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Distinguishing colorings, proper colorings, and covering properties without AC*

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Abstract

We work with simple graphs in ZF (i.e., the Zermelo–Fraenkel set theory without the Axiom of Choice (AC)) and assume that the sets of colors can be either well-orderable or non-well-orderable, to prove that the following statements are equivalent to Kőnig's Lemma:

(a) Any infinite locally finite connected graph G such that the minimum degree of G is greater than k, has a chromatic number for any fixed integer k greater than or equal to 2.

(b) Any infinite locally finite connected graph has a chromatic index.

(c) Any infinite locally finite connected graph has a distinguishing number.

(d) Any infinite locally finite connected graph has a distinguishing index.

The above results strengthen some recent results of Stawiski since he assumed that the sets of colors can be well-ordered. We formulate new conditions for the existence of irreducible proper coloring, minimal edge cover, maximal matching, and minimal dominating set in connected bipartite graphs and locally finite connected graphs, which are either equivalent to AC or Kőnig's Lemma. Moreover, we show that if the Axiom of Choice for families of 2-element sets holds, then the Shelah-Soifer graph has a minimal dominating set.

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1 Introduction

In 1991, Galvin–Komjáth proved that the statements "Any graph has a chromatic number" and "Any graph has an irreducible proper coloring" are equivalent to AC in ZF using Hartogs's theorem (cf. [7]). In 1977, Babai [1] introduced distinguishing vertex colorings under the name asymmetric colorings, and distinguishing edge colorings were introduced by Kalinowski–Pilśniak [14] in 2015. Recently, Stawiski [20] proved that the statements (b)–(d) mentioned in the abstract above and the statement "Any infinite locally finite connected graph has a chromatic number" are equivalent to Kőnig's Lemma (a weak form of AC) by assuming that the sets of colors can be well-ordered (cf. [20, Lemma 3.3 and Section 2]).

1.1 Proper and distinguishing colorings

An infinite cardinal in ZF can either be an ordinal or a set that is not well-orderable. Herrlich–Tachtsis [10, Proposition 23] proved that no Russell graph has a chromatic number in ZF. We refer the reader to [10] for the details concerning Russell graph and Russell sequence. In Theorem 4.2, the first and the second authors study new combinatorial proofs (mainly inspired by the arguments of [10, Proposition 23]) to show that the statements (a)–(d) mentioned in the abstract above are equivalent to Kőnig's Lemma (without assuming that the sets of colors can be well-ordered).[1]

1.2 New equivalents of Kőnig's lemma and AC

The role of AC and Kőnig's Lemma in the existence of graph-theoretic properties like irreducible proper coloring, chromatic numbers, maximal independent sets, spanning trees, and distinguishing colorings were studied by several authors in the past (cf. [2, 3, 5, 6, 7, 11, 19, 20]). We list a few known results apart from the above-mentioned results due to Galvin–Komjáth [7] and Stawiski [20]. In particular, Friedman [6, Theorem 6.3.2, Theorem 2.4] proved that AC is equivalent to the statement "Any graph has a maximal independent set". Höft–Howard [11] proved that the statement "Any connected graph contains a partial subgraph which is a tree" is equivalent to AC. Fix any even integer $m \geq 4$ and any integer $n \geq 2$. Delhommé–Morillon [5] studied the role of AC in the existence of spanning subgraphs and observed that AC is equivalent to "Any connected bipartite graph has a spanning subgraph without a complete bipartite subgraph $K_{n,n}$" as well as "Any connected graph admits a spanning m-bush" (cf. [5, Corollary 1, Remark 1]). They also proved that the statement "Any locally finite connected graph has a spanning tree" is equivalent to Kőnig's lemma in [5, Theorem 2]. Banerjee [2, 3] observed that the statements "Any infi-

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[1] We note that statement (a) mentioned in the abstract is a new equivalent of Kőnig's Lemma. Stawiski's graph from [20, Theorem 3.6] shows that Kőnig's Lemma is equivalent to "Every infinite locally finite connected graph G such that $\delta(G)$ (the minimum degree of G) is 2 has a chromatic number".

nite locally finite connected graph has a maximal independent set" and "Any infinite locally finite connected graph has a spanning m-bush" are equivalent to Kőnig's lemma. However, the existence of maximal matching, minimal edge cover, and minimal dominating set in ZF were not previously investigated. The following table summarizes the new results (cf. Theorem 5.1, Theorem 6.4).[2]

New equivalents of Kőnig's lemma	New equivalents of AC
$\mathcal{P}_{\text{lf,c}}$(irreducible proper coloring)	
$\mathcal{P}_{\text{lf,c}}$(minimal dominating set)	$\mathcal{P}_{\text{c}}$(minimal dominating set)
$\mathcal{P}_{\text{lf,c}}$(maximal matching)	$\mathcal{P}_{\text{c,b}}$(maximal matching)
$\mathcal{P}_{\text{lf,c}}$(minimal edge cover)	$\mathcal{P}_{\text{c,b}}$(minimal edge cover)

In the table, $\mathcal{P}_{\text{lf,c}}$(property X) denotes "Any infinite locally finite connected graph has property X", $\mathcal{P}_{\text{c,b}}$(property X) denotes "Any connected bipartite graph has property X" and $\mathcal{P}_{\text{c}}$(property X) denotes "Any connected graph has property X".

2 Basics

Definition 2.1. Suppose X and Y are two sets. We write:

1. $X \preceq Y$, if there is an injection $f\colon X \to Y$.
2. X and Y are equipotent if $X \preceq Y$ and $Y \preceq X$, i.e., if there is a bijection $f\colon X \to Y$.
3. $X \prec Y$, if $X \preceq Y$ and X is not equipotent with Y.

Definition 2.2. Without AC, a set m is called a *cardinal* if it is the cardinality $|x|$ of some set x, where $|x| = \{y : y \sim x$ and y is of least rank$\}$ where $y \sim x$ means the existence of a bijection $f\colon y \to x$ (see [15, Definition 2.2, page 83] and [13, Section 11.2]).

Definition 2.3. A graph $G = (V_G, E_G)$ consists of a set V_G of vertices and a set $E_G \subseteq [V_G]^2$ of edges.[3] Two vertices $x, y \in V_G$ are *adjacent vertices* if $\{x, y\} \in E_G$, and two edges $e, f \in E_G$ are *adjacent edges* if they share a common vertex. The *degree* of a vertex $v \in V_G$, denoted by $\deg(v)$, is the number of edges emerging from v. We denote by $\delta(G)$ the minimum degree of G. Given a non-negative integer n, a *path of length* n in G is a one-to-one finite sequence $\{x_i\}_{0\leq i\leq n}$ of vertices such that for each $i < n$, $\{x_i, x_{i+1}\} \in E_G$; such a path joins x_0 to x_n.

(1) G is *locally finite* if every vertex of G has a finite degree.

(2) G is *connected* if any two vertices are joined by a path of finite length.

(3) A *dominating set* of G is a set D of vertices of G, such that any vertex of G is either in D, or has a neighbor in D.

(4) An *independent set* of G is a set of vertices of G, no two of which are adjacent vertices. A *dependent set* of G is a set of vertices of G that is not an independent set.

