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Sum-list-colouring of θ-hypergraphs

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Abstract

Given a hypergraph $\mathcal{H}$ and a function $f\colon V(\mathcal{H}) \to \mathbb{N}$, we say that $\mathcal{H}$ is f-choosable if there is a proper vertex coloring ϕ of $\mathcal{H}$ such that $\phi(v) \in L(v)$ for all $v \in V(\mathcal{H})$, where $L\colon V(\mathcal{H}) \to 2^{\mathbb{N}}$ is any assignment of $f(v)$ colors to a vertex v. The sum choice number $\chi_{sc}(\mathcal{H})$ of $\mathcal{H}$ is defined to be the minimum of $\sum_{v\in V(\mathcal{H})} f(v)$ over all functions f such that $\mathcal{H}$ is f-choosable. A trivial upper bound on $\chi_{sc}(\mathcal{H})$ is $|V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. The class Γ_{sc} of hypergraphs that achieve this bound is induced hereditary. We analyze some properties of hypergraphs in Γ_{sc} as well as properties of hypergraphs in the class of forbidden hypergraphs for Γ_{sc}. We characterize all θ-hypergraphs in Γ_{sc}, which leads to the characterization of all θ-hypergraphs that are forbidden for Γ_{sc}.

Keywords: Hypergraphs, sum-list-colouring, θ-hypergraphs.

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1 Introduction

A hypergraph is a very natural generalization of a graph. It always motivates the extension of a problem first posed in the class of graphs to the class of hypergraphs. If it is a vertex colouring problem, then there is additional motivation. Indeed, a lot of scientists consider different concepts of vertex colouring of graphs (for example: list-colouring, sum-colouring, equitable-colouring), starting, in each case, from proper colouring, and next, analyzing some improper variants, in which a graph induced by vertices of a colour class is not necessarily edgeless. If we assume that each colour class has to induce a graph with some property (for example: acyclic, with a bounded degree, and so on) and this property is closed with respect to induced subgraphs, then, in each of these concepts, the problem of improper colouring of a graph is equivalent to the problem of proper colouring of a unique

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hypergraph constructed for this graph. Clearly, the construction of such a hypergraph can be difficult, but this approach gives the possibility to solve the problem. Moreover, in this case, each of the results obtained for hypergraphs can be applied to different variants of the same concept of graph colouring. Consequently, it can produce many special results.

The concept of sum-list-colouring of graphs is motivated by real problems and was first introduced in [6, 8]. Erdős, Rubin and Taylor [6] considered the so called size functions whose values for vertices of a graph represented the sizes of the lists assigned to them. Isaak [8] was the first to analyze the minimum sum of the list sizes that guarantees the existence of any particular proper vertex list colouring if lists are of these sizes. Such an invariant was determined in [11], with the help of Hall's Theorem, for complete graphs, and then, for a few other classes of graphs [9, 12]. In [9] the upper bound on the minimum sum of the list sizes was determined. Graphs that meet this bound, known as *sc-greedy*, led themselves to a very popular line of investigation in the literature [2, 3, 7, 9, 12]. In [5], the authors analyzed sum-list-colouring concept assuming that colour classes need not be edgeless. This investigation shows some differences between proper and improper cases and uses hypergraph theory tools. We continue this consideration herein, focusing on hypergraphs, believing that the following results will be used for various variants of the colouring concept. We extend the sc-greedy notion from graphs to hypergraphs and characterize all θ-hypergraphs that are sc-greedy (Theorem 4.11). This yields the characterization of all θ-hypergraphs that are forbidden for the class of sc-greedy hypergraphs (Corollary 4.12).

2 Preliminaries

In general, we follow the notation and terminology of [1, 4]. A hypergraph $\mathcal{H}$ consists of a non-empty finite set $V(\mathcal{H})$ of *vertices* and a finite set $\mathcal{E}(\mathcal{H})$ of at least 2-element subsets of $V(\mathcal{H})$, called *edges*. A hypergraph is *simple* if none of its edges is a subset of another edge. A hypergraph is *linear* if any two of its edges have no more than one vertex in common.

Let $\mathcal{H}$ be a hypergraph. A hypergraph $\mathcal{H}'$ is a *subhypergraph* of $\mathcal{H}$ if $V(\mathcal{H}') \subseteq V(\mathcal{H})$ and $\mathcal{E}(\mathcal{H}') \subseteq \mathcal{E}(\mathcal{H})$. For $V' \subseteq V(\mathcal{H})$, a *subhypergraph of a hypergraph $\mathcal{H}$ induced by V'*, denoted by $\mathcal{H}[V']$, has the vertex set V' and the edge set $\{E \in \mathcal{E}(\mathcal{H}) : E \subseteq V'\}$. We use $\mathcal{H} - V'$ notation instead of $\mathcal{H}[V(\mathcal{H}) \setminus V']$ and even $\mathcal{H} - v$ instead of $\mathcal{H} - \{v\}$.

Let $\mathcal{H}$ be a hypergraph, $v \in V(\mathcal{H})$ and $\mathcal{E}(v) = \{E \in \mathcal{E}(\mathcal{H}) : v \in E\}$. By $\mathcal{H}(v)$ we denote a hypergraph with the vertex set $\cup_{E\in\mathcal{E}(v)} E$ and with the edge set $\mathcal{E}(v)$. The *degree* of v in $\mathcal{H}$, denoted by $\deg_{\mathcal{H}}(v)$, is defined as the number of edges of $\mathcal{H}(v)$. The *β-degree* of v in $\mathcal{H}$, denoted by $\deg^{\beta}_{\mathcal{H}}(v)$, is the largest number of edges of a linear subhypergraph of $\mathcal{H}(v)$. The $\mathcal{H}_1 \cup \mathcal{H}_2$ symbol denotes the *union* of disjoint hypergraphs $\mathcal{H}_1, \mathcal{H}_2$. By the *identification* of two non-adjacent vertices v_1 and v_2 (in a hypergraph $\mathcal{H}$ into a vertex w) we mean the result of the following operations on $\mathcal{H}$: the removal of vertices v_1, v_2, the addition of a new vertex w, the replacement of each edge containing either v_1 or v_2 by an edge in which w substitutes v_1, v_2, respectively, and the removal of multiple edges if the current hypergraph has such edges. Note that v_1, v_2 can be vertices of different components, say $\mathcal{H}_1, \mathcal{H}_2$, of $\mathcal{H}$. In this case, sometimes, instead of the identification of vertices in $\mathcal{H}_1 \cup \mathcal{H}_2$ we may talk about the identification of vertices of two disjoint hypergraphs $\mathcal{H}_1, \mathcal{H}_2$.

The 1-vertex hypergraph is a *hypertree* without edges. Next, a hypergraph that has one edge consisting of all its vertices is a *hypertree* with one edge. A *hypertree* with m edges

($m \geq 2$) can be constructed from a hypertree $\mathcal{H}_1$ with m_1 edges and a hypertree $\mathcal{H}_2$ with $m - m_1$ edges, $0 < m_1 < m$, by the identification of an arbitrary vertex of $\mathcal{H}_1$ and an arbitrary vertex of $\mathcal{H}_2$. Note that each hypertree is linear.

A hypertree $\mathcal{H}$ is a *hyperpath* if there is an ordering (called *canonical*) of $V(\mathcal{H})$ such that each edge of $\mathcal{H}$ consists of some consecutive vertices (with respect to this ordering). The length of a hyperpath is the number of its edges. By a *hypercycle* we mean a hypergraph obtained from a hyperpath of length of at least three by the identification of the vertex with the first index and the vertex with the last index in an arbitrary canonical ordering of the vertex set of this hyperpath. The length of a hypercycle is the same as the length of a hyperpath that was used in the construction. Moreover, if $v_1, \ldots, v_n$ is a canonical ordering of the vertex set of the hyperpath, then $v_1, \ldots, v_{n-1}$ is a *canonical* ordering of the vertex set of the resulting hypercycle. Let $k \in \mathbb{N}$. By a *k-edge, k^+-edge* (*of a hypergraph $\mathcal{H}$*) we mean an edge of $\mathcal{H}$ consisting of k, at least k vertices, respectively. A hypergraph is *k-uniform* if each of its edges is a k-edge. Thus 2-uniform hypergraphs are *graphs* and especially, 2-uniform hypertrees, hyperpaths, hypercycles are *trees*, *paths*, *cycles*, respectively.

