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Efficient proper embedding of a daisy cube*

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Abstract

For a set X of binary words of length h the daisy cube $Q_h(X)$ is defined as the subgraph of the hypercube Q_h induced by the set of all vertices on shortest paths that connect vertices of X with the vertex 0^h. A vertex in the intersection of all of these paths is a minimal vertex of a daisy cube. A graph G isomorphic to a daisy cube admits several isometric embeddings into a hypercube. We show that an isometric embedding is proper if and only if the label 0^h is assigned to a minimal vertex of G. This result allows us to devise an algorithm which finds a proper embedding of a graph isomorphic to a daisy cube into a hypercube in linear time.

Keywords: Daisy cube, partial cube, isometric embedding, proper embedding.

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1 Introduction

Hypercube is one of the most important interconnection scheme for multicomputers. An obstacle to a direct application of a hypercube is the fact that the number of different hypercubes is very small with respect to the wanted (maximum) number of nodes, that is to say, the number of vertices of a hypercube is always equal to a power of two. For that reason, several other interconnection topologies for multicomputers based on hypercubes have been proposed. These graphs have been devised to preserve a hypercube's most essential properties while allowing more variety of resulting specific graphs. The corresponding families of graphs are mostly various subgraphs of a hypercube, of which its isometric subgraphs, i.e. its induced subgraphs that preserve distances, are of particular importance. A crucial problem in this scope is to find an embedding of a graph of this type to a hypercube (see for example [1, 4, 16]).

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Quite recently, a new concept which led to the class of graphs called daisy cubes has been proposed in [9]. It has been shown that daisy cubes are isometric subgraphs of a hypercube, moreover, they include several other important classes of graphs, some well-known examples are Fibonacci and Lucas cubes (see, for example [2, 5, 8, 11]) as well as some other families of generalized Fibonacci cubes and generalized Lucas cubes [3, 6, 7, 15]. Daisy cubes play an essential role in showing that specific generalized Fibonacci cubes' cube-complements are isometric subgraphs of a hypercube [13]. It is also proven that a class of graphs, which is of significant importance in chemical graph theory, also belongs to daisy cubes [14].

In [12], daisy cubes are characterized in terms of an expansion procedure. For a given graph G isomorphic to a daisy cube, but without the corresponding embedding into a hypercube, an algorithm which finds a proper embedding of G into a hypercube in $O(mn)$ time is also presented.

Several challenging open problems concerning daisy cubes have been proposed [9, 12]. In this paper, we focus our study to the following one.

Problem 1.1. Is there a faster way of finding the vertex 0^h of a daisy cube $Q_h(X)$ than the one provided in [12]?

It is also noted that a positive answer to Problem 1 would give a linear time algorithm for finding a proper embedding of a graph isomorphic to a daisy cube.

The paper is organized as follows. In the next section some basic definitions, concepts and results needed in the sequel are given. In Section 3, a notion of a minimal vertex of a daisy cube is introduced. Some necessary and sufficient conditions that a minimal vertex has to fulfill are also given. In Section 4, it is shown that an isometric embedding of a graph isomorphic to a daisy cube, but without the corresponding embedding into a hypercube, can be constructed in linear time even if a minimal vertex of a daisy cube is unknown. The last section shows that an isometric embedding devised in the Section 4 can be applied in order to find a proper embedding within the same time bound.

2 Preliminaries

Let $B = \{0, 1\}$. If b is a word of length h over B, that is, $b = (b_1, \dots, b_h) \in B^h$, then we will briefly write b as $b_1 \dots b_h$. If $x, y \in B^h$, then the *Hamming distance* $H(x, y)$ between x and y is the number of positions in which x and y differ.

We will use $[n]$ for the set $\{1, 2, \dots, n\}$.

The *hypercube* of order h or simply h-*cube*, denoted by Q_h, is the graph $G = (V, E)$ where the vertex set $V(G)$ is the set of all binary strings $b = b_1 b_2 \dots b_h$, $b_i \in \{0, 1\}$ for all $i \in [h]$, and two vertices $x, y \in V(G)$ are adjacent in Q_h if and only if the Hamming distance between x and y is equal to one.

For a binary string $b = b_1 b_2 \dots b_n$, let $\overline{b}_i = 1 - b_i$ for $i \in [h]$. The *weight* of $u \in B^h$ is $w(u) = \sum_{i=1}^{h} u_i$, in other words, $w(u)$ is the number of 1s in the word u. For the concatenation of bits the power notation will be used, for instance $0^h = 0 \dots 0 \in B^h$.

If G is a connected graph, then the distance $d_G(u, v)$ (or simply $d(u, v)$) between vertices u and v is the length of a shortest u, v-path (that is, a shortest path between u and v) in G. The set of vertices lying on all shortest u, v-paths is called the *interval* between u and v and denoted by $I_G(u, v)$ [10]. We will also write $I(u, v)$ when G will be clear from the context.

If G is a graph and $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G induced by X.

If u is a vertex of a graph G, let $N(u)$ denote the set of neighbors of u. Moreover, let $N[u] = N(u) \cup \{u\}$.

Let $G = (V, E)$ be a graph. A mapping $\alpha : V(G) \to V(Q_h)$ is an *isometric embedding* of G into Q_h if $d_{Q_h}(\alpha(u), \alpha(v)) = d_G(u, v)$ for every $u, v \in V(G)$. If $u \in V(G)$, we will denote the i-th coordinate of $\alpha(u)$ as $\alpha_{(i)}(u)$.

Let G be a connected graph. The isometric dimension of G is the smallest integer h such that G admits an isometric embedding into Q_h. Isometric subgraphs of hypercubes are called *partial cubes*.