[2] We note that Theorem 5.1 is a combined effort of the first and the second authors. Moreover, all remarks in Section 6 including Theorem 6.4 are due to all the authors.

[3] i.e., E_G is a subset of the set of all two-element subsets of V_G.

(5) A *vertex cover* of G is a set of vertices of G that includes at least one endpoint of every edge of the graph G.

(6) A *matching* M in G is a set of pairwise non-adjacent edges.

(7) An *edge cover* of G is a set C of edges such that each vertex in G is incident with at least one edge in C.

(8) A *minimal dominating set (minimal vertex cover, minimal edge cover)* is a dominating set (a vertex cover, an edge cover) that is not a superset of any other dominating set (vertex cover, edge cover). A *maximal independent set (maximal matching)* is an independent set (a matching) that is not a subset of any other independent set (matching).

(9) A *proper vertex coloring* of G with a color set C is a mapping $f\colon V_G \to C$ such that for every $\{x, y\} \in E_G$, $f(x) \neq f(y)$. A *proper edge coloring* of G with a color set C is a mapping $f\colon E_G \to C$ such that for any two adjacent edges e_1 and e_2, $f(e_1) \neq f(e_2)$.

(10) Let $|C| = \kappa$. We say G is *κ-proper vertex colorable* or *C-proper vertex colorable* if there is a proper vertex coloring $f\colon V_G \to C$ and G is *κ-proper edge colorable* or *C-proper edge colorable* if there is a proper edge coloring $f\colon E_G \to C$. The least cardinal κ for which G is κ-proper vertex colorable (if it exists) is the *chromatic number* of G and the least cardinal κ for which G is κ-proper edge colorable (if it exists) is the *chromatic index* of G.

(11) A proper vertex coloring $f\colon V_G \to C$ is a *C-irreducible proper coloring* if $f^{-1}(c_1) \cup f^{-1}(c_2)$ is a dependent set whenever $c_1, c_2 \in C$ and $c_1 \neq c_2$ (cf. [7]).

(12) An automorphism of G is a bijection $\phi\colon V_G \to V_G$ such that $\{u, v\} \in E_G$ if and only if $\{\phi(u), \phi(v)\} \in E_G$. Let f be an assignment of colors to either vertices or edges of G. We say that an automorphism ϕ of G *preserves* f if each vertex of G is mapped to a vertex of the same color or each edge of G is mapped to an edge of the same color. We say that f is a *distinguishing coloring* if the only automorphism that preserves f is the identity. Let $|C| = \kappa$. We say G is *κ-distinguishing vertex colorable* or *C-distinguishing vertex colorable* if there is a distinguishing vertex coloring $f\colon V_G \to C$ and G is *κ-distinguishing edge colorable* or *C-distinguishing edge colorable* if there is a distinguishing edge coloring $f\colon E_G \to C$. The least cardinal κ for which G is κ-distinguishing vertex colorable (if it exists) is the *distinguishing number* of G and the least cardinal κ for which G is κ-distinguishing edge colorable (if it exists) is the *distinguishing index* of G.

(13) The automorphism group of G, denoted by $\mathrm{Aut}(G)$, is the group consisting of automorphisms of G with composition as the operation. Let τ be a group acting on a set S and let $a \in S$. The orbit of a, denoted by $\mathrm{Orb}_\tau(a)$, is the set $\{\phi(a) : \phi \in \tau\}$.

(14) G is *complete* if each pair of vertices is connected by an edge. We denote by K_n, the complete graph on n vertices for any natural number $n \geq 1$.

(15) *Kőnig's Lemma* states that every infinite locally finite connected graph has a ray.

Let ω be the set of natural numbers, $\mathbb{Z}$ be the set of integers, $\mathbb{Q}$ be the set of rational numbers, $\mathbb{R}$ be the set of real numbers, and $\mathbb{Q}+a = \{a+r : r \in \mathbb{Q}\}$ for any $a \in \mathbb{R}$. Shelah–Soifer [17] constructed a graph whose chromatic number is 2 in ZFC and uncountable in some model of ZF (e.g. in Solovay's model from [18, Theorem 1]).

Definition 2.4 (cf. [17])**.** The *Shelah–Soifer Graph* $G = (\mathbb{R}, \rho)$ is defined by $x\rho y \Leftrightarrow (x-y) \in (\mathbb{Q}+\sqrt{2}) \cup (\mathbb{Q}+(-\sqrt{2}))$.

Definition 2.5. A set X is *Dedekind-finite* if it satisfies the following equivalent conditions (cf. [10, Definition 1]):

- $\omega \not\preceq X$,[4]
- $A \prec X$ for every proper subset A of X.

Definition 2.6. For every family $\mathcal{B} = \{B_i : i \in I\}$ of non-empty sets, $\mathcal{B}$ is said to have a *partial choice function* if $\mathcal{B}$ has an infinite subfamily $\mathcal{C}$ with a choice function.

Definition 2.7 (A list of choice forms)**.**

(1) AC_2: Every family of 2-element sets has a choice function.

(2) $\mathsf{AC}_{\mathsf{fin}}$: Every family of non-empty finite sets has a choice function.

(3) $\mathsf{AC}^{\omega}_{\mathsf{fin}}$: Every countably infinite family of non-empty finite sets has a choice function. We recall that $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ is equivalent to Kőnig's Lemma as well as the statement "The union of a countable family of finite sets is countable".

(4) $\mathsf{AC}^{\omega}_{k\times\mathsf{fin}}$ for $k \in \omega\backslash\{0,1\}$: Every countably infinite family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty finite sets, where k divides $|A_i|$, has a choice function.

(5) $\mathsf{PAC}^{\omega}_{k\times\mathsf{fin}}$ for $k \in \omega\backslash\{0,1\}$: Every countably infinite family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty finite sets, where k divides $|A_i|$ has a partial choice function.

Definition 2.8. From the point of view of model theory, the *language of graphs* $\mathcal{L}$ consists of a single binary relational symbol E depicting edges, i.e., $\mathcal{L} = \{E\}$ and a graph is an $\mathcal{L}$-structure $G = \langle V, E\rangle$ consisting of a non-empty set V of vertices and the edge relation E on V. Let $G = \langle V, E\rangle$ be an $\mathcal{L}$-structure, $\phi(x_1, ..., x_n)$ be a first-order $\mathcal{L}$-formula, and let $a_1, ..., a_n \in V$ for some $n \in \omega\backslash\{0\}$. We write $G \models \phi(a_1, ..., a_n)$, if the property expressed by ϕ is true in G for $a_1, ..., a_n$. Let $G_1 = \langle V_{G_1}, E_{G_1}\rangle$ and $G_2 = \langle V_{G_2}, E_{G_2}\rangle$ be two $\mathcal{L}$-structures. We recall that if $j\colon V_{G_1} \to V_{G_2}$ is an isomorphism, $\varphi(x_1, ..., x_r)$ is a first-order $\mathcal{L}$-formula on r variables for some $r \in \omega\backslash\{0\}$, and $a_i \in V_{G_1}$ for each $1 \le i \le r$, then by induction on the complexity of formulae, one can see that $G_1 \models \varphi(a_1, ..., a_r)$ if and only if $G_2 \models \varphi(j(a_1), ..., j(a_r))$ (cf. [16, Theorem 1.1.10]).