3 Sum-choice-number of hypergraphs

Let $\mathcal{H}$ be a hypergraph. A *proper colouring of $\mathcal{H}$* is a mapping $\phi\colon V(\mathcal{H}) \to \mathbb{N}$ such that for every edge E of $\mathcal{H}$ there are at least two different vertices v_1, v_2 in E such that $\phi(v_1) \neq \phi(v_2)$. Given a mapping $L\colon V(\mathcal{H}) \to 2^{\mathbb{N}}$ we call a mapping $\phi\colon V(\mathcal{H}) \to \mathbb{N}$ *an L-colouring of $\mathcal{H}$* if for every vertex $v \in V(\mathcal{H})$ it holds that $\phi(v) \in L(v)$. Let f be a function from $V(\mathcal{H})$ to the set of positive integers, a mapping $L\colon V(\mathcal{H}) \to 2^{\mathbb{N}}$ such that $|L(v)| = f(v)$ for every vertex v in $V(\mathcal{H})$ is called an *f-assignment* for $\mathcal{H}$. The hypergraph $\mathcal{H}$ is *f-choosable* if for each f-assignment L for $\mathcal{H}$ there is a proper L-coloring of $\mathcal{H}$. Thus, $\mathcal{H}$ is f-choosable if $\mathcal{H}$ is properly L-colourable for each f-assignment L for $\mathcal{H}$. The *sum-choice-number* $\chi_{sc}(\mathcal{H})$ of $\mathcal{H}$ is defined as the minimum of $\sum_{v\in V(\mathcal{H})} f(v)$ taken over all f such that $\mathcal{H}$ is f-choosable. Hence

$$\chi_{sc}(\mathcal{H}) = \min_f \left\{ \sum_{v\in V(\mathcal{H})} f(v) : \mathcal{H} \text{ is } f\text{-choosable} \right\}.$$

If $\mathcal{H}$ is f-choosable and $\chi_{sc}(\mathcal{H}) = \sum_{v\in V(\mathcal{H})} f(v)$, then we say that f *realizes* $\chi_{sc}(\mathcal{H})$.

The definition of the sum-choice-number of a hypergraph implies some immediate observations.

Fact 3.1. *If a hypergraph $\mathcal{H}_1$ is f-choosable for some function f with a domain $V(\mathcal{H}_1)$, then each subhypergraph $\mathcal{H}_2$ of $\mathcal{H}_1$ is $f|_{V(\mathcal{H}_2)}$-choosable.*

Fact 3.2. *If $\mathcal{H}_1, \mathcal{H}_2$ are vertex disjoint hypergraphs, then*

$$\chi_{sc}(\mathcal{H}_1 \cup \mathcal{H}_2) = \chi_{sc}(\mathcal{H}_1) + \chi_{sc}(\mathcal{H}_2).$$

Fact 3.3. *If $\mathcal{H}_2$ is a subhypergraph of a hypergraph $\mathcal{H}_1$, then*

$$|V(\mathcal{H}_1)| - |V(\mathcal{H}_2)| + \chi_{sc}(\mathcal{H}_2) \leq \chi_{sc}(\mathcal{H}_1).$$

Applying the reasoning provided in the proof of Theorem 8 from [5] we obtain the following theorem.

Theorem 3.4. *If $\mathcal{H}$ is a hypergraph and $v_1, \ldots, v_n$ is an arbitrary ordering of $V(\mathcal{H})$, then*

$$\chi_{sc}(\mathcal{H}) \leq \sum_{i=1}^{n} \deg^{\beta}_{\mathcal{H}_i}(v_i) + n,$$

where $\mathcal{H}_i = \mathcal{H}[\{v_1, \ldots, v_i\}]$.

Proof. Given the ordering $v_1, \ldots, v_n$, let $f(v_i) = \deg^{\beta}_{\mathcal{H}_i}(v_i) + 1$. To finish the proof we will show that $\mathcal{H}$ is f-choosable. Let L be any f-assignment for $\mathcal{H}$. We colour the vertices of $\mathcal{H}$ greedily, in accordance with the ordering $v_1, \ldots, v_n$. Namely, in the i^{th} step we assign to v_i the least colour from $L(v_i)$ such that for each $a \in \mathbb{N}$ the hypergraph induced by the vertices coloured with a in the hypergraph $\mathcal{H}_i$ is edgeless. Note that such a colouring exists for each of i^{th} steps, $i \in \{1, \ldots, n\}$, since, there are at most $\deg^{\beta}_{\mathcal{H}_i}(v_i)$ colours in $L(v_i)$ for which $\mathcal{H}_i$ has an edge that would be monochromatic if we assigne this colour to v_i. □

Using the reasoning presented in the final part of the above proof we have the following property.

Lemma 3.5. *If $\mathcal{H}$ is a hypergraph and f is a function that realizes $\chi_{sc}(\mathcal{H})$, then $f(v) \leq \deg^{\beta}_{\mathcal{H}}(v) + 1$ for each $v \in V(\mathcal{H})$.*

Proof. Suppose, for contradiction purposes, that f satisfies the assumptions of the lemma and there is a vertex $u \in \mathcal{H}$ such that $f(u) \geq \deg_{\mathcal{H}}(u) + 2$. We will show that $\mathcal{H}$ is f'-choosable for f' defined by $f'(v) = f(v)$ for $v \in V(\mathcal{H}) \setminus \{u\}$ and $f'(u) = f(u) - 1$, giving a contradiction with the assumptions about f. Let $\mathcal{H}' = \mathcal{H} - u$ and let f'-assignment L' for $\mathcal{H}$ be given. Since $f'|_{V(\mathcal{H}')} = f|_{V(\mathcal{H}')}$ we know that there is a proper $L'|_{V(\mathcal{H}')}$-colouring ϕ' of $\mathcal{H}'$, by Fact 3.1. Clearly ϕ' can be extended to a proper L'-colouring of $\mathcal{H}$ since $f'(u) \geq \deg^{\beta}_{\mathcal{H}}(u) + 1$. □

Observe that the bound given in Theorem 3.4 mostly depends on the ordering of vertices. For example, consider a hypergraph $\mathcal{H}$ such that $V(\mathcal{H}) = \{v_1, \ldots, v_5\}$ and $\mathcal{E}(\mathcal{H}) = \{E_1, E_2, E_3\}$, where $E_1 = \{v_1, v_2, v_3, v_4\}$, $E_2 = \{v_1, v_2, v_3, v_5\}$ and $E_3 = \{v_1, v_2, v_4, v_5\}$. Let $\pi_1 \colon v_1, v_2, v_3, v_4, v_5$ and $\pi_2 \colon v_3, v_4, v_5, v_1, v_2$ be two different orderings of $V(\mathcal{H})$. Thus Theorem 3.4 gives the upper bound of 7 on $\chi_{sc}(\mathcal{H})$ when we use π_1 and of 6 when we use π_2. On the other hand $\deg^{\beta}_{\mathcal{H}}(v) \leq \deg_{\mathcal{H}}(v)$ for every vertex v of a hypergraph $\mathcal{H}$. Moreover, for any ordering $v_1, \ldots, v_n$ of vertices of an n-vertex hypergraph $\mathcal{H}$ we have $\sum_{i=1}^{n} \deg_{\mathcal{H}_i}(v_i) = |\mathcal{E}(\mathcal{H})|$, where $\mathcal{H}_i = \mathcal{H}[\{v_1, \ldots, v_i\}]$. Hence Theorem 3.4 implies the following fact.

