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On graphs with the smallest eigenvalue at least $-1-\sqrt{2}$, part II

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Abstract

This is a continuation of the article with the same title. In this paper, the family $\mathscr{H}$ is the same as in the previous paper [11]. The main result is that a minimal graph which is not an $\mathscr{H}$-line graph, is just isomorphic to one of the 38 graphs found by computer.

Keywords: Generalized line graph, spectrum.

Math. Subj. Class.: 05C50

1 Introduction

In the previous paper [11], we proved the uniqueness of strict $\{[H_2],[H_3],[H_5]\}$-cover graphs. This result plays a crucial role in obtaining an upper bound on the number of vertices in a minimal forbidden subgraph.

In this paper, we completely determine minimal forbidden subgraphs for the class of slim $\{[H_2],[H_3],[H_5]\}$-line graphs. By computer, we obtain such graphs (cf. Figure 2). The smallest eigenvalue of the minimal forbidden subgraph $G_{5,2}$ is less than $-1-\sqrt{2}$, and others are greater than or equal to $-1-\sqrt{2}$. We know that the smallest eigenvalues of $\{[H_2],[H_3],[H_5]\}$-line graphs are greater than or equal to $-1-\sqrt{2}$ (cf. Theorem 3.7 of [12]). These mean that, if a graph does not contain subgraphs in Figure 2, then it is a slim $\{[H_2],[H_3],[H_5]\}$-line graph, and has the smallest eigenvalue at least $-1-\sqrt{2}$.

We use the same notation as in [11].

Definition 1.1. A *Hoffman graph* is a graph H with vertex labeling $V(H)\to\{s,f\}$, satisfying the following conditions:

(i) every vertex with label f is adjacent to at least one vertex with label s;

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(ii) vertices with label f are pairwise non-adjacent.

We call a vertex with label s a *slim vertex*, and a vertex with label f a *fat vertex*. We denote by $V_s(H)$ ($V_f(H)$) the set of slim (fat) vertices of H. An ordinary graph without labeling can be regarded as a Hoffman graph without fat vertex. Such a graph is called a *slim graph*. The subgraph of a Hoffman graph H induced on $V_s(H)$ is called the *slim subgraph* of H. We draw Hoffman graphs by depicting vertices as large (small) black dots if they are fat (slim).

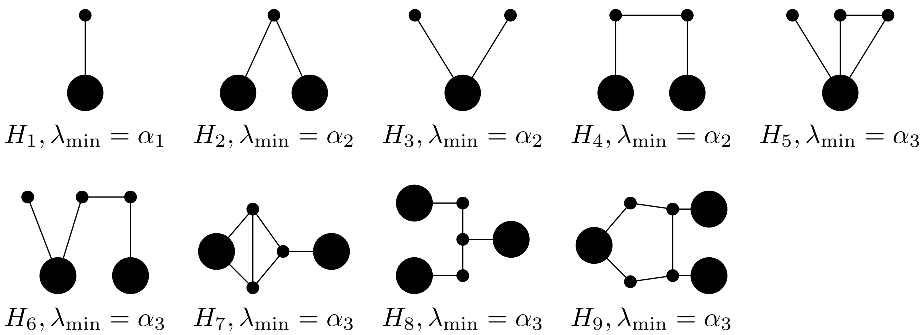


Figure 1:

We denote by $[H]$ the isomorphism class of Hoffman graphs containing H. In the following, all graphs considered are Hoffman graphs and all subgraphs considered are induced subgraphs. For a vertex v of a Hoffman graph H, we denote by $N_H^s(v)$ (resp. $N_H^f(v)$) the set of all slim (resp. fat) neighbours of v, and by $N_H(v)$ the set of all neighbours of v, i.e., $N_H(v) = N_H^s(v) \cup N_H^f(v)$. We write $G \subset H$ if G is an induced subgraph of H. We denote by $\langle S \rangle_H$ the subgraph of H induced on a set of vertices S. For a Hoffman graph H and a subset $S \subset V_s(H)$, let $\langle\langle S \rangle\rangle_H$ denote the subgraph

$$\langle\langle S \rangle\rangle_H = \langle S \cup (\bigcup_{z \in S} N_H^f(z)) \rangle_H.$$

Also, define $H - S$, $H - x$ by $H - S = \langle\langle V_s(H) \setminus S \rangle\rangle_H$, $H - x = H - \{x\}$, respectively, where $x \in V(H)$. Let $\emptyset$ be an empty set, and let ϕ be an empty graph.

Definition 1.2. Let H be a Hoffman graph, and let H^i $(i = 1, 2, \ldots, n)$ be a family of subgraphs of H. The graph H is said to be the *sum* of H^i $(i = 1, 2, \ldots, n)$, denoted

$$H = \biguplus_{i=1}^{n} H^i, \tag{1.1}$$

if the following conditions are satisfied:

(i) $V(H) = \bigcup_{i=1}^n V(H^i)$;

(ii) $V_s(H^i) \cap V_s(H^j) = \emptyset$ if $i \neq j$;

(iii) if $x \in V_s(H^i)$ and $y \in V_f(H)$ are adjacent, then $y \in V(H^i)$;

(iv) if $x \in V_s(H^i)$, $y \in V_s(H^j)$ and $i \neq j$, then x and y have at most one common fat neighbour, and they have one if and only if they are adjacent.

Definition 1.3. Let $\mathscr{H}$ be a family of isomorphism classes of Hoffman graphs. An $\mathscr{H}$-line graph Γ is a subgraph of a graph $H = \biguplus_{i=1}^{n} H^i$ such that $[H^i] \in \mathscr{H}$ for all $i \in \{1, 2, \ldots, n\}$. In this case, we call H an $\mathscr{H}$-cover graph of Γ. If $V_s(\Gamma) = V_s(H)$, then we call H a strict $\mathscr{H}$-cover graph of Γ. Two strict $\mathscr{H}$-covers K and L of Γ are called equivalent, if there exists an isomorphism $\varphi : K \to L$ such that $\varphi|_\Gamma$ is the identity automorphism of Γ.

For the remainder of this section, we assume $\mathscr{H} = \{[H_2], [H_3], [H_5]\}$ (cf. Figure 1). In our previous paper [11], we proved the following theorem:

Theorem 1.4. *Let Γ be a connected slim $\mathscr{H}$-line graph with at least 8 vertices. Then a strict $\mathscr{H}$-cover graph of Γ is unique up to equivalence.*

Every subgraph of an $\mathscr{H}$-line graph is an $\mathscr{H}$-line graph. Thus, it is desirable to determine all minimal slim non $\mathscr{H}$-line graphs. If Γ is a minimal slim non $\mathscr{H}$-line graph with at least 9 vertices, then we can use Theorem 1.4 to derive a contradiction (refer to Section 4 for the details of the proof). Enumerating all the slim non $\mathscr{H}$-line graphs with at most 8 vertices by comupter, we obtain the following theorem which is the main result in this paper:

Theorem 1.5. *If Γ is a minimal slim non $\mathscr{H}$-line graph, then Γ is isomorphic to one of the graphs in Figure 2.*

2 Forbidden graphs found by computer search

In this section, we assume $\mathscr{H} = \{[H_2], [H_3], [H_5]\}$ (cf. Figure 1). Proposition 2.1 is the main result in this section. It is very hard to obtain the propositions without computer search. In this paper, we have computed by the software MAGMA [9]. In order to prove the propositions, we show some lemmas.