Let $\leq$ be the partial order on $V(Q_h)$ defined with $u_1 \dots u_h \leq v_1 \dots v_h$ if $u_i \leq v_i$ holds for all $i \in [h]$. For $X \subseteq V(Q_h)$ the graph induced by the set $\{v \in V(Q_h) \,|\, v \leq x$ for some $x \in X\}$ is a *daisy cube* of Q_h generated by X and denoted by $Q_h(X)$.

Let also $\vee$, $\wedge$ and $\oplus$ denote the bitwise OR, bitwise AND and bitwise exclusive OR operator, respectively.

By a slight abuse of definition, we will say that a graph G is a daisy cube if it is isomorphic to a daisy cube generated by some $X \subseteq V(Q_h)$. If G is a daisy cube $Q_h(X)$, then G may admit more than one isometric embedding of G into the h-cube. Let $X_G \subseteq B^h$ be the set of labels of the vertices of G assigned by an isometric embedding α, i.e. $X_G = \alpha(V(G))$. We say that α is a *proper embedding* of G if G is isomorphic to $Q_h(X_G)$.

Let G be a graph isomorphic to a daisy cube of G_h and let α denote a proper embedding. Note that every permutation of indices of α yields basically the "same" embedding. We say that proper embeddings α and β are *equivalent* if β can be obtained from α by a permutation of its indices.

For a daisy cube $Q_h(X)$, let $\widehat{X}$ denote the antichain consisting of the maximal elements of the poset $(X, \leq)$. It was shown in [9] that $Q_h(X) = Q_h(\widehat{X})$. Hence, for a given set $X \subseteq B_n$ it is enough to consider the antichain $\widehat{X}$. The vertices of $Q_h(X)$ from $\widehat{X}$ are called the *maximal* vertices of $Q_h(X)$. More generally, if G is a daisy cube of Q_h with a proper embedding α such that $\alpha(v) = 0^h$, then $X \subseteq V(G)$ is the set of *maximal vertices of G with respect to* v if $G \cong Q_h(\alpha(X))$ and $\widehat{\alpha(X)} = \alpha(X)$. Moreover, v is the *minimal vertex of G with respect to* α. We also say that v is a *minimal vertex of G* if there exists a proper embedding α such that $\alpha(v) = 0^h$.

The following result shows that a daisy cube is a subgraph of Q_h induced by the union of intervals between 0^h and the vertices from $\widehat{X}$ [9].

Lemma 2.1. *Let $X \subseteq B^h$. Then $Q_h(X) = Q_h[\cup_{x\in\widehat{X}} I(0^h, x)]$.*

3 Minimal vertices of a daisy cube

If $u \in V(Q_h(X))$, then $I(0^n, u)$ induces a $w(u)$-cube in $Q_h(X)$. Note that if $x \in \widehat{X}$, then the cube induced by $I(0^n, x)$ is maximal in $Q_h(X)$, i.e., it is not contained in any other cube that belongs to $Q_h(X)$.

If $x \in B^h$, let S^x denote the set of indices of v with $x_i = 1$, i.e., $S^x = \{i \,|\, x_i = 1 \text{ and } i \in [h]\}$.

Let $v \in B^h$ and let ${}^v\beta : B^h \to B^h$ be the function defined as

$$ {}^v\beta_{(i)}(u) = \begin{cases} u_i, & v_i = 0 \\ \bar{u}_i, & v_i = 1 \end{cases} $$

Lemma 3.1. *Let G be a graph isomorphic to a daisy cube of Q_h with a proper embedding α such that $\alpha(v^0) = 0^h$ and $\widehat{X} \subseteq V(G)$ is its corresponding maximal set. If $v \in \cap_{x\in\widehat{X}} I(v^0, x)$, then*

(i) *${}^v\beta$ restricted to $\alpha(V(G))$ is a bijection that maps to $\alpha(V(G))$,*

(ii) *${}^v\beta \circ \alpha$ is a proper embedding of G with the minimal vertex v and the maximal vertex set $Y = \{y \mid {}^v\beta(\alpha(y)) = \alpha(x) \text{ and } x \in \widehat{X}\}$.*

Proof. (i) We have to show that if $v \in \cap_{x\in\widehat{X}} I(v^0, x)$, then for every $u \in \alpha(V(G)))$ there is exactly one ${}^v\beta(u) \in \alpha(V(G))$. Note that $\alpha^{-1}(u) \in I(v^0, x)$ and $v \in I(v^0, x)$ for some $x \in \widehat{X}$. Thus, $S^u \subseteq S^{\alpha(x)}$ and $S^{\alpha(v)} \subseteq S^{\alpha(x)}$. It follows that $S^{{}^v\beta(u)} \subseteq S^{\alpha(x)}$. Since α is proper, $\alpha(V(G)) = \cup_{x\in\widehat{X}} I(0^h, \alpha(x))$ by Lemma 2.1 and we obtain ${}^v\beta(u) \in V(\alpha(G))$.

In order to see that ${}^v\beta$ is injective, note that ${}^v\beta({}^v\beta(u)) = u$ for every $u \in \alpha(V(G))$. Suppose to the contrary that there exist $u, z \in \alpha(V(G))$, $u \neq z$, such that ${}^v\beta(u) ={}^v \beta(z)$. It follows that ${}^v\beta({}^v\beta(u)) ={}^v\beta({}^v\beta(z))$ and thus $u = z$, which yields a contradiction.

