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Average distance, radius and remoteness of a graph*

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Abstract

Let $G = (V, E)$ be a connected graph on n vertices. Denote by $\bar{l}(G)$ the average distance between all pairs of vertices in G. The *remoteness* $\rho(G)$ of a connected graph G is the maximum average distance from a vertex of G to all others. The aim of this paper is to show that two conjectures in [5] concerned with average distance, radius and remoteness of a graph are true.

Keywords: Distance, radius, eccentricity, proximity, remoteness.

Math. Subj. Class.: 05C12, 05C35

1 Introduction

All graphs considered in this paper are finite and simple. For notation and terminology not defined here, we refer to West [21]. Let $G = (V, E)$ be a finite simple graph with vertex set V and edge set E, $|V|$ and $|E|$ are its order and size, respectively. The distance between vertices u and v is denoted by $d(u, v)$, is the length of a shortest path connecting u and v. The average distance between all pairs of vertices in G is denoted by $\bar{l}(G)$. That is $\bar{l}(G) = \frac{1}{\binom{n}{2}} \sum_{u,v \in V(G)} d(u, v)$, where the summation run over all unordered pairs of vertices. The *eccentricity* $e_G(v)$ of a vertex v in G is the largest distance from v to another vertex of G, i.e. $\max\{d(v, w) |\ w \in V(G)\}$. The *diameter* of G is the maximum eccentricity in G,

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denoted by $diam(G)$. Similarly, the *radius* of G is the minimum eccentricity in G, denoted by $rad(G)$; and the *average eccentricity* of G is denoted by $ecc(G)$. In other words,

$$rad(G) = \min_{v \in V} e_G(v), \quad diam(G) = \max_{v \in V} e_G(v) \text{ and } ecc(G) = \frac{1}{n} \sum_{v \in V} e_G(v).$$

For a connected graph G of order n, $\sigma_G(u)$ denotes the average distance from u to all other vertices of G, that is $\sigma_G(u) = \frac{1}{n-1} \sum_{v \in V(G)} d(u,v)$. The *proximity* $\pi(G)$ of a connected graph G is the minimum average distance from a vertex of G to all others. Similarly, the *remoteness* $\rho(G)$ of a connected graph G is the maximum average distance from a vertex of G to all others. They were recently introduced in [2, 3], that is

$$\pi(G) = \min_{v \in V} \sigma_G(v) \text{ and } \rho(G) = \max_{v \in V} \sigma_G(v).$$

The sum of distances from a vertex of G to all others is known as its transmission. Proximity and remoteness can also be seen as the minimum and maximum normalized transmission in a graph. Indeed, by their definitions

$$\pi(G) \leq rad(G) \leq ecc(G) \leq diam(G) \text{ and } \pi(G) \leq \bar{l}(G) \leq \rho(G) \leq diam(G).$$

There are a number of results which are devoted to the relation between average distance and other graph parameters (see [6-15, 22]). A vertex $u \in V(G)$ with the minimum eccentricity is called a *center* of G. It is well-known that every tree has either exactly one center or two, adjacent centers. The center of graphs have been extensively studied in the literature (see [16]). Some more results on the radius of graphs can be found in [18, 17].

A Soltés or a path-complete graph is the graph obtained from a clique and a path by adding at least one edge between an endpoint of the path and the clique. The Soltés graphs are known to maximize the average distance $\bar{l}$ when the number of vertices and of edges are fixed [20].

In [4] Aouchiche and Hansen established the Nordhaus-Gaddum-type theorem for $\pi(G)$ and $\rho(G)$. In [5] the same authors gave the upper bounds on $rad(G) - \pi(G)$, $diam(G) - \pi(G)$ and $\rho(G) - \pi(G)$, and proposed five related conjectures, two of which are the following.

Conjecture A. (Conjecture 5, [5]) Among all connected graphs G on $n \geq 3$ vertices with average distance $\bar{l}$ and remoteness ρ, the Soltés graphs with diameter $\lfloor \frac{n+1}{2} \rfloor$ maximize $\rho - \bar{l}$.

Conjecture B. (Conjecture 1, [5]) Let G be a connected graph on $n \geq 3$ vertices with remoteness ρ and radius r. Then connected graph G on $n \geq 3$ vertices,

$$\rho - r \geq \begin{cases} \frac{n^2}{4n-4} - \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{3-n}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

The inequality is best possible as shown by the cycle C_n if n is even and of the graph composed by the cycle C_n together with two crossed edges on four successive vertices of the cycle.

The aim of this note is to confirm the validity of the above conjectures.