Fact 3.6. *If $\mathcal{H}$ is a hypergraph, then*

$$\chi_{sc}(\mathcal{H}) \leq |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|.$$

A hypergraph $\mathcal{H}$ is called *sc-greedy* if $\chi_{sc}(\mathcal{H}) = |V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$. In brief, in the following, we denote the number $|V(\mathcal{H})| + |\mathcal{E}(\mathcal{H})|$ by $GB(\mathcal{H})$. The notion of sc-greediness was previously introduced for graphs in [2]. Observe that if $\mathcal{H}$ is an sc-greedy hypergraph, then for every ordering $v_1, \ldots, v_n$ of $V(\mathcal{H})$ it holds that $\deg^{\beta}_{\mathcal{H}_i}(v_i) = \deg_{\mathcal{H}_i}(v_i)$ for each permissible i and $\mathcal{H}_i = \mathcal{H}[\{v_1, \ldots, v_i\}]$. Now suppose that a hypergraph has at least two edges E_1, E_2 that have at least two vertices, in common, say $\{v_1, v_2\} \subseteq E_1 \cap E_2$. We construct an ordering of vertices of $\mathcal{H}$ putting first the vertices from $E_1 \setminus \{v_1, v_2\}$, next the vertices from $E_2 \setminus \{v_1, v_2\}$, next v_1, v_2, and finally the remaining vertices. Clearly, $\deg^{\beta}_{\mathcal{H}_j}(v_2) < \deg_{\mathcal{H}_j}(v_2)$, where $j = |E_1 \cup E_2|$. Hence we conclude the following fact.

Fact 3.7. *Each sc-greedy hypergraph is linear.*

The literature on sc-greediness of graphs is very rich. We try to comment this property in the class of hypergraphs, especially those hypergraphs that are not graphs. Let Γ_{sc} denote the family of all sc-greedy hypergraphs. First note that Γ_{sc} is not closed while taking subhypergraphs. Indeed, the $K_{2,3}$ graph is not sc-greedy, but a graph resulting from $K_{2,3}$ by the addition of an edge, which joins two vertices of degree 3, is sc-greedy [12]. On the other hand, Γ_{sc} is closed while taking induced subhypergraphs (it is a well-known fact for sc-greedy graphs). To see it, suppose that there is an sc-greedy hypergraph $\mathcal{H}$ having an induced subhypergraph $\mathcal{H}'$, which is not sc-greedy. We construct a function f such that $\mathcal{H}$ is f-choosable and $\sum_{v \in V(\mathcal{H})} f(v) \leq GB(\mathcal{H}) - 1$ based on the function f' that realizes $\chi_{sc}(\mathcal{H}')$. Actually, $f|_{V(\mathcal{H}')} = f'$ and for $i \in \{1, \ldots, p\}$ we put $f(v_i) = \deg^{\beta}_{\mathcal{H}_i}(v_i) + 1$, where $v_1, \ldots, v_p$ is an arbitrary ordering of $V(\mathcal{H}) \setminus V(\mathcal{H}')$ and $\mathcal{H}_i = \mathcal{H}[V(\mathcal{H}') \cup \{v_1, \ldots, v_i\}]$. It implies that $\mathcal{H}$ is not sc-greedy, contradicting our assumption. Thus Γ_{sc} is an induced hereditary class and there is a family $\mathcal{C}(\Gamma_{sc})$ of hypergraphs, each of which is not sc-greedy and whose each proper induced subhypergraph is sc-greedy. The elements of $\mathcal{C}(\Gamma_{sc})$ are called *forbidden hypergraphs for* Γ_{sc} and they uniquely determine Γ_{sc}. Note that Γ_{sc} contains only linear hypergraphs. The class $\mathcal{C}(\Gamma_{sc})$ does not have this property. For example, each non-linear hypergraph $\mathcal{H}$ defined by $V(\mathcal{H}) = E_1 \cup E_2$, $\mathcal{E}(\mathcal{H}) = \{E_1, E_2\}$, where $|E_1 \cap E_2| \geq 2$, $E_1 \setminus E_2 \neq \emptyset$, $E_2 \setminus E_1 \neq \emptyset$, is an element of $\mathcal{C}(\Gamma_{sc})$.

In the next part of the paper we focus our attention on linear hypergraphs in $\mathcal{C}(\Gamma_{sc})$.

Lemma 3.8. *Let $\mathcal{H}$ be a linear hypergraph in $\mathcal{C}(\Gamma_{sc})$ and $v \in V(\mathcal{H})$. If f is a function that realizes $\chi_{sc}(\mathcal{H})$, then*

i) $f(v) \leq \deg_{\mathcal{H}}(v)$, *and*

ii) $\deg_{\mathcal{H}}(v) \geq 2$ *implies* $f(v) \geq 2$ *provided that each edge in $\mathcal{E}(\mathcal{H}(v))$ contains in $\mathcal{H}$ at most two vertices of degree greater than one.*

Proof. To show i) suppose that there is at least one vertex u in $V(\mathcal{H})$ such that $f(u) \geq \deg_{\mathcal{H}}(u) + 1$. Lemma 3.5 says that $f(u) = \deg_{\mathcal{H}}(u) + 1$. Now we define $\mathcal{H}' = \mathcal{H} - u$. Clearly $\mathcal{H}'$ is a proper induced subhypergraph of $\mathcal{H}$ and consequently is sc-greedy, by the definition of $\mathcal{C}(\Gamma_{sc})$. From the construction we know that $|V(\mathcal{H}')| = |V(\mathcal{H})| - 1$ and $|\mathcal{E}(\mathcal{H}')| = |\mathcal{E}(\mathcal{H})| - \deg_{\mathcal{H}}(u)$. Thus $\chi_{sc}(\mathcal{H}') = GB(\mathcal{H}') = GB(\mathcal{H}) - (\deg_{\mathcal{H}}(u) + 1)$. As a subhypergraph of $\mathcal{H}$, the hypergraph $\mathcal{H}'$ is $f|_{V(\mathcal{H}')}$-choosable, by Fact 3.1. It follows that $\chi_{sc}(\mathcal{H}') \leq \sum_{v \in V(\mathcal{H}')} f(v) = \sum_{v \in V(\mathcal{H})} f(v) - (\deg_{\mathcal{H}}(u) + 1) \leq GB(\mathcal{H}) - 1 - (\deg_{\mathcal{H}}(u)+1)$. Thus $GB(\mathcal{H})-1-(\deg_{\mathcal{H}}(u)+1) \geq \chi_{sc}(\mathcal{H}') = GB(\mathcal{H})-(\deg_{\mathcal{H}}(u)+1)$, i.e. a contradiction.

To show ii) suppose that $\mathcal{H}$ and $u \in V(\mathcal{H})$ satisfy the assumptions and $f(u) = 1$. If there is an edge $E \in \mathcal{E}(\mathcal{H}(u))$ that contains only one vertex of degree greater than one (only u), then for each vertex in E the value of f is equal to one, and consequently $\mathcal{H}$ is not f-choosable, a contradiction. Let $\{E_1, \ldots, E_k\} = \mathcal{E}(\mathcal{H}(u))$. Thus for each $i \in \{1, \ldots, k\}$ there is exactly one vertex u_i different from u such that $u_i \in E_i$ and $\deg_{\mathcal{H}}(u_i) \geq 2$. Let $\mathcal{H}' = \mathcal{H}[(V(\mathcal{H}) \setminus \cup_{i=1}^{k} E_i) \cup \{u_1, \ldots, u_k\}]$. Note that $|\mathcal{E}(\mathcal{H}')| = |\mathcal{E}(\mathcal{H})| - k$ and $|V(\mathcal{H}')| =$

$|V(\mathcal{H})| - t$ for some $t \in \mathbb{N}$. We define $f' \colon V(\mathcal{H}') \to \mathbb{N}$ such that $f'(v) = f(v)$ if $v \notin \{u_1, \ldots, u_k\}$ and $f'(u_i) = f(u_i) - 1$ for $i \in \{1, \ldots, k\}$. Note that $\sum_{v \in V(\mathcal{H}')} f'(v) \leq \sum_{v \in V(\mathcal{H})} f(v) - k - t$. Since $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$, it holds that $\sum_{v \in V(\mathcal{H})} f(v) \leq GB(\mathcal{H}) - 1$, and consequently $\sum_{v \in V(\mathcal{H}')} f'(v) \leq GB(\mathcal{H}) - 1 - k - t = GB(\mathcal{H}') - 1$. Thus $\mathcal{H}'$ is not f'-choosable. Let L' be an f'-assignment for $\mathcal{H}'$ such that there is no proper L'-colouring of $\mathcal{H}'$ and let $a \notin \bigcup_{v \in V(\mathcal{H}')} L'(v)$. We define an f-assignment L for $\mathcal{H}$ in the following way: $L(v) = L'(v)$ for $v \in V(\mathcal{H}') \setminus \{u_1, \ldots, u_k\}$, next $L(u_i) = L'(u_i) \cup \{a\}$ for $i \in \{1, \ldots, k\}$ and $L(v) = \{a\}$ for $v \in V(\mathcal{H}) \setminus V(\mathcal{H}')$. It is very easy to see that there is no proper L-colouring of $\mathcal{H}$, which means that $\mathcal{H}$ is not f-choosable contradicting the assumption. Hence $f(u) \geq 2$ in this case. □

Let us continue the investigation concerning sc-greedy hypergraphs. We start with the observation that Theorem 1 in [2] (referring to graphs) can be extended to hypergraphs. In fact, the same proof, in which the words graphs are substituted by hypergraphs, works to obtain the following statement.

