

COMPUTATION OF DISPERSION CURVES OF RAYLEIGH-LAMB WAVES BY NEWTON-RAPHSON METHOD

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The paper treats the numerical construction of dispersion curves which belong to the propagation of Rayleigh-Lamb waves in isotropic elastic media. The Newton-Raphson method is applied for computation of real, imaginary and complex wavenumbers in dependence on the frequency of the wave. Due to multivaluedness, unexpected jumps between modes of dispersion curves occur in branch tracing, which prevents successful direct application of the method. This shortcoming is eliminated in the article by introducing the arc-length continuation method. The efficiency of this supplement to the Newton-Raphson method is shown on the example of an aluminum plate with constructions of dispersion curves for real, imaginary and complex wave numbers.

Keywords:

symmetric wave modes,
asymmetric wave
modes,
Helmholtz decoupling,
homotopy,
arc-length continuation



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RAČUNANJE DISPERZIJSKIH KRIVULJ RAYLEIGH-LAMBOVIH VALOV Z NEWTON-RAPHSONOVO METODO

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Članek obravnava numerično konstrukcijo disperzijskih krivulj, ki pripadajo širjenju Rayleigh-Lambovih valov v izotropnih elastičnih sredstvih. Za izračun realnih, imaginarnih in kompleksnih valovnih števil v odvisnosti od frekvence valovanja je uporabljena Newton-Raphsonova metoda. Zaradi večličnosti se pri konstrukciji ob sledenju vej pojavljajo nepričakovani preskoki med načini valovanja disperzijskih krivulj, kar onemogoča uspešno neposredno uporabo metode. Ta pomanjkljivost je v članku odpravljena z uvedbo metode nadaljevanja po ločni dolžini. Učinkovitost te dopolnitve Newton-Raphsonove metode je prikazana na primeru aluminijske plošče s konstrukcijami disperzijskih krivulj za realna, imaginarna in kompleksna valovna števila.

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Ključne besede:

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homotopija,
nadaljevanje po ločni
dolžini

1 Introduction

Guided waves, especially Lamb waves are ideally suited for nondestructive testing of large structures such as metal plates, sheets or composites [3,5]. Although the governing equations of propagation of Rayleigh-Lamb waves in isotropic elastic media have been known for a long time, the construction of dispersion curves for real, imaginary and complex wavenumbers, which these equations produce, is still a challenging problem and different approaches by many researchers were undertaken in the past [2,4]. Newton-Raphson iterative procedure is a classical numerical method having quadratic convergence, which works equally well for real, imaginary and complex wavenumbers, respectively. Due to this property, the method is a natural choice for solving dispersion relations. However, dispersion relations for Rayleigh-Lamb waves have transcendental character, which is the cause of multivaluedness. Thus, in general, many different dispersion curves determining the course of wavenumbers versus frequency will arise in the investigated frequency domain. The direct use of the Newton-Raphson method, which is usually followed by branch tracing through a homotopy procedure, for example by incrementing of the frequency parameter after the iteration procedure in some point of the frequency domain was successfully accomplished, seldom produces the desired result. This is due to sudden and unexpected jumps from one wave mode to another. To prevent such an undesired behaviour and to provide a feasible and powerful method of construction of dispersion curves in the case of transcendental dispersion relations, the Newton-Raphson method must be supplemented by the some kind of the predictor. The role of the predictor is the computation of the starting point, lying inside the actual basin of attraction for performing the next Newton-Raphson iteration step. In this paper, the arc-length continuation as predictor is proposed, which is proven successful for construction of dispersion curves in all aforementioned cases of real, imaginary and complex wavenumbers.

The paper is organized as follows. In the second section, the dispersion relation for propagation of Rayleigh-Lamb waves in an infinitely long homogeneous isotropic elastic plate is derived and the physical meaning of possible appearance of real, imaginary and complex wave numbers, respectively is explained. The third section contains the general formulation of the Newton-Raphson method, which can be applied for computation of the complex wavenumbers as roots of dispersion equation, as well as to computation of real and imaginary roots, respectively, viewed

in numerical sense as special cases. The fourth section is devoted to solving the problem of uncontrolled jumping between wave modes during branch tracing by introducing an auxiliary procedure, called the arc-length continuation. In the fifth section, the construction of dispersion curves for propagation of Rayleigh-Lamb waves in the infinitely long, homogeneous elastic aluminium plate is presented and analyzed. The sixth section summarizes the conclusions and provides directions for further work.

