

On D. G. Higman's note on regular 3-graphs

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Abstract

We introduce the notion of a t -graph and prove that regular 3-graphs are equivalent to cyclic antipodal 3-fold covers of a complete graph. This generalizes the equivalence of regular two-graphs and Taylor graphs. As a consequence, an equivalence between cyclic antipodal distance regular graphs of diameter 3 and certain rank 6 commutative association schemes is proved. New examples of regular 3-graphs are presented.

Keywords: Antipodal graph, association scheme, distance regular graph of diameter 3, Godsil-Hensel matrix, group ring, Taylor graph, two-graph.

Math. Subj. Class.: 05E30, 05B20, 05E18

1 Introduction

This paper is mainly a clarification of [6] — a short draft written by Donald Higman in 1994, entitled “A note on regular 3-graphs”.

The considered generalization of two-graphs was introduced by D. G. Higman in [5]. As in the famous correspondence between two-graphs and switching classes of simple graphs, t -graphs are interpreted as equivalence classes of an appropriate switching relation defined on weights, which play the role of simple graphs.

In his note Higman uses certain association schemes to characterize regular 3-graphs and to obtain feasibility conditions for their parameters. Specifically, he provides a graph theoretic interpretation of a weight and from the resulted graph he constructs a rank 4 symmetric association scheme and a rank 6 fission of it. Furthermore, he proves that rank 6 schemes with parameters as in his construction are equivalent to regular 3-graphs.

During our redetermination of the structure constants of the rank 6 scheme an error in [6] was detected, this miscalculation led Higman to a false restriction on the parameters

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of regular 3-graphs. Our first contribution is the correction of this mistake (see Subsection 5.2). The second contribution is a proof (see Section 4) that in the case of regular 3-graphs, the graph defined by a weight in its switching class is a distance regular cover of the complete graph. Moreover, it is a cyclic antipodal distance regular (ADRG) 3-fold cover of the complete graph in the sense of Godsil and Hensel in [2]. This provides a further restriction on the parameters of regular 3-graphs.

Altogether, in Section 4 and in Section 5 we establish a one-to-one correspondence between regular 3-graphs, cyclic ADRGs of diameter 3 and certain rank 6 association schemes. As a consequence, we provide a new characterization of cyclic antipodal distance regular 3-fold covers of the complete graph in terms of association schemes.

To keep the length of this paper reasonable we did not include all necessary preliminaries. In particular, we assume some knowledge of distance regular graphs, specifically, antipodal distance regular graphs of diameter 3. Also, we assume the reader is familiar with association schemes, in particular, the intersection algebra of an association scheme and its character-multiplicity table. An interested reader may find a more comprehensive consideration of all the diverse links exposed below as well as suggestions for further research in [7].

2 Two-graphs and t -graphs

2.1 Two-graphs and regular two-graphs

Two-graphs have roots originating in diverse areas of combinatorics, geometry and group theory, thus leading to different manifestations in the literature, such as: switching classes of graphs, sets of equidistant points in elliptic geometry, sets of equiangular lines in Euclidean geometry, binary maps of triples with vanishing coboundary, and double coverings of complete graphs (see the celebrated survey [12]). Our focus will be on the last two interpretations and the connection between them. We start with the classical definition and the classical viewpoint of two-graphs as switching classes of simple graphs.

Let X be a set of n elements called *vertices*. For $m \in \mathbb{N}$ denote by $X^{\{m\}}$ the set of all m -subsets of X .

Definition 2.1. A set $\Delta \subseteq X^{\{3\}}$ is a *two-graph* if every 4-subset of X contains an even ($\in \{0, 2, 4\}$) number of members of Δ .

Typically we use the notation (X, Δ) for a two-graph, and call Δ the set of *odd triples*.

Definition 2.2. A two-graph (X, Δ) is called *regular* if every 2-subset $\{x, y\} \in X^{\{2\}}$ is contained in the same number of triples from Δ .

The most famous view of two-graphs is related to a special equivalence relation that is defined on the set of simple (undirected, no loops) graphs. First we remind the reader how to get a two-graph from a graph:

Let $\Gamma = (V, E)$ be a simple graph. The set of triples $\{u, v, w\}$ of vertices, such that the induced subgraph $\Gamma|_{\{u, v, w\}}$ has an odd number of edges, forms a two-graph.

Next, to define the equivalence relation we consider the operation of switching a graph with respect to a set of vertices.

Definition 2.3. Let $X \subseteq V$ be a subset of vertices of a simple graph $\Gamma = (V, E)$. *Switching with respect to X* means interchanging the adjacencies and non-adjacencies between X and its complement $V \setminus X$.

As a more appropriate setting to work with the operation of switching, J. J. Seidel proposed an alternative matrix representation of a simple graph:

Definition 2.4. The *Seidel adjacency matrix* $S = (s_{i,j})$ of a graph $\Gamma = (V, E)$ is a $\{0, -1, 1\}$ -matrix having:

$$s_{i,j} = \begin{cases} 0 & i = j, \\ -1 & \{i, j\} \in E, \\ 1 & \{i, j\} \notin E. \end{cases}$$

In this notation, if the graph Γ' is obtained from Γ by switching with respect to $X \subseteq V$, then its Seidel adjacency matrix S' is obtained from S via a similarity transformation by a diagonal matrix having $\{-1, 1\}$ on its diagonal. Explicitly:

$$S' = DSD,$$

where $D_{i,i} = -1 \iff i \in X$.

