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Half-arc-transitive graphs of order $4p$ of valency twice a prime

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Abstract

A graph is half-arc-transitive if its automorphism group acts transitively on vertices and edges, but not on arcs. Let p be a prime. Cheng and Oxley [On weakly symmetric graphs of order twice a prime, J. Combin. Theory B 42(1987) 196-211] proved that there is no half-arc-transitive graph of order $2p$, and Alspach and Xu [$\frac{1}{2}$-transitive graphs of order $3p$, J. Algebraic Combin. 3(1994) 347-355] classified half-arc-transitive graphs of order $3p$. In this paper we classify half-arc-transitive graphs of order $4p$ of valency $2q$ for each prime $q \geq 5$. It is shown that such graphs exist if and only if $p - 1$ is divisible by $4q$. Moreover, for such p and q a unique half-arc-transitive graph of order $4p$ and valency $2q$ exists and this graph is a Cayley graph.

Keywords: Cayley graph, half-arc-transitive graph, transitive graph.

Math. Subj. Class.: 05C25, 20B25

1 Introduction

Throughout this paper we denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of residue classes modulo n, and by $\mathbb{Z}_n^*$ the multiplicative group of the ring $\mathbb{Z}_n$. Let D_{2n} be the dihedral group of order $2n$, and let A_n and S_n be the alternating and symmetric group of degree n, respectively. All graphs are assumed to be finite, simple and undirected, but with an implicit orientation of the edges when appropriate. For a graph X, let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively. A graph X is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive* (*symmetric*) if $\mathrm{Aut}(X)$ acts transitively on $V(X)$, $E(X)$, or $A(X)$, respectively, and *half-arc-transitive* if X is vertex-transitive and edge-transitive, but not arc-transitive.

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More generally, by a half-arc-transitive action of a subgroup G of $\mathrm{Aut}(X)$ on a graph X we shall mean a vertex-transitive and edge-transitive, but not arc-transitive action of G on X. In this case, we shall say that the graph X is *G-half-arc-transitive*.

The investigation of half-arc-transitive graphs was initiated by Tutte [34] and he proved that a vertex- and edge-transitive graph with odd valency must be arc-transitive. In 1970 Bouwer [4] constructed a $2k$-valent half-arc-transitive graph for every $k \geq 2$ and later more such graphs were constructed (see [10, 15, 17, 18, 33]). Let p, q be odd primes. It is well-known that there are no half-arc-transitive graphs of order p or p^2, and by Cheng and Oxley [6], there are no half-arc-transitive graphs of order $2p$. Alspach and Xu [2] classified half-arc-transitive graphs of order $3p$ and Wang [35] classified half-arc-transitive graphs of order a product of two distinct primes. Despite all of these efforts, however, more classifications of half-arc-transitive graphs with general valencies seem to be very difficult. For example, classification of half-arc-transitive graphs of order $4p$ has been considered for more than 10 years by many authors, but it has still not been completed. Recently, classifications of tetravalent and hexavalent half-arc-transitive graphs of order $4p$ were given in [13] and [37], respectively. In fact, investigation of half-arc-transitive graphs of small valencies is currently an active topic in algebraic graph theory. For more information, see [1, 7, 11, 12, 14, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 38, 39]. In this paper we classify $2q$-valent half-arc-transitive graphs of order $4p$ for each prime $q \geq 5$. It is shown that such graphs are Cayley and exist if and only if $p-1$ is divisible by $4q$. Moreover, for a given order such a graph is unique.

To end this section, we introduce the so called quotient graph of a graph X. Let $\Sigma = \{B_0, B_1, \ldots, B_{n-1}\}$ be a partition of $V(X)$. The *quotient graph X_Σ of X relative to the partition* Σ is defined to have vertex set and edge set as follows:

$$V(X_\Sigma) = \Sigma,$$
$$E(X_\Sigma) = \{\{B_i, B_j\} \mid \text{there exist } v_i \in B_i,\ v_j \in B_j \text{ such that } \{v_i, v_j\} \in E(X)\}.$$

In particular, if $N \leq \mathrm{Aut}(X)$ then the set of orbits of N on $V(X)$ is a partition of $V(X)$. In this case, the quotient graph of X relative to the orbits of N is also called the *quotient graph* of X relative to N, denoted by X_N. It is easy to see that if $N \trianglelefteq G \leq \mathrm{Aut}(X)$ and G is transitive on edges of X then the valency of X_N is a divisor of the valency of X.

2 Preliminary results

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. The following facts about Cayley graph are well known (see [3]). Given $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg, x \in G$. Then *the right regular representation* $R(G) = \{R(g) \mid g \in G\}$ is a regular subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$, and $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of the stabilizer $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$ of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Furthermore, A graph X is isomorphic to a Cayley graph on G if and only if its automorphism group $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices.

A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $\mathrm{Aut}(\mathrm{Cay}(G, S))$ contains $R(G)$ as a normal subgroup. The following proposition is fundamental for normal Cayley graphs.

Proposition 2.1. [38, Proposition 1.5] *Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph on a finite group G with respect to S. Let $A = \mathrm{Aut}(X)$ and let A_1 be the stabilizer of 1 in A. Then X*

is normal if and only if $A_1 = \mathrm{Aut}(G, S)$.

Cheng and Oxley [6] classified the connected symmetric graphs of order $2p$ for a prime p. To extract a classification of connected q-, $2q$- and $4q$-valent symmetric graphs of order $2p$ for a prime $q \geq 5$, we need to define some graphs. Let V and $V^{'}$ be two disjoint copies of $\mathbb{Z}_p$, say $V = \{i \mid i \in \mathbb{Z}_p\}$ and $V' = \{i' \mid i \in \mathbb{Z}_p\}$. Let r be a positive integer dividing $p-1$ and $H(p,r)$ the unique subgroup of $\mathbb{Z}_p^*$ of order r. Define the graph $G(2p, r)$ to have vertex set $V \cup V^{'}$ and edge set $\{xy' \mid x, y \in \mathbb{Z}_p, y-x \in H(p,r)\}$. Clearly, $G(2p, p-1) \cong K_{p,p} - pK_2$, the complete bipartite graph of order $2p$ minus a 1-factor. Furthermore, assume that r is an even integer dividing $p-1$. Then the graph $G(2,p,r)$ is defined to have vertex set $V \cup V^{'}$ and edge set $\{xy, x'y, xy', x'y' \mid x, y \in \mathbb{Z}_p, y-x \in H(p,r)\}$. The *lexicographic product* $X[Y]$ of graph X by graph Y is the graph with vertex set $V(X[Y]) = V(X) \times V(Y)$ and with two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ adjacent whenever x_1 is adjacent to x_2, or x_1=x_2 and y_1 is adjacent to y_2. Clearly, if X is symmetric and Y is a graph with no edge, then $X[Y]$ is symmetric. Moreover, $G(2,p,r)$ is in fact the lexicographic product of a circulant $Cay(\mathbb{Z}_p, H(p,r))$ by $2K_1$.

