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8 Why Nature Made a Choice of Clifford and not Grassmann Coordinates ⋆

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Abstract. This is a discussion on fields, the internal degrees of freedom of which are expressed by either the Grassmann or the Clifford "coordinates". Since both "coordinates" fulfill anticommutation relations, both fields can be second quantized so that their creation and annihilation operators fulfill the requirements of the commutation relations for fermion fields. However, while the internal spin, determined by the generators of the Lorentz group of the Clifford objects S^{ab} and $\tilde{S}^{ab}$ (in the *spin-charge-family* theory S^{ab} determine the spin degrees of freedom and $\tilde{S}^{ab}$ the family degrees of freedom) have half integer spin, have $\mathbf{S}^{ab}$ (expressible with $S^{ab} + \tilde{S}^{ab}$) integer spin. Nature made obviously a choice of the Clifford algebra.

We discuss here the quantization — first and second — of the fields, the internal degrees of freedom of which are functions of the Grassmann coordinates θ and their conjugate momentum, as well as of the fields, the internal degrees of freedom of which are functions of the Clifford γ^a. Inspiration comes from the *spin-charge-family* theory [[1,2,9,3], and the references therein], in which the action for fermions in d-dimensional space is equal to $\int \, d^dx \; E \; \frac{1}{2} \, (\bar{\psi} \, \gamma^a p_{0a} \psi) + \text{h.c.}$, with $p_{0a} = f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E} \, \{p_\alpha, E f^{\alpha}{}_a\}_-$, $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$. We write the basic states of the Grassmann fields and the Clifford fields as a function of products of either Grassmann or Clifford objects, trying to understand *the choice of nature*. We look for the action for free fields which are functions of either the Grassmann coordinates or of the Clifford coordinates in order to understand why Clifford algebra "win" in the competition for the physical degrees of freedom (at least in our observable world).

Povzetek. Avtorja obravnavata polja, pri katerih so notranje prostostne stopnje izražene ali z Grassmannovimi ali pa s Cliffordovimi "koordinatami". Ker obe vrsti "koordinat" zadoščata antikomutacijskim relacijam, lahko za obe vrsti polj naredimo drugo kvantizacijo tako, da kreacijski in anihilacijski operatorji zadoščajo komutacijskim relacijam za fermionska polja. Za razliko od internih spinov, ki jih določajo generatorji Lorentzove grupe Cliffordovih objektov S^{ab} in $\tilde{S}^{ab}$ (v teoriji *spinov-nabojev-družin* S^{ab} določajo spinske prostostne stopnje, $\tilde{S}^{ab}$ pa družinske prostostne stopnje) in imajo polštevilčni spin), imajo $\mathbf{S}^{ab}$ (ki jih lahko izrazimo z $S^{ab} + \tilde{S}^{ab}$) celoštevilski spin. "Narava se je očitno odločila" za Cliffordovo algebro.

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Avtorja obravnavata kvantizacijo — prvo in drugo — za polja, pri katerih so notranje prostostne stopnje funkcije Grassmannovih koordinat θ in ustreznih konjugiranih momentov, pa tudi za polja, kjer so interne prostostne stopnje funkcije Cliffordovih koordinat γ^a. Navdih najdeta v teoriji *spinov-nabojev-družin* [[1,2,9,3], in reference v teh člankih], v kateri je akcija za fermione v d razsežnem prostoru enaka $\int d^dx\, E\, \frac{1}{2}\, (\bar{\psi}\, \gamma^a p_{0a} \psi) + h.c.$, with $p_{0a} = f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E}\, \{p_{\alpha}, E f^{\alpha}{}_a\}_-$, $p_{0\alpha} = p_{\alpha} - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha}$. Da bi razumela “izbiro narave”, zapišeta osnovna stanja Grassmannovih in Cliffordovih polj kot produkte Grassmannovih ali Cliffordovih objektov. Iščeta akcijo za prosta polja, ki so funkcije Grassmannovih ali pa Cliffordovih koordinat, da bi bolje razumela, zakaj Cliffordova algebra “zmaga” v tekmi za fizikalne prostostne stopnje (vsaj v opazljivem svetu).

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8.1 Introduction

This paper is to look for the answers to the questions like: Why our universe ”uses” the Clifford rather than the Grassmann coordinates, although both lead in the second quantization procedure to the anticommutation relations required for fermion degrees of freedom? Does the answer lay on the fact that the Clifford degrees of freedom offers the appearance of the families, the half integer spin and the charges as observed so far for fermions, while the Grassmann coordinates offer the groups of (isolated) integer spin states and to charges in the adjoint representations? Can this explain why the simple starting action of the *spin-charge-family* theory of one of us (N.S.M.B.) [9,3,5,8,4,6,7] is doing so far extremely well in manifesting the observed properties of the fermion and boson fields in the low energy regime?

The working hypothesis is that ”Nature knows” all the mathematics, accordingly therefore ”she knows” for the Grassmann and the Clifford coordinates. To understand why Grassmann space ”was not chosen” – we see that the use of the Dirac γ^a’s enabled to understand the fermions in the first and second quntized theory of fields – or better, to understand why the Clifford algebra (in the *spin-charge-family* theory of two kinds – γ^a’s and $\tilde{\gamma}^a$’s) is succesfully applicable at least in the low enery regime, we work in this paper with both types of spaces.

This work is a part of the project of both authors, which includes the *fermionization* procedure of boson fields or the *bosonization* procedure of fermion fields, discussed in Refs. [10] and in this proceedings for any dimension d (by the authors of this contribution, while one of them, H.B.F.N. [11], has succeeded with another author to do the *fermionization* for $d = (1 + 1)$)), and which would hopefully help to better understand the content and dynamics of our universe.

In the *spin-charge-family* theory [9,3,5,8,4,6,7] — which offers the explanation of all the assumptions of the *standard model*, with the appearance of families, the scalar higgs and the Yukawa couplings included, offering also the explanation for the matter-antimatter asymmetry in our universe and for the appearance of the dark matter — a very simple starting action for massless fermions and bosons in

$d=(1+13)$ is assumed, in which massless fermions interact with only gravity, the vielbeins $f^{\alpha}{}_{a}$ (the gauge fields of momentums p_a) and the two kinds of the spin connections ($\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, the gauge fields of the two kinds of the Clifford algebra objects γ^a and $\tilde{\gamma}^a$, respectively).

$$\mathcal{A} = \int d^dx\, E\, \frac{1}{2}\,(\bar{\psi}\,\gamma^a p_{0a}\psi) + \text{h.c.} + \int d^dx\, E\,(\alpha\, R + \tilde{\alpha}\,\tilde{R})\,, \qquad (8.1)$$

with $p_{0a} = f^{\alpha}{}_{a}p_{0\alpha} + \frac{1}{2E}\{p_\alpha, Ef^{\alpha}{}_{a}\}_-$, $p_{0\alpha} = p_\alpha - \frac{1}{2}S^{ab}\omega_{ab\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}$ and $R = \frac{1}{2}\{f^{\alpha[a}f^{\beta b]}\ (\omega_{ab\alpha,\beta} - \omega_{ca\alpha}\,\omega^{c}{}_{b\beta})\} + \text{h.c.}$, $\tilde{R} = \frac{1}{2}\{f^{\alpha[a}f^{\beta b]}\ (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha}\,\tilde{\omega}^{c}{}_{b\beta})\} + \text{h.c.}$. The two kinds of the Clifford algebra objects, γ^a and $\tilde{\gamma}^a$,

$$\{\gamma^a,\gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a,\tilde{\gamma}^b\}_+\,,$$
$$\{\gamma^a,\tilde{\gamma}^b\}_+ = 0\,. \qquad (8.2)$$

anticommute, $\{\gamma^a,\tilde{\gamma}^b\}_+ = 0$ (γ^a and $\tilde{\gamma}^b$ are connected with the left and the right multiplication of the Clifford objects, there is no third kind of operators). One of the objects, the generators $S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, determine spins and charges of spinors of any families, another, $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$, determine the family quantum numbers. Here [1] $f^{\alpha[a}f^{\beta b]} = f^{\alpha a}f^{\beta b} - f^{\alpha b}f^{\beta a}$. There are correspondingly two kinds of infinitesimal generators of the Lorentz transformations in the internal degrees of freedom - S^{ab} for $SO(13,1)$ and $\tilde{S}^{ab}$ for $\widetilde{SO}(13,1)$, arranging states into representations.

The curvature R and $\tilde{R}$ determine dynamics of the gauge fields — the spin connections and the vielbeins, which manifest in $d=(1+3)$ all the known vector gauge fields as well as the scalar fields [5] which explain the appearance of higgs and the Yukawa couplings, provided that the symmetry breaks from the starting one to $SO(3,1)\times SU(3)\times U(1)$.

The infinitesimal generators of the Lorentz transformations for the gauge fields – the two kinds of the Clifford operators and the Grassmann operators – operate as follows

$$\{S^{ab},\,\gamma^e\}_- = -i\,(\eta^{ae}\,\gamma^b - \eta^{be}\,\gamma^a)\,,$$
$$\{\tilde{S}^{ab},\,\tilde{\gamma}^e\}_- = -i\,(\eta^{ae}\,\tilde{\gamma}^b - \eta^{be}\,\tilde{\gamma}^a)\,,$$
$$\{\mathbf{S}^{ab},\,\theta^e\}_- = -i\,(\eta^{ae}\,\theta^b - \eta^{be}\,\theta^a)\,,$$
$$\{\mathbf{M}^{ab},\,A^{d\dots e\dots g}\}_- = -i\,(\eta^{ae}\,A^{d\dots b\dots g} - \eta^{be}\,A^{d\dots a\dots g})\,, \qquad (8.3)$$

where $\mathbf{M}^{ab}$ are defined by a sum of L^{ab} plus any of S^{ab} or $\tilde{S}^{ab}$, in the Grassmann case $\mathbf{M}^{ab}$ is $L^{ab} + \mathbf{S}^{ab}$, which appear to be $\mathbf{M}^{ab} = L^{ab} + S^{ab} + \tilde{S}^{ab}$, as presented later in Eq. (8.22).

[1] $f^{\alpha}{}_{a}$ are inverted vielbeins to $e^{a}{}_{\alpha}$ with the properties $e^{a}{}_{\alpha}f^{\alpha}{}_{b} = \delta^{a}{}_{b}$, $e^{a}{}_{\alpha}f^{\beta}{}_{a} = \delta^{\beta}_{\alpha}$, $E = \det(e^{a}{}_{\alpha})$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

We discuss in what follows the first and the second quantization of the fields which depend on the Grassmann coordinates θ^a, as well as of the fields which depend on the Clifford coordinates γ^a (or $\tilde{\gamma}^a$) in order to try to understand why "nature has made a choice" of fermions of spins and charges (describable in the *spin-charge-family* theory by subgroups of the Lorentz group expressible with the generators S^{ab}) in the fundamental representations of the groups, which interact in the *spin-charge-family* theory through the boson gauge fields (the vielbeins and the spin connections of two kinds). We choose correspondingly either θ^a's or γ^a's (or $\tilde{\gamma}^a$'s, either γ^a's or $\tilde{\gamma}^a$'s [6,7,9]) to describe the internal degrees of freedom of fields to clarify the "choice of nature" and correspondingly also the meaning of *fermionization* of bosons (or *bosonization* of fermions) discussed in Refs. [10] and in this proceedings for any dimension d.

In all these cases we treat free massless boson and fermion fields; masses of the fields which manifest in $d = (1+3)$ are in the *spin-charge-family* theory due to their interactions with the gravitational fields in $d > 4$, described by the scalar vielbeins or spin connection fields

8.2 Observations which might be of some help when fermionizing boson fields or bosonizing fermion fields

We present in this section properties of fields with the integer spin in d-dimensional space, expressed in terms of the Grassmann algebra objects, and the fermion fields, expressed in terms of the Clifford algebra objects. Since the Clifford algebra objects are expressible with the Grassmann algebra objects (Eqs. (8.14, 8.15)), the norms of both are determined by the integral in the Grassmann space, Eqs. (8.24, 8.27).

a. *Fields with the integer spin in the Grassmann space*

A point in d-dimensional Grassmann space of real anticommuting coordinates θ^a, $(a = 0, 1, 2, 3, 5, \dots, d)$, is determined by a vector $\{\theta^a\} = (\theta^1, \theta^2, \theta^3, \theta^5, \dots, \theta^d)$. A linear vector space over the coordinate Grassmann space has correspondingly the dimension 2^d, due to the fact that $(\theta^{a_i})^2 = 0$ for any $a_i \in (0, 1, 2, 3, 5, \dots, d)$.

Correspondingly are fields in the Grassmann space expressed in terms of the Grassmann algebra objects

$$\mathbf{B} = \sum_{k=0}^{d} a_{a_1 a_2 \dots a_k} \, \theta^{a_1} \theta^{a_2} \dots \theta^{a_k} |\phi_{og}>, \quad a_i \le a_{i+1}, \tag{8.4}$$

where $|\phi_{og}>$ is the vacuum state, here assumed to be $|\phi_{og}>= |1>$, so that $\frac{\partial}{\partial \theta^a}|\phi_{og}>= 0$ for any θ^a. The *Kalb-Ramond* boson fields $a_{a_1 a_2 \dots a_k}$ are antisymmetric with respect to the permutation of indexes, since the Grassmann coordinates anticommute

$$\{\theta^a, \theta^b\}_+ = 0. \tag{8.5}$$

The left derivative $\frac{\partial}{\partial \theta_a}$ on vectors of the space of monomials $\mathbf{B}(\theta)$ is defined as follows

$$\frac{\partial}{\partial \theta_a} \mathbf{B}(\theta) = \frac{\partial \mathbf{B}(\theta)}{\partial \theta_a},$$
$$\left\{ \frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b} \right\}_+ \mathbf{B} = 0, \text{ for all } \mathbf{B}. \tag{8.6}$$

Defining $p^{\theta^a} = i \frac{\partial}{\partial \theta_a}$ it correspondingly follows

$$\{p^{\theta a}, p^{\theta b}\}_+ = 0, \quad \{p^{\theta a}, \theta^b\}_+ = i\eta^{ab}, \tag{8.7}$$

The metric tensor η^{ab} ($= diag(1, -1, -1, \ldots, -1)$) lowers the indexes of a vector $\{\theta^a\}$: $\theta_a = \eta_{ab}\, \theta^b$, the same metric tensor lowers the indexes of the ordinary vector x^a of commuting coordinates.