Theorem 3.9. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two disjoint hypergraphs and $v_1 \in V(\mathcal{H}_1)$, $v_2 \in V(\mathcal{H}_2)$. If $\mathcal{H}$ is the hypergraph obtained by the identification of v_1 and v_2 in $\mathcal{H}_1 \cup \mathcal{H}_2$, then*

$$\chi_{sc}(\mathcal{H}) = \chi_{sc}(\mathcal{H}_1) + \chi_{sc}(\mathcal{H}_2) - 1.$$

Note that each hypertree with one edge is sc-greedy. The recursion used in the definition of a hypertree with an arbitrary number of edges and Theorem 3.9 give us the following consequence.

Corollary 3.10. *Each hypertree is sc-greedy.*

Using Lemma 3.8 concerning hypergraphs in $\mathcal{C}(\Gamma_{sc})$, we have the next result.

Theorem 3.11. *Each hypercycle is sc-greedy.*

Proof. Clearly, if there is a hypercycle $\mathcal{H}$ that is not sc-greedy, then $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$. It follows by Corollary 3.10 and Fact 3.2, since each component of every proper induced subhypergraph of $\mathcal{H}$ is a hypertree (actually it is a hyperpath). Suppose that f realizes $\chi_{sc}(\mathcal{H})$. By Lemma 3.8 i), ii) we have $f(v) \geq \deg_{\mathcal{H}}(v)$ for each vertex of $\mathcal{H}$, and consequently $\sum_{v \in \mathcal{H}} f(v) \geq GB(\mathcal{H})$, a contradiction. □

Let $\mathcal{F}$ be the class of recursively defined hypergraphs such that: all hypercycles and all hypertrees are in $\mathcal{F}$, and, giving any two disjoint hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2$ in $\mathcal{F}$ and vertices $v_1 \in V(\mathcal{H}_1)$, $v_2 \in V(\mathcal{H}_2)$, a hypergraph obtained by the identification of v_1 and v_2 in $\mathcal{H}_1 \cup \mathcal{H}_2$ is also in $\mathcal{F}$.

The following result is a consequence of Corollary 3.10 and Theorems 3.11, 3.9.

Corollary 3.12. *If $\mathcal{H} \in \mathcal{F}$, then $\mathcal{H}$ is sc-greedy.*

4 θ-hypergraphs

Let $k_1, k_2, k_3 \in \mathbb{N}$. By $\theta^h_{k_1,k_2,k_3}$ we denote the hypergraph consisting of two vertices of degree 3 connected by three internally disjoint hyperpaths of lengths k_1, k_2, k_3. In what follows, we sometimes use the notion of a hyperpath of $\theta^h_{k_1,k_2,k_3}$ of length k_i, $i \in \{1, 2, 3\}$, meaning the hyperpath of length k_i, used in the definition of $\theta^h_{k_1,k_2,k_3}$. By a θ-hypergraph we mean an arbitrary hypergraph $\theta^h_{k_1,k_2,k_3}$. Observe that if at least two of the numbers k_1, k_2, k_3 are equal to one, then $\theta^h_{k_1,k_2,k_3}$ is not linear. Additionally, if one of the hyperpaths of length one is created by a 2-edge, $\theta^h_{k_1,k_2,k_3}$ is even not simple. When $\theta^h_{k_1,k_2,k_3}$ is a graph, we can denote it by θ_{k_1,k_2,k_3} since such notation is present in the literature. In [7] Heinold found the values $\chi_{sc}(\theta_{k_1,k_2,k_3})$ for all simple graphs θ_{k_1,k_2,k_3}. We recall here this result.

Theorem 4.1 ([7]). *Let $k_1, k_2, k_3 \in \mathbb{N}$ and at most one of k_1, k_2, k_3 is equal to one. A graph θ_{k_1,k_2,k_3} is not sc-greedy if and only if $k_1 = k_2 = 2$ and k_3 is even. Moreover, if θ_{k_1,k_2,k_3} is not sc-greedy, then*

$$\chi_{sc}(\theta_{k_1,k_2,k_3}) = GB(\theta_{k_1,k_2,k_3}) - 1.$$

Theorem 4.1 shows that the sum-choice-number of each simple graph θ_{k_1,k_2,k_3}, is always less by one or equal to the sum of the numbers of vertices and edges of this graph. Fortunately, θ-hypergraphs have the same property.

Lemma 4.2. *If $k_1, k_2, k_3 \in \mathbb{N}$ and at most two of the numbers k_1, k_2, k_3 are equal to one, then $\chi_{sc}(\theta^h_{k_1,k_2,k_3}) \geq GB(\theta^h_{k_1,k_2,k_3}) - 1$.*

Proof. Let $\mathcal{H} = \theta^h_{k_1,k_2,k_3}$ and let E be an arbitrary edge of $\mathcal{H}$ that is one of the edges of the shortest hyperpath among three hyperpaths that compose $\mathcal{H}$. Next let $\mathcal{H}'$ be a subhypergraph of $\mathcal{H}$ such that $V(\mathcal{H}') = V(\mathcal{H})$ and $\mathcal{E}(\mathcal{H}') = \mathcal{E}(\mathcal{H}) \setminus \{E\}$. Note that $\mathcal{H}'$ is sc-greedy, by Corollary 3.12 and Fact 3.2. It means $\chi_{sc}(\mathcal{H}') \geq GB(\theta^h_{k_1,k_2,k_3}) - 1$. Thus the statement follows from Fact 3.3. □

It is worth mentioning that $\chi_{sc}(\theta^h_{1,1,1}) = GB(\theta^h_{1,1,1}) - 2$. Indeed, let v be one of two vertices of degree 3 in $V(\theta^h_{1,1,1})$ and let $f\colon V(\theta^h_{1,1,1}) \to \mathbb{N}$ be defined by: $f(v) = 2$ and $f(u) = 1$ for every $u \in V(\theta^h_{1,1,1}) \setminus \{v\}$. Clearly, $\theta^h_{1,1,1}$ is f-choosable since, for every f-assignment L, each colouring, in which the colours of v and the other vertex of degree 3 are different, is a proper L-colouring of $\theta^h_{1,1,1}$. Thus, $\chi_{sc}(\theta^h_{1,1,1}) \leq GB(\theta^h_{1,1,1}) - 2 = |V(\theta^h_{1,1,1})| + 1$. Because $\theta^h_{1,1,1}$ has edges, we have $\chi_{sc}(\theta^h_{1,1,1}) \geq |V(\theta^h_{1,1,1})| + 1$.

Let $f\colon V(\mathcal{H}) \to \mathbb{N}$ and L be an f-assignment for $\mathcal{H}$. In what follows, if $L(v) = \{a_1, \ldots, a_{f(v)}\}$, then we always assume that elements $a_1, \ldots, a_{f(v)}$ are pairwise different. Thus, among others, in Lemma 4.3 the integers a, b, c are pairwise different. Furthermore, if $i_1, \ldots, i_p$ are consecutive integers and $i_1 > i_p$, then the set $\{i_1, \ldots, i_p\}$ is empty.