Proposition 2.2. [6, Theorem 2.4 and Table 1] *Let p, q be odd primes with $q \geq 5$ and let X be a connected edge-transitive graph of order $2p$. Then X is symmetric. Furthermore, if X has valency q then one of the following holds:*

(1) $X \cong K_{2p}$, *the complete graph of order $2p$, and $2p-1 = q$;*

(2) $X \cong K_{p,p}$, *the complete bipartite graph of order $2p$, and $p = q$;*

(3) $X \cong G(2p, q)$ *with* $q \mid (p-1)$ *and* $(p,q) \neq (11, 5)$, *and* $\mathrm{Aut}(X) \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$;

(4) $X \cong G(2 \cdot 11, 5)$ *and* $\mathrm{Aut}(X) \cong \mathrm{PSL}(2, 11) \rtimes \mathbb{Z}_2$.

If X has valency $2q$ then X is bipartite and one of the following holds:

(5) *For* $2q < p-1$, $X \cong G(2p, 2q)$ *with* $2q \mid (p-1)$ *and* $\mathrm{Aut}(X) \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_{2q}) \rtimes \mathbb{Z}_2$;

(6) *For* $2q = p-1$, $X \cong K_{p,p} - pK_2$ *and* $\mathrm{Aut}(X) \cong S_p \rtimes \mathbb{Z}_2$.

If X has valency $4q$ then one of the following holds:

(7) *X is non-bipartite,* $X \cong G(2, p, 2q)$ *with* $2q \mid p-1$; *for* $2q < p-1$, $\mathrm{Aut}(X) \cong \mathbb{Z}_2^p \rtimes (\mathbb{Z}_p \rtimes \mathbb{Z}_{2q})$ *and for* $2q = p-1$, $\mathrm{Aut}(X) \cong \mathbb{Z}_2^p \rtimes S_p$;

(8) *X is bipartite and* $X \cong G(2p, 4q)$ *with* $4q \mid (p-1)$; *for* $4q < p-1$, $\mathrm{Aut}(X) \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_{4q}) \rtimes \mathbb{Z}_2$ *and for* $4q = p-1$, $X \cong K_{p,p} - pK_2$ *and* $\mathrm{Aut}(X) \cong S_p \rtimes \mathbb{Z}_2$.

Let G act transitively on a set Ω. Then G induces a natural action on $\Omega \times \Omega$ defined by $(x,y)^g = (x^g, y^g)$ for $(x,y) \in \Omega \times \Omega$ and $g \in G$. The orbits of G on $\Omega \times \Omega$ are called *orbitals* of G. The orbital $\Delta = \{(x,x) \mid x \in \Omega\}$ of G is *trivial* and all other orbitals of G are *nontrivial*. Let O be a nontrivial orbital of G. The pair (Ω, O), denoted by $\mathcal{O}$, is a directed graph with vertex set Ω and directed edge set O, called the *orbital digraph* of G relative to O. For any orbital O of G, it is easy to show that $O^* = \{(\alpha, \beta) \mid (\beta, \alpha) \in O\}$ is also an orbital of G, called the *paired orbital* of O, and O is said to be *self-paired* if $O^* = O$. Clearly, if O is a non-self-paired orbital then the underlying graph of $\mathcal{O}$ is G-half-arc-transitive. Conversely, if X is a half-arc-transitive graph then X is the underlying graph of an orbital digraph $(V(X), O)$ of $\mathrm{Aut}(X)$ relative to a non-self-paired orbital O. In this case, $\mathrm{Aut}(X)$ coincides with the automorphism group of the digraph $(V(X), O)$. This implies the following proposition.

Proposition 2.3. *Let X be a connected half-arc-transitive graph of valency $2n$. Let $A = \text{Aut}(X)$ and let A_u be the stabilizer of $u \in V(X)$ in A. Then each prime divisor of $|A_u|$ is a divisor of $n!$.*

By [10, Lemma 2.2], we have the following proposition (also see [1, 19, 31, 36]).

Proposition 2.4. *The smallest half-arc-transitive graph has order 27. The smallest vertex-primitive half-arc-transitive graph of order kp, with p a prime and $k < p$, has order 253.*

The following proposition is straightforward (also see [13, Propositions 2.1 and 2.2]).

Proposition 2.5. *Let $X = \text{Cay}(G, S)$ be a half-arc-transitive graph. Then, there is no involution in S, and no $\alpha \in \text{Aut}(G, S)$ such that $s^\alpha = s^{-1}$ for some $s \in S$. In particular, there are no half-arc-transitive Cayley graphs on abelian groups.*

Let G be a transitive permutation group on a set Ω. A nonempty subset Δ of Ω is called a *block* for G if for each $g \in G$, either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \phi$. Clearly, the set Ω, the empty set ϕ, and the sets $\{\alpha\}$ consisting of only one point are blocks of G on Ω. We call these *trivial blocks*. A transitive group G is said to be *primitive* if G has only trivial blocks, and *imprimitive* if there is at least one non-trivial block. The *socle* of a finite group G, denoted by $\text{soc}(G)$, is the product of all minimal normal subgroups of G. One may extract the following results from [20, Table 3].

Proposition 2.6. *Let p be a prime and G a primitive group of degree n.*

(1) *For $n = p$, G is either solvable with a normal Sylow p-subgroup or non-solvable with the following table, where d and k denote the degree and transitive multiplicity, respectively.*

soc(G)	d	k	comment
A_p	p	$p-2$	$G = A_p$
A_p	p	p	$G = S_p$
$\text{PSL}(2, 2^{2^s})$	$p = 2^{2^s} + 1$	3	$s > 0$
$\text{PSL}(n, q)$	$p = \frac{q^n - 1}{q - 1}$	2	$n \geq 3$, n odd
$\text{PSL}(2, 11)$	11	2	
M_{11}	11	4	
M_{23}	23	4	

(2) *For $n = 2p$, either G is 2-transitive or $p = 5$.*

(3) *For $n = 4p$, either G is 2-transitive or $p = 7$, 13, or 17.*

One may deduce Proposition 2.6(1) and 2.6(2) also from [9, Corollary 3.5B] and [6, Theorem 1.1], respectively. Moreover, if G is primitive, but not 2-transitive of degree $2p$ then $p = 5$ and $G \cong A_5$ or S_5. If G is primitive, but not 2-transitive of degree $4p$ then $G \cong A_8$, S_8, $\text{PSL}(2, 8)$ or $\text{PGL}(2, 7)$ for $p = 7$, $G \cong \text{Aut}(\text{PSL}(3, 3))$ for $p = 13$, or G is isomorphic to a subgroup between $\text{PSL}(2, 16)$ and $\text{P}\Gamma\text{L}(2, 16)$ for $p = 17$.

