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A classification of the Veldkamp lines of the near hexagon $L_3 \times \mathrm{GQ}(2,2)$

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Abstract

Using a standard technique sometimes (inaccurately) known as Burnside's Lemma, it is shown that the Veldkamp space of the near hexagon $L_3 \times \mathrm{GQ}(2,2)$ features 156 different types of lines. We also give an explicit description of each type of a line by listing the types of the three geometric hyperplanes it consists of and describing the properties of its core set, that is the subset of points of $L_3 \times \mathrm{GQ}(2,2)$ shared by the three geometric hyperplanes in question.

Keywords: Near hexagons, Geometric hyperplanes, Veldkamp spaces.

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1 Introduction

Brouwer *et al.* [1] proved that there are eleven isomorphism types of slim dense near hexagons. Of these eleven, the near hexagons of sizes 27, 45 and 81 are the most promising for physical applications. This paper is devoted to a study of the second of these three examples and its Veldkamp space. The first of the three examples was described in our paper [4], and we plan to study the third case in a future work. The 45 point space we study here is the product $L_3 \times \mathrm{GQ}(2,2)$, where L_3 is the line containing three points and $\mathrm{GQ}(2,2)$ is the generalized quadrangle of order two.

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2 Near polygons, quads, geometric hyperplanes and Veldkamp spaces

In this section we gather all the basic notions and well-established theoretical results that will be needed in the sequel.

A *near polygon* (see, e. g., [3] and references therein) is a connected partial linear space $S = (P, L, I)$, $I \subset P \times L$, with the property that given a point x and a line L, there always exists a unique point on L nearest to x. (Here distances are measured in the point graph, or collinearity graph of the geometry.) If the maximal distance between two points of S is equal to d, then the near polygon is called a near $2d$-gon. A near 0-gon is a point and a near 2-gon is a line; the class of near quadrangles coincides with the class of generalized quadrangles.

A nonempty set X of points in a near polygon $S = (P, L, I)$ is called a subspace if every line meeting X in at least two points is completely contained in X. A subspace X is called geodetically closed if every point on a shortest path between two points of X is contained in X. Given a subspace X, one can define a sub-geometry S_X of S by considering only those points and lines of S that are completely contained in X. If X is geodetically closed, then S_X clearly is a sub-near-polygon of S. If a geodetically closed sub-near-polygon S_X is a non-degenerate generalized quadrangle, then X (and often also S_X) is called a *quad*.

A near polygon is said to have order (s, t) if every line is incident with precisely $s + 1$ points and if every point is on precisely $t + 1$ lines. If $s = t$, then the near polygon is said to have order s. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance two have at least two common neighbours. A near polygon is called *slim* if every line is incident with precisely three points. It is well known (see, e. g., [6]) that there are, up to isomorphism, three slim non-degenerate generalized quadrangles. The (3×3)-grid is the unique generalized quadrangle of order $(2, 1)$, GQ$(2, 1)$. The unique generalized quadrangle of order 2, GQ$(2, 2)$, is the generalized quadrangle of the points and lines of PG(3, 2) that are totally isotropic with respect to a given symplectic form. The points and lines lying on a given nonsingular elliptic quadric of PG$(5, 2)$ define the unique generalized quadrangle of order $(2, 4)$, GQ$(2, 4)$. Any *slim dense* near polygon contains quads, which are necessarily isomorphic to either GQ$(2, 1)$, GQ$(2, 2)$ or GQ$(2, 4)$.

Next, a *geometric hyperplane* of a partial linear space is a proper subspace meeting each line (necessarily in a unique point or the whole line). The set of points at non-maximal distance from a given point x of a dense near polygon S is a hyperplane of S, usually called the *singular* hyperplane (or *perp-set*) with *deepest* point x. Given a hyperplane H (or any subset of points $\mathcal{C}$) of S, one defines the *order* of any of its points as the number of lines through the point that are fully contained in H ($\mathcal{C}$); a point of H ($\mathcal{C}$) is called *deep* if all the lines passing through it are fully contained in H ($\mathcal{C}$). If H is a hyperplane of a dense near polygon S and if Q is a quad of S, then precisely one of the following possibilities occurs: (1) $Q \subseteq H$; (2) $Q \cap H = x^{\perp} \cap Q$ for some point x of Q; (3) $Q \cap H$ is a sub-quadrangle of Q; and (4) $Q \cap H$ is an ovoid of Q. If case (1), case (2), case (3), or case (4) occurs, then Q is called, respectively, *deep*, *singular*, *sub-quadrangular*, or *ovoidal* with respect to H. If S is slim and H_1 and H_2 are its two distinct hyperplanes, then the complement of symmetric difference of H_1 and H_2, $\overline{H_1 \Delta H_2}$, is again a hyperplane; this means that the totality of hyperplanes of a slim near polygon form a vector space over the Galois field with two elements, $\mathbb{F}_2$. In what follows, we shall put $\overline{H_1 \Delta H_2} \equiv H_1 \oplus H_2$ and call it the (Veldkamp) sum of the two hyperplanes.