(ii) By (i), ${}^v\beta$ maps from $\alpha(V(G))$ to $\alpha(V(G))$. Let $x \in \widehat{X}$ and recall that ${}^v\beta({}^v\beta(\alpha(x))) = \alpha(x)$. Thus, if $y \in V(G)$ such that $\alpha(y) = {}^v\beta(\alpha(x))$, we have ${}^v\beta(\alpha(y)) = \alpha(x)$. Moreover, ${}^v\beta(v) = 0^h$. It follows that $Y = \{y \mid {}^v\beta(\alpha(y)) = \alpha(x) \text{ and } x \in \widehat{X}\}$ is the maximal vertex set of G with respect to ${}^v\beta \circ \alpha$, while v is the corresponding minimal vertex. □

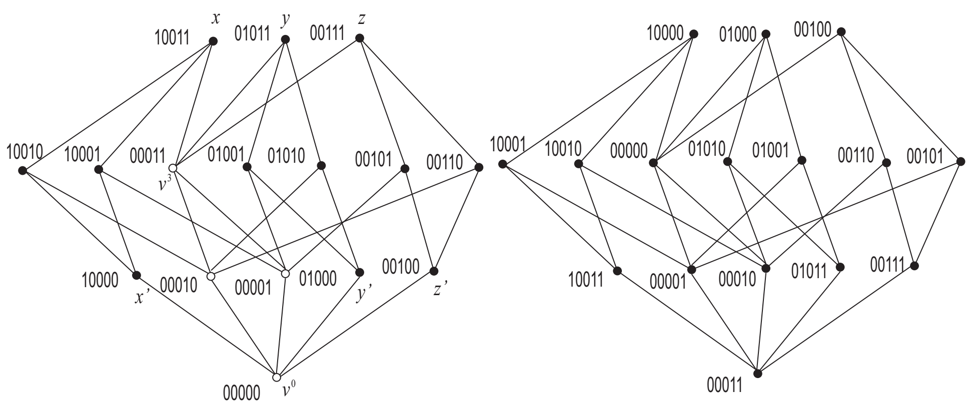

Figure 1: Two proper embeddings of a daisy cube.

Figure 1 shows two proper embeddings of a daisy cube G. The embedding on the left hand side, say α, admits the set of maximal vertices $\widehat{X} = \{x, y, z\}$ with labels $\alpha(x) = 10011$, $\alpha(y) = 01011$ and $\alpha(z) = 00111$. Let $v^0 \in V(G)$ such that $v^0 = \alpha^{-1}(00000)$. Then $I(v^0, x) \cap I(v^0, y) \cap I(v^0, z) = \{v^0, v^1, v^2, v^3\}$, where $\alpha(v^3) = 00011$. The embedding on the right hand side of Figure 1 is ${}^{v^3}\beta \circ \alpha$ with the set of maximal vertices $Y = \{x', y', z'\}$, where the corresponding labels are $\alpha(x') = 10000$, $\alpha(y') = 01000$ and $\alpha(z') = 00100$. Note also that ${}^{v^3}\beta(\alpha(x')) = 10011$, ${}^{v^3}\beta(\alpha(y')) = 01011$ and ${}^{v^3}\beta(\alpha(z')) = 00111$.

Let $u \in V(G)$ where $G = Q_h(X)$ and let X^u be the maximal subset of $\widehat{X}$ with the property $u \in \cap_{x\in X^u} I(0^h, x)$. Let G^u be the graph induced by the set $\cup_{x\in X^u} I(0^h, x)$, i.e. $G^u = G[\cup_{x\in X^u} I(0^h, x)]$. Note that by Lemma 3.1 and Lemma 2.1, G^u is a daisy cube of

Q_h and u is its minimal vertex. Observe for example the graph $Q_4(0111, 1011, 1101, 1110)$ on the right hand side of Figure 2: if $u = 1100$, then $X^u = \{1110, 1101\}$.

As noted in [12], an efficient way of finding a minimal vertex of a daisy cube G would give a linear time algorithm for finding a proper embedding of G. It was also shown that if G is a daisy cube of Q_h, then a minimal vertex of G is of degree h. It is not difficult to see that a vertex of degree h need not to be a minimal vertex of G. Note for example that Q_h^- (that is a vertex deleted Q_h) admits $2^h - h - 1$ vertices of degree h and exactly one minimal vertex (see also Figure 2, where Q_4^- is depicted).

Proposition 3.2. *Let $u \in V(G)$, where $G = Q_h(X)$ and $d(u) = h$. Moreover, let X^u be the maximal subset of $\widehat{X}$ such that $u \in \cap_{x\in X^u} I(0^h, x)$. Then for every proper embedding α, the minimal vertex of G with respect to α belongs to $\cap_{x\in X^u} I(0^h, x)$.*

Proof. Let v be the minimal vertex of G with respect to some proper embedding. Note that for every $x \in \widehat{X}$ and every $u \in I(0^h, x)$ we have $d(v, u) \le |S^x|$. Suppose to the contrary that $v \notin \cap_{x\in X^u} I(0^h, x)$. It follows that there exists $x \in X^u$ such that $v \notin I(0^h, x)$. Since $u \in I(0^h, x)$, it follows that $S^u \subseteq S^x$. Moreover, since $v \notin I(0^h, x)$, there exists an index $j \notin S^x$ such that $v_j = 1$. It follows that the string u defined by

$$u_i = \begin{cases} \bar{v}_i, \ i \in S^x \\ 0, \ \text{otherwise} \end{cases}$$

is a vertex of $I(0^h, x)$ with $d(v, u) > |S^x|$ and we obtain a contradiction. □