3 Main result

Let p, q be odd primes with $q \geq 5$. In this section we classify the $2q$-valent half-arc-transitive graphs of order $4p$. Following the notation of Kwak et al. [16], for any integer ℓ and any positive integer t, define

$$\ell[t] = \ell^{t-1} + \ell^{t-2} + \cdots + 1.$$

Let $G = \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^r \rangle$, where $r^2 \equiv -1 \pmod p$. It is easy to see that r is an element of order 4 in $\mathbb{Z}_p^*$ and the group G is independent of the choice of r. In particular, $p-1$ is divisible by 4. Furthermore, assume that $p-1$ is divisible by q. Then there is a unique subgroup of order q, say $\langle s \rangle$, in $\mathbb{Z}_p^*$. Set $T = \{b, a^{s[1]}b, a^{s[2]}b, a^{s[3]}b, \ldots, a^{s[q-1]}b\}$ and $S = T \cup T^{-1}$. Define

$$\mathcal{C}_{4p}^{2q} = \mathrm{Cay}(G, S). \tag{3.1}$$

Clearly, $s[i] \not\equiv 0 \pmod p$ for each $1 \leq i \leq q-1$, that is $s[i] \in \mathbb{Z}_p^*$, and one may easily show that $\mathcal{C}_{4p}^{2q}$ is a connected graph of order $4p$ and of valency $2q$.

Clearly, s^i $(1 \leq i \leq q-1)$ are all the elements of order q in $\mathbb{Z}_p^*$. Set $S_i = T_i \cup T_i^{-1}$ with $T_i = \{b, a^{s^i[1]}b, a^{s^i[2]}b, a^{s^i[3]}b, \ldots, a^{s^i[q-1]}b\}$. Then $T_1 = T$ and $S_1 = S$. For any given $1 \leq i \leq q-1$, the map $a \mapsto a^{s[i]^{-1}}$, $b \mapsto b$, induces an automorphism of G, say β_i. For each $1 \leq j \leq q-1$, there is $1 \leq k \leq q-1$ such that $s^j = (s^i)^k$. It follows that

$$s[j]s[i]^{-1} = \frac{1-s^j}{1-s}\,\frac{1-s}{1-s^i} = \frac{1-(s^i)^k}{1-s^i} = s^i[k],$$

which means $T_1^{\beta_i} = T_i$. Thus, β_i is an isomorphism from $\mathrm{Cay}(G, S)$ to $\mathrm{Cay}(G, S_i)$. Therefore, $\mathcal{C}_{4p}^{2q}$ is independent of the choice of the element s of order q in $\mathbb{Z}_p^*$.

Denote by α the automorphism of G induced by $a \mapsto a^s$, $b \mapsto ab$. Note that $s[q] = s^{q-1} + s^{q-2} + \cdots + s + 1 = 0 \pmod p$. Then $b^\alpha = ab$, $(a^{s[i]}b)^\alpha = a^{s[i+1]}b$ for each $1 \leq i \leq q-2$, and $(a^{s[q-1]}b)^\alpha = b$. It follows that $\alpha \in \mathrm{Aut}(G, S)$, implying that $\mathrm{Aut}(\mathcal{C}_{4p}^{2q})$ has an edge-transitive subgroup of order $4pq$, that is $R(G) \rtimes \langle \alpha \rangle$. It will be shown in Lemma 3.2 that $\mathrm{Aut}(\mathcal{C}_{4p}^{2q}) = R(G) \rtimes \langle \alpha \rangle$ and hence $\mathcal{C}_{4p}^{2q}$ is half-arc-transitive. The following is the main result of this paper.

Theorem 3.1. *Let p, q be odd primes with $q \geq 5$. Then X is a $2q$-valent half-arc-transitive graph of order $4p$ if and only if $4q \mid (p-1)$ and $X \cong \mathcal{C}_{4p}^{2q}$ with $\mathrm{Aut}(\mathcal{C}_{4p}^{2q}) = R(G) \rtimes \langle \alpha \rangle$.*

The sufficiency is proved as a part of Lemma 3.2 and the necessity is proved in Lemma 3.3. For a graph X and $u \in V(X)$, let $N_d(u)$ denote the set of vertices having distance d from u in X. If X is a directed graph, let $N^+(u)$ denote the set of out-neighbors of u and $N^-(u)$ the set of in-neighbors of u.

Lemma 3.2. *Let p, q be odd primes with $q \geq 5$. Let $X = \mathrm{Cay}(G, S)$ be a $2q$-valent half-arc-transitive Cayley graph on a group G of order $4p$. Then $G \cong \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^r \rangle$ with $r^2 = -1 \pmod p$. Moreover, for each prime p satisfying $4q \mid (p-1)$, the Cayley graph $\mathcal{C}_{4p}^{2q}$ (defined in Eq (3.1)) is half-arc-transitive and $\mathrm{Aut}(\mathcal{C}_{4p}^{2q}) = R(G) \rtimes \langle \alpha \rangle$.*

Proof. Since there are no half-arc-transitive graphs of order p or $2p$ (see [5, 6]), X is connected. Thus, $|S| = 2q$, $S^{-1} = S$ and $\langle S\rangle = G$. By Proposition 2.5, G is non-abelian and by Proposition 2.4, $p \geq 7$. From elementary group theory, all non-abelian groups of order $4p$ for every odd prime $p \geq 7$, up to isomorphism, can be written as follows:

$$\begin{aligned} G_1(p) &= \langle a, b \mid a^{2p} = b^2 = 1, b^{-1}ab = a^{-1}\rangle, \\ G_2(p) &= \langle a, b \mid a^{2p} = 1, b^2 = a^p, b^{-1}ab = a^{-1}\rangle, \\ G_3(p) &= \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^r\rangle,\ r^2 \equiv -1 \pmod p. \end{aligned}$$

Clearly, $G_3(p)$ exists if and only if $p \equiv 1 \pmod 4$, and $G_1(p)$ is the dihedral group D_{4p}. By Proposition 2.5, there is no involution in S, and since $G = \langle S\rangle$, one has $G \neq G_1(p)$. Suppose $G = G_2(p)$. Then S contains at least one element of order 4 and its inverse. Each element of order 4 is of the form a^ib or a^ib^{-1}. The automorphism of $G_2(p)$ induced by $b \mapsto b^{-1}$, $a \mapsto a$, maps a^jb to $(a^jb)^{-1}$ for any integer j and fixes $\langle a\rangle$ pointwise. This is impossible by Proposition 2.5. Thus, $G = G_3(p)$ and $4 \mid p - 1$.

Let $4q \mid (p-1)$, and let s be an element of order q in $\mathbb{Z}_p^*$. Recall that $\mathcal{C}_{4p}^{2q} = \mathrm{Cay}(G, S)$ and $R(G) \rtimes \langle\alpha\rangle \leq \mathrm{Aut}(\mathcal{C}_{4p}^{2q})$, where $S = T \cup T^{-1}$ with $T = \{b, a^{s[1]}b, a^{s[2]}b, a^{s[3]}b, \ldots, a^{s[q-1]}b\}$, and α is the automorphism of G of order q induced by $a \mapsto a^s$, $b \mapsto ab$. To finish the proof, it suffices to show that $\mathrm{Aut}(\mathcal{C}_{4p}^{2q}) = R(G) \rtimes \langle\alpha\rangle$.

