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Cube-contractions in 3-connected quadrangulations

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Abstract

A 3-connected quadrangulation of a closed surface is said to be $\mathcal{K}_3'$-irreducible if no face- or cube-contraction preserves simplicity and 3-connectedness. In this paper, we prove that a $\mathcal{K}_3'$-irreducible quadrangulation of a closed surface except the sphere and the projective plane is either (i) irreducible or (ii) obtained from an irreducible quadrangulation H by applying 4-cycle additions to $F_0 \subseteq F(H)$ where $F(H)$ stands for the set of faces of H. We also determine $\mathcal{K}_3'$-irreducible quadrangulations of the sphere and the projective plane. These results imply new generating theorems of 3-connected quadrangulations of closed surfaces.

Keywords: Quadrangulation, closed surface, generating theorem.

Math. Subj. Class.: 05C10

1 Introduction

In this paper, we only consider simple graphs which have no loops and no multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. We say that $S \subset V(G)$ is a *cut* of G if $G - S$ is disconnected. In particular, S is called a k-*cut* if S is a cut with $|S| = k$. A cycle C of G is said to be *separating* if $V(C)$ is a cut. Similarly, a simple closed curve γ on a closed surface F^2 is said to be *separating* if $F^2 - \gamma$ is disconnected.

A *quadrangulation* G of a closed surface F^2 is a simple graph cellularly embedded on the surface so that each face is quadrilateral; thus, a 2-path on the sphere is not a quadrangulation. We denote the set of faces of G by $F(G)$ throughout the paper. For quadrangulations we consider applying three reductions, called a *face-contraction*, a *4-cycle removal*

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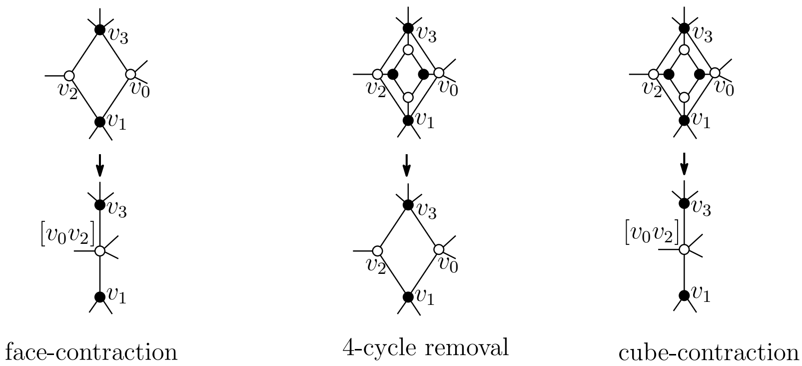

Figure 1: Reductions for quadrangulations.

and a *cube-contraction*, as shown in Figure 1. (Precise definitions of these reductions will be given in the next section.) The corresponding inverse operations are called a *vertex-splitting*, a *4-cycle addition* and a *cube-splitting*, respectively. In particular, the operations of a face-contraction and a 4-cycle removal were first introduced by Batagelj [1]

Irreducible quadrangulations, such that no face-contraction is applicable without making a loop or multiple edges, on a fixed closed surface with low genus were obtained in earlier papers. In [9], it was proven that a 4-cycle is the unique irreducible quadrangulation of the sphere, and that there exist precisely two irreducible quadrangulations of the projective plane shown in Figure 2, where Q_P^1 and Q_P^2 are the unique quadrangular embeddings of K_4 and $K_{3,4}$ on the projective plane, respectively. The irreducible quadrangulations of the torus and the Klein bottle have also been determined in [6, 5]. In [8], it was proven that for any closed surface F^2 there exist only finitely many irreducible quadrangulations of F^2, up to homeomorphism.

A 3-connected quadrangulation G of a closed surface F^2 is said to be $\mathcal{K}_3$*-irreducible* if any of a face-contraction and a 4-cycle removal breaks simplicity or 3-connectedness of G. The following theorem is the starting point of the study of 3-connected quadrangulations. (The definitions of a pseudo double wheel, a Möbius wheel and a double cube are given in the next section.)

