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QUEUE LAYOUTS OF HYPERCUBES

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Queue layouts of hypercubes*

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Abstract

A queue layout of a graph consists of a linear ordering σ of its vertices, and a partition of its edges into sets, called queues, such that in each set no two edges are nested with respect to σ. We show that the n-dimensional hypercube Q_n has a layout into $n - \lfloor \log_2 n \rfloor$ queues for all $n \geq 1$. On the other hand, for every $\varepsilon > 0$ every queue layout of Q_n has more than $(\frac{1}{2} - \varepsilon)n - O(1/\varepsilon)$ queues, and in particular, more than $(n-2)/3$ queues. This improves previously known upper and lower bounds on the minimal number of queues in a queue layout of Q_n. For the lower bound we employ a new technique of out-in representations and contractions which may be of independent interest.

Key words. queue layout, queue-number, hypercube

AMS subject classification. 05C62, 68R10, 94C15

1 Introduction

Let $\sigma : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a linear ordering of vertices in a simple undirected graph G. Two edges $uv, xy \in E(G)$ are *nested* (with respect to the ordering σ) if $\sigma(u) < \sigma(x) < \sigma(y) < \sigma(v)$, see Figure 1. A set $S \subseteq E(G)$ is a *queue* if no two of its edges are nested with respect to σ. A *k-queue layout* of the graph G is a pair of a linear ordering σ of $V(G)$ and a partition of $E(G)$ into k queues. The queue-number $\mathsf{qn}(G)$ of the graph G is the minimum k such that G has a k-queue layout. A graph G is a *k-queue graph* if $\mathsf{qn}(G) \leq k$.

Queue layouts were first introduced by Heath et al. [12, 16]. This concept is analogous to the concept of stack layouts, also known as book embeddings, in which no two edges in the same set are allowed to cross. Applications of queue layouts include sorting permutations, parallel process scheduling, matrix computations, graph drawings, and queue-based computers. See [2, 7, 22] for a comprehensive list of references. If the vertex ordering is fixed, the optimal queue layout can be efficiently determined [7, 16]. But in general, this problem is believed to be intractable. In particular, recognizing k-queue graphs is NP-complete even for $k = 1$ [16]. The class of 1-queue graphs coincides with the class of so called *arched leveled-planar* graphs [16]. Another characterization of 1-queue graphs based on track layouts is given in [5]. Queue layouts of directed graphs [1, 14, 15, 22], posets [13, 22], and several special

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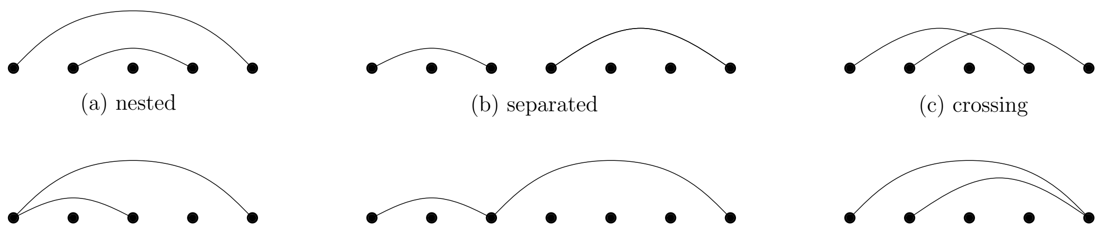

(a) nested (b) separated (c) crossing

(d) incident

Figure 1: All possible relations between two edges in a fixed vertex ordering.

graph classes [6–12,16,18–20,23–26] have also been investigated. For other graph layouts, see the survey [4].

The *n-dimensional hypercube* Q_n is the graph with all binary vectors of $\{0,1\}^n$ as vertices, and edges between every two vectors that differ in exactly one coordinate. The coordinate $i \in [n] = \{1, 2, \ldots, n\}$ in which neighbors u and v differ is called the *direction* of the edge uv. A vertex of Q_n is *even* (*odd*) if it contains even (odd) number of 1's. Even and odd vertices, respectively, form bipartite classes of Q_n. A subgraph of Q_n induced on vertices with fixed $n-k$ coordinates, where $0 \le k \le n$, is called a *k-dimensional subcube*. A vector $w = (w_1, \ldots, w_k) \in \{0,1\}^k$ is a *prefix* of a vector $v = (v_1, \ldots, v_n) \in \{0,1\}^n$, where $0 \le k \le n$, if $w_i = v_i$ for every $1 \le i \le k$.

Heath and Rosenberg [16] showed that Q_n has a layout into $n-1$ queues, that is $\mathsf{qn}(Q_n) \le n-1$, for all $n \ge 2$. Hasunuma and Hirota [11] improved it to $\mathsf{qn}(Q_n) \le n-2$ for all $n \ge 5$. Subsequently, Pai et al. [18] showed that the same upper bound holds also for $n = 4$. Recently, Pai et al. [20] further decreased it to $\mathsf{qn}(Q_n) \le n-3$ for all $n \ge 8$. On the other hand, Heath and Rosenberg [16] showed that the queue-number of every graph is larger than half of its density. In particular, for hypercubes it follows that $\mathsf{qn}(Q_n) > n/4$ [20, 21]. Interestingly, the analogously defined stack-number (better known as the pagenumber) of the hypercube is $\mathsf{pn}(Q_n) = n-1$ for all $n \ge 2$ [3, 17].

In this paper we show that the n-dimensional hypercube Q_n has a layout into $n - \lfloor \log_2 n \rfloor$ queues for all $n \ge 1$. This is the first non-constant improvement. As a corollary, we obtain also an improved upper bound on the queue-number of $2k$-ary hypercubes. Furthermore, we improve also the lower bound by showing that for every $\varepsilon > 0$ every queue layout of Q_n has more than $(\frac{1}{2} - \varepsilon)n - O(1/\varepsilon)$ queues, and in particular, more than $(n-2)/3$ queues. For the lower bound we employ a new technique of out-in representations and contractions which may be of independent interest.