Defining [2]

$$(\theta^a)^\dagger = \frac{\partial}{\partial \theta_a} \eta^{aa} = -i p^{\theta a} \eta^{aa}, \tag{8.8}$$

it follows

$$(\frac{\partial}{\partial \theta_a})^\dagger = \eta^{aa}\, \theta^a, \quad (p^{\theta a})^\dagger = -i\eta^{aa} \theta^a. \tag{8.9}$$

By introducing [2] the generators of the infinitesimal Lorentz transformations in the Grassmann space as

$$\mathbf{S}^{ab} = \theta^a p^{\theta b} - \theta^b p^{\theta a}, \tag{8.10}$$

one finds

$$\{\mathbf{S}^{ab}, \mathbf{S}^{cd}\}_- = i\{\mathbf{S}^{ad}\eta^{bc} + \mathbf{S}^{bc}\eta^{ad} - \mathbf{S}^{ac}\eta^{bd} - \mathbf{S}^{bd}\eta^{ac}\},$$
$$\mathbf{S}^{ab\dagger} = \eta^{aa}\eta^{bb}\mathbf{S}^{ab}. \tag{8.11}$$

The basic states in Grassmann space can be arrange into representations [2] with respect to the Cartan subalgebra of the Lorentz algebra, as presented in App. 8.4. The state in d-dimensional space with all the eigenvalues of the Cartan subalgebra of the Lorentz group of Eq. (8.67) equal to either i or 1 is $(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \cdots (\theta^{d-1} + i\theta^d)|\phi_{og}>$, with $|\phi_{og}> = |1>$.

b. *Fermion fields and the Clifford objects*

Let us present as well the properties of the fermion fields with the half integer spin, expressed by the Clifford algebra objects

$$\mathbf{F} = \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k}\, \gamma^{a_1}\gamma^{a_2} \ldots \gamma^{a_k} |\psi_{oc}>, \quad a_i \le a_{i+1}, \tag{8.12}$$

[2] In Ref. [2] the definition of $\theta^{a\dagger}$ was differently chosen. Correspondingly also the scalar product needed different weight function in Eq. (8.24) is different.

where $|\psi_{oc}>$ is the vacuum state. The *Kalb-Ramond* fields $a_{a_1 a_2 \dots a_k}$ are again in general boson fields, which are antisymmetric with respect to the permutation of indexes, since the Clifford objects have the anticommutation relations

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}\,. \tag{8.13}$$

A linear vector space over the Clifford coordinate space has again the dimension 2^d, due to the fact that $(\gamma^{a_i})^2 = 0$ for any $a_i \in (0, 1, 2, 3, 5, \dots, d)$.

One can see that γ^a are expressible in terms of the Grassmann coordinates and their conjugate momenta as

$$\gamma^a = (\theta^a - i\,p^{\theta a})\,. \tag{8.14}$$

We also find $\tilde{\gamma}^a$

$$\tilde{\gamma}^a = i\,(\theta^a + i\,p^{\theta a})\,, \tag{8.15}$$

with the anticommutation relation of Eq. (8.13) and

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab}\,, \quad \{\gamma^a, \tilde{\gamma}^b\}_+ = 0\,. \tag{8.16}$$

Taking into account Eqs. (8.8, 8.14, 8.15) one finds

$$\begin{aligned} &(\gamma^a)^\dagger = \gamma^a \eta^{aa}\,, \quad (\tilde{\gamma}^a)^\dagger = \tilde{\gamma}^a \eta^{aa}\,, \\ &\gamma^a \gamma^a = \eta^{aa}\,, \quad \gamma^a (\gamma^a)^\dagger = 1\,, \quad \tilde{\gamma}^a \tilde{\gamma}^a = \eta^{aa}\,, \quad \tilde{\gamma}^a (\tilde{\gamma}^a)^\dagger = 1\,. \end{aligned} \tag{8.17}$$

All three choices for the linear vector space – spanned over either the coordinate Grassmann space, over the vector space of γ^a, as well as over the vector space of $\tilde{\gamma}^a$ – have the dimension 2^d.

We can express Grassmann coordinates θ^a and momenta $p^{\theta a}$ in terms of γ^a and $\tilde{\gamma}^a$ as well

$$\begin{aligned} \theta^a &= \frac{1}{2}(\gamma^a - i\tilde{\gamma}^a)\,, \\ \frac{\partial}{\partial \theta_a} &= \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,. \end{aligned} \tag{8.18}$$

It then follows as it should $\frac{\partial}{\partial \theta^b}\theta^a = \frac{1}{2}\,\eta_{bc}(\gamma^c + i\tilde{\gamma^c})\,\frac{1}{2}\,(\gamma^c - i\tilde{\gamma^c}) = \delta^a_b$.

Correspondingly we can use either γ^a as well as $\tilde{\gamma}^a$ instead of θ^a to span the vector space. In this case we change the vacuum from the one with the property $\frac{\partial}{\partial \theta^a}|\phi_{og}>= 0$ to $|\psi_{oc}>$ with the property [2,7,9]

$$\begin{aligned} &<\psi_{oc}|\gamma^a|\psi_{oc}> = 0\,, \quad \tilde{\gamma}^a|\psi_{oc}>= i\gamma^a|\psi_{oc}>\,, \quad \tilde{\gamma}^a\gamma^b|\psi_{oc}>= -i\gamma^b\gamma^a|\psi_{oc}>\,, \\ &\tilde{\gamma}^a\tilde{\gamma}^b|\psi_{oc}>|_{a\neq b} = -\gamma^a\gamma^b|\psi_{oc}>\,, \quad \tilde{\gamma}^a\tilde{\gamma}^b|\psi_{oc}>|_{a=b} = \eta^{ab}|\psi_{oc}>\,. \end{aligned} \tag{8.19}$$

This is in agreement with the requirement

$$\begin{aligned} \gamma^a\,\mathbf{B}(\gamma)\,|\psi_{oc}\ >:= (\,&a_0\,\gamma^a\ + a_{a_1}\,\gamma^a\,\gamma^{a_1} + a_{a_1 a_2}\,\gamma^a\,\gamma^{a_1}\gamma^{a_2} + \dots + \\ &a_{a_1\cdots a_d}\,\gamma^a\,\gamma^{a_1}\cdots\gamma^{a_d}\,)\,|\psi_{oc}\ >\,, \\ \tilde{\gamma}^a\,\mathbf{B}(\gamma)\,|\psi_{oc}\ >:= (\,&i\,a_0\gamma^a - i\,a_{a_1}\gamma^{a_1}\,\gamma^a + i\,a_{a_1 a_2}\gamma^{a_1}\gamma^{a_2}\,\gamma^a + \dots + \\ &i\,(-1)^d\,a_{a_1\cdots a_d}\gamma^{a_1}\cdots\gamma^{a_d}\,\gamma^a\,)\,|\psi_{oc}\ >\,. \end{aligned} \tag{8.20}$$

We find the infinitesimal generators of the Lorentz transformations in the Clifford algebra space

$$S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)\,, \quad S^{ab\dagger} = \eta^{aa}\eta^{bb}S^{ab}\,,$$
$$\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,, \quad \tilde{S}^{ab\dagger} = \eta^{aa}\eta^{bb}\tilde{S}^{ab}\,, \tag{8.21}$$

with the commutation relations for either S^{ab} or $\tilde{S}^{ab}$ of Eq. (8.11), if $\mathbf{S}^{ab}$ is replaced by either S^{ab} or $\tilde{S}^{ab}$, respectively, while

$$\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}\,,$$
$$\{S^{ab}, \tilde{S}^{cd}\}_- = 0\,. \tag{8.22}$$

The basic states in the Clifford space can be arranged in representations, in which any state is the eigenstate of the Cartan subalgebra operators of Eq. (8.67). The state in d-dimensional space with the eigenvalues of either $S^{03}, S^{12}, S^{56}, \ldots, S^{d-1\,d}$ or $\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \ldots, \tilde{S}^{d-1\,d}$ equal to $\frac{1}{2}(i, 1, 1, \ldots, 1)$ is $(\gamma^0 - \gamma^3)(\gamma^1 + i\gamma^2)(\gamma^5 + i\gamma^6)\cdots(\gamma^{d-1} + i\gamma^d)$, where the states are expresses in terms of γ^a. The states of one representation follow from the starting state obtained by S^{ab}, which do not belong to the Cartan subalgebra operators, while $\tilde{S}^{ab}$, which define family, jumps from the starting family to the new one.

8.2.1 Norms of vectors in Grassmann and Clifford space

Let us look for the norm of vectors in Grassmann space

$$\mathbf{B} = \sum_k^d a_{a_1 a_2 \ldots a_k}\, \theta^{a_1}\theta^{a_2}\ldots\theta^{a_k}|\phi_{og}>$$

and in Clifford space

$$\mathbf{F} = \sum_k^d a_{a_1 a_2 \ldots a_k}\, \gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k}|\psi_{oc}>,$$

where $|\phi_{og}>$ and $|\phi_{oc}>$ are the vacuum states in the Grassmann and Clifford case, respectively. In what follows we refer to the Ref. [2].

a. *Norms of the Grassmann vectors*

Let us define the integral over the Grassmann space [2] of two functions of the Grassmann coordinates $<\mathbf{B}|\mathbf{C}>$, $<\mathbf{B}|\theta> = <\theta|\mathbf{B}>^\dagger$, by requiring

$$\{d\theta^a, \theta^b\}_+ = 0, \quad \int d\theta^a = 0\,, \quad \int d\theta^a\theta^a = 1\,,$$
$$\int d^d\theta\; \theta^0\theta^1 \cdot \theta^d = 1\,,$$
$$d^d\theta = d\theta^d \ldots d\theta^0\,, \quad \omega = \Pi^d_{k=0}(\frac{\partial}{\partial\theta_k} + \theta^k)\,, \tag{8.23}$$

with $\frac{\partial}{\partial \theta_a}\theta^c = \eta^{ac}$. The scalar product is defined by the weight function $\omega = \Pi_{k=0}^{d}(\frac{\partial}{\partial \theta_k} + \theta^k)$. It then follows for a scalar product $< \mathbf{B}|\mathbf{C} >$

$$< \mathbf{B}|\mathbf{C} > = \int d^d x d^d \theta^a \;\omega < \mathbf{B}|\theta >< \theta|\mathbf{C} >= \sum_{k=0}^{d} \int d^d x\, b^*_{b_1 \dots b_k} c_{b_1 \dots b_k} \,, \tag{8.24}$$

where according to Eq. (8.8) follows:

$$< \mathbf{B}|\theta >=< \phi_{og}| \sum_{p=0}^{d} (-i)^p \, a^*_{a_1 \dots a_p} p^{\theta a_p} \eta^{a_p a_p} \cdots p^{\theta a_1} \eta^{a_1 a_1} .$$

The vacuum state is chosen to be $|\phi_{og} >= |1 >$, Eq. (8.4).

The norm $< \mathbf{B}|\mathbf{B} >$ is correspondingly always nonnegative.

b. *Norms of the Clifford vectors*

Let us look for the norm ofvectors, expressed with the Clifford objects $\mathbf{F} = \sum_k^d a_{a_1 a_2 \dots a_k} \gamma^{a_1}\gamma^{a_2} \dots \gamma^{a_k}|\psi_{oc} >$, where $|\phi_{og} >$ and $|\psi_{oc} >$ are the two vacuum states when the Grassmann and the Clifford objects are concerned, respectively. By taking into account Eq. (8.17) it follows that

$$(\gamma^{a_1}\gamma^{a_2} \dots \gamma^{a_k})^\dagger = \gamma^{a_k}\eta^{a_k a_k} \dots \gamma^{a_2}\eta^{a_2 a_2}\gamma^{a_1}\eta^{a_1 a_1} \,, \tag{8.25}$$

while $\gamma^a \gamma^a = \eta^{aa}$.

We can use Eqs. (8.23, 8.24) to evaluate the scalar product of two Clifford algebra objects $< \gamma^a|\mathbf{F} >=< (\theta^a - ip^{\theta a})|\mathbf{F} >$ and $< (\theta^b - ip^{\theta b})|\mathbf{G} >$. These expressions follow from Eqs. (8.14, 8.15, 8.17)). We must then choose for the vacuum state the one from the Grassmann case – $|\psi_{oc} >= |\phi_{og} >= |1 >$. We obtain

$$< \mathbf{F}|\mathbf{G} > = \int d^d x d^d \theta^a \;\omega < \mathbf{F}|\gamma >< \gamma|\mathbf{G} >= \sum_{k=0}^{d} \int d^d x\, a^*_{a_1 \dots a_k} b_{b_1 \dots b_k} \,. \tag{8.26}$$

{Similarly we obtain, if we express $\tilde{\mathbf{F}} = \sum_{k=0}^{d} a_{a_1 a_2 \dots a_k} \tilde{\gamma}^{a_1}\tilde{\gamma}^{a_2} \dots \tilde{\gamma}^{a_k}|\phi_{oc} >$ and $\tilde{\mathbf{G}} = \sum_{k=0}^{d} b_{b_1 b_2 \dots b_k} \tilde{\gamma}^{b_1}\tilde{\gamma}^{b_2} \dots \tilde{\gamma}^{b_k}|\phi_{oc} >$ and take $|\psi_{oc} >= |\phi_{og} >= |1 >$, the scalar product

$$< \tilde{\mathbf{F}}|\tilde{\mathbf{G}} > = \int d^d x d^d \theta^a \;\omega < \tilde{\mathbf{F}}|\tilde{\gamma} >< \tilde{\gamma}|\tilde{\mathbf{G}} >= \sum_{k=0}^{d} \int d^d x\, a^*_{a_1 \dots a_k} a_{b_1 \dots b_k} .\} \tag{8.27}$$

Correspondingly we can write

$$(a_{a_1 a_2 \dots a_k} \gamma^{a_1}\gamma^{a_2} \dots \gamma^{a_k})^\dagger (a_{a_1 a_2 \dots a_k} \gamma^{a_1}\gamma^{a_2} \dots \gamma^{a_k}) = a^*_{a_1 a_2 \dots a_k} a_{a_1 a_2 \dots a_k} \,. \tag{8.28}$$

The norm of each scalar term in the sum of **F** is nonnegative.

c. We have learned that in both spaces – Grassmann and Clifford – the norms of basic states can be defined so that the states, which are eigenvectors of the Cartan subalgebra, are orthogonal and normalized using the same integral. Studying the second quantization procedure in Subsect. 8.2.3 we learn that not all 2^d states can be generated by the creation and annihilation operators fullfilling the requirements for the second quantized operators, either for states with integer spins or for states with half integer spin. We also learn that the vacuum state must in the Clifford algebra case be different the one assumed in the first quantization case.

8.2.2 Actions in Grassmann and Clifford space

Let us construct actions for states in the Grassmann space, as well as in the Clifford space. While the action in the Clifford space is well known since long [17], the action in the Grassmann space must be found. In both cases we look for actions for free massless states only.

States in Grassmann space as well as states in Clifford space are organized to be – within each of the two spaces – orthogonal and normalized with respect to Eq. (8.23). We choose the states in each of two spaces to be the eigenstates of the Cartan subalgebra – with respect to $\mathbf{S}^{ab}$ in Grassmann space and with respect to S^{ab} and $\tilde{S}^{ab}$ in Clifford space, Eq. (8.67).