Finally, we shall introduce the notion of the *Veldkamp space*, $\mathcal{V}(\Gamma)$, of a point-line incidence geometry $\Gamma(P, L)$ [2]. Here, $\mathcal{V}(\Gamma)$ is the space in which (i) a point is a geometric hyperplane of Γ and (ii) a line is the collection $H'H''$ of all geometric hyperplanes H of Γ such that $H' \cap H'' = H' \cap H = H'' \cap H$ or $H = H', H''$, where H' and H'' are distinct points of $\mathcal{V}(\Gamma)$. Following [10, 8], we adopt also here the definition of Veldkamp space given by Buekenhout and Cohen [2] instead of that of Shult [11], as the latter is much too restrictive by requiring any three distinct hyperplanes H', H'' and H''' of Γ to satisfy the following two conditions: i) H' is not properly contained in H'' and ii) $H' \cap H'' \subseteq H'''$ implies $H' \subset H'''$ or $H' \cap H'' = H' \cap H'''$. The two definitions differ in the crucial fact that whereas the Veldkamp space in the sense of Shult is *always* a linear space, that of Buekenhout and Cohen needs not be so; in other words, Shult's Veldkamp lines are always of the form $\{H \in \mathcal{V}(\Gamma) \mid H \supseteq H' \cap H''\}$ for certain geometric hyperplanes H' and H''.

3 The near hexagon $\mathbf{L}_3 \times$GQ(2, 2)

The near hexagon $L_3\times$ GQ$(2, 2)$ has recently [9] caught an attention of theoretical physicists due to the fact that its main constituent, the generalized quadrangle GQ$(2, 2)$, reproduces the commutation relations of the 15 elements of the two-qubit Pauli group (see, e. g., [7]), with each of its ten embedded copies of GQ$(2, 1)$ playing, remarkably, the role of the so-called *Mermin magic square* [5] — the smallest configuration of two-qubit observables furnishing a very important proof of contextuality of quantum mechanics. A well-known construction of GQ$(2, 2)$ identifies the points with two-element subsets of $\{1, 2, 3, 4, 5, 6\}$, with two points being collinear if and only if they are equal or disjoint. The natural action of S_6 on this set of size 6 induces automorphisms of GQ$(2, 2)$. In fact, when considered in this way, S_6 turns out to be the full automorphism group.

It is known that every geometric hyperplane of a slim dense near polygon arises from its universal embedding. It can be shown from this that, equipped with the operation of Veldkamp sum, the Veldkamp space $V_{GQ(2,2)}$ is isomorphic to PG$(4, 2)$, the projective space obtained from a 5-dimensional space over $\mathbb{F}_2$ (see also [10]). It follows that GQ$(2, 2)$ has $2^5 - 1 = 31$ geometric hyperplanes, which turn out to be of three types:

(i) 15 perp-sets, with 7 points each;

(ii) 10 grids (copies of GQ$(2, 1)$), with 9 points each;

(iii) 6 ovoids, with 5 points each.

In other words, there are three orbits of geometric hyperplanes under the action of S_6.

Identifying the points of GQ$(2, 2)$ with two-element subsets of the set $\{1, 2, 3, 4, 5, 6\}$ as described earlier, we find that an example of an ovoid is the set

$$e_1 := \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}\}.$$

The other ovoids, $e_2, e_3, \ldots, e_6$ are obtained from e_1 by acting by the transposition $(1, i)$ for $i = 2, 3, \ldots, 6$ respectively.

The Veldkamp sum $e_i + e_j$ (for $1 \leq i < j \leq 6$) is the perp-set of the point $\{i, j\}$. If we have

$$\{1, 2, 3, 4, 5, 6\} = \{i, j, k, l, m, n\}$$

in some order, then the sum $e_i + e_j + e_k$ is the grid whose elements are the nine points

$$\{\{a, b\} : a \in \{i, j, k\} \text{ and } b \in \{l, m, n\}\}.$$

It follows that the six ovoids are a spanning set for $V_{GQ(2,2)}$. Since each point of GQ$(2,2)$ lies in precisely two ovoids, it follows that we have the relation

$$e_1 + e_2 + e_3 + e_4 + e_5 + e_6 = 0,$$

where 0 denotes the subset of GQ$(2,2)$ consisting of all 15 points. Since we have an isomorphism $V_{GQ(2,2)} \cong$ PG(4, 2), it follows by a counting argument that this is the only nontrivial dependence relation between the e_i, and thus that the ovoids $e_1, \ldots, e_5$ form a basis for $V_{GQ(2,2)}$.

The points of the near hexagon $L_3\times$ GQ$(2,2)$ are simply the 45 ordered pairs (p, q) where p is a point of L_3 and q is a point of GQ$(2,2)$. We call a collection of 15 points (p, q) sharing the same value of p a *layer* of the near hexagon. A layer is an example of a quad in the sense of §2. We imagine that the points of L_3 are arranged vertically, and we will sometimes use terms like “the top quad” to refer to one of the layers of the near hexagon.

Two points (p_1, q_1) and (p_2, q_2) of $L_3\times$ GQ$(2,2)$ are collinear if either

(i) $p_1 = p_2$ and q_1 is collinear to q_2, or

(ii) p_1 is collinear to p_2 and $q_1 = q_2$.

The lines of $L_3\times$ GQ$(2,2)$ are of two types. The *type-one* lines are the 15 lines of the form $\{(p, q) : p \in L_3\}$ for a fixed point $q \in$ GQ$(2,2)$. The *type-two* lines are the 45 lines of the form $\{(p, q) : q \in L\}$ for a fixed $p \in L_3$ and some line L of GQ$(2,2)$.