Conjecture 2 in [5] is solved in [19], and Conjecture 4 in [5] is solved in [1]. Up to now, Conjecture 3 in [5] still remains open.

2 Proof of Conjecture A

For convenience, we use some additional definition and notations. Let G be a connected graph. A vertex $u \in V(G)$ is called a $peripheral$ vertex if $\sigma(u) = \rho(G)$. For a vertex $u \in V(G)$, let $V_i(u) = \{v \in V(G) | \, d(u,v) = i\}$ and $n_i(u) = |V_i(u)|$ for each $i \in \{1, \dots, d\}$, where $d = e_G(u)$. In what follows, $V_i(u)$ is simply denoted by V_i for a peripheral vertex u of G.

Lemma 2.1. *Let G be a connected graph of order $n \geq 3$. Let u be a peripheral vertex of G and let $d = e_G(u)$. Let G' be the graph obtained from G by joining each pair of all nonadjacent vertices x, y of G, where $x, y \in V_j \cup V_{j+1}$ for some $j \in \{1, \dots, d-1\}$. We have*

$$\rho(G') - \bar{l}(G') \geq \rho(G) - \bar{l}(G),$$

with equality if and only if $G' = G$.

Proof. It is clear that for any $x \in V(G)$, $d_{G'}(u,x) = d_G(u,x)$ and $d_{G'}(v,w) \leq d_G(v,w)$ for any $v, w \in V(G)$. It follows that $\sigma_{G'}(u) = \sigma_G(u)$ and $\sigma_{G'}(v) \leq \sigma_G(v)$ for any $v \in V(G')$. Combining this with the assumption that u is a peripheral vertex of G, it follows that u is also a peripheral vertex of G'. Thus $\rho(G') = \sigma_{G'}(u) = \sigma_G(u) = \rho(G)$. Moreover, it is obvious that $\bar{l}(G') \leq \bar{l}(G)$, with equality if and only if $G' = G$. So, $\rho(G') - \bar{l}(G') \geq \rho(G) - \bar{l}(G)$, with equality if and only if $G' = G$. □

Lemma 2.2. *Let G be a connected graph of order $n \geq 3$. Let u be a peripheral vertex of G and $e_G(v) = d$. Assume that $G[V_j \cup V_{j+1}]$ is a clique for each $j \in \{0, \dots, d-1\}$. Let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup \{xy : x \in V_{d-2}, y \in V_d\}$. If $d > \lfloor \frac{n+1}{2} \rfloor$, then*

$$\rho(G') - \bar{l}(G') \leq \rho(G) - \bar{l}(G),$$

with equality if and only if n is even and $d = \frac{n}{2} + 1$.

Proof. Note that

$$\sigma_{G'}(x) = \begin{cases} \sigma_G(x) - \frac{1}{n-1} n_d, & \text{if } x \in \bigcup_{j=1}^{d-2} V_j \, \cup \{u\} \\ \sigma_G(x), & \text{if } x \in V_{d-1} \\ \sigma_G(x) - \frac{1}{n-1}(n - n_{d-1} - n_d), & \text{if } x \in V_d \,. \end{cases}$$

Since u is a peripheral vertex of G, $\sigma_{G'}(u) \geq \sigma_{G'}(x)$ for any $x \in \bigcup_{j=1}^{d-2} V_j$. Moreover, since $d > \lfloor \frac{n+1}{2} \rfloor$, $n_{d-1} + n_d \leq \frac{n}{2}$, with equality if and only if n is even and $d = \frac{n}{2} + 1$.

Thus $n - n_{d-1} - n_d \geq \frac{n}{2}$. Again by the assumption that u is a peripheral vertex of G, $\sigma_{G'}(u) \geq \sigma_{G'}(x)$ for any $x \in V_d$. Also, for any $y \in V_{d-1}$, $\sigma_{G'}(y) = \sigma_{G'}(x)$ for any $x \in V_d$. Thus $\sigma_{G'}(u) > \sigma_{G'}(y)$. It means that u is a peripheral vertex of G', and $\rho(G') = \sigma_{G'}(u)$. So, $\rho(G) - \rho(G') = \sigma_{G'}(u) - \sigma_G(u) = \frac{1}{n-1} n_d$. On the other hand, one can see that

$$\bar{l}(G) - \bar{l}(G') = \frac{1}{\binom{n}{2}} \big[(n - n_{d-1} - n_d) n_d \big] = \frac{2}{(n-1)n} \big[(n - n_{d-1} - n_d) n_d \big] \geq \frac{1}{n-1} n_d.$$