Let $G^* = \langle a, b^2\rangle$. Then G^* is isomorphic to D_{2p}. It is easy to see that $\mathcal{C}_{4p}^{2q}$ is bipartite with partite sets $U = G^*$ and $U' = bG^*$. Set $X = \mathcal{C}_{4p}^{2q}$ and $A = \mathrm{Aut}(X)$. Let A^* be the subgroup of A fixing the partite sets U and U' of X setwise. Then A^* is transitive on both U and U' and $|A : A^*| = 2$, implying $A^* \lhd A$. Since $R(G) \leq A$, we have $A = \langle A^*, R(b)\rangle$. Set $R(G^*) = \{R(g) \mid g \in G^*\}$. Then $R(G^*) \leq R(G)$, and $R(G^*)$ fixes U and U' setwise. Hence, $R(G^*) \leq A^*$. Since X has valency $2q$ and $4q \mid (p-1)$, one has $|A| = 4pqn$, where each prime divisor of n is less than p.

Claim 1: A is solvable.

Suppose that A^* is 2-transitive in its action on U. Then $U \setminus \{1\} = N_2(1)$, the set of vertices having distance 2 from 1. Every vertex in $U \setminus \{1\}$ has the same number of neighbors in $N_1(1)$, say k. Then $(2p-1)k = 2q(2q-1)$, and since $(2p-1, 2q) = 1$, one has $k = 2q$, implying $p = q$, contrary to the fact that $4q \mid p-1$. Since $q \geq 5$ and $4q \mid p-1$, we have $p \geq 29$, and by Proposition 2.6, A^* is imprimitive on U.

Let B be a non-trivial block of A^* on U containing 1. Then B is also a block of $R(G^*)$, forcing that B is a subgroup of G^*. Thus, $\Sigma_1 = \{Bg \mid g \in G^*\}$ is a complete block system of A^* on U. Furthermore, $\Sigma_2 = \Sigma_1^{R(b)} = \{Bg \mid g \in bG^*\}$ is a complete block system of A^* on U'. Recall that $A = \langle A^*, R(b)\rangle$. It follows that $\Sigma = \Sigma_1 \cup \Sigma_2 = \{Bg \mid g \in G\}$ is a complete block system of A on $V(X)$. Then the quotient graph X_Σ is bipartite and the edge-transitivity of X implies that X_Σ is edge-transitive. Let K be the kernel of A acting on Σ. Since $|U| = 2p$, one has $|B| = p$ or 2.

Assume $|B| = p$. Then $|\Sigma| = 4$ and since X_Σ is bipartite, X_Σ is a 4-cycle, say $X_\Sigma = (B_0, B_1, B_2, B_3)$ with $B_0 = B$. The induced subgraph $\langle B_i, B_{i+1}\rangle$ of $B_i \cup B_{i+1}$ in X has order $2p$ and valency q, which cannot be isomorphic to $K_{p,p}$ because $4q \mid (p-1)$. Since $p \geq 29$, Proposition 2.2 implies that $\langle B_i, B_{i+1}\rangle \cong G(2p, q)$ and $|\mathrm{Aut}(\langle B_i \cup B_{i+1}\rangle)| = 2pq$. Any Sylow p-subgroup of A is a subgroup of K, and since $|B_i| = p$, K is primitive on B_i. Suppose that K is unfaithful on B_i. Then the kernel of K on B_i is transitive on B_{i+1}, forcing $\langle B_i, B_{i+1}\rangle \cong K_{p,p}$, a contradiction. Thus, K acts faithfully on B_i, implying that

$|K|$ is a divisor of $|\mathrm{Aut}(\langle B_i \cup B_{i+1}\rangle)| = 2pq$. This means that K is solvable. Also, A/K is solvable because $A/K \leq \mathrm{Aut}(X_\Sigma) \cong D_8$. It follows that A is solvable.

Assume $|B| = 2$. Then K is a 2-group and hence solvable. By Proposition 2.2, X_Σ is a connected symmetric graph of order $2p$. Let X_Σ be of valency d and let k be the number of edges between B and Bb in X (note that B is indeed adjacent to Bb in X_Σ). Clearly, $k \leq 4$ and $2 \cdot 2q = k \cdot d$. Since $q \geq 5$, $k \neq 3$ and $d = q$, $2q$, or $4q$. Recall that $4q \mid (p-1)$ and $p \geq 29$. If $4q < p-1$, or $4q = p-1$ and X_Σ has valency q or $2q$ then, by Proposition 2.2, $X_\Sigma \cong G(2p, q)$, $G(2p, 2q)$ or $G(2p, 4q)$ and $\mathrm{Aut}(X_\Sigma) \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$, $(\mathbb{Z}_p \rtimes \mathbb{Z}_{2q}) \rtimes \mathbb{Z}_2$ or $(\mathbb{Z}_p \rtimes \mathbb{Z}_{4q}) \rtimes \mathbb{Z}_2$. In all these cases, $\mathrm{Aut}(X_\Sigma)$ is solvable, and since $A/K \leq \mathrm{Aut}(X_\Sigma)$, A/K is solvable. Thus, A is solvable. Now one may assume $4q = p-1$ and X_Σ has valency $4q$. In this case, $X_\Sigma \cong K_{p,p} - pK_2$ and there is exactly one edge in X between B and Bb, which forces that $K = 1$. It follows that A^* is faithful on Σ_1. Since $|\Sigma_1| = p$, by Proposition 2.6 either $\mathrm{soc}(A^*)$ is non-solvable and 2-transitive on Σ_1 or A^* is solvable.

Suppose that $\mathrm{soc}(A^*)$ is non-solvable and 2-transitive on Σ_1. If $\mathrm{soc}(A^*)$ is not 3-transitive on Σ_1 then, by Proposition 2.6, $\mathrm{soc}(A^*) \cong \mathrm{PSL}(m, r)$ because $p \geq 29$. In this case, $m \geq 3$ is an odd number, r is a prime-power and $p = 1 + r + r^2 + \cdots + r^{m-1}$. Since m is odd, $r(1+r) \mid (p-1)$, which is impossible because $p - 1 = 4q$ and $q \geq 5$. Thus, $\mathrm{soc}(A^*)$ is 3-transitive and hence A^* is 3-transitive on Σ_1. Since $X_\Sigma \cong K_{p,p} - pK_2$, one may let $\Sigma_1 = \{B_i \mid i \in \mathbb{Z}_p\}$ and $\Sigma_2 = \{B'_i \mid i \in \mathbb{Z}_p\}$ such that for $k, \ell \in \mathbb{Z}_p$, B_k is adjacent to B'_ℓ in X_Σ if and only if $k \neq \ell$. Note that for $i, j \in \mathbb{Z}_p$ and $i \neq j$, there is exactly one edge in X between B_i and B'_j. This implies that for any $\alpha \in A$, we have: if α fixes B_i and B'_j setwise then it fixes every vertex in B_i and B'_j because $|B_i| = |B'_j| = 2$. Let H be the subgroup of A^* fixing B_0 and B_1. Clearly, H fixes B'_0 and B'_1, and hence H fixes every vertex in $B_0 \cup B_1 \cup B'_0 \cup B'_1$. By the 3-transitivity of A^* on Σ_1, H is transitive on $\Sigma_1 \backslash \{B_0, B_1\}$, forcing that X has valency $p - 1 = 4q$, contrary to the fact that X has valency $2q$. Thus, A^* is solvable, and since $|A : A^*| = 2$, A is solvable. This completes the proof of Claim 1.