The near hexagon $L_3\times$ GQ$(2,2)$ has a number of obvious automorphisms. One type of automorphism involves permuting the three GQ$(2,2)$-quads, but making no other changes. The subgroup of all such automorphisms is isomorphic to S_3. Another type of automorphism involves acting diagonally on the three GQ$(2,2)$-quads by S_6, the automorphism group of GQ$(2,2)$. This action commutes with the action of S_3 just mentioned, and produces a group of automorphisms isomorphic to $S_6 \times S_3$. It turns out that this is the full automorphism group, as shown by Brouwer *et al.* [1].

From now on, let us denote the Veldkamp space of $L_3\times$ GQ$(2,2)$ by V. Some features of V are close to obvious, which stems from Sec. 2. One of these is that the intersection of one of the three GQ$(2,2)$-quads with a point of V (regarded as a subset of the 45 points) can take one of two forms. Either the GQ$(2,2)$-quad is completely filled in (i. e., it is deep), or takes the form of one of the geometric hyperplanes of GQ$(2,2)$ (i. e., it is singular, sub-quadrangular or ovoidal). Furthermore, the Veldkamp sum of any two of the layers (regarded as subsets of GQ$(2,2)$ under some obvious identification) must be equal to the third layer. It follows from this that V contains $2^{10} - 1 = 1023$ points.

The above discussion shows that, as an $S_6 \times S_3$-module over $\mathbb{F}_2$, V is isomorphic to $M \otimes N$, where M is the 5-dimensional module for S_6 described earlier, and N is the S_3-module obtained by quotienting the 3-dimensional permutation module $\{f_1, f_2, f_3\}$ for S_3 by the submodule spanned by $f_1 + f_2 + f_3$. The set $\{f_1, f_2\}$ then form a basis for N, and the set

$$\{e_i \otimes f_j : 1 \leq i \leq 5,\ 1 \leq j \leq 2\}$$

forms a basis for V. We will write this basis for short as $\{e_1, \ldots e_{10}\}$, where for $1 \leq i \leq 5$, e_i denotes $e_i \otimes f_1$, and for $6 \leq i \leq 10$, e_i denotes $e_{i-5} \otimes f_2$.

4 The classification of hyperplanes

The geometric hyperplanes of $L_3\times$ GQ$(2,2)$ were classified in [9]. Up to automorphisms, there are eight types of them, denoted by H_1 to H_8 and described in detail in [9, Table 2]. We now explain how these eight types can be reconstructed using the results in the previous section.

The description of the hyperplanes of GQ$(2,2)$ above can be used to identify each hyperplane with one of the 31 nontrivial set partitions of a 6-element into two pieces. If S and T are disjoint nonempty sets for which

$$S \cup T = \{1,2,3,4,5,6\},$$

then we identify the pair $\{S,T\}$ with the hyperplane

$$\sum_{i\in S} e_i = \sum_{j\in T} e_j.$$

If $|S| \geq |T|$, we associate the partition $(|S|,|T|)$ of the number 6 to the set partition $\{S,T\}$. Under these identifications, the partitions of 6 given by $(5,1)$, $(4,2)$ and $(3,3)$ correspond, via set partitions, to ovoids, perp sets and grids, respectively.

The Veldkamp sum operation on $V_{GQ(2,2)}$ described in the previous section may now be defined purely in terms of sets: the Veldkamp sum of the two set partitions $\{A|B\}$ and $\{C|D\}$ is given by

$$\{(A\cap C)\cup(B\cap D)|(A\cap D)\cup(B\cap C)\}.$$

This identification extends to a set-theoretic description of the hyperplanes of $L_3\times$ GQ$(2,2)$. The hyperplanes of this larger space may be put into bijection with ordered quadruples of pairwise disjoint sets (A,B,C,D) such that (a) no three of the sets are empty and (b) the union of the four sets is $\{1,2,3,4,5,6\}$. Such a quadruple corresponds to the hyperplane given by the ordered triple of partitions

$$(\{A\cup B|C\cup D\},\{A\cup C|B\cup D\},\{A\cup D|B\cup C\}).$$

Here, the leftmost component of the ordered triple describes the hyperplane of GQ$(2,2)$ appearing in the uppermost GQ$(2,2)$-quad of $L_3\times$ GQ$(2,2)$, and so on. For example, if the sets C and D are empty, the top GQ$(2,2)$-quad will be deep and the other two will be identical to each other, being either singular, sub-quadrangular or ovoidal.

The correspondence between the ordered quadruples and the hyperplanes is four-to-one, because the quadruples (A,B,C,D), (B,A,D,C), (C,D,A,B) and (D,C,B,A) all index the same hyperplane. It follows that acting by an element of the Klein four-group V_4 on an ordered quadruple leaves the corresponding hyperplane invariant. The group $S_6\times S_4$ acts on the quadruples, where S_6 acts diagonally on each of the set partitions A, B, C and D, and S_4 acts by place permutation. This induces an action of $S_6\times S_4$ on the hyperplanes of $L_3\times$ GQ$(2,2)$, and since the action of $V_4 \leq S_4$ is trivial, this in turn induces an action of $S_6\times(S_4/V_4)\cong S_6\times S_3$ on the hyperplanes, thus recovering the full automorphism group of $L_3\times$ GQ$(2,2)$ in which S_3 acts by permuting the GQ$(2,2)$-quads.

