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Automorphism group of the balanced hypercube*

Jin-Xin Zhou †, Jin Ho Kwak
Yan-Quan Feng , Zhen-Lin Wu

Mathematics, Beijing Jiaotong University, Beijing 100044, P.R. China

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Abstract

Huang and Wu in [IEEE Transactions on Computers 46 (1997), pp. 484–490] introduced the balanced hypercube BH_n as an interconnection network topology for computing systems. In this paper, we completely determine the full automorphism group of the balanced hypercube. Applying this, we first show that the n-dimensional balanced hypercube BH_n is arc-transitive but not 2-arc-transitive whenever $n \geq 2$. Then, we show that BH_n is a lexicographic product of an n-valent graph X_n and the null graph with two vertices, where X_n is a $\mathbb{Z}_2^{n-1}$-regular cover of the n-dimensional hypercube Q_n.

Keywords: Automorphism group, balanced hypercube, Cayley graph, arc-transitive.

Math. Subj. Class.: 05C25, 20B25

1 Introduction

The hypercube is widely known as one of the most popular interconnection networks for parallel computing systems. As a variant of the hypercube, the balanced hypercube was proposed by Huang and Wu [8] to enhance some properties of the hypercube. An n-dimensional balanced hypercube, denoted by BH_n, is defined as follows.

Definition 1.1. For $n \geq 1$, BH_n has 4^n vertices, and each vertex has a unique n-component vector on $\{0, 1, 2, 3\}$ for an address, also called an n-bit string. A vertex $(a_0, a_1, \ldots, a_{n-1})$ is connected to the following $2n$ vertices:

$$\begin{cases} ((a_0+1)(\mathrm{mod}\, 4), a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}), \\ ((a_0-1)(\mathrm{mod}\, 4), a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}), \end{cases}$$

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†Corresponding author.

E-mail addresses: jxzhou@bjtu.edu.cn (Jin-Xin Zhou), jinkwak@postech.ac.kr (Jin Ho Kwak), yqfeng@bjtu.edu.cn (Yan-Quan Feng), 10271079@bjtu.edu.cn (Zhen-Lin Wu)

$$\begin{cases} ((a_0+1)(\text{mod } 4), a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0})(\text{mod } 4), a_{i+1}, \ldots, a_{n-1}), \\ ((a_0-1)(\text{mod } 4), a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0})(\text{mod } 4), a_{i+1}, \ldots, a_{n-1}), \end{cases}$$

for $1 \leq i \leq n-1$.

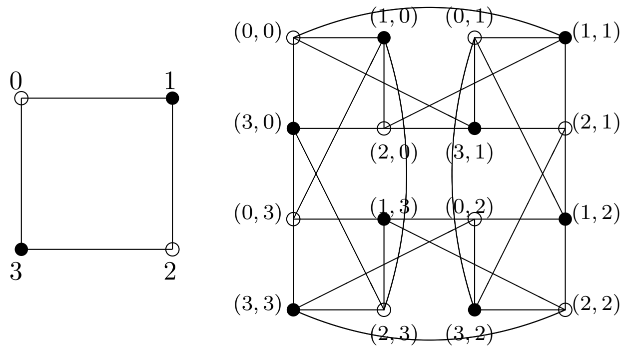

Figure 1: Two balanced hypercubes: BH_1 and BH_2

By now, various properties of the balanced hypercube, such as, Hamiltonian laceability, bipanconnectivity, super connectivity etc. have been extensively investigated in the literature [7, 8, 9, 14, 16, 17, 18, 19]. In many situations, it is highly desired to use interconnection networks which are highly symmetric. This often simplifies the computational and routing algorithms. It has been shown that the balanced hypercube is vertex-transitive and arc-transitive (see [14, 22]). When dealing with the symmetry of graphs, the goal is to gain as much information as possible about the structure of the full automorphism groups. Recently, several publications have been put into investigation of automorphism groups of Cayley graphs having connection with interconnection networks (see, for example, [5, 10, 20, 21]).

In [22], it was proved that BH_n is an arc-transitive Cayley graph.