We believe that the lower bound can be further improved. The upper bound indicates that $\mathsf{qn}(Q_n)$ could asymptotically behave as follows.

Question 1. Is it true that $\mathsf{qn}(Q_n) = n - \Theta(\log_2 n)$?

2 A queue layout with inserted vertices

Heath et al. [12] noticed that $\mathsf{qn}(G \,\square\, K_2) \le \mathsf{qn}(G) + 1$ for every graph G (where $\square$ denotes the cartesian product defined below), hence $\mathsf{qn}(G \,\square\, Q_k) \le \mathsf{qn}(G) + k$ for every $k \ge 1$. In this section we show that a queue layout of $G \,\square\, Q_k$ for $k \ge 2$ can be constructed (with the same additional cost of k queues) from a queue layout of $G - A$ for every set A of $k-1$ independent vertices of G. More precisely, the vertices of A and all incident edges are 'inserted' in the previous known layout of $(G-A)\,\square\, Q_k$ into $\mathsf{qn}(G-A)+k$ queues. This is the key idea in our improvements. It then only suffices to find a feasible set A such that $\mathsf{qn}(G-A)<\mathsf{qn}(G)$.

Our construction is inspired by the construction of Pai et al. [20] where only the vertex $\mathbf{1}=(1,1,\dots,1)$ was removed from $G=Q_{n-2}$ and it was shown that $\mathsf{qn}(Q_n)=\mathsf{qn}(Q_2\,\square\, Q_{n-2})\le \mathsf{qn}(Q_{n-2}-\{\mathbf{1}\})+2$. To describe our construction, let us first recall some definitions and introduce some additional notations.

The *cartesian product* $G\,\square\, H$ of graphs G and H is the graph with vertex set $V(G\,\square\, H)=\{(u,v);\ u\in V(G),\, v\in V(H)\}$ with edges of two types:

- *G-edge*: $(u,v)(w,v)\in E(G\,\square\, H)$ for every edge $uw\in E(G)$ and every vertex $v\in V(H)$,
- *H-edge*: $(u,v)(u,w)\in E(G\,\square\, H)$ for every vertex $u\in V(G)$ and every edge $vw\in E(H)$.

We say that (u,v) is a copy of u that corresponds to v. For the rest of the paper, let us write u^v instead of (u,v), and let G^n denote the n-th cartesian power of the graph G. Note that Q_n can be viewed as K_2^n.

Assume that $H=Q_k$ and recall that $V(Q_k)=\{0,1\}^k$. In this case we can extend our notation as follows. For every $u\in V(G)$, $w\in\{0,1\}^{k-i}$, $0\le i\le k$, and a set $S\subseteq V(G)$ let

$$u^w=\{u^v\in V(G\,\square\, Q_k);\ v\in\{0,1\}^k \text{ has prefix } w\},\qquad S^w=\bigcup_{u\in S}u^w.$$

If $i=k$ then w is the empty string, denoted by λ, and u^w contains all copies of the vertex u. Note that in general, we have $|u^w|=2^i$ and u^w induces a subcube of dimension i in $G\,\square\, Q_k$. Moreover, $u^w=u^{w0}\cup u^{w1}$ if w is of length smaller than k.

Let $[u^w]$ denote the ordering of u^w with respect to the lexicographic ordering of the indices. That is, for every $u^{v_1},u^{v_2}\in u^w$ where $v_1,v_2\in\{0,1\}^k$, we have $u^{v_1}<u^{v_2}$ if and only if $v_1<v_2$ lexicographically.

For orderings $\sigma(A)=(a_1,a_2,\dots,a_k)$ and $\sigma(B)=(b_1,b_2,\dots,b_l)$ of two disjoint sets A and B, let $\sigma(A)\circ\sigma(B)$ denote the *concatenated* ordering of $A\cup B$, and if $k=l$, let $\sigma(A)\bullet\sigma(B)$ denote the *interlaced* ordering of $A\cup B$; that is,

$$\begin{aligned}\sigma(A)\circ\sigma(B)&=(a_1,a_2,\dots,a_k,b_1,b_2,\dots,b_l),\\ \sigma(A)\bullet\sigma(B)&=(a_1,b_1,a_2,b_2,\dots,a_k,b_k)\quad \text{if } k=l.\end{aligned}$$

Lemma 1. *Let A be an independent set of vertices in a graph G and $k=|A|+1\ge 2$. Then,*

$$\mathsf{qn}(G\,\square\, Q_k)\le \mathsf{qn}(G-A)+k.$$

Proof. Let $G=(V,E)$ and $A=\{a_1,a_2,\dots,a_{k-1}\}$. For every $w\in\{0,1\}^{k-i}$, $0\le i\le k$ we define a set of vertices

$$U_i^w=\begin{cases}V^w\setminus A^w=(V(G-A))^w & \text{if } i=0,\\ U_0^{w0}\cup U_0^{w1} & \text{if } i=1,\\ U_{i-1}^{w0}\cup U_{i-1}^{w1}\cup a_{i-1}^w & \text{if } 1<i\le k.\end{cases}$$

Note that the index i in U_i^w is redundant as w is of length $k-i$, but we keep it for the sake of clarity. For $i=k$ we have $w=\lambda$ and $U_k^\lambda=V^\lambda=V(G\,\square\, Q_k)$. See sets U_i^w on Figure 2(b)

