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Complexity of circulant graphs with non-fixed jumps, its arithmetic properties and asymptotics*

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Abstract

In the present paper, we investigate a family of circulant graphs with non-fixed jumps

$$G_n = C_{\beta n}(s_1, \dots, s_k, \alpha_1 n, \dots, \alpha_\ell n),$$
$$1 \le s_1 < \dots < s_k < [\frac{\beta n}{2}],\ 1 \le \alpha_1 < \dots < \alpha_\ell \le [\frac{\beta}{2}].$$

Here n is an arbitrary large natural number and integers $s_1, \dots, s_k, \alpha_1, \dots, \alpha_\ell, \beta$ are supposed to be fixed.

First, we present an explicit formula for the number of spanning trees in the graph G_n. This formula is a product of $\beta s_k - 1$ factors, each given by the n-th Chebyshev polynomial of the first kind evaluated at the roots of some prescribed polynomial of degree s_k. Next, we provide some arithmetic properties of the complexity function. We show that the number of spanning trees in G_n can be represented in the form $\tau(n) = p\, n\, a(n)^2$, where $a(n)$ is an integer sequence and p is a given natural number depending on parity of β and n. Finally, we find an asymptotic formula for $\tau(n)$ through the Mahler measure of the Laurent polynomials differing by a constant from $2k - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$.

Keywords: Spanning tree, circulant graph, Laplacian matrix, Chebyshev polynomial, Mahler measure.

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1 Introduction

The *complexity* of a finite connected graph G, denoted by $\tau(G)$, is the number of spanning trees of G. The famous Kirchhoff's Matrix Tree Theorem [13] states that $\tau(G)$ can be expressed as the product of non-zero Laplacian eigenvalues of G divided by the number of its vertices. Since then, a lot of papers devoted to the complexity of various classes of graphs were published. In particular, explicit formulae were derived for complete multipartite graphs [16], wheels [2], fans [10], prisms [1], anti-prisms [32], ladders [26], Möbius ladders [27], lattices [28] and other families. The complexity of circulant graphs has been the subject of study by many authors [4, 5, 17, 34, 35, 36, 37, 38].

Starting with Boesch and Prodinger [2] the idea to calculate the complexity of graphs by making use of Chebyshev polynomials was implemented. This idea provided a way to find complexity of circulant graphs and their natural generalisations in [4, 14, 19, 25, 36, 38].

Recently, asymptotical behavior of complexity for some families of graphs was investigated from the point of view of so called Malher measure [9, 29, 30]. For general properties of the Mahler measure see, for example [31] and [7]. It worth mentioning that the Mahler measure is related to the growth of groups, values of some hypergeometric functions and volumes of hyperbolic manifolds [3].

For a sequence of graphs G_n, one can consider the number of vertices $v(G_n)$ and the number of spanning trees $\tau(G_n)$ as functions of n. Assuming that $\lim_{n\to\infty}\frac{\log\tau(G_n)}{v(G_n)}$ exists, it is called the thermodynamic limit of the family G_n [20]. This number plays an important role in statistical physics and was investigated by many authors [12, 28, 29, 30, 33].

The purpose of this paper is to present new formulas for the number of spanning trees in circulant graphs with non-fixed jumps and investigate their arithmetical properties and asymptotics. We mention that the number of spanning trees for such graphs was found earlier in [5, 8, 17, 19, 37, 38]. Our results are different from those obtained in the cited papers. Moreover, by the authors opinion, the obtained formulas are more convenient for analytical investigation.

The content of the paper is lined up as follows. Basic definitions and preliminary results are given in Sections 2 and 3. Then, in the Section 4, we present an explicit formula for the number of spanning trees in the undirected circulant graph

$$C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n),$$
$$1 \le s_1 < \ldots < s_k < [\frac{\beta n}{2}],\ 1 \le \alpha_1 < \ldots < \alpha_\ell \le [\frac{\beta}{2}].$$

This formula is a product of $\beta s_k - 1$ factors, each given by the n-th Chebyshev polynomial of the first kind evaluated at the roots of a prescribed polynomial of degree s_k. Through the paper, we will assume that $\beta > 1$ and $\ell > 0$. The case $\beta = 1$ and $\ell = 0$ of the circulant graphs with bounded jumps has been investigated in our previous papers [22, 23].

Next, in the Section 5, we provide some arithmetic properties of the complexity function. More precisely, we show that the number of spanning trees of the circulant graph can be represented in the form $\tau(n) = \beta\, p\, n\, a(n)^2$, where $a(n)$ is an integer sequence and p is a prescribed natural number depending only on parity of n and β. Later, in the Section 6, we use explicit formulas for the number of spanning trees to produce its asymptotics through the Mahler measures of the finite set of Laurent polynomials

$$P_u(z) = 2k - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i}) + 4\sum_{m=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_m}{\beta}),\ u = 0, 1, \ldots, \beta - 1.$$

As a consequence (Corollary 6.2), we prove that the thermodynamic limit of sequence $C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$ as $n \to \infty$ is the arithmetic mean of small Mahler measures of Laurent polynomials $P_u(z),\ u = 0, 1, \ldots, \beta - 1$. In the Section 7, we illustrate the obtained results by a series of examples.

2 Basic definitions and preliminary facts

Consider a connected finite graph G, allowed to have multiple edges but without loops. We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. Given $u, v \in V(G)$, we set a_{uv} to be equal to the number of edges between vertices u and v. The matrix $A = A(G) = \{a_{uv}\}_{u,v \in V(G)}$ is called *the adjacency matrix* of the graph G. The degree $d(v)$ of a vertex $v \in V(G)$ is defined by $d(v) = \sum_{u \in V(G)} a_{uv}$. Let $D = D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv} = d(v)$. The matrix $L = L(G) = D(G) - A(G)$ is called *the Laplacian matrix*, or simply *Laplacian*, of the graph G.

In what follows, by I_n we denote the identity matrix of order n.

Let $s_1, s_2, \ldots, s_k$ be integers such that $1 \le s_1, s_2, \ldots, s_k \le \frac{n}{2}$. The graph $G = C_n(s_1, s_2, \ldots, s_k)$ with n vertices $0, 1, 2, \ldots,\ n-1$ is called *circulant graph* if the vertex $i,\ 0 \le i \le n-1$ is adjacent to the vertices $i \pm s_1, i \pm s_2, \ldots, i \pm s_k \pmod{n}$. All vertices of the graph G have even degree $2k$. If there is i such that $s_i = \frac{n}{2}$ then graph G has multiple edges.