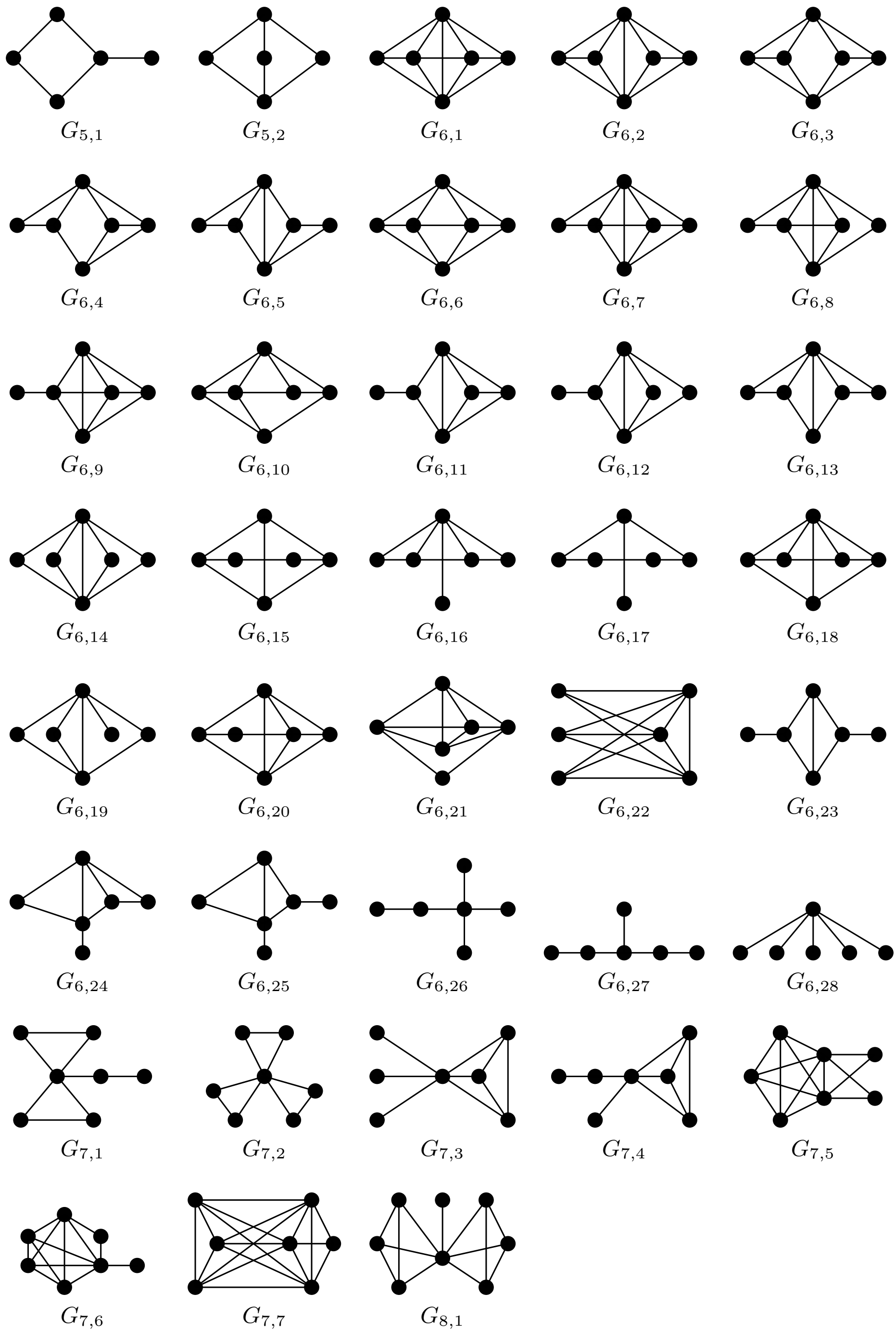
Let $\mathscr{X}_n$ be the family of isomorphism classes of connected slim graphs with n vertices. Brendan McKay gives collections of simple graphs on his web site (cf. [10]). From the data on this web site, we can generate $\mathscr{X}_n$. Let S_n be the family of isomorphism classes of connected slim $\mathscr{H}$-line graphs with n vertices. By computer, we obtain

$$\mathscr{X}_n = S_n \ (n = 1, 2, 3, 4) \text{ and } \mathscr{X}_5 \setminus S_5 = \{[G_{5,1}], [G_{5,2}]\} \text{ (cf. Figure 2).} \tag{2.1}$$

We define $\mathscr{F}_n$ to be the family of isomorphism classes of minimal slim non $\mathscr{H}$-line graphs with n vertices. From (2.1), $\mathscr{F}_i = \emptyset$ $(i = 1, 2, 3, 4)$ and $\mathscr{F}_5 = \{[G_{5,1}], [G_{5,2}]\}$. Removing those graphs which contain $G_{5,1}$ or $G_{5,2}$ from $\mathscr{X}_6 \setminus S_6$, we obtain $\mathscr{F}_6 = \{[G_{6,i}] | \ i = 1, 2, \ldots, 28\}$. Similarly we obtain $\mathscr{F}_7 = \{[G_{7,i}] | \ i = 1, 2, \ldots, 7\}$, $\mathscr{F}_8 = \{[G_{8,1}]\}$, and $\mathscr{F}_9 = \emptyset$ (cf. Figure 2). Hence the following proposition holds:

Proposition 2.1. *Let Γ be a minimal slim non $\mathscr{H}$-line graph. If $|V(\Gamma)| \leq 9$, then $[\Gamma] \in \mathscr{F}_5 \cup \mathscr{F}_6 \cup \mathscr{F}_7 \cup \mathscr{F}_8$.*

Actually, the conclusion of the proposition holds without the assumption $|V(\Gamma)| \leq 9$.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$G_{7,1}$
$G_{7,2}$
$G_{7,3}$
$G_{7,4}$
$G_{7,5}$
$G_{7,6}$
$G_{7,7}$
$G_{8,1}$

Figure 2:

3 Some useful lemmas

A vertex of a graph is called a pendant vertex if it has degree 1.

Lemma 3.1. *Let $H = H^0 \uplus H^1$ be a connected graph. Suppose that $V_f(H^0) \cap V_f(H^1) = \{\alpha\}$ and $N^s_{H^0}(\alpha) = V_s(H^0)$. Then H^1 is connected.*

Proof. Put $F = V_f(H^0) \setminus \{\alpha\}$ and $K = H^0 - F$. Then $F \cap V_f(H^1) = \emptyset$. Hence $H - F = K \uplus H^1$ and $H - F$ is connected. Since α is a unique fat vertex of K which is adjacent to all the slim vertices of K, Lemma 15 of [11] implies that H^1 is connected. □

Lemma 3.2. *Let $\mathscr{H}$ be a family of isomorphism classes of Hoffman graphs, satisfying the following condition:*

$$[H] \in \mathscr{H},\ H \not\cong H_2 \implies |N^f_H(x)| \leq 1 \quad \forall x \in V_s(H).$$

Let H be an $\mathscr{H}$-line graph. Then,

(i) *if $u \in V_s(H)$, then $|N^f_H(u)| \leq 2$,*

(ii) *if u, v are distinct slim vertices of H, then $|N^f_H(u) \cap N^f_H(v)| \leq 1$.*

Proof. See Lemma 23 of [11]. □

From [8, §6, Problem 6(c)], we obtain the following lemma:

Lemma 3.3. *Let Γ be a connected slim graph. If Γ is neither a complete graph nor a cycle, then there exists a non-adjacent pair $\{x, y\}$ in $V(\Gamma)$ such that $\Gamma - \{x, y\}$ is connected.*

For the remainder of this section, we assume $\mathscr{H} = \{[H_2],\ [H_3],\ [H_5]\}$ (cf. Figure 1).