Lemma 4.3. *Let $k \in \mathbb{N}$ and $\mathcal{H}$ be a hypercycle of length $2k$. Next let $v_1, \ldots, v_n$ be an arbitrary canonical ordering of $V(\mathcal{H})$, where $\{v_{i_1} = v_1, \ldots, v_{i_{2k}}\}$ is the set of all vertices of degree two in $\mathcal{H}$ with $i_j < i_l$ for $j < l$.*

If $f\colon V(\mathcal{H}) \to \mathbb{N}$ is defined by $f(v) = \deg_{\mathcal{H}}(v)$ and L is an f-assignment for $\mathcal{H}$ such that

i) $L(v_{i_j}) = \{a, b\}$ *for* $j \in \{1, \ldots, 2k-2\}$, *and*

ii) $L(v_{i_{2k-1}}) = \{b, c\}$, *and*

iii) $L(v_{i_{2k}}) = \{a, c\}$, *and*

iv) $L(v_s) = \{c\}$ *for* $s \in \{i_{2k-1}+1, \ldots, i_{2k}-1\}$, *and*

v) $L(v_s) = \{a\}$ *for* $s \in \{i_{2k}+1, \dots, n\}$ *or for* $s \in \{i_r+1, \dots, i_{r+1}-1\}$ *with odd* r, $1 \leq r \leq 2k-3$, *and*

vi) $L(v_s) = \{b\}$ *for* $s \in \{i_r+1, \dots i_{r+1}-1\}$ *with even* r, $2 \leq r \leq 2k-2$,

then in each proper L*-colouring* ϕ *of* $\mathcal{H}$ *it holds that* $\phi(v_1) = b$.

Proof. Suppose that there is a proper L-colouring ϕ of $\mathcal{H}$ such that $\phi(v_1) = a$. Consequently, it must be $\phi(v_{i_{2p}}) = b$ for all $p \in \{1, \dots, k-1\}$, next $\phi(v_{i_{2k-1}}) = c$ and $\phi(v_{i_{2k}}) = a$. Hence the edge $\{v_{i_{2k}}, \dots, v_n, v_1\}$ is monochromatic in ϕ, a contradiction. □

To avoid repetitions, we skip the simple proof of the next lemma that can be done in the same manner as the proof of Lemma 4.3.

Lemma 4.4. *Let* $k \in \mathbb{N}$ *and* $\mathcal{H}$ *be a hypercycle of length* $2k+1$. *Next let* $v_1, \dots, v_n$ *be an arbitrary canonical ordering of* $V(\mathcal{H})$, *where* $\{v_{i_1} = v_1, \dots, v_{i_{2k+1}}\}$ *is the set of all vertices of degree two in* $\mathcal{H}$ *with* $i_j < i_k$ *for* $j < k$.

If $f\colon V(\mathcal{H}) \to \mathbb{N}$ *is defined by* $f(v) = \deg_{\mathcal{H}}(v)$ *and* L *is an* f*-assignment for* $\mathcal{H}$ *such that*

i) $L(v_{i_j}) = \{a, b\}$ *for* $j \in \{1, \dots, 2k+1\}$, *and*

ii) $L(v_s) = \{a\}$ *for* $s \in \{i_{2k+1}+1, \dots, n\}$ *or for* $s \in \{i_r+1, \dots, i_{r+1}-1\}$ *with odd* r, $1 \leq r \leq 2k-1$, *and*

iii) $L(v_s) = \{b\}$ *for* $s \in \{i_r+1, \dots i_{r+1}-1\}$ *and even* r, $2 \leq r \leq 2k$,

then in each proper L*-colouring* ϕ *of* $\mathcal{H}$ *it holds that* $\phi(v_1) = b$.

Lemma 4.5. *Let* $k_1, k_2, k_3 \in \mathbb{N}$. *If the hyperpath of length* k_2 *of* $\theta^h_{k_1,k_2,k_3}$ *has only* 2*-edges, and either*

i) $k_1 + k_2$ *and* $k_2 + k_3$ *are odd numbers and at least one of the inequalities* $k_1 \geq 2$, $k_3 \geq 2$ *holds, or*

ii) $k_1 + k_2$ *is an odd number and* $k_2 + k_3$ *is an even number and* $k_3 \geq 3$, *or*

iii) $k_1 + k_2$ *is an even number and* $k_2 + k_3$ *is an even number and* $k_1 \geq 3$ *and* $k_3 \geq 3$,

then $\theta^h_{k_1,k_2,k_3}$ *is sc-greedy.*

Proof. Let $\mathcal{H} = \theta^h_{k_1,k_2,k_3}$ and let $\mathcal{H}$ satisfies the assumptions of the lemma. Observe that at most one of integers k_1, k_2, k_3 is equal to one since otherwise, $\mathcal{H}$ does not satisfy the assumptions of the lemma, so $\mathcal{H}$ is linear.

Suppose, for a contradiction, that $\mathcal{H}$ is not sc-greedy. Since each component of each proper induced subhypergraph of $\mathcal{H}$ is in $\mathcal{F}$, we obtain $\mathcal{H} \in \mathcal{C}(\Gamma_{sc})$, by Corollary 3.12 and Fact 3.2. Let f be a function that realizes $\chi_{sc}(\mathcal{H})$. Lemma 3.8 i) implies that $f(v) = 1$ if $\deg_{\mathcal{H}}(v) = 1$ and Lemma 3.8 ii) implies that $f(v) \geq 2$ for each vertex v of degree greater than one. Thus f has values in $\{1, 2\}$ and is fixed, since $\sum_{v \in V(\mathcal{H})} f(v) = GB(\mathcal{H}) - 1$ (see Lemma 4.2).

Now we shall construct an f-assignment L for $\mathcal{H}$ such that $\mathcal{H}$ is not properly L-colourable. Assume that $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are three hyperpaths of lengths k_1, k_2, k_3, respectively, from which $\theta^h_{k_1,k_2,k_3}$ is composed. Next let C_1 be a hypercycle that is a subhypergraph of $\theta^h_{k_1,k_2,k_3}$ composed from vertices and edges of the hyperpaths $\mathcal{P}_1, \mathcal{P}_2$. Similarly, let C_2 be a hypercycle that is a subhypergraph of $\theta^h_{k_1,k_2,k_3}$ composed from vertices and edges of the hyperpaths $\mathcal{P}_2, \mathcal{P}_3$. Thus lengths of C_1, C_2 are $k_1 + k_2$ and $k_2 + k_3$, respectively. Now we define canonical orderings π_1 of C_1 and π_2 of C_2, both starting with the same fixed vertex of degree three in $\mathcal{H}$, say v_1, and both proceeding consecutively, first along the vertices of $\mathcal{P}_2$, and next, along the vertices of either $\mathcal{P}_1$ or $\mathcal{P}_3$, respectively.

Next we construct an f-assignment L_1 for C_1 and π_1 either in accordance with Lemma 4.3 or in accordance with Lemma 4.4, depending on the parity of the length of C_1 (the parity of $k_1 + k_2$). Similarly, we construct an f-assignment L_2 for C_2 and π_2 either in accordance with Lemma 4.3 or Lemma 4.4, but in this case we exchange the meaning of colours a, b. Namely, we substitute a by b and b by a in each value of L_2 (given by the corresponding lemma).

Observe that the assumptions on numbers k_1, k_2, k_3 and the fact that $\mathcal{P}_2$ has only 2-edges imply that $L_1 = L_2$ on vertices of $\mathcal{P}_2$. Define an f-assignment L for $\mathcal{H}$ such that

$$L(v) = \begin{cases} L_1(v), & \text{if } v \in V(C_1) \setminus V(\mathcal{P}_2), \\ L_2(v), & \text{if } v \in V(C_2) \setminus V(\mathcal{P}_2), \\ L_1(v) = L_2(v), & \text{if } v \in V(\mathcal{P}_2). \end{cases}$$

Suppose, for a contradiction, that ϕ is a proper L-colouring of $\mathcal{H}$. Fact 3.1 implies that $\phi|_{V(C_1)}$ is a proper L_1-colouring of C_1 and $\phi|_{V(C_2)}$ is a proper L_2-colouring of C_2. By Lemma 4.3 or Lemma 4.4 (depending on the parity of $k_1 + k_2$) we have $\phi(v_1) = b$ and by one of Lemmas 4.3, 4.4 (depending on the parity of $k_2 + k_3$) we have $\phi(v_1) = a$, a contradiction. □

For forthcoming Lemmas 4.6, 4.7, 4.8, 4.9, 4.10, we introduce the following notations. Let $\mathcal{P}$ be a hyperpath on at least two vertices and let $v_1, \ldots, v_n$ be a canonical ordering of $V(\mathcal{P})$. Next let $f^*\colon V(\mathcal{P}) \to \mathbb{N}$ be defined by $f^*(v) = \deg_{\mathcal{P}}(v)$ for $v \notin \{v_1, v_n\}$ and $f^*(v_1) = f^*(v_n) = 2$.