This approach yields another way to deduce that the number of hyperplanes of $L_3\times$ GQ$(2,2)$ is $2^{10}-1$, as follows. There are 4^6 possible quadruples of pairwise disjoint sets (A,B,C,D) whose union is $\{1,2,3,4,5,6\}$, and four of these quadruples have three

Table 1: A classification of geometric hyperplanes of $L_3\times$ GQ(2, 2).

Name	Partition	Orbit size	Stabilizer	Order
H_1	$(3,3)$	30	$(S_3 \wr \mathbb{Z}_2) \times S_2$	144
H_2	$(4,2)$	45	$S_4 \times S_2 \times S_2$	96
H_3	$(5,1)$	18	$S_5 \times S_2$	240
H_4	$(2,2,1,1)$	270	$S_2 \times S_2 \times S_2 \times S_2$	16
H_5	$(2,2,2)$	90	$S_2 \times S_2 \times S_2 \times S_3$	48
H_6	$(3,1,1,1)$	120	$S_3 \times S_3$	36
H_7	$(3,2,1)$	360	$S_3 \times S_2$	12
H_8	$(4,1,1)$	90	$S_4 \times S_2$	48

empty components. Since the correspondence between quadruples and hyperplanes is four-to-one, the number of hyperplanes is $(4^6 - 4)/4$.

The correspondence described above induces a natural correspondence between $S_6 \times S_4$-orbits (or $S_6 \times S_3$-orbits) of hyperplanes on the one hand, and partitions of 6 into two, three or four parts on the other. There are eight such partitions; they are shown in Table 1, together with their orbit sizes, stabilizers isomorphism types, stabilizer orders, and their name in the $H_1 - H_8$ notation of [9, Table 2].

5 Counting and classifying different types of Veldkamp lines

The orbits of lines in the Veldkamp space V may be enumerated using a standard technique sometimes (inaccurately) known as Burnside's Lemma, which proves the following.

Let G be a finite group acting on a finite set X with t orbits, and for each $g \in G$, let X^g denote the number of elements of X fixed by g. Then we have $t = \frac{1}{|G|}\sum_{g\in G}|X^g|$. Furthermore, if $\mathcal{C}$ is a set of conjugacy class representatives of G, then we have

$$t = \frac{1}{|G|}\sum_{g\in\mathcal{C}}|\mathcal{C}||X^g|.$$

Using this technique, we can recover known results about orbits of lines under the action of the automorphism group S_6 of GQ(2, 2): there are 3 orbits of hyperplanes (Veldkamp points) and 5 orbits of Veldkamp lines. We can also recover the result the Veldkamp space V has 8 orbits of hyperplanes under the automorphism group $S_6 \times S_3$.

The same idea can be adapted to count the orbits of Veldkamp lines of V. The counting argument is more complicated than for the case of Veldkamp points, because it is possible for a line to be fixed by a group element g without the three individual points being fixed. There are three possibilities to consider, which we denote by (1), (2) and (3) in Table 2.

(1) Every point of the Veldkamp line is fixed by g. Such lines lie entirely within the fixed point space of g. Each number in the Fix(1) column is the number of lines in a projective space PG$(d(g) - 1, 2)$, for a suitable integer $d(g)$ depending on the conjugacy class of g.

(2) One point of the Veldkamp line is fixed by g, and the other two are exchanged. To enumerate such lines, we take one point x *outside* the fixed point space of g. The

other two points are the point $g(x)$, and the point collinear with both of them (which is fixed by g). We then divide by 2 to correct for the overcount.

Writing $d(g)$ as above, it follows in each case that the entry in the Fix(2) column of g is given by

$$\frac{1}{2}\left(2^{d(g^2)} - 2^{d(g)}\right).$$

(3) The element g rotates the three points of the Veldkamp line in a 3-cycle. Each entry in the Fix(3) column is a number of the form $(4^k - 1)/3$, and the enumeration of these cases is the most complicated. An ordered Veldkamp line can be thought of as a sequence of 30 binary digits. Typically, some even number, $2k$, of these bits can be chosen arbitrarily, provided that not all of them are zero, and then the rest of the structure is forced. It is then necessary to divide by 3 to correct for an overcount, by identifying an ordered Veldkamp line with each of its cyclic shifts.

We identify the group $S_6 \times S_3$ in the obvious way with the subgroup of S_9 fixing setwise each of the subsets $\{1, 2, 3, 4, 5, 6\}$ and $\{7, 8, 9\}$. Since there are 11 partitions of 6 and 3 partitions of 3, it follows that $S_6 \times S_3$ has 33 conjugacy classes, and it is straightforward to find conjugacy class representatives. Table 2 shows the calculation for the Veldkamp lines of $L_3 \times$ GQ$(2, 2)$. The grand total of

$$673920 = |S_6 \times S_3| \times 156 = 720 \times 6 \times 156$$

proves that there are 156 orbits of Veldkamp lines of the near hexagon.

