THE ART OF DISCRETE AND APPLIED MATHEMATICS

The International Congress on Hypergraphs, Graphs and Designs – HyGraDe 2017

This issue of ADAM – *The Art of Discrete and Applied Mathematics* offers a collection of papers presented at the International Congress on Hypergraphs, Graphs and Designs – HyGraDe 2017, which took place in Sant'Alessio Siculo, Sicily, Italy, June 20 – 24, 2017. HyGraDe 2017 was conceived with the idea of celebrating the 70th birthday of Mario Gionfriddo, a Sicilian mathematician who has devoted his long and successful career to the study of graphs, hypergraphs and designs.

The conference HyGraDe 2017 was organized by Francesco Belardo (University of Naples Federico II) and Giovanni Lo Faro (University of Messina), both serving as chair of the Organizing Committee, Luca Giuzzi (University of Brescia), Enzo M. Li Marzi (University of Messina), Lorenzo Milazzo (University of Catania), Salvatore Milici (University of Catania) and Antoinette Tripodi (University of Messina). The Scientific Committee consisted of Marco Buratti (University of Perugia), Giovanni Lo Faro, Guglielmo Lunardon (University of Naples Federico II) and Martin Milanič (University of Primorska).

The conference brought together scientists working in different disciplines of Combinatorics. There were 85 participants from 13 different countries, 10 invited talks and 35 contributed talks touching on the latest developments in the corresponding research areas. The invited speakers were Richard Brualdi (University of Winsconsin), Marco Buratti (University of Perugia), Charlie Colbourn (Arizona State University), Klavdjia Kutnar (University of Primorska), Josef Lauri (University of Malta), Curt Lindner (University of Auburn), Dragan Marušič (University of Primorska), Alex Rosa (McMaster University), Zsolt Tuza (Hungarian Academy of Sciences) and Vitaly Voloshin (Troy University). Mario Gionfriddo was the honoured participant at the congress. Among the invited speakers, six of them have been Mario's co-authors in several scientific papers.

The Conference Photo

The Organizing Committee thanks all who contributed to the successful organization of this event. In particular the Organizing Committee is grateful to the sponsors of this conference: Universities of Catania, Messina and Naples Federico II, the Department of Mathematics and Informatics of Catania, the Department MIFT of Messina, the INDAM-GNSAGA and the non-profit association Combinatorics 2014. We also recognize the kind support of the University of Primorska, the Department of Mathematics and Applications "R. Caccioppoli" of Naples and the Accademia Peloritana dei Pericolanti.

We are grateful to the Editors-in-Chief, Professors Tomo Pisanski and Dragan Marušič, for this issue of ADAM. We also thank all the colleagues who have participated in this initiative and the referees who have reviewed the papers. We would like to mention that a special issue of the Sicilian scientific journal *Atti Accademia Peloritana dei Pericolanti – AAPP* contains other contributions from the participants of HyGraDe 2017. The special issue of AAPP devoted to HyGraDe 2017 can be accessed at http://cab.unime.it/journals/index.php/AAPP/issue/view/Vol96_Supplement2.

Francesco Belardo

Guest Editor

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Circulant matrices and mathematical juggling*

Richard A. Brualdi
Department of Mathematics, University of Wisconsin, Madison, WI 53706

Michael W. Schroeder
Department of Mathematics, Marshall University, Huntington, WV

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Abstract

Circulants form a well-studied and important class of matrices, and they arise in many algebraic and combinatorial contexts, in particular as multiplication tables of cyclic groups and as special classes of latin squares. There is also a known connection between circulants and mathematical juggling. The purpose of this note is to expound on this connection developing further some of its properties. We also formulate some problems and conjectures with some computational data supporting them.

Keywords: Juggling, permutations, permanent, circulant matrices.

Math. Subj. Class.: 05A05, 05E25, 15A15

1 Introduction

Let n be a positive integer, and let $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ be a sequence of n nonnegative integers. Then $\mathbf{t}$ is a *juggling sequence* of length n provided that

$$1 + t_1, 2 + t_2, \ldots, n + t_n \tag{1.1}$$

are distinct modulo n, implying, in particular, that $t_1 + t_2 + \cdots + t_n \equiv 0 \pmod{n}$. Thus if (1.1) holds and balls are juggled where, at time i, there is at most one ball that lands in the juggler's hand and is immediately tossed so that it lands in t_i time units $(1 \leq i \leq n)$[1], then there are no collisions; that is, juggling balls *with one hand* according to these rules is possible (for a talented juggler!). The number of balls juggled equals $(t_1+t_2+\cdots+t_n)/n$. If we extend $\mathbf{t}$ to a two-way infinite sequence $(t_i : i \in \mathbb{Z})$ where $t_i = t_{i \bmod n}$, then a ball

*We are indebted to a referee who helped improve our exposition.
E-mail addresses: brualdi@math.wisc.edu (Richard A. Brualdi), schroederm@marshall.edu (Michael W. Schroeder)

[1] If $t_i = 0$, then there is no ball to toss at time i.

caught at time i is tossed so that it lands at time $i + t_i$. This defines certain *orbits* of the balls being juggled determined by the times at which a specified ball is caught and then tossed.

The sequence $\mathbf{t}$ is a *minimal juggling sequence* provided that the integers t_i have been reduced modulo n to $0, 1, \ldots, n-1$. In particular, $t_i = n$ (a ball is caught and tossed at time i to land in n time units) is equivalent to $t_i = 0$ (no ball is caught and tossed at time i). For some references on mathematical juggling and related work, see e.g. [1, 4, 10].

We now briefly summarize the contents of this paper. In the next section we introduce many examples and discuss some basic properties of juggling sequences and show how they correspond to decompositions of all 1's matrices. We also show how palindromic juggling sequences correspond to a special graph property. In Section 3, we elaborate on the connection between juggling sequences and circulant matrices as discussed in [3], and relate juggling sequences to the permanent of circulants defined in terms of n indeterminates. In Section 4, we present some calculations concerning the coefficients of the distinct terms in the permanents of these circulants and discuss certain questions and conjectures. Finally, in Section 5 we discuss the existence of juggling sequences with additional properties. Part of the purpose of this paper is to draw attention to a number of directions, questions, and conjectures concerning juggling sequences and the permanent expansion of circulants.

2 Juggling sequences

In this section we introduce some of the basic ideas of juggling sequences with many examples and, in the case of palindromic juggling sequences, establish a connection with matchings in complete graphs.

A theorem of M. Hall, Jr. [8] for abelian groups when restricted to cyclic groups yields the following result concerning juggling sequences.

Theorem 2.1. *Let $U = \{u_1, u_2, \ldots, u_n\}$ be a multiset of n integers. Then there is at least one permutation π of $\{1, 2, \ldots, n\}$ such that $\mathbf{u}_\pi = (u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(n)})$ is a juggling sequence, that is, for which*

$$1 + u_{\pi(1)}, 2 + u_{\pi(2)}, \ldots, n + u_{\pi(n)}$$

are distinct modulo n, if and only if

$$u_1 + u_2 + \cdots + u_n \equiv 0 \pmod{n}. \tag{2.1}$$

In this theorem there is no loss in generality in assuming that $0 \le u_1, u_2, \ldots, u_n \le n-1$.

In view of Theorem 2.1, we call a multiset $U = \{u_1, u_2, \ldots, u_n\}$ of n integers satisfying (2.1) a *juggleable set* of size n. If $u_1, u_2, \ldots, u_n$ have been reduced modulo n, then we have a *minimal juggleable set*. It follows from Theorem 2.1 that $U = \{0, 1, 2, \ldots, n-1\}$ is a (minimal) juggleable set if and only if n is odd.

Given $U = \{u_1, u_2, \ldots, u_n\}$, whether or not U is a juggleable set is independent of which representatives of the equivalence classes modulo n determined by the u_i have been chosen, in particular, whether or not the integers u_i have been reduced modulo n. But if $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ is a juggling sequence for the juggleable set U, the number of balls that are juggled depends on which representatives of the equivalence classes modulo n have been chosen, in particular, on whether or not the integers in U have been reduced modulo n.

A juggling sequence $(t_1, t_2, \ldots, t_n)$ is determined by a unique permutation of $\{1, 2, \ldots, n\}$ and conversely any permutation of $\{1, 2, \ldots, n\}$ determines a unique juggling sequence.

Example 2.2. Let $n = 7$ and consider the permutation σ of $\{1, 2, 3, 4, 5, 6, 7\}$ whose cycle decomposition is $(1, 5, 6)(2, 4, 7, 3)$. (Thus in σ, $1 \to 5 \to 6 \to 1$ and $2 \to 4 \to 7 \to 3 \to 2$). For each $i = 1, 2, \ldots, 7$, define $t_i = \sigma(i) - i \bmod 7$, then $\mathbf{t} = (4, 2, 6, 3, 1, 2, 3)$ is a minimal juggling sequence.

Reversing this procedure, let $n = 9$ and consider the juggling sequence $\mathbf{t} = (1, 5, 3, 4, 8, 3, 3, 6, 3)$. We obtain a permutation σ of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ by calculating and reducing modulo 9:

$$\begin{array}{lll} \sigma(1) = 1 + 1 = 2, & \sigma(2) = 5 + 2 = 7, & \sigma(3) = 3 + 3 = 6, \\ \sigma(4) = 4 + 4 = 8, & \sigma(5) = 8 + 5 = 4, & \sigma(6) = 3 + 6 = 9, \\ \sigma(7) = 3 + 7 = 1, & \sigma(8) = 6 + 8 = 5, & \sigma(9) = 3 + 9 = 3. \end{array}$$

Thus σ is the permutation with cycle decomposition $(1, 2, 7)(3, 6, 9)(4, 8, 5)$. ♢

Example 2.3. Let $n = 3$ and consider $\mathbf{t} = (4, 4, 1)$. Then to juggle according to $\mathbf{t}$ requires three balls and the balls determine three *orbits* of $\mathbb{Z}$:

$$\begin{array}{c} \cdots \to 1 \to 5 \to 9 \to 10 \to 14 \to 18 \to 19 \to \cdots, \\ \cdots \to 2 \to 6 \to 7 \to 11 \to 15 \to 16 \to 20 \to \cdots, \\ \cdots \to 3 \to 4 \to 8 \to 12 \to 13 \to 17 \to 21 \to \cdots. \end{array}$$

(Here, for instance, $2 \to 6$ represents the fact that at time unit 2, a ball is tossed so that it lands in 4 time units in the future, that is, at time unit 6; then the ball is tossed to land in 1 time unit in the future, that is at time unit 7.) Reducing $\mathbf{t} \bmod 3$ to $(1, 1, 1)$ results in only one ball and only one orbit:

$$\cdots \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to \cdots.$$

Let $J_{m,n}$ denote the $m \times n$ matrix of all 1's. Juggling using the juggling sequence $(4, 4, 1)$ gives a decomposition of the matrix $J_{3,3}$ of all 1's whereby any three consecutive matrices sum to $J_{3,3}$. (The first subscript '3' in $J_{3,3}$ represents the number of balls juggled, the second '3' represents the number of terms in the juggling sequence. The ordering of the rows is arbitrary.) This is indicated by

	4	4	1	4	4	1	4	4	1	
	1				1				1	
…		1				1	1			…,
			1	1				1		

giving

$$J_{3,3} = \left[\begin{array}{c|c|c} 1 & & \\ \hline & 1 & \\ \hline & & 1 \end{array}\right] + \left[\begin{array}{c|c|c} & 1 & \\ \hline & & 1 \\ \hline 1 & & \end{array}\right] + \left[\begin{array}{c|c|c} & & 1 \\ \hline 1 & & \\ \hline & 1 & \end{array}\right].$$

Using the $\bmod 3$ reduction $(1, 1, 1)$ of $(4, 4, 1)$ gives the trivial decomposition

$$J_{1,3} = \left[\begin{array}{c|c|c} 1 & 1 & 1 \end{array}\right].$$

♢

Example 2.4. Let $n = 5$ and consider $\mathbf{t} = (3, 3, 4, 4, 1)$. Then juggling (with three balls) using this juggling sequence is indicated by

$$\cdots \begin{array}{|c|c|c|c|c||c|c|c|c|c||c|c|c|c|c|} 3 & 3 & 4 & 4 & 1 & 3 & 3 & 4 & 4 & 1 & 3 & 3 & 4 & 4 & 1 \\ \hline\hline 1 & & & 1 & & & & 1 & & & & 1 & & & 1 \\ \hline & & 1 & & & & 1 & & & 1 & 1 & & & 1 & \\ \hline & 1 & & & 1 & 1 & & & 1 & & & & 1 & & \end{array} \cdots,$$

giving the decomposition

$$J_{3,5} = \left[\begin{array}{c|c|c|c|c} 1 & & & 1 & \\ \hline & & 1 & & \\ \hline & 1 & & & 1 \end{array}\right] + \left[\begin{array}{c|c|c|c|c} & & 1 & & \\ \hline & 1 & & & 1 \\ \hline 1 & & & 1 & \end{array}\right] + \left[\begin{array}{c|c|c|c|c} & 1 & & & 1 \\ \hline 1 & & & 1 & \\ \hline & & 1 & & \end{array}\right].$$

The juggling sequence $\mathbf{t} = (2, 4, 2, 3, 4)$ corresponds to

$$\cdots \begin{array}{|c|c|c|c|c||c|c|c|c|c||c|c|c|c|c|} 2 & 4 & 2 & 3 & 4 & 2 & 4 & 2 & 3 & 4 & 2 & 4 & 2 & 3 & 4 \\ \hline\hline 1 & & 1 & & 1 & & & & 1 & & & 1 & & & \\ \hline & & & 1 & & & 1 & & & & 1 & & 1 & & 1 \\ \hline & 1 & & & & 1 & & 1 & & 1 & & & & 1 & \end{array} \cdots,$$

and gives a different decomposition of $J_{3,5}$. ♢

We call a juggling sequence $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ *decomposable* provided the permutation associated with $\mathbf{t}$ has at least two nontrivial cycles in its cycle decomposition. Equivalently, $\mathbf{t}$ is decomposable provided $\mathbf{t} = \mathbf{r} + \mathbf{s}$ where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$, $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ are juggling sequences such that $\{r_i, s_i\} = \{0, t_i\}$ for $i = 1, 2, \ldots, n$, and $\mathbf{r}, \mathbf{s} \neq \mathbf{t}$. Any juggling sequence can be uniquely written as a sum of indecomposable juggling sequences arising from the unique cycle decomposition of the associated permutation.

Example 2.5. With $n = 9$, $\mathbf{t} = (1, 5, 3, 4, 8, 3, 3, 6, 3)$ is a juggling sequence (4 balls). The corresponding decomposition is not that of $J_{3,9}$ but, after permutation of columns, is

$$J_{1,3} \oplus J_{1,3} \oplus \left(\left[\begin{array}{c|c|c} 1 & 1 & \\ \hline & & 1 \end{array}\right] + \left[\begin{array}{c|c|c} & & 1 \\ \hline 1 & 1 & \end{array}\right] \right).$$ ♢

We summarize this discussion with the following theorem.

Theorem 2.6. *Let $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ be a sequence of n integers. Then $\mathbf{t}$ is a (minimal) juggling sequence if and only if $\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ defined by*

$$\sigma(i) \equiv t_i + i \pmod{n}$$

is a permutation of $\{1, 2, \ldots, n\}$. There is a one-to-one correspondence between minimal juggling sequences of length n and permutations of $\{1, 2, \ldots, n\}$.

Notice that Theorem 2.6 provides an algorithm to determine whether a sequence is a juggling sequence.

Knutson (see [10]) showed how to generate all juggling sequences of length n with k balls ($1 \leq k \leq n$) from the constant juggling sequence $(k, k, \ldots, k)$ of length n. There are two transformations used in the algorithm:

I. Given a juggling sequence $(k_1, k_2, \ldots, k_n)$, the *cyclic shift* $(k_n, k_1, k_2, \ldots, k_{n-1})$ is also a juggling sequence.

II. Given a juggling sequence $(k_1, \ldots, k_i, \ldots, k_j, \ldots, k_n)$, then the *swap* $(k_1, \ldots, (j-i)+k_j, \ldots, -(j-i)+k_i, \ldots, k_n)$ is also a juggling sequence:

$$i + ((j-i)+k_j) = j + k_j \quad \text{and} \quad j + (-(j-i)+k_i) = i + k_i$$

where the balls thrown at times i and j swap landing times.

Theorem 2.7 ([10]). *Any juggling sequence of length n with k balls can be generated from the constant juggling sequence $(k, k, \ldots, k)$ by cyclic shifts and swaps.*

We now consider a special property of juggling sequences that are *palindromic*. In the following argument, we use that a sequence $(t_0, t_1, \ldots, t_{n-1})$ is a juggling sequence of length n if $t_i + i \not\equiv t_j + j \pmod{n}$ for each $i \in \{0, 1, \ldots, n-1\}$, which is a direct consequence of the original definition. That is, if $\mathbf{p}$ is a juggling sequence of length n, then, modulo n, $\mathbf{p} + (0, 1, 2, \ldots, n-1)$ will be a permutation of $\{0, 1, 2, \ldots, n-1\}$.

Let n be odd and $n = 2m+1$. Let $\mathbf{p} = (p_m, p_{m-1}, \ldots, p_1, p_0, p_1, \ldots, p_{m-1}, p_m)$ be a minimal palindromic juggling sequence. For each $i \in \{1, 2, \ldots, m\}$, define

$$x_i = p_i + i \bmod n \quad \text{and} \quad y_i = p_i - i \bmod n.$$

Since $\mathbf{p}$ is a juggling sequence, we have that, modulo n,

$$\begin{aligned}(y_m, \ldots, y_2, y_1, p_0, x_1, x_2, \ldots, x_m) &= \\ (p_m - m, \ldots, p_2 - 2, p_1 - 1, p_0, p_1 + 1, p_2 + 2, \ldots, p_m + m) &= \\ \mathbf{p} + (0, 1, \ldots, n-1) - (m, m, \ldots, m)&.\end{aligned}$$

Hence $\{p_0, x_1, \ldots, x_m, y_1, \ldots, y_m\}$ is a set of distinct values.

Construct a digraph $G(V, E)$ with vertex set $V = \{0, 1, 2, \ldots, n-1\}$ and edge set $E = \{e_1, \ldots, e_m\}$, where $e_i = (x_i, y_i)$ for each $i \in \{1, 2, \ldots, m\}$. We define the *length* of edge e_i to be $y_i - x_i \bmod n$. Hence each e_i has length $n - 2i$; thus G is a directed near 1-factor whose set of edge lengths is $\{1, 3, 5, \ldots, n-2\}$.

Conversely, let $V = \{0, 1, 2, \ldots, n-1\}$ and suppose $G(V, E)$ is a directed near 1-factor whose set of edge lengths is $\{1, 3, 5, \ldots, n-2\}$. Then we may assume $E = \{e_1, \ldots, e_m\}$, where e_i is the directed edge of length $n - 2i$ with $e_i = (x_i, y_i)$. Let p_0 denote the vertex in G not incident to any edge, and for each $i \in \{1, 2, \ldots, m\}$, let $p_i = x_i - i \bmod n$. Then $p_i = y_i + i \bmod n$ for each $i \in \{1, 2, \ldots, m\}$. Define $\mathbf{p} = (p_m, p_{m-1}, \ldots, p_1, p_0, p_1, \ldots, p_{m-1}, p_m)$. Then modulo n we have

$$\mathbf{p} + (0, 1, \ldots, n-1) = (y_m, \ldots, y_2, y_1, p_0, x_1, x_2, \ldots, x_m) + (m, m, \ldots, m).$$

Since all values in $\{p_0, x_1, \ldots, x_m, y_1, \ldots, y_m\}$ are distinct, $\mathbf{p}$ is a juggling sequence.

These two operations which map between minimal palindromic juggling sequences of length n and directed near 1-factors on n vertices whose set of edge lengths is $\{1, 3, 5, \ldots, n-2\}$ are inverses of one another, which leads to the following theorem.

Theorem 2.8. *Let n be an odd positive integer. Then there is a one-to-one correspondence between minimal palindromic juggling sequences of length n and directed near 1-factors on the vertex set $\{0, 1, \ldots, n-1\}$ whose set of edge lengths is $\{1, 3, 5, \ldots, n-2\}$.*

A similar construction gives a result for all positive even integers n.

Theorem 2.9. *Let n be a positive even integer. Then there is a one-to-one correspondence between minimal palindromic juggling sequences of length n and directed 1-factors on the vertex set $\{0, 1, \ldots, n-1\}$ whose set of edge lengths is $\{1, 3, 5, \ldots, n-1\}$.*

Proof. Let $n = 2m$. The proof method is similar to that given for the argument to Theorem 2.8, so in what follows we give only the construction for the correspondence.

Let $\{(x_i, y_i) \mid i \in \{1, 2, \ldots, m\}\}$ be a 1-factor on $\{0, 1, 2, \ldots, n-1\}$ with (x_i, y_i) having length $2i - 1$ for each $i \in \{1, 2, \ldots, m\}$. For each $i \in \{1, 2, \ldots, m\}$, let $p_i = x_i - m + i \bmod n$. Then $p_i = y_i - m - i + 1 \bmod n$. So modulo n,

$$(x_m, \ldots, x_2, x_1, y_1, y_2, \ldots, y_m) - (0, 1, \ldots, n-1) = \\ (p_m, \ldots, p_2, p_1, p_1, p_2, \cdots, p_m).$$

Therefore $(p_m, \ldots, p_2, p_1, p_1, p_2, \ldots, p_m)$ is a minimal palindromic juggling sequence.

Conversely, if $(p_m, \ldots, p_2, p_1, p_1, p_2, \ldots, p_m)$ is a minimal palindromic juggling sequence, then we may define $x_i = p_i + m - i \bmod n$ and $y_i = p_i + m + i - 1 \bmod n$ and have that $\{(x_i, y_i) \mid i \in \{1, 2, \ldots, m\}\}$ is the edge set of a directed 1-factor in which (x_i, y_i) has length $2i - 1$ for each $i \in \{1, 2, \ldots, m\}$. □

Example 2.10. For $n = 6$, $(2, 5, 2, 2, 5, 2)$ is the minimal palindromic juggling sequence corresponding to the directed 1-factor with edge set $\{(4, 5), (0, 3), (2, 1)\}$. Note the edges have lengths 1, 3, and 5, respectively. Similarly for $n = 7$, $(2, 5, 3, 1, 3, 5, 2)$ is the minimal palindromic juggling sequence corresponding to the directed near 1-factor with unused vertex 1 and edge set $\{(4, 2), (0, 3), (5, 6)\}$. In this case, the edges have lengths 5, 3, and 1, respectively. ♢

3 Juggleable sets and circulants

Let $\mathcal{P}_n$ be the set of minimal juggleable sets of size n. For $U \in \mathcal{P}_n$, let $\mathcal{J}_n(U)$ be the set of juggling sequences of length n with U as juggleable set. It follows from [2] that the number of minimal juggleable sets of size n is given by

$$|\mathcal{P}_n| = \frac{1}{n} \sum_{d|n} \binom{2d-1}{d} \phi\left(\frac{n}{d}\right) \tag{3.1}$$

where ϕ is Euler's totient function and the summation extends over all positive integers d dividing n. The number of minimal juggling sequences of length n is $n!$, since for each permutation $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$, we have

$$(i_1 - 1) + (i_2 - 2) + \cdots + (i_n - n) = \sum_{i=1}^{n} i - \sum_{i=1}^{n} i = 0,$$

and hence the multiset $\{i_1 - 1, i_2 - 2, \ldots, i_n - n\}$ of integers taken modulo n, is a juggleable set.

Let n, k, and ν be positive integers. In [7] it is proved that the number of nonnegative integer solutions of

$$u_1 + u_2 + \cdots + u_n = k \quad \text{and} \quad \sum_{i=1}^{n} i u_i \equiv \nu \pmod{n} \tag{3.2}$$

equals the number of nonnegative integer solutions of

$$v_1 + v_2 + \cdots + v_k = n \quad \text{and} \quad \sum_{i=1}^{k} i v_i \equiv \nu \pmod{k}. \tag{3.3}$$

Taking $\nu = 0$, we get the following duality result.

Theorem 3.1. *The number of minimal juggleable sets* $\{u_1, u_2, \ldots, u_n\}$ *with* $u_1 + u_2 + \cdots + u_n = k$ *equals the number of minimal juggleable sets* $\{v_1, v_2, \ldots, v_k\}$ *with* $v_1 + v_2 + \cdots + v_k = n$*.*

The above discussion gives a characterization of the number of juggling sequences corresponding to each minimal juggleable set.

Theorem 3.2. *Let* $U = \{u_1, u_2, \ldots, u_n\}$ *be a minimal juggleable set. The number* $|\mathcal{J}(U)|$ *of juggling sequences with* U *as juggleable set equals the number of permutations* $(j_1, j_2, \ldots, j_n)$ *of* $\{1, 2, \ldots, n\}$ *such that* $j_i \equiv i + r \pmod{n}$ *has* u_i *solutions for each* $r = 0, 1, \ldots, n-1$*.*

Another viewpoint (see [2]) is the following. Consider the $n \times n$ circulant matrix

$$C(x_0, x_1, \ldots, x_{n-1}) = \begin{bmatrix} x_0 & x_1 & \cdots & x_{n-2} & x_{n-1} \\ x_{n-1} & x_0 & \cdots & x_{n-3} & x_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_2 & x_3 & \cdots & x_0 & x_1 \\ x_1 & x_2 & \cdots & x_{n-1} & x_0 \end{bmatrix}. \tag{3.4}$$

Thus

$$C(x_0, x_1, \ldots, x_{n-1}) = x_0 I_n + x_1 P_n + x_2 P_n^2 + \cdots + x_{n-1} P_n^{n-1},$$

where P_n is the $n \times n$ permutation matrix corresponding to the cyclic permutation $(2, 3, \ldots, n, 1)$ (thus $P_n^0 = P_n^n = I_n$). The book [6] contains a thorough discussion of circulants.

Recall that the permanent of an $n \times n$ matrix $A = [a_{ij} : 0 \leq i, j \leq n]$ is

$$\text{per}(A) = \sum_{(i_1, i_2, \ldots, i_n)} a_{1i_1} a_{2i_2} \cdots a_{n,i_n}$$

where the summation extends over all the permutations $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$. Each term $a_{1i_1} a_{2i_2} \cdots a_{n,i_n}$ in the permanent of $C(x_0, x_1, \ldots, x_{n-1})$ is of the form

$$x_0^{k_0} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}$$

where $k_0, k_1, \ldots, k_{n-1}$ are integers such that

$$0 \leq k_i \leq n, \ (0 \leq i \leq n-1) \quad \text{and} \quad k_0 + k_1 + \cdots + k_{n-1} = n,$$

and k_i is the number of integers r with $0 \leq r \leq n-1$ such that $i_r - r \equiv k_i \pmod{n}$ and

$$k_0 \cdot 0 + k_1 \cdot 1 + \cdots + k_{n-1} \cdot (n-1) \equiv 0 \pmod{n}.$$

Thus the number of distinct terms in the permanent of the circulant $C(x_0, x_1, \ldots, x_{n-1})$ equals the number $|\mathcal{P}_n|$ of juggleable sets of size n and thus is given by (3.1). Theorem 2.1 implies that the monomial $x_0 x_1 \ldots x_{n-1}$ is a term in $\operatorname{per}(A)$ if and only if $1 \cdot 0 + 1 \cdot 1 + \cdots + 1 \cdot (n-1) \equiv 0 \pmod n$; since $0 + 1 + \cdots + (n-1) = n(n-1)/2$, $x_0 x_1 \ldots x_{n-1}$ is a term in $\operatorname{per}(A)$ if and only if n is odd. Now let n be even. Then a monomial of the form $x_0^{k_0} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}$ with $k_r = 2$, $k_s = 0$, and all other k_i's equal to 1, is a term in $\operatorname{per}(A)$ if and only if $|r - s| = n/2$.

In [9] it is shown that $|\mathcal{P}_n|$ equals the dimension of a certain symmetric space associated with a cyclic group of order n. See [12] for a comparison with the number of distinct terms occurring in the determinant.

The following corollary is a direct consequence of Theorem 3.2 and the definitions of a circulant matrix and the permanent.

Corollary 3.3. *Two permutations $j_1, j_2, \ldots, j_n$ and $l_1, l_2, \ldots, l_n$ of $\{1, 2, \ldots, n\}$ give the same term in the permanent of $C(x_0, x_1, \ldots, x_{n-1})$ if and only if*

$$|\{i : j_i \equiv i + r \pmod n\}| = |\{i : l_i \equiv i + r \pmod n\}| \quad \text{for each } r = 0, 1, \ldots, n-1.$$

If the common values are $k_0, k_1, \ldots, k_{n-1}$, then the term in the permanent equals $x_0^{k_0} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}$.

Example 3.4. Table 1 gives the minimal juggleable sets of size $n = 4$ and their corresponding terms in $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$, along with the juggling sequences corresponding to each such pattern and their number. ◊

Table 1: Minimal juggleable sets and juggling sequences for $n = 4$.

Juggleable sets U $\{u_0, u_1, u_2, u_3\}$	Corresponding term in the permanent	Corresponding juggling sequences $\mathcal{J}_4(U)$	Cardinalities $\|\mathcal{J}_4(U)\|$ (coefficients)
$\{0, 0, 0, 0\}$	x_0^4	$(0, 0, 0, 0)$	1
$\{1, 1, 1, 1\}$	x_1^4	$(1, 1, 1, 1)$	1
$\{2, 2, 2, 2\}$	x_2^4	$(2, 2, 2, 2)$	1
$\{3, 3, 3, 3\}$	x_3^4	$(3, 3, 3, 3)$	1
$\{0, 0, 2, 2\}$	$x_0^2 x_2^2$	$(0, 2, 0, 2), (2, 0, 2, 0)$	2
$\{1, 1, 3, 3\}$	$x_1^2 x_3^2$	$(1, 3, 1, 3), (3, 1, 3, 1)$	2
$\{0, 0, 1, 3\}$	$x_0^2 x_1 x_3$	$(0, 0, 1, 3), (0, 1, 3, 0), (1, 3, 0, 0), (3, 0, 0, 1)$	4
$\{0, 1, 1, 2\}$	$x_0 x_1^2 x_2$	$(0, 1, 1, 2), (1, 1, 2, 0), (1, 2, 0, 1), (2, 0, 1, 1)$	4
$\{1, 2, 2, 3\}$	$x_1 x_2^2 x_3$	$(1, 2, 2, 3), (2, 2, 3, 1), (2, 3, 1, 2), (3, 1, 2, 2)$	4
$\{0, 2, 3, 3\}$	$x_0 x_2 x_3^2$	$(0, 2, 3, 3), (2, 3, 3, 0), (3, 3, 0, 2), (3, 0, 2, 3)$	4

As in Table 1 for $n = 4$, constant juggleable sets correspond to the n monomials terms $x_0^n, x_1^n, \ldots, x_{n-1}^n$ of the permanent of the matrix $C(x_0, x_1, \ldots, x_{n-1})$ each occuring with coefficient equal to 1.

If $\{u_1, u_2, \ldots, u_n\}$ is a minimal juggleable set of size n, we define $c(u_1, u_2, \ldots, u_n)$ to be the number of juggling sequences of length n whose pattern is given by $\{u_1, u_2, \ldots, u_n\}$. Therefore, $c(u_1, u_2, \ldots, u_n)$ equals the number of permutations $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$ such that $i_r - r \equiv k_i \pmod{n}$ has u_i solutions for $i = 1, 2, \ldots, n$. The permanent of $C(x_0, x_1, \ldots, x_{n-1})$ is then given by the homogeneous polynomial of degree n,

$$\sum_{\{u_1, u_2, \ldots, u_n\} \in \mathcal{P}_n} c(u_1, u_2, \ldots, u_n) x_0^{u_1} x_1^{u_2} \cdots x_n^{u_{n-1}},$$

whose number of terms is given by (3.1). Thus from Table 1 we see that the permanent of $C(x_0, x_1, x_2, x_3)$ equals

$$1x_0^4x_1^0x_2^0x_3^0 + 1x_0^0x_1^4x_2^0x_3^0 + 1x_0^0x_1^0x_2^4x_3^0 + 1x_0^0x_1^0x_2^0x_3^4 + 2x_0^2x_1^0x_2^2x_3^0 + 2x_0^0x_1^2x_2^0x_3^2 + 4x_0^2x_1^1x_2^0x_3^1 + 4x_0^1x_1^2x_2^1x_3^0 + 4x_0^0x_1^1x_2^2x_3^1 + 4x_0^1x_1^0x_2^1x_3^2.$$

As the referee pointed out, $c(u_1, u_2, \ldots, u_n)$ is the number of ways to arrange the multiset consisting of u_1 0's, u_2 1's, …, u_n $(n-1)$'s into a juggling sequence. Some evaluation of these numbers can be found in sequence A006717 [11].

Theorem 3.5. *If $U = \{u_1, u_2, \ldots, u_n\}$ is a minimal juggleable set of size n, then*

$$c(u_1, u_2, \ldots, u_n) \geq 1. \tag{3.5}$$

Equality holds in (3.5) if and only if U is a constant multiset. If n is a prime p and U is not a constant multiset, then p is a divisor of $c(u_1, u_2, \ldots, u_n)$.

Proof. If U is a constant minimal juggleable set $\{k, k, \ldots, k\}$, then x_k^n occurs as a term in the permanent of $C(x_0, x_1, \ldots, x_{n-1})$ corresponding to the positions of the 1's in P_n^k, that is, the positions $(1, k+1), (2, k+2), \ldots, (n, k+n)$ taken modulo n. If $\{u_1, u_2, \ldots, u_n\}$ is a non-constant juggleable set, there is a term in the permanent of $C(x_0, x_1, \ldots, x_{n-1})$ equal to $x_0^{u_1} x_1^{u_2} \cdots x_{n-1}^{u_n}$ not arising solely from the n positions $(1, k+1), (2, k+2), \ldots, (n, k+n)$ modulo n corresponding to the 1's in the permutation matrices $I_n, P_n, P_n^2, \ldots, P_n^{n-1}$.

The $k \times k$ principal submatrix $C[i_1, i_2, \ldots, i_k \mid i_1, i_2, \ldots, i_k] = C(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ of C determined by rows and columns $i_1, i_2, \ldots, i_k$ is cyclically permutation equivalent (row and column indices are taken modulo n) to the submatrix $C[i_1+1, i_2+1, \ldots, i_k+1 \mid i_1+1, i_2+1, \ldots, i_k+1] = C(x_{i_1+1}, x_{i_2+1}, \ldots, x_{i_k+1})$ determined by rows and columns $i_1+1, i_2+1, \ldots, i_k+1$ taken modulo n. Thus if we take a monomial in the permanent corresponding to a permutation $j_1, j_2, \ldots, j_n$, we get $n-1$ other equal monomials by sequentially adding 1 modulo n to each of $j_1, j_2, \ldots, j_n$ and cyclically permuting:

$$\begin{aligned}(j_1, j_2, \ldots, j_n) &\to (j_n+1, j_1+1, \ldots, j_{n-1}+1)\\ &\to \cdots \\ &\to (j_2+(n-1), \ldots, j_n+(n-1), j_1+(n-1)).\end{aligned} \tag{3.6}$$

If U is a non-constant juggleable set, then not all these permutations can be equal. (If e.g. all of these n permutations are equal, then $(j_1, j_2, \ldots, j_n)$ is a cyclic permutation

$a, a+1, a+2, \ldots, a+(n-1)$ modulo n giving the monomial x_i^n with coefficient equal to 1.) This amounts to simultaneously permuting rows and columns of $C(x_0, x_1, \ldots, x_{n-1})$ using the permutation matrix P_n and replacing the permutation (and its corresponding term in the permanent) with the image of $(j_1, j_2, \ldots, j_n)$ under this action. The result is a term in the permanent with the same value; basically we have that the position (i, j) moves into the position $(i+1, j+1)$ (indices taken $\bmod n$) under the action of P_n, so to position $(i+l, j+l)$ (indices taken $\bmod n$) under the action of P^l. So the set of positions in those sets corresponding to powers of P have to be invariant under a cyclic shift by l in order to get another term in the permanent with the same value. If n is a prime this cannot happen unless the term is of the form x_i^n. Since there may be other terms of equal value in the permanent of $C(x_0, x_1, \ldots, x_{n-1})$, we have that $p \mid c(u_1, u_2, \ldots, u_n)$. □

Corollary 3.6. *If n is odd, the coefficient of $c(1, 1, \ldots, 1)$ of $x_0 x_1 \cdots x_{n-1}$ in the permanent $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$ is divisible by n.*

Proof. The corollary follows as in the proof of Theorem 3.5 since the term $x_0 x_1 \cdots x_{n-1}$ comes from the juggleable set $\{0, 1, \ldots, n-1\}$ and whatever order gives a juggling sequence, each of the $(n-1)$ cyclic shifts is different, resulting in a contribution of n to the coefficient. □

4 Coefficients in $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$

We first consider the special case of $n = 5$.

Example 4.1. Let $n = 5$. The formula (3.1) for the number of distinct terms in the permanent of $C(x_0, x_1, x_2, x_3, x_4)$ is

$$\frac{1}{5}\left(\phi(5) + \binom{9}{5}\phi(1)\right) = \frac{1}{5}(4 + 126) = 26.$$

There are five constant terms in the permanent each with coefficient 1 and there are twenty-one terms each with coefficient divisible by 5. So either we have two terms each with coefficient 10 and nineteen terms with coefficient 5, or we have one term with coefficient 15 and twenty terms with coefficient 5.

The term $x_0 x_1 x_2 x_3 x_4$ occurs in each of the following:

$$\left[\begin{array}{c|c|c|c|c} x_0 & & & & \\ \hline & & x_1 & & \\ \hline & & & & x_2 \\ \hline & x_3 & & & \\ \hline & & & x_4 & \end{array}\right], \quad \left[\begin{array}{c|c|c|c|c} x_0 & & & & \\ \hline & & & & x_3 \\ \hline & & & x_1 & \\ \hline & & x_4 & & \\ \hline & x_2 & & & \end{array}\right], \quad \left[\begin{array}{c|c|c|c|c} x_0 & & & & \\ \hline & & & x_2 & \\ \hline & x_4 & & & \\ \hline & & & & x_1 \\ \hline & & x_3 & & \end{array}\right].$$

and thus, by cyclically simultaneously permuting rows and columns (changing the diagonal position in which x_0 occurs by shifting along the main diagonal), appears in the permanent with coefficient at least 15 and therefore exactly 15. Note the positions occupied by the x_i with $i \neq 0$ above:

$$\left[\begin{array}{c|c|c|c|c} & & & & \\ \hline & & x_1 & x_2 & x_3 \\ \hline & x_4 & & x_1 & x_2 \\ \hline & x_3 & x_4 & & x_1 \\ \hline & x_2 & x_3 & x_4 & \end{array}\right].$$

Each x_i with $i \neq 0$ occupies all the positions in the submatrix obtained by striking out row 1 and column 1 that it occupies in $C(x_0, x_1, x_2, x_3, x_4)$. Thus this simple analysis gives

$$\mathrm{per}(C(x_0, x_1, x_2, x_3, x_4)) = \sum_{i=0}^{4} x_i^5 + 5(\text{twenty other terms}) + 15x_0x_1x_2x_3x_4. \qquad \diamondsuit$$

From calculations of $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ using Sage, we found the following information:

- $(n = 5)$: largest coefficient is 15 occuring uniquely for

  $$x_0x_1x_2x_3x_4.$$

  Coefficients are $1, 5, 15$. This confirms the calculations in Example 4.1.
- $(n = 6)$: largest coefficient is 24 occuring for the six terms of the form

  $$x_0^2x_1x_2x_3^0x_4x_5.$$

  Coefficients are $1, 2, 3, 6, 9, 12, 18, 24$.
- $(n = 7)$: largest coefficient is 133 occuring uniquely for

  $$x_0x_1x_2x_3x_4x_5x_6.$$

  Coefficients are $1, 7, 14, 21, 35, 42, 49, 133$.
- $(n = 8)$: largest coefficient is 256 occuring for the 8 terms

  $$\begin{array}{ll} x_0^2x_1x_2x_3x_4^0x_5x_6x_7, & x_0x_1^2x_2x_3x_4x_5^0x_6x_7, \\ x_0x_1x_2^2x_3x_4x_5x_6^0x_7, & x_0x_1x_2x_3^2x_4x_5x_6x_7^0, \\ x_0^0x_1x_2x_3x_4^2x_5x_6x_7, & x_0x_1^0x_2x_3x_4x_5^2x_6x_7, \\ x_0x_1x_2^0x_3x_4x_5x_6^2x_7, & x_0x_1x_2x_3^0x_4x_5x_6x_7^2. \end{array}$$

  For instance, $x_0^2x_1x_2x_3x_4^0x_5x_6x_7$ occurs in the term

  $$\left[\begin{array}{c|c|c|c|c|c|c|c} x_0 & & & & & & & \\ \hline & & & x_2 & & & & \\ \hline & & & & & x_3 & & \\ \hline & x_6 & & & & & & \\ \hline & & & & x_0 & & & \\ \hline & & x_5 & & & & & \\ \hline & & & & & & & x_1 \\ \hline & & & & & & x_7 & \end{array}\right].$$

  There are 810 different terms that occur in $\mathrm{per}(C(x_0, x_1, \ldots, x_7))$. The full set of coefficients in the permanent are

  $$\{1, 2, 4, 6, 8, 12, 16, 20, 24, 32, 40, 48, 56, 64, 72, 80, 96, 128, 160, 256\}.$$

  Note that the differences of consecutive coefficients in this list are:

  $$1, 2, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 16, 32, 32, 96.$$

  Only the last is not a power of 2.