2 The Rayleigh-Lamb dispersion relation

For the reader's convenience, a derivation of the Rayleigh-Lamb dispersion relation is presented in which all detailed algebraic manipulations are omitted for brevity. The derivation starts with governing equations for 3D linear elasticity.

2.1 Equation of motion in 3D elastic solid

The governing equations of 3D linear homogeneous isotropic elastic solid comprise a) the equations of motion of 3D elasticity

$$\sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i, \quad (1)$$

(b) the stress-strain relations, representing the Hooke's law

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}, \quad (2)$$

(c) the strain-displacement (or Cauchy's) relations

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad (3)$$

where u_i are the displacement components, σ_{ij} are components of the stress tensor, ε_{ij} are components of the deformation tensor, ε_{kk} denotes the trace of deformation tensor, f_i are components of volume force, ρ is the density, λ and μ are Lamé coefficients. The Einstein's summation convention is assumed for $i=1,2,3$. Substituting first the strain-displacement relations (3) into Eq. (2) and then the resulting stress-displacement relations into Eq. (1), the Navier's equation of motion is obtained

$$(\lambda + \mu)u_{j,ji} + \mu u_{i,jj} + \rho f_i = \rho \ddot{u}_i. \quad (4)$$

The Navier's equation of motion can be expressed in the compact vector form

$$(\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u} + \rho\mathbf{f} = \rho\ddot{\mathbf{u}}, \quad \mathbf{u} = (u, v, w), \quad (5)$$

where ∇^2 is the Laplace operator. In the derivation of the Rayleigh-Lamb dispersion relation, the body forces are assumed to be zero, giving the reduced form of the equation of motion

$$(\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u} = \rho\ddot{\mathbf{u}}, \quad (6)$$

which is a coupled system of equations in the three displacement components, denoted conventionally by u , v and w . This system can be uncoupled by using the Helmholtz decomposition [1] in which the displacement vector \mathbf{u} is expressed as the sum of the gradient of the scalar potential function φ and curl of the vector potential function Ψ

$$\mathbf{u} = \nabla\varphi + \nabla \times \Psi, \quad \nabla\Psi = 0. \quad (7)$$

Substituting Eq. (7) into equation of motion (6), taking into account the rules $\nabla \cdot \nabla\varphi = \nabla^2\varphi$ and $\nabla \cdot \nabla \times \Psi = 0$, we obtain the equation

$$\nabla \left[(\lambda + 2\mu)\nabla^2\varphi - \rho\ddot{\varphi} \right] + \nabla \times \left[\mu\nabla^2\Psi - \rho\ddot{\Psi} \right] = \mathbf{0}. \quad (8)$$

Equation (7) therefore satisfies the equation of motion (6), if it satisfies the following uncoupled wave equations

$$(\lambda + 2\mu)\nabla^2\varphi - \rho\ddot{\varphi} = 0 \Rightarrow \nabla^2\varphi = \frac{1}{c_L^2}\ddot{\varphi}, \quad c_L^2 = \frac{\lambda + 2\mu}{\rho}, \quad (9)$$

$$\mu\nabla^2\Psi - \rho\ddot{\Psi} = \mathbf{0} \Rightarrow \nabla^2\Psi = \frac{1}{c_T^2}\ddot{\Psi}, \quad c_T^2 = \frac{\mu}{\rho}, \quad (10)$$

where c_L is the longitudinal wave velocity and c_T is the transverse wave velocity. Evidently, the longitudinal wave velocity is greater than the transverse wave velocity, what can be explained by taking into account that shear stiffness has the stiffening effect on the material.

2.2 Rayleigh-Lamb Waves in elastic plate with traction-free surfaces

In the following, we limit the topic to the problem of the propagation of Rayleigh-Lamb waves in a homogeneous isotropic elastic plate. Then it suffices, to solve the wave equations (9) and (10) under plane strain condition and by assumption of traction-free surfaces. The geometry of the plate can be reduced on twodimensional problem in spatial coordinates x and z while the coordinate y is omitted due to invariability of the solution in any plane $y=\text{const}$, see Figure 1.

