ISSN 2590-9770
The Art of Discrete and Applied Mathematics 8 (2025) #P3.05
https://doi.org/10.26493/2590-9770.1733.dc9
(Also available at http://adam-journal.eu)

On extremal (almost) edge-girth-regular graphs

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Received 30 November 2023, accepted 18 July 2024, published online 26 May 2025

Abstract

A k-regular graph of girth g is called an edge-girth-regular graph, or an egr-graph for short, if each of its edges is contained in exactly λ distinct g-cycles. An egr-graph is called extremal for the triple (k, g, λ) if it has the smallest possible order. We prove that some graphs arising from incidence graphs of finite planes are extremal egr-graphs. We also prove new lower bounds on the order of egr-graphs.

Keywords: Edge-girth-regular graph, cage problem, finite biaffine planes.

Math. Subj. Class.: 05C035, 51E20

*Research supported in part by UNAM-DGAPA-PAPIIT IN101821, Proyecto de Intercambio Académico de la Coordinación de la Investigación Científica "Bipartite biregular cages, block designs and generalized quadrangles" and CONAHCyT: CBF 2023-2024-552.

†Corresponding author. This research was supported in part by the Hungarian National Research, Development and Innovation Office OTKA grant no. K 124950.

‡This research was supported in part by the Hungarian National Research, Development and Innovation Office OTKA grant no. SNN 132625.

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1 Introduction

A frequently occurring type of problem in extremal graph theory is the following: we fix some graph parameters or some graph properties and want to deduce the extremal number of another parameter (in many cases, the number of vertices or edges). In this paper, we deal with similar questions, which are also motivated by the classical cage problem. A simple, finite, connected graph G is *k-regular* if each vertex has exactly k neighbors. It is of *girth* g if its smallest cycles have g vertices. A *g-cycle* or *girth cycle* is a cycle of length g. The number of vertices of G is called the *order* of G. A *(k, g)-graph* is a k-regular graph of girth g. If it has minimal order, then it is called a *(k, g)-cage*. For more information about cages we refer to the dynamic survey by Exoo and Jajcay [9].

For $k = 2$ the connected (k, g)-graphs are the g-cycles, so we assume that $k > 2$ throughout this paper. By counting the vertices whose distance from a given vertex (when g is odd) or edge (when g is even) is at most $\lfloor (g-1)/2 \rfloor$ results in the following bound.

Theorem 1.1 (Moore bound)**.** *Let G be a (k, g)-graph with $k > 2$. Then the order of G is at least $n_0(k, g)$, where*

$$n_0(k,g) = \begin{cases} \frac{k(k-1)^{\frac{g-1}{2}}-2}{k-2}, & \text{if } g \text{ is odd;} \\ \frac{2(k-1)^{\frac{g}{2}}-2}{k-2}, & \text{if } g \text{ is even.} \end{cases}$$

Graphs attaining this bound are called Moore cages. In [11] Jajcay, Kiss, and Miklavič recall a nice property of Moore cages: the number of g-cycles through each of its edges is a constant. This observation motivated the following definition.

Definition 1.2. A (k, g)-graph of order n with the property that each of its edges is contained in exactly λ distinct g-cycles is called an *edge-girth-regular graph*, or an *egr-graph* for short, and is denoted by $egr(n, k, g, \lambda)$.

For a given triple (k, g, λ) an $egr(n, k, g, \lambda)$ with minimal order is the analogue of cages among (k, g)-graphs.

Definition 1.3. An $egr(n, k, g, \lambda)$ is called *extremal for the triple* (k, g, λ) if for any $egr(v, k, g, \lambda)$ the inequality $v \geq n$ holds. We denote the order of this extremal graph by $n(k, g, \lambda)$. If G is an extremal bipartite $egr(n, k, g, \lambda)$, then we denote its order by $n_2(k, g, \lambda)$.

Drglin, Filipovski, Jajcay, and Raiman in [7] proved lower bounds on the order of edge-girth-regular graphs.

Theorem 1.4 (Drglin, Filipovski, Jajcay, Raiman)**.** *Let $k \geq 3$ and $g \geq 3$ be a fixed pair of integers, and let $\lambda \leq (k-1)^{\frac{g-1}{2}}$, if g is odd and $\lambda \leq (k-1)^{\frac{g}{2}}$ if g is even. Then*

$$n(k,g,\lambda) \geq n_0(k,g) + \begin{cases} (k-1)^{\frac{g-1}{2}} - \lambda, & \text{if } g \text{ is odd;} \\ \left\lceil 2\frac{(k-1)^{\frac{g}{2}}-\lambda}{k} \right\rceil, & \text{if } g \text{ is even.} \end{cases}$$

Moreover,

$$n_2(k,g,\lambda) \geq n_0(k,g) + 2\left\lceil \frac{(k-1)^{\frac{g}{2}}-\lambda}{k} \right\rceil.$$

For large values of λ, this is the best known general lower bound so far. If λ is relatively small and g is even, then Porupsánszki [15] improved the bounds using eigenvalue techniques.

Theorem 1.5 (Porupsánszki). *Let G be an $egr(n,k,g,\lambda)$ graph, where g is even. If $g \equiv 0$ (mod 4), then*

$$n(k,g,\lambda) \geq \frac{c(g,k)+k\lambda+k^g-2c(\frac{g}{2},k)k^{\frac{g}{2}}}{c(g,k)-c^2(\frac{g}{2},k)+k\lambda},$$

$$n_2(k,g,\lambda) \geq 2\frac{c(g,k)+k\lambda+k^g-2c(\frac{g}{2},k)k^{\frac{g}{2}}}{c(g,k)-c^2(\frac{g}{2},k)+k\lambda}.$$

If $g \equiv 2$ (mod 4), then

$$n(k,g,\lambda) \geq \frac{c(g,k)+k\lambda+k^g}{c(g,k)+k\lambda},$$

$$n_2(k,g,\lambda) \geq \frac{2k^g}{c(g,k)+k\lambda}.$$

Here $c(\ell,k)$ denotes the number of cycle-free closed walks of length ℓ starting at a fixed vertex. These formulas are $\frac{\ell}{2}$th degree polynomials of k.

Let us remark that Theorem 1.4 gives $O\left(k^{\left[\frac{g}{2}\right]-1}\right)$ lower bounds on the order of extremal egr-graphs. On the other hand, if $\lambda = O(k^{\frac{g}{2}-1})$, then the inequalities of Theorem 1.5 give $\Theta\left(k^{\frac{g}{2}}\right)$ lower bounds. Hence if λ is small, then these lower bounds on the order of extremal egr-graphs of even girth are better than the previously known ones.

The aim of the present paper is on the one hand, to introduce new families of egr-graphs and almost egr-graphs ((k,g)-graphs with the property, that each of its edges is contained in λ_1 or λ_2 distinct g-cycles, the exact definition is given at the beginning of Section 3) using cleverly chosen substructures of finite geometries, in particular t-good structures, on the other hand, we prove new lower bounds on the order of (almost) egr-graphs.

In Section 2, we consider graphs arising from incidence graphs of finite planes by the deletion of some specific t-good structures. We prove that these graphs are egr-graphs with small order. In particular, if $t=1$, then we have extremal egr-graphs, if $t=2$ or 3, then their order is only a constant higher than the bound given in Theorem 1.4. Finally, we show that the point-hyperplane incidence graph of $\mathrm{PG}(n,q)$, the n-dimensional finite projective space of order q, is an egr-graph of girth 4 for all $n \geq 3$. Moreover, it is extremal for $n=3$.

The concept of egr-graphs was generalized by Potočnik and Vidali [16], who introduced girth-regular graphs. In Section 3, we consider some girth-regular graphs which are almost egr-graphs. We construct graphs of girth 5 with geometric arguments. It is important to note that this technique of construction has been used previously by a large number of authors. In these papers the authors take the incidence graphs of elliptic semiplanes of type 1 or 2 and introduce two operations that in [4] the authors called reduction and amalgam. The first involves deleting a set of vertices in the incidence graph that correspond to collinear points and parallel lines in the affine plane, and the second consists in adding a set of edges with some special properties in the set of vertices on the incidence graph with the same characteristics.