Theorem 1.1 (Brinkmann et al.[2]). *Any $\mathcal{K}_3$-irreducible quadrangulation of the sphere is isomorphic to a pseudo double wheel.*

Observe that a 3-connected quadrangulation of the sphere corresponds to a 4-regular 3-connected graph on the same surface by taking its dual. Broersma et al. [3] considered the same problem of the dual version with weaker conditions than those of Brinkmann. For the projective plane, Nakamoto proved the following.

Theorem 1.2 (Nakamoto[7]). *Any $\mathcal{K}_3$-irreducible quadrangulation of the projective plane is isomorphic to either a Möbius wheel or Q_P^2.*

Furthermore, the results in [4] imply the following.

Theorem 1.3 (Nagashima et al.[4]). *Let G be a quadrangulation of a closed surface other than the sphere and the projective plane. Then G is $\mathcal{K}_3$-irreducible if and only if G is irreducible.*

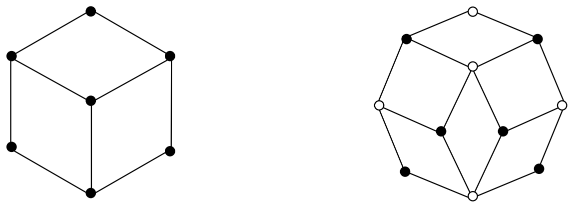

Figure 2: Irreducible quadrangulations on the projective plane.

In this paper, we determine other minimal subsets of 3-connected quadrangulations by replacing 4-cycle removals with cube-contractions. A 3-connected quadrangulation G is said to be $\mathcal{K}'_3$-irreducible if any of a face-contraction and a cube-contraction breaks the simplicity or the 3-connectedness of G. The followings are our main results in the paper. In these statements, $F(H)$ stands for the set of faces of a quadrangulation H.

Theorem 1.4. *Let G be a $\mathcal{K}'_3$-irreducible quadrangulation of a closed surface F^2 other than the sphere and the projective plane. Then, G is either (i) irreducible or (ii) obtained from an irreducible quadrangulation H by applying 4-cycle additions to $F_0 \subseteq F(H)$.*

Theorem 1.5. *Let G be a $\mathcal{K}'_3$-irreducible quadrangulation of the sphere. Then, G is either (i) a pseudo double wheel or (ii) a double cube.*

Theorem 1.6. *Let G be a $\mathcal{K}'_3$-irreducible quadrangulation of the projective plane. Then, G is (i) a Möbius wheel, (ii) Q_P^2 or (iii) obtained from Q_P^1 (resp. Q_P^2) by applying 4-cycle additions to $F_0 \subseteq F(Q_P^1)$ (resp. $F_0 \subseteq F(Q_P^2)$).*

Corollary 1.7. *For any closed surface F^2, there exist only finitely many quadrangulations which are $\mathcal{K}'_3$-irreducible but are not $\mathcal{K}_3$-irreducible, up to homeomorphism.*

This paper is organized as follows. In the next section, we define the reductions used in this paper and introduce typical 3-connected quadrangulations on the sphere and the projective plane called a pseudo double wheel and a Möbius wheel, respectively. In Section 3, we develop some theoretical tools and prove Theorem 1.4. The last section is devoted to prove the planar case and the projective-planar case individually, using some figures.