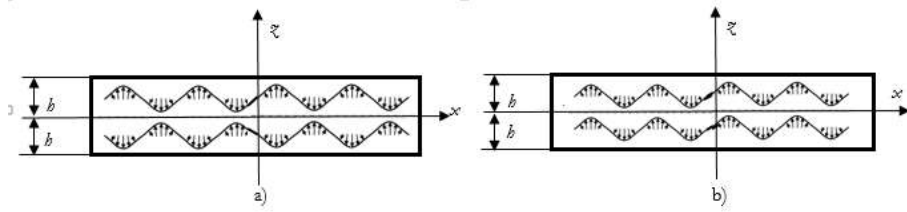


Figure 1: Propagation of a) symmetric and b) asymmetric Rayleigh-Lamb wave modes in a thin homogeneous isotropic elastic plate.

Source: own.

The scalar potential function in two variables is $\varphi = \varphi(x, z)$ and the vector potential function has only one nonzero component $\Psi = (0, \psi_y(x, z), 0)$. If the plane strain condition holds in the xz plane, Eq. (7) reduces to

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi_y}{\partial z}, \quad w = \frac{\partial \varphi}{\partial z} - \frac{\partial \psi_y}{\partial x}, \quad (11)$$

while from Eqs. (2) and (3) we can derive stresses in the form

$$\begin{aligned} \sigma_{xx} &= \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial u}{\partial x}, \\ \sigma_{zz} &= \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial w}{\partial z}, \\ \sigma_{xz} &= \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right). \end{aligned} \quad (12)$$

Further, instead of the vector wave equation (10) for all three components, only the scalar wave equation $\nabla^2 \psi_y(x, z) = \ddot{\psi}_y(x, z)/c_T^2$ needs to be solved. The free wave solution is sought in the form

$$\begin{aligned}\varphi(x, z) &= \mathcal{A}(z)e^{i(kx - \omega t)}, \\ \psi_y(x, z) &= \Theta(z)e^{i(kx - \omega t)},\end{aligned}\tag{13}$$

where k represents the wavenumber, which can be real, imaginary or complex number and ω denotes the angular frequency of the wave. Solutions $\varphi(x, z)$ and $\psi_y(x, z)$ represent traveling waves in the x direction with respect to the function $e^{i(kx - \omega t)}$ and standing waves in the z direction in respect to functions $\mathcal{A}(z)$, $\Theta(z)$. By noting that the wavenumber k appears only in the function $e^{i(kx - \omega t)}$, we can learn more about the nature of wave propagation if we write k as a complex number in the form $k = k_r + i \cdot k_i$ ($i = \sqrt{-1}$) with a real part k_r and an imaginary part $i \cdot k_i$ and examine individual possibilities. If k is a pure real number, $k = k_r$, then wave is harmonic, traveling in the x direction with constant amplitude. If k is a pure imaginary number, $k = i \cdot k_i$, the wave is not traveling wave but is evanescent. If k is complex with nonzero k_r and $i \cdot k_i$ parts, the wave travels in the x direction with a decreasing amplitude. To determine functions $\mathcal{A}(z)$, $\Theta(z)$ in Eq. (13), potential functions are substituted into Eq. (9) and (10), whereby we get

$$\begin{aligned}\mathcal{A}(z) &= A \sin(\alpha z) + B \cos(\alpha z), & \alpha^2 &= \left(\frac{\omega}{c_L}\right)^2 - k^2, \\ \Theta(z) &= C \sin(\beta z) + D \cos(\beta z), & \beta^2 &= \left(\frac{\omega}{c_T}\right)^2 - k^2.\end{aligned}\tag{14}$$

Determination of constants A , B , C and D depend on boundary conditions. Parameters α and β , which are generally complex, influence the course of functions $\mathcal{A}(z)$, $\Theta(z)$ in z direction, quotients $\frac{\omega}{c_L}$ and $\frac{\omega}{c_T}$ denote the wavenumbers of the longitudinal (bending) and transversal (shear) wave, respectively and k denotes the wavenumber along the direction of the wave propagation. The solving of the boundary value problem becomes much more simple, if geometric symmetry around axis $z=0$ is taken into account. The modes of wave propagation in a thin plate may

be separated into symmetric and asymmetric modes, respectively, if instead of Eq. (14) the ansatz

$$A(z) = B \cos(\alpha z), \quad \Theta(z) = C \sin(\beta z) \quad (15)$$

is used for symmetric modes and ansatz

$$A(z) = A \sin(\alpha z), \quad \Theta(z) = D \cos(\beta z) \quad (16)$$

for asymmetric modes. In what follows, only derivation of the dispersion relation for symmetric modes will be presented, because derivation of dispersion relation for asymmetric modes is quite analogous.