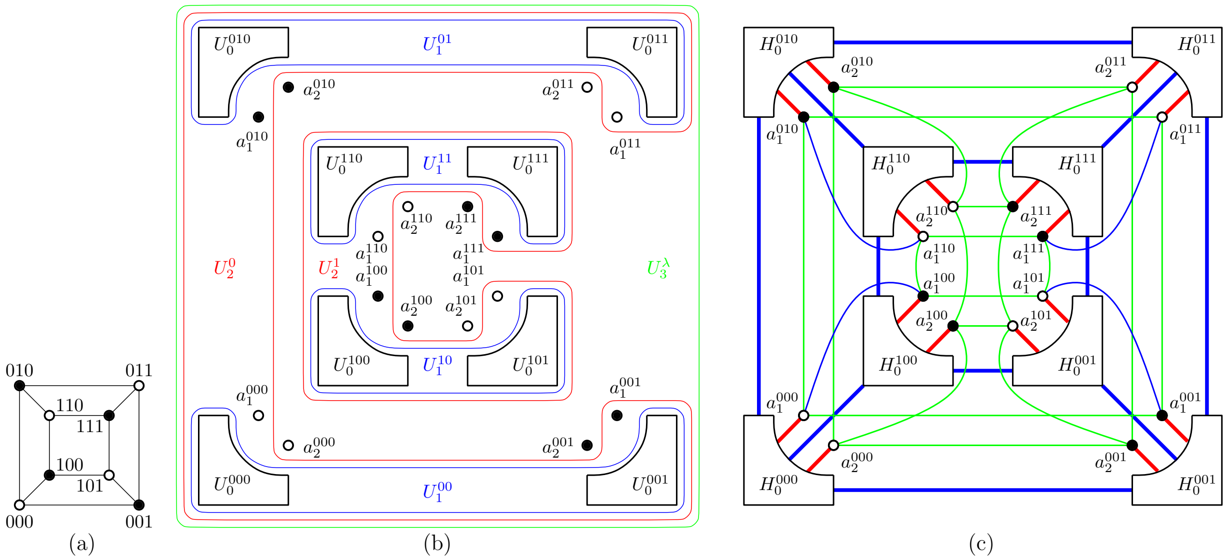

Figure 2: (a) The hypercube Q_3. (b) A scheme of sets U_i^w containing vertices of $G \,\square\, Q_3$. (c) A scheme of edges of $G \,\square\, Q_3$. The inner edges are not shown. The thick (red and blue) lines represent sets of (star and cartesian) edges. The normal (blue and green) lines represent (cartesian and lower) edges.

for an illustration in case $k = 3$. For each vertex $u \in \{0,1\}^3$, the corresponding copy of G has the vertex set $V^u = U_0^u \cup \{a_1^u, a_2^u\}$. The sets U_i^w for every $w \in \{0,1\}^{3-i}$ and $i = 0,1,2,3$ are drawn with black, blue, red, and green color, respectively.

Let H_i^w denote the subgraph of $G \,\square\, Q_k$ induced by the set U_i^w. For two subsets A, B of vertices of a graph H let $E_H(A,B)$ denote the set of edges of H between a vertex of A and a vertex of B. The edges of H_i^w can be recursively partitioned by

$$E(H_i^w) = \begin{cases} (E(G-A))^w & \text{if } i = 0, \\ E(H_{i-1}^{w0}) \cup E(H_{i-1}^{w1}) \cup C(H_i^w) & \text{if } i = 1, \\ E(H_{i-1}^{w0}) \cup E(H_{i-1}^{w1}) \cup C(H_i^w) \cup S(a_{i-1}^w) \cup L(a_{i-1}^w) & \text{if } 1 < i \le k, \end{cases}$$

where we denote

$$C(H_i^w) = E_{H_i^w}(U_{i-1}^{w0}, U_{i-1}^{w1}), \quad S(a_{i-1}^w) = E_{H_i^w}(a_{i-1}^w, U_{i-1}^{w0} \cup U_{i-1}^{w1}), \quad L(a_{i-1}^w) = E_{H_i^w}(a_{i-1}^w, a_{i-1}^w).$$

The edges in $C(H_i^w)$, $S(a_{i-1}^w)$, and $L(a_{i-1}^w)$ are called, respectively, *cartesian*, *star*, and *lower edges*. They are all called *outer edges*, whereas the edges induced by two vertices in the same set U_0^w for some $w \in \{0,1\}^k$ are called *inner edges*. See Figure 3 for an illustration.

Note that inner and star edges are G-edges, cartesian edges are Q_k-edges, and lower edges are only Q_k-edges since A is independent. Furthermore, the (cartesian) edges in $C(H_i^w)$ are of direction $k-i+1$ (that is, the direction i if counted from the right) since $|w| = k-i$. The (lower) edges in $L(a_{i-1}^w)$ are of directions from $k-i+1$ to k (that is, from 1 to i if counted from the right). For $1 \le j \le i$ let $L_j(a_{i-1}^w)$ denote the set of edges of direction $k-j+1$ in the set $L(a_{i-1}^w)$. See Figure 2(c) for an illustration in case $k = 3$.

Recall that $[u^w]$ denotes the lexicographic ordering of u^w and if w is of length at most $k-2$, then $a_i^w = a_i^{w00} \cup a_i^{w01} \cup a_i^{w10} \cup a_i^{w11}$ for every $1 \le i < k$. Let $(u_1, u_2, \ldots, u_l)$ where

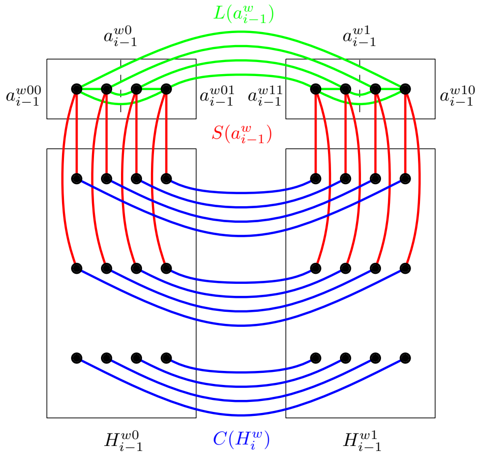

Figure 3: A scheme of edges of H^w_i for $i \geq 2$. The inner edges are not shown. Cartesian, star, and lower edges are blue, red, and green, respectively.