Lemma 3.4. *Let $H = \biguplus_{i=0}^n H^i$ be a Hoffman graph satisfying $[H^j] \in \mathscr{H}$ for $j = 0, 1, \ldots, n$. Let V be a subset of $V_s(H)$, and let $K = \langle\langle V \rangle\rangle_H$. Then there exist subgraphs K^i $(i = 0, 1, \ldots, n')$ of K such that*

$$K = \biguplus_{i=0}^{n'} K^i,\ [K^j] \in \mathscr{H} \cup \{[H_1]\} \text{ for } j = 0, 1, \ldots, n'.$$

Proof. Put $L^i = \langle\langle V \cap V_s(H^i) \rangle\rangle_{H^i}$. Obviously $[L^i] \in \mathscr{H} \cup \{[\phi], [H_1], [H']\}$, where H' is the sum $H_1 \uplus H_1$ of two copies of H_1 sharing a fat vertex. Since $K = \biguplus_{i=0}^n L^i$ by [11, Lemma 12], the lemma holds. □

Lemma 3.5. *Let Γ be a connected slim $\mathscr{H}$-line graph. Then there exists a connected strict $\mathscr{H}$-cover graph $H = \biguplus_{i=0}^n H^i$ of Γ. Conversely, if $H = \biguplus_{i=0}^n H^i$ is a connected graph with $[H^i] \in \mathscr{H}$ and $n > 0$, then $\Gamma = \langle V_s(H) \rangle_H$ is connected.*

Proof. The first part follows from Example 22 of [11]. We prove the second part by induction on n. The assertion is easy to verify when $n = 1$. Suppose $n > 1$, and let $H' = \langle \bigcup_{i=1}^n V(H^i) \rangle_H$. Since H is connected, $V_f(H^0) \cap V_f(H') \neq \emptyset$. Pick $\alpha \in V_f(H^0) \cap V_f(H')$. Then every slim vertex of H^0 is adjacent to α, and hence every slim vertex of H^0 has a slim neighbour in H'. Since $H' = \biguplus_{i=1}^n H^i$ is connected by inductive hypothesis, we see that Γ is connected. □

Lemma 3.6. *If $\biguplus_{i=0}^{m_1} K^i = \biguplus_{i=0}^{m_2} L^i$ and $[K^i], [L^i] \in \mathscr{H}$ for each i, then $m_1 = m_2$, and*

$$\{K^i \mid 0 \le i \le m_1\} = \{L^i \mid 0 \le i \le m_2\}.$$

Proof. It suffices to prove $K^i = L^j$ whenever $V_s(K^i) \cap V_s(L^j) \neq \emptyset$. We may suppose without loss of generality that $i = j = 0$. If $K^0 \cong H_2$, then K^0 has a unique slim vertex, so $V_s(K^0) \subset V_s(L^0)$. By Definition 1.2(iii), we have $K^0 \subset L^0$. This implies $|V_f(L^0)| \ge 2$, hence $L^0 \cong H_2$, and therefore $K^0 = L^0$. The same conclusion holds when $L^0 \cong H_2$, so we suppose $[K^0], [L^0] \in \{[H_3], [H_5]\}$ for the rest of the proof. If $s_1 \in V_s(K^0) \cap V_s(L^0)$, then there exists $s_2 \in V_s(K^0)$ not adjacent to s_1. Since s_1 and s_2 have a common fat neighbour in K^0, Definition 1.2(iv) forces $s_2 \in V_s(L^0)$. This implies $V_s(K^0) \subset V_s(L^0)$ if $K_0 \cong H_3$. If $K^0 \cong H_5$, then consider the third slim vertex s_3 of K^0. We may assume without loss of generality that s_3 is not adjacent to s_1. Since s_1 and s_3 have a common fat neighbour in K^0, Definition 1.2(iv) forces $s_3 \in V_s(L^0)$. Thus $V_s(K^0) \subset V_s(L^0)$. Switching the roles of K^0 and L^0, we obtain $V_s(L^0) \subset V_s(K^0)$. Therefore we conclude $V_s(K^0) = V_s(L^0)$, and hence $K^0 = L^0$. $\square$

Lemma 3.7. *Suppose $H = H^0 \uplus H^1$, $S \subset V_s(H^1)$, and $H^2 = \langle\langle S \rangle\rangle_{H^1}$. Then $\langle V(H^0) \cup V(H^2) \rangle_H = H^0 \uplus H^2$.*

Proof. Routine verification. $\square$

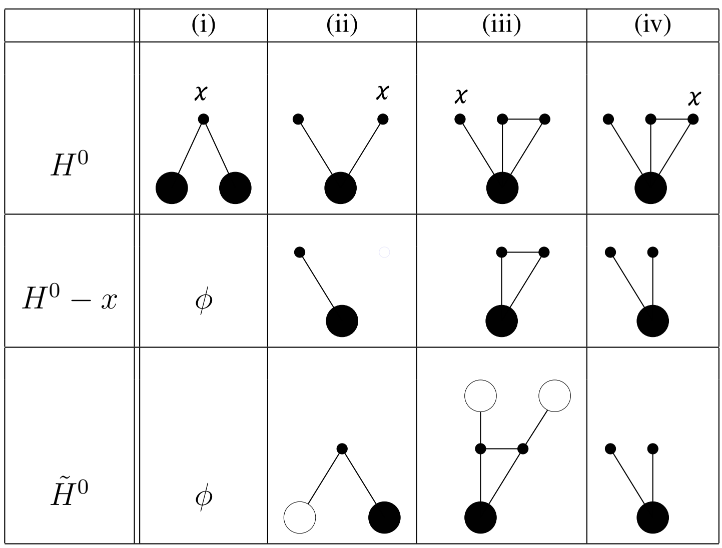

Table 1:

Lemma 3.8. *Let $H = H^0 \uplus H^1$ be a connected Hoffman graph satisfying $[H^0] \in \mathscr{H}$. Let x be a slim vertex of H^0. Then there exists a strict $\mathscr{H}$-cover graph $\tilde{H} = \tilde{H}^0 \uplus H^1$ of $H - x$, and one of the following holds:*

(i) $\tilde{H}^0 = \phi$,

(ii) $\tilde{H}^0 \cong H_2$, *and one of the fat vertices of* $\tilde{H}^0$ *is a pendant vertex in* H,

(iii) $\tilde{H}^0 = K^1 \uplus K^2$, $K^1 \cong K^2 \cong H_2$, K^1 *and* K^2 *have a fat vertex in common, and the other fat vertices of* $\tilde{H}^0$ *are pendant vertices in* H,

(iv) $\tilde{H}^0 \cong H_3$.

Proof. This is shown in the proof of Theorem 31 in [11], using Table 1, Lemma 12 and Lemma 13 in [11]. □

For a Hoffman graph $H = \biguplus_{i=0}^{n} H^i$ and a subset J of $\{0, 1, \ldots, n\}$, we write $H(J) = \biguplus_{i\in J} H^i$.

Lemma 3.9. *Let* $H = \biguplus_{i=0}^{n} H^i$ *be a connected Hoffman graph satisfying* $H^j \cong H_2, H_3$ *or* H_5 *for* $j = 0, 1, \ldots, n$. *Let* V *be a subset of* $V_s(H)$ *such that* $\langle\langle V\rangle\rangle_H$ *is connected. Let* $I = \{i \mid H^i \cong H_2,\ 0 \le i \le n\}$, *and let* $I' = \{i \in I \mid V_s(H^i) \subset V\}$. *Then,*

(i) if $I' \neq \emptyset$, *then* $H(I')$ *is connected, and in particular,* $H(I)$ *is connected,*

(ii) if $I \neq \emptyset$, *then* $V_f(H(I)) = V_f(H)$.