Conjecture 4.2. *Our calculations have shown that for $n = 4$, the largest coefficient in* $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$ *is $4 = 2^2$ occuring 4 times, and for $n = 8$, the largest coefficient is $256 = 2^8$ occuring 8 times. We conjecture that if n is a power of 2, then the largest coefficient is also a power of 2 occuring for the terms of the form $x_0^2 x_1 x_2 \cdots \widehat{x_{n/2}} \cdots x_{n-1}$, and cyclical translates of terms of this form (total number of different terms is n). Unfortunately, the occurrence of these terms does not seem to have a pattern. For instance, with $n = 8$, we have*

$$\begin{bmatrix} \colorbox{lightgray}{x_0} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_7 & \colorbox{lightgray}{x_0} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_6 & x_7 & x_0 & x_1 & x_2 & x_3 & x_4 & \colorbox{lightgray}{x_5} \\ x_5 & x_6 & x_7 & x_0 & \colorbox{lightgray}{x_1} & x_2 & x_3 & x_4 \\ x_4 & x_5 & x_6 & x_7 & x_0 & x_1 & \colorbox{lightgray}{x_2} & x_3 \\ x_3 & x_4 & x_5 & \colorbox{lightgray}{x_6} & x_7 & x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 & x_5 & x_6 & \colorbox{lightgray}{x_7} & x_0 & x_1 \\ x_1 & x_2 & \colorbox{lightgray}{x_3} & x_4 & x_5 & x_6 & x_7 & x_0 \end{bmatrix} \qquad (x_0^2 x_1 x_2 x_3 x_5 x_6 x_7).$$

This corresponds to the permutation σ of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ such that

$$\begin{array}{llll} \sigma(1) = 1, & \sigma(2) = 2, & \sigma(3) = 8, & \sigma(4) = 5, \\ \sigma(5) = 7, & \sigma(6) = 4, & \sigma(7) = 6, & \sigma(8) = 3. \end{array}$$

The coefficient of $x_0^2 x_1 x_2 x_3 x_5 x_6 x_7$ is the number of permutations π of $\{1, 2, 3, 4, 5, 6, 7, 8\}$, such that

$$\pi(i) - i \equiv j \pmod 8$$

has two solutions for $j = 0$, no solutions for $j = 4$, and one solution for $j = 1, 2, 3, 5, 6, 7$. A similar statement holds for all even n, and we seek the number of such solutions.

Conjecture 4.3. *Three* conjectures/problems *for n even (or perhaps just n a power of 2):*

(a) There exists a term $x_0^2 x_1 x_2 \cdots \widehat{x_{n/2}} \cdots x_{n-1}$ in $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$ *arising from every choice of the two x_0's on the main diagonal.*

(b) In reference to (a), the largest number of terms occurs when the x_0's are chosen to be $n/2$ apart (cyclically, the same number of elements on the main diagonal between them). In the case of $n = 8$, there are 16 terms for a choice of x_0's which are 4 apart (5th x_0 on the main diagonal minus 1st x_0 on main diagonal) and 8 terms for all other choices of x_0's.

(c) If n is a power of 2, the coefficient of $x_0^2 x_1 x_2 \cdots \widehat{x_{n/2}} \cdots x_{n-1}$ is a power of 2.

Problem 4.4. The matrix $C(x_0, x_1, \ldots, x_{n-1})$ can be regarded as a special latin square. The coefficient of $x_0 x_1 \cdots x_{n-1}$ in $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$ equals the number of transversals of this latin square. In [5] it is shown that if n is odd and sufficiently large, the coefficient of $x_0 x_1 \cdots x_{n-1}$ in $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$ is greater than $(3.246)^n$. In the cases of $n = 5$ and $n = 7$, the number of latin square transversals of $C(x_0, x_1, \ldots, x_{n-1})$ equals $15 = 5 \times 3$ and $133 = 7 \times 19$, respectively. Since a latin square transversal is mapped into a latin square transversal by multiplying $C(x_0, x_1, \ldots, x_{n-1})$ by the full cycle permutation matrix P_n, it follows that for odd n, the number of latin square transversals, that is, $c(1, 1, \ldots, 1)$ is divisible by n. See also Theorem 3.5 and Corollary 3.6.

If n is odd, the term $x_0x_1\cdots x_{n-1}$ occurs in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1})$ with a nonzero coefficient. A *conjecture* would be that this term has the largest coefficient. Thinking of the x_i as n different colors giving $n!$ multicolored transversals, the conjecture is saying that the number of multicolored transversals with all colors different is greater than the number of multicolored transversals of any other prescribed color type (so at least two colors the same). This coefficient is equal to the number of transversals of $C(x_0, x_1, \ldots, x_{n-1})$ considered as a latin square, so finding this exactly is probably not attainable (see [5]).

Remark 4.5. Concerning Problem 4.4 and the juggleable set $\{1, 2, \ldots, n\}$ with n odd, corresponding to the term $x_0x_1\cdots x_{n-1}$ in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$. A permutation $(i_1, i_2, \ldots, i_n)$ of this juggleable set is a juggling sequence giving the term $x_0x_1\cdots x_{n-1}$ in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ provided $1 + i_1, 2 + i_2, \ldots, n + i_n$ are distinct modulo n. If this is the case, then any cyclic permutation of $(i_1, i_2, \ldots, i_n)$ is also a juggling sequence (since subtracting 1 modulo n from distinct integers modulo n gives distinct integers modulo n, thereby giving n terms in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ equal to $x_0x_1\cdots x_{n-1}$. The difficulty in calculating the coefficient of $x_0x_1\cdots x_{n-1}$ in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ is knowing how many permutations $i_1, i_2, \ldots, i_n$ of the set $\{1, 2, \ldots, n\}$ have the property that $1 + i_1, 2 + i_2, \ldots, n + i_n$ are distinct modulo n. So one might consider the additive group $\mathbb{Z}_n^{(n)} = \mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$ (n copies of $\mathbb{Z}_n$) and the mapping

$$T\colon \mathbb{Z}_n^{(n)} \to \mathbb{Z}_n^{(n)}$$

given by

$$\begin{aligned} T(i_1, i_2, \ldots, i_n) &= (1 + i_1, 2 + i_2, \ldots, n + i_n) \\ &= (1, 2, \ldots, n) + (i_1, i_2, \ldots, i_n) \bmod n. \end{aligned}$$

Unfortunately, this mapping is not a homomorphism and so does not seem useful. But it does seem that for a juggleable set $\{u_1, u_2, \ldots, u_n\}$ with at least one repeat, that is, the number of permutations $(u_1, u_2, \ldots, u_n)$ of this pattern such that $1+u_1, 2+u_2, \ldots, n+u_n$ are distinct modulo n is smaller than when there is no repeat in $\{u_1, u_2, \ldots, u_n\}$. But it seems difficult to make a comparison.

Remark 4.6. Assume n is odd. Then $x_0^1x_1^1x_2^1\cdots x_{n-1}^1$ occurs in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ with a nonzero coefficient. We can think of this term as *generating* other terms that occur in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ as follows:

We increase or decrease (by 1) some of the exponents of this term to get

$$x_0^{1+a_0}x_1^{1+a_1}x_2^{1+a_2}\cdots x_{n-1}^{1+a_{n-1}}$$

where each $a_i \in \{1, 0, -1\}$, and

$$\sum_{i=0}^{n-1} a_i = 0 \tag{4.1}$$

and, in order that the result is a term in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$, we must have

$$\sum_{i=0}^{n-1} ia_i \equiv 0 \pmod{n}. \tag{4.2}$$

(By (4.2), $\sum_{i-0}^{n-1}(1+a_i) = 0$ and $\sum_{i=0}^{n-1} i(a_i+1) \equiv 0 \pmod n$ and thus gives a term in this permanent.) We can do a similar operation on the resulting term but then we need to be sure that the resulting exponents are always between 0 and n. Continuing like this we can generate all terms that occur in this permanent.

So in this operation we increase $s \geq 1$ exponents by 1 and decrease s exponents by -1, so adding $(a_0, a_1, a_2, \ldots, a_{n-1})$, subject to the condition (4.2), to the vector of exponents in a term in our permanent. One line of investigation is to try to determine when this operation increases/decreases the coefficient of the corresponding terms in our permanent. In particular, when with one application starting with the term $x_0^1 x_1^1 x_2^1 \cdots x_{n-1}^1$, does the coefficient decrease? Note that in one application, we must reduce two exponents to 0 in order that we satisfy (4.2); in general there must be at least four changes in exponents. See the following example.

Example 4.7. Let $n = 9$. We start with the term $x_0x_1x_2x_3x_4x_5x_6x_7x_8$. We can change exponents by using the vector $(0,0,1,-1,0,-1,1,0,0)$. Since $1\cdot 2 - 1\cdot 3 + (-1)\cdot 5 + 1\cdot 6 = 0 \equiv 0 \pmod 9$,

$$x_0x_1x_2^2x_4x_6^2x_7x_8$$

is a term in our permanent. ♢

Problem 4.8. If n is even, then we can also ask for the term(s) with the largest coefficient. If $n = 4$, there are four terms that appear with the largest coefficient of 4, namely

$$x_0^2x_1x_3, x_0x_1^2x_2, x_1x_2^2x_3, x_0x_2x_3^2.$$

A *conjecture* might be:

If n is even then the terms in $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$ that occur with the largest coefficient are the terms with the property that x_i occurs with exponent 2, $x_{i+n/2}$ (subscript $\bmod n$) occurs with exponent 0, and all other x_i appear with exponent 1.

Remark 4.9. We have that there is a nonzero term in $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$ with exactly two nonzero exponents (so binomials) if and only if n is not a prime. The reason is as follows: Suppose $x_i^a x_j^b$ occurs with a nonzero coefficient where $0 \leq j < i \leq n-1$ and $i \neq j$, and $a, b \geq 1$, and $a + b = n$ (and so $a, b \leq n-1$). Then by Hall's theorem

$$ai + bj = ai + (n-a)j \equiv 0 \pmod n, \quad \text{that is,} \quad a(i-j) \equiv 0 \pmod n.$$

If n is a prime p, this is a contradiction since $p \nmid a$ and $p \nmid (i-j)$. If n is not a prime, say $n = uv$ where $1 < u, v < n-1$. Then we may choose $a = u$, and i and j so that $i - j = v$, and get a term $x_i^a x_j^{(n-a)}$ with a nonzero coefficient.

In investigating binomials in $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$ it is sufficient to consider binomials of the form $x_0^a x_k^b$ where $1 \leq k \leq n-1$. Thus we consider the terms of $\operatorname{per}(x_0 I_n + x_k P_n^k)$ different from x_0^n and x_k^n. This permanent is easily computed:

$$\operatorname{per}(x_0 I_n + x_k P_n^k) = \sum_{t=0}^{d} \binom{d}{t} x_0^{t\frac{n}{d}} x_k^{(d-t)\frac{n}{d}} \text{ where } d = \gcd(n,k).$$

Thus the largest coefficient of a binomial is $\binom{d}{\frac{d}{2}}$.

More generally, let $H \subseteq \{0, 1, \ldots, n-1\}$. If we set $x_j = 0$ if $j \notin H$, then the permanent of the resulting matrix $C_H(x_0, x_1, \ldots, x_{n-1})$ gives all the terms that occur in

$\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ and their coefficients in which the only x_i that can occur are those with $i \in H$. By also setting $x_i = 1$ for $i \in H$, the permanent equals the number of terms in $\mathrm{per}(C_H(x_0, x_1, \ldots, x_{n-1}))$.

Remark 4.10. Now consider terms in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ where there are exactly three nonzero exponents (so in the juggling context, three different heights in throwing the balls). These terms are then *trinomials*. *Which trinomial has the largest coefficient among all trinomials that occur in* $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$*?* The *conjecture* is that the maximum coefficient occurs when the exponents are as equal as possible; in particular if $n = 3k$, then the trinomial will largest coefficient is conjectured to be $x_0^k x_k^k x_{2k}^k$ and its cyclic permutations. In investigating trinomials in $\mathrm{per}(C(x_0, x_1, \ldots, x_{n-1}))$ it suffices to consider terms of the form $x_0^a x_r^b x_s^c$ where $0 < r < s < n$ and $a + b + c = n$, that is it suffices to consider the trinomials in

$$\mathrm{per}(x_0 I_n + x_r P_n^r + x_s P_n^s).$$

The *conjecture* is that the largest coefficient of a trinomial in this permanent occurs when the exponents are as equal as possible and the powers of P_n, i.e. the subscripts of the x's are as equally spaced as possible (in the cyclic sense). If $n = 3k$, then after permutations $x_0 I_n + x_k P_n^k + x_{2k} P_n^{2k}$ becomes a direct sum of k 3×3 matrices of the form

$$x_0 I_3 + x_k P_3 + x_{2k} P_3^2.$$

5 Juggling sequences with additional properties

Let $U = \{u_1, u_2, \ldots, u_n\}$ be a minimal juggleable set, and let $u_{\tau(1)}, u_{\tau(2)}, \ldots, u_{\tau(n)}$ be a juggling sequence corresponding to U. Thus τ is a permutation of $\{1, 2, \ldots, n\}$ and it is natural to ask about the existence of such permutations τ with additional properties, equivalently, extensions of Theorem 2.1 by imposing additional restrictions on the permutation τ. Juggling sequences correspond to transversals in the circulant $C(x_0, x_1, \ldots, x_{n-1})$ and thus we seeks transversals of $C(x_0, x_1, \ldots, x_{n-1})$ whose pattern has additional properties.

Two natural permutations to consider are *involutions* and *centrosymmetric permutations* of σ of $\{1, 2, \ldots, n\}$. Involutions are permutations σ of $\{1, 2, \ldots, n\}$ where for all i and j, $\sigma(i) = j$ implies $\sigma(j) = i$, and these correspond to transversals of $C(x_0, x_1, \ldots, x_{n-1})$ whose positions have a symmetric matrix pattern, that is, transversal patterns invariant under a reflection about the main diagonal. A permutation σ is centrosymmetric provided that for all i, $\sigma(i) + \sigma(n + 1 - i) = n + 1$ and these correspond to transversals of $C(x_0, x_1, \ldots, x_{n-1})$ whose positions have a centrosymmetric matrix pattern, that is, transversal patterns invariant under a 180 degree rotation. There are permutations that are both symmetric and centrosymmetric.

Example 5.1. Let $n = 4$ and let $\sigma = (2, 1, 4, 3)$. As a permutation matrix, σ equals

$$\left[\begin{array}{c|c|c|c} & 1 & & \\ \hline 1 & & & \\ \hline & & & 1 \\ \hline & & 1 & \end{array}\right]$$

which is invariant under a reflection about the diagonal and a rotation of 180 degrees. Thus σ is both an involution (invariant under a reflection about the main diagonal) and a centrosymmetric permutation (invariant under a 180 degree rotation). Notice that σ is also invariant under reflection about the antidiagonal running from the lower left to the upper right, and this holds in general for permutations that are both symmetric and centrosymmetric. ♢

Let $U = \{u_1, u_2, \ldots, u_n\}$ be a multiset where $u_i \in \{0, 1, \ldots, n-1\}$ for $0 \leq i \leq n-1$. We say that U is *balanced* $\bmod n$ provided that its nonzero elements can be paired as $\{a, b\}$ so that $a + b \equiv 0 \pmod{n}$. Thus if n is even, 0 and $n/2$ each occur an even, possibly zero, number of times, and if n is odd, 0 occurs an odd number of times. If U is balanced $\bmod n$, then it is an immediate consequence of Theorem 2.1 that U is a juggleable set with each x_i with $i \neq 0$ occurring with an even, possibly zero, exponent in $\operatorname{per}(C(x_0, x_1, \ldots, x_{n-1}))$.

Example 5.2. Let $n = 8$ and let $U = \{0, 0, 1, 7, 1, 7, 4, 4\}$. Then U is balanced $\bmod 8$ and hence is a juggleable set. In $C(x_0, x_1, \ldots, x_7)$ below we have realizations

	1	2	3	4	5	6	7	8
1	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
2	x_7	x_0	x_1	x_2	x_3	x_4	x_5	x_6
3	x_6	x_7	x_0	x_1	x_2	x_3	x_4	x_5
4	x_5	x_6	x_7	x_0	x_1	x_2	x_3	x_4
5	x_4	x_5	x_6	x_7	x_0	x_1	x_2	x_3
6	x_3	x_4	x_5	x_6	x_7	x_0	x_1	x_2
7	x_2	x_3	x_4	x_5	x_6	x_7	x_0	x_1
8	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_0

.

corresponding to the term $x_0^2 x_1^2 x_4^2 x_7^2$ in $\operatorname{per}(C(x_0, x_1, \ldots, x_7))$, achieved in the permanent $\operatorname{per}(C(x_0, x_1, \ldots, x_7))$ by an involution (dark gray) and by a centrosymmetric permutation (light gray). ♢

Example 5.3. Let $n = 6$ and consider the multiset $U = \{2, 2, 2, 4, 4, 4\}$ balanced $\bmod 6$ with the pairing $\{2, 4\}$, $\{2, 4\}$, $\{2, 4\}$. In both case we seek a corresponding transversal in

x_0	x_1	x_2	x_3	x_4	x_5
x_5	x_0	x_1	x_2	x_3	x_4
x_4	x_5	x_0	x_1	x_2	x_3
x_3	x_4	x_5	x_0	x_1	x_2
x_2	x_3	x_4	x_5	x_0	x_1
x_1	x_2	x_3	x_4	x_5	x_0

,

consisting of three x_2's and three x_4's. We have indicated such a realization in the centrosymmetric case, but it is straightforward to check that it cannot be attained by an involution. ♢

We have done a substantial amount of calculation with the following consequences:

(i) For $n \leq 19$ a prime, all balanced $\bmod n$ multisets can be achieved by a transversal with a symmetric pattern. When $n = 15$, there are 16 balanced $\bmod 15$ multisets that cannot be achieved by a transversal with a symmetric pattern, e.g. the multiset $\{0, 6, 6, 6, 6, 6, 6, 6, 9, 9, 9, 9, 9, 9, 9\}$ cannot be so achieved. On the other hand, for $n = 18$, there are 48 620 balanced $\bmod 18$ multisets satisfying (2.1) and only 36 195 can be achieved with a symmetric pattern.

(ii) For odd $n \leq 21$, all balanced $\bmod n$ multisets can be achieved by a transversal with a centrosymmetric pattern.

As a consequence of the data obtained we make two conjectures:

Conjecture 5.4. *If n is a prime, then every balanced $\bmod n$ multiset can be achieved by a transversal with a symmetric pattern.*

Conjecture 5.5. *If n is odd, then a balanced $\bmod n$ multiset can be achieved by a transversal with a centrosymmetric pattern. If n is even, then the unachievable balanced $\bmod n$ multisets only have terms with the same parity.*

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On silver and golden optical orthogonal codes*

Marco Buratti

Dipartimento di Matematica e Informatica, Università di Perugia, via Vanvitelli 1

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Abstract

It is several years that no theoretical construction for optimal $(v, k, 1)$ optical orthogonal codes (OOCs) with new parameters has been discovered. In particular, the literature almost completely lacks optimal $(v, k, 1)$-OOCs with $k > 3$ which are not regular. In this paper we will show how some elementary difference multisets allow to obtain three new classes of optimal but not regular $(3p, 4, 1)$-, $(5p, 5, 1)$-, and $(2p, 4, 1)$-OOCs which are describable in terms of Pell and Fibonacci numbers. The OOCs of the first two classes (resp. third class) will be called *silver* (resp. *golden*) since they exist provided that the square of a *silver element* (resp. *golden element*) of $\mathbb{Z}_p$ is a primitive square of $\mathbb{Z}_p$.

Keywords: Silver and golden ratio, Pell and Fibonacci numbers, difference packing, optimal optical orthogonal code, strong difference family, difference multiset.

Math. Subj. Class.: 05B10, 94B25

1 Introduction

The real numbers $1 + \sqrt{2}$ (the *silver ratio*), $\frac{1+\sqrt{5}}{2}$ (the *golden ratio*) and their marvelous properties are very well known. Disregarding their geometrical meaning (see, e.g., [17]), they can be defined in the same algebraic way in any finite field $\mathbb{F}_q$ of an appropriate order q. By the Law of Quadratic Reciprocity (see, e.g., [20]), it is well known that 2 is a non-zero square in $\mathbb{F}_q$ if and only if $q \equiv 1 \text{ or } 7 \pmod 8$ and that 5 is a non-zero square in $\mathbb{F}_q$ if and only if $q \equiv 1 \text{ or } 4 \pmod 5$. Thus, for a prime $p \equiv 1 \text{ or } 7 \pmod 8$, we naturally define the *silver elements of* $\mathbb{Z}_p$ as the two elements $1 + x$ and $1 - x$ of $\mathbb{Z}_p$ where x and $-x$ are the square roots of 2 modulo p. Also, for a prime $p \equiv 1 \text{ or } 4 \pmod 5$, we naturally define the *golden elements of* $\mathbb{Z}_p$ as the two elements $2^{-1}(1 + x)$ and $2^{-1}(1 - x)$ of $\mathbb{Z}_p$ where x and $-x$ are the square roots of 5 modulo p.

We recall that the *Pell sequence* is the integer sequence $\{P_n\}$ defined by $P_0 = 0$, $P_1 = 1$ and by the recursive formula $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$, and that the

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E-mail address: buratti@dmi.unipg.it (Marco Buratti)

Fibonacci sequence is the integer sequence $\{F_n\}$ defined by $F_0 = 0$, $F_1 = 1$ and by the recursive formula $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Two good textbooks on *Pell* and *Fibonacci numbers* are [19] and [18], respectively.

As in the real field, if θ is a silver element of $\mathbb{F}_q$ then we have

$$\theta^n = P_n\theta + P_{n-1} \quad \forall n \geq 1 \tag{1.1}$$

Also, if ϕ is a golden element of $\mathbb{F}_q$ then we have

$$\phi^n = F_n\phi + F_{n-1} \quad \forall n \geq 1 \tag{1.2}$$

An *optical orthogonal code* of *length* v, *weight* k, *auto-correlation* λ_a and *cross-correlation* λ_c – briefly, a $(v, k, \lambda_a, \lambda_c)$-OOC – can be seen as a set of k-subsets (*codeword-sets*) of $\mathbb{Z}_v$ such that

(i) any two distinct translates of a codeword-set share at most λ_a elements (*auto-correlation property*);

(ii) any two translates of two distinct codeword-sets share at most λ_c elements (*cross-correlation property*).

This topic, introduced by Chung, Salehi and Wei [12], has been well studied for a long time in view of its many applications (see, e.g., [13]).

In particular, a $(v, k, 1, 1)$-OOC – briefly, a $(v, k, 1)$-OOC – can be viewed as a set of k-subsets of $\mathbb{Z}_v$ (*codeword-sets*) such that no element of $\mathbb{Z}_v \setminus \{0\}$ can be represented as a difference of two elements of a codeword-set in more than one way. Such an OOC is said to be *optimal* when its *size* (that is the number of its codeword-sets) reaches the upper *Johnson bound* $\lfloor \frac{v-1}{k(k-1)} \rfloor$.

There is a huge literature on optical orthogonal codes but, as far as this author is aware, in the last seven years no theoretical construction for a class of optimal $(v, k, 1)$-OOCs with new parameters has been discovered. In this paper we find three classes of optimal OOCs with new parameters: an optimal $(3p, 4, 1)$-OOC and an optimal $(5p, 5, 1)$-OOC for each prime $p \equiv 7 \pmod 8$ such that the silver elements of $\mathbb{Z}_p$ are generators of $\mathbb{Z}_p^*/\{1, -1\}$ (both these codes will be called silver); an optimal $(2p, 4, 1)$-OOC for each prime $p \equiv 11$ or $29 \pmod{30}$ such that the golden elements of $\mathbb{Z}_p$ are generators of $\mathbb{Z}_p^*/\{1, -1\}$ (this code will be called golden).

The strategy to get our silver/golden OOCs is to use some elementary *difference multisets* (which are *strong difference families* with only one block) but, in the end, we will show that all these codes can be presented in terms of *Pell/Fibonacci numbers*.

2 Difference packings via strong difference families

Given a k-multisubset, in particular a k-subset, $B = \{b_1, \ldots, b_k\}$ of an additive group G, we call list of differences from B the multiset ΔB of all possible differences $b_i - b_j$ with (i, j) an ordered pair of distinct elements of $\{1, \ldots, k\}$. One calls $(G, k, 1)$ *difference packing* any set $\mathcal{D}$ of k-subsets of G (*blocks*) with the property that its list of differences, namely the multiset sum

$$\Delta\mathcal{D} := \biguplus_{B\in\mathcal{D}} \Delta B,$$

does not have repeated elements. It is evident that the size of $\mathcal{D}$ cannot exceed $\lfloor\frac{|G|-1}{k(k-1)}\rfloor$. For this reason, one says that $\mathcal{D}$ is *optimal* when its size reaches this value. The *difference leave* of a $(G, k, 1)$ difference packing $\mathcal{D}$ is defined to be the set of all elements of G not appearing in $\Delta\mathcal{D}$. Thus $\mathcal{D}$ is optimal provided that its difference leave has size less or equal to $k(k-1)$. The difference packing is a *relative difference family* [5] if its difference leave is a subgroup H of G. In this case someone also speaks of a r-*regular* difference packing if H has order r (see, e.g., [24]). Note that a $(\mathbb{Z}_v, k, 1)$ difference packing is nothing but a $(v, k, 1)$-OOC.

The problem of factoring a group into subsets and its variants [23] could play a crucial role in the construction of difference packings. Also, the construction of a $|G|$-regular $(G\times\mathbb{F}_q, k, 1)$ difference packing can be facilitated by a suitable *strong difference family* in G, a concept formally introduced in [6] but implicitly used for a long time. A t-(G, k, μ) *strong difference family* is a t-multiset $\mathcal{S}$ of k-multisubsets (blocks) of a group G such that $\Delta\mathcal{S}$ covers all elements of G exactly μ times. The parameter μ is called the *index* of the strong difference family and a trivial counting shows that it is necessarily equal to $\frac{k(k-1)t}{|G|}$. Of course it is possible to consider, more generally, strong difference families whose blocks have variable sizes [7].

In order to explain why strong difference families might be good to construct relative difference families and more generally OOCs, we have to introduce some notation and terminology.

Denote by $\mathbb{F}_q^*$ the multiplicative group of the field $\mathbb{F}_q$. Given a subset B of a direct product $G\times\mathbb{F}_q$ and given $c\in\mathbb{F}_q^*$, denote by $(1, c)\cdot B$ the subset of $G\times\mathbb{F}_q$ obtained from B by multiplying the second coordinates of all its elements by c and leaving invariant their first coordinates. If $\mathcal{B}$ is a set of subsets of $G\times\mathbb{F}_q$ and $g\in G$, we denote by $\Delta_g\mathcal{B}$ the list of the second coordinates of all elements of $\Delta\mathcal{B}$ whose first coordinate is g so that one can write

$$\Delta\mathcal{B} = \bigcup_{g\in G}\{g\}\times\Delta_g\mathcal{B}.$$

Let us say that two subsets C and Δ of $\mathbb{F}_q^*$ are *companions* if the list $C\cdot\Delta := \{c\delta \mid c\in C; \delta\in\Delta\}$ does not have repeated elements. In this case it is evident that the size of C cannot exceed $\lfloor\frac{q-1}{|\Delta|}\rfloor$. Thus we say that C is an *optimal* companion of Δ when its size reaches this value. In particular, we say that C is a *perfect* or *near-perfect* companion of Δ when its size is exactly equal to $\frac{q-1}{|\Delta|}$ or $\frac{q-2}{|\Delta|}$, respectively. In these last two cases we have $C\cdot\Delta = \mathbb{F}_q^*$ or $C\cdot\Delta = \mathbb{F}_q^*\setminus\{x\}$ for some $x\in\mathbb{F}_q^*$ and one says that $C\cdot\Delta$ is a *factorization* of $\mathbb{F}_q^*$ in the former case and that $\frac{1}{x}C\cdot\Delta$ is a *near factorization* of $\mathbb{F}_q^*$ in the latter (see [23]).

The next proposition is very elementary.

Proposition 2.1. *Let $\mathcal{B} = \{B_1, \ldots, B_t\}$ be a set of k-subsets of $G\times\mathbb{F}_q$ such that all $\Delta_g\mathcal{B}$ are sets admitting a common companion C. Then $\mathcal{D} := \{(1, c)\cdot B \mid c\in C, B\in\mathcal{B}\}$ is a $(G\times\mathbb{F}_q, k, 1)$ difference packing.*

The proof is straightforward; indeed, by assumption, $C\cdot\Delta_g\mathcal{B}$ does not have repeated elements, hence $\Delta\mathcal{D} = \bigcup_{g\in G}\{g\}\times(C\cdot\Delta_g\mathcal{B})$ is also without repeated elements.

Now we show that the above proposition cannot give optimal optical orthogonal codes for arbitrarily high values of q unless the projection of $\mathcal{B}$ on G is a strong difference family.

Proposition 2.2. *Let $\mathcal{D}$ be a difference packing as in Proposition 2.1 and set $\mu = \frac{k(k-1)t}{|G|}$. Then, for $q > k(k-1)\mu$, $\mathcal{D}$ is optimal if and only if the following conditions hold:*

(i) The projection of $\mathcal{B}$ on G is a t-(G,k,μ) strong difference family;

(ii) C is an optimal companion of $\Delta_g\mathcal{B}$ for every $g\in G$;

(iii) the remainder of the Euclidean division of q by μ does not reach $\frac{\mu}{t}$.

Proof. ($\Longrightarrow$): The size of $\mathcal{D}$ is $|C|\cdot t$, therefore we have $|C|\cdot t = \lfloor\frac{|G|q-1}{k(k-1)}\rfloor$ because $\mathcal{D}$ is optimal. This gives $|C|\geq k(k-1)$ in view of the hypothesis $q>k(k-1)\mu$.

For each $g\in G$, let $L_g(\mathcal{B})$ be the complement of $C\cdot\Delta_g\mathcal{B}$ in $\mathbb{F}_q$. We have $|L_g(\mathcal{B})| = q-|C|\cdot|\Delta_g\mathcal{B}|$ for each $g\in G$. This implies that $|L_g(\mathcal{B})|=|L_h(\mathcal{B})|+|C|\cdot(|\Delta_h\mathcal{B}|-|\Delta_g\mathcal{B}|)$ for any pair of elements g and h of G. Thus, if $|\Delta_g\mathcal{B}|<|\Delta_h\mathcal{B}|$ we would have $|L_g(\mathcal{B})|>|C|$ and then $L_g(\mathcal{B})$ would have size greater than $k(k-1)$ in view of the previous paragraph. This is clearly absurd since $\{g\}\times L_g(\mathcal{B})$ is contained in the difference leave of $\mathcal{D}$ whose size is at most $k(k-1)$.

We conclude that $|\Delta_g\mathcal{B}|$ is a constant, i.e., the projection of $\mathcal{B}$ on G is a strong difference family with t blocks of size k. This implies that its index is $\frac{k(k-1)t}{|G|}$ which is equal to μ. Thus $|\Delta_g\mathcal{B}|=\mu$ for every $g\in G$.

Now assume that C is not optimal. In this case the size of $L_g(\mathcal{B})$ would be a constant at least equal to μ, hence the difference leave of $\mathcal{D}$, which is clearly given by $\bigcup_{g\in G}\{g\}\times L_g(\mathcal{B})$, would have size greater than $\mu|G|=tk(k-1)$, therefore greater than $k(k-1)$ contradicting the optimality of $\mathcal{D}$.

If r is the remainder of the Euclidean division of q by μ, then the difference leave of $\mathcal{D}$ has size $r\cdot|G|$. Thus, since $\mathcal{D}$ is optimal, we must have $r\cdot|G|\leq k(k-1)$ which means $r<\frac{\mu}{t}$.

($\Longleftarrow$): Straightforward. □

Note that condition (iii) is certainly satisfied when $t=1$, namely when $\mathcal{B}$ is a singleton $\{B\}$. In this case one says that the projection of B on G, say $\pi(B)$, is a (G,k,μ) *difference multiset* (also called a *difference cover* in [3]) rather than to say that $\{\pi(B)\}$ is a 1-(G,k,μ) strong difference family.

The above proposition suggests the following strategy for getting families of optimal difference packings. Start with a t-(G,k,μ) strong difference family $\mathcal{S}$ which will be used as "skeleton" of the desired optimal difference packing. Then take a prime power $q=\mu n+r$ with $1\leq r\leq\frac{\mu}{t}$ and try to "lift" $\mathcal{S}$ to a suitable t-set $\mathcal{B}$ of k-subsets of $G\times\mathbb{F}_q$ in such a way that all $\Delta_g\mathcal{B}$ admit a common optimal companion C. For $r=1$ this strategy has been used (sometimes implicitly) in many papers to construct relative difference families and, in particular, regular OOCs. The elder constructions are surveyed in [2]. More recent constructions can be found in [8, 9, 11, 14, 15, 21, 22, 25]. Here one often tries to have each $\Delta_g\mathcal{B}$ a complete system of representatives for the cosets of the subgroup of $\mathbb{F}_q^*$ of index μ, namely the group C^μ of non-zero μ-th powers of $\mathbb{F}_q$. Indeed in this case a common companion of each $\Delta_g\mathcal{B}$ is clearly given by C^μ itself.

On the other hand, as far as this author is aware, the above strategy has been never applied with $r>1$ probably because the existence of a common optimal but not perfect companion of all the set $\Delta_g\mathcal{B}$ seems to be almost a miracle. Indeed, the probability that even a single set $\Delta\subset\mathbb{F}_q^*$ admits an optimal companion C diminishes dramatically if $|\Delta|$ is not a divisor of $q-1$. Consider, for instance, that Theorem 2.8 and Theorem 2.9 in [4] imply that for $q\equiv 1 \pmod 3$ the number of 3-subsets of $\mathbb{F}_q^*$ admitting a perfect companion is at least equal to $q(\frac{q-1}{3})^3$, while for $q\equiv 2 \pmod 3$ the number of 3-subsets of $\mathbb{F}_q^*$ admitting a near-perfect companion is less or equal to $q\cdot\Phi(q-1)$ with Φ the *Euler totient function*.

This probably explains why, at the moment, we have only a few known classes of optimal but not regular $(v,k,1)$-OOCs with $k>3$ (see [1, 4, 10]).

Anyway in this paper we manage to find three new classes of optimal but not regular OOCs adopting the strategy described above with $\mathcal{S}$ equal to one the following very elementary strong difference families:

(a) the $(\mathbb{Z}_3,4,4)$ difference multiset $\{0,0,1,1\}$ for getting an optimal $(3p,4,1)$-OOC with $p=8n+7$ a prime whose silver elements are generators of $\mathbb{Z}_p^*/\{1,-1\}$;

(b) the $(\mathbb{Z}_5,5,4)$ difference multiset $\{0,1,1,4,4\}$ for getting an optimal $(5p,5,1)$-OOC with p a prime as above;

(c) the $(\mathbb{Z}_2,4,6)$ difference multiset $\{0,1,1,1\}$ for getting an optimal $(2p,4,1)$-OOC with $p=30n+11$ or $p=30n+29$ a prime whose golden elements are generators of $\mathbb{Z}_p^*/\{1,-1\}$.

3 On the silver $(3p,4,1)$ and $(5p,5,1)$ optical orthogonal codes

Note that the silver elements of $\mathbb{Z}_p$ are precisely the solutions of the congruence $x^2-2x-1\equiv 0 \pmod p$, i.e., the elements θ of $\mathbb{Z}_p$ such that $\theta+1=\theta(\theta-1)$. This property is crucial for getting the following construction.

Theorem 3.1. *Let $p=8n+7$ be a prime and let θ be a silver element of $\mathbb{Z}_p$. If θ is a generator of $\mathbb{Z}_p^*/\{1,-1\}$, then there exists an optimal $(3p,4,1)$-OOC and an optimal $(5p,5,1)$-OOC.*

Proof. By the Chinese Remainder Theorem, $\mathbb{Z}_{3p}$ and $\mathbb{Z}_{5p}$ are isomorphic to $\mathbb{Z}_3\times\mathbb{Z}_p$ and $\mathbb{Z}_5\times\mathbb{Z}_p$, respectively. So it is enough to show that, under the given assumption, there exists an optimal $(\mathbb{Z}_3\times\mathbb{Z}_p,4,1)$ difference packing and an optimal $(\mathbb{Z}_5\times\mathbb{Z}_p,5,1)$ difference packing.

The assumption on θ implies that $\{\theta^i \mid 0\le i\le 4n+2\}$ is a complete system of representatives for the cosets of $\{1,-1\}$ in $\mathbb{Z}_p^*$ so that we have

$$\mathbb{Z}_p^*=\{1,-1\}\cdot\{1,\theta,\theta^2,\dots,\theta^{4n+1},\theta^{4n+2}\}$$

and then $\{\theta^{2i}\mid 0\le i\le 2n\}\cdot\{\pm\theta,\pm\theta^2\}=\mathbb{Z}_p^*\setminus\{1,-1\}$. Thus we can claim that

$$C:=\{\theta^{2i}\mid 0\le i\le 2n\} \text{ is an optimal companion of } \{\pm\theta,\pm\theta^2\}. \tag{3.1}$$

Let us lift the $(\mathbb{Z}_3,4,4)$ difference multiset $\{0,0,1,1\}$ to the following 4-subset of $\mathbb{Z}_3\times\mathbb{Z}_p$

$$B=\{(0,\theta),(0,-\theta),(1,\theta^2),(1,-\theta^2)\}.$$

The *difference table* of B (see Table 1) shows that we can write:

$$\Delta_0B=\{\pm2\theta,\pm2\theta^2\};\quad \Delta_1B=\Delta_2B=\{\pm\theta(\theta-1),\pm\theta(\theta+1)\}. \tag{3.2}$$

Then, recalling that $\theta+1=\theta(\theta-1)$, we have:

$$\Delta_0B=2\{\pm\theta,\pm\theta^2\};\quad \Delta_1B=\Delta_2B=(\theta-1)\{\pm\theta,\pm\theta^2\}.$$

We conclude, by (3.1), that C is an optimal companion of Δ_gB for every $g\in\mathbb{Z}_3$ and then $\mathcal{D}=\{(1,c)\cdot B\mid c\in C\}$ is the desired optimal $(\mathbb{Z}_3\times\mathbb{Z}_p,4,1)$ difference packing by Proposition 2.2.

Table 1: The difference table of $B = \{(0,\theta),(0,-\theta),(1,\theta^2),(1,-\theta^2)\}$.