We call an $n \times n$ matrix *circulant,* and denote it by $circ(a_0, a_1, \ldots, a_{n-1})$ if it is of the form

$$circ(a_0, a_1, \ldots, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\ & \vdots & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}.$$

It easy to chose enumeration of vertices such that adjacency and Laplacian matrices for the circulant graph are circulant matrices. The converse is also true. If the Laplacian matrix of a graph is circulant then the graph is also circulant.

In this paper, we consider a particular class of circulant graphs, namely circulant graphs with *non-fixed jumps.* They are defined as before, with special restrictions on the number of vertices and structure of jumps.

More precisely, we will deal with circulant graphs

$$G_n = C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$$

on βn vertices and jumps $s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n$ satisfying the inequalities $1 \le s_1 < \ldots < s_k < [\frac{\beta n}{2}],\ 1 \le \alpha_1 < \ldots < \alpha_\ell \le [\frac{\beta}{2}]$. Mostly, we are interesting in investigation of such graphs for sufficiently large n. In what follows, the numbers $s_1, s_2, \ldots, s_k, \alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta$ are supposed to be fixed positive integers.

In particular, graph G_n has no multiple edges if $\alpha_\ell < \frac{\beta}{2}$. If $\alpha_\ell = \frac{\beta}{2}$, it has exactly two edges between vertices v_i and $v_{i+\frac{\beta n}{2}}$, where indices are taken $\mod \beta n$. In the latter case, β is certainly an even positive integer. A typical example is graph $C_{2n}(1, n)$ which, under the above agreement, represents a Moebius ladder graph on $2n$ vertices with *double* steps. Circulant graphs with non-fixed jumps have been the subject of investigation in many papers [8, 17, 24, 38].

Warning. In series of papers [5, 22, 23, 37] devoted to circulant graphs with odd degree of vertices the notation $C_{2n}(1, n)$ stands for the Moebius ladder with ordinary steps. The degree of vertices of such graph is three. These families of graphs are outside of consideration in the present paper.

Recall [6] that the eigenvalues of matrix $C = circ(a_0, a_1, \ldots, a_{n-1})$ are given by the following simple formulas $\lambda_j = L(\zeta_n^j)$, $j = 0, 1, \ldots, n-1$, where $L(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$ and ζ_n is a primitive n-th root of unity. Moreover, the circulant matrix $C = L(T)$, where $T = circ(0, 1, 0, \ldots, 0)$ is the matrix representation of the shift operator $T\colon (x_0, x_1, \ldots, x_{n-2}, x_{n-1}) \to (x_1, x_2, \ldots, x_{n-1}, x_0)$.

Let $P(z) = a_0 + a_1 z + \ldots + a_d z^d = a_d \prod_{k=1}^{d}(z - \alpha_k)$ be a non-constant polynomial with complex coefficients. Then, following Mahler [21] its *Mahler measure* is defined to be

$$M(P) := \exp(\int_0^1 \log|P(e^{2\pi i t})|dt), \tag{2.1}$$

the geometric mean of $|P(z)|$ for z on the unit circle. However, $M(P)$ had appeared earlier in a paper by Lehmer [15], in an alternative form

$$M(P) = |a_d| \prod_{|\alpha_k|>1} |\alpha_k|. \tag{2.2}$$

The equivalence of the two definitions follows immediately from Jensen's formula [11]

$$\int_0^1 \log|e^{2\pi i t} - \alpha|dt = \log_+|\alpha|,$$

where $\log_+ x$ denotes $\max(0, \log x)$. We will also deal with the *small Mahler measure* which is defined as

$$m(P) := \log M(P) = \int_0^1 \log|P(e^{2\pi i t})|dt.$$

The concept of Mahler measure can be naturally extended to the class of Laurent polynomials $P(z) = a_0 z^p + a_1 z^{p+1} + \ldots + a_{d-1}z^{p+d-1} + a_d z^{p+d} = a_d z^p \prod_{k=1}^{d}(z - \alpha_k)$, where $a_0, a_d \neq 0$ and p is an arbitrary and not necessarily positive integer.

3 Associated polynomials and their properties

The aim of this section is to introduce a few polynomials naturally associated with the circulant graph

$$G_n = C_{\beta n}(s_1, \ldots, s_k, \alpha_1 n, \ldots, \alpha_\ell n),$$
$$1 \le s_1 < \ldots < s_k < [\frac{\beta n}{2}],\ 1 \le \alpha_1 < \ldots < \alpha_\ell \le [\frac{\beta}{2}].$$

We start with the Laurent polynomial

$$L(z) = 2(k+\ell) - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i}) - \sum_{m=1}^{\ell}(z^{\alpha_m n} + z^{-\alpha_m n})$$

responsible for the structure of Laplacian of graph G_n. More precisely, the Laplacian of G_n is given by the matrix

$$\mathbb{L} = L(T) = 2(k+\ell)I_{\beta n} - \sum_{i=1}^{k}(T^{s_i} + T^{-s_i}) - \sum_{m=1}^{\ell}(T^{\alpha_m n} + T^{-\alpha_m n}),$$

where T the circulant matrix $circ(\underbrace{0, 1, \ldots, 0}_{\beta n})$. We decompose $L(z)$ into the sum of two polynomials $L(z) = P(z) + p(z^n)$, where $P(z) = 2k - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$ and $p(z) = 2\ell - \sum_{m=1}^{\ell}(z^{\alpha_m} + z^{-\alpha_m})$. Now, we have to introduce a family of Laurent polynomials differing by a constant from $P(z)$. They are $P_u(z) = P(z) + p(e^{\frac{2\pi\, i\, u}{\beta}})$, $u = 0, 1, \ldots, \beta-1$. One can check that $P_u(z) = 2k - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i}) + 4\sum_{m=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_m}{\beta})$. In particular, $P_0(z) = P(z)$.