Let P be a Sylow p-subgroup of A. Then $|P| = p$ because $|A| = 4pqn$ with each prime divisor of n less than p.

Claim 2: $P \trianglelefteq A$.

Let N be a minimal normal subgroup of A. By Claim 1, N is elementary abelian, and since $|V(X)| = 4p$, N is a p-group or a 2-group. If N is a p-group then $P = N \trianglelefteq A$. Thus, one may assume that $N \cong \mathbb{Z}_2^r$ for some positive integer r. Since $q \geq 5$, the quotient graph X_N of X relative to N has valency q or $2q$. Let H be the kernel of A acting on $V(X_N)$. Note that X_N is edge-transitive and the orbits of N are of length 2 or 4.

Suppose first that the orbits of N have length 2. Then H is a 2-group and $|X_N| = 2p$. If X_N has valency q, by Proposition 2.2, $X_N \cong G(2p, q)$ is a q-valent symmetric graph of order $2p$ and $A/H \leq \mathrm{Aut}(G(2p, q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. Note that $G = G_3(p)$ and G has no non-trivial normal 2-subgroup. This implies that $R(G) \cap H = 1$ and $R(G) = R(G)/(R(G) \cap H) \cong R(G)H/H \leq A/H$. Since $4 \mid |R(G)|$, one has $4 \mid |\mathrm{Aut}(G(2p, q))|$, a contradiction. Thus, X_N has valency $2q$. In this case, $H_1 = 1$ and $H = N \cong \mathbb{Z}_2$. By Proposition 2.2, $X_N \cong G(2p, 2q)$ and $\mathrm{Aut}(X_N)$ has a normal Sylow p-subgroup. It follows that $PH/H \trianglelefteq A/H$, that is $PH \trianglelefteq A$. Since $|H| = 2$, P is characteristic in PH and hence $P \trianglelefteq A$.

Suppose now that the orbits of N have length 4. Then $|X_N| = p$ and X_N cannot have valency q. Thus, X_N has valency $2q$. In this case, $H_1 = 1$ and $H = N \cong \mathbb{Z}_2^2$. Since

$4q \mid (p-1)$, X_N cannot be a complete graph, implying that A/H cannot be 2-transitive on $V(X_N)$. By Proposition 2.6, A/H has a normal Sylow p-subgroup, that is $PH/H \trianglelefteq A/H$. Since $|H| = 4$, P is characteristic in PH and hence $P \trianglelefteq A$. This completes the proof of Claim 2.

Now we are ready to finish the proof by showing $\mathrm{Aut}(\mathcal{C}_{4p}^{2q}) = R(G) \rtimes \langle\alpha\rangle$. By Claim 2, $P \trianglelefteq A$. Since X is bipartite, the quotient graph X_P of X relative to P is a 4-cycle, say $X_P = (O_0, O_1, O_2, O_3)$. Recall that α is the automorphism of G induced by $a \mapsto a^s$ and $b \mapsto ab$. Since α has order q, α fixes O_i setwise for each $i \in \mathbb{Z}_4$. The induced subgraph $\langle O_i, O_{i+1}\rangle$ of $O_i \cup O_{i+1}$ in X, $i \in \mathbb{Z}_4$, is a q-valent edge-transitive graph of order $2p$. By Proposition 2.2, $\langle O_i, O_{i+1}\rangle \cong G(2p, q)$ and $\mathrm{Aut}(G(2p, q)) \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. Let L be the kernel of A acting on $V(X_P)$. Then $P \leq L$ and $\alpha \in L$, implying that $pq \mid |L|$. Since $|O_i| = p$ ($i \in \mathbb{Z}_4$), L is primitive on O_i. Suppose that L is unfaithful on O_i. Then the kernel of L on O_i is transitive on O_{i+1} because L is primitive on O_{i+1}, which implies that the induced subgraph $\langle O_i, O_{i+1}\rangle$ is isomorphic to $K_{p,p}$. It follows that $2p = 2q$, contrary to the fact that $4q \mid (p-1)$. Thus, L acts faithfully on O_i and $L \lesssim \mathrm{Aut}(G(2p, q))$, implying $|L| \mid 2pq$. It follows that $|L| = pq$ because L is intransitive on $O_i \cup O_{i+1}$. Note that $A/L \leq \mathrm{Aut}(X_P) \cong D_8$. Since $R(G) \rtimes \langle\alpha\rangle \leq A$, one has $|A| = 4pq$ or $8pq$.

Suppose $|A| = 8pq$. Then $A/L = \mathrm{Aut}(X_P) \cong D_8$ and consequently X is symmetric. Furthermore, $R(G) \rtimes \langle\alpha\rangle \trianglelefteq A$ and $|A_1| = 2q$. Noting that $R(G)$ is characteristic in $R(G) \rtimes \langle\alpha\rangle$, we have $R(G) \trianglelefteq A$. By Proposition 2.1, $A_1 = \mathrm{Aut}(G, S)$ and hence $\mathrm{Aut}(G, S) = \langle\alpha, \beta\rangle$, where β is an involution in $\mathrm{Aut}(G, S)$. Recall that $T = \{b, a^{s[1]}b, a^{s[2]}b, a^{s[3]}b, \ldots, a^{s[q-1]}b\}$, $S = T \cup T^{-1}$, $X = \mathrm{Cay}(G, S)$, and α permutes the elements of T cyclically. The arc-transitivity of A implies that β interchanges T and T^{-1}. Thus, there is an i such that $b^{\beta\alpha^i} = b^{-1}$, and since $\langle a\rangle$ is characteristic in G, $a^{\beta\alpha^i} = a^t$ for some $t \in \mathbb{Z}_p^*$. Note that G has an automorphism mapping a^t to a and b to b. It follows that G has an automorphism γ such that $a^\gamma = a$ and $b^\gamma = b^{-1}$. Since $b^{-1}ab = a^r$, one has $b^{-1}ab = bab^{-1}$, that is $b^2a = ab^2$, a contradiction. Thus, $|A| = 4pq$ and hence $A = R(G) \rtimes \langle\alpha\rangle$, as required. □

To finish the proof of Theorem 3.1, we only need to prove the following lemma.