3 Known and basic results

3.1 Known results

Fact 3.1 (ZF)**.** *The following hold:*

[4] i.e., there is no injection $f\colon \omega \to X$.

(1) (Galvin–Komjáth; cf. [7, Lemma 3 and the proof of Lemma 2]). *Any graph based on a well-ordered set of vertices has an irreducible proper coloring and a chromatic number.*

(2) (Delhommé–Morillon; cf. [5, Lemma 1]). *Given a set X and a set A which is the range of no mapping with domain X, consider a mapping $f\colon A \to \mathcal{P}(X)\backslash\{\emptyset\}$ (with values non-empty subsets of X). Then there are distinct a and b in A such that $f(a) \cap f(b) \neq \emptyset$.*

(3) (Herrlich–Rhineghost; cf. [9, Theorem]). *For any measurable subset X of $\mathbb{R}$ with a positive measure there exist $x \in X$ and $y \in X$ with $y - x \in \mathbb{Q} + \sqrt{2}$.*

(4) (Stawiski; cf. [20, proof of Theorem 3.8]). *Any graph based on a well-ordered set of vertices has a chromatic index, a distinguishing number, and a distinguishing index.*

3.2 Basic results

Proposition 3.2 (ZF)**.** *The Shelah-Soifer Graph $G = (\mathbb{R}, \rho)$ has the following properties:*

(1) *If AC_2 holds, then G has a minimal dominating set.*

(2) *Any independent set of G is either non-measurable or of measure zero.*

Proof. First, we note that each component of G is infinite, since $x, y \in \mathbb{R}$ are connected if and only if $x - y = q + \sqrt{2}z$ for some $q \in \mathbb{Q}$ and $z \in \mathbb{Z}$, and G has no odd cycles.

(1). Under AC_2, G has a 2-proper vertex coloring $f\colon V_G \to 2$ (see [9]). This is because, since G has no odd cycles, each component of G has precisely two 2-proper vertex colorings. Using AC_2 one can select a 2-proper vertex coloring for each component, in order to obtain a 2-proper vertex coloring of G. We claim that $f^{-1}(i)$ (which is an independent set of G) is a maximal independent set (and hence a minimal dominating set) of G for any $i \in \{0, 1\}$. Fix $i \in \{0, 1\}$ and assume that $f^{-1}(i)$ is not a maximal independent set. Then $f^{-1}(i) \cup \{v\}$ is an independent set for some $v \in \mathbb{R}\backslash f^{-1}(i) = f^{-1}(1 - i)$ and so $\{v, x\} \notin \rho$ for any $x \in f^{-1}(i)$. Since $f^{-1}(1 - i)$ is an independent set, $\{v, x\} \notin \rho$ for any $x \in f^{-1}(1 - i)$. This contradicts the fact that G has no isolated vertices.

(2). Let M be an independent set of G. Pick any $x, y \in M$ such that $x \neq y$. Then,

$$\neg(y\rho x) \implies (y-x) \notin (\mathbb{Q}+\sqrt{2})\cup(\mathbb{Q}+(-\sqrt{2})) = \{r+\sqrt{2} : r \in \mathbb{Q}\}\cup\{r-\sqrt{2} : r \in \mathbb{Q}\}.$$

Thus, there are no $x, y \in M$ where $x \neq y$ such that $y - x \in \mathbb{Q} + \sqrt{2}$. By Fact 3.1(3), M is not a measurable set of $\mathbb{R}$ with a positive measure. $\square$

Proposition 3.3 (ZF)**.** *The following hold:*

(1) *Any graph based on a well-ordered set of vertices has a minimal vertex cover.*

(2) *Any graph based on a well-ordered set of vertices has a minimal dominating set.*

(3) *Any graph based on a well-ordered set of vertices has a maximal matching.*

(4) *Any graph based on a well-ordered set of vertices with no isolated vertex, has a minimal edge cover.*

Proof. (1). Let $G = (V_G, E_G)$ be a graph based on a well-ordered set of vertices and let $<$ be a well-ordering of V_G. We use transfinite recursion, without invoking any form of choice, to construct a minimal vertex cover. Let $M_0 = V_G$. Clearly, M_0 is a vertex cover. Assume that M_0 is not a minimal vertex cover. Now, assume that for some ordinal number α we have constructed a sequence $(M_\beta)_{\beta<\alpha}$ of vertex covers such that M_β is not a minimal vertex cover for any $\beta < \alpha$. If $\alpha = \gamma + 1$ is a successor ordinal for some ordinal γ, then let $M_\alpha = M_{\gamma+1} = M_\gamma \backslash \{v_\gamma\}$ where v_γ is the $<$-minimal element of the well-ordered set $\{v \in M_\gamma : M_\gamma \backslash \{v\}$ is a vertex cover$\}$. If α is a limit ordinal, we use $M_\alpha = \bigcap_{\beta\in\alpha} M_\beta$. For any ordinal α, if M_α is a minimal vertex cover, then we are done. Since the class of all ordinal numbers is a proper class, it follows that the recursion must terminate at some ordinal stage, say λ. Then, M_λ is the minimal vertex cover.

(2). This follows from (1) and the fact that if I is a minimal vertex cover of G, then $V_G \backslash I$ is a maximal independent set (and hence a minimal dominating set) of G.

(3). If V_G is well-orderable, then $E_G \subseteq [V_G]^2$ is well-orderable as well. Thus, similar to the arguments of (1) we can obtain a maximal matching by using transfinite recursion in ZF and modifying the greedy algorithm to construct a maximal matching.

(4). Let $G = (V_G, E_G)$ be a graph on a well-ordered set of vertices without isolated vertices. Let $\prec'$ be a well-ordering of E_G. By (3), we can obtain a maximal matching M in G. Let W be the set of vertices not covered by M. For each vertex $w \in W$, the set $E_w = \{e \in E_G : e$ is incident with $w\}$ is well-orderable being a subset of the well-orderable set $(E_G, \prec')$. Let f_w be the $(\prec' \restriction E_w)$-minimal element of E_w. Let $F = \{f_w : w \in W\}$ and let $M_1 = \{e \in M :$ at least one endpoint of e is not covered by $F\}$. Then $F \cup M_1$ is a minimal edge cover of G. □

Remark 3.4. We remark that the direct proofs of items (1)–(3) of Proposition 3.3 do not adapt immediately to give a proof of item (4); the issue is in the limit steps, where a vertex of infinite degree might not be covered anymore by the intersection of edge covers.