	$(0,\theta)$	$(0,-\theta)$	$(1,\theta^2)$	$(1,-\theta^2)$
$(0,\theta)$	$\bullet$	$(0,2\theta)$	$(2,\theta-\theta^2)$	$(2,\theta+\theta^2)$
$(0,-\theta)$	$(0,-2\theta)$	$\bullet$	$(2,-\theta-\theta^2)$	$(2,-\theta+\theta^2)$
$(1,\theta^2)$	$(1,\theta^2-\theta)$	$(1,\theta^2+\theta)$	$\bullet$	$(0,2\theta^2)$
$(1,-\theta^2)$	$(1,-\theta^2-\theta)$	$(1,-\theta^2+\theta)$	$(0,-2\theta^2)$	$\bullet$

Now, let us lift the $(\mathbb{Z}_5,5,4)$ difference multiset $\{0,1,1,4,4\}$ to the following 5-subset of $\mathbb{Z}_5 \oplus \mathbb{Z}_p$

$$B = \{(0,0),(1,\theta),(1,-\theta),(4,\theta^2),(4,-\theta^2)\}.$$

Table 2 is its difference table.

Table 2: The difference table of $B = \{(0,0),(1,\theta),(1,-\theta),(4,\theta^2),(4,-\theta^2)\}$.

	$(0,0)$	$(1,\theta)$	$(1,-\theta)$	$(4,\theta^2)$	$(4,-\theta^2)$
$(0,0)$	$\bullet$	$(4,-\theta)$	$(4,\theta)$	$(1,-\theta^2)$	$(1,\theta^2)$
$(1,\theta)$	$(1,\theta)$	$\bullet$	$(0,2\theta)$	$(2,\theta-\theta^2)$	$(2,\theta+\theta^2)$
$(1,-\theta)$	$(1,-\theta)$	$(0,-2\theta)$	$\bullet$	$(2,-\theta-\theta^2)$	$(2,-\theta+\theta^2)$
$(4,\theta^2)$	$(4,\theta^2)$	$(3,\theta^2-\theta)$	$(3,\theta^2+\theta)$	$\bullet$	$(0,2\theta^2)$
$(4,-\theta^2)$	$(4,-\theta^2)$	$(3,-\theta^2-\theta)$	$(3,-\theta^2+\theta)$	$(0,-2\theta^2)$	$\bullet$

Recalling again that $\theta+1=\theta(\theta-1)$, we can write

$$\Delta_0 B = 2\{\pm\theta,\pm\theta^2\},$$
$$\Delta_1 B = \Delta_4 B = \{\pm\theta,\pm\theta^2\},$$
$$\Delta_2 B = \Delta_3 B = (\theta-1)\{\pm\theta,\pm\theta^2\}$$

so that, by (3.1), C is an optimal companion of $\Delta_g B$ for each $g \in \mathbb{Z}_5$. We conclude that $\mathcal{D} = \{(1,c)\cdot B \mid c \in C\}$ is the desired optimal $(\mathbb{Z}_5 \times \mathbb{Z}_p, 5, 1)$ difference packing by Proposition 2.2. The assertion follows. □

The optimal OOCs arising from the above result will be called *silver*. We remark that the assumption on θ is equivalent to ask that θ^2, that is $2\theta+1$, is a primitive square of $\mathbb{Z}_p$ and that this assumption does not depend on the chosen silver element; indeed the product of the two silver elements is -1, hence they have the same orders in $\mathbb{Z}_p^*/\{1,-1\}$. We also note that the difference leaves of the constructed pakings are

$$\{0\}\times\{0,2,-2\} \,\cup\, \{1,2\}\times\{0,\theta-1,1-\theta\}$$

for the $(\mathbb{Z}_3 \times \mathbb{Z}_p, 4, 1)$ difference packing and

$$\{0\}\times\{0,2,-2\} \,\cup\, \{1,4\}\times\{0,1,-1\} \,\cup\, \{2,3\}\times\{0,\theta-1,1-\theta\}$$

for the $(\mathbb{Z}_5 \times \mathbb{Z}_p, 5, 1)$ difference packing.

Among the 2399 primes p congruent to 7 modulo 8 and not exceeding 100 000 we have checked that θ is not a generator of $\mathbb{Z}_p^*/\{1,-1\}$ in "only" 599 cases. Thus, roughly speaking, it seems that the two constructions succeed three times out of four.

Remark 3.2. Using formula (1.1), the optimal difference packings constructed in Theorem 3.1 can be more explicitly written in terms of Pell numbers. They are of the form $\mathcal{D} = \{B_i \mid 0 \le i \le 2n\}$ with

$$B_i = \{(0, P_{2i+1}\theta + P_{2i}),\ (0, -P_{2i+1}\theta - P_{2i}),\ (1, P_{2i+2}\theta + P_{2i+1}),\ (1, -P_{2i+2}\theta - P_{2i+1})\}$$

when $\mathcal{D}$ is a $(\mathbb{Z}_3 \times \mathbb{Z}_p, 4, 1)$ difference packing and with

$$B_i = \{(0,0),\ (1, P_{2i+1}\theta + P_{2i}),\ (1, -P_{2i+1}\theta - P_{2i}),\ (4, P_{2i+2}\theta + P_{2i+1}),\ (4, -P_{2i+2}\theta - P_{2i+1})\}$$

when $\mathcal{D}$ is a $(\mathbb{Z}_5 \times \mathbb{Z}_p, 5, 1)$ difference packing.

By way of illustration we explicitly construct a silver $(141, 4, 1)$-OOC.

We have $141 = 3p$ with $p = 47 = 8n + 7$ prime, $n = 5$. A silver element of $\mathbb{Z}_p$ is $\theta = 8$; indeed we have $8 + 1 \equiv 8^2 - 8 \pmod{47}$. Here the group $\mathbb{Z}_p^*/\{1,-1\}$ has prime order 23, hence θ is certainly a generator of it and Theorem 3.1 can be applied. The reduction modulo p of the Pell sequence up to its 22-nd term is

$$(0, 1, 2, 5, 12, 29, 23, 28, 32, 45, 28, 7, 42, 44, 36, 22, 33, 41, 21, 36, 46, 34, 20).$$

Thus, applying Remark 3.2, the blocks of a $(\mathbb{Z}_3 \times \mathbb{Z}_{47}, 4, 1)$ difference packing are the following:

$$\begin{gathered}
\{(0,\theta), (0,-\theta), (1, 2\theta+1), (1, -2\theta-1)\}\\
\{(0, 5\theta+2), (0, -5\theta-2), (1, 12\theta+5), (1, -12\theta-5)\}\\
\{(0, 29\theta+12), (0, -29\theta-12), (1, 23\theta+29), (1, -23\theta-29)\}\\
\{(0, 28\theta+23), (0, -28\theta-23), (1, 32\theta+28), (1, -32\theta-28)\}\\
\{(0, 45\theta+32), (0, -45\theta-32), (1, 28\theta+45), (1, -28\theta-45)\}\\
\{(0, 7\theta+28), (0, -7\theta-28), (1, 42\theta+7), (1, -42\theta-7)\}\\
\{(0, 44\theta+42), (0, -44\theta-42), (1, 36\theta+44), (1, -36\theta-44)\}\\
\{(0, 22\theta+36), (0, -22\theta-36), (1, 33\theta+22), (1, -33\theta-22)\}\\
\{(0, 41\theta+33), (0, -41\theta-33), (1, 21\theta+41), (1, -21\theta-41)\}\\
\{(0, 36\theta+21), (0, -36\theta-21), (1, 46\theta+36), (1, -46\theta-36)\}\\
\{(0, 34\theta+46), (0, -34\theta-46), (1, 20\theta+34), (1, -20\theta-34)\}
\end{gathered}$$

The isomorphism $f\colon (x,y) \in \mathbb{Z}_3 \times \mathbb{Z}_{47} \to 48y - 47x \in \mathbb{Z}_{141}$ turns the above blocks into the following eleven codeword-sets forming the desired silver $(141, 4, 1)$-OOC with difference leave $\{0, 7, 40, 45, 47, 94, 96, 101, 134\}$:

$$\begin{gathered}
\{102, 39, 64, 124\},\quad \{42, 99, 7, 40\},\quad \{9, 132, 25, 22\},\quad \{12, 129, 49, 139\},\\
\{63, 78, 34, 13\},\quad \{84, 57, 61, 127\},\quad \{18, 123, 97, 91\},\quad \{24, 117, 4, 43\},\\
\{126, 15, 115, 73\},\quad \{27, 114, 28, 19\},\quad \{36, 105, 100, 88\}.
\end{gathered}$$

As far as this author is aware, the above optimal OOC is new but the same cannot be said for its parameters. Indeed it was proved in [16] that there exists a *perfect* $(v,4,1)$ *difference family* for all $v \equiv 1 \pmod{12}$ not exceeding $10\,000$ except $v = 25$ and $v = 37$. Also, according to Remark 1.4 in [1], any perfect $(v,4,1)$ difference family can be also seen as an optimal $(w,k,1)$-OOC for all w's between v and $v + k(k-1)$ included. Thus we have the existence of an optimal $(v,4,1)$-OOC for all v's not exceeding $10\,012$ except $v = 25$ (indeed an optimal $(v,4,1)$-OOC with $26 \le v \le 48$ is known to exist anyway).

4 On the golden $(2p,4,1)$ optical orthogonal codes

Note that the golden elements of $\mathbb{Z}_p$ are precisely the solutions of the congruence $x^2 - x - 1 \equiv 0 \pmod{p}$, i.e., the elements ϕ of $\mathbb{Z}_p$ such that $\phi + 1 = \phi^2$. This property is crucial for getting the following construction.

Theorem 4.1. *Let $p \equiv 11$ or $29 \pmod{30}$ be a prime and let ϕ be a golden element of $\mathbb{Z}_p$. If ϕ is a generator of $\mathbb{Z}_p^*/\{1,-1\}$, then there exists an optimal $(2p,4,1)$-OOC.*

Proof. We have to show that, under the given assumption, there exists an optimal $(\mathbb{Z}_2 \times \mathbb{Z}_p, 4, 1)$ difference packing. Indeed $\mathbb{Z}_2 \times \mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_{2p}$ by the Chinese Remainder Theorem.

We can write $p = 6n + 5$ for a suitable n, hence $\frac{p-1}{2} = 3n + 2$, and the assumption on ϕ implies that we have

$$\mathbb{Z}_p^* = \{1,-1\} \cdot \{1, \phi, \phi^2, \ldots, \phi^{3n}, \phi^{3n+1}\}.$$

It is then clear that

$$C := \{\phi^{3i-1} \mid 1 \le i \le n\} \text{ is an optimal companion of } \{\pm 1, \pm\phi, \pm\phi^2\}. \tag{4.1}$$

Indeed we have $C \cdot \{\pm 1, \pm\phi, \pm\phi^2\} = \mathbb{Z}_p^* \setminus \{\pm 1, \pm\phi\}$.

Let us lift the $(\mathbb{Z}_2, 4, 6)$ difference multiset $\{0,1,1,1\}$ to the 4-subset B of $\mathbb{Z}_2 \oplus \mathbb{Z}_p$

$$B = \{(0,0), (1,1), (1,\phi), (1,\phi^2\}.$$

Looking at the difference table of B (see Table 3) we see that we have

$$\Delta_0 B = \{\pm(\phi - 1), \pm\phi(\phi - 1), \pm(\phi + 1)(\phi - 1)\}; \quad \Delta_1 B = \{\pm 1, \pm\phi, \pm\phi^2\}.$$

Thus, recalling that $\phi + 1 = \phi^2$, we can write

$$\Delta_0 B = (\phi - 1)\{\pm 1, \pm\phi, \pm\phi^2\}, \qquad \Delta_1 B = \{\pm 1, \pm\phi, \pm\phi^2\}$$

so that, by (4.1), C in an optimal companion of $\Delta_g B$ for each $g \in \mathbb{Z}_2$. We conclude that $\mathcal{D} = \{(1,c) \cdot B \mid c \in C\}$ is the desired optimal $(\mathbb{Z}_5 \times \mathbb{Z}_p, 5, 1)$ difference packing by Proposition 2.2. □

The optimal OOCs arising from the above result will be called *golden*. We remark that the assumption on ϕ is equivalent to ask that ϕ^2, that is $\phi + 1$, is a primitive square of $\mathbb{Z}_p$ and it does not depend on the chosen golden element; indeed the product of the two golden elements is -1, hence their orders in $\mathbb{Z}_p^*/\{1,-1\}$ are the same. We also note that the difference leave of the constructed difference packing is

$$\{0\} \times \{0, 1, -1, \phi - 1, 1 - \phi\} \cup \{1\} \times \{0, 1, -1, \phi, -\phi\}.$$

Table 3: The difference table of $B = \{(0,0),(1,1),(1,\phi),(1,\phi^2\}$.

	$(0,0)$	$(1,1)$	$(1,\phi)$	$(1,\phi^2)$
$(0,0)$	$\bullet$	$(1,-1)$	$(1,-\phi)$	$(1,-\phi^2)$
$(1,1)$	$(1,1)$	$\bullet$	$(0,1-\phi)$	$(0,1-\phi^2)$
$(1,\phi)$	$(1,\phi)$	$(0,\phi-1)$	$\bullet$	$(0,\phi-\phi^2)$
$(1,\phi^2)$	$(1,\phi^2)$	$(0,\phi^2-1)$	$(0,\phi^2-\phi)$	$\bullet$

We have checked that in the range $[1,10^5]$, Theorem 4.1 works in 1533 out of 2399 of the cases.

Remark 4.2. Using formula (1.2), the optimal difference packing $\mathcal{D}$ constructed in Theorem 4.1 can be more explicitly written in terms of Fibonacci numbers. Indeed we have $\mathcal{D} = \{B_i \mid 1 \leq i \leq n\}$ with

$$B_i = \{(0,0),\ (1, F_{3i-1}\phi + F_{3i-2}),\ (1, F_{3i}\phi + F_{3i-1}),\ (1, F_{3i+1}\phi + F_{3i})\}.$$

By way of illustration we explicitly construct a golden $(142,4,1)$-OOC using the above remark.

We have $142 = 2p$ with $p = 71 \equiv 11 \pmod{30}$ prime, and we can write $p = 6n+5$ with $n = 11$. A golden element of $\mathbb{Z}_p$ clearly is $\phi = 9$; indeed we have $9^2 \equiv 10 \pmod{71}$. The maximal proper divisors of $(p-1)/2$ are 5 and 7 and neither 10^5 nor 10^7 is $1 \pmod{p}$. This guarantees that 10 has order $(p-1)/2$ in $\mathbb{Z}_p^*$, namely $\phi+1$ is a primitive square of $\mathbb{Z}_p$. Thus Theorem 4.1 can be applied. The reduction modulo p of the Fibonacci sequence up to its 34-th term is

$$(0,1,1,2,3,5,8,13,21,34,55,18,2,20,22,42,64,35,28,63,20,\\ 12,32,44,5,49,54,32,15,47,62,38,29,67,25).$$

Thus, applying Remark 4.2, the blocks of an optimal $(\mathbb{Z}_2 \times \mathbb{Z}_{71}, 4, 1)$ difference packing are the following:

$$\{(0,0),(1,\phi+1),(1,2\phi+1),(1,3\phi+2)\}$$
$$\{(0,0),(1,5\phi+3),(1,8\phi+5),(1,13\phi+8)\}$$
$$\{(0,0),(1,21\phi+13),(1,34\phi+21),(1,55\phi+34)\}$$
$$\{(0,0),(1,18\phi+55),(1,2\phi+18),(1,20\phi+2)\}$$
$$\{(0,0),(1,22\phi+20),(1,42\phi+22),(1,64\phi+42)\}$$
$$\{(0,0),(1,35,\phi+64),(1,28\phi+35),(1,63\phi+28)\}$$
$$\{(0,0),(1,20\phi+63),(1,12\phi+20),(1,32\phi+12)\}$$
$$\{(0,0),(1,44\phi+32),(1,5\phi+44),(1,49\phi+5)\}$$
$$\{(0,0),(1,54\phi+49),(1,32\phi+54),(1,15\phi+32)\}$$
$$\{(0,0),(1,47\phi+15),(1,62\phi+47),(1,38\phi+62)\}$$
$$\{(0,0),(1,29\phi+38),(1,67\phi+29),(1,25\phi+67)\}$$

The isomorphism $f\colon (x,y) \in \mathbb{Z}_2 \times \mathbb{Z}_{71} \to 71x + 72y \in \mathbb{Z}_{142}$ turns the above blocks into the following eleven codeword-sets forming the desired golden $(142,4,1)$-OOC with difference leave $\{0,1,8,9,70,71,72,133,134,141\}$:

$$\{0,81,19,29\}, \quad \{0,119,77,125\}, \quad \{0,131,43,103\}, \quad \{0,75,107,111\},$$
$$\{0,5,45,121\}, \quad \{0,95,3,27\}, \quad \{0,101,57,87\}, \quad \{0,73,89,91\},$$
$$\{0,109,129,25\}, \quad \{0,83,37,49\}, \quad \{0,15,135,79\}.$$

Although the above optimal OOC is probably new, the same cannot be said for its parameters for the same reason explained in the end of Section 3.

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Subspace restrictions and affine composition for covering perfect hash families*

Charles J. Colbourn †, Erin Lanus

Computing, Informatics, and Decision Systems Engineering, Arizona State University, PO Box 878809, Tempe, AZ, 85287-8809, U.S.A.

To Mario Gionfriddo on his seventieth birthday.

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Abstract

Covering perfect hash families provide a very compact representation of a useful family of covering arrays, leading to the best asymptotic upper bounds and fast, effective algorithms. Their compactness implies that an additional row in the hash family leads to many new rows in the covering array. In order to address this, subspace restrictions constrain covering perfect hash family so that a predictable set of many rows in the covering array can be removed without loss of coverage. Computing failure probabilities for random selections that must, or that need not, satisfy the restrictions, we identify a set of restrictions on which to focus. We use existing algorithms together with one novel method, affine composition, to accelerate the search. We report on a set of computational constructions for covering arrays to demonstrate that imposing restrictions often improves on previously known upper bounds.

Keywords: Covering array, covering perfect hash family, affine composition, subspace restriction.

Math. Subj. Class.: 05B40, 05B15, 51E26, 68R05

1 Introduction

We develop effective construction techniques for combinatorial arrays called covering perfect hash families, which form a compact representation of covering arrays. Covering arrays arise in numerous applications in which interactions among options or factors are

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E-mail addresses: colbourn@asu.edu (Charles J. Colbourn), elanus@asu.edu (Erin Lanus)

to be measured; they are used in, for example, software testing [12, 13], hardware testing [10, 20], design of composite materials [2], computational learning [1, 9], and biological networks [14]. Computational methods to construct covering arrays often encounter difficulties when the array has many rows, many columns, or both. To alleviate this concern, covering perfect hash families were introduced in [21] and shown to provide a succinct representation of a class of covering arrays. In [6] they were used to establish the best known asymptotic upper bound on the fewest rows in a covering array. Also in [6], effective and simple algorithms were examined for their construction.

Covering perfect hash families have proved instrumental in obtaining many sizes of covering arrays that are the best currently known. Despite the compactness of the representation that they provide, their use lessens but does not remove the computational burden. We propose and analyze a method, affine composition, to combine small covering perfect hash families to make larger ones; this extends the range of array sizes for which computational methods are feasible. Moreover, the very compactness of the representation severely limits the possible numbers of rows in the covering arrays produced. We develop a method using subspace restrictions to produce covering arrays that are guaranteed to have at least a specified number of duplicated rows, which can be removed without altering the coverage. This provides finer control on the number of rows in the covering array, and hence often improves upon the coarser use of covering perfect hash families without restrictions.

Both of these contribute to the construction of covering arrays with fewer rows than the best previously known, and hence to a reduction in testing and measurement cost when the covering arrays are applied. In order to develop these notions, we first provide formal definitions and background.

Let q be a prime power. Let $\mathbb{F}_q$ be the finite field of order q. Let $\mathcal{R}_{t,q} = \{\mathbf{r}_0, \ldots, \mathbf{r}_{q^t-1}\}$ be the set of all (row) vectors of length t with entries from $\mathbb{F}_q$, and let $\mathcal{T}_{t,q}$ be the set of all column vectors of length t with entries from $\mathbb{F}_q$, not all 0. A vector $\mathbf{x} \in \mathcal{T}_{t,q}$ is a *permutation vector* [21].

Lemma 1.1 (see [21]). *Let $\mathcal{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_t\}$ be a set of vectors from $\mathcal{T}_{t,q}$. The array $A = (a_{ij})$ formed by setting a_{ij} to be the product of $\mathbf{r}_i$ and $\mathbf{x}_j$ is a $q^t \times t$ matrix in which every row is distinct if and only if the $t \times t$ matrix $X = [\mathbf{x}_1 \cdots \mathbf{x}_t]$ is nonsingular.*

When $\mu \in \mathbb{F}_q \setminus \{0\}$, substituting $\mu\mathbf{x}_i$ for $\mathbf{x}_i$ does not change the multiset of rows produced, just their order. Define $\langle \mathbf{x} \rangle = \{\mu\mathbf{x} : \mu \in \mathbb{F}_q, \mu \neq 0\}$. When $\mathbf{x}$ is not all 0, we can select as the representative of $\langle \mathbf{x} \rangle$ the unique vector whose first nonzero coordinate is the multiplicative identity element. Let $\mathcal{V}_{t,q}$ be the set of representatives of the column vectors in $\mathcal{T}_{t,q}$. Let $\mathcal{U}_{t,q}$ be the set of vectors in $\mathcal{V}_{t,q}$ whose first coordinate is not zero. Then $|\mathcal{V}_{t,q}| = \frac{q^t-1}{q-1} = \sum_{i=0}^{t-1} q^i$, and $|\mathcal{U}_{t,q}| = q^{t-1}$.

A *covering perfect hash family* $\mathrm{CPHF}(n; k, q, t)$ is an $n \times k$ array $C = (\mathbf{c}_{ij})$ with entries from $\mathcal{V}_{t,q}$ so that, for every set $\{\gamma_1, \ldots, \gamma_t\}$ of distinct column indices, there is at least one row index ρ of C for which $[\mathbf{c}_{\rho\gamma_1} \cdots \mathbf{c}_{\rho\gamma_t}]$ is nonsingular; call this a *covering t-set* and say that the t-set of columns is *covered*. It is a *Sherwood covering perfect hash family*, $\mathrm{SCPHF}(n; k, q, t)$, if in addition each entry is in $\mathcal{U}_{t,q}$.

Let N, t, k, and v be positive integers with $k \geq t \geq 2$ and $v \geq 2$. A *covering array* $\mathrm{CA}(N; t, k, v)$ is an $N \times k$ array A in which each entry is from a v-ary alphabet Σ, and for every $N \times t$ sub-array B of A and every $\mathbf{x} \in \Sigma^t$, there is a row of B that equals $\mathbf{x}$.

When k is a positive integer, $[k]$ denotes the set $\{1, \ldots, k\}$. A *t-way interaction* is $\{(c_i, a_i) : 1 \leq i \leq t\}$ where $c_i \in [k]$, $c_i \neq c_j$ for $i \neq j$, and $a_i \in \Sigma$. Such an interaction is an assignment of values from Σ to t of the k columns. An $N \times k$ array A *covers* the interaction $\iota = \{(c_i, a_i) : 1 \leq i \leq t,\ c_i \in [k],\ c_i \neq c_j \text{ for } i \neq j, \text{ and } a_i \in \Sigma\}$ if there is a row r in A such that $A(r, c_i) = a_i$ for $1 \leq i \leq t$. When there is no such row in A, ι is *not* covered in A. Hence a $\mathrm{CA}(N; t, k, v)$ covers all the t-way interactions on k columns on an alphabet of v symbols.

Covering arrays are used extensively for interaction testing in complex engineered systems. The k columns represent *factors* and the v values are the *levels* of the factors. The N rows form a *test suite* (each row is a *test*); and the coverage of interactions among the factors is limited to the *strength* t.

Denote by $\mathrm{CAN}(t, k, v)$ the smallest value of N for which a $\mathrm{CA}(N; t, k, v)$ exists. This is a *covering array number*, and for essentially all applications the goal is to minimize it. Applications also require that actual covering arrays be generated, and hence the focus is on explicit, practical construction methods. Online tables at [5] give the least upper bound on $\mathrm{CAN}(t, k, v)$ *by an explicit construction* for $2 \leq t \leq 6$, $2 \leq v \leq 25$, and $k \leq 10\,000$.

The connection between CPHFs and CAs is central in this paper, so we include a standard proof for the following correspondence.

Lemma 1.2 (see [21])**.**

1. *There exists a* $\mathrm{CA}(n(q^t - 1) + 1; t, k, q)$ *if* C *is a* $\mathrm{CPHF}(n; k, q, t)$*; and*
2. *there exists a* $\mathrm{CA}(n(q^t - q) + q; t, k, q)$ *if* C *is an* $\mathrm{SCPHF}(n; k, q, t)$.

Proof. Let C be the covering perfect hash family. Replace each entry $\mathbf{c}_{ij}$ of C by the column vector obtained by multiplying $\mathbf{c}_{ij}$ by each $\mathbf{r}_\ell \in \mathcal{R}_{t,q}$ in the specified order. By Lemma 1.1, this produces a $\mathrm{CA}(nq^t; t, k, q)$. The product of each $\mathbf{c}_{ij}$ with $(0, \ldots, 0) \in \mathcal{R}_{t,q}$ is 0, so the resulting array contains n rows that contain only 0 entries. Remove $n - 1$ of these rows to form the $\mathrm{CA}(n(q^t - 1) + 1; t, k, q)$. Now when $\mathbf{c}_{ij} \in \mathcal{U}_{t,q}$, multiplication by $(\sigma, 0, \ldots, 0) \in \mathcal{R}_{t,q}$ always yields σ. For each $\sigma \in \mathbb{F}_q$, remove $n - 1$ of the rows in which each entry is σ to form the $\mathrm{CA}(n(q^t - q) + q; t, k, q)$. □

In generating the covering array from the CPHF, it may happen that rows are generated that only cover t-way interactions that are also covered by other rows. Such a row is *redundant*, and could be removed. Indeed by tracking coverage as rows of the covering array are generated, one could avoid generating some of these redundant rows. More generally, a post-optimization method [17] may reveal or produce further redundant rows. Because the covering array is typically much larger than the covering perfect hash family, however, effort can be saved by determining in advance certain rows of the covering array that are guaranteed to be redundant, thereby avoiding their generation and subsequent elimination.

The simplest way to ensure that a row is redundant in the covering array generated is that it be identical to another row; then it is *replicated* or *repeated*. Lemma 1.2 already accounts for the redundancy of $n - 1$ rows for CPHFs, and of $(n-1)q$ rows for SCPHFs, by noting that they are replicated. Our goal here is to restrict the CPHF in such a way that many more rows are guaranteed to be replicated, and so reduce the size of the covering array generated without having to analyze its coverage during and after its generation.

Whether restricted or not, CPHFs are needed to apply Lemma 1.2. Few general direct constructions are known [18, 21, 24]; most arise from computation. Computational methods for SCPHFs include backtracking [21] and tabu search [25]. In [6], CPHFs are shown to lead to the best known asymptotic results on the existence of covering arrays. Indeed the probabilistic methods lead to two classes of efficient algorithms for constructing covering arrays for much larger parameters than had been earlier handled, and the best known bounds were improved on for a wide range of parameters as a result.

In [6], a means to restrict the CPHFs to ensure that certain rows are replicated is outlined, and applied for strength $t = 3$. In Section 2, we define restrictions and consider the effect of imposing various restrictions on the expectation that a t-set of columns is covered, in order to determine the types of restrictions that appear to be promising. In Section 3 we develop a recursive composition strategy to accelerate the computational search for restricted CPHFs. In Section 4 we report on new bounds obtained by subspace restrictions, at the same time updating some of the computational results from [6].

2 Subspace restrictions

We limit how entries are placed in a CPHF so that redundant rows are generated in the application of Lemma 1.2; these can be removed. We denote by $\mathcal{F}_{t,p}$ the set of all p-tuples of distinct entries from $\{0, \ldots, t-1\}$. A *subspace restriction* for n rows of *dimension* p and *replication* r is an r-tuple $(x_1, \ldots, x_r)$ of distinct entries from $\{1, \ldots, n\}$ and an r-tuple $(U_1, \ldots, U_r)$ for which each $U_i \in \mathcal{F}_{t,p}$.

Let $A = (a_{ij})$ be a CPHF$(n; k, q, t)$ in which each entry a_{ij} is a permutation vector of length t. Write $a_{ij\ell}$ for the ℓth entry of this vector. Let S be the subspace restriction (for n rows), given by $(x_1, \ldots, x_r)$ and $(U_1, \ldots, U_r)$. Denote by u_{ab} the element of U_a in position b. Then A *satisfies* or *meets* the restriction if, when $1 \leq c, d \leq r$, $a_{x_c,j,u_{c\ell}} = a_{x_d,j,u_{d\ell}}$ for all $1 \leq j \leq k$ and $1 \leq \ell \leq p$; for short, A is *S-restricted*.

Table 1: A CPHF$(4; 18, 3, 4)$. Each permutation vector $(h_0, h_1, h_2, h_3)^T$ is written as $h_0h_1h_2h_3$.

1020	1002	1211	1112	1122	1001	1002	1202	1111	1222	1100	1010	1101	1010	1220	1022	1122	1110
1020	1001	1212	1112	1120	1001	1000	1200	1111	1222	1100	1011	1102	1010	1220	1020	1121	1111
1022	1221	1212	1001	1100	1110	1012	1020	1111	1122	1112	1202	1010	1102	1200	1220	1222	1201
1021	1220	1202	1011	1121	1120	1002	1000	1110	1100	1122	1212	1020	1121	1222	1201	1211	1221

Table 1 shows an example. Verifying that this is a CPHF entails checking, for each 4-set of columns, that in at least one row the four permutation vectors are covering. For instance, checking that the second, fifth, sixth, and seventh columns have a covering 4-set in some row is the same as checking that at least one of the matrices

$$\begin{bmatrix}1&1&1&1\\0&1&0&0\\0&2&0&0\\2&2&1&2\end{bmatrix}, \begin{bmatrix}1&1&1&1\\0&1&0&0\\0&2&0&0\\1&0&1&0\end{bmatrix}, \begin{bmatrix}1&1&1&1\\2&1&1&0\\2&0&1&1\\1&0&0&2\end{bmatrix}, \begin{bmatrix}1&1&1&1\\2&1&1&0\\2&2&2&0\\0&1&0&2\end{bmatrix}$$

is nonsingular over $\mathbb{F}_3$. The first two are not because they repeat columns; the third is not because the sum of the first and fourth rows equals the second. But the fourth matrix is nonsingular, so we have verified that one of the 3 060 possible 4-sets of columns is covering (the diligent reader can verify the rest). Every permutation vector in the CPHF of Table 1 has a 1 in coordinate 0; hence the CPHF satisfies the restriction $(1, 2, 3, 4)$ with $U_1 = U_2 = U_3 = U_4 = \{0\}$. More is true. In the third and fourth rows, in each column the two permutation vectors agree in the first two coordinates, and hence the CPHF satisfies the restriction $(3, 4)$ with $U_3 = U_4 = \{0, 1\}$. Moreover, in the first and second rows, in each column the two permutation vectors agree in the first *three* coordinates, and hence the CPHF satisfies the restriction $(1, 2)$ with $U_1 = U_2 = \{0, 1, 2\}$.

When $\mathcal{S}$ is a set of restrictions for n rows, and a CPHF$(n; k, q, t)$ meets each $S \in \mathcal{S}$, it is *$\mathcal{S}$-restricted*. Now suppose that A is an $\mathcal{S}$-restricted CPHF$(n; k, q, t)$. Suppose that $S \in \mathcal{S}$ consists of $(x_1, \ldots, x_r)$ and $(U_1, \ldots, U_r)$, and that each U_i is a p-tuple. For each $1 \leq i \leq r$, in the q^t rows obtained from the expansion of row x_i, let E_i be the q^p rows that arise using evaluations on t-sets that are 0 on all elements not in U_i. Then $E_1 = \cdots = E_r$; that is, each row in E_1 is replicated $r - 1$ further times in the expansion of A by Lemma 1.2, and hence $(r - 1)q^p$ rows are redundant in the covering array generated (although Lemma 1.2 removes some of them already).

In the example of Table 1, we noted the presence of three restrictions. The restriction of dimension 1 and replication 4 results in $3 \cdot 3 = 9$ redundant rows. The restriction of dimension 2 results in $3^2 = 9$ redundant rows, of which three were already found to be redundant using the restriction of dimension 1. Finally the restriction of dimension 3 results in $3^3 = 27$ redundant rows; three of these were already found to be redundant using the restriction of dimension 1, while none among the remaining 24 are made redudant by the restriction of dimension 2. Hence for our example, we can ensure that at least $9 + 6 + 24 = 39$ rows are redundant; rather than getting $4 \cdot 3^4 - 3 = 321$ rows, we get 285.

A general example of a subspace restriction is straightforward. When the restriction S has $r = n$, $(x_1, \ldots, x_n) = (1, 2, \ldots, n)$, and $U_i = (0)$ for $1 \leq i \leq n$, an S-restricted CPHF$(n; k, q, t)$ is precisely an SCPHF$(n; k, q, t)$. (The entry in coordinate 0 must be nonzero for the array to be a CPHF.) Enforcing restrictions of larger dimension can increase the redundancy [6], but enforcing too many restrictions or restrictions of too large a dimension might result in more rows or fewer columns.

Next we explore the effect of imposing a restriction on the expectation that an array is a CPHF. To simplify the presentation, we fix a strength t and a prime power q, and denote the product $\prod_{i=a}^{b} \frac{q^t - q^i}{q^t}$ by $\pi_{a,b}$. We consider a fixed set of t columns and ask for the probability that the columns are covered within r rows of the CPHF. In the *basic* process, we impose no restriction, choosing each of the coordinates of each of t permutation vectors independently and uniformly at random from $\mathbb{F}_q$ for each of the r rows. The probability that the chosen set of columns is not covered in the basic process is $(1 - \pi_{0,t-1})^r$. In the *restricted process*, we first choose the p entries specified in the restriction independently and uniformly at random for one row, using the same choice for all. The remaining coordinates of each permutation vector are then chosen randomly. The probability that the chosen set of columns is not covered in the restricted process is $(1 - \pi_{0,p-1}) + \pi_{0,p-1} \left[(1 - \pi_{p,t-1})^r\right]$. The restricted process has a larger failure probability; indeed it is larger by an amount equal to

$$(1 - \pi_{p,t-1}) \left[1 - [1 - \pi_{0,t-1}]^{r-1}\right].$$

We consider two cases to examine the effect of the dimension and replication of a restriction. We tabulate failure probabilities within r rows when there is a restriction of size r and dimension p. (When $p = 0$, there is no restriction.) First we give failure probabilities for $q = 25$ and $t = 6$ (see Table 2).

Table 2: Failure probabilities with restrictions for $q = 25$ and $t = 6$.

$p \downarrow r \rightarrow$	2	3	4
0	$.1730551453 \times 10^{-2}$	$.7199076267 \times 10^{-4}$	$.2994808332 \times 10^{-5}$
1	$.1730555215 \times 10^{-2}$	$.7199483800 \times 10^{-4}$	$.2998903189 \times 10^{-5}$
2	$.1730649273 \times 10^{-2}$	$.7209672112 \times 10^{-4}$	$.3101274621 \times 10^{-5}$
3	$.1733000718 \times 10^{-2}$	$.7464379962 \times 10^{-4}$	$.5660560231 \times 10^{-5}$
4	$.1791790606 \times 10^{-2}$	$.1383210872 \times 10^{-3}$	$.6964257727 \times 10^{-4}$
5	$.3263893163 \times 10^{-2}$	$.1730452999 \times 10^{-2}$	$.1669115393 \times 10^{-2}$

As one might expect, restrictions increase the failure probability, but for those of 'low' dimension the increase is modest. Next we examine failure probabilities for $q = 3$ and $t = 5$ (see Table 3).

Table 3: Failure probabilities with restrictions for $q = 3$ and $t = 5$.

$p \downarrow r \rightarrow$	2	3	4
0	.1924749034	.0844425161	.0370465884
1	.1937767036	.0868835477	.0402353448
2	.1977309220	.0942611526	.0498358606
3	.2100498330	.1168880686	.0789332757
4	.2516261575	.1892616707	.1684735084

Naturally, because q is much smaller the failure probabilities are much larger than for $q = 25$, even though the strength here is lower.

Now suppose that our goal is to make $2q^p$ rows redundant in the generated covering array. We can choose a restriction of dimension p and replicate it three times. An alternative is to choose a second restriction of dimension p. If we choose the two restrictions so that they have no rows in common, we can replicate each twice to get the same number of redundant rows that we would get by selecting one with replication three. Which should we prefer? An easy calculation shows that the failure probability after four rows is lower with two restrictions of replication two, than one of replication three along with an unrestricted row. In general when two restrictions of dimension p with replications r_1 and r_2 restrict disjoint sets of rows, failure probabilities are minimized when r_1 and r_2 are as equal as possible. Because setting $r_2 = 1$ imposes no restriction at all, our quick example says that $r_1 = r_2 = 2$ is better than $r_1 = 3$ and $r_2 = 1$. Hence we strive to choose many restrictions with replication two.

This ignores the fact that there may be too few rows to specify the desired number of restrictions. However, the requirement that the restrictions share no rows is too severe. For the analysis, we only require that every two restrictions sharing a row select different coordinates within that row. This ensures that the rows made redundant by one are not those made redundant by the other (with the exception of the all zero row, which is redundant in every case). Moreover, in an analysis of failure probabilities the effects of two restrictions are independent, because the entries in each are chosen independently of one another.

Consider restrictions of dimensions d_1 and d_2 that share s coordinates within a row. The impact is that the q^{d_1} rows made redundant by the first and the q^{d_2} rows made redundant by the second have q^s rows in common (a subspace). Consequently, fewer rows are redundant than if the two restrictions acted independently. When restrictions share the same rows, even when on different coordinates, the effect on the failure probability can be dramatic: When $t = 5$, for example, a restriction of dimension 4 on $(0,1,2,3)$ and a restriction of dimension 2 on $(3,4)$, if replicated on the same two rows, are in fact a restriction of dimension 5, forcing a replicated row in the CPHF itself. Nevertheless, when s is small, the double coverage is also small, and there are cases in which permitting sharing is sensible.

3 Affine composition

Algorithms employed for unrestricted CPHFs from [6, 21, 25] extend in a natural way to search for restricted CPHFs, so we do not repeat them here. Instead we describe an additional approach. When one wants to make a covering array with 'many' columns, computational methods either require too much time, or yield an array with many more rows than anticipated. Yet the same methods can yield arrays with 'few' rows quickly when the number of columns is small. To take advantage of the efficacy of computational methods for smaller arrays, and still construct larger ones, recursive methods have been developed to use small arrays in a cut–and–paste method; for example, see [3, 4, 7, 8, 15, 16, 19]. Roughly speaking, a cut–and–paste (or composition) method starts with an array on k columns, forms m copies of the array written side by side to obtain mk columns, and then uses further rows to cover the as-yet-uncovered interactions.

Writing two copies of a CPHF$(n; k, q, t)$ side by side is equivalent to duplicating each column. But then in the $(n \times 2k)$ array produced, every choice of t columns containing a duplicate leads to a singular matrix for *every* row of the CPHF. Hence exactly $\sum_{s=1}^{\lfloor t/2 \rfloor} \binom{k}{s}\binom{k-s}{t-2s} 2^{t-2s}$ sets of t columns are noncovering. Although the uncovered t-sets are easily counted and characterized, there are many of them.

Let A be a CPHF$(n; k, q, t)$. Consider the effect of multiplying coordinate c of every permutation vector in row r by a nonzero element $\mu_{r,c}$ of the finite field. This does not affect the (non)singularity of any of the $t \times t$ matrices that determine coverage. Indeed we can apply different nonzero multipliers $\mu_{r,c}$ for every $1 \leq r \leq n$ and $1 \leq c \leq t-1$ to form a new array B that is again a CPHF$(n; k, q, t)$. Then we no longer simply duplicate columns, we can change them in a controlled way.

Similarly for any row we can choose a field element m and any two different coordinates c and d; then for each column, add m times the entry in coordinate c to the entry in coordinate d. Again, this does not affect the (non)singularity of any of the $t \times t$ matrices that determine coverage. This is most easily accomplished using an SCPHF, whose permutation vectors always have first coordinate equal to 1. Then setting $c = 0$, this amounts to m being an *adder*, which can be added to the entry of coordinate d of every permutation vector in the row. We can choose adders $\alpha_{r,c}$ for every $1 \leq r \leq n$ and $1 \leq c \leq t-1$, provided that the array is an SCPHF.