In both spaces the requirement that states are obtained by the application of creation operators on vacuum states – $\hat{b}^{\theta}_{i}$ obeying the commutation relations of Eq. (8.40) on the vacuum state $|\phi_{og}>$ for Grassmann space, and $\hat{b}^{\alpha}_{i}$ obeying the commutation relation of Eq. (8.52) on the vacuum states $|\psi_{oc}>$, Eq. (8.59), for Clifford space – reduces the number of states, in the Clifford space more than in the Grassmann space. But while in the Clifford space all physically applicable states are reachable by either S^{ab} or by $\tilde{S}^{ab}$, the states in the Grassmann space, belonging to different representations with respect to the Lorentz generators, seem not to be connected.

a. *Action in Clifford space*

In Clifford space we expect that the action for a free massless object

$$\mathcal{A} = \int d^d x \, \frac{1}{2} \, (\psi^{\dagger}\gamma^0 \, \gamma^a p_a \psi) + h.c. \,, \tag{8.29}$$

is Lorentz invariant, and that it leads to the equations of motion

$$\gamma^a p_a |\psi^{\alpha}_i> = 0 \,, \tag{8.30}$$

which fulfill also the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b |\psi^{\alpha}_i> = p^a p_a |\psi^{\alpha}_i> = 0 \,. \tag{8.31}$$

Correspondingly γ^0 appears in the action since we pay attantion that

$$\begin{aligned} S^{ab\dagger}\gamma^0 &= \gamma^0\, S^{ab}\,, \\ S^{\dagger}\gamma^0 &= \gamma^0 S^{-1}\,, \\ S &= e^{-\frac{i}{2}\omega_{ab}(S^{ab}+L^{ab})}\,. \end{aligned} \tag{8.32}$$

We choose the basic states to be the eigenstates of all the members of the Cartan subalgebra, Eq. (8.67). Correspondingly all the states, belonging to different values of the Cartan subalgebra – at least they differ in one value of either the set of S^{ab} or the set of $\tilde{S}^{ab}$, Eq. (8.67) – are orthogonal with respect to the scalar product for a chosen vacuum state, defined as the integral over the Grassmann coordinates, Eq. (8.23). Correspondingly the states generated by the creation operators, Eq. (8.57), on the vacuum state, Eq. (8.59), are orthogonal as well (both last equations will appear later).

b. *Action in Grassmann space*

In Grassmann space we require – similarly as in the Clifford case – that the action for a free massless object

$$\mathcal{A} = \frac{1}{2}\{\int d^dx\, d^d\theta\, \omega\,(\phi^{\dagger}(1-2\theta^0\frac{\partial}{\partial\theta^0})\,\theta^a p_a\phi)\} + h.c.\,, \tag{8.33}$$

is Lorentz invariant. $p_a = i\,\frac{\partial}{\partial x_a}$. We use the integral also over θ^a coordinates, with the weight function ω from Eq. (8.23). Requiring the Lorentz invariance we add after $\phi^{\dagger}$ the operator $(1-2\theta^0\frac{\partial}{\partial\theta^0})$, which takes care of the Lorentz invariance. Namely

$$\begin{aligned} \mathbf{S}^{ab\dagger}\,(1-2\theta^0\frac{\partial}{\partial\theta^0}) &= (1-2\theta^0\frac{\partial}{\partial\theta^0})\,\mathbf{S}^{ab}\,, \\ \mathbf{S}^{\dagger}\,(1-2\theta^0\frac{\partial}{\partial\theta^0}) &= (1-2\theta^0\frac{\partial}{\partial\theta^0})\,\mathbf{S}^{-1}\,, \\ \mathbf{S} &= e^{-\frac{i}{2}\omega_{ab}(L^{ab}+\mathbf{S}^{ab})}\,. \end{aligned} \tag{8.34}$$

We also require that the action leads to the equations of motion

$$\begin{aligned} \theta^a p_a|\phi^{\theta}_i> &= 0\,, \\ \frac{\partial}{\partial\theta^a}\,p_a\,|\phi^{\theta}_i> &= 0\,, \end{aligned} \tag{8.35}$$

both equations leading to the same solution, and also to the Klein-Gordon equation

$$\{\theta^a p_a, \frac{\partial}{\partial\theta^b}\,p_b\}_+|\phi^{\theta}_i> = p^a p_a|\phi^{\theta}_i> = 0\,. \tag{8.36}$$

c. *We learned*:

In both spaces – in the Clifford and in the Grassmann space – there exists the action, which leads to the equationsof motion and to the corresponding Klein-Gordon equation.

We shall see that creation and annihilation operators in both spaces fulfill the anticommutation relations, required for fermions. But while the Clifford algebra defines spinors with the half integer eigenvalues of the Cartan subalgebra operators of the Lorentz algebra, the Grassmann algebra defines states with the integer eigenvalues of the Cartan subalgebra.

8.2.3 Second quantization of Grassmann vectors and Clifford vectors

States in the Grassmann space as well as states in the Clifford space are organized to be – within each of the two spaces – orthogonal and normalized with respect to Eq. (8.23). All the states in each of spaces are chosen to be eigenstates of the Cartan subalgebra – with respect to $\mathbf{S}^{ab}$ in the Grassmann space, and with respect to S^{ab} and $\tilde{S}^{ab}$ in the Clifford space, Eq. (8.67).

In both spaces the requirement that states are obtained by the application of creation operators on vacuum states – $\hat{b}^{\theta}_{i}$ obeying the commutation relations of Eq. (8.40) on the vacuum state $|\phi_{og}>=|1>$ for the Grassmann space, and $\hat{b}^{\alpha}_{i}$ obeying the commutation relation of Eq. (8.52) on the vacuum states $|\psi_{oc}>$, Eq. (8.59), for the Clifford space – reduces the number of states, in the Clifford space more than in the Grassmann space. But while in the Clifford space all physically applicable states are reachable either by S^{ab} or by $\tilde{S}^{ab}$, the states, belonging to different groups with respect to the Lorentz generators, seems not to be connected by the Lorentz operators in the Grassmann space.

Let us construct the creation and annihilation operators for the cases that we use **a.** the Grassmann vector space, or **b.** the Clifford vector space. We shall see that from 2^d states in either the Grassmann or the Clifford space (all are orthogonal among themselves with respect to the integral, Eq. (8.23)) – separately in each of the two spaces – there are reduced number of sates generated by the corresponding creation and annihilation operators, when products of Grassmann coordinates θ^a's and momenta $\frac{\partial}{\partial\theta^a}$ are required to represent creation and annihilation operators, and only $2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}$, Eq.(8.60), when products of nilpotents and projectors, Eq. (8.46), are chosen to generate creation and annihilation operators.

a. Quantization in Grassmann space

There are 2^d states in Grassmann space, orthogonal to each other with respect to Eq. (8.23). To any coordinate there exists the conjugate momentum. We pay attention in this paper to $2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1}+1)$ states, Eq. (8.43), when products of the superposition of the Grassmann coordinates, which are eigenstates of the Cartan subalgebra operators, are used to represent creation and their Hermitian conjugatde objects the annihilation operators. Let us see how it goes.

If $\hat{b}^{\theta\dagger}_{i}$ is a creation operator, which creates a state in the Grassmann space, when operating on a vacuum state $|\psi_{og}>$ and $\hat{b}^{\theta}_{i}=(\hat{b}^{\theta\dagger}_{i})^{\dagger}$ is the corresponding annihilation operator, then for a set of creation operators $\hat{b}^{\theta\dagger}_{i}$ and the corresponding annihilation operators $\hat{b}^{\theta}_{i}$ it must be

$$
\begin{aligned}
\hat{b}^{\theta}_{i}|\phi_{og}> &= 0\,,\\
\hat{b}^{\theta\dagger}_{i}|\phi_{og}> &\neq 0\,.
\end{aligned}
\tag{8.37}
$$

We first pay attention on only the internal degrees of freedom - the spin.

Choosing $\hat{b}^{\theta}_{a} = \frac{\partial}{\partial \theta^{a}}$ it follows

$$\begin{aligned}
\hat{b}^{\theta\dagger}_{a} &= \theta^{a}\,, \\
\hat{b}^{\theta}_{a} &= \frac{\partial}{\partial \theta^{a}}\,, \\
\{\hat{b}^{\theta}_{a}, \hat{b}^{\theta\dagger}_{b}\}_{+} &= \delta^{a}_{b}\,, \\
\{\hat{b}^{\theta}_{a}, \hat{b}^{\theta}_{b}\}_{+} &= 0\,, \\
\{\hat{b}^{\theta\dagger}_{a}, \hat{b}^{\dagger}_{b}\}_{+} &= 0\,, \\
\hat{b}^{\dagger\theta}_{a}\,|\phi_{og}> &= \theta^{a}|\phi_{og}>\,, \\
\hat{b}^{\theta}_{a}\,|\phi_{og}> &= 0\,.
\end{aligned} \tag{8.38}$$

The vacuum state $|\phi_{og}>$ is in this case $|1>$.

The identity I can not be taken as an creation operator, since its annihilation partner does not fulfill Eq. (8.37).

We can use the products of superposition of θ^{a}'s as creation and products of superposition of $\frac{\partial}{\partial \theta_{a}}$'s as annihilation operators provided that they fulfill the requirements for the creation and annihilation operators, Eq. (8.40), with the vacuum state $|\phi_{og}>=|1>$.

It is convenient to take products of superposition of vectors θ^{a} and θ^{b} to construct creation operators so that each factor is the eigenstate of one of the Cartan subalgebra member of the Lorentz algebra (8.67). We can start with the creation operators as products of $\frac{d}{2}$ states $\hat{b}^{\theta\dagger}_{a_i b_i} = \frac{1}{\sqrt{2}}(\theta^{a_i} \pm \epsilon\theta^{b_i})$. Then the corresponding annihilation operators are $\frac{d}{2}$ factors of $\hat{b}^{\theta}_{a_i b_i} = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta^{a_i}} \pm \epsilon^{*}\frac{\partial}{\partial\theta_{b_i}})$, $\epsilon = i$, if $\eta^{a_i a_i} = \eta^{b_i b_i}$ and $\epsilon = -1$, if $\eta^{a_i a_i} \neq \eta^{b_i b_i}$. Starting with the state $\hat{b}^{\theta\dagger}_{i} = (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^{0} - \theta^{3})(\theta^{1} + i\theta^{2})(\theta^{5} + i\theta^{6})\cdots(\theta^{d-1} + i\theta^{d})$ the rest of states belonging to the same Lorentz representation follows from the starting state by the aplication of the operators $\mathbf{S}^{cf}$, which do not belong to the Cartan subalgebra operators. It follows

$$\begin{aligned}
\hat{b}^{\theta\dagger}_{i} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^{0} - \theta^{3})(\theta^{1} + i\theta^{2})(\theta^{5} + i\theta^{6})\cdots(\theta^{d-1} + i\theta^{d})\,, \\
\hat{b}^{\theta}_{i} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\frac{\partial}{\partial\theta^{d-1}} + i\frac{\partial}{\partial\theta^{d}})\cdots(\frac{\partial}{\partial\theta^{0}} - \frac{\partial}{\partial\theta^{3}})\,,\,, \\
\hat{b}^{\theta\dagger}_{j} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^{0}\theta^{3} + i\theta^{1}\theta^{2})(\theta^{5} + i\theta^{6})\cdots(\theta^{d-1} + i\theta^{d})\,, \\
\hat{b}^{\theta}_{j} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\frac{\partial}{\partial\theta^{d-1}} + i\frac{\partial}{\partial\theta^{d}})\cdots(\frac{\partial}{\partial\theta^{3}}\frac{\partial}{\partial\theta^{0}} - i(\frac{\partial}{\partial\theta^{2}}\frac{\partial}{\partial\theta^{1}})\,. \\
&\cdots
\end{aligned} \tag{8.39}$$

It is taking into account that $\mathbf{S}^{01}$ transforms $(\frac{1}{\sqrt{2}})^{2}(\theta^{0}-\theta^{3})(\theta^{1}+i\theta^{2})$ into $\frac{1}{\sqrt{2}}(\theta^{0}\theta^{3}+i\theta^{1}\theta^{2})$ or any $\mathbf{S}^{ac}$, which does not belong to Cartan subalgebra, Eq.(8.65), transforms $(\frac{1}{\sqrt{2}})^{2}(\theta^{a} + i\theta^{b})(\theta^{c} + i\theta^{d})$ into $i\,\frac{1}{\sqrt{2}}(\theta^{a}\theta^{b} + \theta^{c}\theta^{d})$.

One finds that $\mathbf{S}^{ab}(\theta^{a} \pm \epsilon\theta^{b}) = \mp\,\frac{\eta^{aa}}{\epsilon}(\theta^{a} + \epsilon\theta^{b})$, while $\mathbf{S}^{ab}$ applied on $(\theta^{a}\theta^{b} \pm \epsilon\theta^{c}\theta^{d})$ gives zero.

Although all the states, generated by creation operators, which include one $(I \pm \epsilon\theta^a\theta^b)$ or several $(I \pm \epsilon\theta^{a_1}\theta^{b_1})\cdots(I \pm \epsilon\theta^{a_k}\theta^{a_k})$, are orthogonal with respect to the scalar product, Eq.(8.24), such creation operators do not have appropriate annihilation operators since $(I \pm \epsilon\theta^a\theta^b)$ and $(I \pm \epsilon^* \frac{\partial}{\partial\theta^b}\frac{\partial}{\partial\theta^a})$ (or several $(I \pm \epsilon\theta^{a_1}\theta^{b_1})\cdots(I \pm \epsilon\theta^{a_k}\theta^{b_k})$ and $(I \pm \epsilon^* \frac{\partial}{\partial\theta^{b_k}}\frac{\partial}{\partial\theta^{a_k}})\cdots(I \pm \epsilon^* \frac{\partial}{\partial\theta^{b_1}}\frac{\partial}{\partial\theta^{a_1}})$) do not fulfill Eqs. (8.37, 8.38), since I has no annihilation partner. However, creation operators which are products of one or several, let say n, of the kind $\theta^{a_i}\theta^{b_i}$ (at most $\frac{d}{2}$, each factor of them is the "eigenstate" of one of the Cartan subalgebra operators – $\mathbf{S}^{ab}\theta^a\theta^b|1>=0$), while the rest, $\frac{d}{2}-n$, have the "eigenvalues" either $(+1$ or $-1)$ or $(+i$ or $-i)$, fulfill relations

$$
\begin{aligned}
\{\hat{b}^{\theta}_i, \hat{b}^{\theta\dagger}_j\}_+|\phi_{og}> &= \delta^i_j\,|\phi_{og}>,\\
\{\hat{b}^{\theta}_i, \hat{b}^{\theta}_j\}_+|\phi_{og}> &= 0\,|\phi_{og}>,\\
\{\hat{b}^{\theta\dagger}_i, \hat{b}^{\dagger}_j\}_+|\phi_{og}> &= 0|\phi_{og}>,\\
\hat{b}^{\theta\dagger}_j|\phi_{og}> &= |\phi_j>\\
\hat{b}^{\theta}_j|\phi_{og}> &= 0\,|\phi_{og}>\,.
\end{aligned}
\tag{8.40}
$$

There are in $(d=2)$ two creation $((\theta^0 \mp \theta^1$, for $\eta^{ab}=diag(1,-1))$ and correspondingly two annihilation operators $(\frac{\partial}{\partial\theta^0} \mp \frac{\partial}{\partial\theta^1})$, and one creation operator $\theta^0\theta^1$ and the corresponding annihilation operator $\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^0}$, each belonging to its own group with respect to the Lorentz transformation operators, which fulfill Eq. (8.40), in $(d=4)$ there are two triplets of the kind presented in Eq. (8.39) of creation and correspondingly two triplets of annihilation operators, and four creation operators with one product of $\theta^{a_i}\theta^{b_i}$ multiplied by $(\theta^{c_i} \pm \theta^{d_i})$ and four corresponding annihilation operators as well as the creation operator $\theta^0\theta^3\theta^1\theta^2$ with the corresponding annihilation operator, they all fulfill Eq. (8.40).