4 Proper and distinguishing colorings

Definition 4.1. Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a disjoint countably infinite family of non-empty finite sets and $T = \{t_n : n \in \omega\}$ be a countably infinite sequence disjoint from $A = \bigcup_{n\in\omega} A_n$. Let $G_1(\mathcal{A}, T) = (V_{G_1(\mathcal{A},T)}, E_{G_1(\mathcal{A},T)})$ be the infinite locally finite connected graph such that

$$\begin{aligned} V_{G_1(\mathcal{A},T)} &:= \Big(\bigcup_{n\in\omega} A_n\Big) \cup T, \\ E_{G_1(\mathcal{A},T)} &:= \Big\{\{t_n, t_{n+1}\} : n \in \omega\Big\} \cup \Big\{\{t_n, x\} : n \in \omega, x \in A_n\Big\} \\ &\qquad \cup \Big\{\{x, y\} : n \in \omega, x, y \in A_n, x \neq y\Big\}. \end{aligned}$$

We denote by $\mathcal{C}$ the statement "For any disjoint countably infinite family of non-empty finite sets $\mathcal{A}$, and any countably infinite sequence $T = \{t_n : n \in \omega\}$ disjoint from $A = \bigcup_{n\in\omega} A_n$, the graph $G_1(\mathcal{A}, T)$ has a chromatic number" and we denote by $\mathcal{C}_k$ the statement "Any infinite locally finite connected graph G such that $\delta(G) \geq k$ has a chromatic number".

Theorem 4.2 (ZF)**.** *Fix a natural number $k \geq 3$. The following statements are equivalent:*

(1) *Kőnig's Lemma.*

(2) $\mathcal{C}$.

(3) $\mathcal{C}_k$.

(4) *Any infinite locally finite connected graph has a chromatic number.*

(5) *Any infinite locally finite connected graph has a chromatic index.*

(6) *Any infinite locally finite connected graph has a distinguishing number.*

(7) *Any infinite locally finite connected graph has a distinguishing index.*

Proof. (1)⇒(2)–(7) Let $G = (V_G, E_G)$ be an infinite locally finite connected graph. Pick some $r \in V_G$. Let $V_0(r) = \{r\}$. For each integer $n \geq 1$, define $V_n(r) = \{v \in V_G : d_G(r, v) = n\}$ where "$d_G(r, v) = n$" means there are n edges in the shortest path joining r and v. Each $V_n(r)$ is finite by the local finiteness of G, and $V_G = \bigcup_{n\in\omega} V_n(r)$ since G is connected. By $\mathsf{AC}^{\omega}_{\mathsf{fin}}$, V_G is countably infinite (and hence, well-orderable). The rest follows from Fact 3.1(1), (4) and the fact that $G_1(\mathcal{A}, T)$ is an infinite locally finite connected graph for any given disjoint countably infinite family $\mathcal{A}$ of non-empty finite sets and any countably infinite sequence $T = \{t_n : n \in \omega\}$ disjoint from $A = \bigcup_{n\in\omega} A_n$.

(2)⇒(1) Since $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ is equivalent to its partial version $\mathsf{PAC}^{\omega}_{\mathsf{fin}}$ (Every countably infinite family of non-empty finite sets has an infinite subfamily with a choice function) (cf. [12], [4, the proof of Theorem 4.1(i)] or footnote 5), it suffices to show that $\mathcal{C}$ implies $\mathsf{PAC}^{\omega}_{\mathsf{fin}}$. In order to achieve this, we modify the arguments of Herrlich–Tachtsis [10, Proposition 23] suitably. Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a countably infinite set of non-empty finite sets without a partial choice function. Without loss of generality, we assume that $\mathcal{A}$ is disjoint. Pick a countably infinite sequence $T = \{t_n : n \in \omega\}$ disjoint from $A = \bigcup_{i\in\omega} A_i$ and consider the graph $G_1(\mathcal{A}, T) = (V_{G_1(\mathcal{A},T)}, E_{G_1(\mathcal{A},T)})$ as in Figure 1.

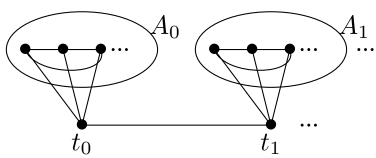

Figure 1: Graph $G_1(\mathcal{A}, T)$, an infinite locally finite connected graph.

Let $f\colon V_{G_1(\mathcal{A},T)} \to C$ be a C-proper vertex coloring of $G_1(\mathcal{A}, T)$, i.e., a map such that if $\{x, y\} \in E_{G_1(\mathcal{A},T)}$ then $f(x) \neq f(y)$. Then for each $c \in C$, the set $M_c = \{v \in f^{-1}(c) : v \in A_i \text{ for some } i \in \omega\}$ must be finite, otherwise M_c will generate a partial choice function for $\mathcal{A}$.

Claim 4.3. *$f[\bigcup_{n\in\omega} A_n]$ is infinite.*

Proof. Otherwise, $\bigcup_{n\in\omega} A_n = \bigcup_{c\in f[\bigcup_{n\in\omega} A_n]} M_c$ is finite since the finite union of finite sets is finite in ZF and we obtain a contradiction. □

Claim 4.4. *$f[\bigcup_{n\in\omega} A_n]$ is Dedekind-finite.*

Proof. First, we note that $\bigcup_{n\in\omega} A_n$ is Dedekind-finite since $\mathcal{A}$ has no partial choice function. For the sake of contradiction, assume that $C = \{c_i : i \in \omega\}$ is a countably infinite subset of $f[\bigcup_{n\in\omega} A_n]$. Fix a well-ordering $<$ of $\mathcal{A}$ (since $\mathcal{A}$ is countable, and hence well-orderable). Define d_i to be the *unique* element of $M_{c_i} \cap A_n$ where n is the $<$-least element of $\{m \in \omega : M_{c_i} \cap A_m \neq \emptyset\}$. Such an n exists since $c_i \in f[\bigcup_{n<\omega} A_n]$ and $M_{c_i} \cap A_n$ has a single element since f is a proper vertex coloring. Then $\{d_i : i \in \omega\}$ is a countably infinite subset of $\bigcup_{n\in\omega} A_n$ which contradicts the fact that $\bigcup_{n\in\omega} A_n$ is Dedekind-finite. □

The following claim states that $\mathcal{C}$ fails.

Claim 4.5. *There is a C_1-proper vertex coloring $f\colon V_{G_1(\mathcal{A},T)} \to C_1$ of $G_1(\mathcal{A},T)$ such that $C_1 \prec C$. Thus, $G_1(\mathcal{A},T)$ has no chromatic number.*

Proof. Fix some $c_0 \in f[\bigcup_{n\in\omega} A_n]$. Then $\text{Index}(M_{c_0}) = \{n \in \omega : M_{c_0} \cap A_n \neq \emptyset\}$ is finite. By Claim 4.3, there exists some $b_0 \in (f[\bigcup_{n\in\omega} A_n] \backslash \bigcup_{m\in\text{Index}(M_{c_0})} f[A_m])$ since the finite union of finite sets is finite. Define a proper vertex coloring $g\colon \bigcup_{n\in\omega} A_n \to (f[\bigcup_{n\in\omega} A_n] \backslash c_0)$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } f(x) \neq c_0, \\ b_0 & \text{otherwise.} \end{cases}$$

Similarly, define a proper vertex coloring $h\colon \bigcup_{n\in\omega} A_n \to (f[\bigcup_{n\in\omega} A_n] \backslash \{c_0, c_1, c_2\})$ for some $c_0, c_1, c_2 \in f[\bigcup_{n\in\omega} A_n]$. Let $h(t_{2n}) = c_0$ and $h(t_{2n+1}) = c_1$ for all $n \in \omega$. Thus, $h\colon V_{G_1} \to (f[\bigcup_{n\in\omega} A_n] \backslash \{c_2\})$ is a $f[\bigcup_{n\in\omega} A_n] \backslash \{c_2\}$-proper vertex coloring of G_1. We define $C_1 = f[\bigcup_{n\in\omega} A_n] \backslash \{c_2\}$. By Claim 4.4, $C_1 \prec f[\bigcup_{n\in\omega} A_n] \preceq C$. □

Similarly, we can see (4)⇒(1).