Theorem 3.3. *If $G = Q_h(X)$ and $\hat{x} = \wedge_{x\in\widehat{X}} x$, then for every proper embedding α, v is the minimal vertex of G with respect to α if and only if $v \in \cap_{x\in\widehat{X}} I(0^h, x) = I(0^h, \hat{x})$.*

Proof. By Lemma 3.1 and Proposition 3.2, v is a minimal vertex of G, if and only if $v \in \cap_{x\in\widehat{X}} I(0^h, x)$. Note that $v \in \cap_{x\in\widehat{X}} I(0^h, x)$ if and only if $S^v \subseteq \cap_{x\in X} S^x$. Since $S^{\hat{x}} = \cap_{x\in X} S^x$, for every $v \in V(G)$ we have $v \in \cap_{x\in\widehat{X}} I(0^h, x)$ if and only if $v \le \hat{x}$. It follows that $\cap_{x\in X} I(0^h, x) = I(0^h, \hat{x})$ and the assertion follows. □

4 Isometric embedding

If v is a vertex of a partial cube G, then $N_G^v(u)$ (or simply $N^v(u)$) is the set of neighbors of u which are closer to v than u, more formally $N_G^v(u) := \{z \,|\, z \in N(u) \text{ and } d(v, z) = d(v, u) - 1\}$,

If G is a graph isomorphic to a hypercube (but without an embedding), then its isometric embedding is easy to obtain as shown in the next result.

Proposition 4.1. *Let G be a graph isomorphic to a h-cube, v an arbitrary vertex of G and $\alpha : V(G) \to V(Q_h)$ a function such that $\alpha(v) = 0^d$, the vertices of $N(v)$ obtain pairwise different labels of the form $0^{i-1}10^{h-i}$, $i \in [h]$, while for the other vertices $u \in V(G)$ ordered by an increasing distance from v, we set $\alpha(u) = \vee_{z\in N^v(u)} \alpha(z)$. Then α is an isometric embedding of G into Q_h. Moreover, when a labeling of vertices in $N[v]$ is chosen, α is unique.*

Proof. Since a hypercube is vertex-transitive, we may choose an arbitrary vertex v of G and set $\alpha(v) = 0^h$. Moreover, for every $u \in V(G)$ with $d(v, u) = s$, $s \ge 1$, we must have $N^v(u) = \{z \mid \alpha_{(i)}(z) = \alpha_{(i)}(u) = 1$ for exactly one $i \in [h]$ and $\alpha_{(j)}(z) = \alpha_{(j)}(u)$ for every $j \in [h] \setminus \{i\}\}$. Thus, $\alpha(u) = \vee_{z \in N^v(u)} \alpha(z)$. It follows that for chosen labeling of vertices in $N[v]$, α is unique. □

Lemma 4.2. *Let G be partial cube of isometric dimension h, u a vertex of degree h in G and let for every $v \in V(G) \setminus N[u]$ it holds that $|N^u(v)| \geq 2$. Define the function $\alpha : V(G) \to V(Q_h)$ such that $\alpha(u) = 0^h$, the vertices of $N(u)$ obtain pairwise different labels of the form $0^{i-1}10^{h-i}$, $i \in [h]$, while for the other vertices $v \in V(G)$ ordered by an increasing distance from u, we set $\alpha(v) = \vee_{z \in N^u(v)} \alpha(z)$. Moreover,*

(i) *α is an isometric embedding of G into Q_h,*

(ii) *when a fixed embedding of vertices in $N[v]$ is chosen, α is unique.*

Proof. Since G is a partial cube of dimension h, we may assume that G is an isometric subgraph of an (unlabeled) h-cube H. Let β be an embedding of H with respect to v as defined in Proposition 4.1 and let α be an embedding of G such that for every $z \in N[u]$ we set $\alpha(z) = \beta(z)$. Since $|N^u_G(v)| \geq 2$ and $N^u_G(v) \subseteq N^u_H(v)$ for every $v \in V(G) \setminus N[u]$, it follows that $\alpha(v) = \beta(v)$ for every vertex $v \in V(G)$. By Proposition 4.1, β is an isometric embedding of H into Q_h. Thus, α is an isometric embedding of H into Q_h. Moreover, by Proposition 4.1, α is unique for a fixed embedding of vertices in $N[v]$. □

Corollary 4.3. *Let G be a graph isomorphic to a daisy cube of order h. If v is a minimal vertex of G and α an isometric embedding with $\alpha(v) = 0^h$, then α is proper.*

Proof. Since v is a minimal vertex of G, there exist a proper embedding, say β, such that $\beta(v) = 0^h$. We may also assume w.l.o.g. that for every $u \in N(v)$ we have $\beta(u) = \alpha(u)$. From Lemma 4.2 then it follows that $\beta(u) = \alpha(u)$ for every $v \in V(G)$. □

Remark 4.4. If G is isomorphic to a daisy cube and α a proper embedding of G, then different selections of labels for vertices of $N(u)$ yield different but equivalent proper embeddings.

If G is a partial cube and α its isometric embedding to Q_h, let $W_i(G)$ denote the set of vertices of G with weight i, i.e. $W_i(G) = \{v \mid w(\alpha(v)) = i\}$.

We will also need the following result.