All 156 types are then listed in Table 3. Here, each type is characterized by its composition (columns 9 to 16) and the properties of the core $\mathcal{C}$ of the line, that is the set of points that are common to all the three geometric hyperplanes of a line of the given type. In particular, for each type (column 1) we list the number of points (column 2) and lines (column 3) of the core as well as the distribution of the orders of its points. The last three columns show the intersection of $\mathcal{C}$ with each of the three GQ$(2, 2)$-quads. Here, 'g-perp' stands for a perp-set in a certain GQ$(2, 1)$ located in the particular GQ$(2, 2)$, and 'unitr/tritr' abbreviates a unicentric/tricentric triad. If two or more types happen to possess the same string of parameters, the distinction between them is given by an explanatory remark/footnote.

Table 2: Orbits of Veldkamp lines of $L_3 \times$ GQ$(2,2)$.

Conjugacy class	Fix(1)	Fix(2)	Fix(3)	Size of class	Product
id	174251	0	0	1	174251
(12)	10795	384	0	15	167685
$(12)(34)$	651	480	0	45	50895
$(12)(34)(56)$	651	480	0	15	16965
(123)	651	0	5	40	26240
$(123)(456)$	1	0	85	40	3440
(1234)	35	24	0	90	5310
$(1234)(56)$	35	24	0	90	5310
$(123)(45)$	35	24	5	120	7680
(12345)	1	0	0	144	144
(123456)	1	0	5	120	720
(78)	155	496	0	3	1953
$(12)(78)$	155	496	0	45	29295
$(12)(34)(78)$	155	496	0	135	87885
$(12)(34)(56)(78)$	155	496	0	45	29295
$(123)(78)$	7	28	1	120	4320
$(123)(456)(78)$	0	1	5	120	720
$(1234)(78)$	7	28	0	270	9450
$(1234)(56)(78)$	7	28	0	270	9450
$(123)(45)(78)$	7	28	1	360	12960
$(12345)(78)$	0	1	0	432	432
$(123456)(78)$	0	1	5	360	2160
(789)	0	0	341	2	682
$(12)(789)$	0	0	85	30	2550
$(12)(34)(789)$	0	0	21	90	1890
$(12)(34)(56)(789)$	0	0	21	30	630
$(123)(789)$	1	0	85	80	6880
$(123)(456)(789)$	35	0	21	80	4480
$(1234)(789)$	0	0	5	180	900
$(1234)(56)(789)$	0	0	5	180	900
$(123)(45)(789)$	1	0	21	240	5280
$(12345)(789)$	0	0	1	288	288
$(123456)(789)$	1	6	5	240	2880
					673920

Table 3: The types of Veldkamp lines of $L_3 \times$ GQ$(2,2)$.

			# of Points of Order					Composition										
Tp	Pt	Ln	0	1	2	3	4	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	1st	2nd	3rd
1	27	27	0	0	0	27	0	3	–	–	–	–	–	–	–	grid	grid	grid
2	25	24	0	0	10	10	5	2	1	–	–	–	–	–	–	full	g-perp	g-perp
3	23	19	0	0	12	11	0	2	–	–	1	–	–	–	–	grid	g-perp	grid
4	21	20	0	0	6	12	3	–	3	–	–	–	–	–	–	full	line	line
5	21	18	0	6	0	12	3	1	1	1	–	–	–	–	–	full	unitr	unitr