(3)⇒(1) Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a disjoint countably infinite set of non-empty finite sets without a partial choice function, such that k divides $|A_n|$ for each $n \in \omega$ and $k \in \omega \backslash \{0, 1\}$. Assume T and $G_1(\mathcal{A}, T)$ as in the proof of (2)⇒(1). Then $\delta(G_1(\mathcal{A}, T)) \geq k$. By the arguments of (2)⇒(1), $\mathcal{C}$ implies $\mathsf{PAC}^{\omega}_{k\times\mathsf{fin}}$. Following the arguments of [4, Theorem 4.1], we can see that $\mathsf{PAC}^{\omega}_{k\times\mathsf{fin}}$ implies $\mathsf{AC}^{\omega}_{\mathsf{fin}}$.[5]

(5)⇒(1) Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a disjoint countably infinite set of non-empty finite sets without a partial choice function and $T = \{t_n : n \in \omega\}$ be a sequence disjoint from $A = \bigcup_{n\in\omega} A_n$. Let H_1 be the graph obtained from the graph $G_1(\mathcal{A}, T)$ of (2)⇒(1) after deleting the edge set $\{\{x, y\} : n \in \omega, x, y \in A_n, x \neq y\}$. Clearly, H_1 is an infinite locally finite connected graph.

Claim 4.6. *H_1 has no chromatic index.*

[5] For the reader's convenience, we write down the proof. First, we can see that $\mathsf{PAC}^{\omega}_{k\times\mathsf{fin}}$ implies $\mathsf{AC}^{\omega}_{k\times\mathsf{fin}}$. Fix a family $\mathcal{A} = \{A_i : i \in \omega\}$ of disjoint nonempty finite sets such that k divides $|A_i|$ for each $i \in \omega$. Then the family

$$\mathcal{B} = \{B_i : i \in \omega\} \text{ where } B_i = \textstyle\prod_{j\leq i} A_j$$

is a disjoint family such that k divides $|B_i|$ and any partial choice function on $\mathcal{B}$ yields a choice function for $\mathcal{A}$.

Finally, fix a family $\mathcal{C} = \{C_i : i \in \omega\}$ of disjoint nonempty finite sets. Then $\mathcal{D} = \{D_i : i \in \omega\}$ where $D_i = C_i \times k$ is a pairwise disjoint family of finite sets where k divides $|D_i|$ for each $i \in \omega$. Thus $\mathsf{AC}^{\omega}_{k\times\mathsf{fin}}$ implies that $\mathcal{D}$ has a choice function f which determines a choice function for $\mathcal{C}$.

Proof. Assume that the graph H_1 has a chromatic index. Let $f\colon E_{H_1} \to C$ be a proper edge coloring with $|C| = \kappa$, where κ is the chromatic index of H_1. Let $B = \{\{t_n, x\} : n \in \omega, x \in A_n\}$. Similar to Claims 4.3, 4.4, and 4.5, $f[B]$ is an infinite, Dedekind-finite set and there is a proper edge coloring $h\colon B \to f[B] \setminus \{c_0, c_1, c_2\}$ for some $c_0, c_1, c_2 \in f[B]$. Finally, define $h(\{t_{2n}, t_{2n+1}\}) = c_0$ and $h(\{t_{2n+1}, t_{2n+2}\}) = c_1$ for all $n \in \omega$. Thus, we obtain a $f[B] \setminus \{c_2\}$-proper edge coloring $h\colon E_{H_1} \to f[B] \setminus \{c_2\}$, with $f[B] \setminus \{c_2\} \prec f[B] \preceq C$ as $f[B]$ is Dedekind-finite, contradicting the fact that κ is the chromatic index of H_1. $\square$

(6)$\Rightarrow$(1) Assume $\mathcal{A}$ and T as in the proof of (5)$\Rightarrow$(1). Let H_1^1 be the graph obtained from H_1 of (5)$\Rightarrow$(1) by adding two new vertices t' and t'' and the edges $\{t'', t'\}$ and $\{t', t_0\}$ (see Figure 2).

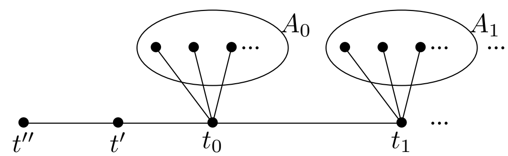

Figure 2: Graph H_1^1, an infinite locally finite connected graph.

It suffices to show that H_1^1 has no distinguishing number. We recall that whenever $j\colon V_{H_1^1} \to V_{H_1^1}$ is an automorphism, $\varphi(x_1, ..., x_r)$ is a first-order $\mathcal{L}$-formula on r variables (where $\mathcal{L}$ is the language of graphs) for some $r \in \omega\backslash\{0\}$ and $a_i \in V_{H_1^1}$ for each $1 \leq i \leq r$, then $H_1^1 \models \varphi(a_1, ..., a_r)$ if and only if $H_1^1 \models \varphi(j(a_1), ..., j(a_r))$ (cf. Definition 2.8).

Claim 4.7. *t', t'', and t_m are fixed by any automorphism for each non-negative integer m.*

Proof. Fix non-negative integers n, m, r. The first-order $\mathcal{L}$-formula

$$\mathsf{Deg}_n(x) := \exists x_0 \ldots \exists x_{n-1} \Big(\bigwedge_{i \neq j}^{n-1} x_i \neq x_j \wedge \bigwedge_{i<n} x \neq x_i \wedge \bigwedge_{i<n} Exx_i \wedge \forall y (Exy \rightarrow \bigvee_{i<n} y = x_i) \Big)$$

expresses the property that a vertex x has degree n, where Eab denotes the existence of an edge between vertices a and b. We define the following first-order $\mathcal{L}$-formula:

$$\varphi(x) := \mathsf{Deg}_1(x) \wedge \exists y (Exy \wedge \mathsf{Deg}_2(y)).$$

It is easy to see the following:

(i) t'' is the unique vertex such that $H_1^1 \models \varphi(t'')$. This means t'' is the unique vertex such that $\deg(t'') = 1$ and t'' has a neighbor of degree 2.

(ii) t' is the unique vertex such that $H_1^1 \models \mathsf{Deg}_2(t')$. So t' is the unique vertex with $\deg(t') = 2$.