$l = |V| - k + 1$ be the ordering of vertices of $V \setminus A$ in a layout of $G - A$ with $\mathsf{qn}(G - A)$ queues. We construct an ordering $\sigma(V(G \,\square\, Q_k))$ recursively as follows. For every $w \in \{0, 1\}^{k-i}$, $0 \leq i \leq k$,

$$\sigma(U^w_i) = \begin{cases} (u^w_1, u^w_2, \ldots, u^w_l) & \text{if } i = 0, \\ \sigma(U^{w0}_0) \circ \sigma(U^{w1}_0) & \text{if } i = 1, \\ \left[a^{w00}_{i-1}\right] \circ \sigma(U^{w0}_{i-1}) \circ \left(\left[a^{w01}_{i-1}\right] \bullet \left[a^{w10}_{i-1}\right]\right) \circ \sigma(U^{w1}_{i-1}) \circ \left[a^{w11}_{i-1}\right] & \text{if } 1 < i \leq k. \end{cases}$$

See the ordering $\sigma(U^\lambda_3)$ in the first row of Figure 4 for an illustration in case $k = 3$.

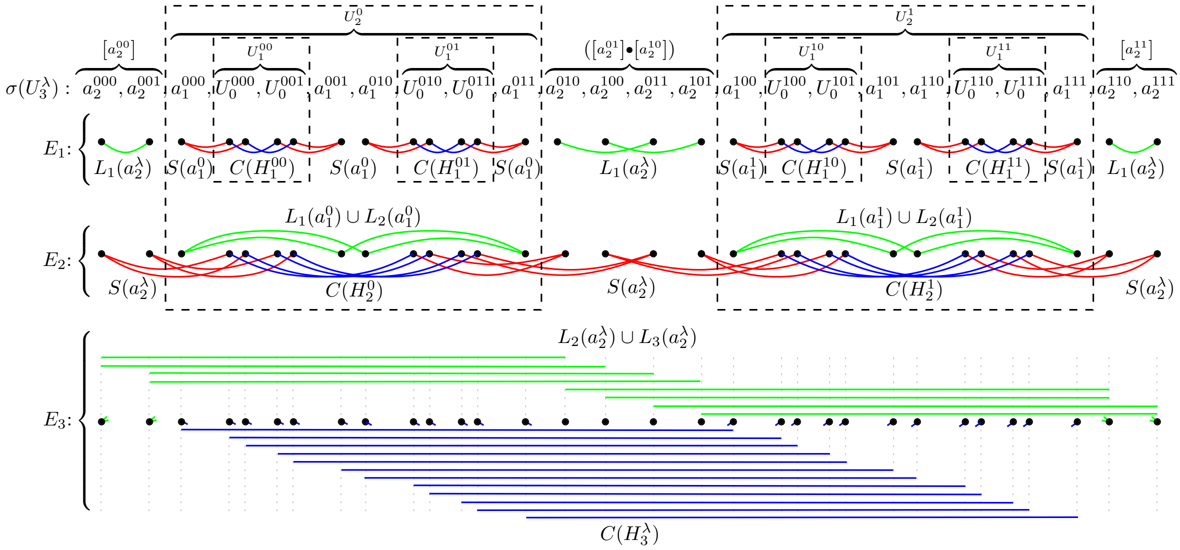

Figure 4: An example of the construction in Lemma 1 for $k = 3$ and $|U^w_0| = |V \setminus A| = 2$. The original $\mathsf{qn}(G - A)$ queues for the inner edges are not shown. The queue E_3 is depicted schematically.

Now, we describe a partition of $E(G \,\square\, Q_k) = E(H^\lambda_k)$ into $\mathsf{qn}(G - A) + k$ queues. For

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the inner edges, that is the edges of H_0^w for all $w \in \{0,1\}^k$, we use $\mathsf{qn}(G-A)$ queues of the (original) layout of $G-A$. For the outer edges of H_i^w where $1 \le i \le k$ we will need i additional queues. The partition is described recursively as follows.

Assume that $E_1^{w0}, E_2^{w0}, \dots, E_{i-1}^{w0}$ and $E_1^{w1}, E_2^{w1}, \dots, E_{i-1}^{w1}$ are partitions of the outer edges of H_{i-1}^{w0} and H_{i-1}^{w1}, respectively, into $i-1$ additional queues. Then we distribute the cartesian, star and lower edges of H_i^w into queues $E_1^w, E_2^w, \dots, E_i^w$ defined by

$$E_j^w = \begin{cases} E_j^{w0} \cup E_j^{w1} \cup L_j(a_{i-1}^w) & \text{if } 1 \le j \le i-2, \\ E_{i-1}^{w0} \cup E_{i-1}^{w1} \cup S(a_{i-1}^w) & \text{if } j = i-1, \\ C(H_i^w) \cup L_{i-1}(a_{i-1}^w) \cup L_i(a_{i-1}^w) & \text{if } j = i. \end{cases}$$

See Figure 4 for an illustration in case $k=3$.

It remains to verify that each set is a queue. A *length* of an edge uv is $|\sigma(u) - \sigma(v)|$. Observe that in the last set E_i^w, every vertex of $[a_{i-1}^{w00}]$, respectively $[a_{i-1}^{w11}]$, connects exactly to a pair of consecutive vertices in $([a_{i-1}^{w01}] \bullet [a_{i-1}^{w10}])$. Moreover, when we contract these pairs of consecutive vertices in $([a_{i-1}^{w01}] \bullet [a_{i-1}^{w10}])$, all edges of E_i^w will be of the same length and independent (up to multiplicity), so they are not nested in E_i^w.

In the penultimate set E_{i-1}^w, every two star edges are separate, crossing, or incident as depicted on Figure 1, and no star edge can be nested with an edge of $E_{i-1}^{w0} \cup E_{i-1}^{w1}$ as every star edge has one vertex inside an 'adjacent' block U_{i-1}^{w0} or U_{i-1}^{w1} and the other vertex outside.