It follows that

$$\bar{l}(G) - \bar{l}(G') \geq \rho(G) - \rho(G'),$$

with equality if n is even and $d = \frac{n}{2} + 1$. □

Lemma 2.3. *Let G be a connected graph of order $n \geq 3$. Let u be a peripheral vertex of G and $e_G(v) = d$. Assume that $G[V_j \cup V_{j+1}]$ is a clique for each $j \in \{0, \ldots, d-1\}$. Let i be the smallest integer in $\{1, \ldots, d\}$ such that $n_i(u) \geq 2$. Let $V_{i-1}(u) = \{u_{i-1}\}$ and v a vertex in $V_i(u)$. Denote by G' the graph with $V(G') = V(G)$ and $E(G') = E(G) \setminus (\{u_{i-1}y : y \in V_i \setminus \{v\}\} \cup A)$, where $A = \{vx : x \in V_{i+1}\}$ if $i \leq d-1$, and $A = \emptyset$ otherwise. If $d < \lfloor \frac{n+1}{2} \rfloor$, then*

$$\rho(G') - \bar{l}(G') > \rho(G) - \bar{l}(G).$$

Proof. One can see that if $i \leq d-1$, then

$$\sigma_{G'}(x) - \sigma_G(x) = \begin{cases} \frac{1}{n-1}(n-i-1), & \text{if } x \in \bigcup_{j=1}^{i-1} V_j \cup \{v, u\} \\ \frac{1}{n-1} i, & \text{if } x \in V_i \setminus \{v\} \\ \frac{1}{n-1}(i+1), & \text{if } x \in \bigcup_{j=i+1}^{d} V_j, \end{cases}$$

if $i = d$, then

$$\sigma_{G'}(x) - \sigma_G(x) = \begin{cases} \frac{1}{n-1}(n-d-1), & \text{if } x \in \bigcup_{j=1}^{d-1} V_j \cup \{v, u\} \\ \frac{1}{n-1} d, & \text{if } x \in V_d \setminus \{v\} \end{cases}$$

If $i \leq d-1$, then by $d \leq \lfloor \frac{n+1}{2} \rfloor - 1$, we have $i+1 \leq \lfloor \frac{n+1}{2} \rfloor - 1$ and $n-i-1 > i+1$. Moreover, since u is a peripheral vertex of G, u is also a peripheral vertex of G'. If $i = d$, then it is trivial to see that u is a peripheral vertex of G'. So, $\rho(G') - \rho(G) = \sigma_{G'}(u) - \sigma_G(u) = \frac{1}{n-1}\big(n-i-1\big)$.

On the other hand, if $i \leq d-1$, then

$$\begin{aligned} \bar{l}(G') - \bar{l}(G) &= \frac{1}{\binom{n}{2}}\big[(i+1)\big(n_{i+1} + n_{i+2} + \cdots + n_d\big) + i\big(n_i - 1\big)\big] \\ &= \frac{1}{\binom{n}{2}}\big[(i+1)\big(n - i - n_i\big) + i\big(n_i - 1\big)\big] \\ &= \frac{2}{(n-1)n}\big[(i+1)n - i^2 - 2i - n_i\big]. \end{aligned}$$

Define a function: $f(i) = (n-i-1) - \frac{2}{n}[(i+1)n - i^2 - 2i - n_i]$. By an easy calculation, one has $f(i) = n - 3(i+1) + \frac{2}{n}(i^2 + 2i + n_i)$ and thus $f'(i) = -3 + \frac{2}{n}(2i+2)$. Since $i \leq d-1$, by $d < \lfloor \frac{n+1}{2} \rfloor$, we have $f'(i) < 0$. Thus $f(i)$ is a decreasing function on $[0, \lfloor \frac{n+1}{2} \rfloor - 2]$, and achieves its minimum value at $\lfloor \frac{n+1}{2} \rfloor - 2$. One can check that

$$f(\lfloor \frac{n+1}{2} \rfloor - 2) > 0.$$

Therefore $f(i) > 0$, and thus $\rho(G') - \rho(G) > \bar{l}(G') - \bar{l}(G)$, the result follows.