6	21	18	0	6	0	12	3	–	3	–	–	–	–	–	–	full	tritr	tritr
7	21	16	0	2	12	6	1	1	1	–	1	–	–	–	–	perp	grid	g-perp
8	21	16	0	0	18	0	3	–	3	–	–	–	–	–	–	perp	perp	perp
9	19	15	0	0	12	7	0	1	–	–	2	–	–	–	–	grid	g-perp	g-perp
10	19	13	0	4	10	5	0	1	–	–	2	–	–	–	–	grid	g-perp	g-perp
11	19	12	0	6	9	4	0	1	1	–	–	–	–	1	–	perp	grid	unitr
12	17	16	0	2	0	14	1	–	1	2	–	–	–	–	–	full	point	point
13	17	12	0	2	12	2	1	–	1	–	2	–	–	–	–	perp	g-perp	g-perp
14	17	12	0	2	11	4	0	–	1	–	2	–	–	–	–	grid	line	g-perp
15	17	10	0	8	6	2	1	1	–	–	1	1	–	–	–	g-perp	g-perp	perp
16	17	10	1	4	10	2	0	1	–	–	1	–	–	1	–	grid	unitr	g-perp
17	17	10	0	8	7	0	2	–	2	–	–	1	–	–	–	perp	line	perp
18	17	10	1	4	10	2	0	–	1	–	2	–	–	–	–	grid	tritr	g-perp
19	17	10	0	8	6	2	1	–	1	–	2	–	–	–	–	perp	g-perp	g-perp
20	17	9	2	6	6	3	0	1	–	1	–	–	–	1	–	ovoid	unitr	grid
21	17	9	0	8	8	1	0	1	–	–	1	–	1	–	–	perp	g-perp	g-perp
22	17	9	0	9	6	2	0	–	2	–	–	–	1	–	–	perp	tritr	perp
23	15	11	0	0	12	3	0	–	–	–	3	–	–	–	–	g-perp	g-perp	g-perp
24	15	9	0	6	6	3	0	1	–	–	–	–	–	2	–	unitr	grid	unitr
25	15	9	0	6	6	3	0	–	–	–	3	–	–	–	–	g-perp[1]	g-perp	g-perp
26	15	9	0	6	6	3	0	–	–	–	3	–	–	–	–	g-perp[1]	g-perp	g-perp
27	15	8	2	4	7	2	0	–	1	–	1	–	–	1	–	grid	tritr	unitr
28	15	8	2	3	9	1	0	–	1	–	1	–	–	1	–	line	grid	unitr
29	15	8	2	4	7	2	0	–	–	1	2	–	–	–	–	grid	unitr	unitr
30	15	8	0	6	9	0	0	–	–	–	3	–	–	–	–	g-perp	g-perp	g-perp
31	15	7	1	8	5	1	0	1	–	–	–	–	1	1	–	perp	g-perp	unitr
32	15	7	4	2	8	1	0	1	–	–	–	–	–	2	–	unitr	grid	unitr
33	15	7	1	8	5	1	0	–	1	–	1	–	–	1	–	perp	unitr	g-perp
34	15	7	0	9	6	0	0	–	–	–	3	–	–	–	–	g-perp	g-perp	g-perp
35	15	6	2	10	1	2	0	1	–	–	–	1	–	1	–	perp	unitr	g-perp
36	15	6	3	6	6	0	0	1	–	–	–	–	–	2	–	ovoid	g-perp	g-perp
37	15	6	2	9	3	1	0	–	1	1	–	–	–	1	–	ovoid	unitr	perp
38	15	5	0	15	0	0	0	–	–	3	–	–	–	–	–	ovoid	ovoid	ovoid
39	13	8	0	4	8	0	1	–	1	–	–	2	–	–	–	perp	line	line
40	13	8	0	3	9	1	0	–	1	–	–	–	–	2	–	line	grid	point
41	13	8	0	4	7	2	0	–	–	–	2	1	–	–	–	line	g-perp	g-perp
42	13	7	2	2	8	1	0	–	–	1	1	–	–	1	–	grid	unitr	point
43	13	6	0	9	3	1	0	–	1	–	–	–	2	–	–	perp	tritr	tritr
44	13	6	0	9	3	1	0	–	1	–	–	–	2	–	–	perp	line	line
45	13	6	4	0	9	0	0	–	1	–	–	–	–	2	–	point	grid	tritr
46	13	6	0	10	2	1	0	–	1	–	–	–	–	2	–	perp	g-perp	point
47	13	6	0	9	3	1	0	–	1	–	–	–	–	2	–	perp	unitr	unitr
48	13	6	1	6	6	0	0	–	–	–	2	–	1	–	–	tritr	g-perp	g-perp
49	13	6	0	8	5	0	0	–	–	–	2	–	1	–	–	line	g-perp	g-perp
50	13	6	1	6	6	0	0	–	–	–	2	–	–	1	–	g-perp	g-perp	unitr