In general, given an SCPHF$(n; k, q, t)$ A, multipliers $\mu_{r,c}$ and adders $\alpha_{r,c}$ for $1 \leq r \leq n$ and $1 \leq c \leq t-1$, we can create a new array by, for each row r, for every column, replacing the entry x in coordinate c by $\mu_{r,c}x + \alpha_{r.c}$, with arithmetic in the field. Hence each coordinate of each row undergoes an affine transformation. No matter how this is done, the resulting array is an SCPHF$(n; k, q, t)$. There are $(q(q-1))^{n(t-1)}$ ways to choose these multipliers and adders. *Affine composition* applied to an SCPHF$(n; k, q, t)$ A_0 selects $m-1$ arrays $A_1, \ldots, A_{m-1}$, each obtained by affine transformations of A.

When affine composition is applied to A_0 to form $A_1, \ldots, A_{m-1}$, not only is each an SCPHF when A_0 is, but each A_i meets *all* of the restrictions that A_0 does. In fact, although $[A_0\ A_1\ \cdots\ A_{m-1}]$ need not be an SCPHF because certain t-sets of columns are not covered, it does meet all restrictions that A_0 does. The question remains: Which affine composition should we apply in order to leave the fewest, or to leave a particular set, of uncovered t-sets of columns?

We consider various SCPHF$(n; k, q, t)$s, taking $m = 2$. Because considering all $(q(q-1))^{n(t-1)}$ affine transformations is too time-consuming, we adopt a greedy strategy. We consider each row in turn, and determine for some or all of the $(q(q-1))^{t-1}$ affine transformations how many uncovered t-sets of columns having at least one column from the original and from the transformed copy remain, if this transformation is applied in the current row. We choose transformations that lead to a smallest number of t-sets yet to cover. After the last row is processed, this smallest number is the number that must be dealt with in additional rows not produced in the composition. Tie-breaking is carried out by choosing the lexicographically first, so the method as implemented is deterministic. Random tie-breaking, or selection methods more clever than greedy, might result in further reductions.

Table 4: Numbers of noncovering t-sets after affine compositions.

$n; k, v, t$	$\times 1 + 0$	$\times 1 + \alpha$	$\times \mu + 0$	$\times \mu + \alpha$	$\times 1 + \alpha_c$	$\times \mu_c + 0$	$\times \mu_c + \alpha_c$
2;12,4,4	2706	548	678	535	486	619	463
3;21,4,4	16170	1876	2015	1739	1572	1964	1452
4;31,4,4	54405	3239	2941	2783	2621	2594	2099
5;45,4,4	171270	4584	4176	3698	4228	3765	3312
6;59,4,4	391819	5161	3911		4835	3546	
2;15,5,4	5565	1016	871	848	915	871	803
3;24,5,4	24564	1795	1611	1409	1634	1442	1236
4;40,5,4	119340	4001	3299	3037	3441	2869	2432
5;59,5,4	391819	5253	3635		4938	3337	
6;88,5,4	1320660	8451	4689		7964	4367	
2;18,7,4	9945	1109	985	940	1081	972	855
3;34,7,4	72369	2883	2421		2619	2202	
4;57,7,4	352716	4627	3140		4386	2983	
2;20,8,4	13870	1296	1242	1117	1224	1172	1069
3;38,8,4	101935	2979	2499		2737	2249	
4;67,8,4	577071	5461	3441		5191	3278	
2;22,9,4	18711	1606	1448	1330	1539	1364	1251
2;11,4,5	11550	2280	1763	1500	2070	1618	1389
3;15,4,5	46410	4856	2706	2350	4550	2706	
2;10,3,6	25320	7984	5081	4573	7524	4766	4274

In Table 4, we report results for the number of uncovered t-sets of columns after affine composition when the allowed affine transformations are limited in various ways. The column $(\times 1 + 0)$ gives numbers when every multiplier is 1 and every adder 0. This, of course, is another way to say that the second array is an exact copy of the first. The column $(\times 1 + \alpha)$ always uses multiplier 1, and selects the *same adder* for every coordinate in the row. The column $(\times \mu + 0)$ always uses adder 0, and selects the *same multiplier* for every coordinate in the row. The column $(\times \mu + \alpha)$ selects the same adder, and the same multiplier, for every coordinate in the row. The column $(\times 1 + \alpha_c)$ always uses multiplier 1, and considers all q^{t-1} ways to select adders for the coordinates in the row. The column $(\times \mu_c + 0)$ always uses adder 0, and considers all $(q-1)^{t-1}$ ways to select multipliers for the coordinates in the row. The column $(\times \mu_c + \alpha_c)$ considers all $(q(q-1))^{t-1}$ ways to select adders and multipliers for the coordinates in the row.

Consider an arbitrary permutation vector in $\mathcal{U}_{t,q}$ and its images under the $(q(q-1))^{t-1}$ affine transformations. Every permutation vector in $\mathcal{U}_{t,q}$ appears precisely $(q-1)^{t-1}$ times among these images. Hence when $q(t-1)$ is large compared to the number of columns, the probability that duplicate columns arise is reduced. Indeed simply making a copy (without any nontrivial transformation) leaves far more uncovered sets of columns than even very restricted sets of affine transformations do. This is part of the explanation for the effectiveness of applying affine transformations.

Choosing the best affine transformation to apply seems impractical; indeed we did not fill in the blank entries in Table 4 because even the greedy strategy has either too many options to consider or too large an array to check repeatedly. Of course, it would be better to determine the most effective transformations without having to conduct a large search, but because this depends heavily on the structure of the SCPHF, we know of no way to do this. Hence we choose them randomly. The expected number of t-sets of columns that remain uncovered after affine composition depends not only on the parameters of the SCPHF and the number of copies made, but also on the structure of the SCPHF. Indeed it depends on the number of rows in which ℓ columns of the SCPHF are linearly independent, for every way to choose ℓ columns with $2 \le \ell < t$. Consider, for example, the situation with strength $t = 3$, and consider two columns. For any row with identical permutation vectors in these two columns, there is no possibility for affine composition to cover a 3-set of columns containing this pair. Then the probability that randomly chosen affine transformations succeed on a particular triple of columns depends upon in how many rows of the CPHF the two have identical entries.

In general, we wish to maximize the number of sets of ℓ columns that are linearly independent for all $\ell \le t$. Because the definition of a CPHF does not include such a strong condition, there can be SCPHFs on which affine composition yields a poor result.

After any affine composition is carried out, we anticipate that not all t-sets of columns will be covered (although within each of the m copies formed. all t-sets of columns are covered). Hence affine composition is not a means to avoid doing any computation, but rather a means to reduce to a substantially smaller problem.

4 Computations with subspace restrictions

Subspace restrictions give a natural way for redundant rows to be formed in the expansion of a CPHF into a covering array. For this to be worthwhile, the restricted CPHF must have more columns than does the unrestricted CPHF with one fewer row. (Otherwise the resulting covering array would in general have more rows without increasing the number of columns.) In the results to follow, we find that restrictions not only reduce the number of rows needed, but in many cases do not reduce the achievable number of columns.

We construct SCPHFs satisfying specific sets of restrictions. We always enforce the restriction $(x_1, \dots, x_n) = (1, 2, \dots, n)$, and $U_i = (0)$ for $1 \le i \le n$; this means we are talking about SCPHFs and not general CPHFs, and hence can employ affine composition as described. We also enforce restrictions of higher dimension. In particular, a restriction on two distinct rows i and j with $U_i = U_j = (0, 1, \dots, p-1)$ is denoted by $\langle p \rangle_{i,j}$ or, more simply, $\langle p \rangle$. When multiple restrictions of this type are enforced, we require that they refer to disjoint sets of rows of the SCPHF; provided that they do, we can simply list the dimensions of the restrictions imposed. We use exponential notation, so that $\langle p \rangle^a$ requires that the restriction $\langle p \rangle$ be imposed on a disjoint pairs of rows.

In order to accelerate the computation, we use affine composition in some cases to make a 'large' fraction of the CPHF. We employ column resampling, random extension, and conditional expectation algorithms from [6]. The adaptation of each to incorporate any number of restrictions is straightforward. There are more intensive search techniques that yield more accurate results, but the simpler methods can be effectively applied for somewhat larger numbers of columns and symbols. So when we report that a certain number of columns can be realized, we fully expect that a more intensive search can find a solution with more columns (and, in some cases, has). Despite this, certain trends are evident, as one expects based on the failure probabilities.

Given a prime power q, number n of rows, strength t, and set $\mathcal{S}$ of restrictions, we tabulate the largest number k of columns found in an $\mathcal{S}$-restricted SCPHF$(n; k, q, t)$. When $q = 23$ and $t = 4$, rows in Table 5 indicate the value of n; columns indicate the restrictions enforced. Here C reports results for CPHFs, while $-$ reports results for SCPHFs (that is, no restrictions beyond the basic one).

Table 5: Improvements (shown in bold) on known covering array numbers when $t = 4$ and $q = 23$.

n	C	$-$	$\langle 2 \rangle$	$\langle 2 \rangle^2$	$\langle 3 \rangle$	$\langle 3 \rangle\langle 2 \rangle$	$\langle 3 \rangle^2$	$\langle 3 \rangle^3$
2	39	39	**39**		**30**			
3	98	98	**98**		**85**			
4	**250**	**245**	**240**	**227**	**196**	**194**	**170**	
5	**603**	**600**	**585**	**569**	497	484	389	
6	**1461**	**1365**	**1333**	**1192**	**1184**	**1174**	**1003**	**874**

Entries shown in bold are those that improve upon the previously best known size of a covering array for these parameters (all from [6]). It may be disappointing that by imposing three $\langle 3 \rangle$ restrictions, our methods construct only 874 columns rather than 1365. However, one must bear in mind that these restrictions force (at least) 36 432 redundant rows in the covering array. This accomplishes what we set out to do. Although we may have fewer columns, we generate fewer rows. Table 6 shows similar results for $t = 5$ and $q = 5$, using a more extensive set of restrictions. Improvements are again frequent.

Table 7 gives a complete set of results for strength $t = 4$ with $4 \le q \le 25$. (Henceforth we do not display every improvement for a covering array number in bold; see [5] to determine when an array is the best known.) Table 8 displays the results for $t = 5$ with $4 \le q \le 25$ having two, three, and four rows, while Table 9 displays results with five and six rows, and Table 10 gives results having seven or more rows.

Table 6: Improvements (shown in bold) on known covering array numbers when $t = 5$ and $q = 5$.

n	C	$-$	$\langle 2\rangle$	$\langle 2\rangle^2$	$\langle 3\rangle$	$\langle 3\rangle\langle 2\rangle$	$\langle 3\rangle^2$	$\langle 4\rangle$	$\langle 4\rangle\langle 2\rangle$	$\langle 4\rangle\langle 3\rangle$	$\langle 4\rangle^2$	$\langle 4\rangle^3$
2	12	12	12		12			**12**				
3	17	17	**17**		16			**16**				
4	24	**24**	23	23	**23**	22	**22**	21	21	21	**21**	
5	32	32	32	32	**32**	**31**	30	30	**30**	28	**28**	
6	44	44	**44**	41	42	40	40	**42**	39	39	**39**	**35**
7	59	59	**59**	58	**58**	**57**	**56**	**54**	52	**52**	**51**	**45**
8	81	81	**81**	**79**	**78**	**76**	**75**	**73**	**70**	**68**	**67**	**62**
9	107	107	**107**	106	106	**106**	**103**	**95**	93	**93**	**91**	**84**
10	**143**	142	**142**	141	**141**	**139**	**138**	**131**	**128**	**126**	**119**	**110**
11	196	196	196	**196**	191	**191**	**187**	**175**	**174**	**171**	**162**	**152**
12	266	266	266	**266**	**262**	**261**	**255**	242	**242**	**239**	**220**	**200**
13	346	346	**346**	**340**	333	333	327	333	**333**	**327**	**286**	**265**

Table 7: Number of columns found for various restrictions and strength four.

q	n	C	$-$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 3\rangle^2$	$\langle 3\rangle^3$
4	2	13	12	11	9		
4	3	21	21	20	17		
4	4	32	31	31	27	23	
4	5	45	45	43	38	34	
5	2	15	15	14	11		
5	3	26	24	24	21		
5	4	41	40	40	34	30	
5	5	60	59	58	54	48	
5	6	90	89	88	83	74	65
5	7	141	138	132	125	110	103
5	8	206	205	198	172	157	157
5	9	306	301	286	265	232	214
5	10	465	457	434	392	355	314
5	11	700	680	657	559	502	480
5	12	1102	1013	1012	820	751	712
5	13	1607	1431	1425	1264	1127	1090
7	2	18	18	17	14		
7	3	34	34	33	30		
7	4	57	57	56	51	45	
7	5	101	99	98	89	79	
7	6	169	169	166	154	134	119
7	7	282	277	275	242	209	186
7	8	475	475	457	394	364	311
7	9	814	764	742	631	551	527
7	10	1338	1336	1334	1064	956	897
8	2	20	20	19	15		
8	3	38	38	37	33		
8	4	67	67	67	60	53	
8	5	122	121	118	109	94	
8	6	220	218	214	177	154	148
8	7	379	370	359	316	277	239
8	8	657	657	611	565	445	415
8	9	1161	1159	1155	918	808	732
9	2	22	22	21	17		
9	3	43	43	41	37		
9	4	79	78	76	69	60	
9	5	148	147	143	133	114	
9	6	270	276	275	223	188	183
9	7	487	484	462	405	356	309
9	8	896	847	845	681	596	556
9	9	1621	1503	1475	1243	1106	1025
11	2	24	24	24	20		
11	3	49	49	47	43		
11	4	100	99	96	89	80	
11	5	207	202	199	166	141	
11	6	388	388	374	321	286	241
11	7	745	735	734	572	494	456
11	8	1508	1507	1474	1112	1009	934
13	2	28	27	27	21		
13	3	57	57	55	51		
13	4	124	124	121	106	94	
13	5	260	265	262	214	177	
13	6	524	524	494	431	371	324
13	7	1102	1027	1002	839	721	654
16	2	31	31	31	24		
16	3	71	71	69	61		
16	4	159	158	157	140	120	
16	5	357	357	344	293	246	
16	6	778	763	745	579	506	468
16	7	1666	1578	1527	1260	1102	1040
17	2	34	34	32	25		
17	3	75	74	71	63		
17	4	175	174	171	152	122	
17	5	382	380	366	320	258	
17	6	873	822	818	654	550	517
17	7	1778	1743			1323	1193
19	2	35	35	35	28		
19	3	83	83	79	72		
19	4	203	199	197	167	136	
19	5	455	455	443	381	301	
19	6	1056	1001	934	802	692	642
23	2	39	39	39	30		
23	3	98	98	98	85		
23	4	250	245	240	196	170	
23	5	603	600	585	497	389	
23	6	1461	1365	1333	1174	1003	874
25	2	42	42	41	33		
25	3	107	107	104	90		
25	4	277	274	265	215	187	
25	5	694	688	668	529	443	
25	6	1706	1584	1575	1286	1115	1037

Table 8: Number of columns found for various restrictions and strength five, $n \in \{2, 3, 4\}$.

	$n=2$					$n=3$					$n=4$					
q	C	–	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	C	–	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	C	–	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 4\rangle^2$
3	12	11	10	10	8	13	13	12	12	12	16	16	15	15	15	14
4	11	11	11	11	10	15	15	15	15	14	20	20	20	19	18	17
5	12	12	12	12	12	17	17	17	16	16	24	24	23	23	21	21
7	14	14	14	14	12	22	21	21	21	19	31	31	31	31	29	27
8	15	15	15	15	13	23	23	23	23	22	36	36	36	35	33	30
9	16	16	15	15	14	25	25	24	24	22	39	39	39	39	36	33
11	17	17	17	16	15	29	29	28	28	26	47	47	47	47	42	38
13	19	19	18	18	16	31	31	31	31	29	55	55	55	53	48	44
16	20	20	20	20	18	38	38	38	36	33	66	66	64	64	56	53
17	21	20	20	20	18	39	39	39	37	34	70	70	70	68	61	56
19	22	22	22	22	19	41	41	40	40	37	78	78	77	75	68	60
23	24	24	24	24	21	48	48	47	46	43	90	90	88	87	81	72
25	25	25	24	24	21	49	49	49	49	42	98	97	96	94	88	76

Table 9: Number of columns found for various restrictions and strength five, $n \in \{5, 6\}$.

	$n=5$						$n=6$						
q	C	–	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 4\rangle^2$	C	–	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 4\rangle^2$	$\langle 4\rangle^3$
3	19	19	19	19	18	17	23	23	23	22	22	21	19
4	26	26	26	25	24	23	34	34	32	32	31	28	27
5	32	32	32	32	30	28	44	44	44	42	42	39	35
7	47	47	47	47	43	39	69	69	68	67	61	59	50
8	55	55	55	55	50	45	83	83	82	82	74	67	60
9	62	62	61	60	55	51	95	95	92	91	88	77	73
11	79	79	79	77	69	61	127	125	125	120	109	100	90
13	93	93	92	88	84	75	157	157	156	153	140	120	106
16	119	119	119	111	103	92	210	207	206	197	185	165	146
17	128	128	128	120	112	97	228	223	219	217	199	181	157
19	143	143	143	136	126	107	269	262	262	258	233	207	180
23	181	178	175	171	155	136	345	344	342	332	299	259	228
25	197	197	194	184	166	147	406	406	391	386	332	288	249

Table 10: Number of columns found for various restrictions and strength five, $n \geq 7$.

q	n	C	$-$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 4\rangle^2$	$\langle 4\rangle^3$
4	7	42	42	41	41	41	38	33
5	7	59	59	59	58	54	51	45
5	8	81	81	81	78	73	67	62
5	9	107	107	107	106	95	91	84
5	10	143	142	142	141	131	119	110
5	11	196	196	196	191	175	162	152
5	12	266	266	266	262	242	220	200
5	13	346	346	346	333	333	288	265
5	14	458	458	455	447	435	394	363
5	15	609	609	606	603	569	527	484
5	16				779	721	672	647
7	7	101	101	99	96	93	82	77
7	8	149	149	148	140	132	121	112
7	9	211	209	205	201	195	179	159
7	10	318	312	312	315	284	253	230
7	11	455	454	451	447	418	361	341
7	12	661	661	660	628	592	546	495
7	13						702	679
8	7	123	123	122	116	109	99	90
8	8	186	185	184	183	163	150	134
8	9	283	283	277	278	252	229	210
8	10	430	429	427	417	371	343	320
8	11	622	622	604	595	569	510	473
8	12					812	749	675
9	7	151	149	149	147	133	118	110
9	8	231	229	229	224	214	184	167
9	9	374	374	355	353	324	283	254
9	10	568	568	546	542	483	445	405
9	11				818	755	690	629
11	7	209	209	209	196	182	161	147
11	8	342	340	332	328	289	248	228
11	9	533	529	524	505	472	425	389
11	10				827	730	680	614
13	7	272	272	271	261	237	209	186
13	8	454	452	450	442	379	349	324
13	9	767	767	764	750	653	567	523
16	7	379	379	376	360	336	284	248
16	8	656	656	650	641	582	515	454
16	9							766
17	7	408	407	407	400	358	309	278
17	8	721	721	718	663	620	555	480
19	7	473	470	466	461	446	373	343
19	8			851	838	723	671	597
23	7	656	656	654	648	564	493	430
23	8							785
25	7	712	712	709	701	636	568	483

We illustrate a further useful application of restrictions using strength $t = 6$. Every time a restriction $\langle t-1 \rangle$ is enforced, $q^{t-1} - q$ additional rows become redundant in the covering array. By enforcing restrictions $\langle t-1 \rangle^q$, $q^t - q^2$ rows are redundant. On the other hand, each row of the SCPHF would employ only a slightly larger number of rows, $q^t - q$. To take advantage of this, compare the four situations with $q = 3$ and $t = 6$ in Table 11. The SCPHF produced is permitted to have n rows, but must satisfy the specified number of $\langle t-1 \rangle$ restrictions. The covering array generated has N rows; notice how close the values of N are. Which should we prefer? By computing failure probabilities after n rows subject to the restrictions, one finds that enforcing more restrictions gives lower failure probability, primarily because more restrictions allow more rows.

Table 11: Four restricted SCPHF$(n; k, 3, 6)$s that yield covering arrays with similar numbers of rows.

n	N	#$\langle 5 \rangle$	Failure	k
15	10893	0	.0000044079	57
16	10899	3	.0000043357	57
17	10905	6	.0000042646	58
18	10911	9	.0000041948	61

We applied the simple computational methods to each; the computed failure probabilities suggest that we should choose more restrictions, and the largest number of columns produced agrees. This is not an isolated example. In Table 12 we report on similar computations for $q = 3$ and $t = 6$ with different numbers of rows and restrictions. In order to read this effectively, an entry ought to be compared with the one that is three columns to the left and one row above, because the resulting covering arrays have comparable number of rows. We reiterate that we have not found the maximum number of columns in general; indeed we may be very far from it. Nevertheless, it is important that when enough restrictions (and the right ones) are enforced, there is a possibility of improving on an unrestricted SCPHF with fewer rows.

Improvements arise frequently when $t = 6$ for larger values of q as well; we summarize the results in Table 13.

Naturally, other search techniques can be applied to make improvements, and other restrictions may prove useful in the construction of covering arrays. What we have shown is that worthwhile restrictions can often be enforced with little penalty in failure probability or in number of columns generated. In order to avoid substantial computaion, it would be of substantial interest to develop further geometric constructions of CPHFs using finite projective or affine spaces, particularly with an eye to which nontrivial restrictions can be enforced.

5 Concluding remarks

Two extensions of research on covering perfect hash families have been developed here. The first provides a flexible recursive technique for making large CPHFs from smaller ones; the remarkable feature of this approach is that rather than simply juxtaposing copies of smaller arrays, each copy can be transformed by affine mappings in order to enhance the coverage obtained. We have demonstrated that different affine transformations can have a

Table 12: Number of columns found in an $\mathrm{SCPHF}(n;k,3,6)$ satisfying $\langle 5\rangle^{\ell}$.

		Number ℓ of $\langle 5\rangle$ restrictions										
n	C	0	1	2	3	4	5	6	7	8	9	10
2	10	10	8									
3	11	11	11									
4	13	13	12									
5	15	16	15	14								
6	18	18	17	16	16							
7	20	21	19	19	18							
8	23	23	21	21	20	19						
9	27	27	24	24	24	23						
10	31	30	28	27	27	25	24					
11	34	33	31	30	30	29	27					
12	39	38	35	35	33	33	31	31				
13	44	43	40	40	37	39	36	35				
14	50	50	50	47	42	44	42	39	39			
15	57	57	54	51	50	48	47	46	44			
16	66	66	62	59	57	55	54	52	51	49		
17	73	71	70	68	66	64	62	58	58	53		
18	85	82	82	77	75	74	72	68	64	63	61	
19	99	98	93	87	86	82	77	73	73	70	69	
20	108	108			101	96	92	85	82	78	77	72
21	123	122						102	97	93	88	85
22	144	142									102	97

Table 13: Number of columns found for various restrictions and strength six.

q	n	C	$-$	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 5\rangle$	$\langle 5\rangle^2$	$\langle 5\rangle^3$
4	2	11	11	11	11	10	9		
4	3	13	13	13	13	13	12		
4	4	16	16	16	16	15	15	14	
4	5	20	19	19	19	19	18	17	
4	6	23	23	23	23	23	22	21	19
4	7	28	28	27	27	26	26	24	22
4	8	33	33	33	33	32	31	31	28
4	9	40	40	40	39	38	38	36	35
4	10	49	49	48	48	46	46	43	43
4	11	58	58	57	57	54	54	53	50
4	12	73	73	73	72	69	66	63	60
4	13	90	86	86	84	82	78	76	74
4	14	107	107	107	103	100	96	93	89
4	15	128	128	128	127	127	119	112	107
4	16	157	156	153	150	146	141	134	131
5	2	11	11	11	11	11	10		
5	3	15	15	14	14	14	13		
5	4	19	19	18	18	18	17	16	
5	5	23	23	23	23	23	22	20	
5	6	30	30	28	27	27	26	25	24
5	7	36	36	35	35	34	33	32	29
5	8	46	45	45	44	44	43	38	37
5	9	59	59	56	56	54	53	48	47
5	10	75	74	74	72	69	68	64	59
5	11	93	93	93	92	90	85	82	76
5	12	121	121	121	120	118	111	103	95
5	13	156	156	149	149	146	142	131	121
7	2	13	13	13	12	12	11		
7	3	17	17	17	17	17	15		
7	4	23	23	22	22	22	21	20	
7	5	31	31	31	31	29	27	27	
7	6	41	41	40	39	38	38	36	33
7	7	55	55	55	52	51	49	46	44
7	8	76	76	76	75	70	68	63	58
7	9	103	103	103	101	95	93	85	80
7	10	140	140	138	135	132	127	116	106
8	2	13	13	13	13	13	12		
8	3	18	18	18	18	17	17		
8	4	26	26	26	25	25	23	22	
8	5	33	33	33	32	32	31	29	
8	6	47	47	47	46	45	42	39	36
8	7	65	65	65	65	61	58	54	48
8	8	94	94	94	91	88	83	77	69
8	9	130	130	130	128	127	118	109	102
9	2	14	14	13	13	13	12		
9	3	20	20	19	19	19	18		
9	4	27	26	26	26	25	24	23	
9	5	38	38	37	37	36	35	33	
9	6	55	55	53	53	51	49	44	39
9	7	78	76	75	75	72	69	64	58
9	8	112	112	112	108	107	99	92	83
11	2	15	15	14	14	14	13		
11	3	22	22	22	22	21	19		
11	4	31	31	31	30	30	28	26	
11	5	46	46	45	44	43	41	35	
11	6	68	67	66	66	61	60	54	47
11	7	100	100	100	97	95	88	83	74
11	8	150	150	147	145	141	132	121	107
13	2	16	16	16	16	15	14		
13	3	23	23	22	22	22	20		
13	4	35	34	34	33	33	30	29	
13	5	54	54	52	52	51	49	41	
13	6	82	81	81	81	77	71	65	59
13	7	124	124	124	124	121	111	102	91
16	2	16	16	16	16	16	15		
16	3	26	26	25	25	25	22		
16	4	41	40	39	39	39	35	33	
16	5	66	66	63	63	58	56	51	
16	6	104	104	104	103	98	89	82	73
16	7	164	163	159	156	150	141	127	113
17	2	17	17	16	16	16	15		
17	3	27	27	25	25	25	23		
17	4	42	42	41	41	41	38	35	
17	5	66	66	65	65	62	57	53	
17	6	108	108	108	106	105	96	86	77
17	7	178	177	176	171	167	156	139	125
19	3	29	29	27	27	27	25		
19	4	46	45	45	45	45	41	37	
19	5	76	74	74	74	70	65	59	
19	6	124	124	124	122	118	110	100	87
23	3	31	31	30	29	29	26		
23	4	52	52	51	51	51	45	41	
23	5	87	87	87	87	85	78	70	
23	6	148	148	146	145	145	133	119	106
25	3	31	31	31	31	30	28		
25	4	56	56	54	54	54	49	43	
25	5	96	95	95	93	91	82	76	
25	6	169	169	166	163	161	147	132	116

dramatic effect on the composition, but theoretical guarantees for the observed improvements are needed.

The second extension provides finer control on the number of rows obtained in the resulting covering array, using the notion of subspace restrictions. Extensive computations demonstrate the effectiveness of such restrictions on reducing the number of rows in the covering arrays produced, over a wide range of parameters. Although the algorithmic methods used are relatively fast, we have repeatedly remarked that there is no reason to believe that they yield sizes that are besr possible in general. More computationally expensive methods such as backtracking [21] and tabu search [25], when feasible, may improve upon the results obtained. Indeed, after this paper was completed, simulated annealing and post-optimization have been used to obtain more accurate sizes for a smaller range of parameters [11, 23].

Certainly further computation will yield further improvements, and the use of subspace restrictions and affine composition accelerate these computations. However, we anticipate that direct constructions (extending those in [18, 22, 24]) would be of most interest, particularly if they accommodate a variety of subspace restrictions.

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Reaction graphs of double Fano planes

Mariusz Meszka
AGH University of Science and Technology, Kraków, Poland

Alexander Rosa
McMaster University, Hamilton, ON, Canada

Dedicated to Mario Gionfriddo on the occasion of his 70th birthday.

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Abstract

We consider various reaction graphs on the set of distinct double Fano planes.

Keywords: Double Fano plane, reaction graph, strongly regular graph.

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The concept of a *reaction graph*, which has its origin in chemistry, has been explored in several papers, for example, [5, 7, 8, 9, 10, 11].

The reaction graph(s) of the Fano plane (i.e. projective plane of order 2, Steiner triple system of order 7, or $\mathrm{BIBD}(7, 3, 1)$) are considered in detail in [8, 9], with some additional comments provided in [7]. The vertices of such reaction graph are the 30 distinct Fano planes (on a fixed 7-element set). The reaction graph is of degree $14, 8$, and 7, respectively, according to how adjacency is defined: namely, whenever two vertices (Fano planes) have one, zero, or three triples in common, respectively. The graph of degree 14 is actually isomorphic to $2K_{15}$, that is, two disjoint complete graphs K_{15} as components. Each of these corresponds to a maximal set of $\mathrm{MAD}\ \mathrm{STS}(7)$s (mutually almost disjoint STSs, cf. [6]).

It is well known that the *simple* $\mathrm{BIBD}(7, 3, 2)$ (i.e. with no repeated blocks) is unique up to an isomorphism and consists of two disjoint Fano planes. It contains 14 blocks (triples) and its automorphism group is of order 42. The blocks (triples) of one such design can be represented as $\{0, 1, 3\}, \{0, 2, 3\} \bmod 7$. We shall call any simple $\mathrm{BIBD}(7, 3, 2)$ a *double Fano plane*. Thus there are $\frac{7!}{42} = 120$ distinct double Fano planes on any 7-element set. A double Fano plane will be denoted (a, b) provided a and b are the two disjoint Fano planes that constitute it.

Let $(a, b), (c, d)$ be two distinct double Fano planes. Due to the structure of the reaction graphs of the single Fano plane, whenever $|\{a, b, c, d\}| = 4$, two of the 4 intersections

E-mail addresses: meszka@agh.edu.pl (Mariusz Meszka), rosa@mcmaster.ca (Alexander Rosa)

between a and c, a and d, b and c, and b and d must contain exactly one triple. Without loss of generality we may assume that the intersection between a and c, and also between b and d both contain one triple. The remaining two intersections, namely between a and d, and between b and c, may

(i) both contain zero triples, or

(ii) both contain three triples, or

(iii) one contains zero and the other contains three triples.

The edges of our reaction graph on K_{120} can now be one of four kinds: either $|\{a, b, c, d\}| = 3$, or it is one of the three types above (see Figure 1).

We shall use the following "colourful" terminology.

A *green* edge joins two double Fano planes $(a, b), (c, d)$ when $|\{a, b, c, d\}| = 3$, that is, one of a, b equals one of c, d. A *yellow, blue* or *red* edge, respectively, joins two double Fano planes $(a, b), (c, d)$ when $|\{a, b, c, d\}| = 4$ and case (i), (ii) or (iii), respectively, occurs.

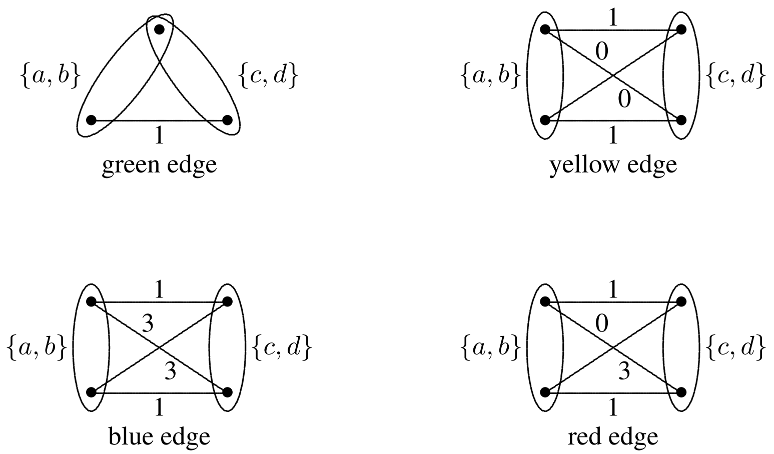

Figure 1: Pairs of double Fano planes.

According to the aforementioned intersections, one may define the following four reaction graphs:

I. The *green* graph (the subgraph of K_{120} induced by the green edges).

This graph is regular of degree 14, and is quasi-strongly regular of grade 3 (cf. [4]) with parameters $(120, 14, 6, (0, 1, 2))$.

II. The *yellow* graph (the subgraph of K_{120} induced by the yellow edges).

This graph is regular of degree 21, and is quasi-strongly regular of grade 2 (cf. [4]) with parameters $(120, 21, 0, (3, 6))$.

III. The *blue* graph (the subgraph of K_{120} induced by the blue edges).

This graph is regular of degree 28 and is quasi-strongly regular of grade 3 with parameters $(120, 28, 6, (4, 6, 12))$.

IV. The *red* graph (the subgraph of K_{120} induced by the red edges).

This graph is regular of degree 56. and is strongly regular with parameters $(120, 56, 28, 24)$. A strongly regular graph with these parameters and automorphism group of order $348\,364\,800$ is known to exist (cf. [3]; see also [12, 13]).

The four coloured graphs together form a 4-class association scheme. The intersection numbers for this scheme can be found at `http://home.agh.edu.pl/~meszka/reaction_graphs.html`.

Next we want to investigate the structure of so-called neighbourhood graphs.

For a vertex $\{a, b\}$ of the reaction graph, a vertex $\{c, d\}$ joined to it by a green edge is called a *green neighbour*, and similarly for yellow, blue or red edges we have *yellow, blue*, or *red* neighbours.

Given a vertex of the reaction graph, the *green neighbourhood graph* is the complete graph K_{14} on its green neighbours. Its edges are coloured green, yellow or red – there are no blue edges. The green edges induce graph consisting of two disjoint K_7's, the yellow edges induce the Heawood graph (cf. [1]) , and the red edges induce the bipartite complement of the Heawood graph. It is well-known that the automorphism group of the Heawood graph is $\mathrm{PGL}(2, 7)$ of order 336. The coloured edges form a 3-class association scheme.

The *yellow neighbourhood graph* of a vertex is the complete graph K_{21} on its yellow neighbours. Its edges are 3-coloured: green, blue and red; there are no yellow edges. The graph induced by the green edges is regular of degree 4 and is distance-transitive with intersection array $[4, 2, 2; 1, 1, 2]$, with automorphism group $\mathrm{PGL}(2, 7)$ of order 336. The graphs induced by the blue and red edges, respectively, both have degree 8, and the same automorphism group as the graph induced by green edges. In this case too the coloured edges form a 3-class association scheme.

The *blue neighbourhood graph* of a vertex is the complete graph K_{28} on its blue neighbours. Its edges are 4-coloured. The graph induced by green and blue edges, respectively, is regular of degree 6, while the graph induced by the yellow edges is cubic, and is actually isomorphic to the Coxeter graph (cf. [2]). The automorphism group of each of these three graphs is again $\mathrm{PGL}(2, 7)$. The graph induced by the red edges is the so-called 8-triangular graph; it is regular of degree 12, and is distance-transitive with intersection array $[12, 5; 1, 4]$. Its automorphism group is S_8 of order $40\,320$. The coloured edges form a 4-class association scheme.

Finally, the *red neighbourhood graph* of a vertex is the complete graph K_{56} on its red neighbours. Its edges are 4-coloured. The graph induced by green edges is of degree 6, and its automorphism group has order $225\,792$. Those induced by the yellow, blue and red edges, respectively, are of degree $9, 12$, and 28, respectively, where the first two of these have automorphism group $\mathrm{PGL}(2, 7)$ of order 336, while the last one has large automorphism group of order $2\,903\,040$. In this case, the coloured edges *do not* form an association scheme.

Let us remark that the graph induced by the union of the green and blue edges is of degree 42, and turns out to be a quasi-strongly regular graph of grade 2 (cf. [4]) with parameters $(120, 42, 18, (6, 15))$. Of course, the graph induced by the union of green, yellow and blue edges is complementary to the red graph, and so is strongly regular with parameters $(120, 63, 30, 36)$.

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Mixed hypergraphs and beyond*

Zsolt Tuza
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, H-1053 Budapest, Reáltanoda u. 13–15, Hungary and
Department of Computer Science and Systems Technology, University of Pannonia, H-8200 Veszprém, Egyetem u. 10, Hungary

Dedicated to Mario Gionfriddo on the occasion of his 70th birthday.

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Abstract

Some open problems are collected on hypergraphs, graphs, and designs, presented at the HyGraDe conference celebrating Mario Gionfriddo's 70th birthday.

Keywords: Mixed hypergraph coloring, stably bounded interval hypergraph, chromatic spectrum, chromatic polynomial, WORM coloring, coloring of Steiner systems.

Math. Subj. Class.: 05C15, 05B05

The conference HyGraDe took its name from HYpergraphs, GRAphs and DEsigns, three important areas of the research activities of Mario Gionfriddo, to whom we happily dedicated all our talks. Those are also the subjects of my collaborations with colleagues in Catania. For the celebration conference I collected some open problems which are related to the coloring theory of mixed hypergraphs; here they are organized in this three-sided structure. The sources of the problems are mentioned in the text, rather than specified inside the statement of each one.

1 Hypergraph coloring

A hypergraph $\mathcal{H}$ is a pair $(X, \mathcal{E})$, where X is the underlying set called *vertex set* and $\mathcal{E}$ is a set system over X, called *edge set*. A hypergraph is *uniform* if all its edges have the same cardinality; more specifically, if $|E| = r$ for all $E \in \mathcal{E}$, then $\mathcal{H}$ is said to be r-*uniform*. (Hence, the 2-uniform hypergraphs are precisely the graphs.) In order to avoid some anomalies, we shall restrict our attention to hypergraphs in which each edge contains at least two vertices.

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E-mail address: tuza@dcs.uni-pannon.hu (Zsolt Tuza)

As a general term, by *coloring* we mean any assignment $\varphi\colon X \to \mathbb{N}$, and call $\varphi(x)$ the *color* of vertex $x \in X$.

The classical notion of *proper coloring* means a coloring such that every edge $E \in \mathcal{E}$ contains two vertices of distinct colors; in other words, no edge is monochromatic. The *chromatic number* of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$, is the smallest possible number of colors in a proper coloring of $\mathcal{H}$.

The opposite side is where each edge $E \in \mathcal{E}$ contains two vertices of the same color; i.e., no edge is multicolored.[1] Motivated by Voloshin's works, we use the term *C-coloring* for a coloring of this type, and if a hypergraph has to be colored in this way, it will be called a *C-hypergraph*. The *upper chromatic number* of a C-hypergraph $\mathcal{H}$, denoted by $\overline{\chi}(\mathcal{H})$, is the largest possible number of colors in a C-coloring of $\mathcal{H}$.

Proper hypergraph coloring is a direct generalization of the fundamental notion of proper graph coloring; research in this direction started in the mid-1960's. On the other hand, C-coloring in graphs is not really interesting as it simply means that each connected component is monochromatic. For hypergraphs, however, such problems become highly nontrivial; the first such questions arose in the first half of the 1970's. But it took two decades until Voloshin independently introduced the notion and also created a model far beyond that, as we shall discuss below.

A comparison of some basic properties of proper colorings and C-colorings is given in Table 1. It is important to emphasize that every number of colors is possible between minimum and maximum; indeed, in a proper coloring it is feasible to split any non-singleton color class into two, while in a C-coloring any two color classes may be united. This simple observation will have a relevance later.

Table 1: Some coloring properties.

	proper coloring	C-coloring
excluded edge type	monochromatic	multicolored
always colorable with	$\lvert X\rvert$ colors (= max)	1 color (= min)
interesting parameter	χ = min # of colors	$\overline{\chi}$ = max # of colors

A general overview on hypergraph colorings — not only these two types — can be found in [13]; and a comprehensive survey on C-coloring is given in [9].