We note that all the above Laurent polynomials are *palindromic*, that is they are invariant under replacement z by $1/z$. Any non-trivial palindromic Laurent polynomial can be represented in the form $\mathcal{P}(z) = a_s z^{-s} + a_{s-1}z^{-(s-1)} + \ldots + a_0 + \ldots + a_{s-1}z^{s-1} + a_s z^s$, where $a_s \neq 0$. We will refer to $2s$ as a *degree* of the polynomial $\mathcal{P}(z)$. Since $\mathcal{P}(z) = \mathcal{P}(\frac{1}{z})$, the following polynomial of degree s is well defined

$$\mathcal{Q}(w) = \mathcal{P}(w + \sqrt{w^2 - 1}).$$

We will call it a *Chebyshev trasform* of $\mathcal{P}(z)$. Since $T_k(w) = \frac{(w+\sqrt{w^2-1})^k + (w+\sqrt{w^2-1})^{-k}}{2}$ is the k-th Chebyshev polynomial of the first kind, one can easy deduce that

$$\mathcal{Q}(w) = a_0 + 2a_1T_1(w) + \ldots + 2a_{s-1}T_{s-1}(w) + 2a_sT_s(w).$$

Also, we have $\mathcal{P}(z) = \mathcal{Q}(\frac{1}{2}(z + \frac{1}{z}))$.

Throughout the paper, we will use the following observation. If $z_1, 1/z_1, \ldots, z_s, 1/z_s$ is the list of all the roots of $\mathcal{P}(z)$, then $w_k = \frac{1}{2}(z_k + \frac{1}{z_k})$, $k = 1, 2, \ldots, s$ are all the roots of the polynomial $\mathcal{Q}(w)$.

By direct calculation, we obtain that the Chebyshev transform of polynomial $P_u(z)$ is

$$Q_u(w) = 2k - 2\sum_{i=1}^{k}T_{s_i}(w) + 4\sum_{m=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_m}{\beta}).$$

In particular, if $z_s(u), 1/z_s(u)$, $s = 1, 2, \ldots, s_k$ are the roots of $P_u(z)$, then $w_s(u) = \frac{1}{2}(z_s(u) + z_s(u)^{-1})$, $s = 1, 2, \ldots, s_k$ are all roots of the algebraic equation $\sum_{i=1}^{k}T_{s_i}(w) = k + 2\sum_{m=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_m}{\beta})$. We also need the following lemma.

Lemma 3.1. *Let* $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta) = 1$. *Suppose that* $P_u(z) = 0$, *where* $0 < u < \beta$. *Then* $|z| \neq 1$.

Proof. Recall that $P_u(z) = P(z) + p(e^{\frac{2\pi\, i\, u}{\beta}})$, where $P(z) = 2k - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i})$ and $p(z) = 2\ell - \sum_{m=1}^{\ell}(z^{\alpha_m} + z^{-\alpha_m})$. We show that $p(e^{\frac{2\pi\, i\, u}{\beta}}) = 4\sum_{m=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_m}{\beta}) > 0$. Indeed, suppose that $p(e^{\frac{2\pi\, i\, u}{\beta}}) = 0$. Then there are integers m_j such that $u\,\alpha_j = m_j\beta$, $j = 1, 2, \ldots, \ell$. Hence

$$B = \gcd(u\,\alpha_1, \ldots, u\,\alpha_\ell, u\,\beta) = u\gcd(\alpha_1, \ldots, \alpha_\ell, \beta) = u < \beta.$$

From the other side

$$B = \gcd(m_1\beta, \ldots, m_\ell\beta, u\,\beta) = \beta \gcd(m_1, \ldots, m_\ell, u) \geq \beta.$$

Contradiction. Now, let $|z| = 1$. Then $z = e^{i\varphi},$ for some $\varphi \in \mathbb{R}$. We have

$$P_u(e^{i\varphi}) = P(e^{i\varphi}) + p(e^{\frac{2\pi\, i\, u}{\beta}}) = 2k - \sum_{j=1}^{k}(e^{is_j\varphi} + e^{-is_j\varphi}) + 4\sum_{m=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_m}{\beta})$$

$$= 2\sum_{j=1}^{k}(1 - \cos(s_j\varphi)) + 4\sum_{m=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_m}{\beta}) > 0.$$

Hence, $P_u(z) > 0$ and lemma is proved. □

4 Complexity of circulant graphs with non-fixed jumps

The aim of this section is to find new formulas for the numbers of spanning trees of circulant graph $C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$ in terms of Chebyshev polynomials. It should be noted that nearby results were obtained earlier by different methods in the papers [5, 8, 17, 19, 37, 38].

Theorem 4.1. *The number of spanning trees in the circulant graph with non-fixed jumps*

$$C_{\beta n}(s_1, \ldots, s_k, \alpha_1 n, \ldots, \alpha_\ell n),\ 1 \leq s_1 < \ldots < s_k < [\frac{\beta n}{2}],\ 1 \leq \alpha_1 < \ldots < \alpha_\ell \leq [\frac{\beta}{2}]$$

is given by the formula

$$\tau(n) = \frac{n}{\beta\, q}\prod_{u=0}^{\beta-1}\prod_{\substack{j=1,\\ w_j(0)\neq 1}}^{s_k}|2T_n(w_j(u)) - 2\cos(\frac{2\pi u}{\beta})|,$$

where for each $u = 0, 1, \ldots, \beta - 1$ the numbers $w_j(u)$, $j = 1, 2, \ldots, s_k$, are all the roots of the equation $\sum_{i=1}^{k} T_{s_i}(w) = k + 2\sum_{m=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_m}{\beta})$, $T_s(w)$ is the Chebyshev polynomial of the first kind and $q = s_1^2 + s_2^2 + \ldots + s_k^2$.