Giving an f^*-assignment L for $\mathcal{P}$ and $(\alpha, \gamma) \in L(v_1) \times L(v_n)$, we say that a pair (α, γ) is *extendable* (*for* $\mathcal{P}$) if there is a proper L-colouring ϕ of $\mathcal{P}$ such that $\phi(v_1) = \alpha$, $\phi(v_n) = \gamma$. The pair $(\alpha, \gamma) \in L(v_1) \times L(v_n)$ that is not extendable for $\mathcal{P}$ is called *forbidden* (*for* $\mathcal{P}$).

The next lemma is a generalization of the lemma that was proven in [3].

Lemma 4.6. *Let $\mathcal{P}$ be a hyperpath on at least two vertices, let $v_1, \ldots, v_n$ be a canonical ordering of $V(\mathcal{P})$ and let $\{v_{i_1}, \ldots, v_{i_k}\}$ be the set of all vertices of degree two in $\mathcal{P}$ with $i_p < i_s$ for $p < s$. If L is an f^*-assignment for $\mathcal{P}$ such that $L(v_1), L(v_{i_1}), \ldots, L(v_{i_k}), L(v_n)$ are not identical, then at most one pair in $L(v_1) \times L(v_n)$ is forbidden for $\mathcal{P}$.*

Proof. We will show that at most one pair in $L(v_1) \times L(v_n)$ is forbidden for the path P, where $V(P) = \{v_1 = v_{i_0}, v_{i_1}, \ldots, v_{i_k}, v_n = v_{i_{k+1}}\}$ and $E(P) = \{v_{i_j} v_{i_{j+1}}\colon j \in \{0, \ldots, k\}\}$. Note that P is a graph. Since every proper $L|_{V(P)}$-colouring of P can be extended to a proper L-colouring of $\mathcal{P}$, the lemma will follow. We prove this statement by the induction on the number of edges in P (the number $k + 1$). Observe that it is true for a path P with one edge. Suppose that $|E(P)| \geq 2$. Since the lists of $v_1, v_{i_1}, \ldots, v_{i_k}, v_n$ are not identical, the lists of $v_1, v_{i_1}, \ldots, v_{i_k}$ or the lists of $v_{i_1}, \ldots, v_{i_k}, v_n$ are not identical either. Say the lists of $v_{i_1}, \ldots, v_{i_k}, v_n$ are not identical. Let $L(v_1) = \{a, b\}$, $L(v_n) =$

$\{c, d\}$, $L(v_{i_1}) = \{\alpha, \beta\}$ and assume that $a \neq \beta$ and $b \neq \alpha$. By inductive assumptions, at least three pairs in $L(v_{i_1}) \times L(v_n)$ are extendable for $P - v_1$, say $(\alpha, c), (\alpha, d), (\beta, c)$. Thus pairs $(b, c), (b, d), (a, c)$ are extendable for P, and hence, they are extendable for $\mathcal{P}$. □

Lemma 4.7. *If $\mathcal{P}$ is a hyperpath with at least one 3^+-edge, $v_1, \ldots, v_n$ is a canonical ordering of $V(\mathcal{P})$ and L is an f^*-assignment for $\mathcal{P}$, then at most one pair in $L(v_1) \times L(v_n)$ is forbidden for $\mathcal{P}$.*

Proof. We prove the assertion by the induction on the number of edges in $\mathcal{E}(\mathcal{P})$. Observe that the lemma trivially holds when $\mathcal{P}$ has one edge. Suppose that $\mathcal{P}$ is a hyperpath with at least one 3^+-edge and $|\mathcal{E}(\mathcal{P})| \geq 2$. Let $\{v_{i_1}, \ldots, v_{i_k}\}$ be the set of all vertices of degree two in $\mathcal{P}$ with $i_p < i_s$ for $p < s$. If the lists of $v_1, v_{i_1}, \ldots, v_{i_k}, v_n$ are not identical, then the statement follows from Lemma 4.6. Thus we may assume that $L(v_1) = L(v_{i_1}) = \cdots = L(v_{i_k}) = L(v_n) = \{a, b\}$. Let $E_1 = \{v_1, v_2, \ldots, v_{i_1}\}$. Renaming vertices, if it is necessary, we may assume that $\mathcal{P}' = \mathcal{P} \setminus \{v_1, \ldots, v_{i_1 - 1}\}$ contains a 3^+-edge. Thus $\mathcal{P}'$ is a hyperpath satisfying inductive assumptions, and so, at least three pairs in $L(v_{i_1}) \times L(v_n)$ are extendable for $\mathcal{P}'$. Note that if we colour v_1 with a and v_{i_1} with b or if we colour v_1 with b and v_{i_1} with a, then we can extend such a colouring to a proper L-colouring of $\mathcal{P}$. Hence, in $L(v_1) \times L(v_n)$ there are three pairs that are extendable for $\mathcal{P}$. □

Lemmas 4.6, 4.7 immediately imply the following fact.

Lemma 4.8. *If $\mathcal{P}$ is a hyperpath on at least two vertices, $v_1, \ldots, v_n$ is a canonical ordering of $V(\mathcal{P})$ and L is an f^*-assignment for $\mathcal{P}$, then at most two pairs in $L(v_1) \times L(v_n)$ are forbidden for $\mathcal{P}$. Moreover,*

i) *exactly two pairs are forbidden for $\mathcal{P}$ if and only if $\mathcal{P}$ contains only 2-edges and $L(v_1) = \cdots = L(v_n)$, and*

ii) *if there are exactly two forbidden pairs for $\mathcal{P}$ and $\mathcal{P}$ is of even length and $L(v_1) = \{a, b\}$, then (a, a) and (b, b) are extendable for $\mathcal{P}$, and*

iii) *if there are exactly two forbidden pairs for $\mathcal{P}$ and $\mathcal{P}$ is of odd length and $L(v_1) = \{a, b\}$, then (a, b) and (b, a) are extendable for $\mathcal{P}$.*

Lemma 4.9. *If $\mathcal{P}$ is a hyperpath of length two, $v_1, \ldots, v_n$ is a canonical ordering of $V(\mathcal{P})$, L is an f^*-assignment for $\mathcal{P}$ and $L(v_1) = L(v_n) = \{a, b\}$, then (a, a) and (b, b) are extendable for $\mathcal{P}$.*

Based on Lemmas 4.6, 4.7, 4.8, 4.9 we have the following result.

Lemma 4.10. *Let $k_1, k_2, k_3 \in \mathbb{N}$. If $\theta^h_{k_1,k_2,k_3}$ is sc-greedy, then one of the hyperpaths of $\theta^h_{k_1,k_2,k_3}$, say the hyperpath of length k_2, has only 2-edges, and, under this assumption, one of the following conditions is satisfied:*

i) *$k_1 + k_2$ and $k_2 + k_3$ are odd numbers and at least one of the inequalities $k_1 \geq 2$, $k_3 \geq 2$ holds, or*

ii) *$k_1 + k_2$ is an odd number and $k_2 + k_3$ is an even number and $k_3 \geq 3$, or*

iii) *$k_1 + k_2$ is an even number and $k_2 + k_3$ is an even number and $k_1 \geq 3$ and $k_3 \geq 3$.*

Proof. Let $\mathcal{H} = \theta^h_{k_1,k_2,k_3}$. Suppose that for each possible permutation $k_{i_1}, k_{i_2}, k_{i_3}$ of numbers k_1, k_2, k_3 either the hyperpath of $\mathcal{H}$ of length k_{i_2} contains 3^+-edge or $\theta^h_{k_{i_1},k_{i_2},k_{i_3}}$ satisfies no of the conditions i), ii), iii) of the lemma. The aim is to prove that $\mathcal{H}$ is not sc-greedy. We may assume that at most one of the numbers k_1, k_2, k_3 is equal to one. Otherwise, $\mathcal{H}$ is not linear and the statement follows by Fact 3.7. We define f so that $f(v) = 1$ if $\deg_{\mathcal{H}}(v) = 1$ and $f(v) = 2$ if $\deg_{\mathcal{H}}(v) \geq 2$. Next, we will show that for each f-assignment L for $\mathcal{H}$ there is a proper L-colouring of $\mathcal{H}$. Since $\sum_{v \in V(\mathcal{H})} f(v) = GB(\mathcal{H}) - 1$, the theorem will follow. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be the hyperpaths of $\mathcal{H}$ of lengths k_1, k_2, k_3, respectively, and let v_1, v_n be vertices of degree 3 in $\mathcal{H}$. Let L be an arbitrary f-assignment for $\mathcal{H}$. First observe that if each hyperpath $\mathcal{P}_i$, $i \in \{1, 2, 3\}$, has at least one 3^+-edge, then for each hyperpath $\mathcal{P}_i$ at most one pair in $L(v_1) \times L(v_n)$ is forbidden, by Lemma 4.7. Since we have four possible pairs in $L(v_1) \times L(v_n)$, at least one pair can be extended to a proper L-colouring of $\mathcal{H}$. Thus at least one hyperpath contains only 2-edges, so it is the path. Moreover, the lists of vertices of this path must be identical, by Lemma 4.8 i). We may assume that each vertex of this path has the list $\{a, b\}$, and so, $L(v_1) = L(v_n) = \{a, b\}$. Let us consider two cases.