Proof. Put $J = \{i \mid 0 \le i \le n,\ V_s(H^i) \cap V \neq \emptyset\}$ so that $I' = I \cap J$. Since $\langle\langle V\rangle\rangle_H$ is connected, so is $H(J)$. Since the removal of $V_s(H^i)$ with $i \in J \setminus I'$ preserves connectivity by Lemma 3.1, we conclude that $H(I')$ is connected.

Suppose $V_f(H(I)) \neq V_f(H)$. Then there exists a fat vertex $f \in V_f(H) \setminus V_f(H(I))$. Since $\langle\langle N_H^s(f)\rangle\rangle_H$ has the unique fat vertex f, it is a connected component of H. But this contradicts the assumption that H is connected and $I \neq \emptyset$. Hence $V_f(H(I)) = V_f(H)$. □

4 Main theorem: The minimal forbidden subgraphs

In this section, we assume $\mathscr{H} = \{[H_2], [H_3], [H_5]\}$ (cf. Figure 1). Let $F_1, F_2, \ldots, F_9$ be the Hoffman graphs depicted in Figure 3.

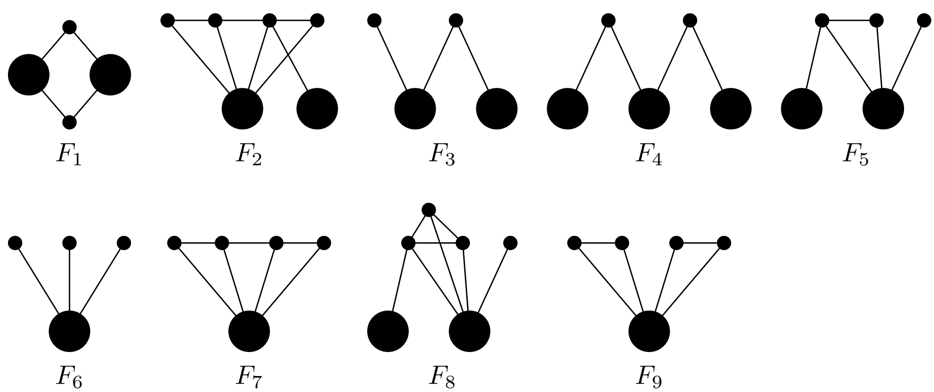

Figure 3:

Let $G = F \uplus K$ be a connected Hoffman graph such that $V_f(F) \subset V_f(K)$ and

$$K = \biguplus_{i=0}^{n} H^i,\ [H^j] \in \mathscr{H} \cup \{[H_1]\} \text{ for } j = 0, 1, \ldots, n. \tag{4.1}$$

When $F \cong F_1, F_3, F_4, F_6, F_7$ or F_9, Table 2 gives a list of slim subgraphs G' guaranteed to exist in G, under some additional assumptions. The assumptions are given in terms of $c(K)$ and $\lvert V_s(K)\rvert$, where $c(K)$ denotes the number of connected components of K. For example, if $F \cong F_1$, $c(K) = 2$, and $\lvert V_s(K)\rvert = 4$, then G has a slim subgraph G' isomorphic to $G_{5,1}$, $G_{5,2}$, $G_{6,3}$, or $G_{6,21}$, while if $F \cong F_3$ and $c(K) = 2$, then Table 2 gives no conclusion. The results in Table 2 were obtained by computer.

	F	$c(K)$	$\lvert V_s(K)\rvert$	G'		
(a)	F_1	1	5	$G_{5,1}$ $G_{5,2}$	$G_{6,3}$ $G_{6,6}$ $G_{6,12}$ $G_{6,14}$ $G_{6,21}$	$G_{7,5}$
(b)		2	4	$G_{5,1}$ $G_{5,2}$	$G_{6,3}$ $G_{6,21}$	
(c)	F_3	1	5	$G_{5,1}$	$G_{6,5}$ $G_{6,7}$ $G_{6,9}$ $G_{6,11}$ $G_{6,12}$ $G_{6,13}$ $G_{6,19}$ $G_{6,17}$ $G_{6,23}$ $G_{6,24}$ $G_{6,25}$ $G_{6,27}$	$G_{7,6}$
(d)	F_4		4	$G_{5,1}$	$G_{6,5}$ $G_{6,8}$ $G_{6,15}$ $G_{6,18}$	
(e)	F_6			$G_{5,2}$	$G_{6,14}$ $G_{6,19}$ $G_{6,22}$ $G_{6,26}$ $G_{6,28}$	$G_{7,3}$
(f)	F_7		2		$G_{6,1}$ $G_{6,6}$ $G_{6,16}$	
(g)	F_9		4		$G_{6,2}$ $G_{6,3}$	$G_{7,1}$ $G_{7,2}$

Table 2:

Lemma 4.1. *Let $G = F \uplus H$ be a Hoffman graph satisfying*

$$H = \biguplus_{i=0}^{n} H^i, \tag{4.2}$$

$$V_f(F) \subset V_f(H), \tag{4.3}$$

$$H^j \cong H_2 \text{ for } j = 0, 1, \ldots, n, \tag{4.4}$$

$$H \text{ is connected.} \tag{4.5}$$

Suppose $F \cong F_i$ for some $i \in \{2, 3, 5, 8\}$, and let F' be a subgraph of F such that $F' \cong F_3$. Let $V_f(F') = \{f_0, f_1\}$. If there is no edge between $N^s_H(f_0)$ and $N^s_H(f_1)$, then G has a slim subgraph isomorphic to $G_{5,1}$, $G_{6,17}$ or $G_{6,27}$.

Proof. First we note $N^s_H(f_0) \cap N^s_H(f_1) = \emptyset$ by Definition 1.2(iv). In particular, we have $n > 0$. From Lemma 3.5, there exists a path in $\langle V_s(H)\rangle_H$ connecting a vertex in $N^s_H(f_0)$ and a vertex in $N^s_H(f_1)$. Let P be such a path with shortest length. The length of P is at least 2 by the assumption. Since G contains $F' \uplus H$ as a subgraph by Lemma 3.7, it suffices to show that $F' \uplus H$ contains a desired slim subgraph. If P has length 2 or 3, then $F' \uplus H$ has a subgraph isomorphic to $G_{5,1}$ or $G_{6,17}$, respectively. If the length of P is at least 4, then $F' \uplus H$ has a subgraph isomorphic to $G_{6,27}$. □

Lemma 4.2. *Let $G = F \uplus H$ be a Hoffman graph satisfying (4.2)–(4.5). Suppose $F \cong F_4$, $V_f(F) = \{f_0, f_1, f_2\}$ with $\lvert N^s_F(f_0)\rvert = 2$. If $\langle N^s_H(f_0) \cup N^s_H(f_1) \cup N^s_H(f_2)\rangle_H$ is not connected, then G has a slim subgraph isomorphic to $G_{5,1}$, $G_{6,17}$, $G_{6,23}$ or $G_{6,27}$.*

Proof. By (4.3), $\lvert V_f(H)\rvert \geq \lvert V_f(F)\rvert = 3$, and therefore $n > 0$. From Lemma 3.5, there exists a path in $\langle V_s(H)\rangle_H$ connecting a vertex in $N^s_H(f_0)$ and a vertex in $N^s_H(f_1) \cup N^s_H(f_2)$ such that the two vertices are not adjacent in H, by the assumption. Let $P = u \sim v \sim \cdots \sim w$ be such a path with shortest length, where $u \in N^s_H(f_1) \cup N^s_H(f_2)$ and $w \in$

$N_H^s(f_0)$. Then $v \notin N_H^s(f_1) \cup N_H^s(f_2)$, and we may assume $u \in N_H^s(f_1)$ without loss of generality. Then $V(P) \cap N_H^s(f_1) = \{u\}$. If $u \sim f_2$, then $N_H^f(u) = \{f_1, f_2\}$, which implies $N_H^f(u) \cap N_H^f(v) = \emptyset$, contradicting $u \sim v$. Thus $u \notin N_H^s(f_2)$.