As was implied above, switching is an equivalence relation on the set of all simple graphs of order n sharing the same prescribed vertex set. Furthermore we note that switching equivalent graphs give rise to the same two-graph, and have the same Seidel spectrum, thus allowing us to define the *eigenvalues* and their *multiplicities* of a two-graph. To sum up we have:

Theorem 2.5. *There is a 1-1 correspondence between two-graphs and switching classes of graphs.*

Theorem 2.6. *A two-graph is regular if and only if it has two distinct (Seidel) eigenvalues $\rho_1 > 0 > \rho_2$, such that $\rho_1\rho_2 = 1 - |X|$.*

The following is an alternative definition of a two-graph. We call it *the cohomological definition* for reasons that will be clear soon.

Definition 2.7. Let U_2 be the group of square roots of unity. A set $\Delta \subseteq X^{\{3\}}$ is a *two-graph* if the function:

$$f : X^{\{3\}} \longrightarrow U_2$$

defined by

$$f(x) = -1 \iff x \in \Delta,$$

satisfies:

$$f(\{x, y, z\}) \cdot f(\{x, y, t\}) \cdot f(\{x, z, t\}) \cdot f(\{y, z, t\}) = 1$$

for any $\{x, y, z, t\} \in X^{\{4\}}$.

Functions satisfying the equation in the above definition are called 3-cocycles (see below).

It is clear that the two definitions are equivalent. Furthermore, we may refer to either (X, Δ) , Δ or the function f as the two-graph.

2.2 The connection with double covers of complete graphs

Two-graphs were originally introduced by Graham Higman to study 2-transitive representations of certain sporadic groups, in his description he used antipodal 2-fold covers of complete graphs. In [16], Taylor and Levingston established a one-to-one correspondence between two-graphs and antipodal 2-fold covers of complete graphs. This correspondence will be described in a more general setting with all details in the next section. Meanwhile we give an overview for the case of two-graphs.

Let Γ be a graph with n vertices in the switching class of the two-graph f and let S_Γ be the Seidel adjacency matrix of Γ . Then by inserting a 2×2 matrix in the place of each entry of S_Γ according to the following rule:

$$0 \longleftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad 1 \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad -1 \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we obtain a $2n \times 2n$ $\{0, 1\}$ -matrix which is the usual adjacency matrix of the corresponding 2-fold cover of K_n . The converse construction is done in a similar manner: substituting each 2×2 block of the adjacency matrix of a 2-fold cover of K_n (writing it in a suitable ordering of the vertices) with an element of $\{0, 1, -1\}$.

A 2-fold cover of K_n , when it is also distance regular, is called a *Taylor graph*, these are distance regular graphs with intersection array

$$\{k, \mu, 1; 1, \mu, k\}.$$

In the above mentioned correspondence, Taylor graphs correspond to regular two-graphs. This will be a particular case of our more general result later on.

2.3 Generalizing two-graphs

Considering the cohomological definition of two-graphs, two very natural generalizations arise:

- t -cocycles into U_2
(functions $f : X^{\{t\}} \rightarrow U_2$ with a similar property as for two-graphs);
- 3-cocycles into U_t
(functions $f : X^{\{3\}} \rightarrow U_t$, where U_t is the group of t -th roots of unity).

Historically, the first of these was indeed the first to be considered. The first appearance of the term t -graph as a t -cocycle over U_2 is due to D. Higman’s generalization (see [4]) of E. Shult’s graph extension theorem (see [13]). Other sources of this (design theoretical) generalization can be found in Mielants [11] or in [1]. In this case, a regular t -graph is a t -cocycle into U_2 which is also a t -design. Here just few examples are known: regular 3-graphs on 8 and 12 points and a regular 5-graph on 12 points (see [10]). Our interest in the current presentation is the second way to generalize two-graphs, i.e. 3-cocycles into U_t . This direction was examined by D. Higman, and the main source of this is [5]. We begin with introducing some very basic elements of cohomology theory, in which terms t -graphs are defined.

2.4 Some cohomology

Let X be a finite set with $|X| = n$. Let ζ be a primitive root of unity of order t , and let $U_t = \langle \zeta \rangle$ denote the cyclic group of t^{th} roots of unity generated by ζ .

Let $x = (x_1, x_2, \dots, x_p) \in X^p$. A function $f : X^p \rightarrow U_t$ is called a p -cochain if:

- (i) $f(x) = \mathbf{1}$ (the identity element of U_t) for all $x \in X^p$ such that $x_i = x_j$ for some $1 \leq i \neq j \leq p$,
- (ii) if y results from x by interchanging x_i and x_j for some $1 \leq i \neq j \leq p$ then $f(y) = (f(x))^{-1}$.

The set of all p -cochains together with pointwise multiplication forms a group denoted by $C^p(X, U_t)$. Define the *coboundary operator*:

$$\delta : C^p(X, U_t) \rightarrow C^{p+1}(X, U_t)$$

by

$$\delta f(x) = \prod_{i=0}^p \sigma^i(f(\hat{x}_i))$$

where $\hat{x}_i \in X^p$ is obtained from $x \in X^{p+1}$ by deleting the i^{th} coordinate x_i , and σ is the inverse operation of U_t .

For $e \in X$ and $p \geq 1$ we have the group homomorphism

$$\Delta_e : C^p(X, U_t) \rightarrow C^{p-1}(X, U_t)$$

defined by

$$\Delta_e f(x) = f(e, x)$$

for $x \in X^{p-1}$.