Fix any automorphism τ. Since every automorphism preserves the properties mentioned in (i)–(ii), t' and t'' are fixed by τ. The vertices t_m are fixed by τ by induction as follows: Since t_i is the unique vertex of path length $i+1$ from t'' such that the degree of t_i is greater than 1, where $i \in \{0, 1\}$, we have that t_0 and t_1 are fixed by τ. Assume that $\tau(t_l) = t_l$ for all $l < m-1$. We show that $\tau(t_m) = t_m$. Now, $\tau(t_m)$ is a neighbour of $\tau(t_{m-1}) = t_{m-1}$ which is of degree at least 2, so $\tau(t_m)$ must be either t_{m-2} or t_m, but $t_{m-2} = \tau(t_{m-2})$ is already taken. So, $\tau(t_m) = t_m$. □

Claim 4.8. *Fix $m \in \omega$ and $x \in A_m$. Then* $\mathrm{Orb}_{\mathrm{Aut}(H_1^1)}(x) = \{g(x) : g \in \mathrm{Aut}(H_1^1)\} = A_m$.

Proof. This follows from the fact that each $y \in \bigcup_{n\in\omega} A_n$ has path length 1 from t_m if and only if $y \in A_m$. □

Claim 4.9. *H_1^1 has no distinguishing number.*

Proof. Assume that the graph H_1^1 has a distinguishing number. Let $f\colon V_{H_1^1} \to C$ be a distinguishing vertex coloring with $|C| = \kappa$, where κ is the distinguishing number of H_1^1. Similar to Claims 4.3 and 4.4, $f[\bigcup_{n\in\omega} A_n]$ is infinite and Dedekind-finite. Consider a coloring $h\colon \bigcup_{n\in\omega} A_n \to f[\bigcup_{n\in\omega} A_n] \backslash \{c_0, c_1, c_2\}$ for some $c_0, c_1, c_2 \in f[\bigcup_{n\in\omega} A_n]$, just as in Claim 4.5. Let $h(t) = c_0$ for all $t \in \{t'', t'\} \cup T$. Then, $h\colon V_{H_1^1} \to (f[\bigcup_{n\in\omega} A_n] \backslash \{c_1, c_2\})$ is a $f[\bigcup_{n\in\omega} A_n] \setminus \{c_1, c_2\}$-distinguishing vertex coloring of H_1^1. Finally, $f[\bigcup_{n\in\omega} A_n] \setminus \{c_1, c_2\} \prec f[\bigcup_{n\in\omega} A_n] \preceq C$ contradicts the fact that κ is the distinguishing number of H_1^1. □

(7)⇒(1) Assume $\mathcal{A}$, T, and H_1^1 as in the proof of (6)⇒(1). By Claim 4.7, every automorphism fixes the edges $\{t'', t'\}$, $\{t', t_0\}$ and $\{t_n, t_{n+1}\}$ for each $n \in \omega$. Moreover, if H_1^1 has a distinguishing edge coloring f, then for each $n \in \omega$ and $x, y \in A_n$ such that $x \neq y$, $f(\{t_n, x\}) \neq f(\{t_n, y\})$.

Claim 4.10. *H_1^1 has no distinguishing index.*

Proof. This follows modifying the arguments of Claims 4.6 and 4.9. □

□

5 Irreducible proper coloring and covering properties

Theorem 5.1 (ZF)**.** *The following statements are equivalent:*

(1) *Kőnig's Lemma.*

(2) *Every infinite locally finite connected graph has an irreducible proper coloring.*

(3) *Every infinite locally finite connected graph has a minimal dominating set.*

(4) *Every infinite locally finite connected graph has a minimal edge cover.*

(5) *Every infinite locally finite connected graph has a maximal matching.*

Proof. Implications (1)⇒(2)–(5) follow from Proposition 3.3, and the fact that $\mathsf{AC}_{\mathsf{fin}}^{\omega}$ implies every infinite locally finite connected graph is countably infinite.

(2)⇒(1) In view of the proof of Theorem 4.2((2)⇒(1)), it suffices to show that the given statement implies $\mathsf{PAC}_{\mathsf{fin}}^{\omega}$. Let $\mathcal{A} = \{A_n : n \in \omega \backslash \{0\}\}$ be a disjoint countably infinite set of non-empty finite sets without a partial choice function. Pick $t \notin \bigcup_{i\in\omega\backslash\{0\}} A_i$. Let

$A_0 = \{t\}$. Consider the following infinite locally finite connected graph $G_2 = (V_{G_2}, E_{G_2})$ (see Figure 3):

$$V_{G_2} := \bigcup_{n\in\omega} A_n,$$

$$E_{G_2} := \Big\{\{t,x\} : x \in A_1\Big\} \cup \Big\{\{x,y\} : n \in \omega\backslash\{0\}, x,y \in A_n, x \neq y\Big\} \cup \Big\{\{x,y\} : n \in \omega\backslash\{0\}, x \in A_n, y \in A_{n+1}\Big\}.$$

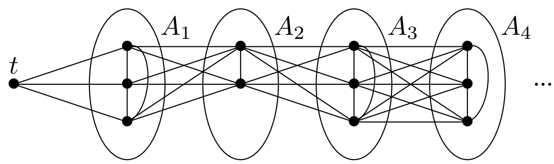

Figure 3: The graph G_2 when $|A_1| = |A_3| = |A_4| = 3$, and $|A_2| = 2$.

Claim 5.2. *G_2 has no irreducible proper coloring.*

Proof. Let $f\colon V_{G_2} \to C$ be a C-irreducible proper coloring of G_2, i.e., a map such that $f(x) \neq f(y)$ if $\{x,y\} \in E_{G_2}$ and $(\forall c_1, c_2 \in C) f^{-1}(c_1) \cup f^{-1}(c_2)$ is dependent. Similar to the proof of Theorem 4.2((2)⇒(1)), $f^{-1}(c)$ is finite for all $c \in C$, and $f[\bigcup_{n\in\omega\backslash\{0\}} A_n]$ is infinite. Fix $c_0 \in f[\bigcup_{n\in\omega\backslash\{0\}} A_n]$. Then $\text{Index}(f^{-1}(c_0)) = \{n \in \omega\backslash\{0\} : f^{-1}(c_0) \cap A_n \neq \emptyset\}$ is finite. So there exists some

$$c_1 \in f[\textstyle\bigcup_{n\in\omega\backslash\{0\}} A_n] \backslash \bigcup_{m\in\text{Index}(f^{-1}(c_0))} (f[A_m] \cup f[A_{m-1}] \cup f[A_{m+1}])$$

as $\bigcup_{m\in\text{Index}(f^{-1}(c_0))}(f[A_m] \cup f[A_{m-1}] \cup f[A_{m+1}])$ is finite. Clearly, $f^{-1}(c_0) \cup f^{-1}(c_1)$ is independent, and we obtain a contradiction. □

(3)⇒(1) Assume $\mathcal{A}$ as in the proof of (2)⇒(1). Let G_2^1 be the infinite locally finite connected graph obtained from G_2 of (2)⇒(1) after deleting t and $\{\{t,x\} : x \in A_1\}$. Consider a minimal dominating set D of G_2^1. The following conditions must be satisfied:

(i) Since D is a dominating set, for each $n \in \omega \setminus \{0,1\}$, there is an $a \in D$ such that $a \in A_{n-1} \cup A_n \cup A_{n+1}$ (otherwise, no vertices from A_n belongs to D or have a neighbor in D).