Definition 1.2. For $n \geq 1$, let H_n be an abelian group defined as follows:

$$H_n \quad = \langle y \rangle \times \langle z_1 \rangle \times \langle z_2 \rangle \times \ldots \times \langle z_{n-1} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4.$$

The *generalized dihedral group of* H_n, denoted by $\text{Dih}(H_n)$, is the semi-direct product of H_n by a group $\langle x \rangle$ of order 2 with the involution x inverting every element in H_n. Let $G_n = \text{Dih}(H_n) = H_n \rtimes \langle x \rangle$ and let $S = \{x, xy, xz_i, xyz_i \mid i = 1, 2, \ldots, n-1\}$. Let Γ_n be the following Cayley graph:

$$\Gamma_n = \text{Cay}(G_n, S). \tag{1.1}$$

Proposition 1.3. [22, Theorem 3.7] *For each* $n \geq 1$, $BH_n \cong \Gamma_n$ *is arc-transitive.*

Definition 1.4. Let L_n be a subgroup of H_n defined by

$$L_n \quad = \langle z_1 \rangle \times \langle z_2 \rangle \times \ldots \times \langle z_{n-1} \rangle \cong \underbrace{\mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4}_{n-1}.$$

Let $T_n = \text{Dih}(L_n) = L_n \rtimes \langle x \rangle$. Clearly, T_n is a subgroup of G_n of index 2. Set $\Omega = \{x, xz_i \mid i = 1, 2, \ldots, n-1\}$, and define X_n as the following Cayley graph:

$$X_n = \text{Cay}(T_n, \Omega). \tag{1.2}$$

For convenience, in what follows we shall always let $\Gamma_n = BH_n$. In [3], the authors proved the following result.

Proposition 1.5. [3, Theorem 3.4] *For each $n \geq 1$, $BH_n \cong X_n[2K_1]$, where X_n is defined as following:*

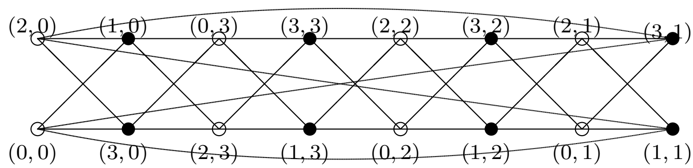

Figure 2: Another layout of BH_2

By Proposition 2.1, it is easy to see that $\text{Aut}(BH_n) = \mathbb{Z}_2 \wr \text{Aut}(X_n)$ *. So, to determine the automorphism group of BH_n, the key is to determine the automorphism group of X_n. In this paper, we prove that X_n is a 2-arc-transitive normal Cayley graph, and $\text{Aut}(X_n) = R(T_n) \rtimes \text{Aut}(T_n, \Omega) \cong T_n \rtimes S_n$.

As the automorphism group of BH_n is known, one may ask: Does BH_n have a stronger symmetry property? In this paper, we show that BH_n is arc-transitive but not 2-arc-transitive.

As another application, we prove that X_n is a $\mathbb{Z}_2^{n-1}$-regular cover of the hypercube Q_n. This, together with the fact $BH_n \cong X_n[2K_1]$, gives a theoretical explanation of the relationship between BH_n and Q_n.

2 Preliminaries

In this section, we shall introduce some notations, terminology and preliminary results. Throughout this paper only undirected simple connected graphs without loops and multiple edges are considered. Unless stated otherwise, we follow Bondy and Murty [2] for terminology and definitions.

Let n be a positive integer. Denote by $\mathbb{Z}_n$ the cyclic group of order n, by S_n the symmetric group of degree n and by $K_{n,n}$ the complete bipartite graph of order $2n$ and valency n, respectively. We also use nK_1, K_n and C_n to denote the null graph, the complete graph and the cycle with n vertices, respectively.

In a parallel computing system, processors are connected based on a specific interconnection network. An interconnection network is usually represented by a graph in which vertices represent processors and edges represent links between processors. Let G be a simple undirected connected graph. We denote by $\text{Aut}(G)$ the full automorphism group of G, and by $V(G)$ and $E(G)$ the sets of vertices and edges of G, respectively. For $u, v \in V(G)$, denote by $\{u, v\}$ the edge incident to u and v in G. For a vertex v in a graph G, use $N_G(v)$ to denote the neighborhood of v, that is, the set of vertices adjacent to v.

*One can also obtain this by using [4, Theorem 5.7]. We thank a referee for pointing out this.