Mixed hypergraphs. A new dimension in the theory of hypergraph coloring was opened in the works of Voloshin [28, 29] where he invented the following complex model. A *mixed hypergraph* has two types of edges, namely C-edges and D-edges; formally we may write $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$. The requirement for a coloring $\varphi\colon X \to \mathbb{N}$ is that every C-edge $E \in \mathcal{C}$ has to contain two vertices with **c**ommon color and every D-edge $E \in \mathcal{D}$ has to contain two vertices with **d**istinct colors. In other words, φ should be a proper coloring of $(X, \mathcal{D})$ and a C-coloring of $(X, \mathcal{C})$ at the same time.

There is no a priori assumption on the relation between $\mathcal{C}$ and $\mathcal{D}$, they may or may not be disjoint. Edges in $\mathcal{C} \cap \mathcal{D}$ are termed *bi-edges*, and if $\mathcal{C} = \mathcal{D}$ then $\mathcal{H}$ is called a *bi-*

[1] By 'multicolored' we mean that the colors of the elements are mutually distinct. Such a set is often called a *rainbow* set in the literature.

hypergraph. A coloring of a bi-hypergraph — termed *bi-coloring* — is a proper coloring and a C-coloring at the same time.

Contrary to proper colorings and C-colorings, which always exist for every hypergraph, a mixed hypergraph may not admit any coloring; in this case it is called *uncolorable*. For instance, the bi-hypergraph whose bi-edges are the ten 3-element subsets of a 5-element vertex set, is uncolorable because either at least three colors occur (violating the condition of C-coloring) or some color occurs on at least three vertices (violating proper coloring).

If a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is colorable, its *lower chromatic number* denoted by $\chi(\mathcal{H})$ is the smallest possible number of colors, and its *upper chromatic number* $\overline{\chi}(\mathcal{H})$ is the largest possible number of colors. The feasible set $\Phi(\mathcal{H})$ of $\mathcal{H}$ is the set of those integers k for which $\mathcal{H}$ admits a coloring with precisely k colors.

A comprehensive account on the first decade of results and methods concerning mixed hypergraphs is the monograph [30].

Stably bounded hypergraphs. A structure more general than mixed hypergraphs was introduced in two steps, in the papers [5, 7] and [6], and studied further in a series of papers. A *stably bounded hypergraph* is a hypergraph $\mathcal{H} = (X, \mathcal{E})$ for which also four functions $s, t, a, b\colon \mathcal{E} \to \mathbb{N}$ are given. The first two of them prescribe lower and upper bounds on the number of colors occurring inside the edges, and the other two prescribe bounds for each edge on the multiplicity of the color occurring most frequently in it. We assume

$$1 \leq s(E) \leq t(E) \leq |E|$$

and

$$1 \leq a(E) \leq b(E) \leq |E|$$

for every $E \in \mathcal{E}$. A coloring φ is feasible if, for each $E \in \mathcal{E}$, we have:

- φ uses at least $s(E)$ colors inside E,
- φ uses at most $t(E)$ colors inside E,
- there exists a color which is assigned to least $a(E)$ vertices of E,
- no color is assigned to more than $b(E)$ vertices of E.

Hence, if E is a C-edge of a mixed hypergraph then its requirements are $t(E) = |E| - 1$ and $a(E) = 2$; and if it is a D-edge, then $s(E) = 2$ and $b(E) = |E| - 1$. In fact one of a and t suffices to describe a C-edge, and one of s and b suffices to describe a D-edge.

Stating the conditions in other words, the functions s and t restrict the sizes of largest multicolored subsets inside the edges, while a and b restrict the sizes of their largest monochromatic subsets.

The lower chromatic number $\chi(\mathcal{H})$, the upper chromatic number $\overline{\chi}(\mathcal{H})$, and the feasible set $\Phi(\mathcal{H})$ are naturally defined in the same way as for mixed hypergraphs.

The conditions $s(E) = 1$, $t(E) = |E|$, $a(E) = 1$, $b(E) = |E|$ put no restriction on the coloring of edge E. We obtain functional subclasses of stably bounded hypergraphs if we prescribe the set of functions which are allowed to be restrictive. For instance, (S,T,A)-hypergraph means that the functions s, t, and a can put restrictions on (some of) the edges, but b must be non-restrictive for all edges. An interesting subclass is that of (S,T)-hypergraphs, termed *color-bounded hypergraphs*. Earlier examples in the literature may be interpreted as B-hypergaphs [1, 24] and S-hypergaphs [15].

Table 2: Coloring restrictions determined by the functions s, t, a, b.

function	meaning
s	at least $s(E)$ colors inside E
t	at most $t(E)$ colors inside E
a	some color at least $a(E)$ times inside E
b	each color at most $b(E)$ times inside E

Interval hypergraphs. A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is called an *interval hypergraph* if its vertex set X admits an ordering $x_1, x_2, \ldots, x_n$ such that every edge $E \in \mathcal{E}$ is a set of consecutive vertices in this order. Interval hypergraphs have many nice properties and admit efficient algorithms for various problems which are intractable on general structures.

Problem 1.1. Determine the time complexity of the following problems over the given functional subclasses of stably bounded *interval* hypergraphs:

1. Colorability of (S,T)-hypergraphs.
2. Lower chromatic number of (S,A)-hypergraphs.
3. Lower chromatic number of (S,T,A)-hypergraphs.
4. Upper chromatic number of (S,T)-hypergraphs.
5. Upper chromatic number of A-hypergraphs.
6. Upper chromatic number of (T,A)-hypergraphs.
7. Upper chromatic number of (T,B)-hypergraphs.
8. Upper chromatic number of (S,T,B)-hypergraphs.

Table 3: Solved and unsolved cases — time complexity of basic coloring problems on seven functional subclasses of stably bounded interval hypergraphs; ??? = open, o = obvious, lin = solvable in linear time, NP-c = NP-complete, NP-h = NP-hard.

	S,T	A / T,A	S,A / S,T,A	T,B / S,T,B
exists?	???	o	NP-c	NP-c
min	lin	o	???	NP-h
max	???	???	NP-h	???

On interval hypergraphs, complexity is known for all the other combinations of the four functions s, t, a, b. Subsets of those results are proved in different papers, the last pieces appearing in [10], where also a detailed summary for several further classes of hypergraphs is given. Note that each of T, A, and (T,A) admits a monochromatic X, whereas each of S, B, and (S,B) admits a multicolored X.

Table 4: The other functional subclasses, time complexity solved completely on interval hypergraphs.

	T	S / B / S,B	A,B / any larger
exists?	o	o	NP-c
min	o	lin	NP-h
max	lin	o	NP-h

Gaps. The feasible set $\Phi(\mathcal{H})$ of a colorable hypergraph $\mathcal{H}$ is called *gap-free* if it is an interval of integers. If this property does not hold, then we say that $\mathcal{H}$ has a *gap* at k (also called 'gap in the chromatic spectrum') if k is an integer such that $\chi(\mathcal{H}) < k < \overline{\chi}(\mathcal{H})$ and $k \notin \Phi(\mathcal{H})$.

If $1 \in \Phi(\mathcal{H})$, then the feasible set is gap-free. On the other hand, for every finite set W of positive integers with $1 \notin W$, in [20] a mixed hypergraph $\mathcal{H}$ is constructed such that $\Phi(\mathcal{H}) = W$. Since $|X| \in \Phi(\mathcal{H})$ also guarantees that $\Phi(\mathcal{H})$ is gap-free, a hypergraph with gaps in $\Phi(\mathcal{H})$ necessarily has both C-edges and D-edges.

It is interesting to investigate which classes of hypergraphs have members with gaps in the chromatic spectrum, and which are completely gap-free. For instance, a gap-free class is that of interval hypergraphs [21], and the property remains valid also for interval (S,T)-hypergraphs. Also, mixed 'hypertrees' — hypergraphs $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ which can be represented over a tree graph such that each hyperedge $E \in \mathcal{C} \cup \mathcal{D}$ induces a subtree — have a gap-free $\Phi(\mathcal{H})$ [23], but this property does not extend for (S,T)-hypertrees [8].

Another famous example is the class of planar mixed hypergraphs, which admit constructions with $\Phi(\mathcal{H}) = \{2, 4\}$, hence a gap at 3 [22].

Problem 1.2.

1. Can interval (S,A)-hypergraphs have gaps?
2. Can interval (T,B)-hypergraphs have gaps?
3. Can interval (A,B)-hypergraphs have gaps?
4. What gaps can occur in stably bounded planar hypergraphs and in their functional subclasses?

The planar case is open also for color-bounded hypergraphs.

Chromatic polynomials. Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph in any of the models above (mixed, stably bounded, etc.), and assume that $\mathcal{H}$ is colorable. For λ running over the natural numbers, it is known that the number of allowed colorings

$$\varphi\colon X \to \{1, \ldots, \lambda\}$$

is a polynomial in λ, more precisely a polynomial of degree $\overline{\chi}(\mathcal{H})$. It is called the *chromatic polynomial* of $\mathcal{H}$, denoted by $P(\mathcal{H}, \lambda)$.

For any class $\mathfrak{H}$ of hypergraphs, one can consider the class

$$\{P(\mathcal{H}, \lambda) \mid \mathcal{H} \in \mathfrak{H}\}$$

of chromatic polynomials. From this point of view, the partial order for functional subclasses of mixed and stably bounded hypergraphs is determined in [6], as illustrated in Figure 1. Also, the chromatic polynomials of non-1-colorable hypergraphs (i.e., of those containing at least one D-edge) is characterized [7], in terms of Stirling numbers of the second kind.

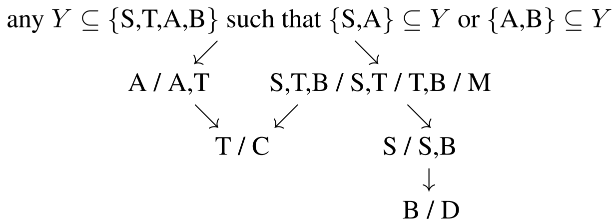

Figure 1: Hierarchy of classes of chromatic polynomials; M = mixed hypergraphs, C = only C-edges, D = only D-edges.

Problem 1.3.

1. Characterize those polynomials which are chromatic polynomials of a given type of 1-*colorable* hypergraphs.
2. Determine the hierarchy analogous to the one exhibited in Figure 1 when the hypergraphs have a structural property (e.g., interval hypergraphs). How does the hierarchy depend on the structure?

The requirement of 1-colorability in Problem 1.3.1 means the restriction to C-hypergraphs for mixed, T-hypergraphs for color-bounded, and A-hypergraphs or (T,A)-hypergraphs for those subclasses of stably bounded hypergraphs which are not color-bounded.

2 Graphs

There are many problems in graph theory which can be interpreted in terms of colorings of mixed hypergraphs. Here we discuss only one of them.

***F*-WORM colorings.** Let F be a fixed graph with at least three vertices. For a graph $G = (V, E)$, a vertex coloring φ is an *F-WORM coloring* if the vertex set of every subgraph isomorphic to F in G is neither monochromatic nor multicolored. ('WORM' abbreviates '**w**ith**o**ut **r**ainbow or **m**onochromatic'.)

The notion was introduced not much time ago, in [19], which actually appeared later than the second paper [18]. Further early works on the subject are [11] and [12].

Three basic coloring problems, also for F-WORM colorings, are whether a given G is colorable, and if it is, then what is the minimum and maximum number of colors in an F-WORM coloring of G. This similarity to the previous section is no surprise because one can observe that F-WORM coloring of G precisely means a feasible coloring of the bi-hypergraph whose bi-edges are the subsets $B \subset V$ such that $|B| = |V(F)|$ and the induced subgraph $G[B]$ contains a subgraph isomorphic to F.

Many aspects of mixed hypergraphs can be raised for F-WORM colorings as well, and also further questions arise. Here we mention only some of the interesting problems.

Problem 2.1. Let F be a connected graph with at least three vertices.

1. Is it NP-complete to decide whether a generic graph G admits an F-WORM coloring?
2. What is the necessary and sufficient condition for F to ensure that the minimum number of colors in an F-WORM coloring is bounded above by a universal constant for all F-WORM colorable graphs?
3. What is the time complexity of computing the minimum number of colors?
4. What is the complexity of deciding whether the feasible set (set of those numbers k of colors for which a generic input graph G admits an F-free coloring with precisely k colors) is gap-free?
5. Can the F-WORM feasible set contain any large gaps?
6. Study similar problems assuming that G belongs to a particular class of graphs.

Partial results are known to these questions, but the case of general F seems to be open.

3 Designs

A *Steiner system* $\mathrm{S}(t, k, v)$ is a k-uniform hypergraph with v vertices, such that each t-tuple of vertices is contained in precisely one edge (also called block). Viewing such systems from the direction of mixed hypergraphs, several interesting approaches arise. For instance, if each block is considered as a C-edge, we obtain a C-$\mathrm{S}(t, k, v)$ system. Another possibility[2] is to assume that each block is a bi-edge; then we have a B-$\mathrm{S}(t, k, v)$ system. Particular types are the systems B-STS(v), C-STS(v), B-SQS(v), C-SQS(v), derived from *Steiner triple* and *quadrulpe systems* (where $(t, k) = (2, 3)$ or $(t, k) = (3, 4)$, respectively), cf. also [27]. Besides, we consider here finite geometries, too.

Finite projective planes. It is proved in [3] that if the points of a projective plane of order q are colored in such a way that no line is multicolored, then the number of colors cannot exceed $q^2 - q - \Theta(q^{1/2})$ as $q \to \infty$; i.e., this function is an upper bound on the upper chromatic number. The bound is tight for an infinite sequence of planes, and it is even proved in [2] that an optimal C-coloring is obtained by making a 'double blocking set' (a set that meets every line in at least two points) monochromatic and assigning a distinct color to every point outside this set, provided that the plane is a Desarguesian plane $\mathrm{PG}(2, q)$ of sufficiently large order.

Problem 3.1.

1. Find a tight general *lower* bound on the upper chromatic number for every finite projective plane of order q.
2. Find estimates on the upper chromatic number of other types of finite geometries.
3. Study further types of colorings of finite geometries.

[2] In fact many more possibilities arise when larger block sizes are considered.

Steiner quadruple systems. It is known that for every fixed $t \geq 2$ the upper chromatic number of a C-S$(t, t+1, v)$ system is at most $c_t \log v$ for some constant c_t [26]. However, a tight estimate is available only for triple systems, as we shall mention below. For quadruple systems of order $v = 2^m$ a repeated application of the 'doubling construction' shows that the upper chromatic number can be at least $m+1$ in general. The method is: start with two vertex-disjoint systems $\mathcal{H}_1 = (X_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (X_2, \mathcal{E}_2)$ of order v, take 1-factorizations $(F_1^i, \dots, F_{v-1}^i)$ of the complete graphs whose vertex set is X_i for $i = 1, 2$; and then the blocks in the system of order $2v$ are those in $\mathcal{H}_1 \cup \mathcal{H}_2$ moreover the 4-tuples of the form $e_j^1 \cup e_k^2$ where $e_j^1 \in F_j^1$ and $e_k^2 \in F_k^2$, for all combinations (j, k) with $1 \leq j, k \leq v-1$.

Problem 3.2.

1. Do there exist uncolorable B-SQS(v) systems?
2. Does every $\mathcal{H} =$ C-SQS(2^m) have $\overline{\chi}(\mathcal{H}) \leq m+1$?
3. Does there exist an infinite sequence of B-SQS(v) systems with unbounded upper chromatic number?

A complete answer to parts 2 and 3 seems to be unknown even for quadruple systems obtained by the repeated application of the doubling construction, starting from a single 4-element block on four vertices. (Such systems always admit a bi-coloring — their feasible set is $\{2, 3\}$ when viewed as bi-hypergraphs — hence they are not relevant concerning part 1.)

Steiner triple systems. The 'doubling plus one' construction builds an STS$(2v+1)$ from an STS(v). The method is: start with a triple system $\mathcal{H} = (X, \mathcal{E})$ of order v, where $X = \{x_1, \dots, x_v\}$; let X' be a set of $v+1$ further vertices, disjoint from X; take a 1-factorization[3] $(F_1, \dots, F_v)$ of the complete graphs whose vertex set is X'; and create the triples of the form $x_i \cup e$ where $e \in F_i$. Together with the edges of $\mathcal{H}$, this yields a Steiner triple system over $X \cup X'$.

The coloring requirement on a B-STS system means that each block (triple) has to contain precisely two colors. In a C-STS system, monochromatic blocks may also occur (and the lower chromatic number is 1).

In [25], the first paper dealing with C-STS and B-STS (and also with SQS) systems, it is proved that if $v < 2^m$, then the upper chromatic number of every STS(v) is at most m; moreover, $\overline{\chi} = m$ is attained for exactly those systems which are obtained by a sequence of doubling plus one constructions starting from the trivial system of order 3 with one triple.

For such B-STS systems, Mario Gionfriddo raised the following attractive problem in [16].

Conjecture 3.3. *If a B-STS$(2^m - 1)$ system $\mathcal{H}$ is obtained from B-STS(3) by a sequence of doubling plus one constructions, then it has $\chi(\mathcal{H}) = \overline{\chi}(\mathcal{H}) = m$.*

In other words, no bi-coloring of B-STS$(2^{m-1}-1)$ can be extended to the B-STS$(2^m - 1)$ without increasing the number of colors, i.e., the latter system does not admit any 'extended bi-coloring'.

[3] Since v is odd — more precisely $v \equiv 1 \vee 3 \pmod 6$ — we have $v+1$ even, therefore the edge set of the complete graph K_{v+1} can be decomposed into 1-factors.

One approach to the conjecture is to assume that a B-STS$(2^m - 1)$, obtained from a B-STS$(2^{m-1} - 1)$ by doubling plus one, admits a bi-coloring with $m - 1$ colors, and to investigate what types of size distributions of the color classes might occur. The first necessary conditions of this kind are given in [14]. The recent paper [4] makes further steps in this direction, and also describes a doubling-plus-one sequence constructed explicitly over $\mathrm{GF}(2)$, which is proved to not admit any extended bi-coloring.

It is important that the construction be started from B-STS(3), because other systems may admit extended bi-colorings [17]. The smallest known example is $v = 13 \to 2v + 1 = 27$, which has an extended bi-coloring with 3 colors.

4 Conclusion

Mixed hypergraph is a great invention. By the combination of two antipodal concepts a new dimension has been opened for coloring theory. The above collection of problems is just an appetizer, lots of interesting further ones remain unsolved, for instance to characterize nice classes of colorable hypergraphs. Moreover, mixed hypergraphs and their generalizations can describe several issues in graph theory as well. WORM coloring considered above is just one example; one can mention areas in Ramsey theory, and more.

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Mario Gionfriddo and mixed hypergraph coloring

Vitaly Voloshin
Troy University, Troy, AL, USA

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Abstract

We give a brief description of the explicit and implicit contribution of Mario Gionfriddo to mixed hypergraph coloring.

Keywords: Graph and hypergraph coloring, mixed hypergraphs, block designs, Steiner systems.

Math. Subj. Class.: 05C15, 05C65

1 A little bit of history

It was the summer of 1992 in new independent state, Republic of Moldova, the very fresh ex-USSR country. Living in Kishinev, the capital city, and desperately looking for any contacts with western mathematicians, one day I went to the library of the Institute of Mathematics and Informatics. That library was good because one could find some additional western mathematical journals compared to the university library. It was an absolutely random event (or was it?): my attention was attracted by the International Mathematical Union Canberra Circular (from Australia!), with the list of mathematical conferences all over the world. At that time the internet was not widely accessible, there was no Google, not even email available. I noticed a very brief, just one paragraph, advertisement that there will be Catania Combinatorial Conference in October 1992, in Italy. And the address of Mario Gionfriddo was simply provided as the contact information.

Since I had nothing to lose, I decided to write a postcard (see both sides in Figure 1). I wrote:

> *Dear Professor Mario Gionfriddo:*
>
> *I would be much obliged to you if you could send me invitation/program/information/ Proceedings of the 3rd Catania Combinatorial Conference. I am a specialist in Graph and Hypergraph Theory, Assistant Professor of the Moldova State University, Kishinev.*

E-mail address: vvoloshin@troy.edu (Vitaly Voloshin)

> *I would like to be friend with you. My report may be entitled :"Conditional Colourings on Hypergraphs".*
>
> *Thank you very much.*
>
> *Sincerely yours V. Voloshin*
>
> *24 July 1992.*

Since I was studying English from zero at that time (my foreign language was French), writing in English was a good exercise. If you look at this postcard, you may also recognize an old Soviet postcard with the stamps of Republic of Moldova over it. But it was beyond any imagination, what significant events were implied by this simple postcard.

Figure 1: Postcard that changed the world.

I was waiting the reply for about two months and, having none, thought that it was one of many cases left without any answer. All of the sudden, one week before the conference, the invitation letter by Mario Gionfriddo arrived. There was no possibility for me to arrange everything (visa etc.) on such short notice (later I learned it required 1–4 months!). So I had to answer that I can't come. Then Mario asked me for my CV (correspondence by email just started in Moldova); it was mailed to Catania. Since then I had no news for a long time and decided that I have to forget it again.

But I will never forget the day of June 26, 1993 when I have received the official invitation by Mario Gionfriddo saying that CNR of Italy has awarded a research grant to me for visiting University of Catania for two months. It was a huge event because it gave me some hope that I can probably survive the hardships of that period. I must confess that there was time when I believed that as mathematician I will not survive. 1992 was the year of war in Moldova, and it felt like we just escaped the Titanic. After arranging multiple problems (like visa, ticket etc.) I arrived to Catania in early October, 1993. I could only recognize Mario at Catania railway station (we never met!) because he was holding the famous book of Claude Berge "Hypergraphs" [3]. It was the best "password" in that moment; and it was the very first application of hypergraph theory in real life for me.

When we started discussions about possible research collaboration, I realized that we have very different backgrounds and not that much in common. Mario mostly worked in block designs but I was not familiar with them except occasional mentioning in the book of Berge. However, at that time I already was developing basic concepts of mixed hypergraph coloring. The basic idea of it was to allow edges that can be monochromatic but must not be polychromatic (all vertices = different colors). At the very beginning they were called "anti-edges" because this term exactly reflected the meaning. Mixing classic edges (non monochromatic subsets) with anti-edges (non polychromatic subsets) lead to the concept of mixed hypergraph coloring. It was completely new at that time (even this circumstance became clear much later!). The very first paper [33] was just published but nobody heard about it in Italy. The main paper [34] was in progress and not even published yet. But because of the generality of graph coloring, the ideas for collaboration came naturally.

It was due to Mario himself and his colleagues Salvatore Milici, Gaetano Quattrocchi, Angelo Lizzio and others. It was due to the Mario's seminar, multiple discussions with international visitors like Zsolt Tuza, Alex Rosa, Curt Lindner, Chris Rodger, Carsten Thomassen, Robin Wilson, and graduate students like Lorenzo Milazzo and many others. Very soon I realized that I got into the very best environment that a mathematician can dream: the international center of active research in graphs, hypergraphs and designs under the leadership of Mario Gionfriddo. The new direction of research aiming at application of mixed hypergraph coloring in coloring of block designs has taken off. It was a matter of not one, not even two years of research collaboration when the very first significant results have been obtained and published.

As I recall, the very first fundamental questions which were worth to work on, were these:

1. What is the upper chromatic number of Steiner triple system (abbreviated by STS for short) considered as C-hypergraph? That is, what is the maximum total number of colors when coloring the vertices of each block with at most two colors?

   At that time, the STS were colored in old classic way: no block was monochromatic. Under this constraint, the problem on maximum number of colors did not exist since the coloring with n (number of points, or vertices in STS) colors was always feasible.

2. If we color each block with precisely two colors, are there uncolorable Steiner triple systems? That is, are there STS which cannot be colored in this way with any number of colors?

Notice that in classic coloring, all systems were trivially colorable with n colors and therefore this problem never arose. However, if the system is colorable under these new constraints, then such concepts as the minimum and maximum number of colors naturally arise; they are called the lower and upper chromatic numbers and denoted by χ and $\bar{\chi}$ respectively.

3. Is the chromatic spectrum of any STS continuous? That is, whether there exist colorings using any intermediate number of colors between χ and $\bar{\chi}$. Otherwise there is a gap in chromatic spectrum meaning there is no coloring with a number of colors k such that $\chi < k < \bar{\chi}$.

When I arrived to Catania for the first time, there were a few preliminary results in the first and second questions regarding some other hypergraph classes like interval mixed hypergraphs. But there was no idea, no approach, not even one fact of any mixed hypergraph with the gap in chromatic spectrum.

Simultaneously, in 1993, in order to find out if the concept of mixed hypergraph coloring was new, I submitted the current version of [34] to Paul Erdős with only one this question. The assumption was that Erdős knew everything; and, to my great satisfaction, the answer was yes. In 1993–1996, I received six letters from Paul Erdős, and in the fifth letter he wrote, see Figure 2:

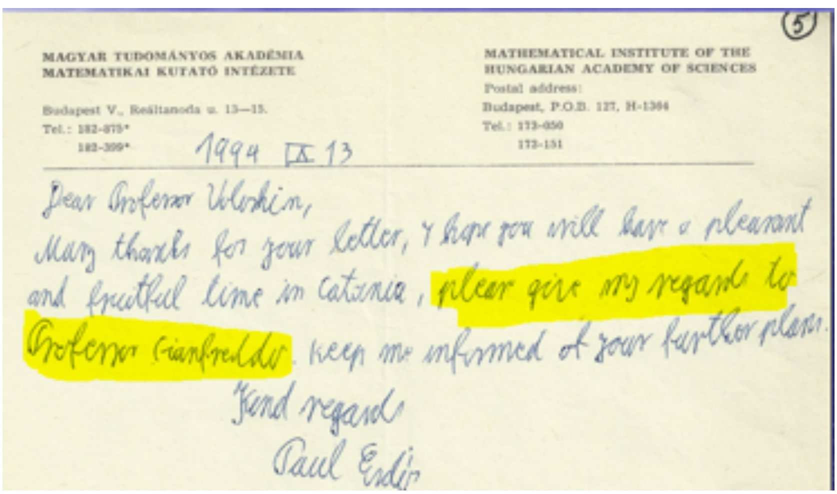
1994 IX 13

Dear Professor Voloshin,

Many thanks for your letter, I hope you will have a pleasant and fruitful time in Catania, please give my regards to Professor Gionfriddo. Keep me informed of your further plans.

Kind regards

Paul Erdős



(5)

MAGYAR TUDOMÁNYOS AKADÉMIA
MATEMATIKAI KUTATÓ INTÉZETE

Budapest V., Reáltanoda u. 13–15.
Tel.: 182-875*
183-399*

MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
Postal address:
Budapest, P.O.B. 127, H-1364
Tel.: 173-050
173-151

1994 IX 13

Dear Professor Voloshin,
Many thanks for your letter, I hope you will have a pleasant and fruitful time in Catania, please give my regards to Professor Gionfriddo. Keep me informed of your further plans.
Kind regards
Paul Erdős

Figure 2: Paul Erdős sends regards to Mario Gionfriddo, 1994.

This letter was an additional evidence of how high was the international recognition of Mario Gionfriddo long before I came to Catania.

2 Mathematical results obtained in Catania

We use the terminology from [4]. Let $V = \{v_1, v_2, \dots, v_n\}$ be a finite set of elements called *vertices*, and let $\mathcal{E} = \{E_1, E_2, \dots, E_m\}$ be a family of subsets of V called *edges* or *hyperedges*. The pair $\mathcal{H} = (V, \mathcal{E})$ is called a *hypergraph* with vertex set $V = V(\mathcal{H})$ and edge-set $\mathcal{E} = \mathcal{E}(\mathcal{H})$. The hypergraph $\mathcal{H} = (V, \mathcal{E})$ is sometimes called a *set system*. If each edge of a hypergraph contains precisely two vertices, then it is a graph.

If every edge of $\mathcal{H}$ is of size r, then $\mathcal{H}$ is called an r*-uniform hypergraph;* evidently, a simple graph is a 2-uniform hypergraph.

Let $\{1, 2, \dots, \lambda\}$ be a set of colors. A *proper* λ*-coloring* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a mapping $c\colon V \to \{1, 2, \dots, \lambda\}$ for which every edge $E \in \mathcal{E}$ has at least two vertices of different colors. The number of proper λ-colorings of $\mathcal{H}$ is a polynomial in λ; it is denoted by $P(\mathcal{H}, \lambda)$ and is called the *chromatic polynomial*. The minimum value of λ for which there exists a proper λ-coloring of a hypergraph $\mathcal{H}$ is called the *chromatic number* of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$. A hypergraph $\mathcal{H}$ is k*-colorable* if $\chi(\mathcal{H}) \le k$.

The concept of a mixed hypergraph coloring was introduced in [33]. Instead of $\mathcal{H} = (V, \mathcal{E})$, the basic idea was to consider a structure $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$, termed a *mixed hypergraph*, with *two* families of subsets called $\mathcal{C}$-edges and $\mathcal{D}$-edges. By definition, a *proper* λ*-coloring* of a mixed hypergraph $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ is a mapping $c\colon V \to \{1, 2, \dots, \lambda\}$ for which two conditions hold:

- every $C \in \mathcal{C}$ has at least two vertices of a $\mathcal{C}$ommon color;
- every $D \in \mathcal{D}$ has at least two vertices of $\mathcal{D}$ifferent colors.

A mixed hypergraph $\mathcal{H}$ is called *colorable* if it admits at least one proper coloring; and it is *uncolorable* if no such colorings exist.

The *chromatic spectrum* is the vector $(r_1, r_2, \dots, r_n)$, where each r_k is the number of partitions of the vertex set induced by proper colorings using precisely k colors. A *gap* in the chromatic spectrum is an integer k for which $\chi(\mathcal{H}) < k < \bar{\chi}(\mathcal{H})$ and $r_k = 0$. A mixed hypergraph $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ is called a *bihypergraph* if the families of $\mathcal{C}$-edges and $\mathcal{D}$-edges coincide, i.e., $\mathcal{C} = \mathcal{D}$.

A *Steiner system* $\mathrm{S}(t, k, v)$ is a k-uniform hypergraph of order v, for which each t-tuple of vertices is contained in precisely one edge. To mention some examples, a system $\mathrm{S}(2, 3, v)$ is a *Steiner triple system* $\mathrm{STS}(v)$, an $\mathrm{S}(3, 4, v)$ is a *Steiner quadruple system* $\mathrm{SQS}(v)$, and an $\mathrm{S}(2, q+1, q^2+q+1)$ is a *finite projective plane* of order q.

The edges are called *blocks*. We may view each block as a $\mathcal{C}$-edge (when an $\mathrm{STS}(v)$ is denoted by $\mathrm{CSTS}(v)$) or as a bi-edge – that is, a $\mathcal{C}$-edge and a $\mathcal{D}$-edge at the same time (when an $\mathrm{STS}(v)$ is denoted by $\mathrm{BSTS}(v)$). The notations $\mathrm{CS}(t, k, v)$, $\mathrm{BS}(t, k, v)$, $\mathrm{CSQS}(v)$ and $\mathrm{BSQS}(v)$ are derived for the respective systems in a similar way.

The study of the upper chromatic number in Steiner triple systems started in Catania in 1993 and resulted in a series of publications, see [26, 27, 28, 29, 31, 30, 32]. For example, in [30] the authors proved that

$$\bar{\chi}(\mathrm{BSTS}(v)) \le \bar{\chi}(\mathrm{CSTS}(v)) \le k,$$

for all $v \le 2^k - 1$. This upper bound on $\bar{\chi}$ is tight for all $k \ge 2$, and the systems attaining equality were also characterized. In particular, in them the cardinalities of color classes are powers of 2. The first PhD Thesis in this field (and in mixed hypergraph coloring in general) was defended under supervision of Mario Gionfriddo by Lorenzo Milazzo in 1997, see [28].

A coloring of a Steiner triple system $\mathrm{STS}(n)$ in a way that every block receives precisely two colors is also called a bicoloring [7]. All bicolorable $\mathrm{STS}(2^h-1)$s have upper chromatic number $\bar{\chi} \leq h$. If $\bar{\chi} = h < 10$, then lower and upper chromatic numbers coincide, i.e., $\chi = \bar{\chi} = h$. In 2003, Mario raised a subtle and challenging conjecture [13] that this equality holds whenever $\bar{\chi} = h \geq 2$. Until today it remains open, intriguing and motivating for further research. Some of the most recent results in this direction discuss extensions of bicolorings of $\mathrm{STS}(v)$ to bicolorings of $\mathrm{STS}(2v+1)$ obtained by using the so called doubling plus one construction, see [5].

The problem of colorability of BSTS was also first formulated in Catania in 1993 though no example of uncolorable BSTS was found. The first such example was constructed by Ganter at TU Dresden, and all uncolorable $\mathrm{BSTS}(15)$s have been characterized by Rosa [35]: $\mathrm{BSTS}(15)$ is colorable if and only if it contains $\mathrm{BSTS}(7)$ as a subsystem. Out of the 80 non-isomorphic $\mathrm{BSTS}(15)$s, only 23 meet this criterion and are therefore colorable. The other 57 are uncolorable. It follows that uncolorable $\mathrm{BSTS}(n)$s exist for each admissible $n \geq 15$.

As to BSQS, the situation is much more difficult. Even though the problem to find at least one uncolorable BSQS was formulated first in Catania in 1993 as well, no one such system has been found. The conjecture is that they exist. The best result related to this problem is by Lo Faro, Milazzo and Tripodi [22]: all $\mathrm{BSQS}(n)$ are colorable for all admissible $n \leq 16$. Therefore, the smallest admissible n for which uncolorable $\mathrm{BSTS}(n)$ may exist is $n = 20$.

There is a significant series of important results and publications by the whole Catania group of mathematicians like these [2, 6, 13, 14, 15, 16, 17, 20, 22, 23, 24, 25, 26, 27, 28, 29, 31, 30, 32] just to name a few. Here is the right point to mention that Catania group is very closely related to Messina group of mathematicians who work in the same direction: Giovanni Lo Faro, Enzo Li Marzi, Corinna Marino and Antoinette Tripodi. Because of this connection, as one can see, they also have published many papers generally speaking in mixed hypergraph coloring.

It was October of 1998. After half year stay in USA, I arrived to Catania where I met Zsolt Tuza as usual, again. We were working on some problems when I received an email from Dhruv Mubayi, a graduate student of Doug West, University of Illinois at Urbana-Champaign. In that email, Dhruv communicated that while looking at a completely different problem, he found an example of mixed hypergraph on 16 vertices, 36 $\mathcal{D}$-edges and 144 (!) $\mathcal{C}$-edges with the gap in the chromatic spectrum. It was a shocking discovery, literally a breakthrough! As I recall, when we realized it, we immediately started searching for the smallest example, and it took just one night for Zsolt to come up next morning with the example depicted in Figure 3. It had 6 vertices $\{1,2,3,4,5,6\}$, 2 $\mathcal{D}$-edges $\{1,6\}$ and $\{2,3,4,5\}$) (solid line and curve), 3 $\mathcal{C}$-edges $\{2,3,4\}$, $\{3,4,5\}$ and $\{2,4,5\}$ (dashed curves), and 4 bi-edges $\{1,2,3\}$, $\{1,4,5\}$, $\{6,2,3\}$, and $\{6,4,5\}$ (bold curves). One can easily see that if vertices 2 and 3 are colored with the same color, say A, then this coloring extends in a unique way to vertices 4 and 5 with color B, vertex 1 with color C and vertex 6 with color D. If, on the contrary, vertices 2 and 3 are colored differently, say colors A and B, then this coloring extends to vertex 1 with color A, vertex 4 with color A, and vertices 5, 6 with color B. Actually, there are four distinct extensions of this coloring, all with colors A and B. So, there is no coloring using 3 colors, and the chromatic spectrum of this example is $R(\mathcal{H}) = (0, 4, 0, 1, 0, 0)$. The chromatic polynomial $P(\mathcal{H}, \lambda) = \lambda(\lambda-1)(\lambda^2-5\lambda+10)$.

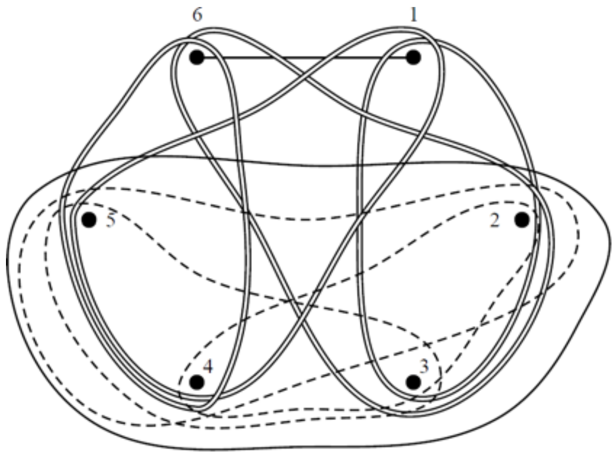


Figure 3: The very first smallest example with gap, Catania, 1998.

The first results have been published in [21]. It was the very beginning of the chase for the gaps in the chromatic spectrum in mixed hypergraphs which with variable success continues until today.

Around the year of 2000, Catania group was reinforced by new researcher, Lucia Gionfriddo. After defending PhD Thesis, she got interested in problems related to mixed hypergraph coloring, namely the gaps in the chromatic spectrum. For an impressively short period of time, Lucia has discovered the first designs, namely P_3-designs with the gaps in the chromatic spectrum. The idea was to consider decompositions of complete graphs into P_3 (a path on 3 vertices) and declare every block as a bi-edge, i.e., colorable with precisely two colors. It is a special case of bi-hypergraph. She proved that there are many such structures with many gaps, in particular, big gaps, even gaps, odd gaps, etc., see [8, 9, 10, 11]. Later other designs, namely P_4-designs with the gaps have been found when considering equicolorings in [1]. Surprisingly, until today, these are the only examples of block designs with the gaps. We do not know anything about continuity of the chromatic spectrum of BSTS or BSQS, for example.

Based on Lucia's results, while my stay in Catania, we were able to carry out some computational experiments and construct the smallest 3-uniform bi-hypergraph with the gap in the chromatic spectrum, see Figure 4: it contains 7 vertices and 9 bi-edges and its chromatic spectrum is $R(\mathcal{H}) = (0, 12, 0, 3, 0, 0, 0)$, see [12]. The chromatic polynomial of this bi-hypergraph $P(\mathcal{H}, \lambda) = 3\lambda(\lambda-1)(\lambda^2-5\lambda+10)$. It is interesting that the chromatic spectra and chromatic polynomials of hypergraphs in Figure 3 and in Figure 4 are related (compare). They were found independently.

There is one more result that is worth to mention. It is about the upper chromatic index of a multi-graph, which represents a type of anti-Vizing theorem. It was first formulated in [34] as Problem 13 and was implied by the duality of mixed hypergraphs. Consider the colorings of the edges of a multi-graph in such a way that every non-pendant vertex is incident to at least two edges of the same color. The maximum number of colors that can be used in such colorings is the upper chromatic index of a multi-graph G, denoted

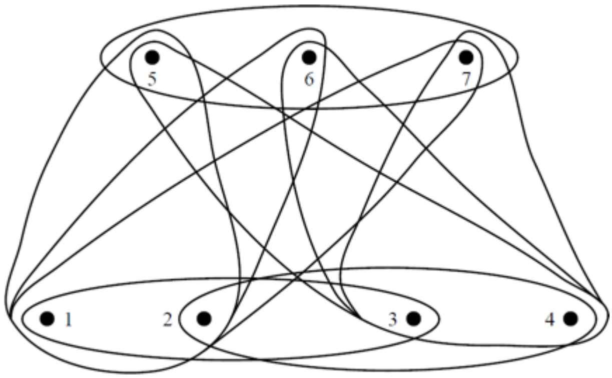


Figure 4: The smallest 3-uniform example with gap, Catania, 2002.

Figure 5: Mixed Hypergraph Coloring encoded in BSTS(7).

by $\bar{\chi}'(G)$. The exact value of it was found in [18]. It was proved that if a multi-graph G has n vertices, m edges, p pendant vertices and maximum number c of disjoint cycles, then $\bar{\chi}'(G) = c + m - n + p$. This result was reported by Lorenzo Milazzo at the Second Lethbridge Workshop on Designs, Codes, Cryptography and Graph Theory, July 9 – 14, 2001.

One of the basic results in applications of mixed hypergraph coloring to block designs, namely, about the cardinality of color classes being powers of 2 in the optimal coloring of BSTS(7), was encoded in the picture of it on the book cover of [19]: one vertex in yellow color, two vertices in red color and four vertices in blue, see Figure 5. In contrast to classic drawing, it is depicted as a hypergraph. Every block is colored with two colors and any other proper coloring has the same distribution of color classes.