Finally, for every $1 \le j \le i-2$, every lower edge from $L_j(a_{i-1}^w)$ is clearly separate with every edge of $E_j^{w0} \cup E_j^{w1}$, and every two lower edges from $L_j(a_{i-1}^w)$ are separate in different 'blocks' $[a_{i-1}^{w00}]$, $([a_{i-1}^{w01}] \bullet [a_{i-1}^{w10}])$, $[a_{i-1}^{w11}]$ or they have the same length. We conclude by induction that every set of the partition is a queue. □

3 A queue layout of the hypercube

Let us first recall the following strengthening of queue layouts that was introduced by Wood [26] for the study of queue layouts of several graph products.

Let σ be a linear ordering of vertices in a graph G. Two edges $uv, xy \in E(G)$ are *overlapping* (with respect to the ordering σ) if $\sigma(u) \le \sigma(x) < \sigma(y) \le \sigma(v)$. A set $S \subseteq E(G)$ is a *strict queue* if no two of its edges are overlapping with respect to σ. The *strict k-queue layout* of the graph G is a pair of a linear ordering σ of $V(G)$ and a partition of $E(G)$ into k strict queues. The strict queue-number $\mathsf{sqn}(G)$ of the graph G is the minimum k such that G has a strict k-queue layout.

Note that nested edges are overlapping. Hence every strict queue is a queue, and consequently, $\mathsf{qn}(G) \le \mathsf{sqn}(G)$ for every graph G. Strict queue-numbers are useful to derive bounds on queue-numbers of a cartesian product, as well as of several other graph products, see [26] for details.

Proposition 1 (Wood [26]). *For all graphs G and H,*

$$\mathsf{qn}(G \,\square\, H) \le \mathsf{qn}(G) + \mathsf{sqn}(H).$$

For the hypercube, it is easy to see that $\mathsf{sqn}(Q_n) = n$ for all $n \ge 1$. Indeed, the lexicographic ordering of $V(Q_n)$ and the partition of $E(Q_n)$ by directions form a strict n-queue layout of Q_n. On the other hand, for every graph G the strict queue-number $\mathsf{sqn}(G)$ is at least the minimum degree in G [26]. Analogously, for the grid P_k^n; that is, the n-th cartesian power of the path P_k on k vertices, it holds $\mathsf{sqn}(P_k^n) = n$ for all $n \geq 1$ and $k \geq 2$ [26].

Theorem 1. *For all $n \geq 3$,*

$$\mathsf{qn}(Q_n) \leq n - \lceil \log_2(n - \lceil \log_2(n-1) \rceil) \rceil .$$

Proof. First, we assume that $n = 2^{d-1} + d + 1$ for some integer $d \geq 1$. Note that $d = \lceil \log_2(n - \lceil \log_2(n-1) \rceil) \rceil$. Let A be the set of all even vertices of Q_d and $k = |A| + 1 = 2^{d-1} + 1$. Thus A is independent, the graph $Q_d - A$ has no edge, and by Lemma 1, we have

$$\mathsf{qn}(Q_n) = \mathsf{qn}(Q_d \,\square\, Q_k) \leq \mathsf{qn}(Q_d - A) + k = k = n - d = n - \lceil \log_2(n - \lceil \log_2(n-1) \rceil) \rceil$$

since $\mathsf{qn}(Q_d - A) = 0$. So the statement holds in this case.

Now, assume that $m = 2^{d-1} + d + 1 < n < 2^d + d + 2$ for some integer $d \geq 1$. Note that $d = \lceil \log_2(n - \lceil \log_2(n-1) \rceil) \rceil$ also in this case. Indeed, we have

$$\lceil \log_2(n - \lceil \log_2(n-1) \rceil) \rceil = \begin{cases} \lceil \log_2(n-d) \rceil = d & \text{if } 2^{d-1} + d + 1 < n \leq 2^d + 1, \\ \lceil \log_2(n-d-1) \rceil = d & \text{if } 2^d + 1 < n < 2^d + d + 2. \end{cases}$$

By Proposition 1,

$$\mathsf{qn}(Q_n) \leq \mathsf{qn}(Q_m) + \mathsf{sqn}(Q_{n-m}) \leq n - d = n - \lceil \log_2(n - \lceil \log_2(n-1) \rceil) \rceil$$

since $\mathsf{qn}(Q_m) \leq m - d$ by the first case and $\mathsf{sqn}(Q_{n-m}) = n - m$. □

It is remarkable that Theorem 1 attains all previously [20] known bounds for $3 \leq n \leq 12$ except $\mathsf{qn}(Q_4) = 2$ [18]. For $n \geq 13$ we obtain better layouts. Altogether, the previously known and new results can be simplified as follows.

Corollary 1. *For all $n \geq 1$,*

$$\mathsf{qn}(Q_n) \leq n - \lfloor \log_2 n \rfloor .$$

Proof. It is easy to see that $\mathsf{qn}(Q_1) = \mathsf{qn}(Q_2) = 1$ and $\mathsf{qn}(Q_3) = 2$. Pai et al. [18] showed that $\mathsf{qn}(Q_4) = 2$. For every $n \geq 5$ it holds that $\lceil \log_2(n - \lceil \log_2(n-1) \rceil) \rceil \geq \lfloor \log_2 n \rfloor$. □

Moreover, from Theorem 1 we also obtain better queue layouts for $2k$-ary hypercubes. A *k-ary n-dimensional hypercube* Q_n^k is the graph with all k-ary vectors of $\{0, 1, \ldots, k-1\}^n$ as vertices, and edges between every two vectors that differ by 1 or $k - 1$ in exactly one coordinate. That is, Q_n^k is the n-th cartesian power of the k-cycle, denoted by C_k^n, and is also known as an n-dimensional toroidal grid.