2 Reductions and typical quadrangulations

Let G be a quadrangulation of a closed surface F^2 and let f be a face of G bounded by a cycle $v_0v_1v_2v_3$. (We also use the notation like $f = v_0v_1v_2v_3$ in this paper.) The *face-contraction* of f at $\{v_0, v_2\}$ in G consists of identification of v_0 and v_2, and replacement of the resulting multiple edges $\{v_0v_1, v_2v_1\}$ and $\{v_0v_3, v_2v_3\}$ with two single edges, respectively. In the resulting graph, let $[v_0v_2]$ denote the vertex arisen by the identification of v_0

and v_2 (see the left-hand side of Figure 1). Similarly, we define the face-contraction of f at $\{v_1, v_3\}$. The inverse operation of a face-contraction is called a *vertex-splitting*. We say that f is *contractible* at $\{v_0, v_2\}$ in G, if the graph obtained from the face-contraction of f at $\{v_0, v_2\}$ is simple. Assume in addition that G is 3-connected. A face f of G is said to be 3-*contractible* at $\{v_0, v_2\}$ if f is contractible at $\{v_0, v_2\}$ and the graph obtained from the face-contraction is still 3-connected.

Let $f = v_0v_1v_2v_3$ be a face of a quadrangulation G of F^2. A 4-*cycle addition* to f consists of inserting a 4-cycle $C = u_0u_1u_2u_3$ inside f in G and joining v_i and u_i for $i = 0, 1, 2, 3$. The inverse operation of a 4-cycle addition is called a 4-*cycle removal* (of C), as shown in the center of Figure 1. We call the subgraph Q isomorphic to a cube with eight vertices u_i, v_i for $i = 0, 1, 2, 3$ an *attached cube*. For an attached cube Q, we call the above 4-cycle C an *inner* 4-*cycle* of Q. In addition, we denote $\partial Q = v_0v_1v_2v_3$. Let G be a 3-connected quadrangulation of a closed surface having an attached cube Q. We say that an inner 4-cycle C of Q (or easily an attached cube Q) is *removable* if the graph obtained from G by applying 4-cycle removal C preserves the 3-connectedness. (Observe that a 4-cycle removable never destroy simplicity of G.)

As mentioned in the introduction, there exist some results of 3-connected quadrangulations (or quadrangulations with minimum degree 3) on surfaces. In those results, the 4-cycle removal is necessary by the following reason: Let $\tilde{G}$ be the graph obtained from a 3-connected quadrangulation G of a closed surface by applying 4-cycle additions to all faces of G. Clearly $\tilde{G}$ is 3-connected, but we cannot apply any face-contraction to $\tilde{G}$ without creating a vertex of degree 2.

Our third reduction of quadrangulations of closed surfaces is defined as a sequence of the above two reductions. Assume that a quadrangulation G has an attached cube Q with an inner 4-cycle C and with $\partial Q = v_0v_1v_2v_3$. A *cube-contraction* of Q at $\{v_0, v_2\}$ in G consists of a 4-cycle removal of C followed by a face-contraction at $\{v_0, v_2\}$ (see the right-hand side of Figure 1). The inverse operation of a cube-contraction is called a *cube-splitting*. We say that an attached cube Q is *contractible* if the graph obtained from G by applying a cube-contraction of Q preserves the simplicity and the 3-connectedness. One might suspect that if an attached cube Q is contractible then Q is removable (and the face that appeared by the removal is contractible). However, this is not true in general since a 4-cycle removal might break the 3-connectedness of the graph.

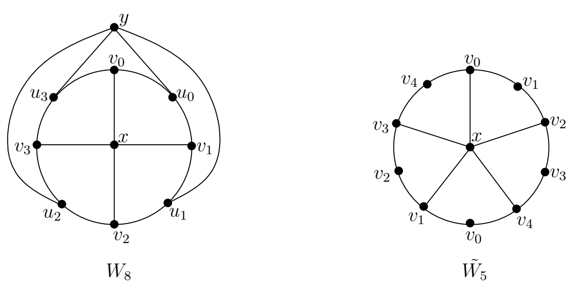

Figure 3: W_8 and $\tilde{W}_5$.