In this paper our arguments are geometric, and we describe a construction for the Hoffman-Singleton graph and for a $(6, 5)$-graph, after which we generalize these constructions. Some of our graphs are graphs originally constructed by Brown [6] and later on by Abreu, Abajo and their co-authors [1, 2, 3, 4]. When q is a power of 2, we generalize a construction originally due to Araujo-Pardo and Leemans [5] who gave this construction for the case $q = 4$. The constructed egr-graphs are related to the cage problem. In some cases, their orders are equal to the order of the smallest known $(k, 5)$-graphs.

Finally, in Section 4, we give lower bounds on the order of girth-regular graphs. These bounds are generalizations of the lower bounds on the order of extremal egr-graphs proved in [7] and [15]. In particular, we present a new, combinatorially proved lower bound on the order of girth-regular graphs having even girth.

2 Edge-girth-regular graphs arising from finite geometries

In this section, we show that some subgraphs of the point-hyperplane incidence graph of a finite projective space are egr-graphs. These graphs are not new. Many of them (and references to their original constructions) can be found in a paper by Gács and Héger [10]; however, previously the authors did not examine the edge-girth-regularity of the graphs. Throughout this paper, if G is an incidence graph, then its vertices will be called points and hyperplanes (lines in the two-dimensional case) according to whether they correspond to a point or a hyperplane of the geometric structure. For a detailed introduction to the concepts from finite geometries we use, we refer the reader to the book by Kiss and Szőnyi [13]. Here we give only the most necessary definitions.

Definition 2.1. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line incidence geometry, $\mathcal{P}_0 \subset \mathcal{P}$ and $\mathcal{L}_0 \subset \mathcal{L}$ be subsets of points and lines in $\mathcal{S}$, respectively. The pair $(\mathcal{P}_0, \mathcal{L}_0)$ is called a *t-good structure* in $\mathcal{S}$ if there are t lines of $\mathcal{L}_0$ through any point not in $\mathcal{P}_0$, and there are t points of $\mathcal{P}_0$ on any line not in $\mathcal{L}_0$,

In [10] the following general construction method was presented.

Theorem 2.2 (Gács, Héger)**.** *Suppose that $(\mathcal{P}_0, \mathcal{L}_0)$ is a t-good structure in a generalized n-gon of order (q, q). Deleting the points and lines of $\mathcal{P}_0$ and $\mathcal{L}_0$, respectively, the incidence graph of the resulting structure is a $(q + 1 - t, g)$-graph with $g \geq 2n$.*

Our first class of egr-graphs is related to Baer subplanes. It is well-known that if a finite projective plane Π_{q^2} of order q^2 contains a Baer subplane Π_q of order q, then the points and lines of Π_q form a 1-good structure in Π_{q^2}.

Theorem 2.3. *Suppose that a finite projective plane Π_{q^2} of order q^2 contains a Baer subplane Π_q of order q. Then there exists an extremal bipartite*

$$egr(2(q^4 - q), q^2, 6, (q^2 - 1)(q^4 - 3q^2 + q + 2)).$$

If q is odd, then it is also an extremal egr-graph.

Proof. Delete all the points and all the lines of Π_q from Π_{q^2}. We claim that the incidence graph G of the remaining points and lines satisfies the condition of the theorem. The points and the lines of Π_q form a 1-good structure, hence G is q^2-regular and its girth is at least 6. We deleted $q^2 + q + 1$ points and the same number of lines of Π_{q^2}, so the order of G is

$$2(q^4 + q^2 + 1) - 2(q^2 + q + 1) = 2(q^4 - q).$$

Now, we show that every edge of G is contained in exactly λ distinct 6-cycles where $\lambda = (q^2-1)(q^4-3q^2+q+2)$. Take an edge u_1v_1 and count the number of 6-cycles $u_1v_1u_2v_2u_3v_3$. Without loss of generality, we may assume that u_1 is a point and v_1 is a line of Π_{q^2}. We have (q^2-1) possible choices for the point u_2. After choosing u_2, the choice of u_3 uniquely determines the lines v_2 and v_3.

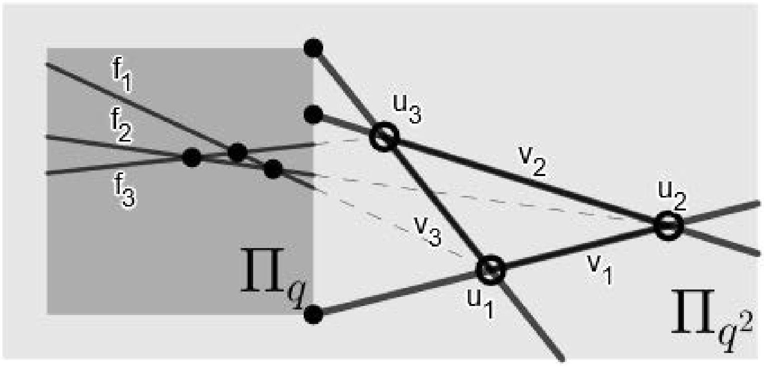


Figure 1: Deleted points, Theorem 2.3.

Let f_i denote the unique line of Π_q that is incident with u_i. A 6-cycle through the edge u_1v_1 contains a deleted element if and only if u_3 is incident with f_1 or f_3. In addition, u_3 cannot be incident with the line v_1, because then $u_1v_1u_2v_2u_3v_3$ would not be a 6-cycle. The union of these three lines and Π_q contains

$$2(q^2-q)+(q^2-2)+(q^2+q+1)=4q^2-q-1$$

points (see Figure 1), therefore the number of possible choices for the point u_3 is

$$(q^4+q^2+1)-(4q^2-q-1)=q^4-3q^2+q+2.$$

So the number of distinct 6-cycles through any edge is exactly λ.

The lower bound in Theorem 1.4 gives

$$\begin{aligned} &n(q^2,6,(q^2-1)(q^4-3q^2+q+2)) \\ &\qquad \geq n_0(q^2,6)+\left\lceil 2\frac{(q^2-1)^3-(q^2-1)(q^4-3q^2+q+2)}{q^2}\right\rceil \\ &\qquad = 2(q^4-q^2+1)+2(q^2-q-2)+\left\lceil\frac{2q+2}{q^2}\right\rceil \geq 2(q^4-q)-1, \end{aligned}$$

and

$$n_2(q^2,6,(q^2-1)(q^4-3q^2+q+2)) \geq 2(q^4-q).$$

Hence G is an extremal bipartite egr-graph for all q. Moreover, if q is odd, then the order of a q-regular graph is even. So in this case,

$$n(q^2,6,(q^2-1)(q^4-3q^2+q+2)) \geq 2(q^4-q),$$

thus G is an extremal egr-graph. $\square$

Let us remark that the points and lines of the union of t disjoint Baer subplanes form a t-good structure in Π_{q^2}. However, for $t > 1$ the corresponding incidence graph is not edge-girth-regular.

The next two classes of egr-graphs result from the deletion of degenerate subplanes.

Theorem 2.4. *Suppose that there exists a finite projective plane Π_q of order $q > 3$. Then there exists a bipartite*

$$egr(2(q^2 - q), q - 1, 6, (q - 2)(q - 3)^2).$$

Proof. Take two points P_1, P_2, the line $e_1 = P_1P_2$, and another line e_2 that is incident with P_1 in Π_q. Let $\mathcal{L}_0$ be the set of lines that are incident with P_1 or P_2, and $\mathcal{P}_0$ be the set of points that are incident with e_1 or e_2. Obviously, $(\mathcal{P}_0, \mathcal{L}_0)$ is a 2-good structure in Π_q. Hence, by Theorem 2.2, after deleting the points and lines of $\mathcal{P}_0$ and $\mathcal{L}_0$ from Π_q, the incidence graph G of the resulting structure is a $(q-1, g)$-graph with $g \geq 6$. The plane Π_q has $q^2 + q + 1$ points and the same number of lines. We deleted $2q + 1$ lines and $2q + 1$ points, so the order of G is

$$2(q^2 + q + 1) - 2(2q + 1) = 2(q^2 - q).$$

Now, we show that every edge is contained in exactly $\lambda = (q - 2)(q - 3)^2$ distinct 6-cycles. Take any edge u_1v_1 of G and count the number of 6-cycles $u_1v_1u_2v_2u_3v_3$. Without loss of generality, we may assume that u_1 is a point and v_1 is a line of Π_q. There are $(q-2)$ possible choices for the point u_2. Again, the choice of u_3 uniquely determines the lines v_2 and v_3.