Pai et al. [19] previously showed that

$$\mathsf{qn}(Q_n^k) \leq \begin{cases} 2n - 3 & \text{if } k = 3,\ n \geq 3, \\ 2n - 2 & \text{if } 4 \leq k \leq 8,\ n \geq 2, \\ 2n - 1 & \text{if } k \geq 9,\ n \geq 1. \end{cases}$$

Corollary 2. *For all $n \geq 1$,*

$$\mathsf{qn}(Q_n^{2k}) \leq \begin{cases} 2n - \lfloor \log_2 n \rfloor - 1 & \text{if } k = 2, \\ 2n - \lfloor \log_2 n \rfloor & \text{if } k \geq 3. \end{cases}$$

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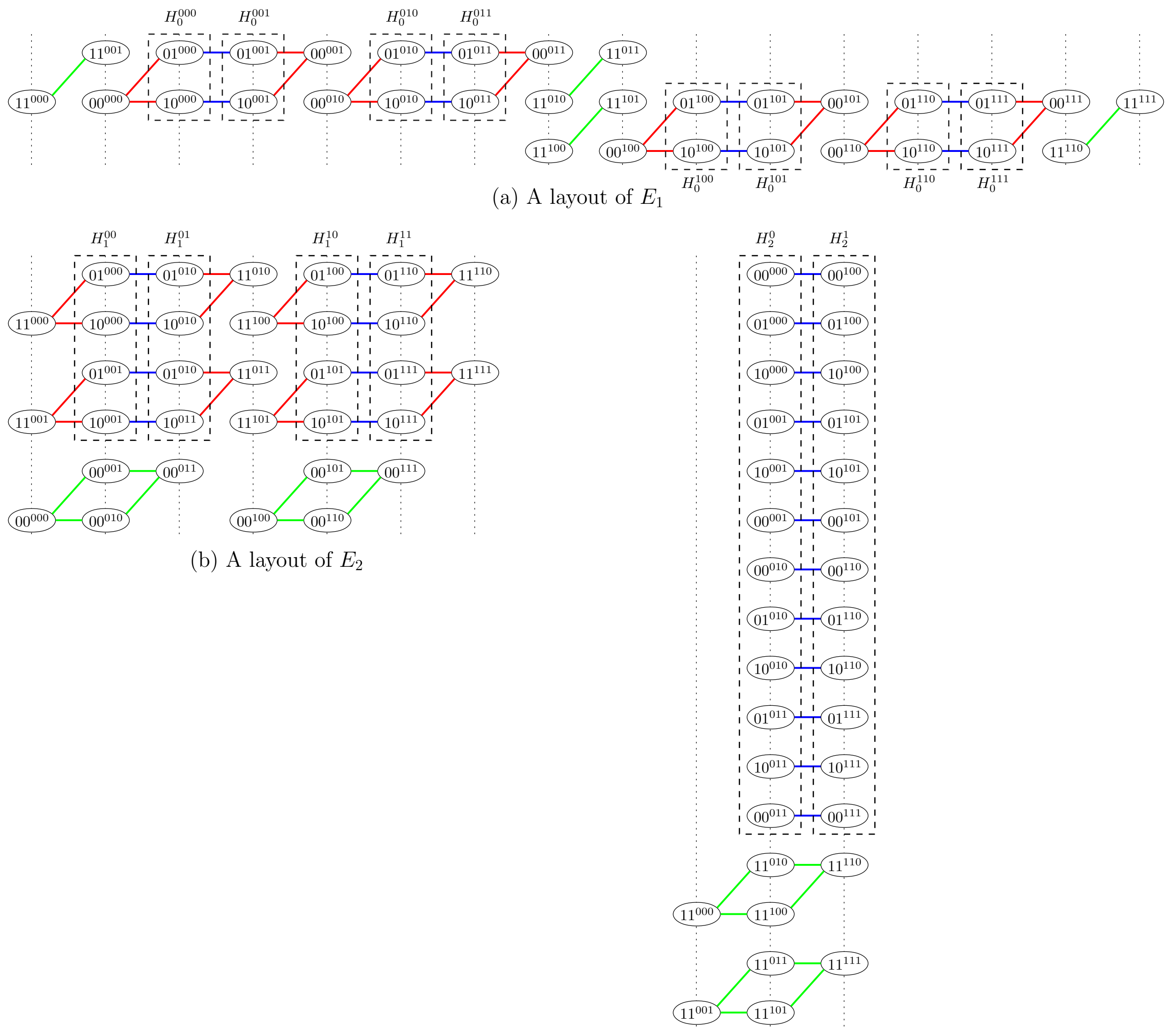

(a) A layout of E_1

(b) A layout of E_2

(c) A layout of E_3

Figure 5: A partition of Q_5 into three leveled planar graphs with the same induced ordering.

Proof. For $k = 2$ we have $Q_n^{2k} \simeq Q_{2n}$ and we directly apply Corollary 1. Now assume that $k \geq 3$. Since C_{2k} is a spanning subgraph of the ladder $P_2 \,\square\, P_k$, it follows that Q_n^{2k} is a spanning subgraph of $(P_2 \,\square\, P_k)^n \simeq Q_n \,\square\, P_k^n$. Therefore, by Proposition 1, Corollary 1, and $\mathsf{sqn}(P_k^n) = n$, we have

$$\mathsf{qn}(Q_n^{2k}) \leq \mathsf{qn}(Q_n \,\square\, P_k^n) \leq \mathsf{qn}(Q_n) + \mathsf{sqn}(P_k^n) \leq 2n - \lfloor \log_2 n \rfloor .$$

$\square$

Remark 1. Theorem 1 also provides a partition of Q_n into $n - \lceil \log_2(n - \lceil \log_2(n-1) \rceil) \rceil$ leveled planar graphs with the same induced ordering. A graph G is *leveled planar* [16] if it has a planar embedding such that vertices are mapped on vertical lines and edges are mapped to straight segments between two vertices on consecutive vertical lines. The *induced ordering* of a leveled planar graph orders its vertices by consecutive vertical lines, and from top to bottom on each line. An example for Q_5 that corresponds to Figure 4 is depicted on Figure 5.