Define the set of p -coboundaries:

$$B^p(X, U_t) = \{ \delta f \mid f \in C^p(X, U_t) \},$$

and the set of p -cocycles:

$$Z^p(X, U_t) = \{ f \in C^p(X, U_t) \mid \delta f = \mathbf{1} \}.$$

Here $\mathbf{1}$ is the identity cochain in $C^{p+1}(X, U_t)$. It is routine to check that $\delta^2 f = \mathbf{1}$ for any $(p-1)$ -cochain f , and thus the coboundary of any $(p-1)$ -cochain is a p -cocycle. Two $(p-1)$ -cochains have the same p -cocycle as their coboundary if and only if their quotient is a $(p-1)$ -cocycle. Thus, p -cocycles correspond to cohomology classes of $(p-1)$ -cochains, as a generalization of Seidel switching we call the cohomology classes *switching classes*.

Along the considered generalization of two-graphs and regular two-graphs we define:

Definition 2.8. A t -graph is a 3-cocycle into U_t .

Definition 2.9. A t -graph is called *regular* if for every pair $x, y \in X$, the number of $z \in X \setminus \{x, y\}$ such that $f(x, y, z) = \alpha$ depends only on $\alpha \in U_t$. This number is denoted $m(\alpha)$.

It is easy to check that in case $t = 2$, the definition of a 3-cocycle into U_2 is compatible with the characterization given in Definition 2.7, and that the above definition of regularity is compatible with Definition 2.2.

2.5 Weights

According to Higman, a *weight on X* with values in U_t is a 2-cochain $w \in C^2(X, U_t)$, from this point onward we will call them simply weights. Thus t -graphs are the coboundaries of weights. A weight w can be represented as a $n \times n$ matrix W with entries from U_t where:

$$(W)_{x,y} = w(x, y).$$

Then W has $\mathbf{1}$ on its diagonal, and $W^* = W$, where W^* is obtained from W by transposing and inverting each entry. We will investigate the matrix representation of a weight with much more detail in the next section where we will focus on the case $t = 3$.

Another way to represent a weight is as an antipodal t -fold cover of K_n .

Definition 2.10. Let $w : X^2 \rightarrow U_t$ be a weight on X . To each element $x \in X$ we associate t vertices x_1, x_2, \dots, x_t and define a graph $\Gamma_w = (V, E)$ on the resulting set V of $t|X|$ vertices by

$$\{x_i, y_j\} \in E \iff w(x, y) = \zeta^{j-i}.$$

The resulting graph is a t -fold cover of the complete graph K_n , and if $w(x, y) = \zeta^i$ then the set of edges between x_1, x_2, \dots, x_t and y_1, y_2, \dots, y_t forms a perfect matching which is given by the i^{th} power of the permutation matrix of $(1, 2, \dots, t)$. Permuting x_1, x_2, \dots, x_t according to some power of the permutation $(1, 2, \dots, t)$ amounts to a change of w in its switching class.

3 Regular 3-graphs

From now on we focus on the case $t = 3$. We will prove that the situation for regular 3-graphs generalizes the case of regular two-graphs. In particular, regular 3-graphs are in $1 - 1$ correspondence with regular (cyclic) $(n, 3, c_2)$ -covers.

3.1 Main conventions

Let $w : X^2 \rightarrow U_3$ be a weight on X and $|X| = n$. The coboundary δw of w is a 3-graph $\Phi \in Z^3(X, U_3)$ on X . Assume that Φ is regular. Recall that this means that for every pair x, y of distinct elements of X and $\alpha \in U_3$, the number $m(\alpha)$ of $z \in X \setminus \{x, y\}$ such that $\Phi(x, y, z) = \alpha$ is independent of the choice of x and y .

Denote

$$\begin{aligned} a &:= m(\mathbf{1}), \\ b &:= m(\zeta) = m(\zeta^2). \end{aligned}$$

We call (n, a, b) the *parameters* of the regular 3-graph Φ . We obtain the first restriction on the parameters by simple counting. Fix two vertices $x, y \in X$, then:

$$\begin{aligned} |X \setminus \{x, y\}| &= |\{z \in X \mid \Phi(x, y, z) = \mathbf{1}\}| \\ &\quad + |\{z \in X \mid \Phi(x, y, z) = \zeta\}| + |\{z \in X \mid \Phi(x, y, z) = \zeta^2\}|. \end{aligned}$$

Thus we have

$$n - 2 = a + 2b.$$

The corresponding graph Γ_w is a 3-fold cover of K_n with exactly 3 types of matchings between fibres:

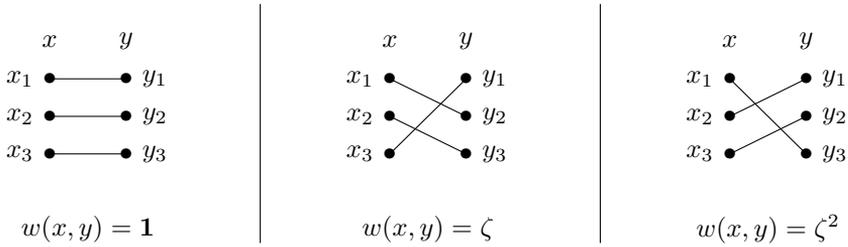


Figure 1: Matchings between fibres of Γ_w

The following subsection serves to remind and fix notation about matrices over the integral group ring, which is the setting in which we characterize regular 3-graphs.