Lemma 3.3. *Let p, q be odd primes with $q \geq 5$ and let X be a $2q$-valent half-arc-transitive graph of order $4p$. Then $4q \mid (p-1)$ and $X \cong \mathcal{C}_{4p}^{2q}$.*

Proof. Since $q \geq 5$, X has valency at least 10, and since there are no half-arc-transitive graphs of order p or $2p$ (see [5, 6]), X is connected. Let $A = \mathrm{Aut}(X)$. Recall that X is an underlying graph of an orbital digraph $\mathcal{O} := (V(X), O)$ of A for some non-self-paired orbital O. Thus, $A = \mathrm{Aut}(\mathcal{O})$ and $\mathcal{O}$ is a directed graph with out- and in-valency equal to q. Furthermore, A is transitive on the directed edges of $\mathcal{O}$. Since $V(X) = V(\mathcal{O})$ and $A = \mathrm{Aut}(\mathcal{O})$, in what follows we change the graph X to $\mathcal{O}$ when it is necessary. Let $u \in V(X)$ and denote by A_u the stabilizer of u in A. Since $|N^+(u)| = |N^-(u)| = q$ is a prime, A_u acts on $N^+(u)$ and $N^-(u)$ primitively, and there exits $\alpha \in A_u$ such that α has order q and permutes the elements in $N^+(u)$ cyclically. This implies that $4pq \mid |A|$, and by Proposition 2.4, $p \geq 7$. By Proposition 2.3, $|A| = 4pqn$, where each prime divisor of n is a divisor of $q!$. Let P be a Sylow p-subgroup of A.

Assume that A has a non-trivial normal 2-subgroup, say N. Then the orbits of N have length 2 or 4 and since $q \geq 5$, the quotient graph X_N of X relative to N has valency q or $2q$. Let L be the kernel of A acting on $V(X_N)$. Then $L \trianglelefteq A$ and $N \leq L$. Since

$|V(X)| = 4p$ and N is a 2-group, by Propostion 2.2, X_N is a symmetric graph of order p or $2p$.

Suppose first that the orbits of N have length 4. Then $|X_N| = p$, X_N cannot have valency q. Thus, X_N has valency $2q$ and the stabilizer L_u of u in L fixes the neighborhood of u in X pointwise because X and X_N have the same valency. Thus, $L_u = 1$ and L acts regularly on each orbit of N, forcing $|L| = 4$. It follows that PL is regular on $V(X)$ and hence X is a Cayley graph on PL. However, Lemma 3.2 implies that L cannot be normal in PL, a contradiction.

Suppose now that the orbits of N are of length 2. Then $|X_N| = 2p$. By Proposition 2.2, X_N is symmetric. If X_N has valency q then $X \cong X_N[2K_1]$, which is symmetric, a contradiction. Thus, X_N has valency $2q$. In this case, $L_u = 1$ and L acts regularly on each orbit of N. It follows that $L = N \cong \mathbb{Z}_2$ and the quotient graph X_N of X relative to N is A/N-half-arc-transitive. Note that PN/N is a Sylow p-subgroup of A/N. By Proposition 2.2, $PN \trianglelefteq A$ or $X_N \cong K_{p,p} - pK_2$ with $2q = p - 1$. For the former, $P \trianglelefteq A$ because $|N| = 2$. For the latter, X_N is bipartite. Let $(A/N)^*$ be the subgroup of A/N fixing the bipartite sets, say Σ and Σ', of X_N setwise. Since $X_N \cong K_{p,p} - pK_2$, $(A/N)^*$ acts faithfully on Σ and Σ', respectively. Since A/N is vertex-transitive on X_N, $|A/N : (A/N)^*| = 2$. Assume $\Delta \in \Sigma$ and $\Delta' \in \Sigma'$ such that Δ is not adjacent to Δ' in X_N, and let $(A/N)_\Delta$ and $(A/N)_{\Delta'}$ be the stabilizers of Δ and Δ' in A/N, respectively. Then $(A/N)_\Delta = (A/N)_{\Delta'} \leq (A/N)^*$. By the half-arc-transitivity of A/N on X_N, the stabilizer $(A/N)_\Delta$ cannot be transitive on $\Sigma' \backslash \{\Delta'\}$. It follows that $(A/N)^*$ is not 2-transitive on Σ and Σ'. Since $|\Sigma| = |\Sigma'| = p$, by Proposition 2.6, $(A/N)^*$ has a normal Sylow p-subgroup, which is also a normal Sylow p-subgroup of A/N because $|A/N : (A/N)^*| = 2$. It follows that $PN \trianglelefteq A$ and hence $P \trianglelefteq A$.

Now assume that A has no non-trivial normal 2-subgroups. Again we prove that $P \trianglelefteq A$. Suppose that A is primitive on $V(X)$. By Proposition 2.6, A is 2-transitive on $V(X)$ provided $p \neq 7, 13$ or 17. Since X is half-arc-transitive, A is not 2-transitive on $V(X)$. It follows that $p = 7, 13$ or 17, which is impossible by Proposition 2.4. Thus, A is imprimitive on $V(X)$. Let B be a non-trivial block of A on $V(X)$. Since $|V(X)| = 4p$, we have $|B| = 2, 4, p$ or $2p$. It follows that $\mathcal{B} = \{B^a \mid a \in A\}$ is a complete block system of A on $V(X)$. Consider the quotient graph $X_{\mathcal{B}}$ relative to $\mathcal{B}$ and let K be the kernel of A on $\mathcal{B}$. Then $A/K \leq \mathrm{Aut}(X_{\mathcal{B}})$. Since X is A-edge-transitive, $X_{\mathcal{B}}$ is A/K-edge-transitive. Let $B' \in \mathcal{B}$ be adjacent to B in $X_{\mathcal{B}}$ and let k be the number of edges in X between B and B'. Then $X_{\mathcal{B}}$ has valency $\frac{2q|B|}{k}$. Assume $u \in B$ and choose $B' \in \mathcal{B}$ such that B' contains an out-neighbor of u in $\mathcal{O}$. Recall that $\alpha \in A_u$ and α permutes the elements in $N^+(u)$ cyclically. Then α fixes B setwise. Since $|N^+(u)| = q$ is a prime, either B' contains exactly one out-neighbor of u in $\mathcal{O}$ or $N^+(u) \subseteq B'$. In particular, if $|B| = p$ or $2p$ then $N^+(u) \subseteq B'$ and $\alpha \in K$. If $|B| = 2$ or 4 then B' contains exactly one out-neighbor of u in $\mathcal{O}$. It follows $K_u = 1$ because K_u fixes each out-neighbor of u in $\mathcal{O}$. Thus, $|K| \leq 4$ and since $|V(X)| = 4p$ and A has no non-trivial normal 2-group, one has $K = 1$, implying $A \leq \mathrm{Aut}(X_{\mathcal{B}})$. Consider the four cases $|B| = 4, 2, p$ or $2p$, respectively.

Case I: $|B| = 4$.