We need to describe two special types of embeddings. Firstly, embed a $2k$-cycle $C = v_0u_0v_1u_1 \dots v_{k-1}u_{k-1}$ ($k \geq 3$) into the sphere, put a vertex x on one side and a vertex y on the other side and add edges xv_i and yu_i for $i = 0, \dots, k-1$. The resulting quadrangulation of the sphere with $2k + 2$ vertices is said to be a *pseudo double wheel* and denoted by W_{2k} (see the left-hand side of Figure 3). The smallest pseudo double wheel is W_6, which is isomorphic to a cube, when the graphs are assumed to be 3-connected. The cycle C of length $2k$ is called the *rim* of W_{2k}. We call a quadrangulation of the sphere obtained from W_6 by a single 4-cycle addition a *double cube*, which is isomorphic to $C_4 \times P_2$.

Secondly, embed a $(2k-1)$-cycle $C = v_0v_1 \dots v_{2k-2}$ $(k \geq 2)$ into the projective plane so that the tubular neighborhood of C forms a Möbius band. Next, put a vertex x on the center of the unique face of the embedding and join x to v_i for all i so that the resulting graph is a quadrangulation. The resulting quadrangulation of the projective plane with $2k$ vertices is said to be a *Möbius wheel* and denoted by $\tilde{W}_{2k-1}$ (see the right-hand side of Figure 3).

3 Lemmas to prove Theorem 1.4

The following lemma holds not only for quadrangulations but also for even embeddings of closed surfaces F^2, that is, for graphs embedded on F^2 with each face bounded by a cycle of even length. Taking a dual of an even embedding and using the odd point theorem, we can easily obtain this lemma.

Lemma 3.1. *An even embedding of a closed surface has no separating closed walk of odd length.*

Let G be a quadrangulation of a closed surface F^2 and let $f = v_0v_1v_2v_3$ be a face of G. Then a pair $\{v_i, v_{i+2}\}$ is called a *diagonal pair* of f in G, where the subscripts are taken modulo 4. A closed curve γ on F^2 is said to be a *diagonal k-curve* for G if γ passes only through distinct k faces $f_0, \dots, f_{k-1}$ and distinct k vertices $x_0, \dots, x_{k-1}$ of G such that for each i, f_i and f_{i+1} share x_i, and that for each i, $\{x_{i-1}, x_i\}$ forms a diagonal pair of f_i of G, where the subscripts are taken modulo k.

Lemma 3.2. *Let G be a quadrangulation of a closed surface F^2 with a 2-cut $\{x, y\}$. Then there exists a separating diagonal 2-curve for G only through x and y.*

Proof. Observe that every quadrangulation of any closed surface F^2 is 2-connected and admits no closed curve on F^2 crossing G at most once. Thus there exists a surface separating simple closed curve γ on F^2 crossing only x and y, since $\{x, y\}$ is a cut of G.

We shall show that γ is a diagonal 2-curve. Suppose that γ passes through two faces f_1 and f_2 meeting at two vertices x and y. If γ is not a diagonal 2-curve, then x and y are adjacent on ∂f_1 or ∂f_2. Since G has no multiple edges between x and y, and since $\{x, y\}$ is a 2-cut of G, we may suppose that x and y are adjacent in ∂f_1, but not in ∂f_2. Here we can take a separating 3-cycle of G along γ. This contradicts Lemma 3.1. □

Lemma 3.3. *Let G be a 3-connected quadrangulation of a closed surface F^2, and let $f = v_0v_1v_2v_3$ be a face of G. If the face-contraction of f at $\{v_0, v_2\}$ breaks 3-connectedness of the graph but preserves simplicity, then G has a separating diagonal 3-curve passing through v_0, v_2 and another vertex $x \in V(G) - \{v_0, v_1, v_2, v_3\}$.*

Proof. Let G' be the quadrangulation of F^2 obtained from G by the face-contraction of f at $\{v_0, v_2\}$. Since G' has connectivity 2, G' has a 2-cut. By Lemma 3.2, G' has a separating diagonal 2-curve γ' passing through two vertices of the 2-cut. Clearly, one of the two vertices must be $[v_0v_2]$ of G', which is the image of v_0 and v_2 by the face-contraction of f. (Otherwise, G would not be 3-connected, a contradiction.) Let x be a vertex of G' on γ' other than $[v_0v_2]$. Note that x is not a neighbor of $[v_0v_2]$ in G'. Now apply the vertex-splitting of $[v_0v_2]$ to G' to recover G. Then a diagonal 3-curve for G passing through only v_0, v_2 and x arises from γ' for G'. □

The next lemma plays an important role in a later argument.