3.2 Matrices over group rings

Let T be a finite group. The elements of the *integral group ring* $\mathbb{Z}[T]$ are expressions of the form

$$\sum_{g \in T} a_g g$$

where $a_g \in \mathbb{Z}$. The ring operations are:

$$\begin{aligned}
 \left(\sum_{g \in T} a_g g \right) + \left(\sum_{g \in T} b_g g \right) &= \sum_{g \in T} (a_g + b_g) g, \\
 \left(\sum_{g \in T} a_g g \right) \cdot \left(\sum_{h \in T} b_h h \right) &= \sum_{g, h \in T} (a_g \cdot b_h) gh.
 \end{aligned}$$

Following the notation of Klin and Pech in [8], for a subset $M \subseteq T$ define the *simple quantity* $\underline{M} \in \mathbb{Z}[T]$:

$$\underline{M} = \sum_{m \in M} 1 \cdot m.$$

When $M = \{g\}$ we will slightly abuse notation and write \underline{g} instead of $\{g\}$. The multiplicative identity of $\mathbb{Z}[T]$ is $\underline{1}$ where $\mathbf{1}$ is the identity of T . The *adjoint* of $\sum_{g \in T} a_g g$ is

$$\left(\sum_{g \in T} a_g g \right)^* = \sum_{g \in T} a_g g^{-1}.$$

The set of $n \times n$ matrices with entries from $\mathbb{Z}[T]$ is denoted by $\mathbb{Z}[T]^{n \times n}$. This set together with usual addition and multiplication of matrices forms a ring with identity. Moreover, $\mathbb{Z}[T]^{n \times n}$ forms a $\mathbb{Z}[T]$ -module, and for a matrix $A = (a_{i,j}) \in \mathbb{Z}[T]^{n \times n}$ we can define the *adjoint* $A^* \in \mathbb{Z}[T]^{n \times n}$, where

$$(a_{i,j})^* = a_{j,i}^*.$$

Recall, that a matrix A is called *self-adjoint* if $A = A^*$.

3.3 Godsil-Hensel matrices

Let Γ be a connected cover of some graph Δ and consider the group T of all automorphisms of Γ that fix each fibre of Γ setwise. Then T acts semi-regularly on $V(\Gamma)$ (cf. [2, Sec. 7]), and in particular on each fibre of Γ . The group T is called the *voltage group* of Γ . If T acts regularly on each fibre, then Γ is called a *regular cover* of Δ .

In [2] Godsil and Hensel studied regular covers in general and in particular gave a characterization of regular antipodal distance regular covers of complete graphs. For this purpose they defined certain matrices over the integral group ring $\mathbb{Z}[T]$, that we will introduce below in the notation used by Klin and Pech in [8].

Let $A = (a_{i,j}) \in \mathbb{Z}[T]^{n \times n}$ be a matrix such that $a_{i,j} \in (\{g \mid g \in T\} \cup \{0\})$, all elements on the diagonal are equal to 0, and such that A is self-adjoint. Then to A we can associate two graphs:

- 1) the underlying graph Δ_A with vertex set $V(\Delta_A) = \{1, \dots, n\}$ and edge set $E(\Delta_A) = \{\{i, j\} \mid a_{i,j} \neq 0\}$,
- 2) the derived graph Γ^A with vertex set $V(\Gamma^A) = \{1, 2, \dots, n\} \times T$ and edge set $E(\Gamma^A) = \{\{(i, g), (j, h)\} \mid a_{i,j} \neq 0, \text{ and } \underline{g} \cdot a_{i,j} = \underline{h}\}$.

Such matrices, when defining connected covers with voltage group T , are called *covering matrices*. When Δ_A is a complete graph K_n , and Γ^A is an (n, r, c_2) -cover of Δ_A then the matrix A is called the *Godsil-Hensel matrix* of the cover.

Theorem 3.1. *Let T be a finite group and let A be a covering matrix of order n over T . Then A is the Godsil-Hensel matrix of a regular antipodal (n, r, c_2) -cover of K_n with voltage group T if and only if*

$$A^2 = (n - 1)I + (n - 2 - rc_2)A + c_2T(J - I). \tag{3.1}$$

4 Main results

Throughout this section we let Φ denote a regular 3-graph, w a weight such that $\delta w = \Phi$. Let Γ_w be the antipodal 3-fold cover of K_n defined by w and let W be the matrix representation of w .

Lemma 4.1. *Let Φ be a regular 3-graph and let w be a weight with $\delta w = \Phi$. Then W satisfies:*

$$W^2 = nI + ((a + 2)\underline{\mathbf{1}} + b\underline{\zeta}, \underline{\zeta}^2) (W - I) \tag{4.1}$$

$$= nI + (a + 2 - b)\underline{\mathbf{1}}(W - I) + b\underline{U}_3(J - I). \tag{4.2}$$

Proof. We calculate $(W^2)_{x,y}$. For $x = y$ we have:

$$\begin{aligned}
 (W^2)_{x,x} &= \sum_{z \in X} (W)_{x,z} \cdot (W)_{z,x} \\
 &= \sum_{z \in X} \underline{w(x,z)} \cdot \underline{w(z,x)} \\
 &= \sum_{z \in X} \underline{w(x,z)} \cdot \underline{w(x,z)}^{-1} \\
 &= \sum_{z \in X} \underline{\mathbf{1}} = n\underline{\mathbf{1}}.
 \end{aligned}$$