Put $S = V(P) \cap N_H^s(f_2)$. Suppose $S = \emptyset$. By Lemma 3.7, $F \uplus \langle\langle V(P)\rangle\rangle_H \subset G$, while f_2 has no slim neighbour in $\langle\langle V(P)\rangle\rangle_H$. This implies $(F - f_2) \uplus \langle\langle V(P)\rangle\rangle_H \subset G$. Since $F - f_2 \cong F_3$, the lemma follows from Lemma 4.1. Suppose $S \neq \emptyset$. Since P is the shortest path, w is adjacent to exactly one vertex s_1 in S, and $|S| = 2$. Put $S \setminus \{s_1\} = \{s_2\}$, and let w' be the neighbour of s_2 different from s_1 in P. Then $\langle V_s(F) \cup S \cup \{w, w'\}\rangle_G \cong G_{6,23}$, and hence G contains a subgraph isomorphic to $G_{6,23}$. □

Lemma 4.3. *Let $G = F \uplus H$ be a Hoffman graph satisfying (4.2), (4.3) and the following conditions:*

$$F \text{ is connected}, \tag{4.6}$$

$$[H^j] \in \mathscr{H} \text{ for } j = 0, 1, \ldots, n. \tag{4.7}$$

Let V is a subset of $V_s(H)$, and let $K = \langle\langle V\rangle\rangle_H$. If $V_f(F) \subset V_f(K)$, and every vertex of V can be joined by a path in K to a fat vertex of F, then G contains a connected subgraph $F \uplus K$ satisfying (4.1).

Proof. From Lemma 12 of [11], $\langle\langle V_s(F) \cup V\rangle\rangle_G = F \uplus K$. Since F is connected and every vertex of V can be joined by a path in K to a fat vertex of F, $F \uplus K$ is connected. From Lemma 3.4, K satisfies (4.1). □

Lemma 4.4. *Let $G = F \uplus H$ be a Hoffman graph satisfying (4.2), (4.3), (4.7), and $F \cong F_i$ for some $i \in \{1, 2, \ldots, 9\}$. Let*

$$m(F) = \begin{cases} 2 & \text{if } F \cong F_7, \\ 4 & \text{if } F \cong F_4, F_6 \text{ or } F_9, \\ 5 & \text{otherwise.} \end{cases}$$

If H is connected and $|V_s(H)| \geq m(F)$, then G has a slim subgraph isomorphic to one of the graphs in Figure 2.

Proof. Let $I = \{i \mid H^i \cong H_2,\ 0 \leq i \leq n\}$. First we suppose $I = \emptyset$. Then, since $H^i \cong H_3$ or H_5, $|V_f(H^i)| = 1$ for all $i \in \{0, 1, \ldots, n\}$. This implies $|V_f(H)| = 1$ since H is connected. Hence $F \cong F_6, F_7$ or F_9 by (4.3). Suppose $F \cong F_7$. Since H_3 is a subgraph of H_5, there exists a subgraph K of H such that $K \cong H_3$. Then G contains $F \uplus K$ as a subgraph from Lemma 3.7. Since $F \uplus K$ satisfies the assumptions of Table 2, the conclusion holds. Suppose $F \cong F_6$ or F_9. Since $|V_s(H)| \geq 4$ and H_3 is a subgraph of H_5, there exists a subgraph K of H isomorphic to the sum $H_3 \uplus H_3$ sharing a fat vertex. Then G contains $F \uplus K$ as a subgraph from Lemma 3.7. Since $F \uplus K$ satisfies the assumptions of Table 2, the conclusion holds. In the remaining part of this proof, we suppose $I \neq \emptyset$. For a subset J of $\{0, 1, \ldots, n\}$, we write $H(J) = \biguplus_{i \in J} H^i$.

Claim 4.5. *The graph $\langle V_s(H)\rangle_H$ is connected.*

Since $|V_s(H)| \geq m(F) \geq 2$ and $I \neq \emptyset$, $n > 0$. Hence, from the last part of Lemma 3.5, $\langle V_s(H)\rangle_H$ is connected.

Claim 4.6. $V_f(F) \subset V_f(H(I))$.

From Lemma 3.9(ii), $V_f(H(I)) = V_f(H)$. By (4.3), $V_f(F) \subset V_f(H(I))$.

Claim 4.7. *Suppose $F \cong F_1, F_3, F_4, F_6, F_7$ or F_9, and that there exists $I' \subset I$ such that $|I'| \le m(F)$, $V_f(F) \subset V_f(H(I'))$ and $H(I')$ is connected. Then the lemma holds.*

If $|I'| = 1$, then obviously $\langle V_s(H(I'))\rangle_H$ is connected. If $|I'| > 1$, then, from the last part of Lemma 3.5, $\langle V_s(H(I'))\rangle_H$ is connected. The graph $\langle V_s(H)\rangle_H$ is also connected from Claim 4.5. Since $|V_s(H(I'))| = |I'| \le m(F) \le |V_s(H)|$, there exists a subset V such that $V_s(H(I')) \subset V \subset V_s(H)$, $|V| = m(F)$ and $\langle V\rangle_H$ is connected. Put $K = \langle\langle V\rangle\rangle_H$. Then K is connected and $V_f(F) \subset V_f(K)$. Hence G contains a connected subgraph $F \uplus K$ satisfying (4.1) by Lemma 4.3. Therefore the assumptions of Table 2 are satisfied. Hence the lemma holds.

Claim 4.8. *If $F \cong F_6, F_7$ or F_9, then the lemma holds.*

From Claim 4.6, there exists $i \in I$ such that the unique fat vertex of F is in $V_f(H^i)$. Then $I' = \{i\}$ satisfies the hypotheses of Claim 4.7, and hence the lemma holds.

Claim 4.9. *If $F \cong F_1$, then the lemma holds.*

Let $V_f(F) = \{f_0, f_1\}$. From Claim 4.6, there exist $i_0, i_1 \in I$ such that $f_k \in V_f(H^{i_k})$ for each $k = 0, 1$. From Definition 1.2(ii), $i_0 \ne i_1$. For each $k = 0, 1$, let s_k be the unique slim vertex of H^{i_k}. Since H is connected and $5 = m(F) \le |V_s(H)|$, there exist disjoint subsets V_0, V_1 of $V_s(H)$ such that $|V_0 \cup V_1| = 5$, $\langle\langle V_k\rangle\rangle_H$ is connected and $s_k \in V_k$ for each $k = 0, 1$. Let $V = V_0 \cup V_1$. Then every vertex of V can be joined by a path in $\langle\langle V\rangle\rangle_H$ to f_0 or f_1.

Suppose $c(\langle\langle V\rangle\rangle_H) = 1$, i.e., $\langle\langle V\rangle\rangle_H$ is connected. Let $I' = \{i \in I \mid V_s(H^i) \subset V\}$. Then $|I'| \le |V| = m(F)$ and $i_0, i_1 \in I'$. Since $I' \ne \emptyset$, $H(I')$ is connected from Lemma 3.9(i). Since $i_0, i_1 \in I'$, $V_f(F) \subset V_f(H(I'))$. Hence I' satisfies the hypotheses of Claim 4.7, and the lemma holds.