A 6-cycle through the edge u_1v_1 contains a deleted element if and only if u_3 is incident with one of the following lines: e_1, e_2, u_1P_1, u_1P_2, u_2P_1 or u_2P_2. In addition, u_3 cannot be incident with the line u_1u_2, because then $u_1v_1u_2v_2u_3v_3$ would not be a 6-cycle.

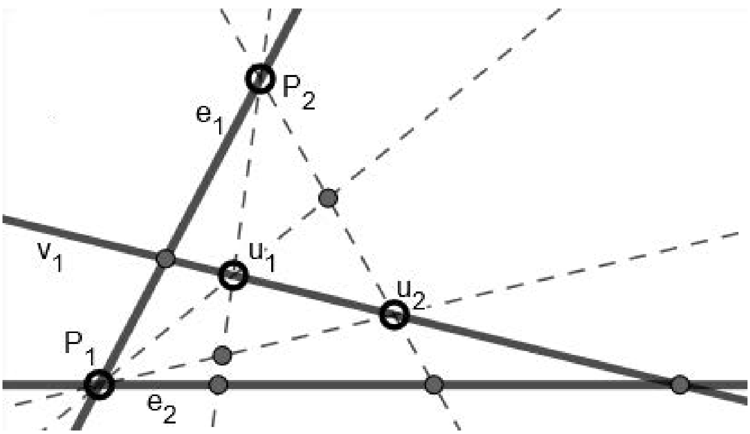


Figure 2: Deleted points, Theorem 2.4.

The union of these seven lines contains $7q - 8$ points (see Figure 2), therefore the number of possible choices for the point u_3 is

$$(q^2 + q + 1) - (7q - 8) = q^2 - 6q + 9 = (q - 3)^2.$$

So the number of distinct 6-cycles through any edge is equal to λ. As $q > 3$, we have $\lambda > 0$, hence the girth of G is 6. Thus G satisfies all conditions of the theorem. $\square$

Theorem 2.5. *Suppose that there exists a finite projective plane Π_q of order $q > 4$. Then there exists a bipartite*

$$egr(2(q-1)^2, q-2, 6, (q-3)(q^2-9q+21)).$$

Proof. The proof is similar to the proof of the previous theorem.

Take three non-collinear points P_1, P_2 and P_3, and the three lines e_1, e_2, and e_3 that they define in Π_q. Let $\mathcal{L}_0$ be the set of lines that are incident with P_1, P_2 or P_3, and $\mathcal{P}_0$ be the set of points that are incident with e_1, e_2 or e_3. Obviously, $(\mathcal{P}_0, \mathcal{L}_0)$ is a 3-good structure in Π_q. Hence, by Theorem 2.2, after deleting the points and lines of $\mathcal{P}_0$ and $\mathcal{L}_0$ from Π_q, the incidence graph G of the resulting structure is a $(q-2, g)$-graph with $g \geq 6$. We deleted $3q$ lines and $3q$ points of Π_q, so the order of G is

$$2(q^2+q+1-3q) = 2(q^2-2q+1) = 2(q-1)^2.$$

Now, we show that every edge is contained in exactly $\lambda = (q-3)(q^2-9q+21)$ distinct 6-cycles. Take any edge u_1v_1 and count the number of $u_1v_1u_2v_2u_3v_3$ 6-cycles. Without loss of generality, we may assume that u_1 is a point and v_1 is a line of Π_q. We have $(q-3)$ possible choices for the point u_2. Again, the choice of u_3 uniquely determines the lines v_2 and v_3.

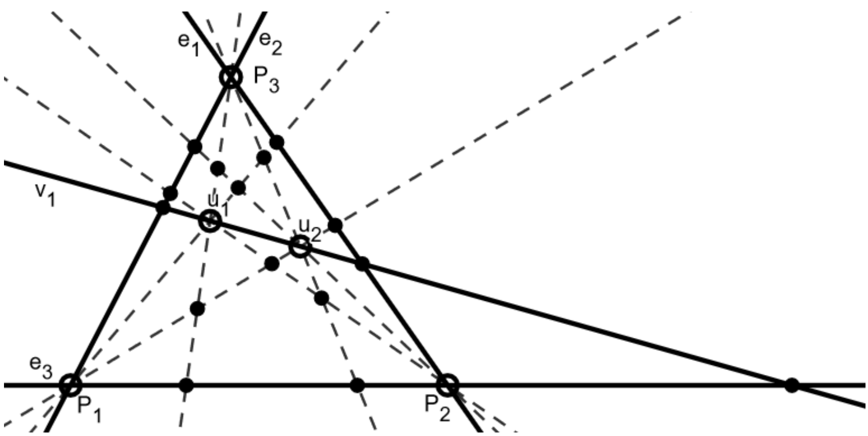


Figure 3: Deleted points, Theorem 2.5.

A 6-cycle through the edge u_1v_1 contains a deleted element if and only if u_3 is incident with one of the following lines: e_1, e_2, e_3, u_1P_1, u_1P_2, u_1P_3, u_2P_1, u_2P_2 and u_2P_3. In addition, u_3 cannot be incident with the line v_1 because then $u_1v_1u_2v_2u_3v_3$ would not be a 6-cycle. The union of these nine lines contains $10q-20$ points of Π_q (see Figure 3), therefore the number of possible choices for the point u_3 is

$$(q^2+q+1)-(10q-20) = q^2-9q+21.$$

So the number of distinct 6-cycles through any edge is λ. The assumption $q > 4$ implies $\lambda > 0$, hence the girth of G is 6. It also implies $k > 2$, thus G satisfies all conditions of the theorem.

Note that if $q = 4$, then the construction results in the union of three disjoint 6-cycles. $\square$

By Theorem 1.4, we have that

$$\begin{aligned} n(q-1, 6, (q-2)(q-3)^2) & \\ &\geq n_0(q-1,6) + \left\lceil 2\frac{(q-2)^3 - (q-2)(q-3)^2}{q-1} \right\rceil \\ &= 2(q^2-3q+3) + 4q - 14 + \left\lceil \frac{6}{q-1} \right\rceil \geq 2q^2 - 2q - 7, \end{aligned}$$

and

$$\begin{aligned} n(q-2, 6, (q-3)(q^2-9q+21) & \\ &\geq n_0(q-2,6) + \left\lceil 2\frac{(q-3)^3 - (q-3)(q^2-9q+21}{q-1} \right\rceil \\ &= 2(q^2-5q+7) + 6q - 30 + \left\lceil \frac{12}{q-2} \right\rceil \geq 2q^2 - 4q - 15. \end{aligned}$$

So the difference between the lower bound and the order of our example is small. It is at most 7 in the case of Theorem 2.4, and it is at most 17 in the case of Theorem 2.5.

All the known finite projective planes have prime power order and if q is any prime power, then the plane $\mathrm{PG}(2,q)$ has order q. As far as we know, the graphs considered in the previous three examples exist if and only if q is a prime power.

The next example of egr-graphs is based on the properties of the Hermitian curve. This is the curve $\mathcal{H}$ with equation $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0$ in the Desarguesian finite projective plane $\mathrm{PG}(2,q^2)$. This curve has q^3+1 points, at each of its points there is a unique tangent line to $\mathcal{H}$, and all other lines of the plane are $(q+1)$-secants of $\mathcal{H}$. The points and the tangent lines of $\mathcal{H}$ form a $(q+1)$-good structure in $\mathrm{PG}(2,q^2)$.