For $x \neq y$ we have:

$$\begin{aligned}
 (W^2)_{x,y} &= \sum_{z \in X} (W)_{x,z} \cdot (W)_{z,y} \\
 &= \sum_{z \in X} w(x,z) \cdot w(z,y) \\
 &= \sum_{z \in X} \delta w(y,x,z) \cdot w(x,y) \\
 &= \left(\sum_{z \in X} \delta w(y,x,z) \right) \cdot \underline{w(x,y)} \\
 &= ((m(\mathbf{1}) + 2)\underline{\mathbf{1}} + m(\underline{\zeta})\underline{\zeta} + m(\underline{\zeta}^2)\underline{\zeta}^2) \cdot \underline{w(x,y)} \\
 &= ((a + 2)\underline{\mathbf{1}} + b\underline{\zeta}, \underline{\zeta}^2) \cdot \underline{w(x,y)}.
 \end{aligned}$$

Summing up we get Equation (4.1).

Using:

$$b\underline{\zeta}, \underline{\zeta}^2(W - I) = b\underline{U}_3 J - b\underline{U}_3 I - b\underline{\mathbf{1}}(W - I)$$

we get Equation (4.2). □

Proposition 4.2. *Every regular 3-graph with parameters (n, a, b) defines a cyclic $(n, 3, b)$ -cover.*

Proof. We prove that the matrix $C = W - I$ is the Godsil-Hensel matrix of the cyclic cover Γ_w . We use Equation (4.2) to prove that C satisfies the condition of Theorem 3.1.

$$\begin{aligned}
 C^2 &= (W - I)^2 = W^2 - 2W + I \\
 &= W^2 - 2(W - I) - I \\
 &= nI + ((a + 2)\underline{\mathbf{1}} + b\underline{\zeta}, \underline{\zeta}^2)(W - I) - 2(W - I) - I \\
 &= (n - 1)I + (a\underline{\mathbf{1}} + b\underline{\zeta}, \underline{\zeta}^2)(W - I) \\
 &= (n - 1)I + (a - b)\underline{\mathbf{1}}(W - I) + b\underline{U}_3(J - I) \\
 &= (n - 1)I + (a - b)\underline{\mathbf{1}}C + b\underline{U}_3(J - I).
 \end{aligned}$$

Plugging in the values

$$c_2 = b, \quad a_1 = a, \quad r = 3$$

we obtain Equation (3.1) in Theorem 3.1. □

The converse is proved similarly:

Proposition 4.3. *Every cyclic $(n, 3, c_2)$ -cover defines a regular 3-graph with parameters (n, a_1, c_2) .*

Proof. Let A be the Godsil-Hensel matrix of a cyclic $(n, 3, c_2)$ -cover. We show that $W = A + I$ is the matrix representation of a weight w in the switching class of a regular 3-graph. We have:

$$\begin{aligned} W^2 &= (A + I)^2 = A^2 + 2A + I \\ &= (n - 1)I + (n - 2 - rc_2)A + c_2\underline{T}(J - I) + 2A + I \\ &= nI + (n - rc_2)A + c_2\underline{T}(J - I) \\ &= nI + (n - rc_2)(W - I) + c_2\underline{T}(J - I). \end{aligned}$$

For the values

$$b = c_2, \quad a = a_1, \quad r = 3$$

W satisfies Equation (4.2) in Lemma 4.1. □

To complete the picture we prove:

Proposition 4.4. *There is a 1 – 1 correspondence between regular 3-graphs and cyclic $(n, 3, c_2)$ -covers.*

Proof. Let w and w' be weights into U_3 . All that needs to be shown is:

$$\delta w = \delta w' \iff \Gamma_w \cong \Gamma_{w'}.$$

As was explained after Definition 2.10, the switching of a weight w is interpreted as a cyclic permutation within the fibres of the corresponding cover Γ_w , thus switching equivalent weights yield isomorphic covers. The converse is straightforward. □

As a consequence, using Theorem 9.2 of Godsil and Hensel in [2], we obtain a restriction on the parameter set of a regular 3-graph.

Corollary 4.5. *If (n, a, b) are the parameters of a regular 3-graph then $3|n$.*

Proof. Since Γ_w is a cyclic $(n, 3, b)$ -cover, then by Theorem 9.2 in [2] we have $3|n$. □

5 Higman’s note: clarification and corrections

5.1 Regular 3-graphs and association schemes

Higman’s first step in [6] is to define $\Gamma_w = (V, E)$, an antipodal 3-fold cover of K_n , with fibre set X . He then constructs a rank 4 symmetric association scheme from Γ_w , this association scheme is (in a different ordering than the one that appears in [6]) the metric association scheme of the ADRG Γ_w . Higman’s key observation is the fact that this rank 4 association scheme admits a rank 6 fission by orienting all the non-edges of Γ_w . We present this construction.

Construction 5.1. Define:

$$\begin{aligned} R_0 &= \text{Id}_V, \\ R_1 &= \{(x_i, x_{i+1 \pmod{3}}) \mid i = 1, 2, 3, x \in X\}, \\ R_2 &= \{(x_i, x_{i+2 \pmod{3}}) \mid i = 1, 2, 3, x \in X\}, \\ R_3 &= E, \\ R_4 &= \{(x_i, y_j) \mid i = 1, 2, 3, \{x_{i+1 \pmod{3}}, y_j\} \in E\}, \\ R_5 &= \{(x_i, y_j) \mid i = 1, 2, 3, \{x_{i+2 \pmod{3}}, y_j\} \in E\}. \end{aligned}$$

Remark 5.2. Notice that the relations R_1, R_2, R_4, R_5 are anti-symmetric, $R_1 = R_2^t$ and $R_4 = R_5^t$; Also $S_1 = R_1 \cup R_2$ is the “distance 3” relation and $S_3 = R_4 \cup R_5$ is the “distance 2” relation with respect to $\Gamma_w = (V, E)$.