Let us count the number of creation operators, when one starts with the creator, which is the product of $\frac{d}{2}$ factors, each with the "eigenvalue" of the Cartan subalgebra operators, Eq. (8.67), equal to either $+i$ or $+1$, Eq. (8.39):

$$\hat{b}^{\theta\dagger}_0 = (\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})(\theta^{d-1}+i\theta^d)\,. \tag{8.41}$$

There are $2^{\frac{d}{2}-1}$ creation operators of this type $\{(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})(\theta^{d-1}+i\theta^d)$, $(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})(\theta^{d-1}+i\theta^5)$, $(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})(\theta^{d-1}+i\theta^d), \cdots, (\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)\cdots(\theta^{d-1}-i\theta^5)\}$ with the eigenvalues of the Cartan subalgebra equal to $\{(+i,+1,+1,\ldots,+1,+1), (-i,-1,+1,\ldots,+1+1), (-i,+1,-1,\ldots,+1,+1),\cdots, (+i,+1,+1\ldots,-1,-1)\}$, each of the operators distinguishing from the others in one pair of factors with the opposite eigenvalues of the Cartan subalgebra operators.

There are in addition $2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1}-1)/2$ Grassmann odd operators obtained when $\mathbf{S}^{ef}$ apply on $(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})(\theta^{d-1}+i\theta^d)$, $(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})(\theta^{d-1}+i\theta^d)$ and on the rest of $2^{\frac{d}{2}-1}-1$ operators. $\mathbf{S}^{01}$ applied on $(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})(\theta^{d-1}+i\theta^d)$, $(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})(\theta^{d-1}+i\theta^d)$ gives $\propto (\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})(\theta^{d-1}+i\theta^d)$, $(\theta^0+\theta^3)(\theta^1-$

$i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^5))$. Each of these operators have two "eigenvalues" of the Cartan subalgebra equal to zero and all the rest equal to either $\pm i$ (if one of the two summands has $\eta^{aa} = 1$) or ± 1 (otherwise). All these creation operators are connected by $\mathbf{S}^{eg}$.

There are correspondingly all together $2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1} + 1)/2$ creation operators and the same number of annihilation operators (they follow from the creation operators by Hermitian conjugation, Eq. (8.8)), belonging to one group, so that all the operators follow from the starting one by the application of $\mathbf{S}^{af}$.

There is additional group of creation and annihilation operators, which follow from the starting one

$$\hat{b}_0^{\theta\dagger} = (\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d)\,. \tag{8.42}$$

(one can chose in the starting creation operator with changed sign in any of factors in the product, in each case the same group will follow). All the rest $2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1} + 1)/2$ creation operators can be obtained from the starting one as in the case of the first group.

There is therefore

$$2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1} + 1) \tag{8.43}$$

creation and the same number of annihilation operators, which are built on two starting states, presented in Eqs. (8.41, 8.42), divided in two groups, each generating or annihilating states belonging to the same representation of the Lorentz algebra.

The rest of creators (and the corresponding annihilators) have opposite Grassmann character than the ones studied so far – like $\theta^0\theta^1$ $(\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^0})$ in $d = (1+1)$ and in $d = (1+3)$ $\theta^0\theta^3(\theta^1 \pm i\theta^2)$ $(\frac{\partial}{\partial\theta^1} \mp i\frac{\partial}{\partial\theta^2})\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0})$, $\theta^1\theta^2(\theta^0 \mp i\theta^3)$ $((\frac{\partial}{\partial\theta^0} \pm i\frac{\partial}{\partial\theta^3})\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^2})$ and $\theta^0\theta^3\theta^1\theta^2$ $(\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0})$, which also fulfill the relations of Eq. (8.40).

All the states $|\phi_i^\theta>$, generated by the creation operators (presented in Eq. (8.40)) on the vacuum state $|\phi_{og}>$ are the eigenstates of the Cartan subalgebra operators and are orthogonal and normalized with respect to the norm of Eq. (8.23)

$$<\phi_i^\theta|\phi_j^\theta> = \delta^i j\,. \tag{8.44}$$

If we now extend the creation and annihilation operators to the ordinary coordinate space, the relation among creation and annihilation operators at one time read

$$\begin{aligned}
\{\hat{b}_i^\theta(\vec{x}), \hat{b}_j^{\theta\dagger}(\vec{x}')\}_+|\phi_{og}> &= \delta_j^i\ \delta(\vec{x} - \vec{x}')|\phi_{og}>\,,\\
\{\hat{b}_i^\theta(\vec{x}), \hat{b}_j^{\theta}(\vec{x}')\}_+|\phi_{og}> &= 0\ |\phi_{og}>\,,\\
\{\hat{b}_i^{\theta\dagger}(\vec{x}), \hat{b}_j^{\theta\dagger}(\vec{x}')\}_+|\phi_{og}> &= 0\ |\phi_{og}>\,,\\
\hat{b}_j^{\theta\dagger}(\vec{x})|\phi_{og}> &= 0\ |\phi_{og}>\\
|\phi_{og}> &= |1>\ .
\end{aligned} \tag{8.45}$$

b. Quantization in Clifford space

In Grassmann space the requirement that products of eigenstates of the Cartan subalgebra operators represent the creation and annihilation operators, obeying the relation Eq. (8.40), reduces the number of states. Let us study what happens, when, let say, γ^a's are used to create the basis and correspondingly also to create the creation and annihilation operators.

Let us point out that γ^a is expressible with θ^a and its its deriative ($\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$), Eq. (8.14), and that we again require that creation (annihilation) operators create (annihilate) states, which are eigenstates of the Cartan subalgebra, Eq. (8.67). We could as well make a choice of $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta_a})$ [3]. We shall follow here to some extend Ref. [15].

Making a choice of the Cartan subalgebra eigenstates of S^{ab}, Eq. (8.67),

$$\stackrel{ab}{(k)}:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \qquad \stackrel{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \tag{8.46}$$

where $k^2 = \eta^{aa}\eta^{bb}$, recognizing that the Hermitian conjugate values of $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$ are

$$\stackrel{ab}{(k)}{}^\dagger = \eta^{aa} \stackrel{ab}{(-k)}, \quad \stackrel{ab}{[k]}{}^\dagger = \stackrel{ab}{[k]}\,, \tag{8.47}$$

while the corresponding eigenvalues of S^{ab}, Eq. (8.48), and $\tilde{S}^{ab}$, Eq. (8.85), are

$$S^{ab} \stackrel{ab}{(k)} = \frac{1}{2}k \stackrel{ab}{(k)}\,, \quad S^{ab} \stackrel{ab}{[k]} = \frac{1}{2}k \stackrel{ab}{[k]}$$
$$\tilde{S}^{ab} \stackrel{ab}{(k)} = \frac{k}{2} \stackrel{ab}{(k)}\,, \quad \tilde{S}^{ab} \stackrel{ab}{[k]} = -\frac{k}{2} \stackrel{ab}{[k]}\,. \tag{8.48}$$

We find in $d = 2(2n+1)$ that from the starting state with products of odd number of only nilpotents

$$|\psi^1_1 >|_{2(2n+1)} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{35}{(+)} \cdots \stackrel{d-3\, d-2}{(+)} \stackrel{d-1\, d}{(+)} |\psi_{oc}>\,, \tag{8.49}$$

having correspondingly an odd Clifford character [4], all the other states of the same Lorentz representation, there are $2^{\frac{d}{2}-1}$ members, follow by the application of S^{cd} [5], which do not belong to the Cartan subalgebra, Eq. (8.67): $S^{cd}\, |\psi^1_1 >|_{2(2n+1)} = |\psi^1_i >|_{2(2n+1)}$. The operators $\tilde{S}^{cd}$, which do not belong to the Cartan subalgebra of

[3] We choose γ^a's, Eq.(8.14) to create the basic states. We could instead make a choice of $\tilde{\gamma}^a$'s, Eq.(8.15) to create the basic states. In the case of this latter choice the role of $\tilde{\gamma}^a$ and γ^a should be correspondingly exchanged in Eq. (8.74).

[4] We call the starting state in $d = 2(2n+1)$ $|\psi^1_1 >|_{2(2n+1)}$, and the starting state in $d = 4n$ $|\psi^1_1 >|_{4n}$.

[5] The smallest number of all the generators S^{ac}, which do not belong to the Cartan subalgebra, needed to create from the starting state all the other members is $2^{\frac{d}{2}-1} - 1$. This is true for both even dimensional spaces – $2(2n+1)$ and $4n$.

$\tilde{S}^{ab}$, Eq. (8.67), generate states with different eigenstates of the Cartan subalgebra ($\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\,d}$), we call the eigenvalues of their eigenstates the "family" quantum numbers. There are $2^{\frac{d}{2}-1}$ families. From the starting new member with a different "family" quantum number the whole Lorentz representation with this "family" quantum number follows by the application of S^{ef}: $S^{ef}\,\tilde{S}^{cd}|\psi^1_1>|_{2(2n+1)} = |\psi^j_i>|_{2(2n+1)}$. All the states of one Lorentz representation of any particular "family" quantum number have an odd Clifford character, since neither S^{cd} nor $\tilde{S}^{cd}$, both with an even Clifford character, can change this character. We shall comment our limitation of states to only those with an odd Clifford character after defining the creation and annihilation operators.

For $d = 4n$ the starting state must be the product of one projector and $4n-1$ nilpotents, since we again limit states to those with an odd Clifford character. Let us start with the state

$$|\psi^1_1>|_{4n} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{35}{(+)}\cdots\stackrel{d-3\,d-2}{(+)}\stackrel{d-1\,d}{[+]}\;|\psi_{oc}>, \qquad (8.50)$$

All the other states belonging to the same Lorentz representation follow again by the application of S^{cd} on this state $|\psi^1_1>|_{4n}$, while a new family starts by the application of $\tilde{S}^{cd}|\psi^1_1>|_{4n}$ and from this state all the other members with the same "family" quantum number can be generated by $S^{ef}\tilde{S}^{cd}$ on $|\psi^1_1>|_{4n}$: $S^{ef}\tilde{S}^{cd}\,|\psi^1_1>|_{4n} = |\psi^j_i>|_{4n}$.

All these states in either $d = 2(2n+1)$ space or $d = 4n$ space are orthogonal with respect to Eq. (8.23).

However, let us point out that $(\gamma^a)^\dagger = \gamma^a\eta^{aa}$. Correspondingly it follows, Eq. (8.47), that $\stackrel{ab}{(k)}^\dagger = \eta^{aa}\stackrel{ab}{(-k)}$, and $\stackrel{ab}{[k]}^\dagger = \stackrel{ab}{[k]}$.

Since any projector is Hermitian conjugate to itself, while to any nilpotent $\stackrel{ab}{(k)}$ the Hermitian conjugated one has an opposite k, it is obvious that Hermitian conjugated product to a product of nilpotents and projectors can not be accepted as a new state [6].

The vacuum state $|\psi_{oc}>$ ought to be chosen so that $<\psi_{oc}|\psi_{oc}>= 1$, while all the states belonging to the physically acceptable states, like $\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\stackrel{78}{[-]}\cdots\stackrel{d-3\,d-2}{(+)}\stackrel{d-1\,d}{(+)}|\psi_{oc}>$, must not give zero for either $d = 2(2n+1)$ or for $d = 4n$. We also want that the states, obtained by the application of ether S^{cd} or $\tilde{S}^{cd}$ or both, are orthogonal. To make a choice of the vacuum it is needed to know the

[6] We could as well start with the state $|\psi^1_1>|_{2(2n+1)} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{35}{(+)}\cdots\stackrel{d-3\,d-2}{(+)}\stackrel{d-1\,d}{(+)}|\psi_{oc}>$ for $d = 2(2n+1)$ and with $|\psi^1_1>|_{4n} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{35}{(+)}\cdots\stackrel{d-3\,d-2}{(+)}\stackrel{d-1\,d}{[+]}|\psi_{oc}>$ in the case of $d = 4n$. Then creation and annihilation operators will exchange their roles.

relations of Eq. (8.71). It must be

$$
\begin{aligned}
&< \psi_{oc}| \cdots \overset{ab}{(k)}^{\dagger} \cdots | \cdots \overset{ab}{(k')} \cdots |\psi_{oc} > = \delta_{kk'}\,,\\
&< \psi_{oc}| \cdots \overset{ab}{[k]}^{\dagger} \cdots | \cdots \overset{ab}{[k']} \cdots |\psi_{oc} > = \delta_{kk'}\,,\\
&< \psi_{oc}| \cdots \overset{ab}{[k]}^{\dagger} \cdots | \cdots \overset{ab}{(k')} \cdots |\psi_{oc} > = 0\,.
\end{aligned}
\tag{8.51}
$$

Our experiences in the case, when states with the integer values of the Cartan subalgebra operators were expressed by Grassmann coordinates, teach us that the requirements, which creation and annihilation operators must fulfill, influence the choice of the number of states, as well as of the vacuum state.