Next suppose $c(\langle\langle V\rangle\rangle_H) > 1$. Since $\langle\langle V_0\rangle\rangle_H$ and $\langle\langle V_1\rangle\rangle_H$ are connected, $c(\langle\langle V\rangle\rangle_H) = 2$. Since $|V_0| + |V_1| = 5$, we may assume $|V_0| \ge 3$ without loss of generality. Let s be a slim vertex of $\langle\langle V_0\rangle\rangle_H$ which has the largest distance from s_0. Then $\langle\langle V_0 \setminus \{s\}\rangle\rangle_H$ is connected. Put $K = \langle\langle V \setminus \{s\}\rangle\rangle_H$. Then $c(K) = 2$. Moreover $V_f(F) \subset V_f(K)$, and every vertex of $V \setminus \{s\}$ can be joined by a path in K to f_0 or f_1. Hence G contains a connected subgraph $F \uplus K$ satisfying (4.1) by Lemma 4.3. Since $|V_s(K)| = |V \setminus \{s\}| = 4$, the assumptions of Table 2 are satisfied. Hence the lemma holds.

Now we consider the remaining cases. Let F' be a subgraph of F such that

$$\begin{cases} F' \cong F_3 & \text{if } F \cong F_2, F_3, F_5 \text{ or } F_8, \\ F' = F & \text{if } F \cong F_4. \end{cases}$$

Obviously $V_f(F') = V_f(F)$. Hence $F' = \langle\langle V_s(F')\rangle\rangle_F$. Thus $\langle V(F') \cup V(H)\rangle_G = F' \uplus H$ from Lemma 3.7, i.e., $F' \uplus H \subset G$. Let f_0 be the unique fat vertex of F' satisfying $|N^s_{F'}(f_0)| = 2$, and let f_1 be a fat vertex of F' different from f_0. Then $f_0, f_1 \in V_f(H(I))$ from Claim 4.6.

Claim 4.10. *If $F \cong F_2, F_3, F_5$ or F_8, then the lemma holds.*

Then $F' \cong F_3$. From Lemma 3.9(i), $H(I)$ is connected. If there is no edge between $N^s_{H(I)}(f_0)$ and $N^s_{H(I)}(f_1)$, then the result follows from Lemma 4.1. Suppose that there exist $s_0 \in N^s_{H(I)}(f_0)$ and $s_1 \in N^s_{H(I)}(f_1)$ such that $s_0 \sim s_1$. For each $k = 0, 1$, there exists $i_k \in I$ such that $V_s(H^{i_k}) = \{s_k\}$. Put $I' = \{i_0, i_1\}$. By Lemma 3.9(i), $H(I')$ is connected. Then I' satisfies the hypotheses of Claim 4.7, and the lemma holds.

Claim 4.11. *If $F \cong F_4$, then the lemma holds.*

Let f_2 be a fat vertex of F different from f_0, f_1. From Lemma 3.9(i), $H(I)$ is connected, and from Claim 4.6, $V_f(F) \subset V_f(H(I))$. Put $N_i = N^s_{H(I)}(f_i)$ for $i = 0, 1, 2$. If $\langle N_0 \cup N_1 \cup N_2 \rangle_{H(I)}$ is not connected, then the result follows from Lemma 4.2. Suppose that $\langle N_0 \cup N_1 \cup N_2 \rangle_{H(I)}$ is connected. Then, for each $i = 1, 2$, there exists an edge $s_i s_0^{(i)}$ between N_i and N_0 such that $s_i \in N_i$ and $s_0^{(i)} \in N_0$. Put $I' = \{i \in I \mid V_s(H^i) \subset \{s_0^{(1)}, s_0^{(2)}, s_1, s_2\}\}$. Since $s_0^{(1)}, s_0^{(2)}, s_1, s_2 \in V_s(H(I))$, $\langle\langle \{s_0^{(1)}, s_0^{(2)}, s_1, s_2\} \rangle\rangle_H = H(I')$. Since f_0 is a common fat neighbour of $s_0^{(1)}$ and $s_0^{(2)}$, $s_0^{(1)}$ and $s_0^{(2)}$ are adjacent, or equivalently in $H(I')$. Thus $H(I')$ is connected. Then I' satisfies the hypotheses of Claim 4.7, and hence the lemma holds. □

The next three lemmas are verified by computer.

Lemma 4.12. *Let F be a fat connected Hoffman graph satisfying the following conditions:*

(i) $|V_s(F)| = 2$,

(ii) the two slim vertices of F are not adjacent,

(iii) $|V_f(F)| \leq 4$,

(iv) every slim vertex has at most 2 fat neighbours,

(v) F is a non $\mathscr{H}$-line graph.

Then F is isomorphic to F_1, F_3 or F_4.

Lemma 4.13. *Let F be a fat connected Hoffman graph satisfying the following conditions:*

(i) $3 \leq |V_s(F)| \leq 4$,

(ii) $|V_f(F)| \leq 2$,

(iii) some slim vertex s of F has 2 fat neighbours,

(iv) some slim vertex s' of F is not adjacent to s, and the others are adjacent to s,

(v) $\langle\langle V_s(F) \setminus \{s\} \rangle\rangle_F \cong H_3$ *or* H_5,

(vi) F is a non $\mathscr{H}$-line graph.

Then F is isomorphic to F_2, F_5 or F_8.

Lemma 4.14. *Let F be a fat connected Hoffman graph satisfying the following conditions:*

(i) $3 \leq |V_s(F)| \leq 6$,

(ii) $|V_f(F)| = 1$,

(iii) *every slim vertex of* F *has* 1 *fat neighbour,*

(iv) *there exist different subsets* V_1 *and* V_2 *of* $V_s(F)$ *such that* $V_1 \cup V_2 = V_s(F)$, $\langle\langle V_1\rangle\rangle_F$ *and* $\langle\langle V_2\rangle\rangle_F$ *are isomorphic to* H_3 *or* H_5*, the vertex of* $V_s(F) \setminus V_2$ *and the vertex of* $V_s(F) \setminus V_1$ *are adjacent to each other except some pair* $\{s_1, s_2\}$ $(s_1 \in V_s(F) \setminus V_2$, $s_2 \in V_s(F) \setminus V_1)$,

(v) F *is a non* $\mathscr{H}$*-line graph.*

Then F *contains a subgraph isomorphic to* F_6, F_7 *or* F_9.

We shall now prove our main result.

Proof of Theorem 1.5. From Proposition 2.1, the theorem holds if $|V(\Gamma)| < 10$. So we prove that, if $|V(\Gamma)| \geq 10$, then Γ has a subgraph isomorphic to one of the graphs in Figure 2.

Since a complete graph and a cycle are $\mathscr{H}$-line graphs, Γ is neither a complete graph nor a cycle. Hence, from Lemma 3.3, there exists a non-adjacent pair $\{x, y\}$ in $V(\Gamma)$ such that $\Gamma - \{x, y\}$ is connected. Then $\Gamma - x$ and $\Gamma - y$ are connected as well. The graphs $\Gamma - x$, $\Gamma - y$ and $\Gamma - \{x, y\}$ are $\mathscr{H}$-line graphs by the minimality of Γ and $|V(\Gamma - \{x, y\})| \geq 8$.