Proposition 5.3 (Higman). $\mathbb{A}_6(\Gamma) := (V, \{R_i\}_{i=0}^5)$ is an association scheme.

Proof. We calculate the intersection matrices of $\mathbb{A}_6(\Gamma)$. For example, we compute p_{44}^4 : let $(x_i, y_j) \in R_4$ and suppose $w(x_i, y_j) = \zeta$ (we may assume so due to switching), thus $j = i + 1 \pmod{3}$. We count the number of $z_k \in V$ such that $(x_i, z_k) \in R_4$ and $(z_k, y_j) \in R_4$: there are 3 types of $z \in X$ which contain such a z_k :

- $k = i \implies \begin{aligned} w(x, z) &= \zeta^2 \\ w(z, y) &= \zeta \end{aligned} \implies \delta w(x, y, z) = \zeta \cdot \zeta \cdot \zeta^2 = \zeta,$
- $k = i + 1 \pmod{3} \implies \begin{aligned} w(x, z) &= \mathbf{1} \\ w(z, y) &= \mathbf{1} \end{aligned} \implies \delta w(x, y, z) = \zeta \cdot \mathbf{1} \cdot \mathbf{1} = \zeta,$
- $k = i - 1 \pmod{3} \implies \begin{aligned} w(x, z) &= \zeta \\ w(z, y) &= \zeta^2 \end{aligned} \implies \delta w(x, y, z) = \zeta \cdot \zeta^2 \cdot \zeta = \zeta.$

Thus $\delta w(x, y, z) = \zeta \iff z$ is one of the above 3 types, hence $p_{44}^4 = b$.

In the same manner we obtain the intersection matrices $\{B_i\}_{i=0}^5$:

$$\begin{aligned} B_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \\ B_3 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ n-1 & 0 & 0 & a & b & b \\ 0 & n-1 & 0 & b & a & b \\ 0 & 0 & n-1 & b & b & a \end{bmatrix}, B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & n-1 & 0 & b & a & b \\ 0 & 0 & n-1 & b & b & a \\ n-1 & 0 & 0 & a & b & b \end{bmatrix}, \\ B_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & n-1 & b & b & a \\ n-1 & 0 & 0 & a & b & b \\ 0 & n-1 & 0 & b & a & b \end{bmatrix}. \end{aligned}$$

□

It turns out that the existence of such a rank 6 association scheme is a sufficient condition:

Proposition 5.4 (Higman). *Every rank 6 association scheme with parameters as in the construction of $\mathbb{A}_6(\Gamma_w)$ arises from a regular 3-graph.*

Proof. Let $(V, \{R_i\}_{i=0}^5)$ be an association scheme with parameters (and notation) as in Construction 5.1. Define $T = R_0 \cup R_1 \cup R_2$, then T is an equivalence relation on V with equivalence classes of size 3. Denote $X = V/T$. We give a labeling of the elements of V as a 3-fold cover of $K_{|X|}$, and then we verify that in this cover we only have matchings of the 3 types shown in Figure 1. Let $a \in X$ be any fibre, and label its elements by a_1, a_2, a_3 so that $(a_1, a_2) \in R_1$. Then

$$(a_i, a_{i+1 \pmod{3}}) \in R_1$$

for $i = 1, 2, 3$. We now label the elements of each fibre $x \neq a$ in X by x_1, x_2, x_3 so that

$$(a_i, x_i) \in R_3$$

for $i = 1, 2, 3$. To prove that (V, R_3) is a 3-fold cover of $K_{|X|}$ with matchings of the 3 permitted types, we prove two things:

- (1) $(x_i, x_{i+1 \pmod{3}}) \in R_1$ for all $x \in X$ and $i = 1, 2, 3$,
- (2) there is no matching such that $(x_i, y_j) \in R_3$ and $(x_j, y_i) \in R_3$, where $i \neq j$.

Proof of (1): Assume that $(x_i, x_{i+1}) \in R_2$. Let k be such that $(a_i, x_{i+1}) \in R_k$. Then we have:

$$(x_i, a_i) \in R_3, \quad (x_i, x_{i+1}) \in R_2, \quad (x_{i+1}, a_i) \in R_{k'}.$$

Therefore:

$$p_{2k'}^3 \neq 0.$$

Examining column 3 in the matrix B_2 , we deduce that $k' = 4$, which means that $k = 5$. Also, we have:

$$(a_i, x_{i+1}) \in R_k, \quad (a_i, a_{i+1}) \in R_1, \quad (a_{i+1}, x_{i+1}) \in R_3.$$

This implies that:

$$p_{13}^k \neq 0.$$

Examining row 3 in the matrix B_1 , we deduce that $k = 4$, which is a contradiction.