Theorem 2.6. *Let q be a prime power. Then there exists a bipartite*

$$egr\left(2(q^4-q^3+q^2), q^2-q, 6, (q^2-q-1)(q^4-3q^3-q^2+5q+3)\right).$$

Proof. Let $\mathcal{H}$ be a Hermitian curve in $\mathrm{PG}(2,q^2)$. Delete the points and the tangent lines of $\mathcal{H}$ from $PG(2,q^2)$. Let G denote the incidence graph of the remaining points and lines. The points and the tangent lines of $\mathcal{H}$ form a $(q+1)$-good structure, hence, by Theorem 2.2, G is (q^2-q)-regular and its girth is at least 6. The order of G is

$$2(q^4+q^2+1) - 2(q^3+1) = 2(q^4-q^3+q^2).$$

Now, we count the number of 6-cycles $u_1v_1u_2v_2u_3v_3$ through an edge u_1v_1. Without loss of generality, we may assume that u_1 is a point and v_1 is a line. We have q^2-q-1 possible choices for the line v_3, and $(q^2-q-1)^2$ possible choices for the pair (u_2,u_3). However, the line v_2 may be a tangent line of $\mathcal{H}$. The tangent lines through the point u_1 intersect $\mathcal{H}$ in exactly $(q+1)$ collinear points and the lines v_1 and v_3 also intersect $\mathcal{H}$ in

$(q+1)$ points. All points of $\mathcal{H}$ have the property that the unique tangent lines at those points intersect the line v_1 and v_3 not in $\mathcal{H}$ and these points of intersection are different from the point u_1. Therefore the exact number of possible choices for the pair (u_2, u_3) is

$$(q^2-q-1)^2-(q^3+1-3(q+1)) = q^4-3q^3-q^2+5q+3.$$

Hence the number of 6-cycles through the edge u_1v_1 is

$$(q^2-q-1)(q^4-3q^3-q^2+5q+3).$$

So G satisfies all conditions of the theorem. □

Remark 2.7. Theorem 1.4 gives

$$n(q^2-q, 6, (q^2-q-1)(q^4-3q^3-q^2+5q+3)) \geq 2(q^4-q^3-3q-1).$$

so the order of G is close to the bound.

Theorem 2.8. *Let G be the point-hyperplane incidence graph of* $\mathrm{PG}(n,q)$, *the n-dimensional finite projective space of order q, where $n \geq 3$. Then G is a bipartite*

$$egr\left(2\frac{q^{n+1}-1}{q-1}, \frac{q^n-1}{q-1}, 4, \frac{q^{2n-1}-q^{n+1}-q^n+q^2}{(q-1)^2}\right).$$

If $n=3$, then G is an extremal bipartite egr-graph.

Proof. The only part we do not obtain immediately from the basic combinatorial properties of $\mathrm{PG}(n,q)$, is the exact value of λ, the number of 4-cycles through an edge. Consider an incident point-hyperplane pair $(P_1, \mathcal{S}_1)$ and count the number of 4-cycles $P_1S_1P_2S_2$ through this edge. For the point P_2 we have $\frac{q^n-1}{q-1}-1$ possible choices. There are $\frac{q^{n-1}-1}{q-1}$ hyperplanes that contain the points P_1 and P_2, one of them is $\mathcal{S}_1$, hence the number of possible choices for the hyperplane $\mathcal{S}_2$ is $\frac{q^{n-1}-1}{q-1}-1$. Therefore the number of girth cycles through any edge is

$$\left(\frac{q^n-1}{q-1}-1\right)\left(\frac{q^{n-1}-1}{q-1}-1\right) = \frac{q^{2n-1}-q^{n+1}-q^n+q^2}{(q-1)^2}.$$

If $n=3$, then Theorem 1.5 gives the lower bound

$$n_2(q^2+q+1, 4, q^3+q^2) \geq 2(q^3+q^2+q+1).$$

So G is an extremal bipartite egr-graph, because its order attains the bound. □

Remark 2.9. The graph G is relatively simple, the interesting part is its extremality for $n=3$, because it shows the strength of Theorem 1.5 for small values of λ. For comparison, Theorem 1.4 gives a significantly weaker lower bound:

$$n_2(q^2+q+1, 4, q^3+q^2) \geq 2(2q^2+q+1).$$

3 Almost edge-girth-regular graphs

The definition of egr-graphs was weakened by Potočnik and Vidali [16] in the following way. One can introduce the signature $(a_1, a_2, \dots, a_k)$ of a vertex as the non-descending sequence of the number of girth cycles containing the edges emanating from the vertex. A graph is called *girth-regular* and its signature is $(a_1, a_2, \dots, a_k)$ if all of its vertices have the signature $(a_1, a_2, \dots, a_k)$. Girth-regular graphs are regular graphs. If the signature of a girth-regular graph G satisfies $a_1 = a_2 = \dots = a_k$, then G is an egr-graph. If the signature of G has exactly two different entries, then we call G an *almost egr-graph*, or an *agr-graph* for short. When an agr-graph G of order v has girth g, the two entries of its signature are a and b which appear k_1 and k_2 times, respectively, then we say that G is an

$$\operatorname{agr}\left(v, k_1 + k_2, g, \left[a_{(k_1)}, b_{(k_2)}\right]\right).$$

We can construct agr-graphs of girth 5 from the incidence graphs of biaffine planes.

Definition 3.1. Let Π be a finite projective plane of order q. A biaffine plane is obtained from Π by choosing a point-line pair (P, ℓ) and deleting P, ℓ, all the lines incident with P and all the points belonging to ℓ. If the point-line pair is incident in Π, then we call the biaffine plane type 1, otherwise, type 2.

Starting from the incidence graph of a biaffine plane, we just need to add some new edges such that the new graph has girth 5, and all new edges are contained in the same number of distinct girth cycles. In this section we always consider $\mathrm{PG}(2, q)$ as $\mathrm{AG}(2, q) \cup \ell_\infty$. We delete the line at infinity and a pencil of lines of $\mathrm{AG}(2, q)$. The resulting biaffine plane is denoted by $\mathcal{B}$, and we coordinatize $\mathcal{B}$ by Cartesian coordinates in the usual way. The incidence graph of $\mathcal{B}$ is denoted by G.

First, we present a general construction method.

Lemma 3.2. *Let $p > 3$ be a prime and $q = p^r$. Then there exists a $(q + 2, 5)$-graph of order $2q^2$.*

Proof. Delete the vertical lines of $\mathrm{AG}(2, q)$. Then $\mathcal{B}$ is a biaffine plane of type 1. Let $\varepsilon \in GF(q) \setminus \{0, \pm 1\}$. We define new edges among the points and lines of $\mathcal{B}$ in the following way. The two neighbors of the point (x, y) are the points $(x, y \pm 1)$ on the same vertical line, and the two neighbors of the line $Y = mX + b$ are the lines $Y = mX + b \pm \varepsilon$ from the same parallel class. As the characteristic of $\mathrm{GF}(q)$ is p, the new edges form p-cycles. Let Γ denote the graph G extended by the new edges. By definition, Γ has $2q^2$ vertices, q^2 points and q^2 lines. Each vertex has 2 neighbors of its type and q neighbors of the other type, so Γ is $(q + 2)$-regular.

We claim that Γ has girth 5. Suppose that Γ contains a cycle $\mathcal{C}$ of length $\ell \leq 4$. Then $\mathcal{C}$ must contain a new edge. A new edge joins either two points on a vertical line of $\mathrm{AG}(2, q)$, or two parallel lines of $\mathrm{AG}(2, q)$. These pairs of vertices have no common neighbors in G, and $p > 3$, so Γ is triangle-free. If $\mathcal{C}$ is a four-cycle $u_1u_2v_2v_1$ where u_1u_2 is a new edge, then there are three possibilities. First, if all four vertices of $\mathcal{C}$ were the same type, then $p = 4$, a contradiction. If three vertices were the same type, then without loss of generality we may assume, that v_2 is the only vertex of the other type. Then u_1 and v_1 were the two same-type neighbors of u_2, so they would be collinear points or parallel lines of $\mathrm{AG}(2, q)$, hence they could not have a common neighbor in the incidence graph of $\mathcal{B}$. Hence v_2v_1 is also a new edge and the types of u_1 and v_1 cannot be the same. We may assume without loss of generality that $u_1 = (c, d)$ and $u_2 = (c, d + 1)$ are points, and v_1 and v_2 are lines having equations $Y = mX + b$ and $Y = mX + b \pm \varepsilon$, respectively. As the point u_i is on the line v_i, we get $d = mc + b$ for $i = 1$, and $d + 1 = mc + b \pm \varepsilon$ for $i = 2$. This is a contradiction because $\varepsilon \neq \pm 1$. Thus the girth of Γ is at least 5. The order of Γ is less than $n_0(q + 2, 6)$, so its girth is 5. □

Corollary 3.3. *The Hoffman-Singleton graph and the* $(6, 5)$*-cage of order* 40 *can be created with the simple, geometric construction described in* Lemma 3.2.