Let $X = \biguplus_{i=0}^{m_1} X^i$ (resp. $Y = \biguplus_{i=0}^{m_2} Y^i$) be a strict $\mathscr{H}$-cover graph of $\Gamma - y$ (resp. $\Gamma - x$). Without loss of generality, we may suppose $x \in V_s(X^0)$ and $y \in V_s(Y^0)$. From Lemma 3.8, there exists a strict $\mathscr{H}$-cover graph $\tilde{X} = \tilde{X}^0 \uplus (\biguplus_{i=1}^{m_1} X^i)$ of $X - x$. Similarly, there exists a strict $\mathscr{H}$-cover graph $\tilde{Y} = \tilde{Y}^0 \uplus (\biguplus_{i=1}^{m_2} Y^i)$ of $Y - y$. Obviously $\tilde{X}$ and $\tilde{Y}$ are strict $\mathscr{H}$-cover graph of $\Gamma - \{x, y\}$. From Theorem 31 of [11], there exists an isomorphism $\varphi : \tilde{Y} \to \tilde{X}$ such that $\varphi|_{(\Gamma - \{x,y\})}$ is the identity automorphism of $\Gamma - \{x, y\}$.

From Lemma 3.8, we can put $\tilde{X}^0 = \tilde{X}_1^0 \uplus \tilde{X}_2^0$ $([\tilde{X}_1^0], [\tilde{X}_2^0] \in \{[\phi], [H_2], [H_3]\})$ and $\tilde{Y}^0 = \tilde{Y}_1^0 \uplus \tilde{Y}_2^0$ $([\tilde{Y}_1^0], [\tilde{Y}_2^0] \in \{[\phi], [H_2], [H_3]\})$, and put $\mathcal{X} = \{\phi, \tilde{X}_1^0, \tilde{X}_2^0\}$ and $\mathcal{Y} = \{\phi, \tilde{Y}_1^0, \tilde{Y}_2^0\}$. Then $\tilde{X} = (\biguplus_{K \in \mathcal{X}} K) \uplus (\biguplus_{i=1}^{m_1} X^i) = (\biguplus_{L \in \mathcal{Y}} \varphi(L)) \uplus (\biguplus_{j=1}^{m_2} \varphi(Y^j))$. From Lemma 3.6, $\{\varphi(L) \mid L \in \mathcal{Y}\} \cup \{\varphi(Y^i) \mid 1 \leq i \leq m_2\} = \mathcal{X} \cup \{X^i \mid 1 \leq i \leq m_1\}$. Put $\mathcal{Z} = \mathcal{X} \cup \{\varphi(L) \mid L \in \mathcal{Y}\}$. Then

$$\tilde{X} = (\biguplus_{Z \in \mathcal{Z}} Z) \uplus H, \tag{4.8}$$

where

$$H = \biguplus_{i \in I} X^i = \biguplus_{j \in J} \varphi(Y^j) \tag{4.9}$$

for some $I \subset \{1, 2, \ldots, m_1\}$ and $J \subset \{1, 2, \ldots, m_2\}$. Obviously

$$X = X^0 \uplus (\biguplus_{Z \in \mathcal{Z} \setminus \mathcal{X}} Z) \uplus H, \quad Y = Y^0 \uplus (\biguplus_{Z \in \mathcal{Z} \setminus \{\varphi(L) | L \in \mathcal{Y}\}} \varphi^{-1}(Z)) \uplus \varphi^{-1}(H), \tag{4.10}$$

Claim 4.15. *The graph* H *is connected.*

Since $\Gamma - \{x, y\}$ is connected, so is $\tilde{X}$. The Hoffman graph $H' = \biguplus_{Z \in \mathcal{Z}} Z$ has the unique fat vertex α satisfying $V_f(H') \cap V_f(H) = \{\alpha\}$ and $N^s_{H'}(\alpha) = V_s(H')$. Using Lemma 3.1 on the decomposition (4.8), We conclude that H is connected.

We define the edge set

$$E_0 = \left(\bigcup_{z \in V_s(X^0)} \{zf \mid f \in V_f(H) \cap N^f_{X^0}(z)\} \right) \cup \left(\bigcup_{z \in V_s(Y^0)} \{z\varphi(g) \mid g \in V_f(\varphi^{-1}(H)) \cap N^f_{Y^0}(z)\} \right)$$

and the Hoffman graph

$$G = (V(\Gamma) \cup V_f(H),\ E(\Gamma) \cup E(H) \cup E_0).$$

Let

$$F = \langle\langle V_s(X^0) \cup V_s(Y^0) \rangle\rangle_G.$$

Obviously the following holds:

$$s \in V_s(F),\ f \in V_f(G),\ sf \in E(G) \Longrightarrow sf \in E_0, \tag{4.11}$$

and

$$\text{(a)} \quad V_f(F) \subset V_f(H), \quad \text{(b)} \quad E_0 \subset E(F), \quad \text{(c)} \quad \Gamma \subset G \text{ and } V_s(\Gamma) = V_s(G). \tag{4.12}$$

Also, from (4.8),

$$\text{(a)} \quad V_s(\Gamma) = V_s(F) \cup V_s(H), \quad \text{(b)} \quad V_s(F) \cap V_s(H) = \emptyset. \tag{4.13}$$

From (4.13),

$$|V_s(H)| \geq 10 - |V_s(F)|. \tag{4.14}$$

By the definition of G,

$$V_f(G) = V_f(H). \tag{4.15}$$

Claim 4.16. $G = F \uplus H$.

Let us check the conditions (i)–(iv) of Definition 1.2.

From (4.12)-(c) and (4.13)-(a), $V_s(G) = V_s(F) \cup V_s(H)$. Moreover, it is $V_f(G) = V_f(H) = V_f(F) \cup V_f(H)$ by (4.12)-(a) and (4.15). Hence the condition (i) is satisfied. Also, by (4.13)-(b), the condition (ii) is satisfied. By the definitions of F and G, the condition (iii) is satisfied.

Let $s_1 \in V_s(F)$, and let $s_2 \in V_s(H)$. Then $s_1 \in V_s(X^0)$ or $s_1 \in V_s(Y^0)$, $s_2 \in V_s(H) \subset V_s(\biguplus_{i=1}^{m_1} X^i)$. By (4.15), $N^f_G(s_2) = N^f_H(s_2)$. First suppose $s_1 \in V_s(X^0)$. Since $N^f_H(s_2) \subset V_f(H)$, $N^f_G(s_1) \cap N^f_G(s_2) = (N^f_{X^0}(s_1) \cap V_f(H)) \cap N^f_H(s_2) = N^f_{X^0}(s_1) \cap N^f_H(s_2)$. Since $N^f_{X^0}(s_1) = N^f_X(s_1)$ and $N^f_H(s_2) = N^f_X(s_2)$, $N^f_G(s_1) \cap N^f_G(s_2) = N^f_X(s_1) \cap N^f_X(s_2)$. Thus, s_1 and s_2 have at most one common fat neighbour in G, and they have one if and only if they are adjacent in X, or equivalently in G. Hence (iv) holds in this case. Next suppose $s_1 \in V_s(Y^0)$. Since $s_2 \in V_s(H)$, $s_2 \in \varphi(V_s(Y^j))$ for some

$j \in \{1, 2, \ldots, m_2\}$. Hence $N^f_G(s_2) = N^f_H(s_2) = N^f_{\varphi(Y^j)}(s_2) = N^f_{\varphi(Y^j)}(\varphi(s_2)) = \varphi(N^f_{Y^j}(s_2)) = \varphi(N^f_Y(s_2))$. Thus

$$\begin{aligned} N^f_G(s_1) \cap N^f_G(s_2) &= \varphi(V_f(\varphi^{-1}(H)) \cap N^f_{Y^0}(s_1)) \cap \varphi(N^f_Y(s_2)) \\ &= \varphi(N^f_Y(s_1) \cap N^f_Y(s_2) \cap V_f(\varphi^{-1}(H))) \\ &= \varphi(N^f_Y(s_1) \cap N^f_Y(s_2)) \end{aligned}$$

since $\varphi^{-1}(H) \subset Y$. A similar argument shows that (iv) holds in this case as well.