Proof. Let $G = C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$. By the celebrated Kirchhoff theorem, the number of spanning trees $\tau(n)$ in G_n is equal to the product of non-zero eigenvalues of the Laplacian of the graph G_n divided by the number of its vertices βn. To investigate the spectrum of Laplacian matrix, we denote by T the $\beta n \times \beta n$ circulant matrix $circ(0, 1, \ldots, 0)$. Consider the Laurent polynomial

$$L(z) = 2(k + \ell) - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i}) - \sum_{m=1}^{\ell}(z^{\alpha_m n} + z^{-\alpha_m n}).$$

Then the Laplacian of G_n is given by the matrix

$$\mathbb{L} = L(T) = 2(k + \ell)I_{\beta n} - \sum_{i=1}^{k}(T^{s_i} + T^{-s_i}) - \sum_{m=1}^{\ell}(T^{\alpha_m n} + T^{-\alpha_m n}).$$

The eigenvalues of the circulant matrix T are $\zeta_{\beta n}^{j}$, $j = 0, 1, \dots, \beta n - 1$, where $\zeta_\ell = e^{\frac{2\pi i}{\ell}}$. Since all of them are distinct, the matrix T is similar to the diagonal matrix $\mathbb{T} = diag(1, \zeta_{\beta n}, \dots, \zeta_{\beta n}^{\beta n-1})$. To find spectrum of $\mathbb{L}$, without loss of generality, one can assume that $T = \mathbb{T}$. Then $\mathbb{L}$ is a diagonal matrix. This essentially simplifies the problem of finding eigenvalues of $\mathbb{L}$. Indeed, let λ be an eigenvalue of $\mathbb{L}$ and x be the respective eigenvector. Then we have the following system of linear equations

$$((2(k+\ell) - \lambda)I_{\beta n} - \sum_{i=1}^{k}(T^{s_i} + T^{-s_i}) - \sum_{m=1}^{\ell}(T^{\alpha_m n} + T^{-\alpha_m n}))x = 0.$$

Let $\mathbf{e}_j = (0, \dots, \underbrace{1}_{j-th}, \dots, 0)$, $j = 1, \dots, \beta n$. The (j,j)-th entry of $\mathbb{T}$ is equal to $\zeta_{\beta n}^{j-1}$. Then, for $j = 0, \dots, \beta n - 1$, the matrix $\mathbb{L}$ has an eigenvalue

$$\lambda_j = L(\zeta_{\beta n}^{j}) = 2(k+\ell) - \sum_{i=1}^{k}(\zeta_{\beta n}^{js_i} + \zeta_{\beta n}^{-js_i}) - \sum_{m=1}^{\ell}(\zeta_{\beta}^{j\alpha_m} + \zeta_{\beta}^{-j\alpha_m}), \tag{4.1}$$

with eigenvector $\mathbf{e}_{j+1}$. Since all graphs under consideration are supposed to be connected, we have $\lambda_0 = 0$ and $\lambda_j > 0$, $j = 1, 2, \dots, \beta n - 1$. Hence

$$\tau(n) = \frac{1}{\beta n}\prod_{j=1}^{\beta n-1} L(\zeta_{\beta n}^{j}). \tag{4.2}$$

By setting $j = \beta t + u$, where $0 \le t \le n-1$, $0 \le u \le \beta - 1$, we rewrite the formula (4.2) in the form

$$\tau(n) = (\frac{1}{n}\prod_{t=1}^{n-1} L(\zeta_{\beta n}^{\beta t}))(\frac{1}{\beta}\prod_{u=1}^{\beta-1}\prod_{t=0}^{n-1} L(\zeta_{\beta n}^{t\beta+u})). \tag{4.3}$$

It is easy to see that $\tau(n)$ is the product of two numbers $\tau_1(n) = \frac{1}{n}\prod_{t=1}^{n-1} L(\zeta_{\beta n}^{\beta t})$ and $\tau_2(n) = \frac{1}{\beta}\prod_{u=1}^{\beta-1}\prod_{t=0}^{n-1} L(\zeta_{\beta n}^{t\beta+u})$.

We note that

$$L(\zeta_{\beta n}^{\beta t}) = 2k - \sum_{i=1}^{k}(\zeta_{\beta n}^{\beta t s_i} + \zeta_{\beta n}^{-\beta t s_i}) = 2k - \sum_{i=1}^{k}(\zeta_{n}^{t s_i} + \zeta_{n}^{-t s_i}) = P(\zeta_n^t),\ 1 \le t \le n-1.$$

The numbers $\mu_t = P(\zeta_n^t)$, $1 \le t \le n-1$ run through all non-zero Laplacian eigenvalues of circulant graph $C_n(s_1, s_2, \dots, s_k)$ with fixed jumps $s_1, s_2, \dots, s_k$ and n vertices. So $\tau_1(n)$ coincide with the number of spanning trees in $C_n(s_1, s_2, \dots, s_k)$. By ([23], Corollary 1) we get

$$\tau_1(n) = \frac{n}{q}\prod_{\substack{j=1,\\ w_j(0)\neq 1}}^{s_k} |2T_n(w_j(0)) - 2|, \tag{4.4}$$

where $w_j(0)$, $j = 1, 2, \dots, s_k$, are all the roots of the equation $\sum_{i=1}^{k} T_{s_i}(w) = k$.

In order to continue the calculation of $\tau(n)$ we have to find the product

$$\tau_2(n) = \frac{1}{\beta}\prod_{u=1}^{\beta-1}\prod_{t=0}^{n-1} L(\zeta_{\beta n}^{t\beta+u}).$$

Recall that $L(z) = P(z) + p(z^n)$. Since $(\zeta_{\beta n}^{\beta t+u})^n = \zeta_{\beta}^{\beta t+u} = \zeta_{\beta}^{u}$, we obtain

$$L(\zeta_{\beta n}^{\beta t+u}) = P(\zeta_{\beta n}^{\beta t+u}) + p(\zeta_{\beta}^{\beta t+u}) = P(\zeta_{\beta n}^{\beta t+u}) + p(\zeta_{\beta}^{u}) = P_u(\zeta_{\beta n}^{\beta t+u}),$$

where $P_u(z) = P(z) + p(\zeta_\beta^u)$. By Section 3, we already know that

$$P_u(z) = -\prod_{j=1}^{s_k}(z - z_j(u))(z - z_j(u)^{-1}),$$

where $w_j(u) = \frac{1}{2}(z_j(u) + z_j(u)^{-1})$, $j = 1, 2, \dots, s_k$ are all roots of the equation $\sum_{i=1}^{k} T_{s_i}(w) = k + 2\sum_{d=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_d}{\beta})$.

We note that $\zeta_{\beta n}^{t\beta+u} = e^{\frac{i(2\pi t+\omega_u)}{n}}$, where $\omega_u = \frac{2\pi u}{\beta}$. Then $\prod_{t=0}^{n-1} L(\zeta_{\beta n}^{t\beta+u}) = \prod_{t=0}^{n-1} P_u(e^{\frac{i(2\pi t+\omega_u)}{n}})$. To evaluate the latter product, we need following lemma.