3 Conclusion

In conclusion, what I witnessed through decades, was a multilateral activity by Mario Gionfriddo which can be summarized in this way (I do not pretend to be complete):

Mario Gionfriddo:

1. Created an outstanding scientific school of researchers in Graphs, Hypergraphs and Designs with many publications in top journals all over the world.

2. Turned University of Catania and University of Messina into major centers of international collaboration. Proved that he is a great teacher, educator, researcher, organizer, and in general, a great leader in contemporary mathematics.

3. Played and still plays an outstanding role in Italian and especially Sicilian Discrete Mathematics, namely, the Theory of Graphs, Hypergraphs and Designs.

4. Played an outstanding explicit and implicit role in developing the theory of Mixed Hypergraph Coloring. Explicit: by personal participation in research and actually proving many theorems. Implicit: by inviting researchers and organizing seminars, workshops and conferences where actual collaboration occurred.

Dear Mario,

I congratulate you on the occasion of 70th anniversary, thank you for your great role in my life and wish you a good health and further achievements in developing Graphs, Hypergraphs and Designs!

Remark. Dear reader! If it were not that postcard, randomly mailed in 1992 (Figure 1), at this very same moment you would read a very different paper.

Figure 6: With Mario Gionfriddo, Messina, 2003.

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An alternate description of a $(q+1,8)$-cage

Marièn Abreu *
Dipartimento di Matematica Informatica ed Economia, Università degli Studi della Basilicata, Viale dell'Ateneo Lucano 10, I-85100 Potenza, Italy

Gabriela Araujo-Pardo †
Instituto de Matemáticas, Universidad Nacional Autónoma de México, México D. F., México

Camino Balbuena
Departament de Matemática Aplicada III, Universitat Politècnica de Catalunya, Campus Nord, Edifici C2, C/ Jordi Girona 1 i 3 E-08034, Barcelona, Spain

Domenico Labbate ‡
Dipartimento di Matematica Informatica ed Economia, Università degli Studi della Basilicata, Viale dell'Ateneo Lucano 10, I-85100 Potenza, Italy

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Abstract

Let $q \geq 2$ be a prime power. In this note we present an alternate description of the known $(q+1,8)$-cages which has allowed us to construct small (k,g)-graphs for $k = q-1, q$ and $g = 7, 8$ in other papers on this same topic.

Keywords: Cages, girth, Moore graphs, perfect dominating sets.

Math. Subj. Class.: 05C35, 05C69, 05B25

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‡Corresponding author. This research was carried out within the activity of INdAM-GNSAGA and supported by the Italian Ministry MIUR. Research supported by PAPIIT-México under Project IN107218.

E-mail addresses: marien.abreu@unibas.it (Marièn Abreu), garaujo@matem.unam.mx (Gabriela Araujo-Pardo), m.camino.balbuena@upc.edu (Camino Balbuena), domenico.labbate@unibas.it (Domenico Labbate)

1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the book by Bondy and Murty [14] for terminology and notation.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of G is the number $g = g(G)$ of edges in a shortest cycle. For every $v \in V$, $N_G(v)$ denotes the *neighbourhood* of v, i.e. the set of all vertices adjacent to v, and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighbourhood* of v. The *degree* of a vertex $v \in V$ is the cardinality of $N_G(v)$. Let $S \subset V(G)$, then we denote by $N_G(S) = \cup_{s\in S} N_G(s) - S$ and by $N_G[S] = S \cup N_G(S)$.

A graph is called *regular* if all its vertices have the same degree. A (k, g)*-graph* is a k-regular graph with girth g. Erdős and Sachs [15] proved the existence of (k, g)-graphs for all values of k and g provided that $k \geq 2$. Since then most work carried out has focused on constructing a smallest (k, g)-graph (cf. e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 18, 20, 21, 22]). A (k, g)*-cage* is a k-regular graph with girth g having the smallest possible number of vertices. Cages have been intensely studied since they were introduced by Tutte [25] in 1947. More details about constructions of cages can be found in the recent survey by Exoo and Jajcay [17].

In this note we are interested in $(k, 8)$-cages. Counting the number of vertices in the distance partition with respect to an edge yields the following lower bound on the order of a $(k, 8)$-cage:

$$n_0(k, 8) = 2(1 + (k - 1) + (k - 1)^2 + (k - 1)^3). \tag{1.1}$$

A $(k, 8)$-cage with $n_0(k, 8)$ vertices is called a Moore $(k, 8)$*-graph* (cf. [14]). These graphs have been constructed as the incidence graphs of generalized quadrangles $Q(4, q)$ and $W(q)$ [12, 17, 24], which are known to exist for q a prime power and $k = q + 1$ and no example is known when $k - 1$ is not a prime power (cf. [11, 13, 19, 27]). Since they are incidence graphs, these cages are bipartite and have diameter 4. Recall also that if q is even, $Q(4, q)$ is isomorphic to the dual of $W(q)$ and viceversa. Hence, the corresponding $(q + 1, 8)$-cages are isomorphic.

In this note we present an alternate description of the known $(q+1, 8)$-cages with $q \geq 2$ a prime power as follows:

Definition 1.1. Let $\mathbb{F}_q$ be a finite field with $q \geq 2$ a prime power and ϱ be a symbol not belonging to $\mathbb{F}_q$. Let $\Gamma_q = \Gamma_q[W_0, W_1]$ denote a bipartite graph with vertex sets $W_i = \mathbb{F}_q^3 \cup \{(\varrho, b, c)_i, (\varrho, \varrho, c)_i : b, c \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_i\}$, $i = 0, 1$, and edge set defined as follows:

For all $a, b, c \in \mathbb{F}_q$

$$N_{\Gamma_q}((a, b, c)_1) = \{(w,\ aw + b,\ a^2w + 2ab + c)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, a, c)_0\};$$
$$N_{\Gamma_q}((\varrho, b, c)_1) = \{(c, b, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, c)_0\};$$
$$N_{\Gamma_q}((\varrho, \varrho, c)_1) = \{(\varrho, c, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\};$$
$$N_{\Gamma_q}((\varrho, \varrho, \varrho)_1) = \{(\varrho, \varrho, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\}.$$

Or equivalently,

For all $i,j,k \in \mathbb{F}_q$

$$N_{\Gamma_q}((i,j,k)_0) = \{(w,\ j-wi,\ w^2i-2wj+k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,j,i)_1\}$$
$$N_{\Gamma_q}((\varrho,j,k)_0) = \{(j,w,k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,j)_1\}$$
$$N_{\Gamma_q}((\varrho,\varrho,k)_0) = \{(\varrho,w,k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_1\};$$
$$N_{\Gamma_q}((\varrho,\varrho,\varrho)_0) = \{(\varrho,\varrho,w)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_1\}.$$

Note that ϱ is just a symbol not belonging to $\mathbb{F}_q$ and no arithmetical operation will be performed with it.

Theorem 1.2. *The graph* Γ_q *given in Definition 1.1 is a Moore* $(q+1,8)$*-graph for each prime power* $q \geq 2$.

The proof of the above theorem shows that the graph Γ_q described in Definition 1.1 is in fact a labelling for a $(q+1,8)$-cage, for each prime power $q \geq 2$. We need to settle this alternate description because it is used in [2, 3, 4] to construct small (k,g)-graphs for $k = q-1, q$ and $g = 7, 8$.

2 Proof of Theorem 1.2

2.1 Preliminaries: the graphs H_q and B_q

In order to prove Theorem 1.2 we will first define two q-regular bipartite graphs H_q and B_q (cf. Definitions 2.1 and 2.4). The graph H_q was also introduced by Lazebnik, Ustimenko and Woldar [20] with a different formulation.

Definition 2.1. Let $\mathbb{F}_q$ be a finite field with $q \geq 2$. Let $H_q = H_q[U_0, U_1]$ be a bipartite graph with vertex set $U_r = \mathbb{F}_q^3$, $r = 0,1$, and edge set $E(H_q)$ defined as follows:

For all $a,b,c \in \mathbb{F}_q$

$$N_{H_q}((a,b,c)_1) = \{(w,\ aw+b,\ a^2w+c)_0 : w \in \mathbb{F}_q\}.$$

Note that throughout the proofs equalities and operations are intended in $\mathbb{F}_q$.

Lemma 2.2. *Let* H_q *be the graph from Definition 2.1. For any given* $a \in \mathbb{F}_q$*, the vertices in the set* $\{(a,b,c)_1 : b,c \in \mathbb{F}_q\}$ *are mutually at distance at least four. And, for any given* $i \in \mathbb{F}_q$*, the vertices in the set* $\{(i,j,k)_0 : j,k \in \mathbb{F}_q\}$ *are mutually at distance at least four.*

Proof. Suppose that there exists a path of length two between distinct vertices of the form $(a,b,c)_1\ (w,j,k)_0\ (a,b',c')_1$ in H_q. By Definition 2.1, $j = aw+b = aw+b'$ and $k = a^2w+c = a^2w+c'$. Combining the equations we get $b = b'$ and $c = c'$ which implies that $(a,b,c)_1 = (a,b',c')_1$ contradicting the assumption that the path has length two. Similarly suppose that there exists a path of length two $(i,j,k)_0\ (a,b,c)_1\ (i,j',k')_0$. Reasoning as before, we obtain $j = ai+b = j'$, and $k = a^2i+c = k'$ yielding $(i,j,k)_0 = (i,j',k')_0$ which is a contradiction. □

Proposition 2.3. *The graph* H_q *from Definition 2.1 is* q*-regular, bipartite, of girth* 8 *and order* $2q^3$.

Proof. For $q = 2$ it can be checked that H_2 consists of two disjoint cycles of length 8. Thus we assume that $q \geq 3$. Clearly H_q has order $2q^3$ and every vertex of U_1 has degree q. Let $(x, y, z)_0 \in U_0$. By definition of H_q,

$$N_{H_q}((x, y, z)_0) = \left\{(a,\ y - ax,\ z - a^2x)_1 : a \in \mathbb{F}_q\right\}. \tag{2.1}$$

Hence every vertex of U_0 has also degree q and H_q is q-regular. Next, let us prove that H_q has no cycles of length smaller than 8. Otherwise suppose that there exists in H_q a cycle

$$C_{2t+2} = (a_0, b_0, c_0)_1\ (x_0, y_0, z_0)_0\ (a_1, b_1, c_1)_1 \cdots (x_t, y_t, z_t)_0\ (a_0, b_0, c_0)_1$$

of length $2t + 2$ with $t \in \{1, 2\}$. By Lemma 2.2, $a_k \neq a_{k+1}$ and $x_k \neq x_{k+1}$ (subscripts being taken modulo $t + 1$). Then

$$\begin{aligned} y_k &= a_k x_k + b_k = a_{k+1} x_k + b_{k+1}, \quad k = 0, \dots, t, \\ z_k &= a_k^2 x_k + c_k = a_{k+1}^2 x_k + c_{k+1}, \quad k = 0, \dots, t, \end{aligned}$$

subscripts k being taken modulo $t + 1$. Summing all these equalities we get

$$\begin{aligned} \sum_{k=0}^{t-1} (a_k - a_{k+1}) x_k &= (a_0 - a_t) x_t, \quad t = 1, 2; \\ \sum_{k=0}^{t-1} (a_k^2 - a_{k+1}^2) x_k &= (a_0^2 - a_t^2) x_t, \quad t = 1, 2. \end{aligned} \tag{2.2}$$

If $t = 1$, then (2.2) leads to $(a_0 - a_1)(x_1 - x_0) = 0$. System (2.2) gives $x_0 = x_1 = x_2$ which is a contradiction to Lemma 2.2. This means that H_q has no squares so that we may assume that $t = 2$. The coefficient matrix of (2.2) has a Vandermonde determinant, i.e.

$$\begin{vmatrix} a_1 - a_0 & a_0 - a_2 \\ a_1^2 - a_0^2 & a_0^2 - a_2^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_0 & a_2 \\ a_1^2 & a_0^2 & a_2^2 \end{vmatrix} = \prod_{0 \leq k < j \leq 2} (a_j - a_k).$$

This determinant is different from zero because by Lemma 2.2, $a_{k+1} \neq a_k$ (the subscripts being taken modulo 3). Using Cramer's rule to solve it we obtain $x_1 = x_0 = x_2$ which is a contradiction to Lemma 2.2.

Hence, H_q has girth at least 8. Furthermore, when $q \geq 3$ the minimum number of vertices of a q-regular bipartite graph of girth greater than 8 must be greater than $2q^3$. Thus we conclude that the girth of H_q is exactly 8. □

Next, we will make use of the following induced subgraph B_q of Γ_q.

Definition 2.4. Let $B_q = B_q[V_0, V_1]$ be a bipartite graph with vertex set $V_i = \mathbb{F}_q^3$, $i = 0, 1$, and edge set $E(B_q)$ defined as follows:

For all $a, b, c \in \mathbb{F}_q$

$$N_{B_q}((a, b, c)_1) = \{(j,\ aj + b,\ a^2j + 2ab + c)_0 : j \in \mathbb{F}_q\}.$$

Lemma 2.5. *The graph B_q is isomorphic to the graph H_q.*

Proof. Let H_q be the bipartite graph from Definition 2.1. Since the map $\sigma\colon B_q \to H_q$ defined by $\sigma((a,b,c)_1) = (a,b,2ab+c)_1$ and $\sigma((x,y,z)_0) = (x,y,z)_0$ is an isomorphism, the result holds. $\square$

Hence, the graph B_q is also q-regular, bipartite, of girth 8 and order $2q^3$.

In what follows, we will obtain the graph Γ_q from the graph B_q by adding some new vertices and edges. We need a preliminary lemma.

Lemma 2.6. *Let B_q be the graph from Definition 2.4. Then the following hold:*

(i) The vertices in the set $\{(a,b,c)_1 : b,c \in \mathbb{F}_q\}$ are mutually at distance at least four for all $a \in \mathbb{F}_q$.

(ii) The vertices in the set $\{(i,j,k)_0 : j,k \in \mathbb{F}_q\}$ are mutually at distance at least four for all $i \in \mathbb{F}_q$.

(iii) The q vertices of the set $\{(x,y,j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least six for all $x,y \in \mathbb{F}_q$.

Proof. The proof of items (i) and (ii) is almost identical to that of Lemma 2.2.

(iii): By (ii), the vertices in $\{(x,y,j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least four. Suppose by contradiction that B_q contains the following path of length four:

$$(x,y,j)_0\ (a,b,c)_1\ (x',y',j')_0\ (a',b',c')_1\ (x,y,j'')_0, \text{ for some } j'' \neq j.$$

Then $y = ax+b = a'x+b'$ and $y' = ax'+b = a'x'+b'$. It follows that $(a-a')(x-x') = 0$, which is a contradiction since, by the previous statements, $a \neq a'$ and $x \neq x'$. $\square$

2.2 The conclusion

Figure 1 shows a spanning tree of Γ_q with the vertices labelled according to Definition 1.1. Note that the lower level of such a tree corresponds to the set of vertices of B_q.

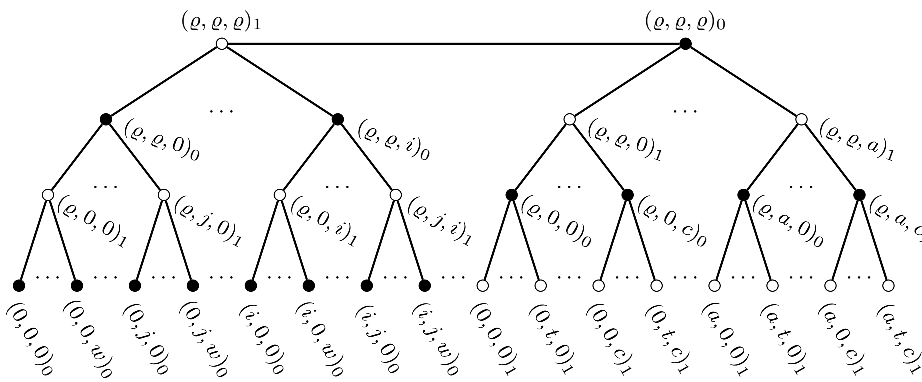

Figure 1: Spanning tree of Γ_q.

We are now ready to prove Theorem 1.2:

Proof of Theorem 1.2. Let $B_q' = B_q'[V_0, V_1']$ be the bipartite graph obtained from $B_q = B_q[V_0, V_1]$ by adding q^2 new vertices to V_1 labeled $(\varrho, b, c)_1$, $b, c \in \mathbb{F}_q$ (i.e., $V_1' = V_1 \cup \{(\varrho, b, c)_1 : b, c \in \mathbb{F}_q\}$), and new edges $N_{B_q'}((\varrho, b, c)_1) = \{(c, b, j)_0 : j \in \mathbb{F}_q\}$ (see Figure 1). Then B_q' has $|V_1'| + |V_0| = 2q^3 + q^2$ vertices, every vertex of V_0 has degree $q+1$, and every vertex of V_1' has still degree q. Note that the girth of B_q' is 8 by Lemma 2.6(iii). The statements from Lemma 2.6 still partially hold in B_q', as stated in the following claim.

Claim 1. *For any given $a \in \mathbb{F}_q \cup \{\varrho\}$, the vertices of the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four in B_q'.*

Proof. For $a = \varrho$, it is clear from Lemma 2.6(i), since the new vertices do not change the distance among the vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$. For $a = \varrho$, the vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four since each vertex of the form $(i, j, k)_0$ has exactly one neighbour in this set, so the result follows from the bipartition of B_q'. □

Claim 2. *For all $a \in \mathbb{F}_q \cup \{\varrho\}$ and for all $c \in \mathbb{F}_q$, the q vertices of the set $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 6 in B_q'.*

Proof. By Claim 1, for all $a \in \mathbb{F}_q \cup \{\varrho\}$ the q vertices of $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 4 in B_q'. Suppose that there exists in B_q' the following path of length four:

$$(a, t, c)_1\ (x, y, z)_0\ (a', t', c')_1\ (x', y', z')_0\ (a, t'', c)_1, \text{ for some } t'' \neq t.$$

If $a = \varrho$, then $x = x' = c$, $y = t$, $y' = t''$ and $a' \neq \varrho$ by Claim 1. Then $y = a'x + t' = a'x' + t' = y'$ yielding that $t = t''$ which is a contradiction. Therefore $a \neq \varrho$. If $a' = \varrho$, then $x = x' = c'$ and $y = y' = t'$. Thus $y = ax + t = ax' + t'' = y'$ yielding that $t = t''$ which is a contradiction. Hence we may assume that $a' \neq \varrho$ and $a \neq a'$ by Claim 1. In this case we have:

$$\begin{aligned} y &= ax + t = a'x + t'; \\ y' &= ax' + t'' = a'x' + t'; \\ z &= a^2x + 2at + c = a'^2x + 2a't' + c'; \\ z' &= a^2x' + 2at'' + c = a'^2x' + 2a't' + c'. \end{aligned}$$

Thus,

$$(a - a')(x - x') = t'' - t; \tag{2.3}$$
$$(a^2 - a'^2)(x - x') = 2a(t'' - t). \tag{2.4}$$

If q is even, (2.4) leads to $x = x'$ and (2.3) leads to $t'' = t$ which is a contradiction with our assumption. Thus assume q is odd. If $a + a' = 0$, then (2.4) gives $2a(t'' - t) = 0$, so that $a = 0$ yielding that $a' = 0$ (because $a + a' = 0$) which is again a contradiction. If $a + a' \neq 0$, multiplying equation (2.3) by $a + a'$ and subtracting both equations we obtain $(2a - (a + a'))(t'' - t) = 0$. Then $a = a'$ because $t'' \neq t$, which is a contradiction to Claim 1. Therefore, Claim 2 holds. □

Let $B_q'' = B_q''[V_0', V_1']$ be the graph obtained from $B_q' = B_q'[V_0, V_1']$ by adding q^2+q new vertices to V_0 labeled $(\varrho, a, c)_0$, $a \in \mathbb{F}_q \cup \{\varrho\}$, $c \in \mathbb{F}_q$, and new edges $N_{B_q''}((\varrho, a, c)_0) = \{(a, t, c)_1 : t \in \mathbb{F}_q\}$ (see Figure 1). Then B_q'' has $|V_1'| + |V_0'| = 2q^3 + 2q^2 + q$ vertices such that every vertex has degree $q+1$ except the new added vertices which have degree q. Moreover the girth of B_q'' is 8 by Claim 2.

Claim 3. *For all $a \in \mathbb{F}_q \cup \{\varrho\}$, the q vertices of the set $\{(\varrho, a, j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least 6 in B_q''.*

Proof. Clearly these q vertices are mutually at distance at least 4 in B_q''. Suppose that there exists in B_q'' the following path of length four:

$$(\varrho, a, j)_0 \ (a, b, j)_1 \ (x, y, z)_0 \ (a, b', j')_1 \ (\varrho, a, j')_0, \text{ for some } j' \neq j.$$

If $a = \varrho$ then $x = j = j'$ which is a contradiction. Therefore $a \neq \varrho$. In this case $y = ax+b = ax+b'$ which implies that $b = b'$. Hence $z = a^2x+2ab+j = a^2x+2ab'+j'$ yielding that $j = j'$ which is again a contradiction. □

Let $B_q''' = B_q'''[V_0', V_1'']$ be the graph obtained from B_q'' by adding $q+1$ new vertices to V_1' labeled $(\varrho, \varrho, a)_1$, $a \in \mathbb{F}_q \cup \{\varrho\}$, and new edges $N_{B_q'''}(\varrho, \varrho, a)_1 = \{(\varrho, a, c)_0 : c \in \mathbb{F}_q\}$, see Figure 1. Then B_q''' has $|V_1''| + |V_0'| = 2q^3 + 2q^2 + 2q + 1$ vertices such that every vertex has degree $q+1$ except the new added vertices which have degree q. Moreover the girth of B_q''' is 8 by Claim 3 and clearly these $q+1$ new vertices are mutually at distance 6. Finally, the graph Γ_q is obtained by adding to B_q''' another new vertex labeled $(\varrho, \varrho, \varrho)_0$ and edges $N_{\Gamma_q}((\varrho, \varrho, \varrho)_0) = \{(\varrho, \varrho, i)_1 : i \in \mathbb{F}_q \cup \{\varrho\}\}$. The graph Γ_q has $2(q^3+q^2+q+1)$ vertices, it is $(q+1)$-regular and has girth 8, so by the uniqueness of a $(q+1,8)$-cage (see e.g. [29]), Γ_q is indeed a $(q+1,8)$ Moore graph. □

Remark 2.7. Coordinatizations of classical generalized quadrangles $Q(4,q)$ and $W(q)$ in four dimensions are discussed in [23, 26, 28]. The alternate description of a Moore $(q+1,8)$-graph given in Theorem 1.2 in three dimensions is equivalent to this coordinatization.

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Perfect blocking sets in P_3-designs

Paola Bonacini , Mario Gionfriddo , Lucia Marino

Department of Mathematics and Computer Science, University of Catania, Catania, Italy

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Abstract

A well-known problem in Design Theory is the study of the possible existence of blocking sets in Steiner systems. In this paper, we introduce the concept of *perfect* blocking sets in G-designs and determine all the possible v for which there exist P_3-designs having perfect blocking sets.

Keywords: 05B05, 05C15

Math. Subj. Class.: Blocking sets, transversals, P_3-designs.

1 Introduction

Let K_v be the complete undirected graph defined in a vertex set X. Given a graph with n vertices, a *G-design* of *order* v (briefly a *$G(v)$-design*), for $v \geq n$, is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge set of K_v into classes generating graphs all isomorphic to G. The classes of $\mathcal{B}$ are said to be the *blocks* of Σ. A K_n-design of order v is a Steiner systems $S(2, n, v)$.

Let $\Sigma = (X, \mathcal{B})$ be a G-design of order v. Following *hypergraph* terminology, a *transversal* T of Σ is a subset of X intersecting every block of Σ. The *transversal number* of Σ is the minimum cardinality of transversals of Σ. A *blocking set* T of Σ is a transversal such that also its complementary C_T is a transversal of Σ: in other words, T is a blocking sets if and only if every block of Σ contains elements of T and elements of $C_X(T)$. For a blocking set T of Σ, the *discrepancy* is the number $\delta(\Sigma) = \big||T| - |C_X T|\big|$ (see [4, 8]). In what follows, we will indicate by $B(\Sigma)$ the set of all possible $p \in N$ for which there exist in Σ blocking sets of cardinality p. Therefore: $\delta(\Sigma) = |B(\Sigma)|$. Note that *there exist blocking sets* in a system if and only if the system is 2*-vertex-colourable*.

The problem to determine the existence of possible *blocking-sets* in Steiner systems has been studied by many authors [1, 2, 7] , expecially for $S(2, 4, v)$ [3, 9, 10, 11, 12], and for G-designs [6]. Interesting results can be found also in [5].

E-mail addresses: bonacini@dmi.unict.it (Paola Bonacini), gionfriddo@dmi.unict.it (Mario Gionfriddo), lmarino@dmi.unict.it (Lucia Marino)

In this paper, we introduce the concept of *perfect blocking-set* of a G-design and determine all possible v for which there exist P_3-designs having perfect blocking sets.

In what follows, b will always indicate the number of blocks in a G-design. Observe that, in the case of systems with $b = 1$ (Steiner systems $S(h, k, v)$ with $v = k$), the research of blocking sets is trivial. Therefore, in what follows, we will consider always systems with $b > 1$. It is known that:

Theorem 1.1. *A P_3-design of order v exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \geq 4$.*

Observe that if a path P_3 has vertices x, y, z and edges $\{x, y\}, \{y, z\}$, we will denote it by $[x, y, z]$.

2 Transversals and blocking sets in G-designs

The following results were proved in [4, 8]:

Theorem 2.1. *If $\Sigma = (X, \mathcal{B})$ is a G-design of order v and T is a blocking set of cardinality p of Σ such that $p \leq \frac{v-1}{r}$, then:*

$$\binom{p}{2} + p \cdot \left[\frac{v-1}{r} - (p-1)\right] \geq |\mathcal{B}|.$$

Theorem 2.2. *If $\Sigma = (X, \mathcal{B})$ is a P_3-design of order v and T is a transversal of cardinality p of Σ, then:*

$$\binom{p}{2} + p \cdot (v - p) \geq v(v-1)/4.$$

where the inequality is the best possible.

Proof. To see that the inequality is the best possible, consider the system $\Sigma = (X, \mathcal{B})$, defined in $X = \{1, 2, \ldots, 8\}$, and having for blocks:

$$\begin{aligned} \mathcal{B}\colon \quad & [1,2,3],\ [1,3,4],\ [1,4,2],\ [5,6,7],\ [5,7,8],\ [5,8,6], \\ & [1,5,3],\ [2,5,4],\ [1,6,3],\ [2,6,4], \\ & [1,7,3],\ [2,7,4],\ [1,8,3],\ [2,8,4]. \end{aligned}$$

We can see that Σ is a P_3-design of order $v = 8$ and that $T = \{1, 2, 5\}$ is a blocking set of Σ. Further, we can verify that, from Theorem 2.2, the minimum possible value of p for $v = 8$ is $p = 3$. □

Observe that the minimum cardinality of a blocking set depends on the system and not only on its order. The following two systems $\Sigma_1 = (X, \mathcal{B}_1)$ and $\Sigma_2 = (X, \mathcal{B}_2)$, are defined both in $X = \{1, 2, \ldots, 9\}$. Therefore their order is $v = 9$. However, the minimum cardinality of a blocking set in them is different:

$$\begin{aligned} \mathcal{B}_1\colon \quad & [1,2,4],\ [1,3,4],\ [2,3,5], \\ & [1,4,7],\ [1,5,4],\ [1,6,4],\ [1,7,8],\ [1,8,4],\ [1,9,4], \\ & [2,5,8],\ [2,6,5],\ [2,7,5],\ [2,8,9],\ [2,9,5], \\ & [3,6,9],\ [3,7,6],\ [3,8,6],\ [3,9,7]. \end{aligned}$$

$$\begin{aligned} \mathcal{B}_2: \quad & [1,2,3],\ [1,3,4],\ [1,4,2],\ [5,6,7],\ [5,7,8],\ [5,8,6],\\ & [1,5,3],\ [2,5,4],\ [1,6,3],\ [2,6,4],\\ & [1,7,3],\ [2,7,4],\ [1,8,3],\ [2,8,4],\\ & [1,9,7],\ [2,9,6],\ [3,5,9],\ [4,9,8]. \end{aligned}$$

We can see that:

- the minimum possible cardinality in Σ_1 is exactly $p = 3$ and that $T_1 = \{1, 2, 3\}$ is a blocking set of Σ_2;
- the minimum possible cardinality in Σ_2 is $p = 4$ and that $T_2 = \{1, 2, 5, 9\}$ is a blocking set of Σ_1.

Definition 2.3. Let $\Sigma = (X, \mathcal{B})$ be a G-design. We say that a blocking set T of Σ is *perfect* if there exists a constant $C \in N$ such that in every block $B \in \mathcal{B}$ there are exactly C edges having an extreme in T and the other extreme in $C_X T$.

Observe that the blocking set T_1 of the P_3-design Σ_1, defined above, is *perfect*; while the blocking set T_2 of Σ_2 is not *perfect*.

3 Perfect blocking sets in P_3-designs

We see the exact cardinality of any perfect blocking set in in P_3-designs.

Theorem 3.1. *If T is a perfect blocking set of any P_3-design of order v, then*

$$|T| = \frac{v \pm \sqrt{v}}{2},$$

and v must be a square.

Proof. Let $\Sigma = (X, \mathcal{B})$ be a P_3-design of order v and let T be a perfect blocking set of Σ. Observe that, from the definition of *perfect blocking set*, every block of Σ contains exactly one edge having an extreme in T and the other extreme in $C_X T$. Therefore, since $|T| = p$ and $|C_X T| = v - p$, it follows:

$$p(v-p) = \frac{v(v-1)}{4},$$

hence:

$$p = \frac{v \pm \sqrt{v}}{2}.$$

Since p is a positive integer, it follows that v must be a square. □

From Theorem 3.2, if we consider a P_3-design of order v having perfect blocking set, since $v \equiv 0 \text{ or } 1 \pmod 4$, then there is a positive integer k such that $v = (2k)^2$ or $v = (2k+1)^2$.

Theorem 3.2. *Let T be a perfect blocking set of any P_3-design of order v. If $|T| \leq |C_X T|$, then:*

(i) if $v \equiv 0 \pmod 4$, then $v = (2k)^2$ and $|T| = k(2k-1)$, for any $k \in N$;

(ii) if $v \equiv 1 \pmod 4$, then $v = (2k+1)^2$ and $|T| = k(2k+1)$, for any $k \in N$.

Proof. Let $\Sigma = (X, \mathcal{B})$ be a P_3-design of order v and let T be a perfect blocking set of Σ, such that $|T| \leq |C_X T|$.

(i): If $v \equiv 0 \pmod 4$, then there exists $k \in N$ such that $v = (2k)^2$. Further:

$$|T| = \frac{v - \sqrt{v}}{2} = \frac{4k^2 - 2k}{2} = k(2k-1).$$

(ii): If $v \equiv 1 \pmod 4$, then there exists $k \in N$ such that $v = (2k+1)^2$. Further:

$$|T| = \frac{v - \sqrt{v}}{2} = \frac{(4k^2 + 4k + 1) - (2k+1)}{2} = k(2k+1). \qquad \square$$

4 Main results

In this section we determine the spectrum of P_3-designs having perfect blocking sets.

Theorem 4.1. *There exist P_3-designs of order v having perfect blocking sets if and only if v is a square.*

Proof. Let $\Sigma = (X, \mathcal{B})$ be a P_3-design of order v and let T be a perfect blocking set of Σ. From Theorems 3.1 and 3.2, it follows that v must be a square.

Therefore, let v be an *odd* [resp. *even*] square number. This implies that $v = (2k+1)^2$ and $p = |T| = k(2k+1)$ [resp. $v = (2k)^2$ and $p = |T| = k(2k-1)$].

If X is a set of cardinality v, partition X into 3 classes X_1, X_2, X_3, defined as follows:

$$\begin{array}{ll} X_1 = \{a_1, a_2, \ldots, a_{k(2k+1)}\} & \text{[resp.: } X_1 = \{a_1, a_2, \ldots, a_{k(2k-1)}\}]; \\ X_2 = \{b_1, b_2, \ldots, b_{k(2k+1)}\} & \text{[resp.: } X_2 = \{b_1, b_2, \ldots, b_{k(2k-1)}\}]; \\ X_3 = \{c_1, c_2, \ldots, c_{2k+1}\} & \text{[resp.: } X_3 = \{c_1, c_2, \ldots, c_{2k}\}]. \end{array}$$

To simplify, indicate by q the cardinality of X_3, i.e. $q = 2k+1$ [resp. $q = 2k$], and of course $p = k(2k+1)$ [resp. $p = k(2k-1)$].

Define in X the following families of paths P_3:

$$\begin{array}{ll} \mathcal{F}: & [a_1, a_2, b_1],\ [a_1, a_3, b_1],\ \ldots,\ [a_1, a_p, b_1], \\ & [a_2, a_3, b_2],\ [a_2, a_4, b_2],\ \ldots,\ [a_2, a_p, b_2], \\ & \quad \vdots \\ & [a_{p-2}, a_{p-1}, b_{p-2}],\ [a_{p-2}, a_p, b_{p-2}], \\ & [a_{p-1}, a_p, b_{p-1}]; \end{array}$$

$\mathcal{G}_{1,2}$: $[a_1, b_1, c_1],\ [a_2, b_2, c_1],\ \dots,\ [a_{q-1}, b_{q-1}, c_1],$
$[a_q, b_q, c_2],\ [a_{q+1}, b_{q+1}, c_2],\ \dots,\ [a_{2q-3}, b_{2q-3}, c_2],$
(the last index $2q-3$ is because of $2q-3 = (q-1)+(q-2)$),
$[a_{2q-2}, b_{2q-2}, c_3],\ [a_{2q-1}, b_{2q-1}, c_3],\ \dots,\ [a_{3q-6}, b_{3q-6}, c_3],$
(the last index $3q-6$ is because of $2q-3 = (q-1)+(q-2)+(q-3)$),
$\vdots$
$[a_{p-2}, b_{p-2}, c_{q-2}],\ [a_{p-1}, b_{p-1}, c_{q-2}],$
$[a_p, b_p, c_{q-1}];$

$\mathcal{G}_{2,2}$: $[a_1, c_1, c_2],\ [a_2, c_1, c_3],\ [a_3, c_1, c_4],\ \dots,\ [a_{q-1}, c_1, c_q],$
$[a_q, c_2, c_3],\ [a_{q+1}, c_2, c_4],\ [a_{q+2}, c_2, c_5],\ \dots,\ [a_{2q-3}, c_2, c_{2q-3}],$
$\vdots$
$[a_{p-2}, c_{q-2}, c_{q-1}],\ [a_{p-1}, c_{q-2}, c_q],$
$[a_p, c_{q-1}, c_q];$

$\mathcal{H}_1$: $[a_1, b_2, b_1],\ [a_1, b_3, b_1],\ \dots,\ [a_1, b_p, b_1],$
$[a_2, b_3, b_2],\ [a_2, b_4, b_2],\ \dots,\ [a_2, b_p, b_2],$
$[a_3, b_4, b_3],\ [a_3, b_5, b_3],\ \dots,\ [a_3, b_p, b_3],$
$\vdots$
$[a_{q-1}, b_q, b_{q-1}],\ [a_{q-1}, b_{q+1}, b_{q-1}],\ \dots,\ [a_{q-1}, b_p, b_{q-1}],$
$[a_q, b_{q+1}, b_q],\ [a_q, b_{q+2}, b_q],\ \dots,\ [a_q, b_p, b_q],$
$[a_{q+1}, b_{q+2}, b_{q+1}],\ [a_{q+1}, b_{q+3}, b_{q+1}],\ \dots,\ [a_{q+1}, b_p, b_{q+1}],$
$\vdots$
$[a_{2q-3}, b_{2q-2}, b_{2q-3}],\ [a_{2q-3}, b_{2q-1}, b_{2q-3}],\ \dots,\ [a_{2q-3}, b_p, b_{2q-3}],$
$\vdots$
$[a_{p-2}, b_{p-1}, b_{p-2}],\ [a_{p-2}, b_p, b_{p-2}],$
$[a_{p-1}, b_p, b_{p-1}];$

$\mathcal{H}_2$: $[a_1, c_2, b_1],\ [a_1, c_3, b_1],\ \dots,\ [a_1, c_q, b_1],$
$[a_2, c_2, b_2],\ [a_2, c_3, b_2],\ \dots,\ [a_2, c_q, b_2],$
$[a_3, c_2, b_3],\ [a_3, c_3, b_3],\ \dots,\ [a_3, c_q, b_3],$
$\vdots$
$[a_{q-1}, c_2, b_{q-1}],\ [a_{q-1}, c_3, b_{q-1}],\ \dots,\ [a_{q-1}, c_q, b_{q-1}],$
$[a_q, c_1, b_q],\ [a_q, c_3, b_q],\ \dots,\ [a_q, c_q, b_q],$

$$[a_{q+1}, c_1, b_{q+1}],\ [a_{q+1}, c_3, b_{q+1}],\ \ldots,\ [a_{q+1}, c_q, b_{q+1}],$$
$$\vdots$$
$$[a_{2q-3}, c_1, b_{2q-3}],\ [a_{2q-3}, c_3, b_{2q-3}],\ \ldots,\ [a_{2q-3}, c_q, b_{2q-3}],$$
$$\vdots$$
$$[a_{p-2}, c_1, b_{p-2}],\ [a_{p-2}, c_2, b_{p-2}],\ [a_{p-2}, c_3, b_{p-2}],\ \ldots,\ [a_{p-2}, c_{q-3}, b_{p-2}],\ [a_{p-2}, c_{q-1}, b_{p-2}],\ [a_{p-2}, c_q, b_{p-2}],$$
$$[a_{p-1}, c_1, b_{p-1}],\ [a_{p-1}, c_2, b_{p-1}],\ [a_{p-1}, c_3, b_{p-1}],\ \ldots,\ [a_{p-1}, c_{q-3}, b_{p-1}],\ [a_{p-1}, c_{q-1}, b_{p-1}],\ [a_{p-1}, c_q, b_{p-1}],$$
$$[a_p, c_1, b_p],\ [a_p, c_2, b_p],\ [a_p, c_3, b_p],\ \ldots,\ [a_p, c_{q-3}, b_p],\ [a_p, c_{q-2}, b_p],\ [a_p, c_q, b_p].$$

If $X = X_1 \cup X_2 \cup X_3$ and $\mathcal{B} = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{H}_1 \cup \mathcal{H}_2$, then it is possible to verify that $\Sigma = (X, \mathcal{B})$ is a P_3-design of order $v = (2k+1)^2$ [resp. $v = (2k)^2$], for any $k \in N$, and that X_1 having is a *perfect* blocking set of Σ of cardinality $k(2k+1)$ [resp. $v = k(2k-1)$].

Indeed, observe that:

1. the family $\mathcal{F}$ has cardinality $|\mathcal{F}| = \binom{p}{2}$ and its blocks contain exactly an edge having both extremes in X_1; no block of $\mathcal{B} - \mathcal{F}$ contains two elements of X_1; further they contain all the edges $\{a_j, b_i\}$, for every $i = 1, 2, \ldots, p-1$ and $j = i+1, i+2, \ldots, p$;
2. the family $\mathcal{G}_1$ contains all the blocks of type $[a_i, b_i, c_j]$, where:
   - $j = 1$, for $i = 1, 2, \ldots, q-1$;
   - $j = 2$, for $i = q, q+1, \ldots, 2q-3$;

     $\vdots$
   - $j = q-1$, for $i = p = \binom{q}{2} = k(2k+1)$ [resp. $= k(2k-1)$];
3. the family $\mathcal{G}_2$ contains blocks of type $\{a, c', c''\}$, where $\{a, c'\} \in X_1 \times X_3$ and $\{c', c''\} \in X_3 \times X_3$;
4. the family $\mathcal{H}_1$ contains all the blocks of type $[a_i, b_j, b_i]$, for every $i = 1, 2, \ldots, p-1$ and $j = i+1, \ldots, p$;
5. the family $\mathcal{H}_2$ contains all the blocks of type $[a_i, c_j, b_i]$, for every $i = 1, 2, \ldots, p$ and $j = 1, \ldots, q$, with exception for:
   - $j = 1$, for $i = 1, 2, \ldots, q-1$;
   - $j = 2$, for $i = q, q+1, \ldots, 2q-3$;

     $\vdots$
   - $j = q-1$, for $i = p = q$. □

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Parallelism in Steiner systems*

Maria Di Giovanni , Mario Gionfriddo

Department of Mathematics and Computer Science, University of Catania, Catania, Italy

Antoinette Tripodi

Department of Mathematical and Computer Science, Physical Sciences and Earth Sciences, University of Messina, Italy

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Abstract

The authors give a survey about the problem of parallelism in Steiner systems, pointing out some open problems.