Putting Eqs. (15) into Eq. (13) and forming required partial derivatives, which appear in Eqs. (11) and (12), we get the system of equations:

$$\begin{aligned} u &= [ikB \cos(\alpha z) + \beta C \cos(\beta z)] e^{i(kx - \omega t)}, \\ w &= [\alpha A \cos(\alpha z) - ikC \sin(\beta z)] e^{i(kx - \omega t)}, \\ \sigma_{xx} &= \left\{ -\left[\lambda(\alpha^2 + k^2) + 2\mu k^2 \right] B \cos(\alpha z) + 2i\mu k \beta C \cos(\beta z) \right\} e^{i(kx - \omega t)}, \\ \sigma_{zz} &= \left\{ -\left[\lambda(\alpha^2 + k^2) + 2\mu \alpha^2 \right] B \cos(\alpha z) - 2i\mu k \beta C \cos(\beta z) \right\} e^{i(kx - \omega t)}, \\ \sigma_{xz} &= \mu \left[(k^2 - \beta^2) C \sin(\beta z) - 2ik\alpha B \sin(\alpha z) \right] e^{i(kx - \omega t)}. \end{aligned} \quad (17)$$

By applying the traction free boundary conditions $\sigma_{xz} = \sigma_{zx} = 0$ for $z = \pm h$, we obtain from last two equations of the system (17) the following system

$$\begin{bmatrix} -\left[\lambda(\alpha^2 + k^2) + 2\mu \alpha^2 \right] \cos(\alpha h) & -2i\mu k \beta \cos(\beta h) \\ -2i\mu k \alpha \sin(\alpha h) & \mu(k^2 - \beta^2) \sin(\beta h) \end{bmatrix} \begin{Bmatrix} B \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (18)$$

Since the system of Equation (18) is homogeneous, nontrivial solution for constants B and C is possible, if determinant vanishes, which results in the desired dispersion relation. For this purpose, let's first prove the following identity

$$\lambda(\alpha^2 + k^2) + 2\mu\alpha^2 = \mu(\beta^2 - k^2) \quad (19)$$

and the Rayleigh-Lamb dispersion relation is

$$\frac{\tan(\beta h)}{\tan(\alpha h)} - \frac{4k^2\alpha\beta}{(k^2 - \beta^2)^2} = 0, \quad (20)$$

which is evidently transcendental. It is fully clear, that the Rayleigh-Lamb dispersion relation for propagation of asymmetric modes can be derived in an analogous way. The dispersion relation is similar to Eq. (20) with the only difference that the second quotient on the left side of the equation is reciprocal. Thus, the dispersion relation for symmetric and asymmetric modes can be written with one equation

$$\frac{\tan(\beta h)}{\tan(\alpha h)} - \left[\frac{4k^2\alpha\beta}{(k^2 - \beta^2)^2} \right]^{\pm 1} = 0, \quad (21)$$

where the value of the exponent $+1$ belongs to the symmetric modes, and the value of the exponent -1 to the asymmetric modes.

3 The application of the Newton-Raphson method for construction of the dispersion curves

The Newton-Raphson method is an iterative algorithm for finding the roots of an equation $f(x)=0$, where the argument x can be real, imaginary or complex. The method has been in past widely used for solving non-linear equations with single or multiple variables. It should be noted, that the dispersion relation (21) is of the form $f(\omega, k)=0$, that is, solutions are pairs of numbers if the wavenumber k is real or

imaginary, and are even triples of numbers if the wavenumber is complex. We shall present this method in complex and vector space for complex arguments, while the functions of the real valued or imaginary valued arguments, respectively can be treated as special cases. For a function $f(\omega, k): C \rightarrow C$ with a complex argument $k = k_r + i \cdot k_i \in C$, ($i = \sqrt{-1}$), where $k_r, k_i \in R$ and where first two derivatives $f'(\omega, k) = \frac{df(\omega, k)}{dk}$, $f''(\omega, k) = \frac{d^2f(\omega, k)}{dk^2}$ are assumed to be continuous functions in the vicinity of the roots, we create a sequence of iterations