If $i = d$, then $\rho(G') - \rho(G) = \frac{1}{n-1}\big(n-d-1\big)$, and

$$\bar{l}(G') - \bar{l}(G) = \frac{2d(n_d - 1)}{(n-1)n} = \frac{2d(n-d-1)}{(n-1)n}.$$

Since $d \leq \lfloor \frac{n+1}{2} \rfloor - 1$,

$$\frac{\rho(G') - \rho(G)}{\bar{l}(G') - \bar{l}(G)} = \frac{n}{2d} > 1.$$

□

Lemma 2.4. *Let G be a connected graph of order $n \geq 3$. Let u be a peripheral vertex of G and $e_G(v) = d$. Assume that $G[V_j \cup V_{j+1}]$ is a clique for each $j \in \{0, \ldots, d-1\}$ and that $n_i(u) \geq 2$ for some $i \in \{1, \ldots, d-1\}$. Further, assume that i is the minimum subject to the above condition. Let v be a vertex in $V_i(u)$ and $V_{i-1} = \{u_{i-1}\}$. Let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup A \setminus \{vu_{i-1}\}$, where $A = \{vy : \; y \in V_{i+2}\}$ if $i \leq d-2$, and $A = \emptyset$ otherwise. If $d = \lfloor \frac{n+1}{2} \rfloor$, then*

$$\rho(G') - \bar{l}(G') > \rho(G) - \bar{l}(G).$$

Proof. Note that if $i \leq d-2$, then

$$\sigma_{G'}(x) - \sigma_G(x) = \begin{cases} \frac{1}{n-1}, & \text{if } x \in \bigcup_{j=1}^{i-1} V_j \; \cup \{u\} \\ 0, & \text{if } x \in V_i \cup V_{i+1} \\ -\frac{1}{n-1}, & \text{if } x \in \bigcup_{j=i+2}^{d} V_j \; , \end{cases}$$

if $i = d-1$, then

$$\sigma_{G'}(x) - \sigma_G(x) = \begin{cases} \frac{1}{n-1}, & \text{if } x \in \bigcup_{j=1}^{d-2} V_j \; \cup \{u\} \\ 0, & \text{if } x \in V_{d-1} \cup V_d \; . \end{cases}$$

Thus $\rho(G') = \sigma_{G'}(u) = \sigma_G(u) + \frac{1}{n-1} = \rho(G) + \frac{1}{n-1}$. On the other hand, since $i \leq d-1 = \lfloor \frac{n-1}{2} \rfloor < \frac{n}{2}$, we have

$$\begin{aligned} \bar{l}(G') - \bar{l}(G) &= \frac{1}{\binom{n}{2}} \big[i - \big(n_{i+2} + n_{i+3} + \cdots + n_d \big) \big] \\ &= \frac{2}{(n-1)n} \big(n_i + n_{i+1} + 2i - n \big) \\ &\leq \frac{2}{(n-1)n} i \\ &< \frac{1}{n-1}. \end{aligned}$$

The results follows. □

The statement of Conjecture A is refined as follows.

Theorem 2.5. *Among all connected graphs G on $n \geq 3$ vertices with average distance $\bar{l}$ and remoteness ρ, the maximum value of $\rho - \bar{l}$ is attained by the Soltés graphs with diameter d, where*

$$\begin{cases} d = \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ d \in \{\frac{n}{2}, \frac{n}{2}+1\} & \text{if } n \text{ is even} \; . \end{cases}$$

Proof. It is immediate from Lemmas 2.1-2.4. □

In the remaining sections, we prove Conjecture B.

3 Some preparations

Let G be a connected graph. Recall that for a vertex $u \in V(G)$, let $V_i(u) = \{v \in V(G) |\, d(u,v) = i\}$ and $n_i(u) = |V_i(u)|$ for each $i \in \{1, \ldots, d\}$, where $d = diam(G)$.

Lemma 3.1. *Let G be connected graph with order n and radius $r \geq 2$. If u is a center of G, then $n_i(u) \geq 2$ for all $i \in \{1, \ldots, r-1\}$.*

Proof. By contradiction, suppose that $n_i(u) = 1$ for some $1 \leq i \leq r-1$ and let $V_i(u) = \{w\}$. Let P be a shortest path connecting u and w in G, v be the neighbor of u on P. For a vertex $x \in V(G) \setminus \{v\}$,

$$d(v,x) \begin{cases} = d(u,x) - 1, & \text{if } d(u,x) \geq i \\ \leq d(u,x) + 1, & \text{if } d(u,x) < i\ . \end{cases}$$

It follows that $ecc(v) = r - 1$, a contradiction. □

Corollary 3.2. *If G is a connected graph with order n and radius r, then $r \leq \frac{n}{2}$.*

Proof. Let u be a center of G. By Lemma 3.1, $n_i(u) \geq 2$ for all $i \in \{1, \ldots, r-1\}$. So, $n \geq 1 + \sum_{i=1}^{r} n_i(u) \geq 2r$, the result then follows. □

For a graph G, $p(G)$ denotes the maximum cardinality of a subset of vertices that induce a path in G.

