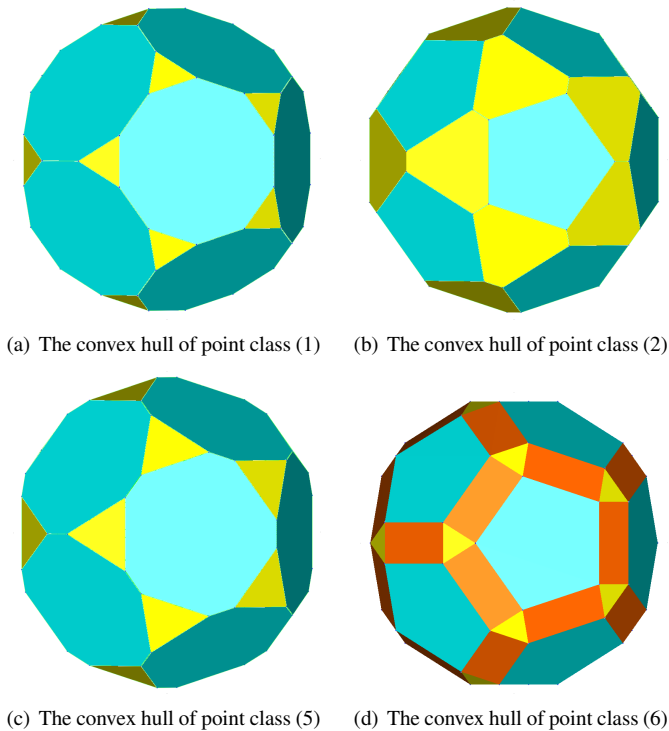


Figure 13: Supporting polytopes for the $(360_2, 60_{12})$ configuration.

The 360 points can be partitioned into 6 classes with respect to their distance from the origin, each containing 60 points; they also form orbits under the action of the group I_h . Because of this latter property, the convex hull of each of them is a vertex-transitive polytope (in fact, in each case it is combinatorially equivalent to an Archimedean solid, see Figure 13). In this way these polytopes form a nested sequence, and are particularly suitable for visualizing the structure of our $(360_2, 60_{12})$ configuration. Hence we call them

³Euler 3D developed by Tamás Petró.

<http://www.mozaik.info.hu/Homepage/Mozaportal/MPeuler3d.php>

supporting polytopes of this configuration. In Figure 13 they are shown in the order of growing size. The convex hull of class (3) is just the RID we started from; and class (4) is a homothetic copy of class (2) (actually, τ times larger), so we did not repeat the corresponding polytope in the figure. Note that these six polytopes fall by two into three combinatorial types.

Observe that these 6 classes of points can be switched in and out independently of each other; hence, all but last of the five types in list (4.4) above can be realized as more than one geometrically distinct configuration. (Among them, isomorphism may occur; we did not investigate this possibility.) By raising them to a suitable power, one obtains balanced configurations of the following types:

$$\begin{aligned}
 \bullet (120_2, 60_4)^2 &= (14\,400_4) && \subset \mathbb{E}^6 \\
 \bullet (180_2, 60_6)^3 &= (5\,832\,200_6) && \subset \mathbb{E}^9 \\
 \bullet (240_2, 60_8)^4 &= (3\,317\,760\,000_8) && \subset \mathbb{E}^{12} \\
 \bullet (300_2, 60_{10})^5 &= \left((2.43 \cdot 10^{10})_{10} \right) && \subset \mathbb{E}^{15} \\
 \bullet (360_2, 60_{12})^6 &= \left((2^{18} \cdot 3^{12} \cdot 5^6)_{12} \right) && \subset \mathbb{E}^{18}.
 \end{aligned} \tag{4.5}$$

Due to the geometric differences we mentioned just above, a number of geometrically distinct cases occur here as well. For example, even for the $(14\,400)_4$ configuration, this amounts to 125 geometrically distinct cases (possibly not all combinatorially distinct).

5 An incidence conjecture

Recall our definition of a complete n -lateral in the preceding section (examples of Class 1). For the case $n = 5$ we use the term *complete pentalateral*. The points of this configuration we shall also call *vertices*. The following properties of complete pentalaterals are well known (cf. [21], pp. 85–86, Satz 21 and Aufgabe 3b).