$$k_{n+1} = k_n - \frac{f(\omega, k_n)}{f'(\omega, k_n)} \quad (22)$$

that by selecting a good enough initial guess k_0 leads to the approximate solution of equation $f(\omega, k) = 0$. Newton-Raphson method in the complex space is characterized by basins of attraction with accompanied fractal structure for many functions and by divergent iterations for some starting points. Typically for transcendental functions as they appear in dispersion relation (21), which contains expressions $\alpha = \sqrt{(\omega/c_L)^2 - k^2}$ and $\beta = \sqrt{(\omega/c_T)^2 - k^2}$ are discontinuities in $f(\omega, k)$ and $f'(\omega, k)$ where $\arg(\omega, k) = (2j+1)\pi/2$ and $j \in Z$.

Instead of solving the original problem (22), the two-dimensional vector space Newton-Raphson method is presented in this research work for simplicity and ease of use. The equation $f(\omega, k) = 0$ implies that equations $\text{Re}[f(\omega, k)] = 0$, $\text{Im}[f(\omega, k)] = 0$ hold simultaneously and variables $k_r, k_i \in R$ are independent. Then the vector space Newton-Raphson method can be formulated in R^2 by introduction of the vector argument $\mathbf{k} = (k_r, k_i)$ and vector function $\mathbf{f} = (f_r, f_i)$, where the notation $f_r = \text{Re}[f(\omega, \mathbf{k})]$, $f_i = \text{Im}[f(\omega, \mathbf{k})]$ is used. Two equations of the vector space problem then are $f_r(\omega, \mathbf{k}_n) = 0$, $f_i(\omega, \mathbf{k}_n) = 0$, which are iteratively solved by means of the equation

$$\mathbf{k}_{n+1} = \mathbf{k}_n - \mathbf{J}^{-1}(\omega, \mathbf{k}_n) \cdot \mathbf{f}(\omega, \mathbf{k}_n), \quad (23)$$

where $\mathbf{J}(\omega, \mathbf{k})$ is the Jacobean matrix, defined as

$$\mathbf{J}(\omega, \mathbf{k}) = \begin{bmatrix} \frac{\partial f_r}{\partial k_r} & \frac{\partial f_r}{\partial k_i} \\ \frac{\partial f_i}{\partial k_r} & \frac{\partial f_i}{\partial k_i} \end{bmatrix}. \quad (24)$$

The vector method fails if $\mathbf{J}(\omega, \mathbf{k}_n)$ is a singular matrix. In respect to the dispersion relation (21) the vector equation (23) is reduced to

$$k_{r,n+1} = k_{r,n} - \frac{f(\omega, k_{r,n})}{f'(\omega, k_{r,n})}, \quad k_{i,n+1} = k_{i,n} - \frac{f(\omega, k_{i,n})}{f'(\omega, k_{i,n})} \quad (25)$$

for real or imaginary wavenumbers, respectively. Similarly as in the vector problem for $\mathbf{k}=(k_r, k_i)$, the iterations (25) fail, if either $f'(\omega, k_{r,n}) = 0$ or $f'(\omega, k_{i,n}) = 0$.

4 The arc-length continuation

The construction of the dispersion curves can be in principle performed by using homotopy. This principle is depicted in Fig. 2, where the attempt to construct the curve of the first asymmetric mode for real wavenumbers is tried. The homotopy procedure is flexible enough to start at the top or bottom of the curve. In Fig. 2, for example, it starts at the top of the first asymmetric mode, where the first (highest lying) point is computed by Newton-Raphson iterative procedure. The homotopy procedure then decrements the real wave number k_r for a prescribed value (of course, the incrementing the real wavenumber k_r would be applied, when starting from the bottom) and uses the computed value of frequency ω as the starting value for the next iterative step. In Fig. 2, the homotopy procedure is successful in 10 subsequent points, but then it follows an unexpected jump to the curve of the next asymmetric mode. After two points on the right curve are computed, the Newton-Raphson iterative process no longer converges and the construction must be terminated.