Lemma 4.2. *Let $H(z) = \prod_{s=1}^{m}(z - z_s)(z - z_s^{-1})$ and ω be a real number. Then*

$$\prod_{t=0}^{n-1} H(e^{\frac{i(2\pi t+\omega)}{n}}) = (-e^{i\omega})^m \prod_{s=1}^{m}(2T_n(w_s) - 2\cos(\omega)),$$

where $w_s = \frac{1}{2}(z_s + z_s^{-1})$, $s = 1, \dots, m$ and $T_n(w)$ is the n-th Chebyshev polynomial of the first kind.

Proof of Lemma 4.2. We note that $\frac{1}{2}(z^n + z^{-n}) = T_n(\frac{1}{2}(z + z^{-1}))$. By the substitution $z = e^{i\,\varphi}$, this follows from the evident identity $\cos(n\varphi) = T_n(\cos\varphi)$. Then we have

$$\begin{aligned}
\prod_{t=0}^{n-1} H(e^{\frac{i(2\pi t+\omega)}{n}}) &= \prod_{t=0}^{n-1}\prod_{s=1}^{m}(e^{\frac{i(2\pi t+\omega)}{n}} - z_s)(e^{\frac{i(2\pi t+\omega)}{n}} - z_s^{-1}) \\
&= \prod_{s=1}^{m}\prod_{t=0}^{n-1}(-e^{\frac{i(2\pi t+\omega)}{n}} z_s^{-1})(z_s - e^{\frac{i(2\pi t+\omega)}{n}})(z_s - e^{-\frac{i(2\pi t+\omega)}{n}}) \\
&= \prod_{s=1}^{m}(-e^{i\omega} z_s{}^{-n}) \prod_{t=0}^{n-1}(z_s - e^{\frac{i(2\pi t+\omega)}{n}})(z_s - e^{-\frac{i(2\pi t+\omega)}{n}}) \\
&= \prod_{s=1}^{m}(-e^{i\omega} z_s{}^{-n})(z_s^{2n} - 2\cos(\omega) z_s^n + 1) \\
&= \prod_{s=1}^{m}(-e^{i\omega})(2\,\frac{z_s^n + z_s^{-n}}{2} - 2\cos(\omega)) \\
&= (-e^{i\omega})^m \prod_{s=1}^{m}(2T_n(w_s) - 2\cos(\omega)).
\end{aligned}$$

□

Since $P_u(z) = -H_u(z)$, where $H_u(z) = \prod_{j=1}^{s_k}(z - z_j(u))(z - z_j(u)^{-1})$, by Lemma 4.2 we get

$$\prod_{t=0}^{n-1} P_u(e^{\frac{i(2\pi t+\omega_u)}{n}}) = (-1)^n(-e^{\frac{2\pi\, i\, u}{\beta}})^{s_k} \prod_{j=1}^{s_k}(2T_n(w_j(u)) - 2\cos(\frac{2\pi\, u}{\beta})).$$

Then,

$$\begin{aligned}
\tau_2(n) &= \frac{1}{\beta}\prod_{u=1}^{\beta-1}\prod_{t=0}^{n-1} L(\zeta_{\beta n}^{\beta t+u}) = \frac{1}{\beta}\prod_{u=1}^{\beta-1}\prod_{t=0}^{n-1} P_u(e^{\frac{i(2\pi j+\omega_u)}{n}}) \\
&= \frac{(-1)^{n(\beta-1)}}{\beta}\prod_{u=1}^{\beta-1}(-e^{\frac{2\pi\, i\, u}{\beta}})^{s_k}\prod_{j=1}^{s_k}(2T_n(w_j(u)) - 2\cos(\frac{2\pi\, u}{\beta})) \qquad (4.5)\\
&= \frac{(-1)^{n(\beta-1)}}{\beta}\prod_{u=1}^{\beta-1}\prod_{j=1}^{s_k}(2T_n(w_j(u)) - 2\cos(\frac{2\pi\, u}{\beta})).
\end{aligned}$$

Since the number $\tau_2(n)$ is a product of positive eigenvalues of G_n divided by β, from (4.5) we have

$$\tau_2(n) = \frac{1}{\beta}\prod_{u=1}^{\beta-1}\prod_{j=1}^{s_k}|2T_n(w_j(u)) - 2\cos(\frac{2\pi\, u}{\beta})|. \qquad (4.6)$$

Combining Equations (4.4) and (4.6) we finish the proof of the theorem. □

As the first consequence from Theorem 4.1 we have the following result obtained earlier by Justine Louis [19] in a slightly different form.

Corollary 4.3. *The number of spanning trees in the circulant graphs with non-fixed jumps $C_{\beta n}(1, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$, where $1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_\ell \le [\frac{\beta}{2}]$ is given by the formula*

$$\tau(n) = \frac{n\, 2^{\beta-1}}{\beta}\prod_{u=1}^{\beta-1}(T_n(1 + 2\sum_{m=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_m}{\beta})) - \cos(\frac{2\pi\, u}{\beta})),$$

where $T_n(w)$ is the Chebyshev polynomial of the first kind.

Proof. Follows directly from the theorem. □

The next corollary is new.

Corollary 4.4. *The number of spanning trees in the circulant graphs with non-fixed jumps $C_{\beta n}(1, 2, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$, where $1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_\ell \le [\frac{\beta}{2}]$ is given by the formula*

$$\tau(n) = \frac{nF_n^2}{\beta}\prod_{u=1}^{\beta-1}\prod_{j=1}^{2}|2T_n(w_j(u)) - 2\cos(\frac{2\pi\, u}{\beta})|,$$

where F_n is the n-th Fibonacci number, $T_n(w)$ is the Chebyshev polynomial of the first kind and $w_{1,2}(u) = \left(-1 \pm \sqrt{25 + 16\sum_{m=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_m}{\beta})}\right)/4$.

We note that nF_n^2 is the number of spanning trees in the graph $C_n(1,2)$.