Keywords: Steiner system, (partial) parallel class.

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1 Introduction

A *Steiner system* $S(h, k, v)$ is a k-uniform hypergraph $\Sigma = (X, \mathcal{B})$ of order v, such that every subset $Y \subseteq X$ of cardinality h has degree $d(Y) = 1$ [4]. In the language of classical design theory, an $S(h, k, v)$ is a pair $\Sigma = (X, \mathcal{B})$ where X is a finite set of cardinality v, whose elements are called *points* (or *vertices*), and $\mathcal{B}$ is a family of k-subsets $B \subseteq X$, called *blocks*, such that for every subset $Y \subseteq X$ of cardinality h there exists exactly one block $B \in \mathcal{B}$ containing Y.

Using more modern terminology, if K_n^u denotes the complete u-uniform hypergraph of order n, then a Steiner system $S(h, k, v)$ is a K_k^h*-decomposition* of K_v^h, i.e. a pair $\Sigma = (X, \mathcal{B})$, where X is the vertex set of K_v^h and $\mathcal{B}$ is a collection of hypergraphs all isomorphic to K_k^h (blocks) such that every edge of K_v^h belongs to exactly one hypergraph of $\mathcal{B}$. An $S(2, 3, v)$ is usually called *Steiner Triple System* and denoted by STS(v); it is well-kown that an STS(v) exists if and only if $v \equiv 1, 3 \pmod 6$, and contains $v(v-1)/6$ triples. An $S(3, 4, v)$ is usually called *Steiner Quadruple System* and denoted by SQS(v);

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E-mail address: mariadigiovanni1@hotmail.com (Maria Di Giovanni), gionfriddo@dmi.unict.it (Mario Gionfriddo), atripodi@unime.it (Antoinette Tripodi)

it is well-kown that an SQS(v) exists if and only if $v \equiv 2, 4 \pmod 6$, and contains $v(v-1)(v-2)/24$ quadruples.

Given a Steiner system $\Sigma = (X, \mathcal{B})$, two distinct blocks $B', B'' \in \mathcal{B}$ are said to be *parallel* if $B' \cap B'' = \emptyset$. A *partial parallel class* of Σ is a family $\Pi \subseteq \mathcal{B}$ of parallel blocks. If Π is a partition of X, then it is said to be a *parallel class* of Σ. Of course not every Steiner system $S(h, k, v)$ has a parallel class (for example, when v is not a multiple of k) and so it is of considerable interest to determine in general just how large a partial parallel class a Steiner system $S(h, k, v)$ must have.

Open problem 1.1 (Parallelism Problem)**.** Let $2 \leq h < k < v$, determine the maximum integer $\pi(h, k, v)$ such that any $S(h, k, v)$ has at least $\pi(h, k, v)$ distinct parallel blocks.

In this paper we will survey known results on the *parallelism problem* and give some open problems, including Brouwer's conjecture.

2 A result of Lindner and Phelps

In [6] C. C. Lindner and K. T. Phelps proved the following result.

Theorem 2.1. *Any Steiner system $S(k, k+1, v)$, with $v \geq k^4 + 3k^3 + k^2 + 1$, has at least $\lceil \frac{v-k+1}{k+2} \rceil$ parallel blocks.*

Proof. Let $\Sigma = (X, \mathcal{B})$ be a Steiner system $S(k, k+1, v)$, with $v \geq k^4 + 3k^3 + k^2 + 1$. Let Π be a partial parallel class of maximum size, say t, and denote by P the set of vertices belonging to the blocks of Π. Since Π is a partial parallel class of maximum size, every $Y \subseteq X - P$, $|Y| = k$, is contained in one block $B \in \mathcal{B}$ which intersects P in exactly one vertex. Denote by Ω the set of all blocks having k elements in $X - P$ (and so the remaining vertex in P) and by A the set of all vertices belonging to P and to some block of Ω:

$$\Omega = \{B \in \mathcal{B} : |B \cap (X - P)| = k\},$$
$$A = \{x \in P : x \in B, B \in \Omega\}.$$

For every $x \in A$, set

$$T(x) = \{B - \{x\} : B \in \Omega\}.$$

We can see that $\Sigma' = (X - P, T(x))$ is a partial Steiner system of type $S(k-1, k, v - (k+1)t)$, with

$$|T(x)| \leq \frac{\binom{v-(k+1)t}{k-1}}{k},$$

and $\{T(x)\}_{x \in A}$ is a partition of $\mathcal{P}_k(X - P)$, i.e., the set of all k-subsets of $X - P$. Observe that, if B is a block of Π containing at least two vertices of A, then for each $x \in A \cap B$ we must have

$$|T(x)| \leq \frac{k\binom{v-(k+1)t-1}{k-2}}{k-1}.$$

Indeed, otherwise, let y be any other vertex belonging to $A \cap B$ and B_1 be a block of $T(y)$. Since at most $k\binom{v-(k+1)t-1}{k-2}/(k-1)$ of the blocks in $T(x)$ can intersect the block B_1, then $T(x)$ must contain a block B_2 such that $B_1 \cap B_2 = \emptyset$. Hence, the family $\Pi' =$

$(\Pi - \{B\}) \cup \{B_1, B_2\}$ is a partial parallel class of blocks having size $|\Pi'| > |\Pi|$, a contradiction. It follows that, for every block $B \in \Pi$ containing at least two vertices of A,

$$\sum_{x \in A \cap B} |T(x)| \leq \frac{(k+1)k\binom{v-(k+1)t-1}{k-2}}{k-1}.$$

Therefore, if we denote by r the number of blocks of Π containing at most one vertex of A and by s the number of blocks of Π containing at least two vertices of A, then

$$\binom{v-(k+1)t}{k} = \sum_{x \in A} |T(x)| \leq \left\lfloor \frac{(k+1)k\binom{v-(k+1)t-1}{k-2}}{k-1} \right\rfloor r + \left\lfloor \frac{\binom{v-(k+1)t}{k-1}}{k} \right\rfloor s.$$

Now, consider the following two cases:

Case 1. $\left\lfloor (k+1)k\binom{v-(k+1)t-1}{k-2}/(k-1) \right\rfloor \leq \binom{v-(k+1)t}{k-1}/k.$

It follows

$$\binom{v-(k+1)t}{k} = \sum_{x \in A} |T(x)| \leq \frac{(r+s)\binom{v-(k+1)t}{k-1}}{k} \leq t \left\lfloor \frac{\binom{v-(k+1)t}{k-1}}{k} \right\rfloor,$$

from which $t \geq \frac{v-k+1}{k+2}$.

Case 2. $\left\lfloor (k+1)k\binom{v-(k+1)t-1}{k-2}/(k-1) \right\rfloor > \binom{v-(k+1)t}{k-1}/k.$

In this case, it follows $t \geq \left(v - k^3 - k^2\right)/(k+1)$ and so

$$t \geq \frac{v - k^3 - k^2}{k+1} \geq \frac{v-k+1}{k+2},$$

for $v \geq k^4 + 3k^3 + k^2 + 1$.

Combining Cases 1 and 2 completes the proof of the theorem. □

For Steiner triple and quadruple systems Theorem 2.1 gives the following result.

Corollary 2.2.

(i) Any STS(v), with $v \geq 45$, has at least $\lceil \frac{v-1}{4} \rceil$ parallel blocks.

(ii) Any SQS(v), with $v \geq 172$, has at least $\lceil \frac{v-2}{5} \rceil$ parallel blocks.

Regarding STS(v)s, the cases of $v < 45$ has been studied by C. C. Lindner and K. T. Phelps in [6] and by G. Lo Faro in [7, 8], while for SQS(v)s, the cases of $v < 172$ has been examined by G. Lo Faro in [9]. Collecting together their results gives the following theorem.

Theorem 2.3.

(i) Any STS(v), with $v \geq 9$, has at least $\lceil \frac{v-1}{4} \rceil$ parallel blocks.

(ii) Any SQS(v) has at least $\lceil \frac{v-2}{5} \rceil$ parallel blocks, with the possible exceptions for $v = 20, 28, 34, 38$.

The following result due to D. E. Woolbright [12] improves the inequality of Lindner-Phelps for Steiner triple systems of order $v \geq 139$.

Theorem 2.4. *Any STS(v) has at least $\frac{3v-70}{10}$ parallel blocks.*

For large values of v (greater then $v' \approx 10000$), the above result in turn is improved by the following theorem which is due to A. E. Brouwer [1] and is valid for every admissible $v \geq 127$.

Theorem 2.5. *Any Steiner triple system of sufficiently large order v has at least $\left\lceil \frac{v-5v^{2/3}}{3} \right\rceil$ parallel blocks.*

In 1981 A. E. Brouwer stated the following open problem.

Open problem 2.6 (Brouwer's Conjecture)**.** Any STS(v) has at least $\left\lceil \frac{v-c}{3} \right\rceil$ parallel blocks, for a constant $c \in N$.

By similar arguments as in Theorem 2.1, C. C. Lindner and R. C. Mullin [11] proved a further result for an arbitrary Steiner system $S(h, k, v)$.

Theorem 2.7. *Any Steiner system $S(h, k, v)$, with*

$$v \geq \frac{2k[2k(k-1)^2(k-h) - (h-1)(k-h-1)] + h - 1}{k^2 - kh - h + 1},$$

has at least $\frac{2(v-h+1)}{(k+1)(k-h+1)}$ parallel blocks.

3 A result on parallelism in $S(k, k+1, v)$, for $k \geq 3$

For $k \geq 3$, in [3] (for $k = 3$) and in [2] (for $k > 3$) the author proved the following result.

Theorem 3.1. *Any Steiner system $S(k, k+1, v)$, with $k \geq 3$, has at least $\left\lfloor \frac{v+2}{2k} \right\rfloor$ parallel blocks.*

Proof. Let $\Sigma = (X, \mathcal{B})$ be a Steiner system $S(k, k+1, v)$, with $k \geq 3$, and Π be a family of parallel blocks of Σ such that

$$P = \bigcup_{B \in \Pi} B$$

and

$$|X - P| \geq (k-1)|\Pi| + 2(k-1),$$

which implies $v \geq 2k(|\Pi|+1) - 2$. We will prove that Σ has a family Π' of parallel blocks such that $|\Pi'| > |\Pi|$. This is trivial if there exists a block $B \in \Sigma$ such that $B \subseteq X - P$. Therefore, we suppose that for every block $B \in \mathcal{B}$, $B \nsubseteq X - P$.

Note that, for any $Y \subseteq X - P, |Y| = k-1$, if $R = (X-P) - Y$, then there exists an injection $\varphi\colon R \to P$ defined as follows: for every $x \in R$, $\varphi(x)$ is the element of P such that $Y \cup \{x, \varphi(x)\} \in \mathcal{B}$. Now let

$$\{a_{i,1}, a_{i,2}, \ldots, a_{i,k+1}\} \in \Pi, \text{ for } i = 1, 2, \ldots, r,$$

such that

$$\{a_{i,1}, a_{i,2}, \ldots, a_{i,k}\} \subseteq \varphi(R);$$

let

$$\{b^j_{i,1}, b^j_{i,2}, \ldots, b^j_{i,k+1}\} \in \Pi, \text{ for } j = 1, 2, \ldots, k-1 \text{ and } i = 1, 2, \ldots, p_j,$$

such that

$$\{b^j_{i,1}, b^j_{i,2}, \ldots, b^j_{i,j}\} \subseteq \varphi(R) \quad \text{and} \quad \{b^j_{i,j+1}, \ldots, b^j_{i,k+1}\} \cap \varphi(R) = \emptyset;$$

and let

$$\{c_{i,1}, c_{i,2}, \ldots, c_{i,k+1}\} \in \Pi, \text{ for } i = 1, 2, \ldots, h,$$

such that

$$\{c_{i,1}, c_{i,2}, \ldots, c_{i,k+1}\} \cap \varphi(R) = \emptyset.$$

Necessarily,

$$(k+1)r + \sum_{i=1}^{k-1} ip_i \geq |\varphi(R)| = |X - P| - (k-1) \geq (k-1)t + k - 1.$$

Since $t = r + \sum_{i=1}^{k-1} p_i + h$, it follows that

$$(k+1)r + \sum_{i=1}^{k-1} ip_i \geq (k-1)r + (k-1)\sum_{i=1}^{k-1} p_i + (k-1)h + k - 1,$$

and so

$$r \geq \frac{1}{2}\left[\sum_{i=1}^{k-2} p_i(k-1-i) + h(k-1) + (k-1)\right].$$

Let $x_{i,j} \in R$ such that $\varphi(x_{i,j}) = a_{i,j}$ and let $y^j_{i,u} \in R$ such that $\varphi(y^u_{i,j}) = b^j_{i,u}$.

Case 1. Suppose $a_{i,k+1} \notin \varphi(R)$, for each $i = 1, 2, \ldots, r$. It follows that

$$|X - P| - (k-1) = \sum_{i=1}^{k-1} ip_i + kr.$$

Since $|X - P| - (k-1) \geq (k-1)t + (k-1)$ and $t = h + r + \sum_{i=1}^{k-1} p_i$, it follows

$$\sum_{i=1}^{k-1} ip_i + kr \geq (k-1)t + k - 1 = (k-1)h + (k-1)r + (k-1)\sum_{i=1}^{k-1} p_i + k - 1,$$

hence

$$r \geq \sum_{i=1}^{k-2} p_i(k-1-i) + h(k-1) + (k-1).$$

Now, consider the injection $\psi\colon R' \to P$, where $R' = \{x_{i,j} \in R : i \neq 1\}$, such that for all $x_{i,j} \in R', \psi(x_{i,j})$ is the element of $\varphi(R)$ satisfying the condition $\{x_{1,1}, x_{1,2}, \ldots, x_{1,k-1}, x_{i,j}, \psi(x_{i,j})\} \in \mathcal{B}$.

If Γ is the family of the blocks $\{x_{1,1}, x_{1,2}, \ldots, x_{1,k-1}, x_{i,j}, \psi(x_{i,j})\}$ and

$$L = \{c_{i,j} : i = 1, 2, \ldots, h,\ j = 1, 2, \ldots, k+1\} \cup \{b^1_{i,1} : i = 1, 2, \ldots, p_1\} \cup \{a_{1,k}\},$$

then $|\Gamma| = k(r-1)$ and $|L| = (k+1)h + p_1 + 1$, with

$$\begin{aligned}|\Gamma| &= k(r-1) = kr - k \\ &\geq \sum_{i=1}^{k-2} p_i(k-i-1) + hk(k-1) + k^2 - 2k > (k+1)h + p_1 + 1 = |L|,\end{aligned}$$

where we used the following inequalities, which hold for $k \geq 3$,

$$\begin{aligned} r &\geq \sum_{i=1}^{k-2} p_i(k-i-1) + h(k-1) + k - 1, \\ hk(k-1) &> h(k+1), \\ k^2 - k &> 1. \end{aligned}$$

Then, it is possible to find an element $x \in P - L$ such that

$$\{x_{1,1}, x_{1,2}, \ldots, x_{1,k-1}, \psi^{-1}(x), x\} \in \mathcal{B}.$$

Further, there exists at least an element $y \in \varphi(R)$, $y \neq x$, with x and y belonging to the same $B_{x,y} \in \Pi$. If

$$\Pi' = \Pi - \{B_{x,y}\} \cup \{\{x_{1,1}, x_{1,2}, \ldots, x_{1,k-1}, \psi^{-1}(x), x\}, Y \cup \{\varphi^{-1}(y), y\}\},$$

then Π' is a family of parallel blocks of $\mathcal{B}$ with $|\Pi'| > |\Pi|$.

Case 2. Suppose there is at least one element $a_{i,k+1}$ such that $\{a_{i,1}, a_{i,2}, \ldots, a_{i,k+1}\} \subseteq \varphi(R)$. Assume that

$$\{a_{i,1}, a_{i,2}, \ldots, a_{i,k+1}\} \subseteq \varphi(R), \text{ for each } i = 1, 2, \ldots, r'$$

and

$$\{a_{i,1}, a_{i,2}, \ldots, a_{i,k}\} \subseteq \varphi(R), \ a_{i,k+1} \notin \varphi(R), \text{ for each } i = r'+1, \ldots, r.$$

If $r \geq 2$, consider the injection $\mu \colon R'' \to P$, where

$$R'' = \left\{x_{i,j} \in R : (i,j) \neq (1,1), (1,2), \ldots, \left(1, \left\lceil \tfrac{k-1}{2} \right\rceil\right), (2,1), (2,2), \ldots, \left(2, \left\lceil \tfrac{k-1}{2} \right\rceil\right)\right\},$$

such that for every $x_{i,j} \in R''$, $\mu(x_{i,j})$ is the element of $\varphi(R)$ satisfying the condition

$$\left\{x_{1,1}, x_{1,2}, \ldots, x_{1,\left\lceil \frac{k-1}{2} \right\rceil}, x_{2,1}, x_{2,2}, \ldots, x_{2,\left\lfloor \frac{k-1}{2} \right\rfloor}, x_{i,j}, \mu(x_{i,j})\right\} \in \mathcal{B}.$$

If Γ' is the family of blocks

$$\left\{x_{1,1}, x_{1,2}, \ldots, x_{1,\left\lceil \frac{k-1}{2} \right\rceil}, x_{2,1}, x_{2,2}, \ldots, x_{2,\left\lfloor \frac{k-1}{2} \right\rfloor}, x_{i,j}, \mu(x_{i,j})\right\}$$

and

$$L' = \{c_{i,j} : i = 1, 2, \ldots, h, \ j = 1, 2, \ldots, k+1\} \cup \{b^1_{i,1} : i = 1, 2, \ldots, p_1\},$$

it follows that

$$
\begin{aligned}
|\Gamma'| &\geq (k-1)t - \sum_{i=1}^{k-1} ip_i = (k-1)(r + \sum_{i=1}^{k-1} p_i + h) - \sum_{i=1}^{k-1} ip_i \\
&= (k-1)r + (k-1)h + \sum_{i=1}^{k-2}(k-1-i)p_i \\
&\geq \frac{k+1}{2}\sum_{i=1}^{k-2}(k-1-i)p_i + \frac{h(k^2-1)}{2} + \frac{(k-1)^2}{2} \\
&> (k+1)h + p_1 + 1 = |L'| + 1,
\end{aligned}
$$

where we used

$$
\begin{aligned}
&t = r + h + \sum_{i=1}^{k-1} p_i, \\
&r \geq \frac{1}{2}\left[\sum_{i=1}^{k-2} p_i(k-i-1) + h(k-1) + (k-1)\right], \\
&k \geq 3.
\end{aligned}
$$

Therefore, it is possible to find at least two distinct elements x', x'' belonging to two distinct blocks B', B'' of Γ':

$$
\begin{aligned}
B' &= \left\{x_{1,1}, x_{1,2}, \ldots, x_{1,\lceil \frac{k-1}{2}\rceil}, x_{2,1}, x_{2,2}, \ldots, x_{2,\lfloor \frac{k-1}{2}\rfloor}, \mu^{-1}(x'), x'\right\}, \\
B'' &= \left\{x_{1,1}, x_{1,2}, \ldots, x_{1,\lceil \frac{k-1}{2}\rceil}, x_{2,1}, x_{2,2}, \ldots, x_{2,\lfloor \frac{k-1}{2}\rfloor}, \mu^{-1}(x''), x''\right\},
\end{aligned}
$$

such that $x', x'' \in P - L'$. Since $x' \neq x''$, we can suppose that

$$x' \neq a_{2,\lceil \frac{k-1}{2}\rceil}.$$

Therefore, it is possible to find an element $y \in \varphi(R)$, $y \neq x'$, with x' and y belonging to the same block $B_{x,y}$ of Π. It follows that there exists a family Π' of parallel blocks with $|\Pi'| = |\Pi| + 1$.

If $r = 1$, then $r' = r = 1$. Since

$$r \geq \frac{1}{2}\left[\sum_{i=1}^{k-2} p_i(k-i-1) + h(k-1) + (k-1)\right],$$

then

$$\sum_{i=1}^{k-2} p_i(k-i-1) + h(k-1) + (k-1) \leq 2.$$

It follows necessarily: $k = 3$, $h = 0$, $p_1 = 0$. Hence $t = p_2 + 1$, $|X - P| = 2p_2 + 6$, and $v = 6p_2 + 10$.

If $p_2 = 0$, then $v = 10$ and $t = 1$, and it is well-known that the unique STS(10) has two parallel blocks.

If $p_2 \geq 1$, consider the blocks

$$B' = \{x_{1,1}, \varphi^{-1}(b^2_{1,1}), X_{1,2}, x'\},$$
$$B'' = \{x_{1,1}, \varphi^{-1}(b^2_{1,1}), X_{1,3}, x''\},$$

where $x', x'' \in \varphi(R)$. Since $x' \neq x''$, we can assume $x' \neq b^2_{1,2}$ and by applying the same technique as the previous cases we can find a family Π' of parallel blocks with $|\Pi'| = |\Pi| + 1$.

Therefore, it is proved that if $\Sigma = (X, \mathcal{B})$ is any $S(k, k+1, v)$, with $k \geq 3$, and Π is a family of parallel blocks of Σ such that $|\Pi| = t$ and $|X - P| \geq (k-1)t + 2(k-1)$, where $P = \bigcup_{B \in \Pi} B$, then Σ has a partial parallel class Π' of cardinality $|\Pi'| > |\Pi|$. It follows that, if $t = \lfloor \frac{v-2(k-1)}{2k} \rfloor$, then Σ has a partial parallel class of cardinality $t' = t + 1 = \lfloor \frac{v+2}{2k} \rfloor$. □

By applying the same technique used in the previous proof, M. C. Marino and R. S. Rees [10] improved the lower bound stated by Theorem 3.1 to $\left\lfloor \frac{2(v+2)}{3(k+1)} \right\rfloor$.

4 Open problems

(a) *Remove the exceptions of Theorem 2.3.*

It is known that $\pi(3, 4, v) = \lfloor \frac{v}{4} \rfloor$ for $v = 4, 8, 10, 14$. In [5] by means of an exhaustive computer search the authors classified the Steiner quadruple systems of order 16 up to isomorphism; following a private conversation, it turned out that the computer search showed that every SQS(16) has a parallel class and so $\pi(3, 4, 16) = 4$.

(b) *Determine the smallest v such that $\pi(3, 4, v) \neq \lfloor \frac{v}{4} \rfloor$.*

Concerning the parallelism in Steiner systems, an interesting question arises when we consider resolvable systems. A Steiner system $\Sigma = (X, \mathcal{B})$ is said to be *resolvable* provided $\mathcal{B}$ admits a partition $\mathcal{R}$ (*resolution*) into parallel classes. A resolvable Steiner triple system is called *Kirkman Triple System* (KTS, in short). It is well-known that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$ (any resolution contains $(v-1)/2$ parallel classes of size $v/3$).

(c) **Problem of A. Rosa** (1978)**:** *Let $\Sigma = (X, \mathcal{B})$ be any KTS(v) and $\mathcal{R}$ be a resolution of Σ. Determine a lower bound for the size of partial parallel classes of Σ in which no two triples come from the same parallel class of $\mathcal{R}$.*

The problem of A. Rosa can be posed for any resolvable Steiner systems $S(h, k, v)$:

(c') **Problem of A. Rosa:** *Let $\Sigma = (X, \mathcal{B})$ be any Steiner system $S(h, k, v)$ and $\mathcal{R}$ be a resolution of Σ. Determine a lower bound for the size of partial parallel classes of Σ in which no two blocks come from the same parallel class of $\mathcal{R}$.*

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Open problems in the spectral theory of signed graphs*

Francesco Belardo †

Department of Mathematics and Applications, University of Naples Federico II, I-80126 Naples, Italy

Sebastian M. Cioabă

Department of Mathematical Sciences, University of Delaware, USA

Jack Koolen

School of Mathematical Sciences, University of Science and Technology of China, Wen-Tsun Wu Key Laboratory of the Chinese Academy of Sciences, Anhui, 230026, China

Jianfeng Wang

School of Mathematics and Statistics, Shandong University of Technology, Zibo, China

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Abstract

Signed graphs are graphs whose edges get a sign $+1$ or -1 (the signature). Signed graphs can be studied by means of graph matrices extended to signed graphs in a natural way. Recently, the spectra of signed graphs have attracted much attention from graph spectra specialists. One motivation is that the spectral theory of signed graphs elegantly generalizes the spectral theories of unsigned graphs. On the other hand, unsigned graphs do not disappear completely, since their role can be taken by the special case of balanced signed graphs.

Therefore, spectral problems defined and studied for unsigned graphs can be considered in terms of signed graphs, and sometimes such generalization shows nice properties which cannot be appreciated in terms of (unsigned) graphs. Here, we survey some general results

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†Corresponding author.

on the adjacency spectra of signed graphs, and we consider some spectral problems which are inspired from the spectral theory of (unsigned) graphs.

Keywords: Signed graph, adjacency matrix, eigenvalue, unbalanced graph.

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1 Introduction

A *signed graph* $\Gamma = (G, \sigma)$ is a graph $G = (V, E)$, with vertex set V and edge set E, together with a function $\sigma\colon E \to \{+1, -1\}$ assigning a positive or negative sign to each edge. The (unsigned) graph G is said to be the underlying graph of Γ, while the function σ is called the signature of Γ. Edge signs are usually interpreted as ± 1. In this way, the adjacency matrix $A(\Gamma)$ of Γ is naturally defined following that of unsigned graphs, that is by putting $+1$ or -1 whenever the corresponding edge is either positive or negative, respectively. One could think about signed graphs as weighted graphs with edges of weights in $\{0, 1, -1\}$, however the two theories are very different. In fact, in signed graphs the product of signs has a prominent role, while in weighted graphs it is the sum of weights that is relevant. A walk is positive or negative if the product of corresponding weights is positive or negative, respectively. Since cycles are special kinds of walks, this definition applies to them as well and we have the notions of positive and negative cycles.

Many familiar notions related to unsigned graphs directly extend to signed graphs. For example, the degree d_v of a vertex v in Γ is simply its degree in G. A vertex of degree one is said to be a pendant vertex. The diameter of $\Gamma = (G, \sigma)$ is the diameter of its underlying graph G, namely, the maximum distance between any two vertices in G. Some other definitions depend on the signature, for example, the positive (resp., negative) degree of a vertex is the number of positive (negative) edges incident to the vertex, or the already mentioned sign of a walk or cycle. A signed graph is *balanced* if all its cycles are positive, otherwise it is *unbalanced*. Unsigned graphs are treated as (balanced) signed graphs where all edges get a positive sign, that is, the *all-positive signature*.

An important feature of signed graphs is the concept of *switching* the signature. Given a signed graph $\Gamma = (G, \sigma)$ and a subset $U \subseteq V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signs of the edges in the cut $[U, V(G)\backslash U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G)\backslash U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be (switching) equivalent to Γ and σ_{Γ^U} to σ_Γ, and we write $\Gamma^U \sim \Gamma$ or $\sigma_{\Gamma^U} \sim \sigma_\Gamma$. It is not difficult to see that each cycle in Γ maintains its sign after a switching. Hence, Γ^U and Γ have the same positive and negative cycles. Therefore, the signature is determined up to equivalence by the set of positive cycles (see [82]). Signatures equivalent to the all-positive one (the edges get just the positive sign) lead to balanced signed graphs: all cycles are positive. By $\sigma \sim +$ we mean that the signature σ is equivalent to the all-positive signature, and the corresponding signed graph is equivalent to its underlying graph. Hence, all signed trees on the same underlying graph are switching equivalent to the all-positive signature. In fact, signs are only relevant in cycles, while the edge signs of bridges are irrelevant.

Note that (unsigned) graph invariants are preserved under switching, but also by vertex

E-mail addresses: fbelardo@unina.it (Francesco Belardo), cioaba@udel.edu (Sebastian M. Cioabă), koolen@ustc.edu.cn (Jack Koolen), jfwang@sdut.edu.cn (Jianfeng Wang)

permutation, so we can consider the isomorphism class of the underlying graph. If we combine switching equivalence and vertex permutation, we have the more general concept of switching isomorphism of signed graphs. For any not given notation and basic results in the theory of signed graphs, the reader is referred to Zaslavsky [82] (see also the dynamic surveys [80, 81]).

We next consider matrices associated to signed graphs. For a signed graph $\Gamma = (G, \sigma)$ and a graph matrix $M = M(\Gamma)$, the M-polynomial is $\phi_M(\Gamma, x) = \det(xI - M(\Gamma))$. The spectrum of M is called the M-spectrum of the signed graph Γ. Usually, M is the adjacency matrix $A(\Gamma)$ or the Laplacian matrix $L(\Gamma) = D(G) - A(\Gamma)$, but in the literature one can find their normalized variants or other matrices. In the remainder, we shall mostly restrict to M being the adjacency matrix $A(\Gamma)$. The adjacency matrix $A(\Gamma) = (a_{ij})$ is the symmetric $\{0, +1, -1\}$-matrix such that $a_{ij} = \sigma(ij)$ whenever the vertices i and j are adjacent, and $a_{ij} = 0$ otherwise. As with unsigned graphs, the Laplacian matrix is defined as $L(\Gamma) = D(G) - A(\Gamma)$, where $D(G)$ is the diagonal matrix of vertices degrees (of the underlying graph G). In the sequel we will mostly restrict to the adjacency matrix.

Switching has a matrix counterpart. In fact, let Γ and Γ^U be two switching equivalent graphs. Consider the matrix $S_U = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ such that

$$s_i = \begin{cases} +1, & i \in U; \\ -1, & i \in \Gamma \setminus U. \end{cases}$$

The matrix S_U is the *switching matrix*. It is easy to check that

$$A(\Gamma^U) = S_U\, A(\Gamma)\, S_U \qquad \text{and} \qquad L(\Gamma^U) = S_U\, L(\Gamma)\, S_U.$$

Hence, signed graphs from the same switching class share similar graph matrices by means of signature matrices (signature similarity). If we also allow permutation of vertices, we have signed permutation matrices, and we can speak of (switching) isomorphic signed graphs. Switching isomorphic signed graphs are cospectral, and their matrices are signed-permutationally similar. From the eigenspace viewpoint, the eigenvector components are also switched in signs and permuted. Evidently, for each eigenvector, there exists a suitable switching such that all components become nonnegative.

In the sequel, let $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$ denote the eigenvalues of the adjacency matrix $A(\Gamma)$ of the signed graph Γ of order n; they are all real since $A(\Gamma)$ is a real symmetric matrix. The largest eigenvalue $\lambda_1(\Gamma)$ is sometimes called the *index* of Γ. If Γ contains at least one edge, then $\lambda_1(\Gamma) > 0 > \lambda_n(\Gamma)$ since the sum of the eigenvalues is 0. Note that in general, the index $\lambda_1(\Gamma)$ does not equal the spectral radius $\rho(\Gamma) = \max\{|\lambda_i| : 1 \leq i \leq n\} = \max\{\lambda_1, -\lambda_n\}$ because the Perron-Frobenius Theorem is valid only for the all-positive signature (and those equivalent to it). For example, an all -1 signing (all-negative signature) of the complete graph on $n \geq 3$ vertices will have eigenvalues $\lambda_1 = \cdots = \lambda_{n-1} = 1$ and $\lambda_n = -(n-1)$.

We would like to end this introduction by mentioning what may be the first paper on signed graph spectra [82]. In that paper, Zaslavsky showed that 0 appears as an L-eigenvalue in connected signed graphs if and only if the signature is equivalent to the all-positive one, that is, Γ is a balanced signed graph.

For notation not given here and basic results on graph spectra, the reader is referred to [23, 22], for some basic results on the spectra of signed graphs, to [83], and for some applications of spectra of signed graphs, to [27].

In Section 2, we survey some important results on graph spectra which are valid in terms of the spectra of signed graphs. In Section 3 we collect some open problems and conjectures which are open at the writing of this note.

2 What do we lose with signed edges?

From the matrix viewpoint, when we deal with signed graphs we have symmetric $\{0, 1, -1\}$-matrices instead of just symmetric $\{0, 1\}$-matrices. Clearly, the results coming from the theory of nonnegative matrices can not be applied directly to signed graphs. Perhaps the most important result that no longer holds for adjacency matrices of signed graphs is the Perron-Frobenius theorem. We saw one instance in the introduction and we will see some other consequences of the absence of Perron-Frobenius in the next section. Also, the loss of non-negativity has other consequences related to counting walks and the diameter of the graph (Theorem 3.10). On the other hand, all results based on the symmetry of the matrix will be still valid in the context of signed graphs with suitable modifications. In this section, we briefly describe how some well-known results are (possibly) changed when dealing with matrices of signed graphs.

We start with the famous *Coefficient Theorem*, also known as *Sachs Formula*. This formula, perhaps better than others, describes the connection between the eigenvalues and the combinatorial structure of the signed graph. It was given for unsigned graphs in the 1960s independently by several researchers (with different notation), but possibly first stated by Sachs (cf. [23, Theorem 1.2] and the subsequent remark). The signed-graph variant can be easily given as follows. An elementary figure is the graph K_2 or C_n ($n \geq 3$). A basic figure (or linear subgraph, or sesquilinear subgraph) is the disjoint union of elementary figures. If B is a basic figure, then denote by $\mathcal{C}(B)$ the class of cycles in B, with $c(B) = |\mathcal{C}(B)|$, and by $p(B)$ the number of components of B and define $\sigma(B) = \prod_{C \in \mathcal{C}(B)} \sigma(C)$. Let $\mathcal{B}_i$ be the set of basic figures on i vertices.

Theorem 2.1 (Coefficient Theorem)**.** *Let Γ be a signed graph and let $\phi(\Gamma, x) = \sum_{i=0}^{n} a_i x^{n-i}$ be its adjacency characteristic polynomial. Then, $a_0 = 1$ and, for $i > 0$,*

$$a_i = \sum_{B \in \mathcal{B}_i} (-1)^{p(B)} 2^{c(B)} \sigma(B).$$

Another important connection between the eigenvalues and the combinatorial structure of a signed graph is given by the forthcoming theorem. If we consider unsigned graphs, it is well known that the k-th spectral moment gives the number of closed walks of length k (cf. [22, Theorem 3.1.1]). Zaslavsky [83] observed that a signed variant holds for signed graphs as well, and from his observation we can give the subsequent result.

Theorem 2.2 (Spectral Moments)**.** *Let Γ be a signed graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. If $W_k^{\pm}$ denotes the difference between the number of positive and negative closed walks of length k, then*

$$W_k^{\pm} = \sum_{i=1}^{n} \lambda_i^k.$$

Next, we recall another famous result for the spectra of graphs, that is, the *Cauchy Interlacing Theorem*. Its general form holds for principal submatrices of real symmetric matrices (see [22, Theorem 1.3.11]). It is valid in signed graphs without any modification to the formula. For a signed graph $\Gamma = (G, \sigma)$ and a subset of vertices U, then $\Gamma - U$ is the signed graph obtained from Γ by deleting the vertices in U and the edges incident to them. For $v \in V(G)$, we also write $\Gamma - v$ instead of $\Gamma - \{v\}$. Similar notation applies when deleting subsets of edges.

Theorem 2.3 (Interlacing Theorem for Signed Graphs). *Let $\Gamma = (G, \sigma)$ be a signed graph. For any vertex v of Γ,*

$$\lambda_1(\Gamma) \geq \lambda_1(\Gamma - v) \geq \lambda_2(\Gamma) \geq \lambda_2(\Gamma - v) \geq \cdots \geq \lambda_{n-1}(\Gamma - v) \geq \lambda_n(\Gamma).$$

In the context of subgraphs, there is another famous result which is valid in the theory of signed graphs. In fact, it is possible to give the characteristic polynomial as a linear combination of vertex- or edge-deleted subgraphs. Such formulas are known as *Schwenk's Formulas* (cf. [22, Theorem 2.3.4], see also [5]). As above, $\Gamma - v$ ($\Gamma - e$) stands for the signed graph obtained from Γ in which the vertex v (resp., edge e) is deleted. Also, to make the formulas consistent, we set $\phi(\emptyset, x) = 1$.

Theorem 2.4 (Schwenk's Formulas). *Let Γ be a signed graph and v (resp., $e = uv$) one of its vertices (resp., edges). Then*

$$\phi(\Gamma, x) = x\phi(\Gamma - v, x) - \sum_{u \sim v} \phi(\Gamma - u - v, x) - 2 \sum_{C \in \mathcal{C}_v} \sigma(C)\phi(\Gamma - C, x),$$

$$\phi(\Gamma, x) = \phi(\Gamma - e, x) - \phi(\Gamma - u - v, x) - 2 \sum_{C \in \mathcal{C}_e} \sigma(C)\phi(\Gamma - C, x),$$

where $\mathcal{C}_a$ denotes the set of cycles passing through a.

Finally, a natural question is the following: if we fix the underlying graph, how much can the eigenvalues change when changing the signature? Given a graph with cyclomatic number ξ, then there are at most 2^ξ nonequivalent signatures as for each independent cycle one can assign either a positive or a negative sign. However, among the 2^ξ signatures, some of them might lead to switching isomorphic graphs, as we see later. In general, the eigenvalues coming from each signature cannot exceed in modulus the spectral radius of the underlying graph, as is shown in the last theorem of this section.

Theorem 2.5 (Eigenvalue Spread). *For a signed graph $\Gamma = (G, \sigma)$, let $\rho(\Gamma)$ be its spectral radius. Then $\rho(\Gamma) \leq \rho(G)$.*

Proof. Clearly, $\rho(\Gamma)$ equals $\lambda_1(\Gamma)$ or $-\lambda_n(\Gamma)$. Let A be the adjacency matrix of $(G, +)$, and A_σ be the adjacency matrix of $\Gamma = (G, \sigma)$. For a vector $X = (x_1, \ldots, x_n)^T$, let $|X| = (|x_1|, \ldots, |x_n|)^T$.

If X is a unit eigenvector corresponding to $\lambda_1(A_\sigma)$, by the Rayleigh quotient we get

$$\lambda_1(G, \sigma) = X^T A_\sigma X \leq |X|^T A |X| \leq \max_{z: z^T z = 1} z^T A z = \lambda_1(G, +).$$

Similarly, if X is a unit eigenvector corresponding to the least eigenvalue $\lambda_n(A_\sigma)$, by the Rayleigh quotient we get

$$\lambda_n(G, \sigma) = X^T A_\sigma X \geq |X|^T (-A) |X| \geq \min_{z: z^T z = 1} z^T (-A) z = \lambda_n(G, -) = -\lambda_1(G, +).$$

By gluing together the two inequalities, we get the assertion. □

It is evident from the preceding results that the spectral theory of signed graphs well encapsulates and extends the spectral theory of unsigned graphs. Perhaps, we can say that adding signs to the edges just gives more variety to the spectral theory of graphs. This fact was already observed with the Laplacian of signed graphs, which nicely generalizes the results coming from the Laplacian and signless Laplacian theories of unsigned graphs. It is worth mentioning that thanks to the spectral theory it was possible to give matrix-wise definitions of the signed graph products [29], line graphs [6, 83] and subdivision graphs [6].

3 Some open problems and conjectures

In this section we consider some open problems and conjectures which are inspired from the corresponding results in the spectral theory of unsigned graphs. We begin with the intriguing concept of "sign-symmetric graph" which is a natural signed generalization of the concept of bipartite graph.

3.1 Symmetric spectrum and sign-symmetric graphs

One of the most celebrated results in the adjacency spectral theory of (unsigned) graphs is the following.

Theorem 3.1.

1. *A graph is bipartite if and only if its adjacency spectrum is symmetric with respect to the origin.*
2. *A connected graph is bipartite if and only if its smallest eigenvalue equals the negative of its spectral radius.*

For the first part, one does not need Perron-Frobenius theorem. To the best of our knowledge, Perron-Frobenius is crucial for the second part (see [10, Section 3.4] or [33, Section 8.8] or [76, Chapter 31]).