Proof of (2): Assume that $(x_i, y_j) \in R_3$ and $(x_j, y_i) \in R_3$ for some $i \neq j$. Let k be such that $(x_i, y_i) \in R_k$. W.l.o.g we may assume that $j = i + 1 \pmod{3}$. Then we have:

$$(x_i, y_i) \in R_k, \quad (x_i, x_{i+1}) \in R_1, \quad (x_{i+1}, y_i) \in R_3.$$

Therefore:

$$p_{13}^k \neq 0.$$

Examining row 3 in the matrix B_1 we see $k = 4$. Also, we have:

$$(y_i, x_i) \in R_{k'}, \quad (y_i, y_{i+1}) \in R_1, \quad (y_{i+1}, x_i) \in R_3.$$

Thus we obtain:

$$p_{13}^{k'} \neq 0,$$

which implies that $k' = 4$ and $k = 5$, a contradiction.

It follows that all the matchings of the graph (V, R_3) are of the 3 types shown in Figure 1, and we can define a weight w on X by $w(x, x) = 1$ and $w(x, y) = 1, \zeta$ or ζ^2 for $x \neq y$ according to as the matching is of the first, second or third type. It is straightforward to verify that δw is regular. \square

5.2 Characterization and feasibility conditions

We now sum up the results of the previous sections with our characterization of regular 3-graphs. Notice that the equivalence of (ii) and (iii) is a characterization of cyclic $(n, 3, c_2)$ -covers in terms of association schemes.

Corollary 5.5. *Let Γ be an antipodal 3-fold cover of K_n . The following are equivalent:*

- (i) Γ defines a regular 3-graph with parameters (n, a, b) ;
- (ii) Γ is a cyclic $(n, 3, b)$ -cover;
- (iii) $\mathbb{A}_6(\Gamma)$ is an association scheme.

Using this characterization we would like to obtain feasibility restrictions on the parameters (n, a, b) of regular 3-graphs. We begin by calculating the character-multiplicity tables of $\mathbb{A}_4(\Gamma_w)$ and $\mathbb{A}_6(\Gamma_w)$. We used the well-known computer software Mathematica to calculate these tables, the program code is presented in [7].

The character-multiplicity table of $\mathbb{A}_4(\Gamma_w)$ is:

$$\begin{bmatrix} 1 & 2 & n-1 & 2(n-1) \\ 1 & 2 & -1 & -2 \\ 1 & -1 & \alpha & -\alpha \\ 1 & -1 & \beta & -\beta \end{bmatrix} \begin{bmatrix} 1 \\ n-1 \\ z_1 \\ z_2 \end{bmatrix}.$$

Here:

- α and β are the roots of $x^2 - (a-b)x - (n-1) = 0$,
- $z_1 = \frac{2n\beta}{\beta-\alpha}$,
- $z_2 = 2n - z_1 = \frac{2n\alpha}{\alpha-\beta}$.

If $z_1 = z_2 = n$ then we have $\alpha = -\beta$, and $\alpha, \beta = \pm\sqrt{n-1}$.

Otherwise, $z_2 - z_1$ is a non-zero integer, and we have:

$$z_2 - z_1 = 2n \left(\frac{\alpha}{\alpha-\beta} - \frac{\beta}{\beta-\alpha} \right) = 2n \left(\frac{\alpha+\beta}{\alpha-\beta} \right).$$

This means that $\alpha - \beta = \sqrt{(a-b)^2 + 4(n-1)}$ is rational, i.e. $(a-b)^2 + 4(n-1)$ is a square, which implies that α and β are rational algebraic integers, and thus are integers.

The character-multiplicity table of $\mathbb{A}_6(\Gamma_w)$ is:

$$\begin{bmatrix} 1 & 1 & 1 & n-1 & n-1 & n-1 \\ 1 & \zeta & \zeta^2 & \alpha & \alpha\zeta & \alpha\zeta^2 \\ 1 & \zeta^2 & \zeta & \alpha & \alpha\zeta^2 & \alpha\zeta \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & \zeta & \zeta^2 & \beta & \beta\zeta & \beta\zeta^2 \\ 1 & \zeta^2 & \zeta & \beta & \beta\zeta^2 & \beta\zeta \end{bmatrix} \begin{bmatrix} 1 \\ z_1/2 \\ z_1/2 \\ n-1 \\ z_2/2 \\ z_2/2 \end{bmatrix}.$$

Here:

- $\alpha\zeta$ and $\alpha\zeta^2$ are the roots of $x^2 + \alpha x + \alpha^2 = 0$,
- $\beta\zeta$ and $\beta\zeta^2$ are the roots of $x^2 + \beta x + \beta^2 = 0$.

Remark 5.6. In Higman’s note appeared the equations:

- $x^2 - \alpha x + \left(\frac{3(n-1)}{2} + \alpha^2\right) = 0$,
- $x^2 - \beta x + \left(\frac{3(n-1)}{2} + \beta^2\right) = 0$,

which led him to the false conclusion that n must be odd. These equations are the result of a miscalculation of the intersection matrices B_3 and B_5 of $\mathbb{A}_6(\Gamma_w)$ (compare these matrices from our paper with those from the note [6]).

Summing up all the considered restrictions we obtain:

Proposition 5.7. *Necessary conditions for the set (n, a, b) of parameters of a regular 3-graph are:*

- (i) $n = a + 2b + 2$,
- (ii) $3|n$,
- (iii) *The roots α and β of the equation $x^2 - (a - b)x - (n - 1) = 0$ are integers,*
- (iv) $\alpha - \beta$ divides $n\alpha$.