Let us first repeat therefore the requirements which the creation and annihilation operators must fulfill

$$
\begin{aligned}
\{\hat{b}^{\alpha\gamma}_{i}, \hat{b}^{\beta\gamma\dagger}_{k}\}_{+}|\psi_{oc}> &= \delta^{\alpha}_{\beta}\,\delta^{i}_{k}|\psi_{oc}>\,,\\
\{\hat{b}^{\alpha\gamma}_{i}, \hat{b}^{\beta\gamma}_{k}\}_{+}|\psi_{oc}> &= 0|\psi_{oc}>\,,\\
\{\hat{b}^{\alpha\gamma\dagger}_{i}, \hat{b}^{\beta\gamma\dagger}_{k}\}_{+}|\psi_{oc}> &= 0|\psi_{oc}>\,,\\
\hat{b}^{\alpha\gamma\dagger}_{i}|\psi_{oc}> &= 0|\psi_{oc}>\,,
\end{aligned}
\tag{8.52}
$$

paying attention at this stage only at the internal degrees of freedom of the states, that is on their spins. Here $(\alpha, \beta, \dots)$ represent the family quantum number determined by $\tilde{S}^{ac}$ and $(i, j, \dots)$ the quantum number of one representation, determined by S^{ac}. From Eqs. (8.49, 8.50) is not difficult to extract the creation operators which, when applied on the two vacuum states, generate the starting states.

i. *One Weyl representation*

We define the creation $\hat{b}^{1\dagger}_{1}$ – and the corresponding annihilation operator $\hat{b}^{1}_{1}$, $(\hat{b}^{1\dagger}_{1})^{\dagger} = \hat{b}^{1}_{1}$ – which when applied on the vacuum state $|\psi_{oc}>$ create a vector of one of the two equations (8.49, 8.50), as follows

$$
\begin{aligned}
\hat{b}^{1\dagger}_{1} &:= \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} \cdots \overset{d-1\;d}{(+)}\,,\\
\hat{b}^{1}_{1} &:= \overset{d-1\;d}{(-)} \cdots \overset{56}{(-)}\overset{12}{(-)}\overset{03}{(-i)}\,,\\
&\text{for } d = 2(2n+1)\,,\\
\hat{b}^{1\dagger}_{1} &:= \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} \cdots \overset{d-3\;d-2}{(+)}\overset{d-1\;d}{[+]}\,,\\
\hat{b}^{1}_{1} &:= \overset{d-1,d}{[+]}\overset{d-2\;d-3}{(-)} \cdots \overset{56}{(-)}\overset{12}{(-)}\overset{03}{(-i)}\,,\\
&\text{for } d = 4n\,.
\end{aligned}
\tag{8.53}
$$

We shall call this vector the starting vector of the starting "family".

Now we can make a choice of the vacuum state for this particular "family" taking into account Eq. (8.71)

$$
\begin{aligned}
|\psi_{oc}> &= \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]}\cdots\overset{d-1\,d}{[-]}\;|0>, \\
&\quad \text{for } d = 2(2n+1), \\
|\psi_{oc}> &= \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]}\cdots\overset{d-3\,d-2}{[-]}\;\overset{d-1\,d}{[+]}\;|0>, \\
&\quad \text{for } d = 4n,
\end{aligned} \tag{8.54}
$$

n is a positive integer, so that the requirements of Eq. (8.52) are fulfilled. We see: The creation and annihilation operators of Eq. (8.53) (both are nilpotents, $(\hat{b}^{1\dagger}_1)^2 = 0$ and $(\hat{b}^{1}_1)^2 = 0$), $\hat{b}^{1\dagger}_1$ (generating the vector $|\psi^1_1>$ when operating on the vacuum state) gives $\hat{b}^{1\dagger}_1|\psi_{oc}>\neq 0$, while the annihilation operator annihilates the vacuum state $\hat{b}^1_1|\psi_0>= 0$, giving $\{\hat{b}^1_1, \hat{b}^{1\dagger}_1\}_+|\psi_{oc}>=|\psi_{oc}>$, since we choose the appropriate normalization, Eq. (8.46).

All the other creation and annihilation operators, belonging to the same Lorentz representation with the same family quantum number, follow from the starting ones by the application of particular S^{ac}, which do not belong to the Cartan subalgebra (8.65).

We call $\hat{b}^{1\dagger}_2$ the one obtained from $\hat{b}^{1\dagger}_1$ by the application of one of the four generators ($S^{01}, S^{02}, S^{31}, S^{32}$). This creation operator is for $d = 2(2n+1)$ equal to $\hat{b}^{1\dagger}_2 = \overset{03}{[-i]}\overset{12}{[-]}\overset{35}{(+)}\cdots\overset{d-1\,d}{(+)}$, while it is for $d = 4n$ equal to $\hat{b}^{1\dagger}_2 = \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)}\cdots\overset{d-1\,d}{[+]}$. All the other family members follow from the starting one by the application of different S^{ef}, or by the product of several S^{gh}.

We accordingly have

$$
\begin{aligned}
\hat{b}^{1\dagger}_i &\propto S^{ab}..S^{ef}\hat{b}^{1\dagger}_1, \\
\hat{b}^{1}_i &\propto \hat{b}^{1}_1 S^{ef}..S^{ab},
\end{aligned} \tag{8.55}
$$

with $S^{ab\dagger} = \eta^{aa}\eta^{bb}S^{ab}$. We shall make a choice of the proportionality factors so that the corresponding states $|\psi^1_1>=\hat{b}^{1\dagger}_i|\psi_{oc}>$ will be normalized.
We recognize that [15]:
i.a. $(\hat{b}^{1\dagger}_i)^2 = 0$ and $(\hat{b}^{1}_i)^2 = 0$, for all i.

To see this one must recognize that S^{ac} (or S^{bc}, S^{ad}, S^{bd}) transforms $\overset{ab}{(+)}\overset{cd}{(+)}$ to $\overset{ab}{[-]}\overset{cd}{[-]}$, that is an even number of nilpotents $(+)$ in the starting state is transformed into projectors $[-]$ in the case of $d = 2(2n+1)$. For $d = 4n$, S^{ac} (or S^{bc}, S^{ad}, S^{bd}) transforms $\overset{ab}{(+)}\overset{cd}{[+]}$ into $\overset{ab}{[-]}\overset{cd}{(-)}$. Therefore for either $d = 2(2n+1)$ or $d = 4n$ at least one of factors, defining a particular creation operator, will be a nilpotent. For $d = 2(2n+1)$ there is an odd number of nilpotents, at least one, leading from the starting factor $(\overset{dg}{(+)})$ in the creator. For $d = 4n$ a nilpotent factor can also be $\overset{d-1\,d}{(-)}$ (since $\overset{d-1\,d}{[+]}$ can be transformed by $S^{e\,d-1}$, for example into $\overset{d-1\,d}{(-)}$). A square of at least one nilpotent factor (we started with an odd number of nilpotents, and oddness can not be changed by S^{ab}), is enough to guarantee that the square of the corresponding $(\hat{b}_i^{1\dagger})^2$ is zero. Since $\hat{b}_i^1 = (\hat{b}_i^{1\dagger})^\dagger$, the proof is valid also for annihilation operators.

i.b. $\hat{b}_i^{1\dagger}|\psi_{oc}> \neq 0$ and $\hat{b}_i^1|\psi_{oc}> = 0$, for all i.

To see this in the case $d = 2(2n+1)$ one must recognize that $\hat{b}_i^{1\dagger}$ distinguishes from $\hat{b}_1^{1\dagger}$ in (an even number of) those nilpotents $(+)$, which have been transformed into $[-]$. When $\stackrel{ab}{[-]}$ from $\hat{b}_i^{1\dagger}$ meets $\stackrel{ab}{[-]}$ from $|\psi_{oc}>$, the product gives $\stackrel{ab}{[-]}$ back, and correspondingly a nonzero contribution. For $d = 4n$ also the factor $\stackrel{d-1\,d}{[+]}$ can be transformed. It is transformed into $\stackrel{d-1\,d}{(-)}$ which, when applied to a vacuum state, gives again a nonzero contribution ($\stackrel{d-1\,d}{(-)}\stackrel{d-1\,d}{[+]} = \stackrel{d-1\,d}{(-)}$, Eq.(8.71)).

In the case of $\hat{b}_i^1$ we recognize that in $\hat{b}_i^{1\dagger}$ at least one factor is nilpotent; that of the same type as in the starting $\hat{b}_1^\dagger$ – $(+)$ – or in the case of $d = 4n$ it can be also $\stackrel{d-1\,d}{(-)}$. Performing the Hermitian conjugation $(\hat{b}_i^{1\dagger})^\dagger$, $(+)$ transforms into $(-)$, while $\stackrel{d-1\,d}{(-)}$ transforms into $\stackrel{d-1\,d}{(+)}$ in $\hat{b}_i^1$. Since $(-)[-]$ gives zero and $\stackrel{d-1\,d}{(+)}\stackrel{d-1\,d}{[+]}$ also gives zero, $\hat{b}_i^1|\psi_{oc}> = 0$.

i.c. $\{\hat{b}_i^{1\dagger}, \hat{b}_j^{1\dagger}\}_+ = 0$, for each pair (i, j).

There are several possibilities, which we have to discuss. A trivial one is, if both $\hat{b}_i^{1\dagger}$ and $\hat{b}_j^{1\dagger}$ have a nilpotent factor (or more than one) for the same pair of indexes, say $\stackrel{kl}{(+)}$. Then the product of such two $\stackrel{kl}{(+)}\stackrel{kl}{(+)}$ gives zero. It also happens, that $\hat{b}_i^{1\dagger}$ has a nilpotent at the place (kl) ($\stackrel{03}{[-]} \cdots \stackrel{kl}{(+)} \cdots \stackrel{mn}{[-]} \cdots$) while $\hat{b}_j^{1\dagger}$ has a nilpotent at the place (mn) ($\stackrel{03}{[-]} \cdots \stackrel{kl}{[-]} \cdots \stackrel{mn}{(+)} \cdots$). Then in the term $\hat{b}_i^{1\dagger}\hat{b}_j^{1\dagger}$ the product $\stackrel{mn}{[-]}\stackrel{mn}{(+)}$ makes the term equal to zero, while in the term $\hat{b}_j^{1\dagger}\hat{b}_i^{1\dagger}$ the product $\stackrel{kl}{[-]}\stackrel{kl}{(+)}$ makes the term equal to zero. There is no other possibility in $d = 2(2n+1)$. In the case that $d = 4n$, it might appear also that $\hat{b}_i^{1\dagger} = \stackrel{03}{[-]} \cdots \stackrel{ij}{(+)} \cdots \stackrel{d-1\,d}{[+]}$ and $\hat{b}_j^{1\dagger} = \stackrel{03}{[-]} \cdots \stackrel{ij}{[-]} \cdots \stackrel{d-1\,d}{(-)}$. Then in the term $\hat{b}_i^{1\dagger}\hat{b}_j^{1\dagger}$ the factor $\stackrel{d-1\,d}{[+]}\stackrel{d-1\,d}{(-)}$ makes it zero, while in $\hat{b}_j^{1\dagger}\hat{b}_i^{1\dagger}$ the factor $\stackrel{ij}{[-]}\stackrel{ij}{(+)}$ makes it zero. Since there are no further possibilities, the proof is complete.

i.d. $\{\hat{b}_i^1, \hat{b}_j^1\}_+ = 0$, for each pair (i, j).

The proof goes similarly as in the case with creation operators. Again we treat several possibilities. $\hat{b}_i^1$ and $\hat{b}_j^1$ have a nilpotent factor (or more than one) with the same indexes, say $\stackrel{kl}{(-)}$. Then the product of such two $\stackrel{kl}{(-)}\stackrel{kl}{(-)}$ gives zero. It also happens, that $\hat{b}_i^1$ has a nilpotent at the place (kl) ($\cdots \stackrel{mn}{[-]} \cdots \stackrel{kl}{(-)} \cdots \stackrel{03}{[-]}$) while $\hat{b}_j^1$ has a nilpotent at the place (mn) ($\cdots \stackrel{mn}{(-)} \cdots \stackrel{kl}{[-]} \cdots \stackrel{03}{[-]}$). Then in the term $\hat{b}_i^1\hat{b}_j^1$ the product $\stackrel{kl}{(-)}\stackrel{kl}{[-]}$ makes the term equal to zero, while in the term $\hat{b}_j^1\hat{b}_i^1$ the product $\stackrel{mn}{(-)}\stackrel{mn}{[-]}$ makes the term equal to zero. In the case that $d = 4n$, it appears also that

$\hat{b}^1_i = \stackrel{d-1\, d}{[+]} \cdots \stackrel{ij}{(-)} \cdots \stackrel{03}{[-]}$ and $\hat{b}^1_j = \stackrel{d-1\, d}{(+)} \cdots \stackrel{ij}{[-]} \cdots \stackrel{03}{[-]}$. Then in the term $\hat{b}^1_i \hat{b}^1_j$ the factor $\stackrel{ij}{(-)}\stackrel{ij}{[-]}$ makes it zero, while in $\hat{b}^1_j \hat{b}^1_i$ the factor $\stackrel{d-1\, d}{(+)}\stackrel{d-1\, d}{[+]}$ makes it zero.

i.e. $\{\hat{b}^1_i, \hat{b}^{1\dagger}_j\}_+ |\psi oc >= \delta_{ij} |\psi_{oc}>$.

To prove this we must recognize that $\hat{b}^1_i = \hat{b}_1 S^{ef}..S^{ab}$ and $\hat{b}^{1\dagger}_i = S^{ab}..S^{ef}\hat{b}_1$. Since any $\hat{b}^1_i |\psi_{oc}>= 0$, we only have to treat the term $\hat{b}^1_i \hat{b}^{1\dagger}_j$. We find $\hat{b}^1_i \hat{b}^{1\dagger}_j \propto \cdots \stackrel{lm}{(-)} \cdots \stackrel{03}{(-)} S^{ef} \cdots S^{ab} S^{lm} \cdots S^{pr} \stackrel{03}{(+)} \cdots \stackrel{lm}{(+)} \cdots$. If we treat the term $\hat{b}^1_i \hat{b}^{1\dagger}_i$, generators $S^{ef} \cdots S^{ab} S^{lm} \cdots S^{pr}$ are proportional to a number and we normalize $< \psi_0 | \hat{b}^1_i \hat{b}^{1\dagger}_i |\psi_{oc}>$ to one. When $S^{ef} \cdots S^{ab} S^{lm} \cdots S^{pr}$ are proportional to several products of S^{cd}, these generators change $\hat{b}^{1\dagger}_1$ into $\stackrel{03}{(+)} \cdots \stackrel{kl}{[-]} \cdots \stackrel{np}{[-]} \cdots$, making the product $\hat{b}^1_i\, \hat{b}^{1\dagger}_j$ equal to zero, due to factors of the type $\stackrel{kl}{(-)}\stackrel{kl}{[-]}$. In the case of $d = 4n$ also a factor $\stackrel{d-1\, d}{[+]}\stackrel{d-1\, d}{(-)}$ might occur, which also gives zero.

We saw and proved that for the definition of the creation and annihilation operators in Eqs.(8.49,8.50) all the requirements of Eq. (8.52) are fulfilled, provided that creation and correspondingly also the annihilation operators have an odd Clifford character, that is that the number of nilpotents in the product is odd.

For an even number of factors of the nilpotent type in the starting state and accordingly in the starting $\hat{b}^{1\dagger}_1$*, an annihilation operator* $\hat{b}^1_i$ *would appear with all factors of the type* $[-]$*, which on the vacuum state (Eq.(8.54)) would not give zero.*

ii. *Families of Weyl representations*

Let $\hat{b}^{\alpha\dagger}_i$ be a creation operator, fulfilling Eq. (8.52), which creates one of the $(2^{d/2-1})$ Weyl basic states of an $\alpha-$th "family", when operating on a vacuum state $|\psi_{oc}>$ and let $\hat{b}^{\alpha}_i = (\hat{b}^{\alpha\dagger}_i)^\dagger$ be the corresponding annihilation operator. We shall now proceed to define $\hat{b}^{\alpha\dagger}_i$ and $\hat{b}^{\alpha}_i$ from a chosen starting state (8.49, 8.50), which $\hat{b}^{1\dagger}_1$ creates on the vacuum state $|\psi_{oc}>$.