4 Lower bound

In this section we improve the lower bound on the queue-number of the hypercube. First, we recall general concepts of rainbows and midpoints [7, 16] for establishing lower bounds on queue-numbers. Let $\sigma : V(G) \to \{1, 2, \dots, |V(G)|\}$ be a fixed vertex ordering of a graph G. A *k-rainbow* is a matching $\{u_i v_i \in E(G);\ 1 \le i \le k\}$ such that

$$\sigma(u_1) < \sigma(u_2) < \cdots < \sigma(u_k) < \sigma(v_k) < \sigma(v_{k-1}) < \cdots < \sigma(v_1).$$

Heath and Rosenberg [16] and then Dujmović and Wood [7] in a simpler argument showed that the size of a largest rainbow determines the number of queues in a queue layout of G with the ordering σ.

Lemma 2 (Heath and Rosenberg [16]). *The vertex ordering σ admits a k-queue layout of G if and only if it has no $(k+1)$-rainbow.*

The *midpoint* of an edge uv is $(\sigma(u) + \sigma(v))/2$. We use the following key observation.

Observation 1 (Dujmović and Wood [7]). *If k distinct edges share the same midpoint, they form a k-rainbow.*

As Dujmović and Wood [7] noticed, Observation 1 together with Lemma 2 immediately implies the following lemma, originally proved by Heath and Rosenberg [16]. Indeed, if we denote $m = |V(G)|$, all midpoints are in a set $\{\frac{3}{2}, \frac{4}{2}, \dots, \frac{2m-1}{2}\}$, which is of size $2m - 3$.

Lemma 3 (Heath and Rosenberg [16]). *Every k-queue graph on m vertices has at most $k(2m-3)$ edges.*

Recall that the *density* of a graph G is $\eta(G) = |E(G)|/|V(G)|$.

Corollary 3 (Heath and Rosenberg [16]). *For every graph G,*

$$\mathsf{qn}(G) > \eta(G)/2.$$

For the hypercube we obtain $\mathsf{qn}(Q_n) > n/4$ as $|V(Q_n)| = 2^n$ and $|E(Q_n)| = n2^{n-1}$, which was mentioned by Pai et al. [20]. Our improvement in Proposition 2 and Theorem 2 is based on two tools.

The first tool is the following representation of a linear layout of the graph G which is equivalent regarding nesting of edges. Let G' denote the graph obtained from G by replacing every vertex u with a pair of vertices u_{out}, u_{in}, and every edge uv with the edge $u_{\mathsf{out}} v_{\mathsf{in}}$ if $\sigma(u) < \sigma(v)$. Furthermore, let σ' be the vertex ordering of G' given by

$$\sigma'(u_{\mathsf{out}}) = \sigma(u), \quad \sigma'(u_{\mathsf{in}}) = \sigma(u) + m$$

for every $u \in V(G)$. We say that the pair (G', σ') is an *out-in representation* of (G, σ). See Figure 6(a)-(c) for an illustration.

Observation 2. *Two edges of G are nested (with respect to σ) if and only if their corresponding edges of G' are nested (with respect to σ').*

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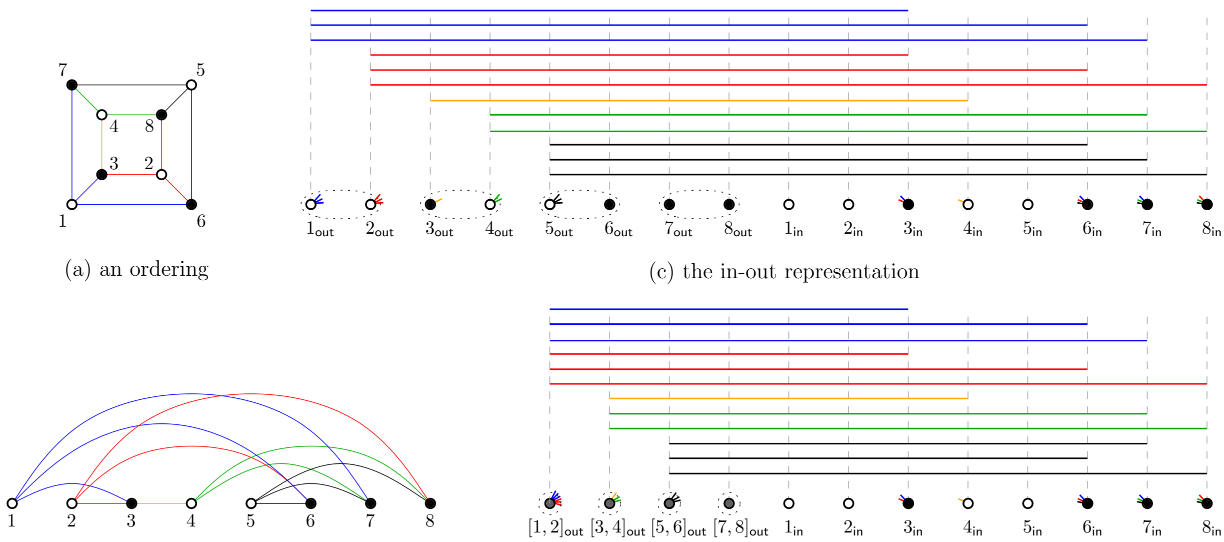

(a) an ordering

(c) the in-out representation

(b) the linear layout

(d) the contraction

Figure 6: (a) An example of an ordering σ of Q_3, (b) the linear layout of Q_3 with respect to σ, (c) the out-in representation Q_3' and σ', (d) the contraction Q_3^*. The colors distinguish edges from distinct out vertices.

Note that the midpoints of edges of G' are in a set $\{\frac{m+3}{2}, \frac{m+4}{2}, \ldots, \frac{3m-1}{2}\}$, which is, again, of size $2m-3$. In particular, note that the first and last possible midpoints are not $\frac{m+2}{2}$ and $\frac{3m}{2}$, respectively, as the vertices 1_{in} and m_{out} are isolated.