Proposition 4.5. *If G is a partial cube, α its isometric embedding to Q_h and $v \in V(G)$ such that $w(\alpha(v)) = i$, then $|N(v) \cap W_{i-1}(G)| \leq i$.*

Proof. Since α is isometric embedding of G to Q_h, for every $v \in V(G)$ with $w(\alpha(v)) = i$, we have $N_G(v) \subseteq N_{Q_h}(v)$. Moreover, $|N(v) \cap W_{i-1}(Q_h)| = i$ and therefore $|N(v) \cap W_{i-1}(G)| \leq i$. □

Proposition 4.6. *Let $G = Q_h(X)$, $x, y \in \widehat{X}$ and $x \neq y$. If $u \in I(0^h, x)$ and $v \in I(0^n, y)$ such that $u, v \notin I(0^n, x) \cap I(0^h, y)$ then $uv \notin E(G)$.*

Proof. Suppose to the contrary that there exist $u \in I(0^h, x)$ and $v \in I(0^h, y)$ such that $u, v \notin I(0^h, x) \cap I(0^h, y)$ and $d(u, v) = 1$. Since $\widehat{X}$ is maximal, there exist at least two indices $i, j \in [h]$, such that $x_i \neq y_i$ and $x_j \neq y_j$ (otherwise we have either $x \leq y$ or $y \leq x$). Suppose w.l.o.g. $x_i = 1$, $y_j = 1$ and $u_k = v_k$ for every $k \in [h] \setminus \{i, j\}$. If $u_i = 0$ (resp. $v_j = 0$), then $u \in I(0^h, y)$ (resp. $v \in I(0^h, x)$). It follows that $u_i = v_j = 1$. But then $u = v$ and we obtain a contradiction. □

Proposition 4.7. *Let $G = Q_h(X)$, X^u be the maximal subset of $\widehat{X}$ such that $u \in \cap_{x \in X^u} I(0^h, x)$ and $G^u = G[\cup_{x \in X^u} I(0^h, x)]$. If $u \in V(G)$ and $d(u) = h$, then $N(u) \subseteq V(G^u)$.*

Proof. Suppose to the contrary that there exists $v \in N(u)$ such that $v \notin \cup_{x \in X^u} I(0^h, x)$. It follows that there exists $y \in \widehat{X} - X^u$ such that $v \in I(0^h, y)$. Since $u \in I(0^h, x)$ for some $x \in \widehat{X}$ and $x \neq y$, Proposition 4.6 yields a contradiction. □

Proposition 4.8. *Let $G = Q_h(X)$, $u \in V(G)$ and X^u be the maximal subset of $\widehat{X}$ such that $u \in \cap_{x \in X^u} I(0^h, x)$. If $d(u) = h$, then $| \cup_{x \in X^u} S^x| = h$.*

Proof. Suppose $| \cup_{x \in X^u} S^x| < h$. It follows that there exist $j \in [h]$ such that for all $v \in \cup_{x \in X^u} I(0^h, x)$ we have $v_j = 0$. Since $d(u) = h$, there exists $z \in N(u)$ such that $z_j = 1$. It follows that $z \notin \cup_{x \in X^u} I(0^h, x)$. Thus, there exists $y \in \widehat{X} - X^u$ such that $v \in I(0^h, y)$. Proposition 4.7 yields a contradiction. □

Lemma 4.9. *Let $G = Q_h(X)$ and $u \in V(G)$ such that $d(u) = h$. Then $|N^u(v)| \geq 2$ for every $v \in V(G) \setminus N[u]$.*

Proof. Let X^u be the maximal subset of $\widehat{X}$ with the property $u \in \cap_{x \in X^u} I(0^h, x)$ and $G^u = G[\cup_{x \in X^u} I(0^h, x)]$. By Lemma 3.1 and Lemma 2.1, G^u is a daisy cube and u its minimal vertex. It follows that the lemma holds for every $v \in V(G^u)$. Suppose then that $v \notin \cup_{x \in X^u} I(0^h, x)$. Thus, there exists $y \in \widehat{X} - X^u$, such that $v \in I(0^h, y)$. Note that $S^u \subseteq \cap_{x \in X^u} S^x$.

Let $S^{u+} = \{i \,|\, u_i = 1 \text{ and } v_i = 0\}$ and $S^{u-} = \{i \,|\, v_i = 1 \text{ and } u_i = 0\}$.

We first show that $|S^{u-}| \neq 1$. Suppose to the contrary that there exists exactly one index $i \in [h] \setminus S^{u+}$, such that $v_i = 1$ and $u_i = 0$. Since $d(u) = h$, by Proposition 4.8, there exists $x \in X^u$ such that $x_i = 1$. Note also that $S^u \subseteq S^x$ and since $x_i = 1$, we have $S^v \subseteq S^x$. It follows that $v \leq x$ and we obtain a contradiction.

If $|S^{u+}| = 0$, then vertices of $I(u, v)$ induce a $|S^{u-}|$-cube in G. Thus, v admits $|S^{u-}|$ neighbors at distance $d(u, v) - 1$ from u. Clearly, $|S^{u+}| = 0$ implies $|S^{u-}| > 0$. Moreover, since we show above that $|S^{u-}| \neq 1$, we have $|S^{u-}| \geq 2$ and the case is settled.

If $|S^{u+}| > 0$, we may find $i, j \in S^{u-}$ such that $i \neq j$. Let z and z' be vertices obtained from v by setting the i-th and j-th coordinate to zero, respectively. Obviously, $z, z' \in N^u(v)$.

Since we show that we obtain $|N^u(v)| \geq 2$ for every value of $|S^{u+}|$, the lemma holds for every $v \in V(G) \setminus N[u]$. This assertion concludes the proof. □

Lemma 4.9 is the basis for the next algorithm which finds an isometric embedding for an unlabeled graph isomorphic to a daisy cube of dimension h.