In this case, $K = 1$ and $A \leq \mathrm{Aut}(X_{\mathcal{B}})$. Since $|\mathcal{B}| = p$, Proposition 2.6 implies that either $P \trianglelefteq A$ or A is 2-transitive on $\mathcal{B}$. First suppose that A is 2-transitive on $\mathcal{B}$. Then $X_{\mathcal{B}} \cong K_p$ and $(p-1)k = 4 \cdot 2q = 8q$, where k is the number of edges in X between B and B'. It follows that $k = 1$, 2 or 4. The 2-transitivity of A on $\mathcal{B}$ implies that the number of directed edges in $\mathcal{O}$ with direction from B to B' is equal to that of directed edges with direction from B' to B. Thus, half-arc-transitivity of X implies that $k \neq 1$, and hence $k = 2$ or 4 and $p - 1 = 2q$ or $4q$. Again by Proposition 2.6, $\mathrm{soc}(A) \cong A_p$, $\mathrm{PSL}(2, 2^{2^s})$ with $p = 2^{2^s} + 1$, $\mathrm{PSL}(m, r)$ with $p = \frac{r^m-1}{r-1}$, $\mathrm{PSL}(2, 11)$, M_{11} or M_{23}. If $\mathrm{soc}(A) \cong A_p$ then $|A|$ is divisible by $\frac{1}{2}p!$. From elementary number theory it is well known that there exists a prime t between q and $2q$. Since $p - 1 = 2q$ or $4q$, one has $t \mid |A|$, which is impossible because $|A| = 4pqn$ where each prime divisor of n is a divisor of $q!$. If $\mathrm{soc}(A) \cong \mathrm{PSL}(2, 2^{2^s})$ then $p - 1 = 2^{2^s} \neq 2q$ or $4q$, which is clearly impossible. If $\mathrm{soc}(A) \cong \mathrm{PSL}(m, r)$ then $p = \frac{r^m-1}{r-1} = r^{m-1} + \cdots + r + 1$ and $m \geq 3$ is odd. It follows that $r(1 + r) \mid (p - 1)$, which is also impossible because $p - 1 = 2q$ or $4q$ and $q \geq 5$ (note that $4 \cdot 5 + 1 = 21$ is not a prime). Thus, $\mathrm{soc}(A) \cong \mathrm{PSL}(2, 11)$, M_{11} or M_{23}. It follows that $A \cong \mathrm{PSL}(2, 11)$, $\mathrm{PGL}(2, 11)$, M_{11} or M_{23} because $|\mathrm{Out}(\mathrm{PSL}(2, 11))| = 2$, and $|\mathrm{Out}(M_{11})| = |\mathrm{Out}(M_{23})| = 1$ (see [8]). Note that $|\mathrm{PSL}(2, 11)| = 2^2 \cdot 3 \cdot 5 \cdot 11$, $|M_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $|M_{23}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Since $|V(X)| = 4p$, if $A \cong \mathrm{PSL}(2, 11)$ or $\mathrm{PGL}(2, 11)$ then A_u is a subgroup of A of order 15 or 30, respectively, which is not true by [8]. Similarly, $A \not\cong M_{11}$ or M_{23} because M_{11} and M_{23} have no subgroups of order $2^2 \cdot 3^2 \cdot 5$ and $2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, respectively. Thus, $P \trianglelefteq A$.

Case II: $|B| = 2$.

In this case $K = 1$ and $A \leq \mathrm{Aut}(X_{\mathcal{B}})$. By Proposition 2.2, $X_{\mathcal{B}}$ is symmetric. Let $B = \{u, v\}$ and $B' = \{u', v'\}$, and assume that (u, u') is a directed edge in $\mathcal{O}$. Suppose that u has two neighbors in X in B'. Since B' contains exactly one out-neighbor of u in $\mathcal{O}$, (v', u) is a directed edge in $\mathcal{O}$. Since A is transitive on the directed edges in $\mathcal{O}$, A contains an element mapping (u, u') to (v', u), forcing that $k \geq 3$. Recall that $X_{\mathcal{B}}$ has valency $\frac{2q|B|}{k}$. It follows that $k = 4$ and hence $X = X_{\mathcal{B}}[2K_1]$, which is symmetric, a contradiction. Thus, there are exactly 2 or 1 edges in X between B and B', meaning that $X_{\mathcal{B}}$ has valency $2q$ or $4q$. Recall that A has no non-trivial normal 2-subgroup and then, by Proposition 2.2, $P \trianglelefteq A$, or $X_{\mathcal{B}} \cong K_{p,p} - pK_2$ with $p - 1 = 2q$ or $4q$, or $X_{\mathcal{B}} \cong G(2, p, 2q)$ with $p - 1 = 2q$.

Assume that $X_{\mathcal{B}} \cong K_{p,p} - pK_2$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the bipartite sets of $X_{\mathcal{B}}$ such that $B \in \mathcal{B}_1$ and $B' \in \mathcal{B}_2$, and let A^* be the subgroup of A fixing $\mathcal{B}_1$ and $\mathcal{B}_2$ setwise. Then $|A : A^*| = 2$ and $A^* \trianglelefteq A$. There is a unique block $C \in \mathcal{B}_2$ which is not adjacent to B in $X_{\mathcal{B}}$. Let A_B and A_C be the block stabilizers of B and C in A, that is the subgroups of A fixing B and C, respectively. Then $A_B = A_C \leq A^*$ and A^* is faithful on $\mathcal{B}_1$ and $\mathcal{B}_2$. Clearly, $P \leq A^*$. By Proposition 2.6, $P \trianglelefteq A^*$ or A^* is 2-transitive on $\mathcal{B}_1$ and $\mathcal{B}_2$.

Suppose that A^* is 2-transitive on $\mathcal{B}_1$ and $\mathcal{B}_2$. Then $A_B = A_C$ is transitive on $\mathcal{B}_1 \backslash \{B\}$ and $\mathcal{B}_2 \backslash \{C\}$, respectively. Note that $p - 1 = 4q$ or $2q$. If $p - 1 = 4q$ then there is exactly one directed edge in $\mathcal{O}$ between B and B'. Since X is half-arc-transitive, A_B has two orbits on $\mathcal{B}_2 \backslash \{C\}$, a contradiction. Thus, $p - 1 = 2q$ and there are exactly two directed edges in $\mathcal{O}$ between B and B'. These two directed edges have different direction, that is one from B to B' and the other from B' to B, because A_B is transitive on $\mathcal{B}_2 \backslash \{C\}$. This means that the permutation β on $V(X)$ interchanging the two vertices in each block of $\mathcal{B}$ cannot be an automorphism of $\mathcal{O}$. On the other hand, since the induced subgraph $\langle B \cup B' \rangle$ of $B \cup B'$ in X is a matching, one has $\beta \in A$, contrary to the fact that $A \leq \mathrm{Aut}(\mathcal{O})$. Thus, $P \trianglelefteq A^*$ and hence $P \trianglelefteq A$.

Assume that $X_{\mathcal{B}} \cong G(2, p, 2q)$ with $p - 1 = 2q$. By Proposition 2.2, $\mathrm{Aut}(X_{\mathcal{B}}) \cong \mathbb{Z}_2^p \rtimes S_p$. Thus, $\mathrm{Aut}(X_{\mathcal{B}})$ has a normal subgroup N such that $N \cong \mathbb{Z}_2^p$. Since $|V(X_{\mathcal{B}})| = 2p$, the orbits of N have size 2, which are blocks of $\mathrm{Aut}(X_{\mathcal{B}})$ (in fact $X_{\mathcal{B}} \cong K_p[2K_1]$). It follows that A has blocks of size 4 on $V(X)$. By Case I, $P \trianglelefteq A$.

Case III: $|B| = p$.