The second tool is the contraction of consecutive vertices. Let G^* be a multigraph obtained by contractions of some pairwise-disjoint sets of consecutive vertices of G. Here consecutive means with respect to the ordering σ. Furthermore, let σ^* be the vertex ordering of G^* inherited from σ. See Figure 6(d) for an illustration. Note that G^* may contain loops in general (even with higher multiplicity), but in Theorem 2 this will not be the case.

Observation 3. *If G^* contains a k-rainbow (with respect to σ^*), then G contains a k-rainbow (with respect to σ).*

To improve the lower bound, the key idea is to contract large number of consecutive vertices in order to decrease the number of midpoints, but at the same time, to have only a small number of multiple edges. Our preliminary lower bound is as follows.

Proposition 2. *For every $n \geq 1$,*

$$\mathsf{qn}(Q_n) > (n-2)/3.$$

Proof. Let σ be a vertex ordering of Q_n in a layout into $\mathsf{qn}(Q_n)$ queues. Our aim is to show that Q_n contains a rainbow of size more than $(n-2)/3$. Let (Q_n', σ') be the out-in representation of (Q_n, σ), and let Q_n^* be the graph obtained from Q_n' by contraction of the following 2^{n-1} pairwise-disjoint pairs of consecutive out-vertices

$$(u_{\mathsf{out}}, v_{\mathsf{out}}) \text{ such that } \sigma'(u_{\mathsf{out}}) = 2i-1,\ \sigma'(v_{\mathsf{out}}) = 2i \text{ for every } 1 \leq i \leq 2^{n-1}.$$

See Figure 6(d) for an illustration.

It is well-known that every two vertices of Q_n have 0 or 2 neighbors in common. Hence, there are at most 2 multiple edges from each contracted vertex. Thus, the number of distinct edges of Q_n^* is at least $(n-2)2^{n-1}$. On the other hand, all midpoints of edges of Q_n^* are in a set $\{\frac{2^{n-1}+3}{2}, \frac{2^{n-1}+4}{2}, \ldots, \frac{2^{n+1}}{2}\}$, which is of size $3 \cdot 2^{n-1} - 2$. Note that the smallest midpoint cannot be $\frac{2^{n-1}+2}{2}$ as the in-copy of the first vertex is isolated in Q_n^*. Hence by Observation 1, the graph Q_n^* contains a rainbow larger than $(n-2)/3$. By Observations 2 and 3 it follows that also Q_n contains a rainbow larger than $(n-2)/3$. Therefore, the statement follows from Lemma 2. □

In what follows we extend the above approach by contracting more vertices together instead of pairs. We define the *multiplicity index* of a vertex v in a multigraph G to be the number of edges incident with v minus the number of neighbors of v. The multiplicity index $m(S)$ of a set S of vertices is defined as the multiplicity index of the vertex obtained by contraction of S.

Lemma 4. *For every $d \geq 2$, $n \geq 1$, and every d-set S of vertices in Q_n it holds $m(S) \leq 2\binom{d}{2}$.*

Proof. Every pair of vertices of S contributes by at most 2 to $m(S)$ as they have at most two common neighbors. As there are $\binom{d}{2}$ pairs, the bound follows. □

Let us define $c(d)$ to be the maximal multiplicity index of a d-set S of vertices in some Q_n (with at least d vertices). We have shown that $c(d) \leq 2\binom{d}{2}$. On the other hand, consider the set S consisting of d neighbors of a single vertex v. After their contraction, there will be d edges to v from S. Moreover, each pair of vertices of S has another distinct common neighbor. Thus we have $m(S) = \binom{d}{2} + d - 1 = \frac{d^2+d-2}{2}$.

Question 2. Is it true that $c(d) = \frac{d^2+d-2}{2}$ for every d?

Now we employ the idea of contracting every d consecutive out-vertices together.

Lemma 5. *Let σ be a vertex ordering of Q_n and $d = 2^k$, $1 < k < n$. Then σ contains a rainbow larger than $\frac{dn-2c(d)}{2d+2}$.*

Proof. Similarly as in the proof of Proposition 2, we take the out-in representation and we contract every d consecutive out-vertices. Thus we get 2^{n-k} contracted out-vertices, and $2^n + 2^{n-k} - 2$ midpoints: $\frac{2^{n-k}+3}{2}, \frac{2^{n-k}+4}{2}, \ldots, \frac{2^{n-k+1}+2^n}{2}$. On the other hand, the number of distinct edges is at most $n2^{n-1} - 2^{n-k}c(d)$. Hence by Observation 1, in the contracted out-in representation there exists a rainbow of size at least

$$\frac{n2^{n-1} - 2^{n-k}c(d)}{2^n + 2^{n-k} - 2} > \frac{n2^{k-1} - c(d)}{2^k + 1} = \frac{dn - 2c(d)}{2d+2}.$$

By Observations 2 and 3 it follows that also σ contains a rainbow larger than $\frac{dn-2c(d)}{2d+2}$. □

Since $c(d)$ is bounded independently on n by Lemma 4, we obtain an improved lower bound. It shows that we can get arbitrarily close to the factor 1/2 instead of 1/3 in Proposition 2.

Theorem 2. *For all $\varepsilon > 0$, for every sufficiently large n,*

$$\mathsf{qn}(Q_n) > \left(\frac{1}{2} - \varepsilon\right) n - O(1/\varepsilon).$$

Proof. Let σ be the vertex ordering in an optimal queue-layout of Q_n (where n is large) and

$$d = 2^{\lceil \log_2(\frac{1}{2\varepsilon}-1)\rceil},$$

so $d = O(1/\varepsilon)$. Then by Lemma 5, the ordering σ contains a rainbow larger than

$$\frac{dn - 2c(d)}{2d+2} \geq \left(\frac{1}{2} - \varepsilon\right) n - \frac{2c(d)}{2d+2}.$$

Since $c(d) = O(d^2)$ by Lemma 4, the statement follows by Lemma 2. □

Remark 2. One of the anonymous referees suggested generalizations of the lower bounds in Proposition 2 and in Theorem 2 that might be applicable to other graph classes. We leave his suggestion as a possible direction for further research.

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