Lemma 3.4. *Let G be a 3-connected quadrangulation on a closed surface F^2. If G has a separating 4-cycle $C = x_0x_1x_2x_3$ and a face f of G such that*

(i) one of the diagonal pairs of f is $\{x_i, x_{i+2}\}$ for some i, and

(ii) f has a separating diagonal 3-curve γ intersecting C only at x_i and x_{i+2} transversely,

then there exists a 3-contractible face in G.

Proof. Suppose that G has a separating 4-cycle $C = x_0x_1x_2x_3$ and a face f bounded by ax_1cx_3. Since C is separating, G has two subgraphs G_R and G_L such that $G_R \cup G_L = G$ and $G_R \cap G_L = C$. Suppose that f is contained in G_R. Furthermore, we assume that G_R contains as few vertices of G as possible.

Since C is separating, we have $\partial f \neq C$. By (ii), f has a separating diagonal 3-curve γ through x_1, x_3 and some vertex x. Note that $x \in V(G_L) - V(C)$ by the condition (ii) in the lemma. Now assume that f is not 3-contractible at $\{a, c\}$. Observe that γ (or the 3-cut $\{x_1, x, x_3\}$) separates a from c. Further, G does not have both of edges ax and cx since $\partial f \neq C$. Therefore, there is no path of G of length at most 2 joining a and c other than ax_1c and ax_3c. Moreover, if $\{a, c\} \cap \{x_0, x_2\} = \emptyset$, then f has no separating diagonal 3-curve joining a and c. This contradicts our assumption by Lemma 3.3 and so we may suppose that $a = x_0$ and $c \neq x_2$, and f has a separating diagonal 3-curve, say γ', through a $(= x_0)$ and c.

Since γ' separates x_1 and x_3 and since x_2 is a common neighbors of x_1 and x_3, γ' must pass through x_2, and hence we can find a face f' of G_R one of whose diagonal pair is $\{c, x_2\}$. Let C' be the 4-cycle $x_1x_2x_3c$ of G. Since $\deg(c) \geq 3$, we have $\partial f' \neq C'$, and hence C' is a separating 4-cycle in G_R such that $C' \neq C$. Moreover, γ' and C' cross transversely at x_2 and c. Therefore, C' and f' are a 4-cycle and a face which satisfy the assumption of the lemma, and moreover, C' can cut a strictly smaller graph than G_R from G. Therefore, this contradicts the choice of C. □

Lemma 3.5. *Let G be a 3-connected quadrangulation of a closed surface F^2. If G is $\mathcal{K}_3$-irreducible then G is $\mathcal{K}'_3$-irreducible.*

Proof. Let G be a 3-connected quadrangulation of a closed surface. Assume that G is not $\mathcal{K}'_3$-irreducible. Then, G has either a 3-contractible face or a contractible cube. If G has a 3-contractible face, then G is not $\mathcal{K}_3$-irreducible. Therefore, we suppose that G has no 3-contractible face but has a contractible cube Q with an inner 4-cycle C in the following argument.