Claim 4.17. *For any $s \in V_s(F)$,*

$$|N^f_G(s)| \leq \begin{cases} |V_f(X^0)| & \text{if } s \in V_s(X^0), \\ |V_f(Y^0)| & \text{otherwise.} \end{cases}$$

By (4.11), $sf \in E_0$ for each $f \in N^f_G(s)$. Suppose $s \in V_s(X^0)$. Then $sf \in E(X^0)$. Hence $|N^f_G(s)| \leq |N^f_{X^0}(s)| \leq |V_f(X^0)|$. Suppose $s \in V_s(Y^0)$. Then $s\varphi^{-1}(f) \in E(Y^0)$. Hence $|N^f_G(s)| \leq |N^f_{Y^0}(s)| \leq |V_f(Y^0)|$.

Claim 4.18. $|V_f(F)| \leq |V_f(X^0)| + |V_f(Y^0)|$.

From Claim 4.16, $V_f(F) = V_f(\langle\langle V_s(X^0) \cup V_s(Y^0)\rangle\rangle_{F \uplus H})$. By (4.12)-(a), $V_f(F) = V_f(\langle\langle V_s(X^0) \cup V_s(Y^0)\rangle\rangle_{F \uplus H}) \cap V_f(H)$, i.e.,

$$V_f(F) = \big(V_f(X^0) \cap V_f(H)\big) \cup \big(\varphi(V_f(Y^0) \cap \varphi^{-1}(V_f(H)))\big).$$

Hence

$$\begin{aligned} |V_f(F)| &\leq |V_f(X^0) \cap V_f(H)| + |V_f(Y^0) \cap \varphi^{-1}(V_f(H))| \\ &\leq |V_f(X^0)| + |V_f(Y^0)|. \end{aligned}$$

Claim 4.19. *The Hoffman graph F is a non $\mathscr{H}$-line graph.*

The Hoffman graph H is a strict $\mathscr{H}$-cover graph of itself. Suppose that F is an $\mathscr{H}$-line graph. Then there exists a strict $\mathscr{H}$-cover graph of F (cf. Example 22 of [11]). Hence G has a strict $\mathscr{H}$-cover graph from Lemma 20 of [11]. Since $\Gamma \subset G$, Γ is an $\mathscr{H}$-line graph, a contradiction.

Claim 4.20. *If X^0 or Y^0 is isomorphic to H_2, the theorem holds.*

If X^0 or Y^0 is isomorphic to H_2, then $V_s(X^0) \cap V_s(Y^0) = \emptyset$, and each slim vertex of F has at most 2 fat neighbours by Claim 4.17. First suppose that X^0 and Y^0 are isomorphic to H_2. Then $|V_s(F)| = |\{x, y\}| = 2$ and $|V_f(F)| \leq 4$ by Claim 4.18. Hence the hypotheses of Lemma 4.12 hold by Claim 4.19. Thus $F \cong F_1, F_3$ or F_4, and $|V_s(H)| \geq 8$ by (4.14). Next suppose otherwise. Then $3 \leq |V_s(F)| = |V_s(X^0) \cup V_s(Y^0)| \leq 4$, and $|V_f(F)| \leq 3$ by Claim 4.18. If $|V_f(F)| = 3$, then $V_f(X^0) \cap V_f(Y^0) = \emptyset$, and therefore F is an $\mathscr{H}$-line graph since $V_s(X^0) \cap V_s(Y^0) = \emptyset$, a contradiction to Claim 4.19. Obviously the hypotheses (v) and (iv) of Lemma 4.13 hold. Hence the hypotheses of Lemma 4.13 hold by Claim 4.19. Thus $F \cong F_2$, F_5 or F_8, and $|V_s(H)| \geq 6$ by (4.14). Hence the theorem holds from Lemma 4.4 if X^0 or Y^0 is isomorphic to H_2.

For the remainder of this proof, we assume that X^0 and Y^0 are isomorphic to H_3 or H_5. Then $3 \le |V_s(X^0) \cup V_s(Y^0)|(= |V_s(F)|) \le 6$. Hence the condition (i) of Lemma 4.14 holds. Suppose $V_f(X^0) \cap \varphi(V_f(Y^0)) = \emptyset$. Then $V_f(\tilde{X}^0) \cap V_f(\varphi(\tilde{Y}^0)) = \emptyset$. Hence $V(\tilde{X}^0) \cap V(\varphi(\tilde{Y}^0)) = \emptyset$ by (4.8) since $\tilde{X} = \varphi(\tilde{Y})$. Thus $V_s(X^0) \cap V_s(Y^0) = \emptyset$, and therefore $F = \langle\langle V_s(X^0) \cup V_s(Y^0)\rangle\rangle_G = \langle\langle V_s(X^0)\rangle\rangle_G \uplus \langle\langle V_s(Y^0)\rangle\rangle_G$. Obviously $\langle\langle V_s(X^0)\rangle\rangle_G$ and $\langle\langle V_s(Y^0)\rangle\rangle_G$ are isomorphic to H_3 or H_5. Hence F is an $\mathscr{H}$-line graph. But this contradicts Claim 4.19. Thus $V_f(X^0) \cap \varphi(V_f(Y^0)) \neq \emptyset$, i.e., φ maps the unique fat vertex of Y^0 to the unique fat vertex of X^0, and $|V_f(F)| = 1$. Hence the conditions (ii) and (iii) of Lemma 4.14 hold. Moreover the condition (v) of Lemma 4.14 holds by Claim 4.19.

Put $V_1 = V_s(X^0)$ and $V_2 = V_s(Y^0)$, and put $s_1 = x$ and $s_2 = y$. Then

- $(V_s(F)\backslash V_2)\backslash\{s_1\} = V_s(\tilde{X}^0)\backslash V_s(Y^0) \subset V_s(\biguplus_{Z\in\mathcal{Z}\backslash\{\varphi(L)|L\in\mathcal{Y}\}} \varphi^{-1}(Z))$ by (4.10),
- $V_s(F) \setminus V_1 = V_s(Y^0) \setminus V_s(X^0) \subset V_s(Y^0)$,
- $(V_s(F) \setminus V_1) \setminus \{s_2\} = V_s(\tilde{Y}^0) \setminus V_s(X^0) \subset V_s(\biguplus_{Z\in\mathcal{Z}\setminus\mathcal{X}} Z)$ by (4.10),
- $V_s(F) \setminus V_2 = V_s(X^0) \setminus V_s(Y^0) \subset V_s(X^0)$.

Hence the vertex of $V_s(F) \setminus V_2$ and the vertex of $V_s(F) \setminus V_1$ are adjacent to each other except the pair $\{s_1, s_2\}$ ($s_1 \in V_s(F) \setminus V_2$, $s_2 \in V_s(F) \setminus V_1$). Thus the conditions (iv) of Lemma 4.14 holds. Therefore F has a subgraph isomorphic to F_6, F_7 or F_9. Let F' be a subgraph isomorphic to F_6, F_7 or F_9, of F. Then $F' \uplus H \subset G$ from Lemma 3.7. Now $|V_f(F')| = 1$, $|V_s(H)| \ge 4$ by (4.14). Moreover $V_f(F') = V_f(F) \subset V_f(H)$ from (4.12)-(a). Hence the hypothesis of Lemma 4.4 is satisfied. Thus $F' \uplus H$ has a slim subgraph isomorphic to one of the graphs in Figure 2, and so does G. □

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