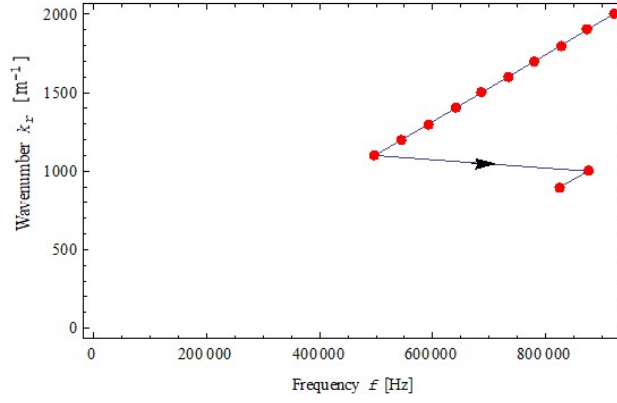


Figure 2: The fail of the direct use of the Newton-Raphson method for construction of dispersion curve for real wavenumbers by homotopy. The jump from first asymmetric mode curve to the next asymmetric mode curve is shown. Below the second point of the right curve, the convergence cannot be reached.

Source: own.

In order to prevent such sudden jumps and in the same time to reduce the number of iterations required that the solution converges, an arc-length continuation combined with cubic extrapolation to predict the next point of the dispersion curve is applied. For the sake of brevity, let $\mathbf{p}_0=(\omega_0, \mathbf{k}_0)$, $\mathbf{p}_1=(\omega_1, \mathbf{k}_1)$, $\mathbf{p}_2=(\omega_2, \mathbf{k}_2)$ and $\mathbf{p}_3=(\omega_3, \mathbf{k}_3)$ be known points and predict the next point $\mathbf{p}_4=(\omega_4, \mathbf{k}_4)$ by extrapolation of the prescribed arc length Δs . To achieve this, the arc length l is used as parameter so that l_i corresponds to \mathbf{p}_i , $i=0,1,\dots,3$ and corresponding arc lengths are l_0 , $l_1=s_1$, $l_2=l_1+s_2$, $l_3=l_2+s_3$, $l_4=l_3+\Delta s$. The extrapolated point \mathbf{p}_4 is then calculated by using the formula

$$\mathbf{p}_4 = \sum_{i=0}^3 \left(\prod_{\substack{j=0, \\ j \neq i}}^3 \frac{l_4 - l_j}{l_i - l_j} \right) \mathbf{p}_i, \quad (26)$$

whereby it is

$$s_i = |\mathbf{p}_i - \mathbf{p}_{i-1}| = \sqrt{\sum_{j=1}^{N_f+1} [\mathbf{p}_i(j) - \mathbf{p}_{i-1}(j-1)]^2} \quad (27)$$

and N_f denotes the number of components of the wavenumber vector \mathbf{k} . It is obvious that $N_f=1$ in the case of real or imaginary wavenumbers and $N_f=2$ for complex wavenumbers. The choice of the incremental length Δs is determined by the principal curvature of the hyperspace curve.

5 The results

The presented method of construction of dispersion curves for real, imaginary and complex wavenumbers is applied for aluminium plate with the longitudinal wave velocity $c_L=6300$ m/s, the transverse wave velocity $c_T=3100$ m/s and half- thickness of the plate $b=4$ mm. The construction of dispersion curves is carried out in the frequency range 0 - 1 MHz, whereby the circular frequency ω is converted into frequency f using equation $\omega=2\pi f$. In all figures, the symmetric modes of Lamb waves are shown by continuous and asymmetric modes by dashed curves, respectively. The dispersion curves are computed in Mathematica programming environment, using the strong capabilities of symbolic computing.

5.1 Dispersion curves for real wavenumbers

In general, several dispersion curves appear at each selected frequency interval, the calculation of which is time consuming. In order to limit the processing time, the computer program prescribes the number of symmetric and corresponding asymmetric modes of Lamb waves that we want to calculate. In the Fig. 3, first three symmetric and corresponding asymmetric modes of real wavenumbers are shown.

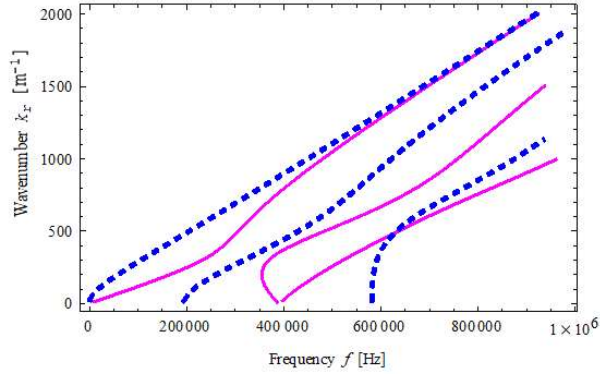


Figure 3: Dispersion curves of Lamb waves for real wavenumbers. Symmetric modes are presented by continuous curves, asymmetric modes by dashed curves.
Source: own.