Now, we apply a 4-cycle removal of C to G and let G' be the resulting quadrangulation. Let $f' = \partial Q$ be the new face of G' into which C was inserted. If G' is 3-connected, G is not $\mathcal{K}_3$-irreducible by the definition, and we are done. Therefore, we assume that G' is not 3-connected. By Lemma 3.2, there is a diagonal 2-curve γ passing through f' and another face f''; otherwise, G would have a 2-cut, contrary to our assumption. Note that f'' is also a face in G. Now ∂Q and f'' satisfy the conditions of Lemma 3.4, and hence there exists a 3-contractible face in G. However, this contradicts the above assumption. Thus, the lemma follows. □

In the following argument, we denote the set of $\mathcal{K}_3$-irreducible (resp. $\mathcal{K}_3'$-irreducible) quadrangulations of a closed surface F^2 by $\mathcal{K}_3\mathcal{I}(F^2)$ (resp. $\mathcal{K}_3'\mathcal{I}(F^2)$).

Lemma 3.6. *Let G be a 3-connected quadrangulation of F^2. If $G \in \mathcal{K}_3'\mathcal{I}(F^2) \setminus \mathcal{K}_3\mathcal{I}(F^2)$, then G has an attached cube Q such that the graph obtained from G by applying a 4-cycle removal of Q is in $\mathcal{K}_3'\mathcal{I}(F^2)$.*

Proof. Let G be in $\mathcal{K}_3'\mathcal{I}(F^2) \setminus \mathcal{K}_3\mathcal{I}(F^2)$. By the definition, G has an attached cube Q with an inner 4-cycle C which is removable, but is not contractible. We apply a 4-cycle removal of C and let G^- be the resulting quadrangulation. We denote the new face of G^- by f^-, where $f^- = \partial Q$.

First, we confirm that G^- is 3-connected. Otherwise, G^- has a 2-cut and has a separating diagonal 2-curve γ on F^2 by Lemma 3.2. If γ does not pass through f^- then γ would also be a diagonal 2-curve in G, a contradiction. Let f_0 be the other face passed by γ. Here, f_0 and ∂Q in G satisfy the conditions in Lemma 3.4 and there exists a 3-contractible face, contrary to G being $\mathcal{K}_3'$-irreducible.

By way of contradiction, assume that G^- is not in $\mathcal{K}_3'\mathcal{I}(F^2)$. That is, G^- has either (a) a 3-contractible face or (b) a contractible cube. First, we assume (a) and let f be a 3-contractible face in G^-. If $f^- = f$, the attached cube Q in G would be contractible, contrary to G being $\mathcal{K}_3'$-irreducible. Thus, suppose $f^- \neq f$. In this case, let G' be the resulting 3-connected quadrangulation after applying a face-contraction of f in G^-. Since any 4-cycle addition doesn't break the 3-connectedness of a quadrangulation, the graph obtained from G' by a 4-cycle addition to f^- is clearly 3-connected. This means that f is also 3-contractible in G, a contradiction.

Next, suppose (b) and let Q' be such a contractible cube with $\partial Q' = v_0v_1v_2v_3$. If Q' does not contain f^- as one of its five faces, Q' is also contractible in G and G would not be $\mathcal{K}_3'$-irreducible by the similar argument as above. Thus, we assume that Q' contains f^-. Let $C = u_0u_1u_2u_3$ denotes the inner 4-cycle of Q' where $u_iv_i \in E(Q')$ for $i = 0, 1, 2, 3$. We consider the following two cases up to symmetry; (b-1) $f^- = C$ and (b-2) $f^- = v_0u_0u_1v_1$. At first, suppose (b-1). Here, we apply a face-contraction of $f_1 = v_0u_0u_1v_1$ at $\{u_0, v_1\}$ to G. If the above face-contraction breaks the 3-connectedness of G, there exists a face $f_2 = v_1xv_3y$ in the outside of Q' by Lemma 3.3; note that it clearly preserves the simplicity of the graph since $v_1 \neq v_3$. Now, a separating diagonal 3-curve passing through $\{v_1, u_0, v_3\}$ satisfies the conditions of Lemma 3.4 and hence G is not $\mathcal{K}_3'$-irreducible, contrary to our assumption. In fact, an analogous proof is valid for (b-2) if we try to apply a face contraction at $\{v_1, u_2\}$ to G. Therefore the lemma follows. □