Proof. Let $q = 5$ and $\varepsilon = 2$. Then the graph Γ constructed in Lemma 3.2 is a $(7, 5)$-graph of order 50, so it is the Hoffman-Singleton graph, which is a Moore cage.

The five points on the vertical line $X = 0$ and the five horizontal lines of $\mathrm{AG}(2, 5)$ form a 1-good structure in $\mathcal{B}$. Although $\mathcal{B}$ is not a generalized polygon, it is clear that Theorem 2.2 holds in this case as well. Deleting these points and lines from Γ, the resulting graph is a $(6, 5)$-graph of order 40. The uniqueness of this graph was proven by Wong [17], so it is the $(6, 5)$-cage. □

Remark 3.4. A computer-assisted calculation shows that the $(6, 5)$-cage is an $\mathrm{egr}(40, 6, 5, 22)$. Its order attains the bound of Theorem 1.4, so it is an extremal egr-graph. We can also prove this fact using purely combinatorial arguments. The proof is a straightforward long counting, we omit it.

Theorem 3.5. *Let* $p > 5$ *be a prime and* $q = p^r \geq 11$ *be a prime power. Then there exists an*

$$\mathrm{agr}\left(2q^2, q + 2, 5, \left[8(q - 1)_{(q)}, (q^2 - q)_{(2)}\right]\right).$$

Proof. As $q \geq 11$, we can choose ε so that $\varepsilon \notin \{0, 1, \pm 2, \pm\frac{1}{2}\}$. Consider the $(q + 2, 5)$-graph Γ constructed in Lemma 3.2. We claim that Γ is an agr-graph.

First, we show that a 5-cycle in Γ cannot contain three consecutive vertices of the same type. The assumption $p > 5$ implies that a 5-cycle cannot contain five vertices of the same type. If exactly four vertices were of the same type in a 5-cycle $u_1u_2u_3u_4v_1$, then u_1 and u_4 would be two points on a vertical line of $\mathrm{AG}(2, q)$, or two parallel lines of $\mathrm{AG}(2, q)$. Thus v_1 would be a vertical line, or a point on the line at infinity, respectively. But these elements were deleted from the incidence graph of $\mathrm{PG}(2, q)$. If exactly three consecutive vertices were of the same type in a 5-cycle $u_1u_2u_3v_1v_2$, then we may assume without loss of generality that $u_1 = (c, d)$ and $u_3 = (c, d+2)$ are points, and v_1 and v_2 are lines having equations $Y = mX + b$ and $Y = mX + b \pm \varepsilon$, respectively. The same calculations as in the proof of Lemma 3.2 show that these imply $\varepsilon = \pm 2$, while the assumption that u_1 and u_3 are lines and v_1 and v_2 are points implies $\varepsilon = \pm\frac{1}{2}$.

Now, we count the number of 5-cycles through an edge. First, consider an edge u_1v_1 where u_1 is a point and v_1 is a line. There are four types of 5-cycles through the edge u_1v_1: $u_1v_1u_2u_3v_2$, $u_1v_1u_2v_2v_3$, $u_1v_1u_2v_2u_3$, and $u_1v_1v_2u_2v_3$, where u_i denotes a point and v_j denotes a line of $\mathcal{B}$. Consider the first type of 5-cycles. We have $q - 1$ possible choices for the point u_2, and two possible choices for the point u_3. The line v_2 is determined, it is the line joining the points u_1 and u_3. In the second case, we have $q-1$ possible choices for the line v_3, and two possible choices for the line v_2. The point u_2 is the point of intersection of the lines v_1 and v_2. In the third type of 5-cycles, we have two possible choices for the point u_3. In this case, we have $q - 1$ possible choices for the point u_2 and the line v_2 is the line joining the points u_2 and u_3. Finally, consider a 5-cycle of the fourth type. We have

$q-1$ possible choices for the line v_3, and two possible choices for the line v_2. The point u_2 is the point of intersection of the lines v_2 and v_3. In summary, there are $8(q-1)$ distinct girth cycles through the edge u_1v_1.

Now, consider the girth cycles through a new edge joining the points u_1 and u_2. This time, we have only one type of 5-cycles: $u_1u_2v_1u_3v_3$. The cycle is uniquely determined by u_3, which is an arbitrary point not on the vertical line through u_1 and u_2. Hence there are q^2-q distinct girth cycles through the edge u_1u_2. Similarly, the new edge joining the lines v_1 and v_2 is contained in exactly q^2-q distinct 5-cycles.

As every vertex of Γ has exactly q neighbors of the same type and two neighbors of the other type, the signature of Γ is

$$\left[8(q-1)_{(q)}, (q^2-q)_{(2)}\right]. \qquad \square$$

Remark 3.6. For $q=7$ and $\varepsilon=2$, Lemma 3.2 results in a $(9,5)$-graph of order 98. It has 2 more vertices than the smallest known $(9,5)$-graph, which was constructed by Jørgensen [12]. However, in this case, there are 5-cycles containing three consecutive points and 5-cycles containing three consecutive lines. Any of these possibilities breaks the symmetry of points and lines, so the constructed graph is not an agr-graph.

Theorem 3.7. *Let $q=5^r$ be a power of 5. Then there exists an*

$$\operatorname{agr}\left(2q^2, q+2, 5, \left[8q-4_{(q)}, (q+1)^2_{(2)}\right]\right).$$

Proof. Let $\varepsilon=2$ and consider the $(q+2,5)$-graph Γ constructed in Lemma 3.2. We claim that Γ is an agr-graph.

First, consider an edge u_1v_1, such that u_1 is a point and v_1 is a line. There are four types of 5-cycles through u_1v_1 that have at most two consecutive vertices of the same type: $u_1v_1u_2u_3v_2$, $u_1v_1u_2v_2v_3$, $u_1v_1u_2v_2u_3$, and $u_1v_1v_2u_2v_3$, where u_i denotes a point and v_j denotes a line of $\mathcal{B}$. From the proof of the previous theorem, we know that the number of these girth cycles is $8(q-1)$. Unlike the previous theorem, we have girth cycles that contain exactly three consecutive points or lines. Assume that $u_1=(c,d)$ and the line v_1 has equation $Y=mX+b$. There are four distinct girth cycles through the edge u_1v_1 that contain three consecutive points or lines:

1. $v_2\colon Y=mX+b+2,\quad v_3\colon Y=mX+b+4,\quad u_2=(c,d-1)$,
2. $v_2\colon Y=mx+b-2,\quad v_3\colon Y=mX+b-4,\quad u_2=(c,d+1)$,
3. $v_2\colon Y=mX+b-2,\quad u_2=(c,d-2),\quad u_3=(c,d-1)$,
4. $v_2\colon Y=mX+b+2,\quad u_2=(c,d+2),\quad u_3=(c,d+1)$.

There are no girth cycles with exactly four points or lines because $3\cdot(\pm 2)\neq 0$, hence the number of girth cycles through the edge u_1v_1 is exactly

$$8(q-1)+4=8q-4.$$

The other type of edges comes from the cycles. Consider the girth cycles through the edge u_1u_2. Similarly to the proof of the previous theorem, the edge u_1u_2 is contained in exactly q^2-q distinct 5-cycles that do not contain three or more consecutive points or lines. We have three types of 5-cycles through the edge u_1u_2 that have exactly three consecutive vertices of the same type: $u_1u_2u_3v_1v_2$, $u_1u_2v_1v_2u_3$, and $u_1u_2v_1v_2v_3$. Consider the first type of these 5-cycles. There are q possible choices for the line v_2, and the point u_2 and line v_1 are uniquely determined by the other components of the cycle. In the second case, we have q possible choices for the line v_1, while the rest of the cycle is again uniquely determined. In the third type of 5-cycles we have q possible choices for the line v_1. In this case, the lines v_2 and v_3 are uniquely determined. Since the characteristic of the field is 5, we also have 5-cycles that contain only one type of vertices. Every edge joining two vertices of the same type is contained in exactly one such cycle. In summary, the number of girth cycles containing the edge u_1u_2 is

$$q(q-1)+3q+1=(q+1)^2.$$

Every vertex of Γ has exactly q neighbors of the same type and two of the other type, hence the signature of Γ is

$$\left[8q-4_{(q)},(q+1)^2_{(2)}\right].$$

In particular, if $q=5$, then $8q-4=(q+1)^2$, so Γ is an egr-graph, as we have already seen in Corollary 3.3. □

Theorem 3.8. *Let $q \geq 11$ be a prime power. Then there exists an*

$$agr\left(2(q^2-1),q+2,5,\left[8(q-1)_{(q)},(q^2-q)_{(2)}\right]\right).$$

Proof. Delete the lines of $\mathrm{AG}(2,q)$ through the origin. Then $\mathcal{B}$ is a biaffine plane of type 2. Choose two generators, ε and η of the multiplicative group $GF(q)^*$ so that $\varepsilon \neq \eta^{\pm 1}, \varepsilon \neq \eta^{\pm 2}, \eta \neq \varepsilon^{\pm 2}$. Such generators exist because $q>8$.