Theorem 3.3. *(Erdős, Saks, Sós [18]) For any connected graph G, $p(G) \geq 2rad(G) - 1$.*

Corollary 3.4. *Let G be a connected graph of order $n \geq 3$. For an even n, $rad(G) = \frac{n}{2}$ if and only if $G \cong P_n$ or $G \cong C_n$.*

Proof. The sufficiency is obvious. Next we prove its necessity. By Theorem 3.3, $p(G) \geq n-1$. Let $P = v_1 \ldots v_{n-1}$ be an induced path of G, and let v_n be the remaining vertex of G. We consider the vertex $v_{\frac{n}{2}}$. Since $d(v_{\frac{n}{2}}, v_i) < \frac{n}{2}$ for each $i \neq \frac{n}{2}$, and $rad(G) = \frac{n}{2}$, we have $d(v_{\frac{n}{2}}, v_n) = \frac{n}{2}$. So, $v_n v_i \notin E(G)$ for each $2 \leq i \leq n-2$, and thus $N(v_n) \subseteq \{v_1, v_n\}$. It implies that $G \in \{P_n, C_n\}$. □

For an odd integer $n \geq 5$, we define some special graphs of order n with $rad(G) = \frac{n-1}{2}$: $C_{n-1}(1)$ is the graph obtained from C_{n-1} by adding a new vertex which joins two adjacent vertices of C_{n-1}; $C_{n-1}(2)$ is the graph obtained from C_{n-1} by adding a new vertex which joins two vertices with distance two on C_{n-1}; $C_{n-1}(3)$ is the graph obtained from C_{n-1} by adding a new vertex which joins three consecutive vertices of C_{n-1}. One can see that $p(C_n) = p(C_{n-1}(1)) = n-1$ and $p(C_{n-1})(2)) = p(C_{n-1}(3)) = n-2$.

The construction of the following graphs are illustrated in Figure 1. For an $i \in \{1, \ldots, n-1\}$, $P_{n-1}(i-1, i, i+1)$ is the graph obtained from P_{n-1} by adding a new vertex which is adjacent to the vertices v_{i-1}, v_i, v_{i+1}; $P_{n-1}(i-1, i+1)$ is the graph obtained from P_{n-1} by adding a new vertex which is adjacent to the vertices v_{i-1}, v_{i+1}; $P_{n-1}(i, i+1)$ is the graph obtained from P_{n-1} by adding a new vertex which is adjacent to the vertices v_i, v_{i+1}; For $j \in \{2, \ldots, n-2\}$, $P_{n-1}(j)$ is the graph obtained from P_{n-1} by adding a new vertex which adjacent to v_j, where $i-1, i+1$ are taken modulo $n-1$.

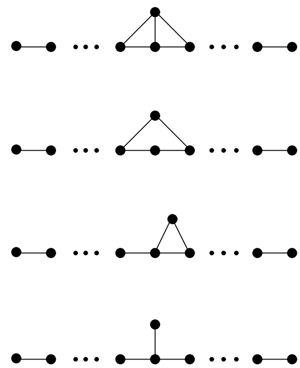

Figure 1. Graphs with an odd order n, $rad(G) = \frac{n-1}{2}$ and $p(G) = n-1$

Note that $P_{n-1}(n-1, n, n+1) = P_{n-1}(n-1, 1, 2) \cong C_n(1)$ and $P_{n-1}(n-1, n) \cong C_n$. It is easy to see that $p(P_{n-1}(i-1, i, i+1)) = p(P_{n-1}(i-1, i+1)) = p(P_{n-1}(i, i+1)) = n-1$ for each $i \in \{1, \dots, n-1\}$, and $p(P_{n-1}(j)) = n-1$ for each $j \in \{2, \dots, n-2\}$.

The result of Lemma 3.5 is straightforward. But its proof is somewhat tedious and will be given in Section 4.

Lemma 3.5. *Let G be a connected graph of order $n \geq 5$. If n is odd and $rad(G) = \frac{n-1}{2}$, then*

(1) $p(G) = n$ if and only if $G \cong P_n$

(2) $p(G) = n-1$ if and only if $G \in \{P_{n-1}(i-1, i, i+1), P_{n-1}(i-1, i+1), P_{n-1}(i, i+1) : i \in \{1, \dots, n-1\}\}$, or $G \cong P_{n-1}(j)$ for some $j \in \{2, \dots, n-2\}$.

(3) $p(G) = n-2$ if and only if $G \in \{C_{n-1}(2), C_{n-1}(3)\}$.