Proof. In this case, $k=2$, $s_1=1$, $s_2=2$ and $q=s_1^2+s_2^2=5$. Given u we find $w_j(u)$, $j=1,2$ as the roots of the algebraic equation

$$T_1(w)+T_2(w)=2+2\sum_{m=1}^{\ell}\sin^2(\frac{\pi\,u\,\alpha_m}{\beta}),$$

where $T_1(w)=w$ and $T_2(w)=2w^2-1$. For $u=0$ the roots are $w_1(0)=1$ and $w_2(0)=-3/2$. Hence, by (4.4), $\tau_1(n)=\frac{n}{5}|2T_n(-\frac{3}{2})-2|=\frac{n}{5}|(\frac{-3+\sqrt{5}}{2})^n+(\frac{-3-\sqrt{5}}{2})^n-2|=nF_n^2$ gives the well-known formula for the number of spanning trees in the graph $C_n(1,2)$. (See, for example, [2], Theorem 4). For $u>0$ the numbers $w_1(u)$ and $w_2(u)$ are roots of the quadratic equation

$$2w^2+w-3-2\sum_{m=1}^{\ell}\sin^2(\frac{\pi\,u\,\alpha_m}{\beta})=0.$$

By (4.6) we get $\tau_2(n)=\frac{1}{\beta}\prod_{u=1}^{\beta-1}\prod_{j=1}^{2}|2T_n(w_j(u))-2\cos(\frac{2\pi\,u}{\beta})|$. Since $\tau(n)=\tau_1(n)\tau_2(n)$, the result follows. □

5 Arithmetic properties of the complexity for circulant graphs

It was noted in the series of paper [14, 22, 23, 25] that in many important cases the complexity of graphs is given by the formula $\tau(n)=p\,n\,a(n)^2$, where $a(n)$ is an integer sequence and p is a prescribed constant depending only on parity of n.

The aim of the next theorem is to explain this phenomena for circulant graphs with non-fixed jumps. Recall that any positive integer p can be uniquely represented in the form $p=q\,r^2$, where p and q are positive integers and q is square-free. We will call q the *square-free part* of p.

Theorem 5.1. *Let $\tau(n)$ be the number of spanning trees of the circulant graph*

$$G_n=C_{\beta n}(s_1,s_2,\ldots,s_k,\alpha_1 n,\alpha_2 n,\ldots,\alpha_\ell n),$$

where $1\le s_1<s_2<\ldots<s_k<[\frac{\beta n}{2}]$, $1\le\alpha_1<\alpha_2<\ldots,\alpha_\ell\le[\frac{\beta}{2}]$.

Denote by p and q the number of odd elements in the sequences $s_1,s_2,\ldots,s_k$ and $\alpha_1,\alpha_2,\ldots,\alpha_\ell$, respectively. Let r be the square-free part of p and s be the square-free part of $p+q$. Then there exists an integer sequence $a(n)$ such that

1^0 $\tau(n)=\beta\,n\,a(n)^2$, *if n and β are odd;*

2^0 $\tau(n)=\beta\,r\,n\,a(n)^2$, *if n is even;*

3^0 $\tau(n)=\beta\,s\,n\,a(n)^2$, *if n is odd and β is even.*

Proof. The number of odd elements in the sequences $s_1,s_2,\ldots,s_k$ and $\alpha_1,\alpha_2,\ldots,\alpha_\ell$, respectively is counted by the formulas $p=\sum_{i=1}^{k}\frac{1-(-1)^{s_i}}{2}$ and $q=\sum_{i=1}^{\ell}\frac{1-(-1)^{\alpha_i}}{2}$. We already know that all non-zero Laplacian eigenvalues of the graph G_n are given by the formulas $\lambda_j=L(\zeta_{\beta n}^j)$, $j=1,\ldots,\beta n-1$, where $\zeta_{\beta n}=e^{\frac{2\pi i}{\beta n}}$ and

$$L(z)=2(k+l)-\sum_{i=1}^{k}(z^{s_i}+z^{-s_i})-\sum_{m=1}^{\ell}(z^{n\alpha_m}+z^{-n\alpha_m}).$$

We note that $\lambda_{\beta n-j}=L(\zeta_{\beta n}^{\beta n-j})=L(\zeta_{\beta n}^{j})=\lambda_j$.

By the Kirchhoff theorem we have $\beta n\,\tau(n)=\prod_{j=1}^{\beta n-1}\lambda_j$. Since $\lambda_{\beta n-j}=\lambda_j$, we obtain $\beta n\,\tau(n)=(\prod_{j=1}^{\frac{\beta n-1}{2}}\lambda_j)^2$ if βn is odd and $\beta n\,\tau(n)=\lambda_{\frac{\beta n}{2}}(\prod_{j=1}^{\frac{\beta n}{2}-1}\lambda_j)^2$ if βn is even. We note that each algebraic number λ_j comes into the above products together with all its Galois conjugate [18]. So, the numbers $c(n)=\prod_{j=1}^{\frac{\beta n-1}{2}}\lambda_j$ and $d(n)=\prod_{j=1}^{\frac{\beta n}{2}-1}\lambda_j$ are integers. Also, for even n we have

$$\lambda_{\frac{\beta n}{2}}=L(-1)=2(k+l)-\sum_{i=1}^{k}((-1)^{s_i}+(-1)^{-s_i})-\sum_{m=1}^{\ell}((-1)^{n\alpha_m}+(-1)^{-n\alpha_m})$$
$$=2k-\sum_{i=1}^{k}((-1)^{s_i}+(-1)^{-s_i})=4\sum_{i=1}^{k}\frac{1-(-1)^{s_i}}{2}=4p.$$

If n is odd and β is even, the number $\frac{\beta n}{2}$ is integer again. Then we obtain

$$\lambda_{\frac{\beta n}{2}}=L(-1)=2(k+l)-\sum_{i=1}^{k}((-1)^{s_i}+(-1)^{-s_i})-\sum_{m=1}^{\ell}((-1)^{\alpha_m}+(-1)^{-\alpha_m})$$
$$=4\sum_{i=1}^{k}\frac{1-(-1)^{s_i}}{2}+4\sum_{m=1}^{\ell}\frac{1-(-1)^{\alpha_m}}{2}=4p+4q.$$

Therefore, $\beta\,n\,\tau(n)=c(n)^2$ if β and n are odd, $\beta\,n\,\tau(n)=4p\,d(n)^2$ if n is even and $\beta\,n\,\tau(n)=4(p+q)\,d(n)^2$ if n is odd and β is even. Let r be the square-free part of p and s be the square-free part of $p+q$. Then there are integers u and v such that $p=ru^2$ and $s=(p+q)v^2$. Hence,

1° $\dfrac{\tau(n)}{\beta\,n}=\left(\dfrac{c(n)}{\beta\,n}\right)^2$ if n and β are odd,

2° $\dfrac{\tau(n)}{\beta\,n}=r\left(\dfrac{2\,u\,d(n)}{\beta\,n}\right)^2$ if n is even and

3° $\dfrac{\tau(n)}{\beta\,n}=s\left(\dfrac{2\,v\,d(n)}{\beta\,n}\right)^2$ if n is odd and β is even.