We define a cycle on the $q-1$ points on each deleted line of $\mathrm{AG}(2,q)$, and a cycle on the $q-1$ lines in each parallel class of lines of $\mathcal{B}$ in the following way: if $P=(x,y)$ is a point on a line ℓ through the origin, then the set of the $q-1$ points of $\mathcal{B}$ on ℓ is

$$\{P^i=(x\varepsilon^i,y\varepsilon^i)\colon i=1,2,\dots,q-1\}.$$

Join the points P^i and P^{i+1} for all i, where the superscripts are taken modulo $q-1$. As ε is a generator, these new edges form a cycle of length $q-1$. If $e\colon AX+BY+1=0$ is a line of $\mathcal{B}$ in a parallel class P_e, then the set of the $q-1$ lines of $\mathcal{B}$ in P_e is

$$\{e^i : AX+BY+\eta^i=0\colon i=1,2,\dots,q-1\}.$$

Join the lines e^i and e^{i+1} for all i where the superscripts are taken modulo $q-1$. As η is a generator, these new edges again form a cycle of length $q-1$. Let Γ denote the graph G extended by the edges of these $2(q+1)$ cycles. By definition, Γ has $2(q^2-1)$ vertices, q^2-1 points and q^2-1 lines. Each vertex has 2 neighbors of its type and q neighbors of the other type, so Γ is $(q+2)$-regular.

We claim that Γ has girth 5. Suppose that Γ contains a cycle $\mathcal{C}$ of length $\ell \leq 4$. Then $\mathcal{C}$ must contain a new edge. If all four vertices of $\mathcal{C}$ were of the same type, then the order of ε or η would be $4<q-1$, a contradiction. If three vertices were of the same type, then the same reasoning as in the proof of Lemma 3.2 would work. Finally, if u_i are points and v_i are lines, then without loss of generality we may assume that $u_1=(c,d)$ and

$u_2 = (c\varepsilon, d\varepsilon)$ are points, and v_1 and v_2 are lines having equations $AX + BY + F = 0$ and $AX + BY + F\eta^{\pm 1} = 0$, respectively. As the point u_i is on the line v_i, we get

$$Ac + Bd + \eta^i = 0 \text{ and } Ac\varepsilon + Bd\varepsilon + \eta^{i\pm 1} = 0.$$

Hence $\varepsilon\eta^i - \eta^{i\pm 1} = 0$, so $\varepsilon = \eta^{\pm 1}$, a contradiction again. Thus the girth of Γ is at least 5. The order of Γ is less than $n_0(q+2, 6)$, so its girth is 5.

Now, we count the number of girth cycles through each edge. First, consider an edge u_1v_1, such that u_1 is a point and v_1 is a line. In any cycle there are at most two consecutive vertices of the same type (due to the choice of ε and η), hence we have four types of 5-cycles through the edge u_1v_1: $u_1v_1u_2u_3v_2$, $u_1v_1u_2v_2v_3$, $u_1v_1u_2v_2u_3$, and $u_1v_1v_2u_2v_3$, where u_i denotes a point and v_j denotes a line of $\mathcal{B}$.

The counting is very similar to that presented in Theorem 3.5. Consider the first type of 5-cycles. We have $q-1$ possible choices for the point u_2, and two possible choices for the point u_3. Then v_2 is the line joining the points u_1 and u_3. In the second case, we have $q-1$ possible choices for the line v_3, and two possible choices for the line v_2. Then u_2 is the point of intersection of v_1 and v_2. In the third type of 5-cycles we have two possible choices for the point u_3. In this case, we have $q-1$ possible choices for the point u_2 and v_2 is the line joining the points u_2 and u_3. Finally, consider a 5-cycle of the fourth type. We have $q-1$ possible choices for the line v_3, and two possible choices for the line v_2. The point u_2 is the point of intersection of the lines v_2 and v_3. In summary, there are $8(q-1)$ distinct girth cycles through the edge u_1v_1.

The counting of the girth cycles through a new edge is exactly the same as in the proof of Theorem 3.5. So each new edge is contained in $q^2 - q$ distinct 5-cycles. Every vertex of Γ has exactly q neighbors of the same type and two of the other type. Hence the signature of Γ is

$$\left[8(q-1)_{(q)}, (q^2-q)_{(2)}\right]. \qquad \square$$

Remark 3.9. Again, this construction works and results in a $(2q^2-2, 5)$-graph if there are two distinct generators of the multiplicative group $GF(q)^*$ such that their product is not the unit element, so for all $q \geq 8$. The problem is the same as in Theorem 3.5: there are 5-cycles with three vertices of the same type next to each other. This breaks the symmetry of points and lines and we obtain a graph that is not an agr-graph. For $q = 8$ the order of the constructed graph is 126. It has 2 more vertices than the smallest known $(9,5)$-graph, which was discovered by Exoo [8]. He found it by a computer search starting from the incidence graph of $\mathrm{PG}(2, 11)$.

In the next two constructions, the new edges are perfect matchings of the points and of the lines.

Theorem 3.10. *Let $q > 2$ be a power of two. Then there exists an*

$$\operatorname{agr}\left(2q^2, q+1, 5, \left[4(q-1)_{(q)}, (q^2-q)_{(1)}\right]\right).$$

Proof. Delete the vertical lines of $\mathrm{AG}(2,q)$. Then $\mathcal{B}$ is a biaffine plane of type 1. Let $\varepsilon \in GF(q) \setminus \{0,1\}$. Now, we define a perfect matching of the points of $\mathcal{B}$:

$$(x, y) \longleftrightarrow (x, y+1),$$

and a perfect matching of the lines of $\mathcal{B}$:

$$Y = mX + b \longleftrightarrow Y = mX + b + \varepsilon.$$

These are bijections because q is even. Let Γ denote the graph G extended by the new edges of these two matchings. We claim that Γ satisfies the conditions of the theorem.

Since Γ is a $(q+1)$-regular subgraph of the graph presented in Theorem 3.5, we only need to count the number of 5-cycles through each edge. First, consider an edge u_1v_1, such that u_1 is a point and v_1 is a line. In any cycle there are at most two consecutive vertices of the same type, hence we have four types of 5-cycles through the edge u_1v_1: $u_1v_1u_2u_3v_2$, $u_1v_1u_2v_2v_3$, $u_1v_1u_2v_2u_3$, and $u_1v_1v_2u_2v_3$, where u_i denotes a point and v_j denotes a line of $\mathcal{B}$. There are $q-1$ distinct 5-cycles for each type. The other type of edge comes from the matchings. Again, any edge of this type is contained in exactly q^2-q distinct 5-cycles.

Every vertex of Γ has exactly q neighbors of the same type and one neighbor of the other type. Hence the signature of Γ is

$$\left[4(q-1)_{(q)}, (q^2-q)_{(1)}\right]. \qquad \square$$

Remark 3.11. If $q = 4$, then Γ is the extremal $egr(32, 5, 5, 12)$ graph constructed by Araujo-Pardo and Leemans [5].