Case 1. The numbers k_1, k_2, k_3 are all of the same parity. Definitely, $\mathcal{H}$ does not fulfill neither the property i) nor ii). Since $\mathcal{H}$ does not have the property iii) either, at least one of integers k_1, k_2, k_3 is less than or equal to two. Assume that $k_1 \leq 2$.

Subcase 1.1. All of the numbers k_1, k_2, k_3 are even. Thus $k_1 = 2$. If $k_2 \geq 3$ and $k_3 \geq 3$, then $\mathcal{P}_1$ has to contain at least one 3^+-edge. Otherwise, $\mathcal{H}$ satisfies iii) for a permutation $k_{i_1}, k_{i_2}, k_{i_3}$, where $k_{i_2} = k_1$. By our initial assumption $\mathcal{P}_2$ or $\mathcal{P}_3$ has only 2-edges and pairs $(a, a), (b, b)$ are extendable for this path, by Lemma 4.8 ii). Without loss of generality assume that $\mathcal{P}_2$ contains only 2-edges and pairs $(a, a), (b, b)$ are extendable for $\mathcal{P}_2$. Note that pairs $(a, a), (b, b)$ are also extendable for $\mathcal{P}_1$, by Lemma 4.9. From Lemma 4.8, at most two pairs are forbidden for $\mathcal{P}_3$, and if exactly two pairs are forbidden for $\mathcal{P}_3$, then $(a, a), (b, b)$ are extendable for $\mathcal{P}_3$. Thus both pairs $(a, a), (b, b)$ can be extended to a proper L-colouring of $\mathcal{H}$. If at most one pair is forbidden for $\mathcal{P}_3$, then at least one of pairs $(a, a), (b, b)$ can be extended to a proper L-colouring of $\mathcal{H}$.

Now suppose that at least two hyperpaths have lengths two, say $k_1 = 2$ and $k_2 = 2$. From Lemma 4.9, (a, a) and (b, b) are extendable for both $\mathcal{P}_1$ and $\mathcal{P}_2$. From Lemma 4.8, at most two pairs are forbidden for $\mathcal{P}_3$. If exactly two pairs are forbidden for $\mathcal{P}_3$, then again by Lemma 4.8 ii), both pairs $(a, a), (b, b)$ are extendable for $\mathcal{P}_3$. Otherwise, at most one pair is forbidden for $\mathcal{P}_3$. Thus (a, a) or (b, b) is extendable for all $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$.

Subcase 1.2. All of the numbers k_1, k_2, k_3 are odd. Thus $k_1 = 1$. Since $\mathcal{H}$ is linear, we have $k_2 \geq 3$ and $k_3 \geq 3$. Furthermore, $\mathcal{P}_1$ is a 3^+-edge, otherwise, $\mathcal{H}$ satisfies iii). Again, without loss of generality, assume that $\mathcal{P}_2$ contains only 2-edges and pairs $(a, b), (b, a)$ are extendable for $\mathcal{P}_2$, by Lemma 4.8 iii). Thus $(a, b), (b, a)$ are extendable for both $\mathcal{P}_1$, $\mathcal{P}_2$. If there is at most one pair forbidden for $\mathcal{P}_3$, then one of pairs $(a, b), (b, a)$ is extendable for all $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$. Otherwise, two pairs are forbidden for $\mathcal{P}_3$, however, then both pairs $(a, b), (b, a)$ are extendable for $\mathcal{P}_3$, by Lemma 4.8 iii). So, both pairs $(a, b), (b, a)$ can be extended to a proper L-colouring of $\mathcal{H}$.

Case 2. The numbers k_1, k_2, k_3 are not of the same parity. In this case either one hyperpath in $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ is of odd length and two of them are of even length or one hyperpath in $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ is of even length and two are of odd length. So, $\mathcal{H}$ does not have the property iii). The hyperpath of odd length in the first case and the hyperpath of even length in the second case has a 3^+-edge, since otherwise, $\mathcal{H}$ has the property i). Let us consider the following two possible subcases.

Subcase 2.1. The number k_1 is odd and $\mathcal{P}_1$ has a 3^+-edge, k_2, k_3 are even. Since $\mathcal{H}$ does not have the property ii), $k_2 = 2$ or $k_3 = 2$. Say $k_3 = 2$. If $k_2 > 2$, then $\mathcal{P}_3$ has 3^+-edge, as otherwise, $\mathcal{H}$ has the property ii). Thus $\mathcal{P}_2$ must have only 2-edges and pairs $(a, a), (b, b)$ are extendable for $\mathcal{P}_2$, by Lemma 4.8 ii). From Lemma 4.9, pairs $(a, a), (b, b)$ are extendable for $\mathcal{P}_3$, so, $(a, a), (b, b)$ are extendable for both $\mathcal{P}_2$, $\mathcal{P}_3$. By Lemma 4.7, at most one of these pairs is forbidden for $\mathcal{P}_1$, so there is a proper L-colouring of $\mathcal{H}$. Suppose now that $k_2 = 2$, so we have $k_2 = k_3 = 2$. From Lemma 4.9 both pairs $(a, a), (b, b)$ are extendable for both $\mathcal{P}_2$, $\mathcal{P}_3$. Since at most one of these pairs is forbidden for $\mathcal{P}_1$, the statement follows.

Subcase 2.2. The number k_1 is even and $\mathcal{P}_1$ has a 3^+-edge, k_2, k_3 are odd. Since $\mathcal{H}$ does not have the property ii), $k_2 = 1$ or $k_3 = 1$. Say $k_3 = 1$. Furthermore, the previous consideration leads to $k_2 \geq 3$. If $\mathcal{P}_3$ has exactly 2-edge, then $\mathcal{H}$ has the property ii). Thus $\mathcal{P}_3$ must have a 3^+-edge, and so, $\mathcal{P}_2$ must have only 2-edges. Hence, pairs $(a, b), (b, a)$ are extendable for $\mathcal{P}_2$. Thus (a, b) and (b, a) are extendable for both $\mathcal{P}_2$, $\mathcal{P}_3$. Furthermore, at most one of pairs is forbidden for $\mathcal{P}_1$, and so, at least one of the pairs $(a, a), (b, b)$ is extendable for all $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, which creates a proper L-colouring of $\mathcal{H}$ and proves the lemma. □

Now, we are in a position to present the main result of the paper that immediately follows from Lemmas 4.5, 4.10. Next, the consequence of this result is formulated.

Theorem 4.11. *Let $k_1, k_2, k_3 \in \mathbb{N}$. A hypergraph $\theta^h_{k_1,k_2,k_3}$ is sc-greedy if and only if one of the hyperpaths of $\theta^h_{k_1,k_2,k_3}$, say the hyperpath of length k_2, has only 2-edges and, under this assumption, one of the following conditions holds:*

i) *$k_1 + k_2$ and $k_2 + k_3$ are odd numbers and at least one of the inequalities $k_1 \geq 2$, $k_3 \geq 2$ holds, or*

ii) *$k_1 + k_2$ is an odd number and $k_2 + k_3$ is an even number and $k_3 \geq 3$, or*

iii) *$k_1 + k_2$ is an even number and $k_2 + k_3$ is an even number and $k_1 \geq 3$ and $k_3 \geq 3$.*

Note that for arbitrary parameters k_1, k_2, k_3 such that $(k_1, k_2, k_3) \neq (1, 1, 1)$ and arbitrary vertex of $\theta^h_{k_1,k_2,k_3}$ each component of $\theta^h_{k_1,k_2,k_3} - v$ is in $\mathcal{F}$. Hence the following fact is valid.