An *s-arc* in a graph G is an ordered $(s+1)$-tuple $(v_0, v_1, \dots, v_{s-1}, v_s)$ of vertices of G such that v_{i-1} is adjacent to v_i for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i \le s-1$. A graph G is said to be *s-arc-transitive* if $\mathrm{Aut}(G)$ is transitive on the set of s-arcs in G. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph G is *edge-transitive* if $\mathrm{Aut}(G)$ acts transitively on $E(G)$. Clearly, every arc-transitive graph is both edge-transitive and vertex-transitive.

2.1 Wreath products of groups

For a set V and a group G with identity element 1, an *action* of G on V is a mapping $V \times G \to V$, $(v, g) \mapsto v^g$, such that $v^1 = v$ and $(v^g)^h = v^{gh}$ for $v \in V$ and $g, h \in G$. The *kernel* of G acting on V is the subgroup of G fixing V pointwise. For two groups K, H, if H acts on K (as a set) such that $(xy)^h = x^h y^h$ for any $x, y \in K$ and $h \in H$, then H is said to act on K as a group. In this case, we use $K \rtimes H$ to denote the *semi-direct product* of K by H with respect to the action.

Let H be a permutation group on a finite set Δ. For convenience, let $\Delta = \{1, 2, \cdots, n\}$. Let G be a permutation group on a finite set Φ, and let

$$N = \underbrace{G \times G \times \cdots \times G}_{n \text{ times}}.$$

We define the action of H on N as following:

$$\forall h \in H, (g_1, g_2 \cdots, g_n)^h = (g_{1^{h^{-1}}}, g_{2^{h^{-1}}}, \cdots, g_{n^{h^{-1}}}), g_i \in G, i = 1, 2, \cdots, n.$$

Then the semi-direct product of N by H with respect to this action is called the *wreath product* of G and H, denoted by $G \wr H$. Clearly,

$$G \wr H = \{(g_1, g_2, \cdots, g_n; h) \mid g_i \in G, h \in H\}.$$

Moreover, $G \wr H$ can be viewed as a permutation group on $\Phi \times \Delta$ as following:

$$(x, i)^{(g_1, g_2, \cdots, g_n; h)} = (x^{g_i}, i^h).$$

Let G and H be two graphs. The *lexicographic product* $G[H]$ is defined as the graph with vertex set $V(G) \times V(H)$ and for any two vertices $(u_1, v_1), (u_2, v_2) \in V(G) \times V(H)$, they are adjacent in $G[H]$ if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in H, or u_1 is adjacent to u_2 in G. In view of [13, Theorem.], we have the following.

Proposition 2.1. [13, Theorem.] *Let X and Y be two graphs. Then* $\mathrm{Aut}(X[Y]) = \mathrm{Aut}(Y) \wr \mathrm{Aut}(X)$ *if and only if*

(1) *if there are two distinct vertices $u, v \in V(X)$ such that $N_X(u) = N_X(v)$, then Y is connected;*

(2) *if there are two distinct vertices $u, v \in V(X)$ such that $N_X(u) \cup \{u\} = N_X(v) \cup \{v\}$, then the complement $\overline{Y}$ of Y is connected.*

2.2 Cayley graphs

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if

$G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. A Cayley graph $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg, x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a permutation group isomorphic to G. It is well-known that $R(G) \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$. So, $\mathrm{Cay}(G, S)$ is vertex-transitive. In general, a vertex-transitive graph X is isomorphic to a Cayley graph on a group G if and only if its automorphism group has a subgroup isomorphic to G, acting regularly on the vertex set of X (see [1, Lemma 16.3]).

For two inverse closed subsets S and T of a group G not containing the identity 1, if there is an $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$ then S and T are said to be *equivalent*, denoted by $S \equiv T$. The following proposition is easy to obtain.

Proposition 2.2. *If S and T are equivalent then* $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$.

A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ (see [15]). Let $\mathrm{Cay}(G, S)$ be a Cayley graph on a group G with respect to a subset S of G. Set $A = \mathrm{Aut}(\mathrm{Cay}(G, S))$ and $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$.