On the other hand, in the larger context of signed graphs the symmetry of the spectrum is not a privilege of bipartite and balanced graphs. A signed graph $\Gamma = (G, \sigma)$ is said to be *sign-symmetric* if Γ is switching isomorphic to its negation, that is, $-\Gamma = (G, -\sigma)$. It is not difficult to observe that the signature-reversal changes the sign of odd cycles but leaves unaffected the sign of even cycles. Since bipartite (unsigned) graphs are odd-cycle free, it happens that bipartite graphs are a special case of sign-symmetric signed graphs, or better to say, if a signed graph $\Gamma = (G, \sigma)$ has a bipartite underlying graph G, then Γ and $-\Gamma$ are switching equivalent. In Figure 1 we depict an example of a sign-symmetric graph. Here and in the remaining pictures as well negative edges are represented by heavy lines and positive edges by thin lines.

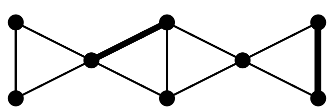

Figure 1: A sign-symmetric signed graph.

If Γ is switching isomorphic to $-\Gamma$, then A and $-A$ are similar and we immediately get:

Theorem 3.2. *Let Γ be a sign-symmetric graph. Then its adjacency spectrum is symmetric with respect to the origin.*

The converse of the above theorem is not true, and counterexamples arise from the theory of Seidel matrices. The Seidel matrix of a (simple and unsigned) graph G is $S(G) = J - I - 2A$, so that adjacent vertices get the value -1 and non-adjacent vertices the value $+1$. Hence, the Seidel matrix of an unsigned graph can be interpreted as the adjacency matrix of a signed complete graph. The signature similarity becomes the famous Seidel switching. The graph in Figure 2 belongs to a triplet of simple graphs on 8 vertices sharing the same symmetric Seidel spectrum but not being pairwise (Seidel-)switching isomorphic. In [25, p. 253], they are denoted as A_1, its complement $\bar{A}_1$ and A_2 (note, A_2 and its complement $\bar{A}_2$ are Seidel switching isomorphic). In fact, A_1 and its complement $\bar{A}_1$ are cospectral but not Seidel switching isomorphic. In terms of signed graphs, the signed graph A_1' whose adjacency matrix is $S(A_1)$ has symmetric spectrum but it is not sign-symmetric.

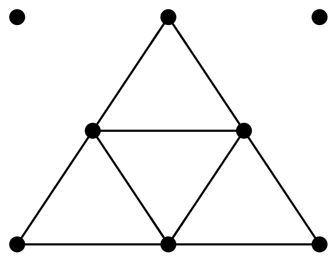

Figure 2: The graph A_1.

Note that the disjoint union of sign-symmetric graphs is again sign-symmetric. Since the above counterexamples involve Seidel matrices which are the same as signed complete graphs, the following is a natural question.

Problem 3.3. Are there non-complete connected signed graphs whose spectrum is symmetric with respect to the origin but they are not sign-symmetric?

Observe that signed graphs with symmetric spectrum have odd-indexed coefficients of the characteristic polynomial equal to zero and all spectral moments of odd order are also zero. A simple application of Theorems 2.1 and 2.2 for $i = 3$ or $k = 3$, respectively, leads to equal numbers of positive and negative triangles in the graph. When we consider $i = 5$ or $k = 5$, we cannot say that the numbers of positive and negative pentagons are the same. The following corollary is an obvious consequence of the latter discussion (cf. also [25, Theorem 1]).

Corollary 3.4. *A signed graph containing an odd number of triangles cannot be sign-symmetric.*

Remark 3.5. As we mentioned in Section 2, a signed graph with cyclomatic number ξ has exactly 2^{ξ} not equivalent signatures (see also [55]). On the other hand, the symmetries, if any, in the structure of the underlying graph can make several of those signatures lead to isomorphic signed graphs.

3.2 Signed graphs with few eigenvalues

There is a well-known relation between the diameter and the number of distinct eigenvalues of an unsigned graph (cf. [22, Theorem 3.3.5]). In fact, the number of distinct eigenvalues cannot be less than the diameter plus 1. With signed graphs, the usual proof based on the minimal polynomial does not hold anymore. Indeed, the result is not true with signed graphs. As we can see later, it is possible to build signed graphs of any diameter having exactly two distinct eigenvalues.

For unsigned graphs, the identification of graphs with a small number of eigenvalues is a well-known problem. The unique connected graph having just two distinct eigenvalues is the complete graph K_n. If a graph is connected and regular, then it has three distinct eigenvalues if and only if it is strongly regular (see [22, Theorem 3.6.4]). At the 1995 British Combinatorial Conference, Haemers posed the problem of finding connected graphs with three eigenvalues which are neither strongly regular nor complete bipartite. Answering Haemers' question, van Dam [71, 72] and Muzychuk and Klin [58] described some constructions of such graphs. Other constructions were found by De Caen, van Dam and Spence [24] who also noticed that the first infinite family nonregular graphs with three eigenvalues already appeared in the work of Bridges and Mena [9]. The literature on this topic contains many interesting results and open problems. For example, the answer to the following intriguing problem posed by De Caen (see [73, Problem 9]) is still unknown.

Problem 3.6. Does a graph with three distinct eigenvalues have at most three distinct degrees?

Recent progress was made recently by van Dam, Koolen and Jia [70] who constructed connected graphs with four or five distinct eigenvalues and arbitrarily many distinct degrees. These authors posed the following *bipartite* version of De Caen's problem above.

Problem 3.7. Are there connected bipartite graphs with four distinct eigenvalues and more than four distinct valencies?

For signed graphs there are also some results. In 2007, McKee and Smyth [57] considered symmetric integral matrices whose spectral radius does not exceed 2. In their nice paper, they characterized all such matrices and they further gave a combinatorial interpretation in terms of signed graphs. They defined a signed graph to be *cyclotomic* if its spectrum is in the interval $[2, -2]$. The maximal cyclotomic signed graphs have exactly two distinct eigenvalues. The graphs appearing in the following theorem are depicted in Figure 3.

Theorem 3.8. *Every maximal connected cyclotomic signed graph is switching equivalent to one of the following:*

- *For some* $k = 3, 4, \ldots$, *the* $2k$*-vertex toroidal tessellation* T_{2k}.
- *The* 14*-vertex signed graph* S_{14}.
- *The* 16*-vertex signed hypercube* S_{16}.

Further, every connected cyclotomic signed graph is contained in a maximal one.

It is not difficult to check that all maximal cyclotomic graphs are sign-symmetric. Note that for k even T_{2k} has a bipartite underlying graph, while for k odd T_{2k} has not bipartite underlying graph but it is sign-symmetric, as well. The characteristic polynomial to T_{2k} is $(x-2)^k(x+2)^k$, so T_{2k} is an example of a signed graph with two distinct eigenvalues and diameter $\lfloor \frac{k}{2} \rfloor$.

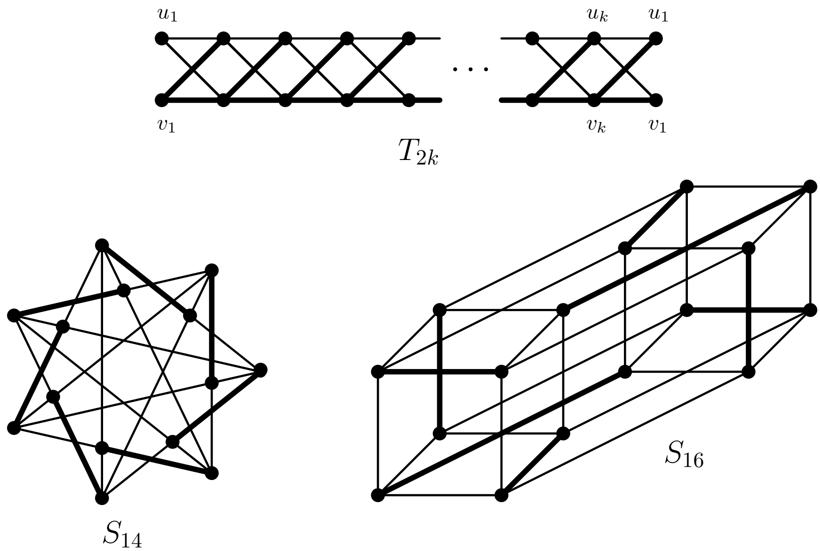

Figure 3: Maximal cyclotomic signed graphs.

Problem 3.9 (Signed graphs with exactly 2 distinct eigenvalues)**.** Characterize all connected signed graphs whose spectrum consists of two distinct eigenvalues.

In the above category we find the complete graphs with homogeneous signatures $(K_n, +)$ and $(K_n, -)$, the maximal cyclotomic signed graphs T_{2k}, S_{14} and S_{16}, and that list is not complete (for example, the unbalanced 4-cycle C_4^- and the 3-dimensional cube whose cycles are all negative must be included). There is already some literature on this problem, and we refer the readers to see [30, 62]. All such graphs have in common the property that positive and negative walks of length greater than or equal to 2 between two different and non-adjacent vertices are equal in number. In this way we can consider a *signed* variant of the diameter. In a connected signed graph, two vertices are at signed distance k if they are at distance k and the difference between the numbers of positive and negative walks of length k among them is nonzero, otherwise the signed distance is set to 0. The signed diameter of Γ, denoted by $\mathrm{diam}^{\pm}(\Gamma)$, is the largest signed distance in Γ. Recall that the (i, j)-entry of A^k equals the difference between the numbers of positive and negative walks of length k among the vertices indexed by i and j. Then we have the following result (cf. [22, Theorem 3.3.5]):

Theorem 3.10. *Let Γ be a connected signed graph with m distinct eigenvalues. Then* $\mathrm{diam}^{\pm}(\Gamma) \leq m - 1$.

Proof. Assume the contrary, so that Γ has vertices, say s and t, at signed distance $p \geq m$. The adjacency matrix A of Γ has minimal polynomial of degree m, and so we may write $A^p = \sum_{k=0}^{m-1} a_k A^k$. This yields the required contradiction because the (s, t)-entry on the right is zero, while the (s, t)-entry on the left is non-zero. □

Recently, Huang [47] constructed a signed adjacency matrix of the n-dimensional hypercube whose eigenvalues are $\pm\sqrt{n}$, each with multiplicity 2^{n-1}. Using eigenvalue interlacing, Huang proceeds to show that the spectral radius (and therefore, the maximum degree) of any induced subgraph on $2^{n-1}+1$ vertices of the n-dimensional hypercube, is at least $\sqrt{n}$. This led Huang to a breakthrough proof of the Sensitivity Conjecture from theoretical computer science. We will return to Huang's construction after Theorem 3.23.

3.3 The largest eigenvalue of signed graphs

In the adjacency spectral theory of unsigned graphs the spectral radius is the largest eigenvalue and it has a prominent role because of its algebraic features, its connections to combinatorial parameters such as the chromatic number, the independence number or the clique number and for its relevance in applications. There is a large literature on this subject, see [11, 20, 43, 45, 60, 68, 78] for example.

As already observed, the presence of negative edges leads invalidates of the Perron-Frobenius theorem, and we lose some nice features of the largest eigenvalue:

- The largest eigenvalue may not be the spectral radius although by possibly changing the signature to its negative, this can be achieved.
- The largest eigenvalue may not be a simple eigenvalue.
- Adding edges might reduce the largest eigenvalue.

Therefore one might say that it not relevant to study signed graphs in terms of the magnitude of the spectral radius. In this respect, Theorem 2.3 and Theorem 2.5 are helpful because the spectral radius does not decrease under the addition of vertices (together with some incident edges), and the spectral radius of the underlying graph naturally limits the magnitude of the eigenvalues of the corresponding signed graph. For the same reason, the theory of limit points for the spectral radii of graph sequences studied by Hoffman in [43, 45] is still valid in the context of signed graphs.

The *Hoffman program* is the identification of connected graphs whose spectral radii do not exceed some special limit points established by A. J. Hoffman [45]. The smallest limit point for the spectral radius is 2 (the limit point of the paths of increasing order), so the first step would be to identify all connected signed graphs whose spectral radius does not exceed 2. The careful reader notices that the latter question has already been completely solved by Theorem 3.8. Therefore, the problem jumps to the next significant limit point, which is $\sqrt{2+\sqrt{5}} = \tau^{\frac{1}{2}} + \tau^{-\frac{1}{2}}$, where τ is the golden mean. This limit point is approached from above (resp., below) by the sequence of positive (resp., negative) cycles with exactly one pendant vertex and increasing girth.

In [11, 19], the authors identified all connected unsigned graphs whose spectral radius does not exceed $\sqrt{2+\sqrt{5}}$. Their structure is fairly simple: they mostly consist of paths with one or two additional pendant vertices. Regarding signed graphs, we expect that the family is quite a bit larger than that of unsigned graphs. A taste of this prediction can be seen by comparing the family of Smith Graphs (the unsigned graphs whose spectral radius is 2, cf. Figure 2.4 in [23]) with the graphs depicted in Figure 3. On the other hand, the graphs identified by Cvetković et al. acts as a "skeleton" (that is, appear as subgraphs) of the signed graphs with the same bound on the spectral radius.

Problem 3.11 (Hoffman Program for Signed Graphs)**.** Characterize all connected signed graphs whose spectral radius does not exceed $\sqrt{2+\sqrt{5}}$.

3.4 The smallest eigenvalue of signed graphs

Unsigned graphs with smallest eigenvalue at least -2 have been characterized in a veritable tour de force by several researchers. We mention here Cameron, Goethals, Seidel and Shult [15], Bussemaker and Neumaier [13] who among other things, determined a complete list of minimal forbidden subgraphs for the class of graphs with smallest eigenvalue at least -2. A monograph devoted to this topic is [21] whose Chapter 1.4 tells the history about the characterization of graphs with smallest eigenvalue at least -2.

Theorem 3.12. *If G is a connected graph with smallest eigenvalue at least -2, then G is a generalized line graph or has at most 36 vertices.*

In the case of unsigned graphs, their work was extended, under some minimum degree condition, from -2 to $-1-\sqrt{2}$ by Hoffman [44] and Woo and Neumaier [79] and more recently, to -3 by Koolen, Yang and Yang [51].

For signed graphs, some of the above results were extended by Vijayakumar [77] who showed that any connected signed graph with smallest eigenvalue less than -2 has an induced signed subgraph with at most 10 vertices and smallest eigenvalue less than -2. Chawathe and Vijayakumar [17] determined all minimal forbidden signed graphs for the class of signed graphs whose smallest eigenvalue is at least -2. Vijayakumar's result [77, Theorem 4.2] was further extended by Koolen, Yang and Yang [51, Theorem 4.2] to signed matrices whose diagonal entries can be 0 or -1. These authors introduced the notion of s-integrable graphs. For an unsigned graph G with smallest eigenvalue $\lambda_{\min}$ and adjacency matrix A, the matrix $A-\lfloor\lambda_{\min}\rfloor I$ is positive semidefinite. For a natural number s, G is called s-integrable if there exists an integer matrix N such that $s(A-\lfloor\lambda_{\min}\rfloor I)=NN^T$. Note that generalized line graphs are exactly the 1-integrable graphs with smallest eigenvalue at least -2. In a straightforward way, the notion of s-integrabilty can be extended to signed graphs. Now we can extend Theorem 3.12 to the class of signed graphs with essentially the same proof.

Theorem 3.13. *Let Γ be a connected signed graph with smallest eigenvalue at least -2. Then Γ is 2-integrable. Moreover, if Γ has at least 121 vertices, then Γ is 1-integrable.*

As E_8 has 240 vectors of (squared) norm 2, one can take from each pair of such a vector and its negative exactly one to obtain a signed graph on 120 vertices with smallest eigenvalue -2 that is not 1-integrable. Many of these signed graphs are connected.

Koolen, Yang and Yang [51] proved that if a connected unsigned graph has smallest eigenvalue at least -3 and valency large enough, then G is 2-integrable. An interesting direction would be to prove a similar result for signed graphs.

Problem 3.14. Extend [51, Theorem 1.3] to signed graphs.

An interesting related conjecture was posed by Koolen and Yang [52].

Conjecture 3.15. *There exists a constant c such that if G is an unsigned graph with smallest eigenvalue at least -3, then G is c-integrable.*

Koolen, Yang and Yang [51] also introduced (-3)-maximal graphs or maximal graphs with smallest eigenvalue -3. These are connected graphs with smallest eigenvalue at least -3 such any proper connected supergraph has smallest eigenvalue less than -3. Koolen and Munemasa [50] proved that the join between a clique on three vertices and the complement of the McLaughlin graph (see Goethals and Seidel [36] or Inoue [48] for a description) is (-3)-maximal.

Problem 3.16. Construct maximal signed graphs with smallest eigenvalue at least -3.

Woo and Neumaier [79] introduced the notion of Hoffman graphs, which has proved an essential tool in many results involving the smallest eigenvalue of unsigned graphs (see [51]). Perhaps a theory of signed Hoffman graphs is possible as well.

Problem 3.17. Extend the theory of Hoffman graphs to signed graphs.

3.5 Signatures minimizing the spectral radius

As observed in Section 2, an unsigned graph with cyclomatic number ξ gives rise to at most 2^{ξ} switching non-isomorphic signed graphs. In view of Theorem 2.5, we know that, up to switching equivalency, the signature leading to the maximal spectral radius is the all-positive one. A natural question is to identify which signature leads to the minimum spectral radius.

Problem 3.18 (Signature minimizing the spectral radius)**.** Let Γ be a simple and connected unsigned graph. Determine the signature(s) $\bar{\sigma}$ such that for any signature σ of Γ, we have $\rho(\Gamma, \bar{\sigma}) \leq \rho(\Gamma, \sigma)$.

This problem has important connections and consequences in the theory of expander graphs. Informally, an expander is a sparse and highly connected graph. Given an integer $d \geq 3$ and λ a real number, a λ-expander is a connected d-regular graph whose (unsigned) eigenvalues (except d and possibly $-d$ if the graph is bipartite) have absolute value at most λ. It is an important problem in mathematics and computer science to construct, for fixed $d \geq 3$, infinite families of λ-expanders for λ *small* (see [8, 46, 56] for example). From work of Alon-Boppana (see [18, 46, 61]), we know that $\lambda = 2\sqrt{d-1}$ is the best bound we can hope for and graphs attaining this bound are called Ramanujan graphs.

Bilu and Linial [8] proposed the following combinatorial way of constructing infinite families of d-regular Ramanujan graphs. A double cover (sometimes called 2-lift or 2-cover) of a graph $\Gamma = (G = (V, E), \sigma)$ is the (unsigned) graph Γ' with vertex set $V \times \{+1, -1\}$ such that (x, s) is adjacent to $(y, s\sigma(xy))$ for $s = \pm 1$. It is easy to see that if Γ is d-regular, then Γ' is d-regular. A crucial fact is that the spectrum of the unsigned adjacency matrix of Γ' is the union of the spectrum of the unsigned adjacency matrix $A(G)$ and the spectrum of signed adjacency matrix $A_\sigma = A(\Gamma)$, where $A_\sigma(x, y) = \sigma(x, y)$ for any edge xy of Γ and 0 otherwise (see [8] for a short proof). Note that this result can be deduced using the method of equitable partitions (see [10, Section 2.3]), appears in the mathematical chemistry literature in the work of Fowler [26] and was extended to other matrices and directed graphs by Butler [14].

The spectral radius of a signing σ is the spectral radius $\rho(A_\sigma)$ of the signed adjacency matrix A_σ. Bilu and Linial [8] proved the important result

Theorem 3.19 (Bilu-Linial [8])**.** *Every connected d-regular graph has a signing with spectral radius at most $c \cdot \sqrt{d \log^3 d}$, where $c > 0$ is some absolute constant.*

and made the following conjecture.

Conjecture 3.20 (Bilu-Linial [8]). *Every connected d-regular graph G has a signature σ with spectral radius at most $2\sqrt{d-1}$.*

If true, this conjecture would provide a way to construct or show the existence of an infinite family of d-regular Ramanujan graphs. One would start with a base graph that is d-regular Ramanujan (complete graph K_{d+1} or complete bipartite graph $K_{d,d}$ for example) and then repeatedly apply the result of the conjecture above. Recently, Marcus, Spielman and Srivastava [56] made significant progress towards solving the Bilu-Linial conjecture.

Theorem 3.21. *Let G be a connected d-regular graph. Then there exists a signature σ of G such that the largest eigenvalue of A_σ is at most $2\sqrt{d-1}$.*

As mentioned before, A_σ may have negative entries and one cannot apply the Perron-Frobenius theorem for it. Therefore, the spectral radius of A_σ is not always the same as the largest eigenvalue of A_σ. In more informal terms, the Bilu-Linial conjecture is about bounding all the eigenvalues of A_σ by $-2\sqrt{d-1}$ and $2\sqrt{d-1}$ while the Marcus-Spielman-Srivastava result shows the existence of a signing where all the eigenvalues of A_σ are at most $2\sqrt{d-1}$. By taking the negative of the signing guaranteed by Marcus-Spielman-Srivastava, one gets a signed adjacency matrix where all eigenvalues are at least $-2\sqrt{d-1}$, of course.

There are several interesting ingredients in the Marcus-Spielman-Srivastava result. The first goes back to Godsil and Gutman [35] who proved the remarkable result that the average of the characteristic polynomials of the all the signed adjacency matrices of a graph Γ equals the matching polynomial of Γ. This is defined as follows. Define $m_0 = 1$ and for $k \geq 1$, let m_k denote the number of matchings of Γ consisting of exactly k edges. The matching polynomial $\mu_\Gamma(x)$ of Γ is defined as

$$\mu_\Gamma(x) = \sum_{k\geq 0} (-1)^k m_k x^{n-2k}, \tag{3.1}$$

where n is the number of vertices of Γ. Heilmann and Lieb [42] proved the following results regarding the matching polynomial of a graph. See Godsil's book [34] for a nice, self-contained exposition of these results.

Theorem 3.22. *Let Γ be a graph.*

1. *Every root of the matching polynomial $\mu_\Gamma(x)$ is real.*
2. *If Γ is d-regular, then every root of $\mu_\Gamma(x)$ has absolute value at most $2\sqrt{d-1}$.*

If Γ is a d-regular graph, then the average of the characteristic polynomials of its signed adjacency matrices equals its matching polynomial $\mu_\Gamma(x)$ whose roots are in the desired interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. As Marcus-Spielman-Srivastava point out, just because the average of certain polynomials has roots in a certain interval, does not imply that one of the polynomials has roots in that interval. However, in this situation, the characteristic polynomials of the signed adjacency matrices form an *interlacing family of polynomials* (this is a term coined by Marcus-Spielman-Srivastava in [56]). The theory of such polynomials is developed in [56] and it leads to an existence proof that one of the signed adjacency matrices of G has the largest eigenvalue at most $2\sqrt{d-1}$. As mentioned in [56],

> *The difference between our result and the original conjecture is that we do not control the smallest new eigenvalue. This is why we consider bipartite graphs.*

Note that the result of Marcus, Spielman and Srivastava [56] implies the existence of an infinite family of d-regular bipartite Ramanujan graphs, but it does not provide a recipe for constructing such family. As an amusing exercise, we challenge the readers to solve Problem 3.18 by finding a signature of the Petersen graph (try it without reading [84]) or of their favorite graph that minimizes the spectral radius.

A weighing matrix of weight k and order n is a square $n \times n$ matrix W with $0, +1, -1$ entries satisfying $WW^T = kI_n$. When $k = n$, this is the same as a Hadamard matrix and when $k = n - 1$, this is called a conference matrix. Weighing matrices have been well studied in design and coding theory (see [28] for example). Examining the trace of the square of the signed adjacency matrix, Gregory [39] proved the following.

Theorem 3.23. *If σ is any signature of Γ, then*

$$\rho(\Gamma, \sigma) \geq \sqrt{k} \tag{3.2}$$

where k is the average degree of Γ. Equality happens if and only if Γ is k-regular and A_σ is a symmetric weighing matrix of weight k.

This result implies that $\rho(K_n, \sigma) \geq \sqrt{n-1}$ for any signature σ with equality if and only if a conference matrix of order n exists. By a similar argument, one gets that $\rho(K_{n,n}, \sigma) \geq \sqrt{n}$ with equality if and only if there is a Hadamard matrix of order n. Note also that when $k = 4$, the graphs attaining equality in the previous result are known from McKee and Smyth's work [57] (see Theorem 3.8 above). Using McKee and Smyth characterization and the argument below, we can show that the only 3-regular graph attaining equality in Theorem 3.23 is the 3-dimensional cube.

Let Q_n denote the n-dimensional hypercube. Huang [47] constructed a signed adjacency matrix A_n of Q_n recursively as follows:

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad A_{n+1} = \begin{bmatrix} A_n & I_{2^n} \\ I_{2^n} & -A_n \end{bmatrix},$$

for $n \geq 1$. It is not too hard to show that $(A_n)^2 = nI_{2^n}$ for any $n \geq 1$ and thus, A_n attains equality in Theorem 3.23. We remark that Huang's method can be also used to produce infinite families of regular graphs and signed adjacency matrices attaining equality in Theorem 3.23. If G is a k-regular graph of order N with signed adjacency matrix A_s such that $\rho(A_s) = \sqrt{k}$, then define the $k+1$-regular graph H by taking two disjoint copies of G and adding a perfect matching between them and a signed adjacency matrix for H as

$$B = \begin{bmatrix} A_s & I_N \\ I_N & -A_s \end{bmatrix}.$$

Because $A_s^2 = kI_N$, we can get that $B^2 = (k+1)I_{2N}$. Thus, using any 4-regular graph G from McKee and Smyth [57] (see again Theorem 3.8) with a signed adjacency matrix A_s satisfying $A_s^2 = 4I$, one can construct a 5-regular graph H with signed adjacency matrix B such that $B^2 = 5I$. The following is a natural question.

Problem 3.24. Are there any other 5-regular graphs attaining equality in Theorem 3.23?

If the regularity assumption on G is dropped, Gregory considered a the following variant of Conjecture 3.20.

Conjecture 3.25 ([39]). *If Δ is the largest vertex degree of a nontrivial graph G, then there exists a signature σ such that $\rho(G,\sigma) < 2\sqrt{\Delta - 1}$.*

Gregory came to the above conjecture by observing that in view of Theorem 3.22 the bound in the above conjecture holds for the matching polynomial of G and by noticing that

$$\mu_G(x) = \frac{1}{|C|}\sum_{C\in\mathcal{C}} \phi(G,\sigma;x),$$

where $\mathcal{C}$ is the set of subgraphs of G consisting of cycles and $|C|$ is the number of cycles of C. Since the matching polynomial of G is the average of polynomials of signed graphs on G, one could expect that there is at least one signature $\bar{\sigma}$ such that $\rho(G,\bar{\sigma})$ does not exceed the spectral radius of $\mu_G(x)$. As observed in [39], for odd unicyclic signed graphs the spectral radius of the matching polynomial is always less than the spectral radius of the corresponding adjacency polynomial, but the conjecture still remains valid. We ask the following question whose affirmative answer would imply Conjecture [39].

Problem 3.26. If ρ is the spectral radius of a connected graph G, then is there a signature σ such that $\rho(G,\sigma) < 2\sqrt{\rho - 1}$?

In view of the above facts, we expect that the signature minimizing the spectral radius is the one balancing the contributions of cycles so that the resulting polynomial is as close as possible to the matching polynomial. For example, we can have signatures whose corresponding polynomial equals the matching polynomial, as in the following proposition.

Proposition 3.27. *Let Γ be a signed graph consisting of $2k$ odd cycles of pairwise equal length and opposite signs. Then $\rho(\Gamma) < 2\sqrt{4k-1}$.*

Is the signature in Proposition 3.27 the one minimizing the spectral radius? We leave this as an open problem (see also [85]).

We conclude this section by observing that for a general graph, it is not known whether Problem 3.18 is NP-hard or not. However, progress is made in [16] where the latter mentioned problem is shown to be NP-hard when restricted to arbitrary symmetric matrices. Furthermore, the problems described in this subsection can be considered in terms of the largest eigenvalue λ_1, instead of the spectral radius.

3.6 Spectral determination problems for signed graphs

A graph is said to be determined by its (adjacency) spectrum if cospectral graphs are isomorphic graphs. It is well-known that in general the spectrum does not determine the graph, and this problem has pushed a lot of research in spectral graph theory, also with respect to other graph matrices. In general, we can say that there are three kinds of research lines:

(1) Identify, if any, cospectral non-isomorphic graphs for a given class of graphs.

(2) Routines to build cospectral non isomorphic graphs (e.g., Godsil-McKay switching).

(3) Find conditions such that the corresponding graphs are determined by their spectrum.

Evidently, the same problems can be considered for signed graphs with respect to switching isomorphism. On the other hand, when considering signed graphs, there are many more possibilities for getting pairs of switching non-isomorphic cospectral signed graphs. For example, the paths and the cycles are examples of graphs determined by their spectrum, but the same graphs as signed ones are no longer determined by their spectrum since they admit cospectral but non-isomorphic mates [1, 3].

Hence, the spectrum of the adjacency matrix of signed graphs has less control on the graph invariants. In view of the spectral moments we get the following proposition:

Proposition 3.28. *From the eigenvalues of a signed graph* Γ *we obtain the following invariants:*

- *number of vertices and edges;*
- *the difference between the number of positive and negative triangles (*$\frac{1}{6}\sum \lambda_i^3$*);*
- *the difference between the number of positive and negative closed walks of length* p *(*$\sum \lambda_i^p$*).*

Contrarily to unsigned graphs, from the spectrum we cannot decide any more whether the graph has some kind of signed regularity, or it is sign-symmetric. For the former, we note that the co-regular signed graph $(C_6, +)$ (it is a regular graph with net regular signature) is cospectral with $P_2 \cup \tilde{Q}_4$ (cf. Figure 4). For the latter, we observe that the signed graphs A_1 and A_2 are cospectral but A_1 is not sign-symmetric while A_2 is sign-symmetric.

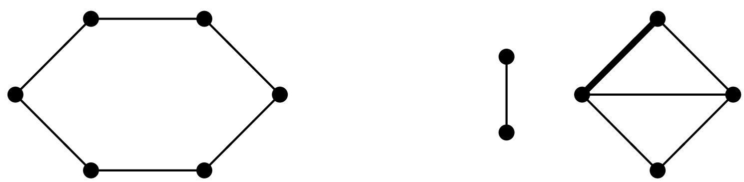

Figure 4: The cospectral pair $(C_6, +)$ and $P_2 \cup \tilde{Q}_4$.

3.7 Operations on signed graphs

In graph theory we can find several operations and operators acting on graphs. For example, we have the complement of a graph, the line graph, the subdivision graph and several kind of products as the cartesian product, and so on. Most of them have been ported to the level of signed graph, in a way that the resulting underlying graph is the same obtained from the theory of unsigned graphs, while the signatures are given in order to preserve the balance property, signed regularities, and in many cases also the corresponding spectra. However, there are a few operations and operators which do not yet have a, satisfactory, 'signed' variant.

One operator that is missing in the signed graph theory is the complement of a signed graph. The complement of signed graph should be a signed graph whose underlying graph is the usual complement, however the signature has not been defined in a satisfactory way yet. What we can ask from the signature of the complement of a signed graph? One could expect some nice features on the spectrum, as for the Laplacian, so that the spectra of the two signed graphs Γ and $\bar{\Gamma}$ are complementary to the spectrum of the obtained complete graph.

Problem 3.29. Given a graph $\Gamma = (G, \sigma)$, define the complement $\bar{\Gamma} = (\bar{G}, \bar{\sigma})$ such that there are nice (spectral) properties derived from the complete signed graph $\Gamma \cup \bar{\Gamma}$.

In terms of operators, in the literature we have nice definitions for subdivision and line graphs of signed graphs [6, 83]. The signed total graph has been recently considered and defined in [7].

From the product viewpoint, most standard signed graph products have been defined and considered in [29] and the more general NEPS (or, Cvetković product) of signed graphs have been there considered. In [66] the lexicographic product was also considered, but the given definition is not stable under the equivalence switching classes.

However, there are some graph products which do not have a signed variant yet. As an example, we mention here the wreath product and the co-normal product.

3.8 Seidel matrices

The Seidel matrix of a graph Γ on n vertices is the adjacency matrix of a signed complete graph K_n in which the edges of Γ are negative (-1) and the edges not in Γ are positive $(+1)$. More formally, the Seidel matrix $S(\Gamma)$ equals $J_n - I_n - 2A(\Gamma)$. Zaslavsky [83] confesses that

> *This fact inspired my work on adjacency matrices of signed graphs.*

Seidel matrices were introduced by van Lint and Seidel [75] and studied by many people due to their interesting properties and connections to equiangular lines, two-graphs, strongly regular graphs, mutually unbiased bases and so on (see [10, Section 10.6] and [4, 37, 64] for example). The connection between Seidel matrices and equiangular lines is perhaps best summarized in [10, p. 161]:

> *To find large sets of equiangular lines, one has to find large graphs where the smallest Seidel eigenvalue has large multiplicity.*

Let d be a natural number and $\mathbb{R}^d$ denote the Euclidean d-dimensional space with the usual inner product $\langle \cdot, \cdot \rangle$. A set of $n \geq 1$ lines (represented by unit vectors) $v_1, \dots, v_n \in \mathbb{R}^d$ is called equiangular if there is a constant $\alpha > 0$ such that $\langle v_i, v_j \rangle = \pm\alpha$ for any $1 \leq i < j \leq n$. For given α, let $N_\alpha(d)$ be the maximum n with this property. The Gram matrix G of the vectors $v_1, \dots, v_n$ is the $n \times n$ matrix whose (i, j)-th entry equals $\langle v_i, v_j \rangle$. The matrix $S := (G - I)/\alpha$ is a symmetric matrix with 0 diagonal and ± 1 entries off-diagonal. It is therefore the Seidel matrix of some graph Γ and contains all the relevant parameters of the equiangular line system. The multiplicity of the smallest eigenvalue $-1/\alpha$ of S is the smallest dimension d where the line system can be embedded into $\mathbb{R}^d$.

Lemmens and Seidel [53] (see also [4, 37, 49, 54, 59] for more details) showed that $N_{1/3}(d) = 2d - 2$ for d sufficiently large and made the following conjecture.

Conjecture 3.30. *If* $23 \leq d \leq 185$, $N_{1/5}(d) = 276$. *If* $d \geq 185$, *then* $N_{1/5}(d) = \lfloor 3(d-1)/2 \rfloor$.

The fact that $N_{1/5}(d) = \lfloor 3(d-1)/2 \rfloor$ for d sufficiently large was proved by Neumaier [59] and Greaves, Koolen, Munemasa and Szöllősi [37]. Recently, Lin and Yu [54] made progress in this conjecture by proving some claims from Lemmens and Seidel [53]. Note that these results can be reformulated in terms of Seidel matrices with smallest eigenvalue -5. Seidel and Tsaranov [65] classified the Seidel matrices with smallest eigenvalue -3.

Neumann (cf. [53, Theorem 3.4]) proved that if $N_\alpha(d) \geq 2d$, then $1/\alpha$ is an odd integer. Bukh [12] proved that $N_\alpha(d) \leq c_\alpha d$, where c_α is a constant depending only on α. Balla, Dräxler, Keevash and Sudakov [4] improved this bound and showed that for d sufficiently large and $\alpha \neq 1/3$, $N_\alpha(d) \leq 1.93d$. Jiang and Polyanskii [49] further improved these results and showed that if $\alpha \notin \{1/3, 1/5, 1/(1+2\sqrt{2})\}$, then $N_\alpha(d) \leq 1.49d$ for d sufficiently large. When $1/\alpha$ is an odd integer, Glazyrin and Yu [32] obtained a general bound $N_\alpha(d) \leq \left(2\alpha^2/3 + 4/7\right) d + 2$ for all n.

Bukh [12] and also, Balla, Dräxler, Keevash and Sudakov [4] conjecture the following.

Conjecture 3.31. *If $r \geq 2$ is an integer, then $N_{\frac{1}{2r-1}}(d) = \frac{r(n-1)}{r-1} + O(1)$ for n sufficiently large.*

When $1/\alpha$ is not a totally real algebraic integer, then $N_\alpha(d) = d$. Jiang and Polyanskii [49] studied the set $T = \{\alpha \mid \alpha \in (0,1),\ \limsup_{d\to\infty} N_\alpha(d)/d > 1\}$ and showed that the closure of T contains the closed interval $[0, 1/\sqrt{\sqrt{5}+2}]$ using results of Shearer [67] on the spectral radius of unsigned graphs.

Seidel matrices with two distinct eigenvalues are equivalent to regular two-graphs and correspond to equality in the *relative bound* (see [10, Section 10.3] or [37] for example). It is natural to study the combinatorial and spectral properties of Seidel matrices with three distinct eigenvalues, especially since for various large systems of equiangular lines, the respective Seidel matrices have this property. Recent work in this direction has been done by Greaves, Koolen, Munemasa and Szöllősi [37] who determined several properties of such Seidel matrices and raised the following interesting problem.

Problem 3.32. Find a combinatorial interpretation of Seidel matrices with three distinct eigenvalues.

A classification for the class of Seidel matrices with exactly three distinct eigenvalues of order less than 23 was obtained by Szöllősi and Östergård [69]. Several parameter sets for which existence is not known were also compiled in [37]. Greaves [38] studied Seidel matrices with three distinct eigenvalues, observed that there is only one Seidel matrix of order at most 12 having three distinct eigenvalues, but its switching class does not contain any regular graphs. In [38], he also showed that if the Seidel matrix S of a graph Γ has three distinct eigenvalues of which at least one is simple, then the switching class of Γ contains a strongly regular graph. The following question was posed in [38].

Problem 3.33. Do there exist any Seidel matrices of order at least 14 with precisely three distinct eigenvalues whose switching class does not contain a regular graph?

The switching class of conference graph and isolated vertex has two distinct eigenvalues. If these two eigenvalues are not rational, then the switching class does not contain a regular graph. So we suspect that there must be infinitely many graphs whose Seidel matrix has exactly three distinct eigenvalues and its switching graph does not contain a regular graph. A related problem also appears in [38].

Problem 3.34. Does every Seidel matrix with precisely three distinct rational eigenvalues contain a regular graph in its switching class?

The Seidel energy $\mathcal{S}(\Gamma)$ of a graph Γ is the sum of absolute values of the eigenvalues of the Seidel matrix S of Γ. This parameter was introduced by Haemers [41] who proved that

$\mathcal{S}(\Gamma) \le n\sqrt{n-1}$ for any graph Γ of order n with equality if and only S is a conference matrix. Haemers [41] also conjectured that the complete graphs on n vertices (and the graphs switching equivalent to them) minimize the Seidel energy.

Conjecture 3.35. *If Γ is a graph on n vertices, then $\mathcal{S}(\Gamma) \ge \mathcal{S}(K_n) = 2(n-1)$.*

Ghorbani [31] proved the Haemers' conjecture in the case $\det(S) \ge n-1$ and very recently, Akbari, Einollahzadeh, Karkhaneei and Nematollah [2] finished the proof of the conjecture. Ghorbani [31, p. 194] also conjectured that the fraction of graphs on n vertices with $|\det S| < n-1$ goes to 0 as n tends to infinity. This conjecture was also recently proved by Rizzolo [63].

It is known that if Γ has even order, then its Seidel matrix S is full-rank. If a graph Γ has odd order n, then $\operatorname{rank}(S) \ge n-1$. There are examples such C_5 for example where $\operatorname{rank}(S) = n-1$. Haemers [40] posed the following problem which is still open to our knowledge.

Problem 3.36. If $\operatorname{rank}(S) = n-1$, then there exists an eigenvector of S corresponding to 0 that has only ± 1 entries?

Recently, van Dam and Koolen [74] determined an infinitely family of graphs on n vertices whose Seidel matrix has rank $n-1$ and their switching class does not contain a regular graph.

4 Conclusions

Spectral graph theory is a research field which has been very much investigated in the last 30–40 years. Our impression is that the study of the spectra of signed graphs is very far from the level of knowledge obtained with unsigned graphs. So the scope of the present note is to promote investigations on the spectra of signed graphs. Of course, there are many more problems which can be borrowed from the underlying spectral theory of (unsigned) graphs. Here we just give a few of them, but we have barely scratched the surface of the iceberg.

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