Corollary 4.12. *If $k_1, k_2, k_3 \in \mathbb{N}$ and at most two of the numbers k_1, k_2, k_3 are equal to one, then $\theta^h_{k_1,k_2,k_3} \in \mathcal{C}(\Gamma_{sc})$ if and only if $\theta^h_{k_1,k_2,k_3} \notin \Gamma_{sc}$.*

5 Concluding remarks and open problems

A connected hypergraph is 2-*connected* if it cannot be a result of identification of a vertex of $\mathcal{H}_1$ and a vertex of $\mathcal{H}_2$, where $\mathcal{H}_1$, $\mathcal{H}_2$ are some disjoint hypergraphs, each on at least two vertices.

Note that, based on Fact 3.2 and Theorem 3.9, both, the union operation and the identification operation (applied to vertices of two disjoint hypergraphs) keep sc-greediness of hypergraphs. Hence, analyzing sc-greediness of hypergraphs it is enough to focus on 2-connected ones. Additionally, each sc-greedy hypergraph must be linear, by Fact 3.7. Thus the following question seems to be interesting.

Problem 5.1. How to characterize all hypergraphs in Γ_{sc} that are 2-connected and linear?

To support the discussion this question we start with some notes concerning graphs. The famous theorem that characterizes all linear (equivalently, simple) 2-connected graphs can be found in [4]. To cite it we need the following notion.

Let G be a graph on at least two vertices. By *adding* a G-path to G, we mean the result of two operations of identification applied to the graph G and an arbitrary path P with a canonical ordering $v_1, \ldots, v_n$ of $V(P)$, $n \geq 2$ (G and P are disjoint). More precisely, it is a result of identification of v_1 and x and also v_n and y, where x, y are two different vertices of G.

Lemma 5.2 ([4]). *A simple graph is* 2*-connected if and only if it can be constructed from a cycle by successively adding* G*-paths to graphs* G *already constructed.*

Observe that each cycle is 2-connected and sc-greedy. Next, adding a G-path to the cycle G we obtain a θ-graph. Theorem 4.1 characterizes all θ-graphs that are in Γ_{sc}. The question is, whether we should expect that an sc-greedy graph be obtained by adding a G-path to an sc-greedy θ-graph. In [3] the authors proved that $\chi_{sc}(G) + 2s \leq \chi_{sc}(G_1) \leq \chi_{sc}(G) + 2s + 1$ if G_1 is the result of adding a G-path on $s + 2$ vertices to an arbitrary simple graph G, assuming that $s \geq 1$. They did not consider the case when a G-path has 2 vertices, which seems to be very important. However, based on this observation, they gave the characterization of all graphs in Γ_{sc} that are generalized θ-graphs. For $r \geq 3$, a *generalized* θ*-graph* is a simple graph, denoted by $\theta_{k_1,k_2,\ldots,k_r}$, consisting of two vertices connected by r internally vertex disjoint paths of lengths $k_1, k_2, \ldots, k_r$. In [3] the authors showed that $\theta_{k_1,k_2,\ldots,k_r}$ is not sc-greedy if $r \geq 5$. Moreover, θ_{k_1,k_2,k_3,k_4} is not sc-greedy if and only if it contains an induced subgraph $\theta_{2,2,k_i}$ with even k_i and $i \in \{3, 4\}$ or, if all numbers k_1, k_2, k_3, k_4 have the same parity. It follows that starting with a θ-graph G and adding a G-path two times, in this special case (to obtain a generalized θ-graph), we always obtain a graph that is not sc-greedy. On the other hand, the graph presented in Figure 1 that needs to be added a G-path 3 times is still sc-greedy (see the graph $G_{10,12}$ in [10]). It leads to formulating the following problem.

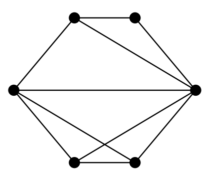

Figure 1: An sc-greedy graph that needs the application of adding a G-path 3 times.

Problem 5.3. Under which conditions can we obtain an sc-greedy graph by adding a G-path to an sc-greedy graph G?

In a way similar for graphs we can define the operation of adding an $\mathcal{H}$-path to a hypergraph $\mathcal{H}$ and pose the problem similar to Problem 5.3 in the class of hypergraphs.

Problem 5.4. Under which conditions can we obtain an sc-greedy hypergraph by adding an $\mathcal{H}$-path to an sc-greedy hypergraph $\mathcal{H}$?

Unfortunately, there are 2-connected linear hypergraphs that cannot be obtained from a hypercycle by successively adding $\mathcal{H}$-paths.

On the other hand, there is a relatively large subclass of the class of 2-connected linear hypergraphs, for which, the problem of belonging to Γ_{sc} can be solved, with help of consideration concerning graphs.

A hypergraph $\mathcal{G}$ is a *blown of a graph* G if $\mathcal{G}$ is a result of a substitution of each edge of G by a 3^+-edge of $\mathcal{G}$ that contains vertices of substituted edge of G (see for example Figure 2). Let $\mathcal{F}'$ be the class consisting of all hypergraphs that are all possible blowns of 2-connected graphs, except cycles. Clearly, every hypergraph in $\mathcal{F}'$ is 2-connected, by definition.

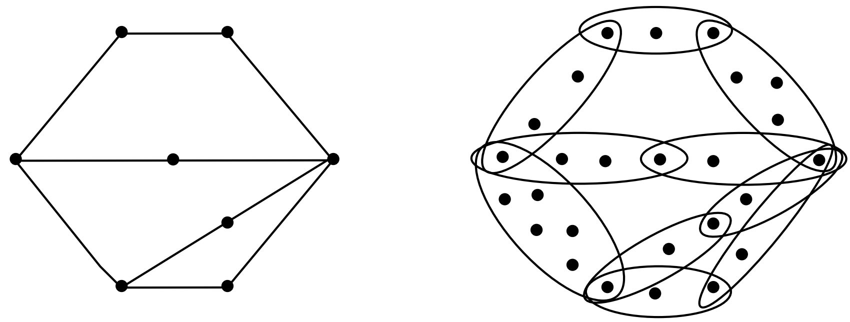

Figure 2: A graph and its blown.

Theorem 5.5. *No hypergraph in $\mathcal{F}'$ is sc-greedy.*

Proof. First observe that if $\mathcal{H} \in \mathcal{F}'$, then $\mathcal{H}$ is a blown of some 2-connected graph H that is not a cycle. From Lemma 5.2, H contains a subgraph that is a θ-graph. By the definition of $\mathcal{F}'$ we know that $\mathcal{H}$ contains an induced subhypergraph $\mathcal{H}'$ that is a θ-hypergraph, and moreover, $\mathcal{H}'$ has no hyperpath having only 2-edges. Thus $\mathcal{H}'$ is not sc-greedy, by Lemma 4.10. Finally, $\mathcal{H}$ is not sc-greedy, since Γ_{sc} is closed when taking induced subhypergraphs. □

It is worth mentioning that hypergraphs in $\mathcal{F}'$ can or cannot be blowns of sc-greedy graphs. It follows that probably, to be an sc-greedy hypergraph is a relatively rare feature in the class of hypergraphs with only 3^+-edges. It motivates the following separate subproblem of Problem 5.1.

Problem 5.6. How to characterize all hypergraphs in Γ_{sc} that are 2-connected, linear and have only 3^+-edges.

Finally, observe that each hypergraph in $\mathcal{C}(\Gamma_{sc})$ is 2-connected, but this class includes linear and non-linear hypergraphs. Thus, referring to the $\mathcal{C}(\Gamma_{sc})$ class, we obtain the following open question and its subquestion.

Problem 5.7. How to characterize all hypergraphs in $\mathcal{C}(\Gamma_{sc})$?

Problem 5.8. How to characterize all non-linear hypergraphs in $\mathcal{C}(\Gamma_{sc})$?

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