Corollary 3.6. *Let G be a connected graph of order $n \geq 5$. If n is odd and $rad(G) = \frac{n-1}{2}$, then $\rho(G) \geq \frac{n+1}{4}$, with equality if and only if*

$$G \in \{C_n, C_n(1), C_n(2), C_n(3)\}.$$

Proof. By Lemma 3.5, we consider the following cases. If $G \cong P_n$, then

$$\rho(G) = \frac{1}{n-1}\sum_{i=1}^{n-1} i = \frac{n}{2} > \frac{n+1}{4}.$$

Assume that either $G \cong P_{n-1}(1, 2)$ or $G \in \{P_{n-1}(i-1, i, i+1), P_{n-1}(i-1, i+1), P_{n-1}(i, i+1), P_{n-1}(i) : i \in \{2, \dots, n-2\}$. Let $P = v_1 \dots v_{n-1}$ be the induced path of G, and v_n be the new vertex, added to P in the construction of G. Since $n \geq 5$,

$$\rho(G) \geq \rho(v_1) > \frac{1}{n-1}\sum_{i=1}^{n-2} i = \frac{n-2}{2} \geq \frac{n+1}{4}.$$

We saw that $P_{n-1}(n-1, 1) \cong C_n$, $P_{n-1}(n-1, n, n+1) \cong C_n(1) \cong P_{n-1}(n-1, 1, 2)$. It is easy to check that $\rho(G) = \frac{n+1}{4}$ for $G \in \{C_n, C_n(1), C_n(2), C_n(3)\}$ and $\rho(P_{n-1}(2, n-1)) = \rho(P_{n-1}(n-2, 1)) > \frac{n+1}{4}$. □

Now we are ready to prove Conjecture B.

Theorem 3.7. *Let G be a connected graph on $n \geq 3$ vertices with remoteness ρ and radius r. Then*

$$\rho - r \geq \begin{cases} \frac{n^2}{4n-4} - \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{3-n}{4}, & \text{if } n \text{ is odd,} \end{cases}$$

with equality if and only if

$$\begin{cases} G \cong C_n, & \text{if } n \text{ is even,} \\ G \in \{C_n, C_{n-1}(1), C_{n-1}(2), C_{n-1}(3)\}, & \text{if } n \text{ is odd .} \end{cases}$$

Proof. If $n = 3$, then $G \cong P_3$ or $G \cong K_3$. Since $\rho(P_3) = \frac{3}{2}$, $\rho(K_3) = 1$, and $rad(P_3) = rad(K_3) = 1$,

$$\rho - r \geq 0,$$

the result holds. Next we assume that $n \geq 5$, and consider $r - \rho$, instead of $\rho - r$. Let u be a center of G, and $n_i = n_i(u)$ for each $i \in \{1, \ldots, r\}$.

Define a function $f(r) = r - \frac{1}{n-1}(n - 2r + r^2)$. By Corollary 3.2, since $r \leq \frac{n}{2}$, $f'(r) = 1 - \frac{1}{n-1}(2r - 2) > 0$. Thus $f(r)$ is a strictly increasing function on the interval $[1, \frac{n}{2}]$, and achieves its maximum value $\frac{n}{2} - \frac{n^2}{4n-4}$ at $r = \frac{n}{2}$.

Case 1. n is even

By Lemma 3.1, $n_i \geq 2$ for each $i \in \{1, \ldots, r-1\}$. Therefore,

$$\begin{aligned} r - \rho &\leq r - \frac{1}{n-1}\sum_{i=1}^{r} in_i \\ &\leq r - \frac{1}{n-1}\Big((n - 2r + 2) + \sum_{i=2}^{r-1} 2i + r\Big) \\ &= r - \frac{1}{n-1}(n - 2r + r^2) \\ &\leq \frac{n}{2} - \frac{n^2}{4n-4}. \end{aligned}$$

By Corollary 3.4, it is easy to check that $r - \rho = \frac{n}{2} - \frac{n^2}{4n-4}$ if and only if $G \cong C_n$.

Case 2. n is odd

By the similar argument as in Case 1, we have

$$r - \rho \leq r - \frac{1}{n-1}(n - 2r + r^2) = f(r).$$

Since $f(r)$ is a strictly increasing function on the interval $[1, \frac{n-1}{2}]$, if $r \leq \frac{n-1}{2} - 1$, then for $n \geq 5$,

$$\begin{aligned} f(\frac{n-1}{2}-1) &= (\frac{n-1}{2}-1) - \frac{1}{n-1}(3+(\frac{n-1}{2}-1)^2) \\ &= \frac{n-1}{4} - \frac{2n-6}{n-1} \\ &< \frac{n-3}{4}. \end{aligned}$$

So, it remains to consider the case when $r = \frac{n-1}{2}$. By Corollary 3.6, since $\rho(G) \geq \frac{n+1}{4}$,

$$r - \rho \leq \frac{n-1}{2} - \frac{n+1}{4} \leq \frac{n-3}{4},$$

with equality if and only if $G \in \{C_n, C_n(1), C_n(2), C_n(3)\}$. □

4 Proof of Lemma 3.5

(1) is trivial.