When treating more than one Weyl representation, that is, more than one "family", we must take into account that: **i.** The vacuum state chosen to fulfill requirements for second quantization of the starting family might not and it will not be the correct one when all the families are taken into account. **ii.** The products of $\tilde{S}^{ab}$, which do not belong to the Cartan subalgebra set of the generators $\tilde{S}^{ab}$ $(2^{d/2-1} - 1$ of them), when being applied on the starting family ψ^1_1, generate the starting members ψ^{α}_1 of all the rest of the families. There are correspondingly the same number of "families" as there is the number of vectors of one Weyl representation, namely $2^{d/2-1}$. Then the whole Weyl representations of a particular family ψ^{α}_1 follows again with the application of S^{ef}, which do not belong to the Cartan subalgebra of S^{ab} on this starting family.

Any vector $|\psi^{\alpha}_i>$ follows from the starting vector (Eqs.8.49, 8.50) by the application of either $\tilde{S}^{ef}$, which change the family quantum number, or S^{gh}, which change the member of a particular family (as it can be seen from Eqs. (8.73, 8.86)) or with the corresponding product of S^{ef} and $\tilde{S}^{ef}$

$$|\psi^{\alpha}_i> \propto \tilde{S}^{ab} \cdots \tilde{S}^{ef} |\psi^1_i> \propto \tilde{S}^{ab} \cdots \tilde{S}^{ef} S^{mn} \cdots S^{pr} |\psi^1_1> . \qquad (8.56)$$

Correspondingly we define $\hat{b}_i^{\alpha\dagger}$ (up to a constant) to be

$$\begin{aligned} \hat{b}_i^{\alpha\dagger} &\propto \tilde{S}^{ab}\cdots\tilde{S}^{ef} S^{mn}\cdots S^{pr}\hat{b}_1^{1\dagger} \\ &\propto S^{mn}\cdots S^{pr}\hat{b}_1^{1\dagger} S^{ab}\cdots S^{ef}\,. \end{aligned} \tag{8.57}$$

This last expression follows due to the property of the Clifford object $\tilde{\gamma}^a$ and correspondingly of $\tilde{S}^{ab}$, presented in Eqs. (8.74, 8.75).

For $\hat{b}_i^{\alpha} = (\hat{b}_i^{\alpha\dagger})^\dagger$ we accordingly have

$$\hat{b}_i^{\alpha} = (\hat{b}_i^{\alpha\dagger})^\dagger \propto S^{ef}\cdots S^{ab}\hat{b}_1^{1} S^{pr}\cdots S^{mn}\,. \tag{8.58}$$

The proportionality factor will be chosen so that the corresponding states $|\psi_i^{\alpha}>= \hat{b}_i^{\alpha\dagger}|\psi_{oc}>$ will be normalized.

We ought to generalize the vacuum state from Eq. (8.54) so that $\hat{b}_i^{\alpha\dagger}\,|\psi_{oc}>\neq 0$ and $\hat{b}_i^{\alpha}\,|\psi_{oc}>=0$ for all the members i of any family α. Since any $\tilde{S}^{eg}$ changes $\stackrel{ef}{(+)}\stackrel{gh}{(+)}$ into $\stackrel{ef}{[+]}\stackrel{gh}{[+]}$ and $(\stackrel{ab}{[+]})^\dagger=\stackrel{ab}{[+]}$, while $(\stackrel{ab}{(+)})^\dagger\ \stackrel{ab}{(+)}=\stackrel{ab}{[-]}$, the vacuum state $|\psi_{oc}>$ from Eq. (8.54) must be replaced by

$$\begin{aligned} &|\psi_{oc}>= \\ &\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]}\cdots\stackrel{d-1\,d}{[-]} + \cdots|0>, \\ &\text{for } d=2(2n+1), \\ &|\psi_{oc}>= \\ &\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]} + \cdots|0>, \\ &\text{for } d=4n, \end{aligned} \tag{8.59}$$

n is a positive integer. There are $2^{\frac{d}{2}-1}$ summands. since we step by step replace all possible pairs of $\stackrel{ab}{[-]}\cdots\stackrel{ef}{[-]}$ in the starting part $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-1\,d}{[-]}$ (or $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]}$) into $\stackrel{ab}{[+]}\cdots\stackrel{ef}{[+]}$ and include new terms into the vacuum state so that the last $2n+1$ summands have for $d=2(2n+1)$ case, n is a positive integer, only one factor $[-]$ and all the rest $[+]$, each $[-]$ at different position. For $d=4n$ also the factor $\stackrel{d-1\,d}{[+]}$ in the starting term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]}$ changes to $\stackrel{d-1\,d}{[-]}$. The vacuum state has then the normalization factor $1/\sqrt{2^{d/2-1}}$. There is therefore

$$2^{\frac{d}{2}-1}\,2^{\frac{d}{2}-1} \tag{8.60}$$

number of creation operators, defining the orthonormalized states when applaying on the vacuum state of Eqs. (8.59) and the same number of annihilation operators, which are defined by the creation operators on the vacuum state of Eqs. (8.59). $\tilde{S}^{ab}$ connect members of different families, S^{ab} generates all the members of one family.

We recognize that:

ii.a. The above creation and annihilation operators are nilpotent – $(\hat{b}_i^{a\dagger})^2=0=$

$(\hat{b}^a_i)^2$ – since the "starting" creation operator $\hat{b}^{1\dagger}_1$ and annihilation operator $\hat{b}^a_i$ are both made of the product of an odd number of nilpotents, while products of either S^{ab} or $\tilde{S}^{ab}$ can change an even number of nilpotents into projectors. Any $\hat{b}^{a\dagger}_i$ is correspondingly a factor of an odd number of nilpotents (at least one) (and an even number of projectors) and its square is zero. The same is true for $\hat{b}^a_i$.
ii.b. All the creation operators operating on the vacuum state of Eq.(8.59) give a non zero vector – $\hat{b}^{a\dagger}_i|\psi_{oc}>\neq 0$ – while all the annihilation operators annihilate this vacuum state – $\hat{b}^a_i|\psi_0>$ for any α and any i.
It is not difficult to see that $\hat{b}^a_i|\psi_{oc}>=0$, for any α and any i. First we recognize that whatever the set of factors $S^{mn}\cdots S^{pr}$ appear on the right hand side of the annihilation operator $\hat{b}^1_1$ in Eq.(8.58), it lives at least one factor $[-]$ unchanged. Since $\hat{b}^1_1$ is the product of only nilpotents $(-)$ and since $\stackrel{ab}{(-)}\stackrel{ab}{[-]}=0$, this part of the proof is complete.
Let us prove now that $\hat{b}^{\alpha\dagger}_i|\psi_{oc}>\neq 0$ for each α, i. According to Eq.(8.57) the operation S^{mn} on the left hand side of $\hat{b}^{1\dagger}_1$, with m, n, which does not belong to the Cartan subalgebra set of indices, transforms the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{[-]}\cdots\stackrel{nk}{[-]}\cdots\stackrel{d-1\,d}{[-]}$ (or the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{[-]}\cdots\stackrel{nk}{[-]}\cdots\cdots\stackrel{d-1\,d}{[+]}$) into the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{(+)}\cdots\stackrel{nk}{(+)}\cdots\stackrel{d-1\,d}{[-]}$ (or the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{(+)}\cdots\stackrel{nk}{(+)}\cdots\cdots\stackrel{d-1\,d}{[+]}$) and $\hat{b}^{1\dagger}_1$ on such a term gives zero, since $\stackrel{lm}{(+)}\stackrel{lm}{(+)}=0$ and $\stackrel{nk}{(+)}\stackrel{nk}{(+)}=0$. Let us first assume that S^{mn} is the only term on the right hand side of $\hat{b}^{1\dagger}_1$ and that none of the operators from the left hand side of $\hat{b}^{1\dagger}_1$ in Eq.(8.57) has the indices m, n. It is only one term among all the summands in the vacuum state (Eq.8.59), which gives non zero contribution in this particular case, namely the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{[+]}\cdots\stackrel{nk}{[+]}\cdots\stackrel{d-1\,d}{[-]}$ (or the term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{lm}{[+]}\cdots\stackrel{nk}{[+]}\cdots\cdots\stackrel{d-1\,d}{[+]}$). S^{mn} transforms the part $\cdots\stackrel{lm}{[+]}\cdots\stackrel{nk}{[+]}\cdots$ into $\cdots\stackrel{lm}{(-)}\cdots\stackrel{nk}{(-)}\cdots$ and since $\stackrel{lm}{(+)}\stackrel{lm}{(-)}$ gives $\eta^{ll}\stackrel{lm}{[+]}$, while for the rest of factors it was already proven that such a factor on $\hat{b}^{1\dagger}_1$ forms a $b^{1\dagger}_i$ giving non zero contribution on the vacuum (8.54).
We also proved that what ever other S^{ab} but S^{mn} operate on the left hand side of $\hat{b}^{1\dagger}_1$ the contribution of this particular part of the vacuum state is nonzero. If the operators on the left hand side have the indexes m or n or both, the contribution on this term of the vacuum will still be nonzero, since then such a S^{mp} will transform the factor $\stackrel{lm}{(+)}$ in $\hat{b}^{1\dagger}_1$ into $\stackrel{lm}{[-]}$ and $\stackrel{lm}{[-]}\stackrel{lm}{(-)}$ is nonzero, Eq. (8.71).
The vacuum state has a term which guarantees a non zero contribution for any possible set of $S^{mn}\cdots S^{pr}$ operating from the right hand side of $\hat{b}^{1\dagger}_1$ (that is for each family) (which we achieved just by the transformation of all possible pairs of $\stackrel{cd}{[-]},\stackrel{gh}{[-]}$ into $\stackrel{cd}{[+]},\stackrel{gh}{[+]}$), the proof that $\hat{b}^{\alpha\dagger}_i$ operating on the vacuum $|\psi_{oc}>$ of Eq. (8.59) gives nonzero contribution. Among $[-]$ also $\stackrel{03}{[-i]}$ is understood.
It is not difficult to see that for each "family" of $2^{\frac{d}{2}-1}$ families it is only one term among all the summands in the vacuum state $|\psi_{oc}>$ of Eq. (8.59), which give a nonzero contribution, since when ever $[+]$ appears on a wrong position, that

is on the position, so that the product of $\stackrel{ab}{(+)}$ from $\hat{b}^{1\dagger}$ and $\stackrel{ab}{[+]}$ from the vacuum summand appears, the contribution is zero.
ii.b. Any two creation operators anti commute — $\{\hat{b}^{\alpha\dagger}_i, \hat{b}^{\beta\dagger}_j\}_+ = 0$.
According to Eq.8.57 we can rewrite $\{\hat{b}^{\alpha\dagger}_i, \hat{b}^{b\dagger}_j\}_+$, up to a factor, as

$$\{S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_1 S^{ab}\cdots S^{ef}, S^{m'n'}\cdots S^{p'r'}\hat{b}^{1\dagger}_1 S^{a'b'}\cdots S^{e'f'}\}_+.$$

Whatever the product $S^{ab}\cdots S^{ef}S^{m'n'}\cdots S^{p'r'}$ (or $S^{a'b'}\cdots S^{e'f'}S^{mn}\cdots S^{pr}$) is, it always transforms an even number of $(+)$ in $\hat{b}^{1\dagger}_1$ into $[-]$. Since an odd number of nilpotents $(+)$ (at least one) stays unchanged in this right $\hat{b}^{1\dagger}_1$, after the application of all the S^{ab} in the product in front of it or $\stackrel{d-1\, d}{[+]}$ transforms into $\stackrel{d-1\, d}{(-)}$, and since the left $\hat{b}^{1\dagger}_1$ is a product of only nilpotents $(+)$ or an odd number of nilpotents and $[+]$ for $d = 2(2n+1)$ and $d = 4n$, n is an integer, respectively, while $\stackrel{d-1\, d}{[+]}\stackrel{d-1\, d}{(-)} = 0$, the anticommutator for any two creation operators is zero.
ii.c.. Any two annihilation operators anticommute – $\{\hat{b}^{\alpha}_i, \hat{\beta}^{b}_j\}_+ = 0$.
According to Eq.8.58 we can rewrite $\{\hat{b}^{\alpha}_i, \hat{b}^{\beta}_j\}_+$, up to a factor, as

$$\{S^{ab}\cdots S^{ef}\hat{b}^{1}_1 S^{mn}\cdots S^{pr}, S^{a'b'}\cdots S^{e'f'}\hat{b}^{1}_1 S^{m'n'}\cdots S^{p'r'}\}_+.$$

What ever the product $S^{mn}\cdots S^{pr}S^{a'b'}\cdots S^{e'f'}$ (or $S^{m'n'}\cdots S^{p'r'}S^{ab}\cdots S^{ef}$) is, it always transforms an even number of $(-)$ in $\hat{b}^1_1$ into $[+]$. Since an odd number of nilpotents $(-)$ (at least one) stays unchanged in this $\hat{b}^1_1$, after the application of all the S^{ab} in the product in front of it or $\stackrel{d-1\, d}{[+]}$ transforms into $\stackrel{d-1\, d}{(-)}$, and since $\hat{b}^1_1$ in the left hand side is a product of only nilpotents $(-)$ or an odd number of nilpotents and $[+]$ for $d = 2(2n+1)$ and $d = 4n$, n is an integer, respectively, while $\stackrel{ab}{(-)}\stackrel{ab}{(-)}= 0$ and $\stackrel{ab}{[+]}\stackrel{ab}{[-]}= 0$, the anti commutator of any two annihilation operators is zero.
ii.d. For any creation and any annihilation operators it follows: $\{\hat{b}^{\alpha}_i, \hat{b}^{\beta\dagger}_j\}_+ |\psi_- >= \delta^{ab}\delta_{ij}|\psi_0>$.
Let us prove this. According to Eqs. (8.57,8.58) we may rewrite $\{\hat{b}^{\alpha}_i, \hat{b}^{\beta\dagger}_j\}_+$ up to a factor as $\{S^{ab}\cdots S^{ef}\hat{b}^1_1 S^{mn}\cdots S^{pr}, S^{m'n'}\cdots S^{p'r'}\hat{b}^{1\dagger}_1 S^{a'b'}\cdots S^{e'f'}\}_+$. We distinguish between two cases. It can be that both $S^{mn}\cdots S^{pr}S^{m'n'}\cdots S^{p'r'}$ and $S^{a'b'}\cdots S^{e'f'}S^{ab}\cdots S^{ef}$ are numbers. This happens when $\alpha = \beta$ and $i = j$. Then we follow **i.b.**. We normalize the states so that $< \psi^{\alpha}_i|\psi^{\alpha}_i >= 1$.
The second case is that at least one of

$$S^{mn}\cdots S^{pr}S^{m'n'}\cdots S^{p'r'} \text{ and } S^{a'b'}\cdots S^{e'f'}S^{ab}\cdots S^{ef}$$

is not a number. Then the factors like $\stackrel{ab}{(-)}\stackrel{ab}{[-]}$ or $\stackrel{ab}{[+]}\stackrel{ab}{(-)}$ or $\stackrel{ab}{(+)}\stackrel{ab}{[+]}$ make the anticommutator equal zero. And the proof is completed.