5.2 Dispersion curves for imaginary wavenumbers

In third section it was shown that dispersion curves for imaginary wavenumbers can be constructed as the special case of the vector space method. However, there exists an alternative way, which is actually applied in this paper. If wavenumber is pure imaginary, it can be written as $k = i \cdot k_i$. Inserting this substitution into Eq. (14) for a and β and rename them as α_i and β_i , we get

$$\alpha_i = \sqrt{\left(\frac{\omega}{c_L}\right)^2 + k_i^2}, \quad \beta_i = \sqrt{\left(\frac{\omega}{c_T}\right)^2 + k_i^2},$$

$$\frac{\tan(\beta_i h)}{\tan(\alpha_i h)} + \left\{ \frac{4k_i^2 \alpha_i \beta_i}{\left[2k_i^2 + \left(\frac{\omega}{c_T}\right)^2\right]^2} \right\}^{\pm 1} = 0. \quad (28)$$

By using Eq. (28), the construction of dispersion curves proceeds in the same way as for real wavenumbers.

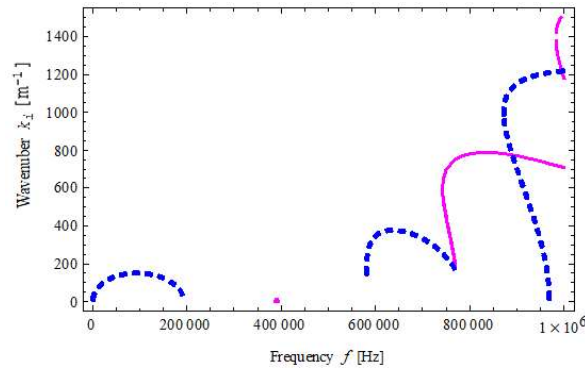


Figure 4: Dispersion curves of Lamb waves for imaginary wavenumbers. Symmetric modes are presented by continuous curves, asymmetric modes by dashed curves.

Source: own.

5.3 Dispersion curves for complex wavenumbers

The presented method can easily be used for the construction of dispersion curves for complex wavenumbers. Of course, in this case, it is necessary to separately present the course of the real and imaginary part of the complex wavenumber as a function of frequency, as shown, for example, in Fig. 5. Figures 5.a and 5.b show only individual symmetric (depicted by continuous line) and asymmetric wave mode (depicted by dashed line). Unlike the finite number of wave modes for real and imaginary wavenumbers, there are infinitely many wave modes for complex wave numbers [6].

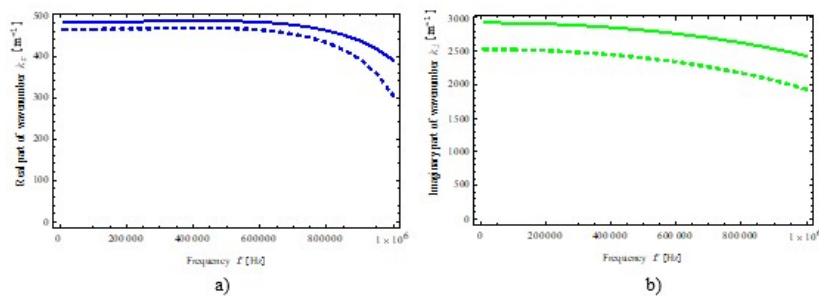


Figure 5: Dispersion curves of Lamb waves for complex wavenumbers: a) real part of complex wavenumber, b) imaginary part of complex wavenumber. Symmetric modes are presented by continuous curves, asymmetric modes by dashed curves.

Source: own.

6 The conclusions

In the paper, the necessary prediction capability is added to the classical Newton-Raphson method to enable successful construction of curves for transcendental dispersion relations, which is the characteristic of Rayleigh-Lamb waves. The prediction is performed by arc length continuation combined with cubic extrapolation to successfully prevent unexpected and undesired jumps between wave modes during the branch tracing. It is proved that the proposed supplement to the Newton-Raphson method works equally well in the cases of construction of dispersion curves for real, imaginary and complex wave numbers. In the future, we hope that the proposed method can be extended to be used in multilayer plates.

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