The sufficiency of (2) is obvious by the construction of those graphs. To show the necessity of (2), let $P = v_1 \dots v_{n-1}$ be an induced path of G and v_n be the remaining vertex of G.

Claim 1. If v_n has two neighbors $v_i, v_j \in N(v_n)$ with $i, j \in \{1, \dots, n-1\}$, then $|i-j| \leq 2$ or $|i-j| \geq n-3 = (n-1)-2$.

Proof of Claim 1. If $n = 5$, the cliam holds trivially. Next we show the claim by contradiction for $n \geq 7$. Suppose that there exist two vertices $v_i, v_j \in N(v_n)$ with $i, j \in \{1, \dots, n-1\}$ such that $3 \leq |i-j| \leq n-4 = (n-1)-3$. Without loss of generality, let $i < j$.

Case 1. $i \geq \frac{n-1}{2}$ or $j \leq \frac{n+1}{2}$

By the symmetry, we just consider the case when $i \geq \frac{n-1}{2}$. Note that

$$d_P(v_{\frac{n-1}{2}}, v_k) < \frac{n-1}{2}$$

for each $k \in \{1, \dots, n-2\}$, $d_P(v_{\frac{n-1}{2}}, v_{n-1}) = \frac{n-1}{2}$, and

$$d_G(v_{\frac{n-1}{2}}, v_n) \leq d_G(v_{\frac{n-1}{2}}, v_i) + 1.$$

Since $3 \leq |i-j| \leq n-4 = (n-1)-3$, we have $d_G(v_{\frac{n-1}{2}}, v_i) \leq \frac{n-1}{2} - 3$, and $d_G(v_{\frac{n-1}{2}}, v_n) \leq \frac{n-1}{2} - 2$. Furthermore

$$d_G(v_{\frac{n-1}{2}}, v_{n-1}) \leq d_P(v_{\frac{n-1}{2}}, v_i) + 2 + d_P(v_j, v_{n-1}) < \frac{n-1}{2}.$$

This proves that $ecc(v_{\frac{n-1}{2}}) < \frac{n-1}{2}$, which contradicts $rad(G) = \frac{n-1}{2}$.

Case 2. $i < \frac{n-1}{2} < \frac{n+1}{2} < j$

We show that $ecc(v_n) < \frac{n-1}{2}$. Let C be the cycle obtained from the segment of P between v_j and v_j adding the vertex v_n and joining it to v_i and v_j. It is clear that the length of C is at most $n-2$. So, for any vertex v on C, $d(v_n, v) \leq \frac{|C|}{2} < \frac{n-1}{2}$. To prove $d(v_n, w) < \frac{n-1}{2}$, it suffices to show that $\max\{d(v_n, v_1), d(v_n, v_{n-1})\} < \frac{n-1}{2}$. This holds, because

$$d_G(v_n, v_1) \leq d_P(v_{\frac{n-1}{2}}, v_1) < \frac{n-1}{2}, \quad d_G(v_n, v_{n-1}) \leq d_P(v_{\frac{n-1}{2}}, v_{n-1}) < \frac{n-1}{2}.$$

So, $ecc(v_n) < \frac{n-1}{2}$, which contradicts $rad(G) = \frac{n-1}{2}$. □

By Claim 1 and $p(G) = n-1$, one has $d(v_n) \leq 3$. Furthermore, if $d(v_n) = 3$, then $N(v_n) = \{v_{i-1}, v_i, v_{i+1}\}$ for some $i \in \{1, \ldots, n-1\}$, and thus $G \cong P_{n-1}(i-1, i, i+1)$. Also, if $d(v_n) = 2$, then $1 \leq |i-j| \leq 2$, and thus $G \in \{P_{n-1}(i, i+1), P_{n-1}(i-1, i+1)\}$ for some $i \in \{1, \ldots, n-1\}$. If $d(v_n) = 1$, then by $p(G) = n-1$, $G \cong P_{n-1}(j)$ for some $j \in \{2, \ldots, n-2\}$. This completes the proof of (2).

The sufficiency of (3) is trivial. Next we show its necessity. By Theorem 2.3, let $P = v_1 \ldots v_{n-2}$ be an induced path of G, and v_{n-1}, v_n the remaining two vertices of G.