Proposition 2.3. [15, Proposition 1.5] *The Cayley graph* $\mathrm{Cay}(G, S)$ *is normal if and only if $A_1 = \mathrm{Aut}(G, S)$, where A_1 is the stabilizer of the identity 1 of G in A.*

2.3 Covers of graphs

An important tool in studying symmetry properties of graphs is the covering technique. An epimorphism $\wp : \widetilde{X} \to X$ of graphs is called a *regular covering projection* if there is a semiregular subgroup $\mathrm{CT}(\wp)$ of the automorphism group $\mathrm{Aut}(\widetilde{X})$ of $\widetilde{X}$ whose orbits in $V(\widetilde{X})$ coincide with the *vertex fibers* $\wp^{-1}(v)$, $v \in V(X)$, and the arc and edge orbits of $\mathrm{CT}(\wp)$ coincide with the *arc fibers* $\wp^{-1}((u, v))$, $u \sim v$, and the *edge fibers* $\wp^{-1}(\{u, v\})$, $u \sim v$, respectively. In particular, we call the graph $\widetilde{X}$ a regular *cover* of the graph X. The semiregular group $\mathrm{CT}(\wp)$ is the *covering transformation group*. If $\mathrm{CT}(\wp)$ is isomorphic to an abstract group N then we speak of $\widetilde{X}$ as a regular *N-cover* of X. For more results on the covering of graphs, we refer the reader to [6, 12].

Let X be a connected k-valent graph and let $G \leq \mathrm{Aut}(X)$ act transitively on the 2-arcs of X. Let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N in $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [11, Theorem 9], we have the following.

Proposition 2.4. *If N has more than two orbits in $V(X)$, then N is semiregular on $V(X)$, X_N is a k-valent graph with G/N as a 2-arc-transitive group of automorphisms, and X is a regular N-cover of X_N.*

3 Automorphism group of the balanced hypercube

In this section, we shall determine the full automorphism group of the balanced hypercube. From Proposition 1.5 we know that $\Gamma_n \cong X_n[2K_1]$, and by Proposition 2.1, $\mathrm{Aut}(\Gamma_n) \cong \mathbb{Z}_2 \wr \mathrm{Aut}(X_n)$. So, the key step is to determine the automorphism group of X_n.

Lemma 3.1. *For $n \geq 1$, X_n is a 2-arc-transitive normal Cayley graph, and furthermore,* $\text{Aut}(X_n) = R(T_n) \rtimes \text{Aut}(T_n, \Omega)$, *where* $R(T_n) \cong T_n = \text{Dih}(\mathbb{Z}_4^{n-1})$ *and* $\text{Aut}(T_n, \Omega) \cong S_n$.

Proof. Clearly, $X_1 \cong K_2$ and $X_2 \cong C_8$. It is easy to see that the statement is true for these two cases. In what follows, assume that $n \geq 3$. We first prove the following two claims.

Claim 1 $\text{Aut}(T_n, \Omega) \cong S_n$.

Since Ω generates T_n, $\text{Aut}(T_n, \Omega)$ acts faithfully on Ω, and hence $\text{Aut}(T_n, \Omega) \leq S_n$.

It is easy to verify that $xz_1, z_1^{-1}z_i (2 \leq i \leq n-1), z_1^{-1}$ generate T_n and they satisfy the same relations as $x, z_i (1 \leq i \leq n-2), z_{n-1}$. This implies that the map

$$\alpha : x \mapsto xz_1, z_i \mapsto z_1^{-1}z_{i+1} (1 \leq i \leq n-2), z_{n-1} \mapsto z_1^{-1},$$

induces an automorphism of T_n. Clearly, for each $1 \leq i \leq n-2$, $(xz_i)^\alpha = xz_{i+1}$, and $x \mapsto xz_1$ and $(xz_{n-1})^\alpha = x$. This implies that α cyclicly permutates the elements in Ω, and so $\alpha \in \text{Aut}(T_n, \Omega)$.

Similarly, for each $2 \leq i \leq n-1$, we define a map as the following:

$$\beta_i : x \mapsto x, z_1 \mapsto z_i, z_i \mapsto z_1, z_j \mapsto z_j (1 \leq i, j \leq n-1 \text{ and } i \neq j).$$

Then β_i induces an automorphism of T_n, and furthermore, β_i interchanges xz_1 and xz_i and fixes all other elements in Ω. Hence, $\beta_i \in \text{Aut}(T_n, \Omega)$ and by elementary group theory, we obtain that the subgroup generated by $\beta_i (2 \leq i \leq n-1)$ is isomorphic to S_{n-1}. Since S_{n-1} is maximal in S_n, one has $\langle \alpha, \beta_i \mid 2 \leq i \leq n-1 \rangle \cong S_n$. It follows that $\text{Aut}(T_n, \Omega) = \langle \alpha, \beta_i \mid 2 \leq i \leq n-1 \rangle \cong S_n$.