Theorem 3.12. *Let $q > 5$ be an odd prime power. Then there exists an*

$$\mathrm{agr}\left(2(q^2-1), q+1, 5, \left[4(q-1)_{(q)}, (q^2-q)_{(1)}\right]\right).$$

Proof. Delete the lines through the origin of $\mathrm{AG}(2,q)$. Then $\mathcal{B}$ is a biaffine plane of type 1. Choose two generators ε and η of the multiplicative group $GF(q)^*$ such that $\varepsilon \neq \eta^{\pm 1}$. We define a perfect matching on the $q-1$ points on each deleted line of $\mathrm{AG}(2,q)$, and a perfect matching on the $q-1$ lines in each parallel class of lines of $\mathcal{B}$ in the following way. If $P = (x, y)$ is a point on a line ℓ through the origin, then the set of the $q-1$ points of $\mathcal{B}$ on ℓ is

$$\{P^i = (x\varepsilon^i, y\varepsilon^i)\colon i = 1, 2, \ldots, q-1\}.$$

Join the points P^i and P^{i+1} for all even i where the superscripts are taken modulo $q-1$. As ε is a generator, we get a perfect matching.

If $e\colon AX + BY + 1 = 0$ is a line of $\mathcal{B}$ in a parallel class P_e, then the set of the $q-1$ lines of $\mathcal{B}$ in P_e is

$$\{e^i : AX + BY + \eta^i = 0\colon i = 1, 2, \ldots, q-1\}.$$

Join the lines e^i and e^{i+1} for all even i where the superscripts are taken modulo $q-1$. As η is a generator, we get a perfect matching.

Let Γ denote the graph G extended by the new edges of these $2(q+1)$ cycles. We claim that Γ satisfies the conditions of the theorem. We can apply the same argument as in the proof of Theorem 3.10. Hence the signature of Γ is

$$\left[4(q-1)_{(q)}, (q^2-q)_{(1)}\right]. \qquad \square$$

4 Lower bounds on the order of girth-regular graphs

In this section, we present lower bounds on the order of girth-regular graphs. We present a natural extension of the known bounds on the order of extremal egr-graphs to girth-regular graphs and give a purely combinatorial proof on the lower bound for even girth. Let $gr(n,k,g,\mathbf{a})$ denote a girth-regular graph of order n, valency k, girth g and signature $\mathbf{a}=(a_1,a_2,\dots,a_k)$.

Similarly to the edge-girth-regular case, we denote the minimal order of a girth-regular (k,g)-graph with signature $\mathbf{a}$ by $n(k,g,\mathbf{a})$. The minimal order of a bipartite (k,g)-graph with signature $\mathbf{a}$ is denoted by $n_2(k,g,\mathbf{a})$.

Theorem 4.1. *Let G be a $gr(n,k,g,\mathbf{a})$ graph, where $g=2h$ is an even number. Then*

$$n \geq 2\frac{(k-1)^h-1}{k-2}+\left\lceil 2\frac{(k-1)^h-a_1}{k}\right\rceil .$$

Proof. Same argument as in Theorem 1.4 but with an edge that is contained in exactly a_1 distinct girth cycles. □

Theorem 4.2. *Let G be a $gr(n,k,g,\mathbf{a})$ graph, where $g=2h+1$ is an odd number. Then*

$$n \geq \frac{k(k-1)^h-2}{k-2}+\left\lceil \frac{k(k-1)^h-\sum_{i=1}^k a_i}{k}\right\rceil .$$

Proof. Same argument as in Theorem 1.4 with changing $k\lambda$ to $\sum_{i=1}^k a_i$. □

Theorem 4.3. *Let G be an $sgr(n,k,g,\mathbf{a})$ graph, where g is even.*
If $g\equiv 0 \pmod 4$, then

$$n(k,g,\mathbf{a}) \geq \frac{c(g,k)+\sum_{i=1}^k a_i+k^g-2c(\frac{g}{2},k)k^{\frac{g}{2}}}{c(g,k)-c^2(\frac{g}{2},k)+\sum_{i=1}^k a_i},$$

$$n_2(k,g,\mathbf{a}) \geq 2\frac{c(g,k)+\sum_{i=1}^k a_i+k^g-2c(\frac{g}{2},k)k^{\frac{g}{2}}}{c(g,k)-c^2(\frac{g}{2},k)+\sum_{i=1}^k a_i}.$$

If $g\equiv 2 \pmod 4$, then

$$n(k,g,\mathbf{a}) \geq \frac{c(g,k)+\sum_{i=1}^k a_i+k^g}{c(g,k)+\sum_{i=1}^k a_i},$$

$$n_2(k,g,\mathbf{a}) \geq \frac{2k^g}{c(g,k)+\sum_{i=1}^k a_i}.$$

Proof. Again, the proof is the same as the proof of Theorem 1.5. We only need to change $k\lambda$ to $\sum_{i=1}^k a_i$. □

Theorem 4.4. *Let G be a $gr(n,k,g,\mathbf{a})$ graph, where $g=2h$ is an even number. Then*

$$\begin{aligned} n \geq{}& 2\frac{(k-1)^h-1}{k-2} \\ &+\max_{1\leq i\leq k}\left\lceil \frac{\left((k-1)^h-a_i\right)^2}{\sum_{j=1}^k a_j-3a_i+(k-1)^h-2\max\left\{0,\left\lceil a_i\frac{a_i-(k-1)^{h-1}}{2(k-1)^{h-1}}\right\rceil\right\}}\right\rceil . \end{aligned} \tag{4.1}$$

Proof. Choose an arbitrary edge uv that is contained in exactly λ distinct g-cycles. We define the set D_u of vertices as follows: $w \in D_u$ if the length of the shortest uw-path is $h-1$, and the length of the shortest vw-path is h. Similarly, $w \in D_v$ if and only if the length of the shortest vw-path is $h-1$, and the length of the shortest uw-path is h.

First, count the number of edges between D_u and D_v. Suppose that there is an edge ab between D_u and D_v. Then the edges ab, uv, and the unique $(h-1)$-paths au and bv form a g-cycle. Since G has girth g, this is a bijection between the edges between D_u and D_v and the g-cycles through uv. Hence the number of edges between D_u and D_v is exactly λ.

Now, count the number of girth cycles through u that does not contain the edge uv. The graph is k-regular, hence u lies on exactly k edges and every edge is contained in exactly a_i distinct g-cycles, but this way, every g-cycle through u is counted twice. Therefore the number of g-cycles through u is $\frac{1}{2}\sum a_i$. For simplicity, define the variable s as the sum of all a_i. So the number of girth cycles through u that does not contain the edge uv is $\frac{s}{2}-\lambda$. There are two types of these cycles: the ones that reach D_v and the ones that have a vertex at distance h from u and outside of D_v. Denote the set of these vertices with M (see Figure 4).

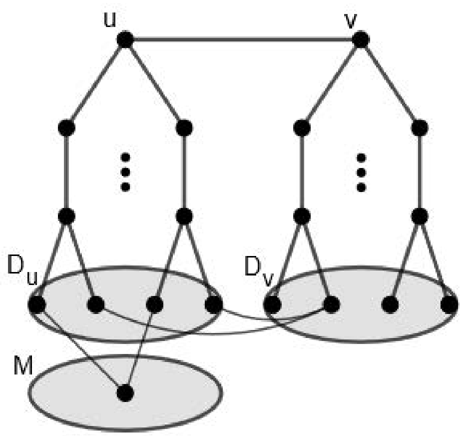


Figure 4: Girth cycles through u.

We give a lower bound for the number of the first type of these cycles to give an upper bound for the number of the second type of these cycles. There are $(k-1)^{h-1}$ vertices in D_v=$\{v_1,\dots,v_{(k-1)^{h-1}}\}$. Each vertex v_i has y_i neighbours in D_u. Clearly, $\sum y_i = \lambda$. In each vertex v_i, we have $\binom{y_i}{2}$ possible choices to form a girth cycle through u that does not contain the edge uv. So the number of girth cycles of the first type is

$$\sum_{i=1}^{(k-1)^{h-1}} \binom{y_i}{2} = \frac{1}{2}\sum_{i=1}^{(k-1)^{h-1}} y_i^2 - \frac{1}{2}\sum_{i=1}^{(k-1)^{h-1}} y_i = \frac{1}{2}\sum_{i=1}^{(k-1)^{h-1}} y_i^2 - \frac{\lambda}{2}.$$

The inequality between the arithmetic and quadratic means gives

$$\frac{\sum_{i=1}^{(k-1)^{h-1}} y_i^2}{(k-1)^{h-1}} \geq \frac{\left(\sum_{i=1}^{(k-1)^{h-1}} y_i\right)^2}{(k-1)^{2(h-1)}} = \frac{\lambda^2}{(k-1)^{2(h-1)}},$$

hence we obtain the following lower bound:

$$\frac{1}{2}\sum_{i=1}^{(k-1)^{h-1}} y_i^2 - \frac{\lambda}{2} \geq \frac{\lambda^2}{2(k-1)^{(h-1)}} - \frac{\lambda}{2} = \lambda\frac{\lambda-(k-1)^{h-1}}{2(k-1)^{h-1}}.$$