Claim 2. Either $N(v_{n-1}) \setminus \{v_n\} = \{v_1, v_{n-2}\}$ or $N(v_n) \setminus \{v_{n-1}\} = \{v_1, v_{n-2}\}$.

Proof of Claim 2. By contradiction, suppose that Claim 2 is not true. If there exist $i, j \in \{2, \ldots, n-3\}$ such that $v_i \in N(v_{n-1})$ and $v_j \in N(v_n)$, $d(v_{\frac{n-1}{2}}, v_k) \leq \frac{n-1}{2} - 1$ for $k \in \{n-1, n\}$. Together this with $d(v_{\frac{n-1}{2}}, v_k) \leq \frac{n-1}{2} - 1$ for $k \in \{1, \ldots, n-2\}$, we have $ecc(v_{\frac{n-1}{2}}) \leq \frac{n-1}{2} - 1$, a contradiction. Hence,

$$either\ N(v_{n-1}) \setminus \{v_n\} \subseteq \{v_1, v_{n-2}\}\ or\ N(v_n) \setminus \{v_{n-1}\} \subseteq \{v_1, v_{n-2}\}.$$

Without loss of generality, assume that $N(v_{n-1}) \setminus \{v_n\} \subseteq \{v_1, v_{n-2}\}$. Since $N(v_{n-1}) \setminus \{v_n\} \neq \{v_1, v_{n-2}\}$ and $p(G) = n-2$, we have $N(v_{n-1}) \setminus \{v_n\} = \emptyset$. Moreover, since G is connected, we conclude that

$$N(v_{n-1}) = \{v_n\}\ and\ N(v_n) \setminus \{v_{n-1}, v_1, v_{n-2}\} \neq \emptyset.$$

If there exists $i \in \{3, \ldots, n-4\}$ such that $v_i \in N(v_n) \setminus \{v_{n-1}, v_1, v_{n-2}\}$, then it follows $d(v_{\frac{n-1}{2}}, v_n) \leq \frac{n-1}{2} - 2$ and thereby $d(v_{\frac{n-1}{2}}, v_{n-1}) \leq \frac{n-1}{2} - 1$. So, $ecc(v_{\frac{n-1}{2}}) < \frac{n-1}{2}$, which contradicts $rad(G) = \frac{n-1}{2}$. This means that

$$N(v_n) \setminus \{v_{n-1}, v_1, v_{n-2}\} \subseteq \{v_2, v_{n-3}\}.$$

Since $N(v_n) \setminus \{v_{n-1}, v_1, v_{n-2}\} \neq \emptyset$, let $v_2 \in N(v_n)$, without loss of generality. If $n = 5$, then by $p(G) = 3$, $v_1, v_3 \in N(v_5)$, and thus $e(v_5) = 1$, a contradiction. For $n \geq 7$, since $p(G) = n-2$, $v_{n-3} \in N(v_n)$ or $v_{n-2} \in N(v_n)$. In both cases, one can see that $ecc(v_n) \leq \max\{\frac{n-3}{2}, 2\} < \frac{n-1}{2}$. This proves Claim 2. □

By Claim 2, let $N(v_{n-1}) \setminus \{v_n\} = \{v_1, v_{n-2}\}$. Since $P = v_1 \ldots v_{n-2}$ is an induced path, $G[\{v_1, \ldots, v_{n-1}\}] \cong C_{n-1}$.

Claim 3. If v_n has two neighbors $v_i, v_j \in N(v_n)$ with $i, j \in \{1, \ldots, n-1\}$, then $|i-j| \leq 2$ or $|i-j| \geq n-3 = (n-1)-2$.

Proof of Claim 3. By contradiction, suppose that v_n has two neighbors $v_i, v_j \in N(v_n)$ with $i, j \in \{1, \ldots, n-1\}$ and $3 \leq |i-j| \leq n-4$. One can see that, for any vertex v_k, $d(v_n, v_k) \leq \max\{\frac{|i-j|+1}{2}, \frac{n-1-|i-j|+1}{2}\} \leq \frac{n-3}{2} < \frac{n-1}{2}$, it means that $ecc(v_n) < \frac{n-1}{2}$, a contradiction. □

By Claim 3 and $p(G) = n-2$, one has $d(v_n) \leq 3$. Furthermore, if $d(v_n) = 3$, then $N(v_n) = \{v_{i-1}, v_i, v_{i+1}\}$ for some $i \in \{1, \ldots, n-1\}$, and thus $G \cong C_{n-1}(3)$. Also, if $d(v_n) = 2$, then $|i-j| = 2$, and thus $G \cong C_{n-1}(2)$. This completes the proof of the necessity of (3).

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