In this case, $N^+(u) \subseteq B'$ and $\alpha \in K$, where $\alpha \in A_u$ permutes the elements in $N^+(u)$ cyclically. Since $|V(X_{\mathcal{B}})| = 4$, any element of order p in A fixes each vertex of $X_{\mathcal{B}}$. Thus, $pq \mid |K|$ and $X_{\mathcal{B}}$ is a 4-cycle, say $V(X_{\mathcal{B}}) = (B_0, B_1, B_2, B_3)$, where B_i is adjacent to B_{i+1} for each $i \in \mathbb{Z}_4$. Let $Y = \langle B_0 \cup B_1 \rangle$ be the subgraph induced by $B_0 \cup B_1$ in X. Then Y is a q-valent edge-transitive graph of order $2p$, and all edges in Y have the same direction either from B_0 to B_1 or from B_1 to B_0 in $\mathcal{O}$, forcing $A/K \cong \mathbb{Z}_4$. If $Y \cong K_{p,p}$ then $X \cong C_4[pK_1]$, which is symmetric, a contradiction. One may thus assume that $Y \ncong K_{p,p}$. If K is unfaithful on B_0 then the kernel of K on B_0 is transitive on B_1 because $|B_1| = p$, forcing that $Y \cong K_{p,p}$, a contradiction. Thus, $K \leq \mathrm{Aut}(Y)$. By Proposition 2.2, either $Y \cong G(2p, q)$ with $q \mid (p-1)$ and $(p, q) \neq (11, 5)$, and $\mathrm{Aut}(Y) \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$, or $Y \cong G(2 \cdot 11, 5)$ and $\mathrm{Aut}(Y) \cong \mathrm{PSL}(2, 11) \rtimes \mathbb{Z}_2$. For the former, since K fixes B_0 and B_1 setwise and $pq \mid |K|$, one has $|K| = pq$. For the latter, $Y \cong G(2 \cdot 11, 5)$ and $\mathrm{Aut}(Y) \cong \mathrm{PSL}(2, 11) \rtimes \mathbb{Z}_2$. Since K fixes B_0 and B_1 setwise, $K \lesssim \mathrm{PSL}(2, 11)$. Suppose that $K \cong \mathrm{PSL}(2, 11)$. Let $C_A(K)$ be the centralizer of K in A. Then $C_A(K) \cap K = 1$ because K is non-abelian simple. Since $C_A(K) \cong C_A(K)/(K \cap C_A(K)) \cong KC_A(K)/K \leq A/K \cong \mathbb{Z}_4$ and A has no non-trivial normal 2-subgroup, one has $C_A(K) = 1$. It follows that $A = A/C_A(K) \lesssim \mathrm{Aut}(\mathrm{PSL}(2, 11)) \cong \mathrm{PGL}(2, 11)$, contrary to the fact that $A/K \cong \mathbb{Z}_4$. This implies that K is isomorphic to a proper subgroup of $\mathrm{PSL}(2, 11)$ of order divisible by 55. By [8], $|K| = 55$. Thus, we always have $|K| = pq$. Since $|A| = 4pq$, $P \trianglelefteq K$ and hence $P \trianglelefteq A$.

Case IV: $|B| = 2p$.

In this case, X is a bipartite graph with bipartite sets B and B'. If $p = q$, then $X \cong K_{2p,2p}$ is symmetric, a contradiction. Thus $q < p$. Let A^* be the subgroup of A fixing B and B' setwise. Then $|A : A^*| = 2$ and $A^* \trianglelefteq A$. Suppose that A^* is 2-transitive on B. Then every vertex in $B \setminus \{u\}$ has the same number of in-neighbors and the same number of out-neighbors in $N^+(u)$ in $\mathcal{O}$, say m_1 and m_2, respectively. It follows that $q^2 = (2p-1)m_1$ and $q(q-1) = (2p-1)m_2$, forcing $m_1 - m_2 = 1$ and $q = 2p - 1$, contrary to the fact that $q < p$. Since $|B| = 2p$, by Proposition 2.6 A^* is imprimitive on B. Note that A^* is the block stabilizer of B in A. Thus, every non-trivial block of A^* on B is also a block of A, which has size 2 or p. By Case II and Case III, $P \trianglelefteq A$.

Now we are ready to prove the lemma. Since $P \trianglelefteq A$, the orbits of P are blocks of A of length p. By the proof in Case III, $|A| = 4pq$ and hence A is solvable. Clearly, the Hall $\{2, p\}$-subgroup of A acts regularly on $V(X)$, implying that X is a Cayley graph. By Lemma 3.2, one may let $X = \mathrm{Cay}(G, S)$, where $G = \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^r \rangle$ with $r^2 \equiv -1 \pmod{p}$. Thus, $4q \mid (p-1)$.

Since $P \trianglelefteq A$, one has $P \leq R(G)$. Set $C = C_A(P)$. Since $R(G)$ is a Hall $\{2, p\}$-subgroup of A and there is no involution in G commuting with a, C is a p-group or a $\{q, p\}$-group. If $q \mid |C|$ then $|C| = pq$ and C has a normal Sylow q-subgroup, which is normal in A because $C \trianglelefteq A$. This implies that $A_u \trianglelefteq A$, forcing $A_u = 1$, a contradiction. It follows that $P = C$, and since $A/C = A/P \lesssim \mathrm{Aut}(P) \cong \mathbb{Z}_{p-1}$, one has $R(G)/P \trianglelefteq A/P$, that is $R(G) \trianglelefteq A$. Thus $X = \mathrm{Cay}(G, S)$ is a normal Cayley graph. By Proposition 2.1, $A = R(G) \rtimes \mathrm{Aut}(G, S)$ and since $|A| = 4pq$, there is an element δ of order q in $\mathrm{Aut}(G, S)$.

The connectivity of X implies $\langle S \rangle = G$. Since G has a unique normal subgroup of order $2p$, S contains elements of order 4. Each element of order 4 in G is of the form a^ib or a^ib^{-1} for some integer i. It is easy to see that $\mathrm{Aut}(G)$ is transitive on the set $\{a^ib \mid 0 \leq i \leq$

$p-1\}$. Thus, one may assume $b \in S$. Since there is no automorphism of G mapping b to b^{-1}, one may further assume $a^\delta = a^s$ and $b^\delta = a^k b$ ($s \neq 0 \pmod p$), and since $S^\delta = S$, we have that $E = \{b, b^\delta, \ldots, b^{\delta^{q-1}}\} = \{b, a^{ks[1]}b, a^{ks[2]}b, a^{ks[3]}b, \ldots, a^{ks[q-1]}b\}$ is a subset of S, where $s[i] = s^{i-1} + \cdots + s + 1$ for $1 \leq i \leq q-1$ and $s[q] = s^{q-1} + \cdots + s + 1 = 0 \pmod p$. It follows that $S = E \cup E^{-1}$ because $S = S^{-1}$. Clearly, $k \neq 0 \pmod p$. Since the map $a \mapsto a^k$, $b \mapsto b$, induces an automorphism of G, we have $X \cong \mathcal{C}_{4p}^{2q}$. □

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