Table 3: (Continued.)

Tp	Pt	Ln	# of Points of Order 0	1	2	3	4	Composition H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	1st	2nd	3rd
51	13	5	2	8	2	1	0	–	1	–	–	1	1	–	–	perp	line	tritr
52	13	5	2	8	2	1	0	–	–	1	1	–	1	–	–	perp	unitr	unitr
53	13	5	2	8	2	1	0	–	–	–	2	1	–	–	–	tritr	g-perp	g-perp
54	13	5	0	11	2	0	0	–	–	–	2	1	–	–	–	line	g-perp	g-perp
55	13	5	2	7	4	0	0	–	–	–	2	–	1	–	–	tritr	g-perp	g-perp
56	13	5	2	8	2	1	0	–	–	–	2	–	–	1	–	g-perp	g-perp	unitr
57	13	5	2	7	4	0	0	–	–	–	2	–	–	1	–	unitr	g-perp	g-perp
58	13	4	4	8	0	0	1	1	–	–	–	1	–	–	1	perp	unitr	unitr
59	13	4	4	8	0	0	1	–	1	1	–	–	–	–	1	perp	ovoid	point
60	13	4	4	8	0	0	1	–	1	–	1	–	–	–	1	perp	unitr	unitr
61	13	4	4	8	0	0	1	–	1	–	–	2	–	–	–	perp	tritr	tritr
62	13	4	4	7	1	1	0	–	1	–	–	–	2	–	–	tritr	tritr	perp
63	13	4	4	7	1	1	0	–	1	–	–	–	–	2	–	line	g-perp	ovoid
64	13	4	4	7	1	1	0	–	1	–	–	–	–	2	–	perp	unitr	unitr
65	13	4	4	6	3	0	0	–	1	–	–	–	–	2	–	tritr	g-perp	ovoid
66	13	4	4	8	0	0	1	–	–	1	1	1	–	–	–	perp	unitr	unitr
67	13	3	6	6	0	1	0	1	–	–	–	–	1	–	1	perp	unitr	unitr
68	13	3	6	6	0	1	0	1	–	–	–	–	–	1	1	ovoid	g-perp	unitr
69	11	6	2	0	9	0	0	–	–	1	–	–	–	2	–	grid	point	point
70	11	5	0	7	4	0	0	–	–	–	1	–	–	2	–	g-perp	g-perp	point
71	11	4	2	7	1	1	0	–	–	1	–	1	–	1	–	perp	unitr	point
72	11	4	2	7	1	1	0	–	–	–	1	1	–	1	–	line	g-perp	unitr
73	11	4	2	6	3	0	0	–	–	–	1	1	–	1	–	line	unitr	g-perp
74	11	4	2	6	3	0	0	–	–	–	1	–	1	1	–	unitr	tritr	g-perp
75	11	4	2	6	3	0	0	–	–	–	1	–	1	1	–	line	unitr	g-perp
76	11	4	2	6	3	0	0	–	–	–	1	–	–	2	–	g-perp[2]	unitr	unitr
77	11	4	2	6	3	0	0	–	–	–	1	–	–	2	–	g-perp[2]	unitr	unitr
78	11	4	1	8	2	0	0	–	–	–	1	–	–	2	–	point	g-perp	g-perp
79	11	3	4	6	0	1	0	–	1	–	–	–	–	1	1	perp	point	unitr
80	11	3	4	6	0	1	0	–	–	1	–	–	1	1	–	perp	unitr	point
81	11	3	2	9	0	0	0	–	–	1	–	–	–	2	–	unitr	unitr	ovoid
82	11	3	4	6	0	1	0	–	–	–	2	–	–	–	1	unitr	g-perp	unitr
83	11	3	4	6	0	1	0	–	–	–	1	1	–	1	–	tritr	unitr	g-perp
84	11	3	4	5	2	0	0	–	–	–	1	–	1	1	–	tritr	g-perp	unitr
85	11	3	3	7	1	0	0	–	–	–	1	–	1	1	–	line	g-perp	unitr
86	11	3	4	6	0	1	0	–	–	–	1	–	–	2	–	unitr[3]	g-perp	unitr
87	11	3	4	6	0	1	0	–	–	–	1	–	–	2	–	unitr[3]	g-perp	unitr
88	11	3	4	5	2	0	0	–	–	–	1	–	–	2	–	g-perp[4]	unitr	unitr
89	11	3	4	5	2	0	0	–	–	–	1	–	–	2	–	g-perp[4]	unitr	unitr
90	11	2	6	4	1	0	0	–	1	–	–	–	–	1	1	line	unitr	ovoid
91	11	2	6	4	1	0	0	–	–	–	2	–	–	–	1	unitr	g-perp	unitr
92	11	2	6	4	1	0	0	–	–	–	1	1	–	1	–	tritr	unitr	g-perp
93	11	2	6	4	1	0	0	–	–	–	1	–	1	1	–	tritr	g-perp	unitr
94	11	2	6	4	1	0	0	–	–	–	1	–	–	2	–	g-perp	unitr	unitr
95	11	1	8	3	0	0	0	–	–	2	–	–	–	–	1	ovoid	point	ovoid
96	11	1	8	3	0	0	0	–	–	1	–	–	–	2	–	unitr	unitr	ovoid
97	11	0	11	0	0	0	0	1	–	–	–	–	–	–	2	unitr	unitr	ovoid
98	11	0	11	0	0	0	0	–	1	–	–	–	–	1	1	tritr	ovoid	unitr
99	9	6	0	0	9	0	0	–	–	–	–	3	–	–	–	line	line	line
100	9	4	0	8	0	0	1	–	1	–	–	–	–	–	2	perp	point	point

Table 3: (Continued.)