(ii) By the minimality of D, we have $|A_n \cap D| \leq 1$ for each $n \in \omega \setminus \{0\}$.

Clearly, (i) and (ii) determine a partial choice function over $\mathcal{A}$, contradicting the assumption that $\mathcal{A}$ has no partial choice function.

(4)⇒(1) Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a disjoint countably infinite set of non-empty finite sets and let $A = \bigcup_{n\in\omega} A_n$. Consider a countably infinite family $(B_i, <_i)_{i\in\omega}$ of well-ordered sets such that the following hold (cf. the proof of [5, Theorem 1, Remark 6]):

(i) $|B_i| = |A_i| + k$ for some fixed $1 \leq k \in \omega$ and thus, there is no mapping with domain A_i and range B_i.

(ii) for each $i \in \omega$, B_i is disjoint from A and the other B_j's.

Let $B = \bigcup_{i\in\omega} B_i$. Pick a countably infinite sequence $T = \{t_i : i \in \omega\}$ disjoint from A and B and consider the following infinite locally finite connected graph $G_3 = (V_{G_3}, E_{G_3})$ (see Figure 4):

$$V_{G_3} := A \cup B \cup T,$$
$$E_{G_3} := \Big\{\{t_i, t_{i+1}\} : i \in \omega\Big\} \cup \Big\{\{t_i, x\} : i \in \omega, x \in A_i\Big\}$$
$$\cup \Big\{\{x, y\} : i \in \omega, x \in A_i, y \in B_i\Big\}.$$

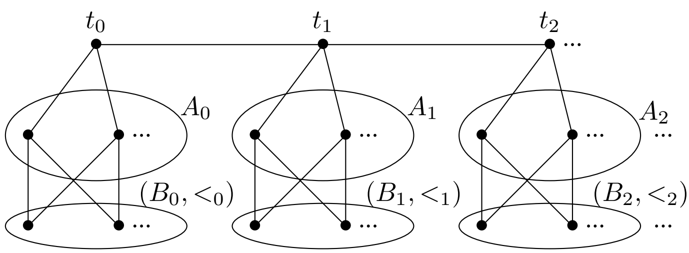

Figure 4: Graph G_3.

By assumption, G_3 has a minimal edge cover, say G'_3. For each $i \in \omega$, let $f_i \colon B_i \to \mathcal{P}(A_i)\backslash\{\emptyset\}$ map each vertex of B_i to its neighborhood in G'_3.

Claim 5.3. *Fix $i \in \omega$. For any two distinct ϵ_1 and ϵ_2 in B_i, $|f_i(\epsilon_1) \cap f_i(\epsilon_2)| \leq 1$.*

Proof. This follows from the fact that G'_3 does not contain a complete bipartite subgraph $K_{2,2}$. In particular, each component of G'_3 has at most one vertex of degree greater than 1. If any edge $e \in G'_3$ has both of its endpoints incident on edges of $G'_3\backslash\{e\}$, then $G'_3\backslash\{e\}$ is also an edge cover of G_3, contradicting the minimality of G'_3. □

By Fact 3.1(2) and (i), there are tuples $(\epsilon'_1, \epsilon'_2) \in B_i \times B_i$ such that $f_i(\epsilon'_1) \cap f_i(\epsilon'_2) \neq \emptyset$. Consider the first such tuple $(\epsilon''_1, \epsilon''_2)$ with respect to the lexicographical ordering of $B_i \times B_i$. Then $\{f_i(\epsilon''_1) \cap f_i(\epsilon''_2) : i \in \omega\}$ is a choice function of $\mathcal{A}$ by Claim 5.3.

(5)⇒(1) Assume $\mathcal{A}$, and A as in the proof of (4)⇒(1). Let $R = \{r_n : n \in \omega\}$ and $T = \{t_n : n \in \omega\}$ be two disjoint countably infinite sequences disjoint from A. We define the following locally finite connected graph $G_4 = (V_{G_4}, E_{G_4})$ (see Figure 5):

$$V_{G_4} := (\bigcup_{n\in\omega} A_n) \cup R \cup T,$$
$$E_{G_4} := \Big\{\{t_n, t_{n+1}\} : n \in \omega\Big\} \cup \Big\{\{t_n, x\} : n \in \omega, x \in A_n\Big\}$$
$$\cup \Big\{\{r_n, x\} : n \in \omega, x \in A_n\Big\}.$$

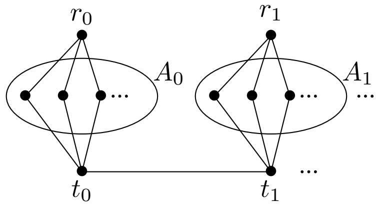

Figure 5: Graph G_4.

Let M be a maximal matching of G_4. For all $i \in \omega$, there is at most one $x \in A_i$ such that $\{r_i, x\} \in M$ since M is a matching and there is at least one $x \in A_i$ such that $\{r_i, x\} \in M$ since M is maximal. These unique $x \in A_i$ determine a choice function for $\mathcal{A}$.

This concludes the proof of the Theorem. □

6 Remarks on new equivalents of AC

Remark 6.1. We remark that the statement "Any connected graph has a minimal dominating set" implies AC.[6] Consider a family $\mathcal{A} = \{A_i : i \in I\}$ of pairwise disjoint non-empty sets. For each $i \in I$, let $B_i^0 = A_i \times \{0\}$ and $B_i^1 = A_i \times \{1\}$. Pick $t \notin \bigcup_{i \in I} B_i^0 \cup \bigcup_{i \in I} B_i^1$ and consider the following connected graph $G_5 = (V_{G_5}, E_{G_5})$ in Figure 6:

$$V_{G_5} := \{t\} \cup \bigcup_{i \in I} B_i^0 \cup \bigcup_{i \in I} B_i^1,$$

$$\begin{aligned} E_{G_5} := &\Big\{\{x,t\} : i \in I, x \in B_i^0\Big\} \cup \Big\{\{x,y\} : i \in I, x \in B_i^0, y \in B_i^1\Big\} \\ &\cup \Big\{\{x,y\} : i \in I, x, y \in B_i^0, x \neq y\Big\} \cup \Big\{\{x,y\} : i \in I, x, y \in B_i^1, x \neq y\Big\}. \end{aligned}$$

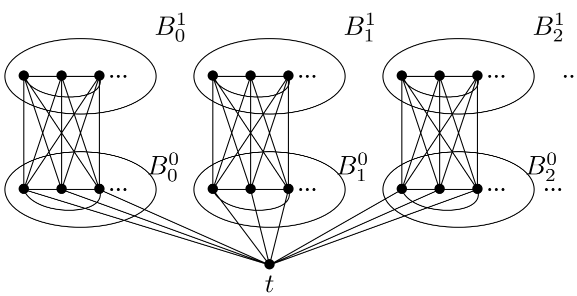

Figure 6: Graph G_5, a connected graph. If each A_i is finite, then G_5 is rayless.

[6]The authors are very thankful to one of the referees for pointing out to us an error that appeared in this remark in a former version of the paper and especially for guiding us to eliminate the error.