This lower bound is negative if λ is small enough. But we know that the number of girth cycles of the first type is a non-negative integer, therefore we have the following lower bound:

$$\sum_{i=1}^{(k-1)^{h-1}} \binom{y_i}{2} \geq \max\left\{0, \left\lceil \lambda\frac{\lambda-(k-1)^{h-1}}{2(k-1)^{h-1}}\right\rceil\right\}.$$

Suppose that there are m vertcies in $M = \{w_1, \dots, w_m\}$ and each w_i has x_i neighbors in D_u. We obtain a girth circle if we choose such a vertex w_i, its two neighbors in D_u, and their unique $(h-1)$-paths to the vertex u. Therefore the number of girth cycles of the second type is exactly $\sum_{i=1}^{m}\binom{x_i}{2}$. Now, we have the following upper bound of the number of these circles:

$$\sum_{i=1}^{m}\binom{x_i}{2} = \frac{s}{2} - \lambda - \sum_{i=1}^{(k-1)^{h-1}}\binom{y_i}{2} \leq \frac{s}{2} - \lambda - \max\left\{0, \left\lceil \lambda\frac{\lambda-(k-1)^{h-1}}{2(k-1)^{h-1}}\right\rceil\right\}.$$

We use this inequality to give a lower bound for m. But first, we need to rearrange the terms. We also use the fact that $\sum_{i=1}^{m} x_i = (k-1)^h - \lambda$.

Now, we have that

$$2\sum_{i=1}^{m}\binom{x_i}{2} = \sum_{i=1}^{m} x_i^2 - \sum_{i=1}^{m} x_i \leq s - 2\lambda - 2\max\left\{0, \left\lceil \lambda\frac{\lambda-(k-1)^{h-1}}{2(k-1)^{h-1}}\right\rceil\right\},$$
$$\sum_{i=1}^{m} x_i^2 \leq (k-1)^h + s - 3\lambda - 2\max\left\{0, \left\lceil \lambda\frac{\lambda-(k-1)^{h-1}}{2(k-1)^{h-1}}\right\rceil\right\}.$$

By using the inequality between the arithmetic and quadratic means, we get a lower bound for m:

$$m \geq \frac{\left(\sum_{i=1}^{m} x_i\right)^2}{\sum_{i=1}^{m} x_i^2} \geq \frac{\left((k-1)^h - \lambda\right)^2}{s - 3\lambda + (k-1)^h - 2\max\left\{0, \left\lceil \lambda\frac{\lambda-(k-1)^{h-1}}{2(k-1)^{h-1}}\right\rceil\right\}}.$$

Since G is a k-regular graph of girth g, it has at least $n_0(k,g)$ vertices, and with the lower bound of m, we also give a lower bound for the additional vertices. We add it to the Moore bound, take the maximum in the signature, and obtain our lower bound for the order of girth-regular graphs of even girth:

$$\begin{aligned} n \geq & 2\frac{(k-1)^h - 1}{k-2} \\ & + \max_{1\leq i\leq k}\left\lceil \frac{\left((k-1)^h - a_i\right)^2}{\sum_{j=1}^{k} a_j - 3a_i + (k-1)^h - 2\max\left\{0, \left\lceil a_i\frac{a_i-(k-1)^{h-1}}{2(k-1)^{h-1}}\right\rceil\right\}}\right\rceil. \quad \square \end{aligned}$$

Corollary 4.5. *If G is an $egr(n,k,g,\lambda)$ graph with even girth $g=2h$, then*

$$n \geq 2\frac{(k-1)^h-1}{k-2} + \left\lceil \frac{((k-1)^h-\lambda)^2}{(k-3)\lambda+(k-1)^h-2\max\left\{0,\left\lceil\lambda\frac{\lambda-(k-1)^{h-1}}{2(k-1)^{h-1}}\right\rceil\right\}} \right\rceil.$$

Remark 4.6. As M. Macaj [14] pointed out, in the case of bipartite graphs, a better lower bound can be obtained by slightly modifying our reasoning. In this case, we can repeat the proccess for the vertex v, too. We define the set M_v similarly to M_u. We get exactly the same lower bound for the size of M_v as of the size of M_u. Since the graph is bipartite, M_u and M_v are disjoint sets, hence we can double the second summand in the right hand side of Inequality 4.1. This results in the following statements.

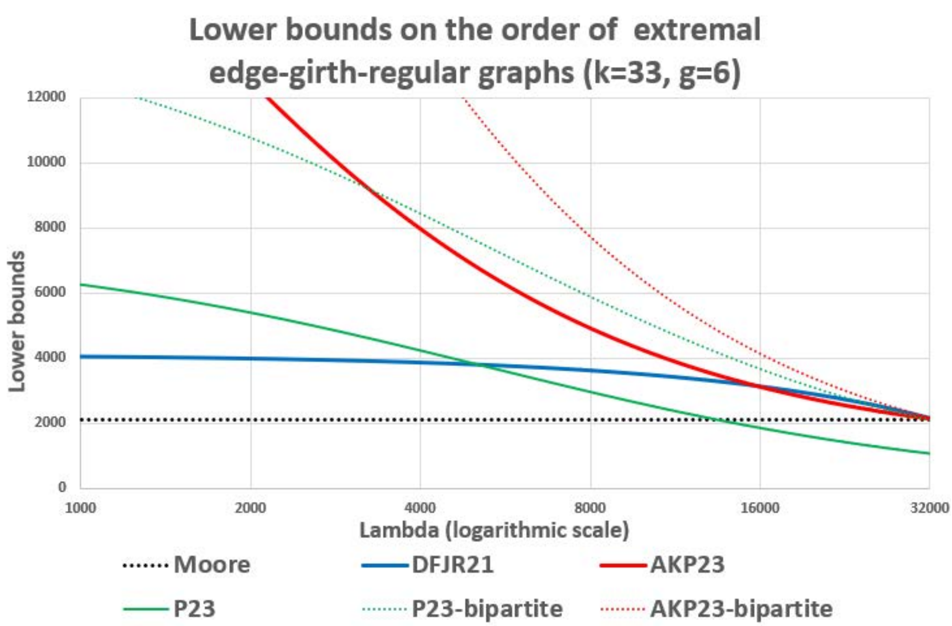
Corollary 4.7. *Let G be a bipartite $gr(n,k,g,\mathbf{a})$ graph, where $g=2h$ is an even number. Then*

$$\begin{aligned} n \geq & 2\frac{(k-1)^h-1}{k-2} \\ & + 2\max_{1\leq i\leq k}\left\lceil \frac{\left((k-1)^h-a_i\right)^2}{\sum_{j=1}^{k} a_j - 3a_i + (k-1)^h - 2\max\left\{0,\left\lceil a_i\frac{a_i-(k-1)^{h-1}}{2(k-1)^{h-1}}\right\rceil\right\}} \right\rceil. \end{aligned}$$

If G is a bipartite $egr(n,k,g,\lambda)$ graph with even girth $g=2h$, then

$$n \geq 2\frac{(k-1)^h-1}{k-2} + 2\left\lceil \frac{((k-1)^h-\lambda)^2}{(k-3)\lambda+(k-1)^h-2\max\left\{0,\left\lceil\lambda\frac{\lambda-(k-1)^{h-1}}{2(k-1)^{h-1}}\right\rceil\right\}} \right\rceil.$$

Finally, we compare the existing lower bounds (DFJR21 [7], P23 [15]) on the order of extremal edge-girth-regular graphs of even girth with the one in Corollary 4.7 (AKP23).



When λ is close to its upper bound $(k-1)^{\frac{g}{2}}$, then all three lower bounds are close to the Moore-bound, and DFJR21 is always the best one. But it falls back when λ is small, as we have seen in Remark 2.9. Our result and P23 are of the same order of magnitude, namely $\Theta(k^{\frac{g}{2}})$ when λ is of the order of magnitude $O(k^{\frac{g}{2}-1})$; however our result is better with a constant multiplier.

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