Procedure Embedding(G, h, β, u);

1. u is a vertex of degree h in G;
2. $\beta(u) := 0^h$;
3. $i := 1$;
4. $Q := \emptyset$; $\{Q$ is an empty queue$\}$
5. **for all** $v \in V(G)$ **do** $p(v) := 0$;
6. **for all** $v \in N(u)$ **do begin**
   $\beta(v) := 0^{i-1}10^{h-i}$;

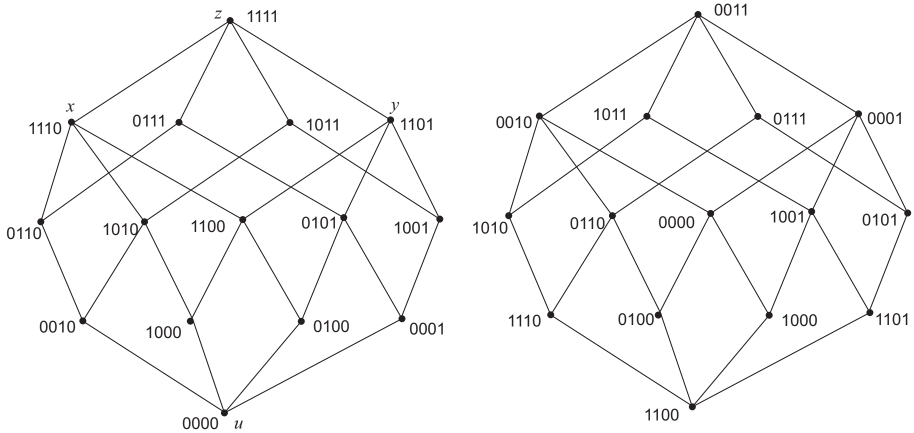

Figure 2: An isometric (left) and proper (right) embedding of a daisy cube isomorphic to Q_4^-.

$i := i + 1$;
$p(v) := u$;
Insert v in the end of Q;
end;
7. **while** $Q \neq \emptyset$ **do begin**
7.1 Remove the first vertex v from Q;
7.2. **for all** $z \in N(v)$ **do**
if $p(z) = 0$ **then begin**
$p(z) := v$;
Append z to the end of Q;
end
else $\beta(z) := \beta(v) \vee \beta(p(z))$;
end.

Theorem 4.10. *If G is a daisy cube, then an isometric embedding of G can be found in linear time.*

Proof. Note first that Lemma 4.2 defines the procedure to construct an isometric embedding of G into Q_h. Let α and β be isometric embeddings as defined in Lemma 4.2 and algorithm Embedding, respectively. Suppose that u is the vertex being labeled 0^h both by the algorithm and by the construction of Lemma 4.2. Clearly, for every v in $N[u]$ we could have $\alpha(v) = \beta(v)$. Note also that in the essence the algorithm performs a BFS search in G (see for example [4, Section 17.3]). Thus, for every $z \in N(v)$ of Step 7.2 we have $d(u, z) = d(u, p(z)) + 1 = d(u, v) + 1$. It follows that $v, p(z) \in N^u(z)$. By Lemma 4.9, since $d(u) = h$, for every $v \in V(G) \setminus N[u]$ we have $|N_G^u(v)| \geq 2$. Therefore, $\alpha(z) = \beta(z)$ for every $z \in V(G) \setminus N[u]$.

For the time complexity of the algorithm, note that the number of the executions of the body of the loop in Step 7.2 is bounded by the number of edges of a graph. Since the time complexity of the body of the loop is constant, the overall number of step of the algorithm is linear in the number of the edges of the graph. □

5 Proper embedding

Lemma 5.1. *Let G be a daisy cube of Q_h, v a minimal vertex of G and u a vertex of degree h of G. If β is an isometric embedding of G such that $\beta(u) = 0^h$, then ${}^v\beta \circ \beta$ is a proper embedding of G.*

Proof. Note that ${}^v\beta(\beta(v)) = 0^h$. Since β is isometric, it is easy to see that ${}^v\beta \circ \beta$ is also isometric. Corollary 4.3 now yields the assertion. □

Let u be a vertex of degree h of $G = Q_h(X)$. Let X^u be the maximal subset of $\widehat{X}$ with the property $u \in \cap_{x \in X^u} I(0^h, x)$ and $G^u = G[\cup_{x \in X^u} I(0^h, x)]$. Recall that G^u is a daisy cube of Q_h and u its minimal vertex. If β is an isometric embedding of G such that $\beta(u) = 0^h$, let Y^u be the set of maximal vertices of G^u with respect to u and let Z^u be the set of vertices z of $V(G) \setminus V(G^u)$ with the property $N^u(z) = N(z)$.

Proposition 5.2. *Let u be a vertex of degree h of $G = Q_h(X)$. If β is an isometric embedding of G such that $\beta(u) = 0^h$, then $Y^u = \{y \,|\, \beta(y) = x \text{ and } x \in X^u\}$.*

Proof. As noted above, G^u is a daisy cube of Q_h and u its minimal vertex. Since u is of degree h and $\beta(u) = 0^h$, the restriction of β to $V(G^u)$ is a proper embedding of G^u. Moreover, since every permutation of indices of a proper embedding yields an equivalent embedding, we may assume w.l.o.g. that for every $z \in N(u)$ we have $\beta(z) = 0^{i-1}10^{h-i}$ if and only if $u_i \neq z_i$. It follows that for every $w \in N(0^h)$ we have ${}^u\beta(\beta(w)) = w$. By Lemma 3.1, ${}^u\beta \circ \beta$ is proper. Moreover, by Lemma 4.2, ${}^u\beta(\beta(v)) = v$ for every $v \in V(G^u)$. From Lemma 3.1 then follows that $Y^u = \{y \,|\, \beta(y) = x \text{ and } x \in X^u\}$. □