Lemma 3.7. *Let G be a 3-connected quadrangulation of a closed surface F^2. If $G \in \mathcal{K}_3'\mathcal{I}(F^2) \setminus \mathcal{K}_3\mathcal{I}(F^2)$, then G can be obtained from $H \in \mathcal{K}_3\mathcal{I}(F^2)$ by applying 4-cycle additions to $F_0 \subseteq F(H)$.*

Proof. Assume that $G \in \mathcal{K}_3'\mathcal{I}(F^2) \setminus \mathcal{K}_3\mathcal{I}(F^2)$. By the previous lemma, there exists a sequence of $\mathcal{K}_3'$-irreducible quadrangulations $G = G_0, G_1, \ldots, G_k$ such that G_{i+1} is obtained from G_i by a single 4-cycle removal of C_i, where $G_k \in \mathcal{K}_3\mathcal{I}(F^2)$. (Since the number of vertices of G is finite, $G_k \in \mathcal{K}_3\mathcal{I}(F^2)$.) Let Q_i denote an attached cube in G_i with an inner 4-cycle C_i.

For a contradiction, we assume that there exists $l \in \{0, \ldots, k-2\}$ such that G_l is obtained from G_{l+1} by a 4-cycle addition which is put on a face not of $F(G_k)$; this l should be maximal. This implies that C_l is put on a face of Q_{l+1} as one of its five faces. Then the same argument as the proof of Lemma 3.6 holds and hence G_l would not be $\mathcal{K}_3'$-irreducible, contrary to our assumption. Thus for each $i \in \{0, \ldots, k-1\}$, G_i is obtained from G_{i+1} by a 4-cycle addition which is put on a face of $F(G_k)$. □

Proof of Theorem 1.4. By Lemma 3.5, we have $\mathcal{K}_3\mathcal{I}(F^2) \subseteq \mathcal{K}_3'\mathcal{I}(F^2)$. Furthermore, by Theorem 1.3 and Lemma 3.7, we obtain (i) and (ii) in the statement. Thus, we have got a conclusion. □

4 Spherical and projective-planar cases

In this section, we discuss the spherical case and the projective-planar case.

Proof of Theorem 1.5. Let G be a $\mathcal{K}_3'$-irreducible quadrangulation of the sphere. We have $\mathcal{K}_3\mathcal{I}(S^2) \subseteq \mathcal{K}_3'\mathcal{I}(S^2)$ by Lemma 3.5, where S^2 stands for the sphere.

If G is $\mathcal{K}_3$-irreducible, then G is isomorphic to a pseudo double wheel by Theorem 1.1. If G is in $\mathcal{K}_3'\mathcal{I}(S^2) \setminus \mathcal{K}_3\mathcal{I}(S^2)$, G can be obtained from a pseudo double wheel W_{2k} $(k \geq 3)$ by some 4-cycle additions to faces of W_{2k} by Lemma 3.7. However if $k \geq 4$, G has a 3-contractible face (or a contractible cube), as shown in the first operation in Figure 4. (For example, the entire Figure 4 presents a sequence of a face-contraction and a cube-contraction which deforms W_8 with an attached cube Q into W_6, preserving the 3-connectedness.)

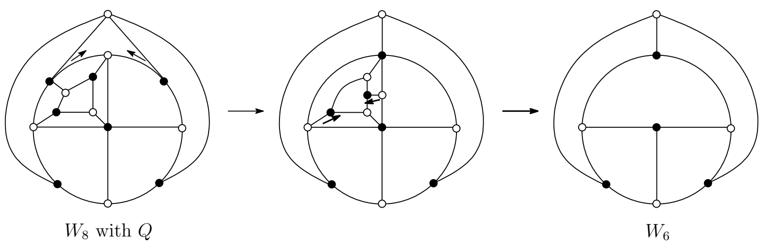

Figure 4: W_8 with an attached cube Q deformed into W_6.