Consider an automorphism group $\mathbb{Z}_{\beta n}=\langle g\rangle$ of the graph G_n generated by the element g circularly permuting vertices $v_0,v_1,\ldots,v_{\beta\,n-1}$ by the rule $v_i\to v_{i+1}$ and the addition in the indices is done modulo βn. The action of such a group is uniquely defined on the set of all edges of G_n, except for those that connect diametrically opposite vertices. Consider separately two cases $\alpha_\ell=\beta/2$ and $\alpha_\ell<\beta/2$.

In the first case, we have two parallel edges between the diametrically opposite vertices v_i and $v_{i+\frac{\beta\,n}{2}}$, where the indices are taken mod $\beta\,n$. To distinguish them, we orient one of this edges by the arrow from v_i and $v_{i+\frac{\beta\,n}{2}}$ and the other one by the arrow from $v_{i+\frac{\beta\,n}{2}}$ to v_i. As a result, we get exactly $\beta\,n$ oriented edges. Denote the edge oriented from $v_{i+\frac{\beta\,n}{2}}$ to v_i by e_i and define the action of g on such edges by the rule $e_i\to e_{i+1}$, where i is taken $\bmod\ \beta\,n$.

In the second case, we have $\alpha_\ell < \frac{\beta}{2}$ and $s_k < \frac{\beta\, n}{2}$. Therefore, all jumps $\alpha_1 n, \ldots, \alpha_\ell n$ and $s_1, \ldots, s_k$ of the graph G_n are strictly less then $\frac{\beta\, n}{2}$ and G_n has no edges between the diametrically opposite edges. That is, the action of group $\mathbb{Z}_{\beta n}$ is well defined on its edges.

So, one can conclude that group $\mathbb{Z}_{\beta n}$ acts fixed point free on the set vertices and on the set of edges of G_n.

We are aimed to show that it also acts freely on the set of the spanning trees in the graph. Indeed, suppose that some non-trivial element γ of $\mathbb{Z}_{\beta n}$ leaves a spanning tree A in the graph G_n invariant. Then γ fixes the center of A. The center of a tree is a vertex or an edge. The first case is impossible, since γ acts freely on the set of vertices. In the second case, γ permutes the endpoints of an edge connecting the opposite vertices of G_n. This means that $\beta\, n$ is even, and γ is the unique involution in the group $\mathbb{Z}_{\beta n}$. This is also impossible, since the group is acting without fixed edges.

So, the cyclic group $\mathbb{Z}_{\beta n}$ acts on the set of spanning trees of the graph G_n fixed point free. Therefore $\tau(n)$ is a multiple of $\beta\, n$ and their quotient $\frac{\tau(n)}{\beta\, n}$ is an integer.

Setting $a(n) = \frac{c(n)}{\beta\, n}$ in the case 1°, $a(n) = \frac{2\, u\, d(n)}{\beta\, n}$ in the case 2° and $a(n) = \frac{2\, v\, d(n)}{\beta\, n}$ in the case 3° we conclude that number $a(n)$ is always integer and the statement of the theorem follows. □

6 Asymptotic for the number of spanning trees

In this section, we give asymptotic formulas for the number of spanning trees for circulant graphs. It is interesting to compare these results with those in papers [5, 8, 17, 19, 37], where the similar results were obtained by different methods.

Theorem 6.1. *Let* $\gcd(s_1, s_2, \ldots, s_k) = d$ *and* $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta) = 1$. *Then the number of spanning trees in the circulant graph*

$$C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n),$$
$$1 \le s_1 < s_2 < \ldots < s_k < [\frac{\beta n}{2}],\ 1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_\ell \le [\frac{\beta}{2}],$$

has the following asymptotic

$$\tau(n) \sim \frac{n\, d^2}{\beta\, q} A^n, \text{ as } n \to \infty \text{ and } (n, d) = 1,$$

where $q = s_1^2 + s_2^2 + \ldots + s_k^2$, $A = \prod_{u=0}^{\beta-1} M(P_u)$ *and* $M(P_u) = \exp(\int_0^1 \log|P_u(e^{2\pi i t})|dt)$ *is the Mahler measure of Laurent polynomial*

$$P_u(z) = 2k - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i}) + 4\sum_{m=1}^{\ell} \sin^2(\frac{\pi u \alpha_m}{\beta}).$$

Proof. By Theorem 4.1, $\tau(n) = \tau_1(n)\tau_2(n)$, where $\tau_1(n)$ is the number of spanning trees in $C_n(s_1, s_2, \ldots, s_k)$ and $\tau_2(n) = \frac{1}{\beta}\prod_{u=1}^{\beta-1}\prod_{j=1}^{s_k} |2T_n(w_j(u)) - 2\cos(\frac{2\pi\, u}{\beta})|$. By ([23], Theorem 5) we already know that

$$\tau_1(n) \sim \frac{n\, d^2}{q} A_0^n, \text{ as } n \to \infty \text{ and } (n, d) = 1,$$

where A_0 is the Mahler measure of Laurent polynomial $P_0(z)$. So, we have to find asymptotics for $\tau_2(n)$ only.