iii. *We learned*:
iii.a. From 2^d internal states expressed with Grassmann coordinates, which are all orthogonal with respect to the scalar product, Eq.(8.24), not all of 2^d fulfill requirements that the states should be written as product of Grassmann coordinates

on the vacuum state. We payed particular attention on $2^{\frac{d}{2}-1}$ $(2^{\frac{d}{2}-1}+1)$, states, Eqs. (8.41, 8.42). To these creation operators the same number, $(2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1}+1))$, of the corresponding annihilation operators belong, fulfilling the relation for the creation and annihilation operators (8.40), for which we expect that the creation and annihilation operators have to. These states form two (separate) groups of the Lorentz representation: The members of each group are reachable by $\mathbf{S}^{ab}$ (which do not belong to the Cartan subalgebra (8.65)) from one of the state of each group, each with $(2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1}+1))/2$ members. The second quantized states have in $d = 4n$ an even Grassmann caharacter, while in $d = 2(2n+1)$ they have an odd Grassmann character. There are in addition creation operators of opposite Grassmann character then these $2^{\frac{d}{2}-1}(2^{\frac{d}{2}-1}+1))$ states either in $d = 4n$ or in $d = 2(2n+1)$. They are products of two, four or at most product of $d\,\theta^a$.
iii.b. From 2^d internal states expressed with Clifford coordinates, which again are orthogonal with respect to the scalar product, Eq.(8.24), only $2^{\frac{d}{2}-1}$ $(2^{\frac{d}{2}-1})$ fulfill requirements that the second quantized states are expressed by products of nilpotents and projectors, which apply on the vacuum state. The products of nillpotents and projectors have to have an odd Clifford character in either $d = 4n$ or $d = 2(2n+1)$. They form creation operators and annihilation operators, fullfilling Eq.(8.52), for which we expect that the creation and annihilation operators have to.
The corresponding states form families of states. Each family members are reachable from any one by S^{ab}, while any family can be reached by $\tilde{S}^{ab}$.
iii.c. We pay attention on even-dimensional spaces only.

8.3 Conclusions

We have started the present study to understand, why "nature made a choice" of the Clifford algebra, rather than the Grassmann algebra, to describe the internal degrees of freedom of fermion fields, although both spaces enable the second quantization of the internal degrees of freedom of the fermion type. We study as well how to fermionize boson fields (or bosonize fermion fields) in any d (the reader can find the corresponding contribution in this proceedings) to better understand why and how "nature made choices of the theories and models" in the expansion of the universe.

The creation and annihilation operators fulfill anticommutation relations, desired for fermions either in Grassmann space or in Clifford space, although states in Grassmann space carry integer spins, what leads in the *spinn-charge-family* theory (since spins in $d \geq 5$ manifest as charges in $d = (1+3)$) to the charges in the adjoint representations of the charge groups (the subgroups of the Lorentz group $SO(1,13)$) while states in the Clifford space carry half integer spin and correspondingly are all the charges in the fundamental representations of the groups.

We want to understand as well how does this choice of whether taking Grassmann or Clifford space, manifest in the breaking of the starting symmetry in d-dimension down to $d = (1+3)$. The *spin-charge-family* theory namely starts at $d = (1+13)$ with the simple action in which massless fermions carry only two kinds of spin described by two kinds of the Clifford algebra objects – γ^a and $\tilde{\gamma}^a$ – and interact with the gravity only – through vielbeins, the gauge fields of the Poincaré algebra and the two kinds of the spin connection fields, the gauge fields of these two kinds of the Clifford algebra objects. The theory offers the explanation for all the assumptions of the *standard model* of elementary fields, fermions and bosons, with the appearance of families including, explaining also the phenomena like the existence of the dark matter, of the matter-antimatter asymmetry, offering correspondingly the next step beyond both standard models – cosmological one and the one of the elementary fields.

To come to the low energy regime the symmetry must break, first from $SO(13,1)$ to $SO(7,1) \times SU(3) \times U(1)$ and then further to $SO(3,1) \times SU(3) \times U(1)$. Further study is needed to understand whether the "nature could start" at all with Grassmann space while "recognizing", when breaking symmetry in steps, the "advantage" of the Clifford degrees of freedom with respect to the Grassmann ones: The covariant momentum of the starting action of the *spin-charge-family* theory, Eq. (8.1), would in the case that the Grassmann coordinates describe the internal degrees of freedom of massless objects with the anticommutation relation of the creation and annihilation operators (Eq.(8.40)) read: $p_{0\alpha} = p_\alpha - \frac{1}{2}\,\mathbf{S}^{ab}\Omega_{ab\alpha}$, where $\Omega_{ab\alpha}$ are the spin connection gauge fields of $\mathbf{S}^{ab}$ (of the generators of the Lorentz transformations in the Grassmann space) and $f^{\alpha}{}_{a}\, p_{0\alpha}$ would replace the ordinary momentum, when massless objects start to interact with the gravitational field, through the vielbeins and the spin connections in Eq. (8.33).

This contribution is a step towards understanding better the open problems of the elementary particle physics and cosmology.

Although we have not yet learned enough to be able to answer the four questions – **a.** Why is the simple starting action of the *spin-charge-family* theory doing so well in manifesting the observed properties of the fermion and boson fields? **b.** Under which condition can more general action lead to the starting action of Eq. (8.1)? **c.** What would more general action, if leading to the same low energy physics, mean for the history of our Universe? **d.** Could the fermionization procedure of boson fields or the bosonization procedure of fermion fields, discussed in this Proceedings for any dimension d (by the authors of this contribution, while one of them, H.B.F.N. [11], has succeeded with another author to do the fermionization for $d = (1+1)$), tell more about the "decisions" of the universe in the history.

8.4 APPENDIX: Lorentz algebra and representations in Grassmann and Clifford space

A Lorentz transformation on vector components θ^a, γ^a, or $\tilde{\gamma}^a$, which are used to describe internal degrees of freedom of fields with the fermion nature, and on vector components x^a, which are real (ordinary) commuting coordinates:
$\theta'^a = \Lambda^a{}_b\, \theta^b, \gamma'^a = \Lambda^a{}_b\, \gamma^b, \tilde{\gamma}'^a = \Lambda^a{}_b\, \tilde{\gamma}^b$ and $x^a = \Lambda^a{}_b\, x^b$,
leaves forms

$$a_{a_1 a_2 \dots a_i} \theta^{a_1} \theta^{a_2} \dots \theta^{a_i},\, a_{a_1 a_2 \dots a_i} \gamma^{a_1} \gamma^{a_2} \dots \gamma^{a_i},\, a_{a_1 a_2 \dots a_i} \tilde{\gamma}^{a_1} \tilde{\gamma}^{a_2} \dots \tilde{\gamma}^{a_i}$$

and

$$b_{a_1 a_2 \dots a_i} x^{a_1} x^{a_2} \dots x^{a_i}, i = (1, \dots, d)$$

invariant.

While $b_{a_1 a_2 \dots a_i}$ $(= \eta_{a_1 b_1} \eta_{a_2 b_2} \cdots \eta_{a_i b_i} b^{b_1 b_2 \dots b_i})$ is a symmetric tensor field, $a_{a_1 a_2 \dots a_i}$ $(= \eta_{a_1 b_1} \eta_{a_2 b_2} \cdots \eta_{a_i b_i} a^{b_1 b_2 \dots b_i})$ are antisymmetric tensor *Kalb-Ramond* fields. The requirements that $x'^a x'^b \eta_{ab} = x^c x^d \eta_{cd}$, $\theta'^a \theta'^b \varepsilon_{ab} = \theta^c \theta^d \varepsilon_{cd}$, $\gamma'^a \gamma'^b \varepsilon_{ab} = \gamma^c \gamma^d \varepsilon_{cd}$ and $\tilde{\gamma}'^a \tilde{\gamma}'^b \varepsilon_{ab} = \tilde{\gamma}^c \tilde{\gamma}^d \varepsilon_{cd}$, where the metric tensor η^{ab} (in our case $\eta^{ab} = diag(1, -1, -1, \dots, -1)$) lowers the indices of vectors $\{x^a\}$ $(= \eta^{ab} x_b)$, $\{\theta^a\}$: $(\theta^a = \eta^{ab} \theta_b)$, $\{\gamma^a\}$: $(\gamma^a = \eta^{ab} \gamma_b)$ and $\{\tilde{\gamma}^a\}$: $(\tilde{\gamma}^a = \eta^{ab} \tilde{\gamma}_b)$, ε_{ab} is the antisymmetric tensor, lead to $\Lambda^a{}_b \Lambda^c{}_d \eta_{ac} = \eta_{bd}$. An infinitesimal Lorentz transformation for the case with $\det\Lambda = 1, \Lambda^0{}_0 \geq 0$ can be written as $\Lambda^a{}_b = \delta^a_b + \omega^a{}_b$, where $\omega^a{}_b + \omega_b{}^a = 0$.

According to Eqs. (8.14, 8.15, 8.21) one finds

$$\begin{aligned}
\{\gamma^a, \tilde{S}^{cd}\}_- &= 0 = \{\tilde{\gamma}^a, S^{cd}\}_- \,, \\
\{\gamma^a, \mathbf{S}^{cd}\}_- &= \{\gamma^a, S^{cd}\}_- = \frac{i}{2} (\eta^{ac} \gamma^d - \eta^{ad} \gamma^c) \,, \\
\{\tilde{\gamma}^a, \mathbf{S}^{cd}\}_- &= \{\tilde{\gamma}^a, \tilde{S}^{cd}\}_- = \frac{i}{2} (\eta^{ac} \tilde{\gamma}^d - \eta^{ad} \tilde{\gamma}^c) \,.
\end{aligned} \tag{8.61}$$

Comments: In the cases with either the basis θ^a or with the basis of γ^a or $\tilde{\gamma}^a$ the scalar products — the norms — $< \mathbf{B}|\mathbf{B} > < \mathbf{F}|\mathbf{F} >$ are non negative and equal to $\sum_{k=0}^{d} \int d^d x \, b^*_{b_1 \dots b_k} b_{b_1 \dots b_k}$.

To have the norm which would have fields with the positive and the negative norm one could define the norm as $< \phi_0 | b_{b_1 \dots b_k} \gamma^{b_k} \dots \gamma^{b_1} c_{c_1 \dots c_k} \gamma^{c_1} \dots \gamma^{c_k} | \phi_0 >$, as it is used in Ref. [21] to obtain the generalized Stueckelberg equation.

8.4.1 Lorentz properties of basic vectors

What follows is taken from Ref. [2] and Ref. [9], Appendix B.

Let us first repeat some properties of the anticommuting Grassmann coordinates.

An infinitesimal Lorentz transformation of the proper ortochronous Lorentz group is then

$$\begin{aligned}
\delta\theta^c &= -\frac{i}{2} \omega_{ab} \mathbf{S}^{ab} \theta^c = \omega^c{}_a \theta^a \,, \\
\delta\gamma^c &= -\frac{i}{2} \omega_{ab} S^{ab} \gamma^c = \omega^c{}_a \gamma^a \,, \\
\delta\tilde{\gamma}^c &= -\frac{i}{2} \omega_{ab} \tilde{S}^{ab} \tilde{\gamma}^c = \omega^c{}_a \tilde{\gamma}^a \,, \\
\delta x^c &= -\frac{i}{2} \omega_{ab} L^{ab} x^c = \omega^c{}_a x^a \,,
\end{aligned} \tag{8.62}$$

where ω_{ab} are parameters of a transformation and γ^a and $\tilde{\gamma}^a$ are expressed by θ^a and $\frac{\partial}{\partial \theta_a}$ in Eqs. (8.14, 8.15).

Let us write the operator of finite Lorentz transformations as follows

$$\mathcal{U} = e^{\frac{i}{2} \omega_{ab} (\mathbf{S}^{ab} + L^{ab})} \,. \tag{8.63}$$

We see that the Grassmann θ^a and the ordinary x^a coordinates and the Clifford objects γ^a and $\tilde{\gamma}^a$ transform as vectors Eq.(8.63)

$$\theta'^c = e^{-\frac{i}{2}\omega_{ab}(\mathbf{S}^{ab}+L^{ab})}\,\theta^c\, e^{\frac{i}{2}\,\omega_{ab}(\mathbf{S}^{ab}+L^{ab})}$$
$$= \theta^c - \frac{i}{2}\omega_{ab}\{\mathbf{S}^{ab},\theta^c\}_- + \cdots = \theta^c + \omega^c{}_a\theta^a + \cdots = \Lambda^c{}_a\theta^a\,,$$
$$x'^c = \Lambda^c{}_a x^a\,,\quad \gamma'^c = \Lambda^c{}_a\gamma^a\,,\quad \tilde{\gamma}'^c = \Lambda^c{}_a\tilde{\gamma}^a\,. \tag{8.64}$$

Correspondingly one finds that compositions like $\gamma^a p_a$ and $\tilde{\gamma}^a p_a$, here p_a are p^x_a $(= i\frac{\partial}{\partial x^a})$, transform as scalars (remaining invariants), while $S^{ab}\,\omega_{abc}$ and $\tilde{S}^{ab}\,\tilde{\omega}_{abc}$ transform as vectors: $\mathcal{U}^{-1}\,S^{ab}\,\omega_{abc}\,\mathcal{U} = \Lambda_c{}^d S^{ab}\,\omega_{abd}, \mathcal{U}^{-1}\,\tilde{S}^{ab}\,\tilde{\omega}_{abc}\,\mathcal{U} = \Lambda_c{}^d\tilde{S}^{ab}\,\tilde{\omega}_{abd}$.

Also objects like

$$R = \frac{1}{2}\,f^{\alpha[a}f^{\beta b]}\,(\omega_{ab\alpha,\beta} - \omega_{ca\alpha}\,\omega^c{}_{b\beta})$$

and

$$\tilde{R} = \frac{1}{2}\,f^{\alpha[a}f^{\beta b]}\,(\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha}\tilde{\omega}^c{}_{b\beta})$$

from Eq. (8.1) transform with respect to the Lorentz transformations as scalars.

Making a choice of the Cartan subalgebra set of the algebra $\mathbf{S}^{ab}$, S^{ab} and $\tilde{S}^{ab}$, Eqs. (8.10, 8.14, 8.15),

$$\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\;d}\,,$$
$$S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d}\,,$$
$$\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d}\,, \tag{8.65}$$

one can arrange the basic vectors so that they are eigenstates of the Cartan subalgebra, belonging to representations of $\mathbf{S}^{ab}$, or of S^{ab} and $\tilde{S}^{ab}$.