Proposition 5.3. *Let u be a vertex of degree h of $G = Q_h(X)$ and $z \in Z^u$. If β is an isometric embedding of G and $\beta(u) = 0^h$, then there exists $y \in \widehat{X} - X^u$ such that $z \in I(0^h, y)$. Moreover,*

$$\beta_{(i)}(z) = \begin{cases} 0, & i \in S^u \\ y_i, & i \notin S^u \end{cases}$$

Proof. Let X^u be the maximal subset of $\widehat{X}$ with the property $u \in \cap_{x \in X^u} I(0^h, x)$. By Lemma 2.1, since $z \notin \cup_{x \in X^u} I(0^h, x)$, there must be $y \in \widehat{X} - X^u$ such that $z \in I(0^h, y)$. By $N^u(z) = N(z)$, we have $d(u, z) \geq d(u, v)$ for every $v \in I(0^h, y)$. If $v_i = 1$ for some $i \in S^u$, then let v' be the vertex of G such that $v'_j = v_j$ for every $j \neq i$ and $v'_i = 0$. Obviously, $v' \leq y$, thus $v' \in I(0^h, y)$. Moreover, since $\beta_{(i)}(v') = 1$, we have $d(u, v') > d(u, v)$ and we obtain a contradiction. It follows that the assertion holds for every $i \in S^u$. If $i \notin S^u$, then $\beta_{(i)}(v) = v_i$ for every $v \in I(0^h, y)$. Since y is maximal in $I(0^h, y)$, the assertion follows. □

Theorem 5.4. *Let u be a vertex of degree h of $G = Q_h(X)$. If β is an isometric embedding of G such that $\beta(u) = 0^h$, $\hat{y} = \wedge_{y \in Y^u} \beta(y)$, $\hat{z} = \wedge_{z \in Z^u} \beta(z)(\wedge 1^h)$ and $v = \beta^{-1}(\hat{y} \wedge \hat{z})$, then v is the minimal vertex of G with respect to ${}^v\beta \circ \beta$.*

Proof. Note first that $\beta = \beta^{-1}$, thus, for every $b \in B^h$ and every $i \in [h]$ it holds

$$\beta_{(i)}(b) = \beta^{-1}_{(i)}(b) = \begin{cases} \bar{b}_i, & i \in S^u \\ b_i, & i \notin S^u \end{cases} \tag{5.1}$$

Let $\hat{x} = \wedge_{x \in X^u} x$. By Proposition 5.2, we have $Y^u = \{y \mid \beta(y) = x \text{ and } x \in X^u\}$. Thus, $\hat{x} = \hat{y}$. Note that by Proposition 3.2, every minimal vertex of G belongs to $I(0^h, \hat{x})$.

If $X^u = \widehat{X}$, then $Z^u = \emptyset$ and we get $\beta^{-1}(\hat{y} \wedge \hat{z}) = \beta^{-1}(\hat{y}) = \beta^{-1}(\hat{x})$. By equation (5.1), we have $\beta^{-1}(\hat{x}) \leq x$. It follows that $\beta^{-1}(\hat{x}) \in I(0^h, \hat{x})$ and we are done.

Otherwise, let $z \in Z^u$ be such that $z \in I(0^h, y)$ for some $y \in \widehat{X} - X^u$. We have to show that $\beta^{-1}(\hat{x} \wedge \beta(z))$ is a minimal vertex of $\cup_{x \in X^u} I(0^h, x) \cup I(0^h, y)$, i.e. $S^{\beta^{-1}(\hat{x} \wedge \beta(z))} \subseteq S^{\hat{x} \wedge y}$.

By Proposition 5.3, we have

$$\beta_{(i)}(z) = \begin{cases} 0, & i \in S^u \\ y_i, & i \notin S^u \end{cases}$$

Since $S^u \subseteq S^{\hat{x}}$, we have

$$(\hat{x} \wedge \beta(z))_i = \begin{cases} y_i, & i \notin S^{\hat{x}} \setminus S^u \\ 0, & \text{otherwise} \end{cases}$$

By equation (5.1), we have $\beta_i^{-1}(\hat{x} \wedge \beta(z)) = 0$ for every $i \in [h] \setminus S^{\hat{x} \wedge y}$. Since we can repeat the above discussion for every $z \in Z^u$, we showed that $\beta^{-1}(\hat{x} \wedge \hat{z}) = \beta^{-1}(\hat{y} \wedge \hat{z})$ is a minimal vertex of G. Moreover, since by Lemma 5.1 it follows that ${}^{\beta^{-1}(\hat{y} \wedge \hat{z})}\beta \circ \beta$ is a proper embedding of G, the proof is complete. □

Figure 2 shows two embeddings of a daisy cube G isomorphic to Q_4^-. The embedding β on the left hand side is determined such that $\beta(u) = 0000$ (note that $d(u) = 4$). Since u is not minimal in G, the embedding β is isometric but not proper. From $X^u = Y^u = \{x, y\}$ and $Z^u = \{z\}$ we get $\hat{y} = 1110 \wedge 1101 = 1100$, $\hat{z} = 1111$ and $\hat{y} \wedge \hat{z} = 1100 \wedge 1111 = 1100$. Moreover, the minimal vertex of G is $v = \beta^{-1}(1100)$ and ${}^{v}\beta \circ \beta$ is the proper embedding of G as described in Lemma 5.1. That is to say, we obtain the proper embedding of G by assigning $\beta(w) \oplus 1100$ to every $w \in V(G)$.