By Lemma 3.1, for any integer u, $0 < u < \beta$ we obtain $T_n(w_j(u)) = \frac{1}{2}(z_j(u)^n + z_j(u)^{-n})$, where the $z_j(u)$ and $1/z_j(u)$ are roots of the polynomial $P_u(z)$ satisfying the inequality $|z_j(u)| \neq 1$, $j = 1, 2, \ldots, s_k$. Replacing $z_j(u)$ by $1/z_j(u)$, if necessary, we can assume that $|z_j(u)| > 1$ for all $j = 1, 2, \ldots, s_k$. Then $T_n(w_j(u)) \sim \frac{1}{2} z_j(u)^n$, as n tends to ∞. So $|2T_n(w_j(u)) - 2\cos(\frac{2\pi u}{\beta})| \sim |z_j(u)|^n$, $n \to \infty$. Hence

$$\prod_{j=1}^{s_k} |2T_n(w_j(u)) - 2\cos(\frac{2\pi\, u}{\beta})| \sim \prod_{s=1}^{s_k} |z_j(u)|^n = \prod_{P_u(z)=0,\,|z|>1} |z|^n = A_u^n,$$

where $A_u = \prod_{P_u(z)=0,\,|z|>1}|z|$ coincides with the Mahler measure of $P_u(z)$. As a result,

$$\tau_2(n) = \frac{1}{\beta}\prod_{u=1}^{\beta-1}\prod_{j=1}^{s_k} |2T_n(w_j(u)) - 2\cos(\frac{2\pi\, u}{\beta})| \sim \frac{1}{\beta}\prod_{u=1}^{\beta-1} A_u^n.$$

Finally, $\tau(n) = \tau_1(n)\tau_2(n) \sim \frac{n\,d^2}{\beta\, q}\prod_{u=0}^{\beta-1} A_u^n$, as $n \to \infty$ and $(n, d) = 1$. Since $A_u = M(P_u)$, the result follows. $\square$

As an immediate consequence of above theorem we have the following result obtained earlier in ([8], Theorem 3) by completely different methods.

Corollary 6.2. *The thermodynamic limit of the sequence* $C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n)$ *of circulant graphs is equal to the arithmetic mean of small Mahler measures of Laurent polynomials* $P_u(z)$, $u = 0, 1, \ldots, \beta - 1$. *More precisely,*

$$\lim_{n\to\infty}\frac{\log\tau(C_{\beta n}(s_1, s_2, \ldots, s_k, \alpha_1 n, \alpha_2 n, \ldots, \alpha_\ell n))}{\beta\, n} = \frac{1}{\beta}\sum_{u=0}^{\beta-1} m(P_u),$$

where $m(P_u) = \int_0^1 \log|P_u(e^{2\pi i t})|dt$ *and* $P_u(z) = 2k - \sum_{i=1}^{k}(z^{s_i} + z^{-s_i}) + 4\sum_{m=1}^{\ell}\sin^2(\frac{\pi\, u\, \alpha_m}{\beta})$.

7 Examples

1. **Graph** $C_{2n}(1, n)$. (Möbius ladder with double steps). By Theorem 4.1, we have $\tau(n) = \tau(C_{2n}(1, n)) = n\,(T_n(3) + 1)$. Compare this result with ([38], Theorem 4). Recall [2] that the number of spanninig trees in the Möbius ladder with single steps is given by the formula $n\,(T_n(2) + 1)$.

2. **Graph** $C_{2n}(1, 2, n)$. We have $\tau(n) = 2nF_n^2|T_n(\frac{-1-\sqrt{41}}{4}) - 1||T_n(\frac{-1+\sqrt{41}}{4}) - 1|$. By Theorem 5.1, one can find an integer sequence $a(n)$ such that $\tau(n) = 2n\,a(n)^2$ if n is even and $\tau(n) = na(n)^2$ if n is odd.

3. **Graph** $C_{2n}(1, 2, 3, n)$. Here $\tau(n) = \frac{8n}{7}(T_n(\theta_1)-1)(T_n(\theta_2)-1)\prod_{p=1}^{3}(T_n(\omega_p)+1)$, where $\theta_1 = \frac{-3+\sqrt{-7}}{4}$, $\theta_2 = \frac{-3-\sqrt{-7}}{4}$ and ω_p, $p = 1, 2, 3$ are roots of the cubic equation $2w^3 + w^2 - w - 3 = 0$. We have $\tau(n) = 6na(n)^2$ is n is odd and $\tau(n) = 4n\,a(n)^2$ is n is even. Also, $\tau(n) \sim \frac{n}{28}A^n$, $n \to \infty$, where $A \approx 42.4038$.

4. **Graph** $C_{3n}(1,n)$. We have

$$\tau(n)=\frac{n}{3}(2\,T_n(\frac{5}{2})+1)^2=\frac{n}{3}((\frac{5+\sqrt{21}}{2})^n+(\frac{5-\sqrt{21}}{2})^n+1)^2.$$

See also ([38], Theorem 5). We note that $\tau(n)=3n\,a(n)^2$, where $a(n)$ satisfies the recursive relation $a(n)=6a(n-1)-6a(n-2)+a(n-3)$ with initial data $a(1)=2,\ a(2)=8,\ a(3)=37$.

5. **Graph** $C_{3n}(1,2,n)$. By Theorem 4.1, we obtain

$$\tau(n)=\frac{n}{3}F_n^2(2\,T_n(\omega_1)+1)^2(2\,T_n(\omega_2)+1)^2,$$

where $\omega_1=\frac{-1+\sqrt{37}}{4}$ and $\omega_2=\frac{-1-\sqrt{37}}{4}$. By Theorem 5.1, $\tau(n)=3n\,a(n)^2$ for some integer sequence $a(n)$.

6. **Graph** $C_{6n}(1,n,3n)$. Now, we get

$$\tau(n)=\frac{n}{3}(2\,T_n(\frac{5}{2})+1)^2(2\,T_n(\frac{7}{2})-1)^2(T_n(5)+1).$$

For a suitable integer sequence $a(n)$, one has $\tau(n)=6n\,a(n)^2$ if n is even and $\tau(n)=18n\,a(n)^2$ if n is odd.

7. **Graph** $C_{12n}(1,3n,4n)$. In this case

$$\tau(n)=\frac{2n}{3}T_n(2)^2(2\,T_n(\frac{5}{2})+1)^2(T_n(3)+1)(4\,T_n(\frac{7}{2})^2-3)^2(2\,T_n(\frac{9}{2})-1)^2.$$

By Theorem 5.1, one can conclude that $\tau(n)=3n\,a(n)^2$ if n is even and $\tau(n)=6n\,a(n)^2$ if n is odd, for some sequence $a(n)$ of even numbers.

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