Let D be a minimal dominating set of G_5. Define $M_i = (B_i^0 \cup B_i^1) \cap D$ for every $i \in I$. We claim that for every $i \in I$, $|M_i| = 1$.

Case (i): If there exists an $i \in I$ such that $M_i = \emptyset$, then any member of B_i^1 is neither in D nor it has a neighbour in D. This contradicts the fact that D is a dominating set of G_5.

Case (ii): If there exists an $i \in I$ such that $|M_i| \geq 2$, then pick $x, y \in M_i$.

- Case (ii(a)): If $x, y \in B_i^0$, or $x, y \in B_i^1$, then $D \backslash \{x\}$ is a dominating set, which contradicts the minimality of D.

- Case (ii(b)): If $x \in B_i^0$, and $y \in B_i^1$, then $D \backslash \{y\}$ is a dominating set, which contradicts the minimality of D. Similarly, we can obtain a contradiction if $y \in B_i^0$, and $x \in B_i^1$.

Let $M_i = \{a_i\}$ for every $i \in I$. Define,

$$g(i) = \begin{cases} p_i^1(a_i) & \text{if } a_i \in B_i^1 \cap D, \\ p_i^0(a_i) & \text{if } a_i \in B_i^0 \cap D, \end{cases}$$

where for $m \in \{0,1\}$, $p_i^m \colon B_i^m \to A_i$ is the projection map to the first coordinate for each $i \in I$. Then, g is a choice function for $\mathcal{A}$.

Remark 6.2. The statement "Any connected bipartite graph has a minimal edge cover" implies AC. Assume $\mathcal{A} = \{A_i : i \in I\}$ as in the proof of Remark 6.1. Consider a family $\{(B_i, <_i) : i \in I\}$ of well-ordered sets with fixed well-orderings such that for each $i \in I$, B_i is disjoint from $A = \bigcup_{i \in I} A_i$ and the other B_j's, and there is no mapping with domain A_i and range B_i (see the proofs of [5, Theorem 1] and Theorem 5.1((4)⇒(1))). Let $B = \bigcup_{i \in I} B_i$. Then given some $t \notin B \cup (\bigcup_{i \in I} A_i)$, consider the following connected bipartite graph $G_6 = (V_{G_6}, E_{G_6})$ in Figure 7:

$$V_{G_6} := \{t\} \cup B \cup (\bigcup_{i \in I} A_i),$$

$$E_{G_6} := \Big\{\{x, t\} : i \in I, x \in A_i\Big\} \cup \Big\{\{x, y\} : i \in I, x \in A_i, y \in B_i\Big\}.$$

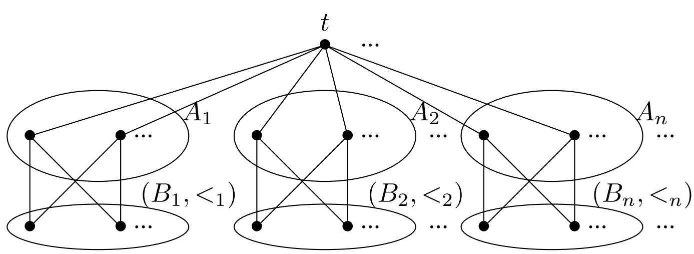

Figure 7: Graph G_6, a connected bipartite graph. If each A_i is finite, then G_6 is rayless.

The rest follows from the arguments of the implication (4)⇒(1) in Theorem 5.1.

Remark 6.3. The statement "Any connected bipartite graph has a maximal matching" implies AC. Assume $\mathcal{A}$ as in the proof of Remark 6.1. Pick a sequence $T = \{t_n : n \in I\}$ disjoint from $\bigcup_{i\in I} A_i$, a $t \notin \bigcup_{i\in I} A_i \cup T$ and consider the following connected bipartite graph $G_7 = (V_{G_7}, E_{G_7})$ in Figure 8:

$$V_{G_7} := \bigcup_{i\in I} A_i \cup T \cup \{t\}, \qquad E_{G_7} := \Big\{\{t_i, x\} : x \in A_i\Big\} \cup \Big\{\{t, t_i\} : i \in I\Big\}.$$

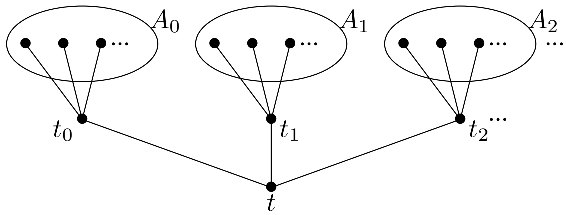

Figure 8: Graph G_7, a connected rayless bipartite graph.

Let M be a maximal matching of G_7. Clearly, $S = \{i \in I : \{t_i, t\} \in M\}$ has at most one element and for each $j \in I\backslash S$, there is exactly one $x \in A_j$ (say x_j) such that $\{x, t_j\} \in M$. Let $f(A_j) = x_j$ for each $j \in I\backslash S$. If $S \neq \emptyset$, pick any $r \in A_i$ if $i \in S$, since selecting an element from a set does not involve any form of choice. Let $f(A_i) = r$. Clearly, f is a choice function for $\mathcal{A}$.

Theorem 6.4 (ZF)**.** *The following statements are equivalent:*

(1) AC

(2) *Any connected graph has a minimal dominating set.*

(3) *Any connected bipartite graph has a maximal matching.*

(4) *Any connected bipartite graph has a minimal edge cover.*

Proof. Implications (1)⇒(2)–(4) are straightforward (cf. Proposition 3.3). The other directions follow from Remarks 6.1, 6.2, and 6.3. □

Remark 6.5. The locally finite connected graphs forbid those graphs that contain vertices of infinite degrees but may contain rays. There is another class of connected graphs that forbid rays but may contain vertices of infinite degrees. For a study of some properties of the class of rayless connected graphs, the reader is referred to Halin [8].

(1). We can see that the statement "Every connected rayless graph has a minimal dominating set" implies $\mathsf{AC}_{\mathsf{fin}}$. Consider a non-empty family $\mathcal{A} = \{A_i : i \in I\}$ of pairwise disjoint finite sets and the graph G_5 from Remark 6.1. Clearly, G_5 is connected and rayless. The rest follows by the arguments of Remark 6.1.

(2). By applying Remark 6.3 and Proposition 3.3, we can see that the statement "Every connected rayless graph has a maximal matching" is equivalent to AC.

(3). The statement "Every connected rayless graph has a minimal edge cover" implies $\mathsf{AC}_{\mathsf{fin}}$. Let $\mathcal{A} = \{A_i : i \in I\}$ be as in (1) and G_6 be the graph from Remark 6.2. Then G_6 is connected and rayless. By the arguments of Remark 6.2, the rest follows.

7 Questions

Question 7.1. Do the following statements imply AC (without assuming that the sets of colors can be well-ordered)?

(1) Any graph has a chromatic index.

(2) Any graph has a distinguishing number.

(3) Any graph without a component isomorphic to K_1 or K_2 has a distinguishing index.

Stawiski [20, Theorem 3.8] proved that the statements (1)–(3) mentioned above are equivalent to AC by assuming that the sets of colors can be well-ordered.

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