Proof. Item (i) appears in the beginning of Section 3. Item (ii) is Corollary 4.5. Item (iii) comes from the latter analysis of the character-multiplicity table of $\mathbb{A}_4(\Gamma_w)$, and (iv) is just the integrality of the multiplicity $\frac{z_2}{2} = \frac{n\alpha}{\alpha - \beta}$ of $\mathbb{A}_6(\Gamma_w)$. \square

These feasibility conditions provide a list of just 64 feasible parameter sets with $n \leq 1000$. We refer to [7] for the complete list and details about known constructions for some of them.

5.3 The symplectic example

D. G. Higman provided an infinite family of regular 3-graphs which is described briefly below.

In [5], Higman considers a more general cohomological setting, and presents several group theoretic examples of regular 3-cocycles (here cochains are functions into a monoid with the appropriate conditions). These examples are mainly extensions of examples by

D. E. Taylor in [15]. We mention one of them. In this example, we consider weights with values in the additive group of the field $GF(q)$, thus we will use additive notation: C_+^2 instead of C^2 , δ_+ instead of δ . etc.

Let V be a $2m$ -dimensional vector space over $GF(q)$. Let B be a non-degenerate alternating bilinear form on V . Then $B \in C_+^2(V, GF(q))$ is a weight on V with values in $GF(q)$. In case q is a prime, this 3-cocycle is a q -graph. To see that in this case $\Phi = \delta_+ B$ is a regular q -graph we consider the symplectic group $Sp(2m, q)$. It acts transitively on the non-zero vectors of V , thus the subgroup $H := VSp(2m, q)$ of the affine group on V acts 2-transitively on the vectors of V . The coboundary $\Phi = \delta_+ B$ is invariant under translations and is therefore invariant under the action of H on V . This provides an infinite family of regular q -graphs for every prime q .

6 New constructions

The equivalence of regular 3-graphs with parameters (n, a, b) and cyclic $(n, 3, b)$ -covers provides a rich source of new examples of regular 3-graphs.

In their recent paper [8], Klin and Pech present a construction of cyclic $(m^2, 3, \frac{m^2}{3})$ -covers from generalized Hadamard matrices of order m over the cyclic group of order 3; the set of such matrices is denoted by $\text{gH}(U_3, m)$. Their method takes as input any generalized Hadamard matrix $H \in \text{gH}(U_3, m)$ and produces a so-called *skew* generalized Hadamard matrix $W \in \text{gH}(U_3, m^2)$ of order m^2 ; such matrices correspond to cyclic $(m^2, 3, \frac{m^2}{3})$ -covers, this is the Godsil-Hensel matrix of the cover.

We used classifications of generalized Hadamard matrices with suitable parameters (see [3], [9] and [14]) to construct all the corresponding non-isomorphic cyclic covers using the Klin-Pech method, which provide different regular 3-graphs. A summary of our new constructions of regular 3-graphs:

- 1 new example with parameters $(36, 10, 12)$,
- 1 new example with parameters $(45, 19, 12)$ (exceptional),
- 1 new example with parameters $(81, 25, 27)$,
- 1 new example with parameters $(144, 46, 48)$,
- 28 new examples with parameters $(324, 106, 108)$.

For the complete list of feasible parameter sets with $n \leq 1000$, and details about the above examples see [7].

7 Extension to regular t -graphs with $t \geq 4$

The theory outlined in this paper can be extended to regular t -graphs with any $t \geq 4$ only if we impose certain restrictions on the parameters of the regular t -graph. For example, when

t is odd, the parameters of a regular t -graph are:

$$\begin{aligned} &n, \\ &m(1), \\ &m(\zeta) = m(\zeta^{t-1}), \\ &\vdots \\ &m(\zeta^{t-1/2}) = m(\zeta^{t+1/2}). \end{aligned}$$

A graph Γ_w defined by a regular t -graph will be distance regular only if most parameters of the regular t -graph are equal. Explicitly, in the case that t is odd we demand:

$$m(\zeta) = m(\zeta^2) = \dots = m(\zeta^{t-1/2}).$$

In this case, these will also be cyclic covers since for any t we have

$$C_{S_t}(C_t) \cong C_t.$$

Here we use the notation $C_t \leq S_t$ for the cyclic group $C_t = \langle (1, 2, \dots, t) \rangle$.

Higman’s theory also extends to regular t -graphs with $t \geq 4$ in the case of equal parameters (as described above). The construction of $\mathbb{A}_4(\Gamma_w)$ is exactly the same, and it has a rank $2t$ refinement which completely determines the weight w (analogously to $\mathbb{A}_6(\Gamma_w)$ in the case of regular 3-graphs). Thus, the extension of our theory to $t \geq 4$ is described schematically in Figure 2:

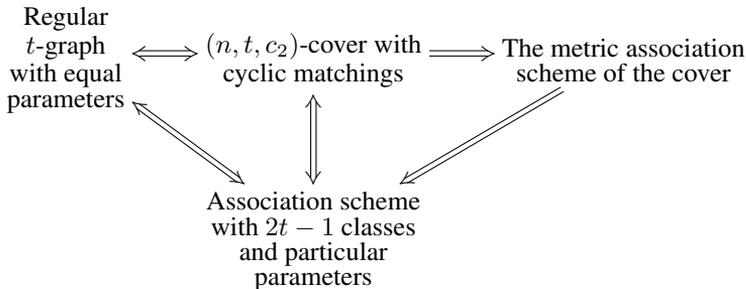


Figure 2: Extension to $t \geq 4$

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