8.5 APPENDIX: Technique to generate spinor representations in terms of Clifford algebra objects

We shall briefly repeat the main points of the technique for generating spinor representations from Clifford algebra objects, following the reference[12]. We ask the reader to look for details and proofs in this reference.

We assume the objects γ^a, Eq. (8.14), which fulfill the Clifford algebra, Eq (8.13).

$$\{\gamma^a, \gamma^b\}_+ = I\;\; 2\eta^{ab}, \quad \text{for} \quad a, b \quad \in \{0, 1, 2, 3, 5, \cdots, d\}, \tag{8.66}$$

for any d, even or odd. I is the unit element in the Clifford algebra, while $\{\gamma^a, \gamma^b\}_\pm = \gamma^a\gamma^b \pm \gamma^b\gamma^a$.

We accept the "Hermiticity" property for γ^a's, Eq. (8.17), $\gamma^{a\dagger} = \eta^{aa}\gamma^a$. leading to $\gamma^{a\dagger}\gamma^a = I$.

The Clifford algebra objects S^{ab} close the Lie algebra of the Lorentz group of Eq. (8.21) $\{S^{ab}, S^{cd}\}_- = i(\eta^{ad}S^{bc} + \eta^{bc}S^{ad} - \eta^{ac}S^{bd} - \eta^{bd}S^{ac})$. One finds from Eq.(8.17) that $(S^{ab})^\dagger = \eta^{aa}\eta^{bb}S^{ab}$ and that $\{S^{ab}, S^{ac}\}_+ = \frac{1}{2}\eta^{aa}\eta^{bc}$.

Recognizing that two Clifford algebra objects S^{ab}, S^{cd} with all indexes different commute, we select (out of infinitely many possibilities) the Cartan sub algebra set of the algebra of the Lorentz group as follows

$$\begin{aligned} &S^{0d}, S^{12}, S^{35}, \cdots, S^{d-2\; d-1}, \quad \text{if} \quad d = 2n, \\ &S^{12}, S^{35}, \cdots, S^{d-1\; d}, \quad \text{if} \quad d = 2n+1. \end{aligned} \tag{8.67}$$

To make the technique simple, we introduce the graphic representation[12] as follows

$$\begin{aligned} \stackrel{ab}{(k)} &:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b), \\ \stackrel{ab}{[k]} &:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b), \end{aligned} \tag{8.68}$$

where $k^2 = \eta^{aa}\eta^{bb}$. One can easily check by taking into account the Clifford algebra relation (Eq.8.66) and the definition of S^{ab} that if one multiplies from the left hand side by S^{ab} the Clifford algebra objects $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$, it follows that

$$\begin{aligned} S^{ab} \stackrel{ab}{(k)} &= \frac{1}{2} k \stackrel{ab}{(k)}, \\ S^{ab} \stackrel{ab}{[k]} &= \frac{1}{2} k \stackrel{ab}{[k]}\, . \end{aligned} \tag{8.69}$$

This means that $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$ acting from the left hand side on anything (on a vacuum state $|\psi_0\rangle$, for example) are eigenvectors of S^{ab}.

We further find

$$\begin{aligned} \gamma^a \stackrel{ab}{(k)} &= \eta^{aa} \stackrel{ab}{[-k]}, \\ \gamma^b \stackrel{ab}{(k)} &= -ik \stackrel{ab}{[-k]}, \\ \gamma^a \stackrel{ab}{[k]} &= \stackrel{ab}{(-k)}, \\ \gamma^b \stackrel{ab}{[k]} &= -ik\eta^{aa} \stackrel{ab}{(-k)} \end{aligned} \tag{8.70}$$

It follows that $S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{(k)} = -\frac{i}{2}\eta^{aa}\eta^{cc} \stackrel{ab}{[-k]}\stackrel{cd}{[-k]}$, $S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{[k]} = \frac{i}{2} \stackrel{ab}{(-k)}\stackrel{cd}{(-k)}$, $S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{[k]} = -\frac{i}{2}\eta^{aa} \stackrel{ab}{[-k]}\stackrel{cd}{(-k)}$, $S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{(k)} = \frac{i}{2}\eta^{cc} \stackrel{ab}{(-k)}\stackrel{cd}{[-k]}$. It is useful to deduce the following relations

$$\begin{array}{llll} \stackrel{ab}{(k)}\stackrel{ab}{(k)} = 0, & \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa} \stackrel{ab}{[k]}, & \stackrel{ab}{(-k)}\stackrel{ab}{(k)} = \eta^{aa} \stackrel{ab}{[-k]}, & \stackrel{ab}{(-k)}\stackrel{ab}{(-k)} = 0 \\ \stackrel{ab}{[k]}\stackrel{ab}{[k]} = \stackrel{ab}{[k]}, & \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0, & \stackrel{ab}{[-k]}\stackrel{ab}{[k]} = 0, & \stackrel{ab}{[-k]}\stackrel{ab}{[-k]} = \stackrel{ab}{[-k]} \\ \stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0, & \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}, & \stackrel{ab}{(-k)}\stackrel{ab}{[k]} = \stackrel{ab}{(-k)}, & \stackrel{ab}{(-k)}\stackrel{ab}{[-k]} = 0 \\ \stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}, & \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0, & \stackrel{ab}{[-k]}\stackrel{ab}{(k)} = 0, & \stackrel{ab}{[-k]}\stackrel{ab}{(-k)} = \stackrel{ab}{(-k)}\, . \end{array} \tag{8.71}$$

We recognize in the first equation of the first row and the first equation of the second row the demonstration of the nilpotent and the projector character of the Clifford algebra objects $\overset{ab}{(k)}$ and $\overset{ab}{[k]}$, respectively.

Whenever the Clifford algebra objects apply from the left hand side, they always transform $\overset{ab}{(k)}$ *to* $\overset{ab}{[-k]}$, *never to* $\overset{ab}{[k]}$, *and similarly* $\overset{ab}{[k]}$ *to* $\overset{ab}{(-k)}$, *never to* $\overset{ab}{(k)}$.

We define in Eq. (8.59) a vacuum state $|\psi_0>$ so that one finds

$$< \overset{ab}{(k)}{}^{\dagger} \overset{ab}{(k)} >= 1,$$
$$< \overset{ab}{[k]}{}^{\dagger} \overset{ab}{[k]} >= 1. \qquad (8.72)$$

Taking the above equations into account it is easy to find a Weyl spinor irreducible representation for d-dimensional space, with d even or odd. (We advise the reader to see the reference[12].)

For d even, we simply set the starting state as a product of d/2, let us say, only nilpotents $\overset{ab}{(k)}$, one for each S^{ab} of the Cartan sub algebra elements (Eq.(8.67)), applying it on an (unimportant) vacuum state[12]. Then the generators S^{ab}, which do not belong to the Cartan sub algebra, applied to the starting state from the left hand side, generate all the members of one Weyl spinor.

$$\begin{array}{c}
\overset{0d}{(k_{0d})}\overset{12}{(k_{12})}\overset{35}{(k_{35})}\cdots\overset{d-1\ d-2}{(k_{d-1\ d-2})}\ \psi_0 \\
\overset{0d}{[-k_{0d}]}\overset{12}{[-k_{12}]}\overset{35}{(k_{35})}\cdots\overset{d-1\ d-2}{(k_{d-1\ d-2})}\ \psi_0 \\
\overset{0d}{[-k_{0d}]}\overset{12}{(k_{12})}\overset{35}{[-k_{35}]}\cdots\overset{d-1\ d-2}{(k_{d-1\ d-2})}\ \psi_0 \\
\vdots \\
\overset{0d}{[-k_{0d}]}\overset{12}{(k_{12})}\overset{35}{(k_{35})}\cdots\overset{d-1\ d-2}{[-k_{d-1\ d-2}]}\ \psi_0 \\
\overset{od}{(k_{0d})}\overset{12}{[-k_{12}]}\overset{35}{[-k_{35}]}\cdots\overset{d-1\ d-2}{(k_{d-1\ d-2})}\ \psi_0 \\
\vdots
\end{array} \qquad (8.73)$$

8.5.1 Technique to generate "families" of spinor representations in terms of Clifford algebra objects

When all 2^d states are considered as a Hilbert space, we recognize that for d even there are $2^{d/2}$ "families" and for d odd $2^{(d+1)/2}$ "families" of spinors [12,13,9]. We shall pay attention of only even d.

One Weyl representation form a left ideal with respect to the multiplication with the Clifford algebra objects. We proved in Ref.[9], and the references therein that there is the application of the Clifford algebra object from the right hand side, which generates "families" of spinors.

Right multiplication with the Clifford algebra objects namely transforms the state of one "family" into the same state with respect to the generators S^{ab} (when the multiplication from the left hand side is performed) of another "family".

We defined in refs.[13] the Clifford algebra objects $\tilde{\gamma}^a$'s as operations which operate formally from the left hand side (as γ^a's do) on any Clifford algebra object A as follows

$$\tilde{\gamma^a} A = i(-)^{(A)} A \gamma^a, \tag{8.74}$$

with $(-)^{(A)} = -1$, if A is an odd Clifford algebra object and $(-)^{(A)} = 1$, if A is an even Clifford algebra object.

Then it follows that $\tilde{\gamma^a}$ obey the same Clifford algebra relation as γ^a.

$$(\tilde{\gamma^a}\tilde{\gamma^b} + \tilde{\gamma^b}\tilde{\gamma^a})A = -ii((-)^{(A)})^2 A(\gamma^a\gamma^b + \gamma^b\gamma^a) = 2\eta^{ab}A \tag{8.75}$$

and that $\tilde{\gamma^a}$ and γ^a anticommute

$$(\tilde{\gamma^a}\gamma^b + \gamma^b\tilde{\gamma^a})A = i(-)^{(A)}(-\gamma^b A\gamma^a + \gamma^b A\gamma^a) = 0. \tag{8.76}$$

We may write

$$\{\tilde{\gamma^a}, \gamma^b\}_+ = 0, \quad \text{while} \quad \{\tilde{\gamma^a}, \tilde{\gamma^b}\}_+ = 2\eta^{ab}. \tag{8.77}$$

One accordingly finds

$$\tilde{\gamma^a} \overset{ab}{(k)}: = -i \overset{ab}{(k)} \gamma^a = -i\eta^{aa} \overset{ab}{[k]}, \tag{8.78}$$

$$\tilde{\gamma^b} \overset{ab}{(k)}: = -i \overset{ab}{(k)} \gamma^b = -k \overset{ab}{[k]},$$

$$\tilde{\gamma^a} \overset{ab}{[k]}: = \ i \overset{ab}{[k]} \gamma^a = \ i \overset{ab}{(k)}, \tag{8.79}$$

$$\tilde{\gamma^b} \overset{ab}{[k]}: = \ i \overset{ab}{[k]} \gamma^b = -k\eta^{aa} \overset{ab}{(k)}. \tag{8.80}$$

If we define

$$\tilde{S}^{ab} = \frac{i}{4}\,[\tilde{\gamma}^a, \tilde{\gamma}^b] = \frac{1}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a), \tag{8.81}$$

it follows

$$\tilde{S}^{ab}A = A\frac{1}{4}(\gamma^b\gamma^a - \gamma^a\gamma^b), \tag{8.82}$$

manifesting accordingly that $\tilde{S}^{ab}$ fulfil the Lorentz algebra relation as S^{ab} do. Taking into account Eq.(8.74), we further find

$$\{\tilde{S}^{ab}, S^{ab}\}_- = 0, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0, \quad \{S^{ab}, \tilde{\gamma}^c\}_- = 0. \tag{8.83}$$

One also finds

$$\{\tilde{S}^{ab}, \Gamma\}_- = 0, \quad \{\tilde{\gamma}^a, \Gamma\}_- = 0, \quad \text{for d even},$$

$$\{\tilde{S}^{ab}, \Gamma\}_- = 0, \quad \{\tilde{\gamma}^a, \Gamma\}_+ = 0, \quad \text{for d odd}, \tag{8.84}$$

which means that in d even transforming one "family" into another with either $\tilde{S}^{ab}$ or $\tilde{\gamma}^a$ leaves handedness Γ unchanged. (The transformation to another "family" in d odd with $\tilde{\gamma}^a$ changes the handedness of states, namely the factor $\frac{1}{2}(1 \pm \Gamma)$ changes to $\frac{1}{2}(1 \mp \Gamma)$ in accordance with what we know from before: In spaces with odd d changing the handedness means changing the "family".)

We advise the reader also to read [2]where the two kinds of Clifford algebra objects follow as two different superpositions of a Grassmann coordinate and its conjugate momentum.

We present for $\tilde{S}^{ab}$ some useful relations

$$\begin{aligned}
\tilde{S}^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}, \\
\tilde{S}^{ab} \stackrel{ab}{[k]} &= -\frac{k}{2} \stackrel{ab}{[k]}, \\
\tilde{S}^{ac} \stackrel{ab}{(k)}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{aa}\eta^{cc} \stackrel{ab}{[k]}\stackrel{cd}{[k]}, \\
\tilde{S}^{ac} \stackrel{ab}{[k]}\stackrel{cd}{[k]} &= -\frac{i}{2} \stackrel{ab}{(k)}\stackrel{cd}{(k)}, \\
\tilde{S}^{ac} \stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa} \stackrel{ab}{[k]}\stackrel{cd}{(k)}, \\
\tilde{S}^{ac} \stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc} \stackrel{ab}{(k)}\stackrel{cd}{[k]} \,.
\end{aligned} \tag{8.85}$$

We transform the state of one "family" to the state of another "family" by the application of $\tilde{\gamma}^a$ or $\tilde{S}^{ac}$ (formally from the left hand side) on a state of the first "family" for a chosen a or a, c. To transform all the states of one "family" into states of another "family", we apply $\tilde{\gamma}^a$ or $\tilde{S}^{ac}$ to each state of the starting "family". It is, of course, sufficient to apply $\tilde{\gamma}^a$ or $\tilde{S}^{ac}$ to only one state of a "family" and then use generators of the Lorentz group (S^{ab}), and for d even also γ^a's, to generate all the states of one Dirac spinor.

One must notice that nilpotents $\stackrel{ab}{(k)}$ and projectors $\stackrel{ab}{[k]}$ are eigenvectors not only of the Cartan subalgebra S^{ab} but also of $\tilde{S}^{ab}$. Accordingly only $\tilde{S}^{ac}$, which do not carry the Cartan subalgebra indices, cause the transition from one "family" to another "family".

The starting state of Eq.(8.73) can change, for example, to

$$\stackrel{0d}{[k_{0d}]}\stackrel{12}{[k_{12}]}\stackrel{35}{(k_{35})} \cdots \stackrel{d-1\; d-2}{(k_{d-1\; d-2})}, \tag{8.86}$$

if $\tilde{S}^{01}$ was chosen to transform the Weyl spinor of Eq.(8.73) to the Weyl spinor of another "family".

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