Proposition 5.1. *There is a unique complete pentalateral in the projective plane \mathbb{P}^2 up to combinatorial equivalence. It decomposes \mathbb{P}^2 into one pentagonal, five quadrangular and five triangular regions.*

We shall call the vertices of the complete pentalateral belonging to the pentagonal region *internal vertices*, while the other *external vertices*. The existence and uniqueness of the pentagon guarantees that such a distinction is indeed possible:

Proposition 5.2. *The partition of the set of vertices of the complete pentalateral to internal and external vertices is well-defined.*

The structure of the tiling of \mathbb{P}^2 just described is shown in Figure 14. Figure 14a also illustrates that the group of a complete pentalateral is isomorphic to D_5 , i.e. to the symmetry group of a regular pentagon (the latter in Euclidean sense). Recall that the *group of a configuration* is defined as the group of the permutations (both the points and lines among themselves) preserving incidences [7].

We have seen that squaring a complete pentalateral results in a configuration (100_4) in projective 4-space (cf. Class 1 in the preceding section). This configuration can nicely be visualized by projecting it into three dimensions and restricting ourselves to Euclidean space. To this end, a useful tool is the Schlegel diagram [14, 26]. In fact, there are 10 copies

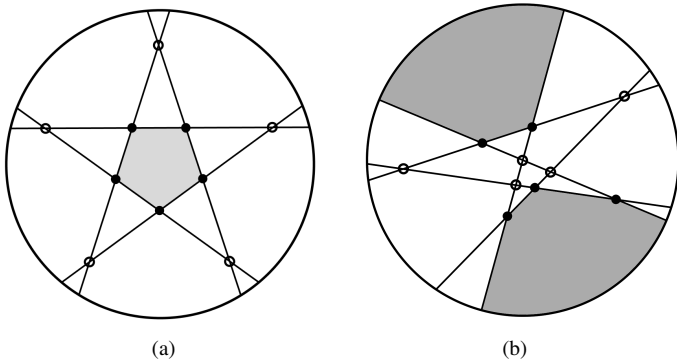


Figure 14: The decomposition of the projective plane by a complete pentalateral, in two versions, with the pentagonal region shaded. The internal and external vertices are indicated by black and white vertices, respectively.

of the complete pentalateral in the configuration (100_4) such that they can be inscribed in the 10 pentagonal 2-faces of the Cartesian product of two pentagons, which is a 4-polytope. The Schlegel diagram of this latter polytope is depicted in Figure 15, while the image of the (100_4) configuration is shown in Figure 16.

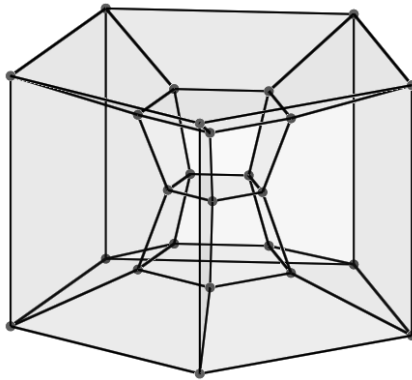


Figure 15: Schlegel diagram of the Cartesian product of two pentagons.

The following conjecture is motivated by the three-dimensional image of the (100_4) configuration. We will denote a complete pentalateral determined by lines l_1, \dots, l_5 by $P(l_1, \dots, l_5)$.

Conjecture 5.3. *Let be given in the projective space \mathbb{P}^3 25 lines, a_{ij} ($i, j = 1, \dots, 5$) such that they form five complete pentalaterals:*

$$A_1 = P(a_{11}, \dots, a_{15}), \dots, A_5 = P(a_{51}, \dots, a_{55}).$$

Assume that the following conditions hold:

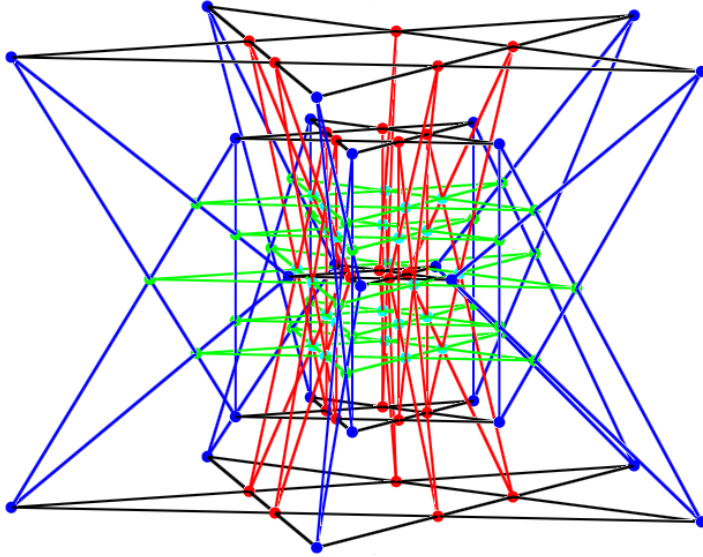


Figure 16: Twenty pentalaterals in \mathbb{E}^3 : a (100_4) configuration.

1. the external vertices of the pentalaterals A_i form the external vertices of complete pentalaterals $B_j = P(b_{1j}, \dots, b_{5j})$, as follows:

$$a_{ij} \cap a_{i,j+2} = b_{ij} \cap b_{i+2,j};$$

2. the internal vertices of the pentalaterals A_i form the external vertices of complete pentalaterals $C_j = P(c_{1j}, \dots, c_{5j})$, as follows:

$$a_{ij} \cap a_{i,j+1} = c_{ij} \cap c_{i+2,j}$$

(indexing is meant modulo 5).

Then there is a quintuple of complete pentalaterals D_i such that their vertices coincide with the internal vertices of the pentalaterals B_j and C_j , as follows:

$$b_{ij} \cap b_{i+1,j} = d_{ij} \cap d_{i,j+2} \quad \text{and} \quad c_{ij} \cap c_{i+1,j} = d_{ij} \cap d_{i,j+1}.$$

In some particular cases this conjecture is known to be true. In these cases the pentalaterals are embedded in Euclidean 3-space, every A_i is in distinct and pairwise parallel planes (these planes can simply be conceived as “horizontal planes”), while every B_j and C_j are in planes perpendicular to the former ones (thus they can be conceived as being in “vertical position”). In addition, every D_i is in “horizontal” planes, too. The cases are as follows:

Case A.

The pentagons determined by the A_i s and D_i s are all regular (in Euclidean sense), and they have a common axis of rotation (of order five). In this case the conditions of the

conjecture can easily be satisfied by suitably scaling the A_i s and by suitably chosen shapes and sizes of the B_j s. Just this case is shown in our Figure 16 above. Here the lines of the pentalaterals A_i , B_j , C_j and D_i , are distinguished by black, blue, red and green colour, respectively. Observe that each of these colour classes represents 100 incidences. Thus, our conjecture can also be formulated that the incidences belonging to any three of the colour classes imply the remaining 100 incidences.

Case B.

All the pentalaterals A_i are homothetic copies of a pentalateral A_0 . Furthermore, the external vertices of A_0 (hence those of all the A_i s) are inscribed in a circle. This case is visualized in an interactive model made using *Mathematica* [20]. In this model it is possible to move the external vertices of A_0 (and simultaneously, all the corresponding vertices of the A_i s) along a circle, while all the incidences required by the conjecture are preserved.

These cases provide some support for the conjecture. We remark that any projective transformation preserves the conjecture. We also remark that Case A also illustrates the fact that the automorphism group of this configuration is larger than or isomorphic to the group $D_5 \times D_5 \times C_2$. Here the first factor corresponds to the group of the pentalaterals A_i and D_i , the second factor to that of the pentalaterals B_j and C_j , while the last term is responsible for interchanging the “horizontal” and “vertical” quintuples of pentalaterals.

More generally, one expects that given a configuration \mathcal{C} with group G , the group of its p th power is larger than or isomorphic to the semi-direct product $G^p \rtimes S_p$, where the first term is a direct power of G , and the second term is the symmetric group of degree p .

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