Theorem 5.4 is the basis for the next algorithm, which finds a proper embedding of a graph isomorphic to a daisy cube of dimension h.

Procedure Proper(G, h, α);
1. Embedding(G, h, β, u);
2. **for** $i := 1$ **to** $h + 1$ **do** $W_i := \emptyset$;
3. **for all** $v \in V(G)$ **do** $W_{w(\beta(v))} := W_{w(\beta(v))} \cup \{v\}$;
4. **for all** $v \in V(G)$ **do** $q(v) := 0$;
5. **for** $i := 1$ **to** h **do begin**
   5.1. **for all** $x \in W_i$ **do**
      5.1.1 **if** $\sum_{y \in N(x) \cap W_{i-1}} q(y) = i(i-1)$ **then begin**
         $q(x) := i$;
         for all $y \in N(x) \cap W_{i-1}$ **do** $q(y) := 0$;
      end
      5.1.2 **else if** $N(x) \cap W_{i+1} = \emptyset$ **then** $q(x) := i$
6. $s := 1^h$;
7. **for all** $v \in V(G)$ **do**
   7.1. **if** $q(v) \neq 0$ **then** $s := s \wedge \beta(v)$;
8. **for all** $v \in V(G)$ **do** $\alpha(v) := s \oplus \beta(v)$;

end.

Theorem 5.5. *A proper embedding of an unlabeled graph isomorphic to a daisy cube can be found in linear time.*

Proof. We first show that the algorithm Proper finds a proper embedding of G. As shown in Theorem 4.10, embedding β provided by the algorithm Embedding is isometric. With respect to Theorem 5.4 and Step 7, we have to show that if $q(v) \neq 0$, then either $v \in Y^u$ or $v \in Z^u$. Clearly, in Step 3, all vertices at distance i from u are inserted in W_i, while in Step 4, $q(v)$ is set to 0 for every $v \in V(G)$. The value of $q(v)$ is altered either in Step 5.1.1 or in Step 5.1.2.

Let $w(x) = i$. We show that $q(x) = i$ in the i-th iteration of for loop if and only if either $I(u, x)$ induces an i-cube or $x \in Z^u$. Note that $I(u, x)$ induces an i-cube, if and only $|N(x) \cap W_{i-1}| = i$ and for every $y \in N(x) \cap W_{i-1}$ the set $I(u, y)$ induces a $(i-1)$-cube. Moreover, if $x \in Y^u$, then $I(u, x)$ induces a maximal i-cube in G^u.

In the first iteration of Step 5, for every vertex of W_1 the value of q is set to 1. In the next iteration, when a vertex x of W_2 is considered, these values for two vertices of W_1, say y and y', are set to zero if $\{u, y, y', x\}$ induce a 2-cube. Thus, for every $x, y \in W_1 \cup W_2$ we have

- $q(y) = 1$ if and only if $x \in N(u)$ and there is no vertex $y \in W_2$ such that $I(u, y) \subseteq I(u, x)$ and $I(u, x)$ induces Q_2.

- $q(x) = 2$ if and only $I(u, x)$ induces Q_2.

Suppose now that for $i \geq 3$ and $y \in W_{i-1}$ it holds that $q(y) = i-1$ if and only if $I(u, y)$ induces a maximal cube in $G[W_1 \cup W_2 \ldots \cup W_{i-1}]$ or $N^u(y) = N(y)$; otherwise, $q(y) = 0$. Let $w(x) = i$. Note that $|N(x) \cap W_{i-1}| \leq i$ by Proposition 4.5. Thus, the condition of the if statement in Step 5.1.1 is fulfilled if and only if for every $y \in N(x) \cap W_{i-1}$ we have $q(y) = i - 1$, i.e. for every $y \in N(x) \cap W_{i-1}$ the set $I(u, y)$ induces an $(i-1)$-cube. If the condition of the if statement returns true, then $q(x)$ obtains the value i while for every $y \in N(x) \cap W_{i-1}$ the value of $q(y)$ is set to 0. If the condition of the if statement returns false, then $q(x)$ is set to i if and only if $N(x) \cap W_{i+1} = \emptyset$, i.e. $x \in Z^u$. Thus, we showed that in the i-th iteration of the for loop $q(x) = i$ if and only if either $I(u, x)$ induces an i-cube or $x \in Z^u$. Since the claim holds for every i, we showed that if $q(v) \neq 0$, $v \in V(G)$, then either $v \in Y^u$ or $v \in Z^u$. From Theorem 5.4 then it follows that the string s computed in Step 7 is equal to $\hat{y} \wedge \hat{z}$, where $\hat{y} = \wedge_{y \in Y^u} \beta(y)$ and $\hat{z} = \wedge_{z \in Z^u} \beta(z)$. By Theorem 5.4, $\beta^{-1}(s) = v$ is a minimal vertex of G while the embedding α obtained in Step 8 is equal to ${}^v\beta \circ \beta$. Moreover, α is proper by Lemma 5.1.

In order to consider the time complexity of the algorithm, note first that all steps of the algorithm except Step 5 can be executed in $O(m)$ time, where m is the number of edges of G. For the time complexity of Step 5 it is convenient to store the weights of vertices in a vector, which allows that the weight of a vertex and therefore its inclusion in a set W_i can be determined in constant time. Thus, the time complexity of Steps 5.1.1 and 5.1.2 is linear in the number of edges incident with the vertex x. Since Step 5 is performed for every vertex of the graph, the total number of steps is bounded by the number of edges of G. This assertion concludes the proof. □

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