Tp	Pt	Ln	# of Points of Order 0	1	2	3	4	Composition H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	1st	2nd	3rd
101	9	3	2	6	0	1	0	–	–	1	–	–	1	–	1	perp	point	point
102	9	3	2	6	0	1	0	–	–	–	1	–	–	1	1	point	g-perp	unitr
103	9	3	0	9	0	0	0	–	–	–	–	3	–	–	–	line	line	line
104	9	3	2	5	2	0	0	–	–	–	–	2	1	–	–	line	tritr	line
105	9	3	0	9	0	0	0	–	–	–	–	1	2	–	–	line	line	line
106	9	3	2	5	2	0	0	–	–	–	–	1	–	2	–	tritr	g-perp	point
107	9	3	1	7	1	0	0	–	–	–	–	1	–	2	–	point	g-perp	line
108	9	3	0	9	0	0	0	–	–	–	–	–	3	–	–	tritr	tritr	tritr
109	9	3	1	7	1	0	0	–	–	–	–	–	1	2	–	point	g-perp	line
110	9	3	0	9	0	0	0	–	–	–	–	–	–	3	–	unitr	unitr	unitr
111	9	2	4	4	1	0	0	–	–	–	1	–	1	–	1	line	unitr	unitr
112	9	2	4	4	1	0	0	–	–	–	1	–	–	1	1	g-perp	point	unitr
113	9	2	4	4	1	0	0	–	–	–	–	1	–	2	–	line	unitr[5]	unitr
114	9	2	4	4	1	0	0	–	–	–	–	1	–	2	–	line	unitr[5]	unitr
115	9	2	4	4	1	0	0	–	–	–	–	1	2	–	–	tritr	tritr	line
116	9	2	3	6	0	0	0	–	–	–	–	–	3	–	–	line	line	tritr
117	9	2	4	4	1	0	0	–	–	–	–	–	1	2	–	tritr	g-perp	point
118	9	2	3	6	0	0	0	–	–	–	–	–	1	2	–	tritr	unitr	unitr
119	9	2	4	4	1	0	0	–	–	–	–	–	–	3	–	point[6]	g-perp	unitr
120	9	2	4	4	1	0	0	–	–	–	–	–	–	3	–	point[6]	g-perp	unitr
121	9	1	6	3	0	0	0	–	–	–	1	1	–	–	1	unitr	line	unitr
122	9	1	6	3	0	0	0	–	–	–	–	3	–	–	–	tritr	tritr	line
123	9	1	6	3	0	0	0	–	–	–	–	1	2	–	–	line	tritr	tritr
124	9	1	6	3	0	0	0	–	–	–	–	1	–	2	–	line	unitr	unitr
125	9	1	6	3	0	0	0	–	–	–	–	–	3	–	–	tritr	tritr	tritr
126	9	1	6	3	0	0	0	–	–	–	–	–	1	2	–	line	unitr	unitr
127	9	1	6	3	0	0	0	–	–	–	–	–	1	2	–	tritr	unitr	unitr
128	9	1	6	3	0	0	0	–	–	–	–	–	–	3	–	unitr	unitr	unitr
129	9	0	9	0	0	0	0	–	1	–	–	–	–	–	2	tritr	point	ovoid
130	9	0	9	0	0	0	0	–	–	1	–	–	–	1	1	ovoid	unitr	point
131	9	0	9	0	0	0	0	–	–	–	1	1	–	–	1	tritr	unitr	unitr
132	9	0	9	0	0	0	0	–	–	–	1	–	1	–	1	tritr	unitr	unitr
133	9	0	9	0	0	0	0	–	–	–	1	–	–	1	1	unitr[7]	unitr	unitr
134	9	0	9	0	0	0	0	–	–	–	1	–	–	1	1	unitr[7]	unitr	unitr
135	9	0	9	0	0	0	0	–	–	–	–	2	1	–	–	tritr	tritr	tritr
136	9	0	9	0	0	0	0	–	–	–	–	1	–	2	–	tritr	unitr	unitr
137	9	0	9	0	0	0	0	–	–	–	–	–	1	2	–	tritr	unitr	unitr
138	9	0	9	0	0	0	0	–	–	–	–	–	–	3	–	unitr	unitr	unitr
139	7	2	2	4	1	0	0	–	–	–	–	1	–	1	1	point	unitr	line
140	7	2	2	4	1	0	0	–	–	–	–	–	–	2	1	point	g-perp	point
141	7	1	4	3	0	0	0	–	–	1	–	–	–	–	2	ovoid	point	point
142	7	1	4	3	0	0	0	–	–	–	–	–	1	1	1	line	unitr	point
143	7	1	4	3	0	0	0	–	–	–	–	–	–	2	1	unitr[8]	unitr	point
144	7	1	4	3	0	0	0	–	–	–	–	–	–	2	1	point	unitr[8]	unitr
145	7	0	7	0	0	0	0	–	–	–	1	–	–	–	2	unitr	unitr	point
146	7	0	7	0	0	0	0	–	–	–	–	1	–	1	1	tritr	point	unitr
147	7	0	7	0	0	0	0	–	–	–	–	–	1	1	1	tritr	point[9]	unitr
148	7	0	7	0	0	0	0	–	–	–	–	–	1	1	1	tritr	point[9]	unitr
149	7	0	7	0	0	0	0	–	–	–	–	–	–	2	1	point[10]	unitr	unitr
150	7	0	7	0	0	0	0	–	–	–	–	–	–	2	1	point[10]	unitr[11]	unitr

Table 3: (Continued.)

Tp	Pt	Ln	# of Points of Order 0	1	2	3	4	Composition H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	1st	2nd	3rd
151	7	0	7	0	0	0	0	–	–	–	–	–	–	2	1	point[10]	unitr[11]	unitr
152	5	1	2	3	0	0	0	–	–	–	–	1	–	–	2	line	point	point
153	5	0	5	0	0	0	0	–	–	–	–	–	1	–	2	tritr	point	point
154	5	0	5	0	0	0	0	–	–	–	–	–	–	1	2	unitr	point	point
155	3	1	0	3	0	0	0	–	–	–	–	–	–	–	3	point	point	point
156	3	0	3	0	0	0	0	–	–	–	–	–	–	–	3	point	point	point

Explanatory remarks:

[1]Two (25) or no two (26) of the g-perps are such that their centers are joined by a type-one line.

[2]The center of the g-perp does (77) or does not (76) lie on the type-one line passing through the center of one of the two unicentric triads.

[3]The centers of the two unicentric triads are (86) or are not (87) joined by a type-one line.

[4]One line (88) or no line (89) of the g-perp is incident with the type-one line passing through the center of one of the two unicentric triads.

[5]The five type-one lines through the points of the two triads do (114) or do not (113) cut a doily-quad in an ovoid.

[6]One line (120) or no line (119) of type-two through the point is incident with the type-one line through the center of the g-perp.

[7]One (133) or none (134) of the unicentric triads is such that the type-one lines through two of its points pass through the centers of the other two triads.

[8]The centers of the two unicentric triads are (143) or are not (144) joined by a type-one line.

[9]The point does (147) or does not (148) lie on the type-one line passing through a center of the tricentric triad.

[10]The point does (149) or does not (150 and 151) lie on the type-one line passing through the center of one of the two unicentric triads.

[11]The centers of the two unicentric triads do (150) or do not (151) belong to the same grid-quad.

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