Therefore, we only consider the case of $k = 3$ in the following argument. Assume that G is obtained from W_6 by at least two 4-cycle additions to faces of W_6. Similarly to the above argument, G would have a 3-contractible face (or a contractible cube) , as in Figure 5, contrary to G being $\mathcal{K}_3'$-irreducible; note that it suffices to discuss these two cases, up to symmetry. Therefore, we conclude that G is obtained from W_6 by exactly one 4-cycle addition. This is nothing but a double cube; observe that a double cube has no 3-contractible face and no contractible cube. □

To conclude with, we prove the projective-planar case.

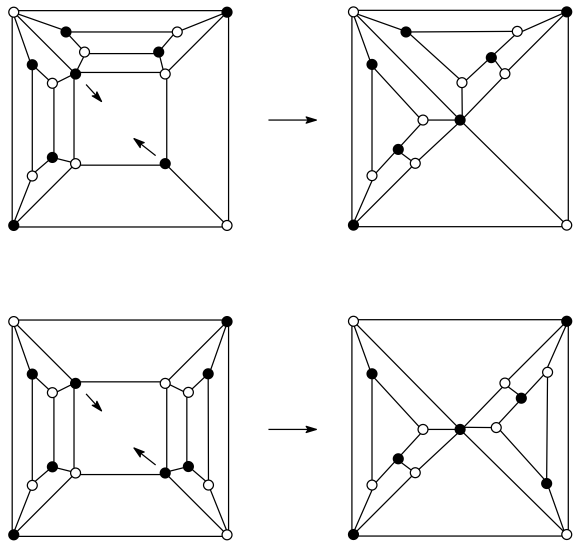

Figure 5: W_6 with two attached cubes can be reduced.

Proof of Theorem 1.6. In this case, we use Möbius wheels $\tilde{W}_k (k \geq 3)$ and Q_P^2 as base graphs by Theorem 1.2.

First we consider the former case. Similarly to the previous proof (and see Figure 6), we consider only a Möbius wheel $\tilde{W}_3$ as a base to which we apply some 4-cycle additions. However, $\tilde{W}_3$ $(= Q_P^1)$ is isomorphic to the complete graph with four vertices, and hence it is irreducible. This fact implies that every G obtained from $\tilde{W}_3$ by applying at most three 4-cycle additions is $\mathcal{K}_3'$-irreducible since any face-contraction and any cube-contraction to G destroys the simplicity of the graph, or results in a vertex of degree 2. From this case, we obtain exactly three quadrangulations in $\mathcal{K}_3'\mathcal{I}(P^2) \setminus \mathcal{K}_3\mathcal{I}(P^2)$, up to homeomorphism, where P^2 stands for the projective plane.

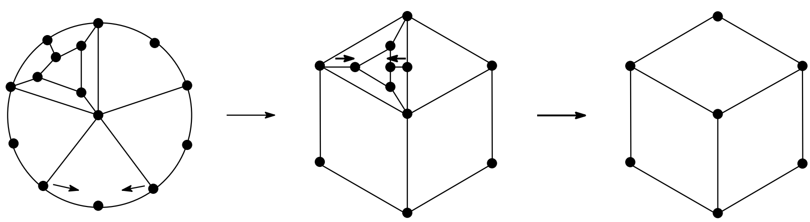

Figure 6: $\tilde{W}_5$ with an attached cube Q deformed into $\tilde{W}_3$.

Similarly, as the latter case, we obtain the other ten quadrangulations in $\mathcal{K}_3'\mathcal{I}(P^2) \setminus \mathcal{K}_3\mathcal{I}(P^2)$ from Q_P^2; consider all the way to put attached cubes into faces of Q_P^2, up to symmetry. As a result, we have $|\mathcal{K}_3'\mathcal{I}(P^2) \setminus \mathcal{K}_3\mathcal{I}(P^2)| = 13$ in total. □

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