Claim 2 *For any xz_i, there are $(n-2)$ 6-cycles in X_n passing through the 2-arc $(x, 1, xz_i)$, namely, $C^{i,j} = (1, x, z_j^{-1}, xz_iz_j^{-1}, z_j^{-1}z_i, xz_i, 1)$ with $j \neq i$ and $1 \leq j \leq n-1$.*

By Claim 1, $\text{Aut}(T_n, \Omega)$ acts 2-transitively on Ω. It is well-known that a vertex-transitive graph is 2-arc-transitive if and only if the vertex-stabilizer $\text{Aut}(X_n)_v$ is 2-transitive on the set of vertices adjacent to v. So, X_n is 2-arc-transitive. To prove the claim, it suffices to show that the statement is true for the case when $i = 1$.

First, for any $2 \leq j \leq n-1$, one may easily check that $C^{1,j} = (1, x, z_j^{-1}, xz_1z_j^{-1}, z_1z_j^{-1}, xz_1, 1)$ is a 6-cycle passing through the 2-arc $(x, 1, xz_1)$. Let C' be an arbitrary 6-cycle passing through $(x, 1, xz_1)$. Then there exist $s_1, s_2, t_1, t_2 \in \Omega$ such that $C' = (1, x, s_1x, s_2s_1x = t_2t_1xz_1, t_1xz_1, xz_1, 1)$, where $s_1 \neq x$, $s_2 \neq s_1$, $t_1 \neq xz_1$ and $t_1 \neq t_2$. Clearly, $s_1 = xz_j$ for some $1 \leq j \leq n-1$. In the rest of the proof of Claim 2 the following well-known fact will be frequently used.

Fact *Every element in $\langle z_1 \rangle \times \langle z_2 \rangle \times \ldots \times \langle z_{n-1} \rangle$ can be uniquely written in the following form*

$$z_1^{a_1} z_2^{a_2} \ldots z_{n-1}^{a_{n-1}}, a_i \in \mathbb{Z}_4 (1 \leq i \leq n-1).$$

If $s_2 = x$, then $xxz_jx = t_2t_1xz_1$. It follows that $z_jx = t_2t_1xz_1$ and hence $z_jz_1 = t_2t_1$. If $t_2 = x$, then $t_1 = xz_k$ for some $1 \leq k \leq n-1$, and so $z_jz_1 = z_k$. By Fact, this is impossible. If $t_2 = xz_\ell$ for some $1 \leq \ell \leq n-1$, then we have either $t_1 = x$ or $t_1 = xz_p$ for some $1 \leq p \leq n-1$. For the former, we have $z_jz_1 = z_\ell^{-1}$, and for the latter, we have $t_2t_1 = xz_\ell xz_p = z_\ell^{-1}z_p = z_jz_1$. From the above Fact, both of these cannot happen.

If $s_2 = xz_i$ for some $1 \le i \le n-1$, then $xz_ixz_jx = t_2t_1xz_1$. It follows that $z_i^{-1}z_jx = t_2t_1xz_1$ and hence $z_i^{-1}z_jz_1 = t_2t_1$. If $t_1 = xz_k$ and $t_2 = xz_p$ for some $1 \le k, p \le n-1$, then $t_2t_1 = z_p^{-1}z_k = z_i^{-1}z_jz_1$. This is also impossible. If $t_1 = x$ and $t_2 = xz_p$ for some $1 \le p \le n-1$, then $t_2t_1 = z_p^{-1} = z_i^{-1}z_jz_1$. This is also impossible. So, we must have $t_1 = xz_k$ and $t_2 = x$ for some $1 \le k \le n-1$. Then $t_2t_1 = z_k = z_i^{-1}z_jz_1$. Clearly, $s_1 \ne s_2$. Then $z_k = z_j$ and $z_i = z_1$. That is $s_2 = xz_1, t_2 = x, t_1 = s_1 = xz_j$. It follows that $C' = C^{1,j} = (1, x, z_j^{-1}, xz_1z_j^{-1}, z_j^{-1}z_1, xz_1, 1)$.

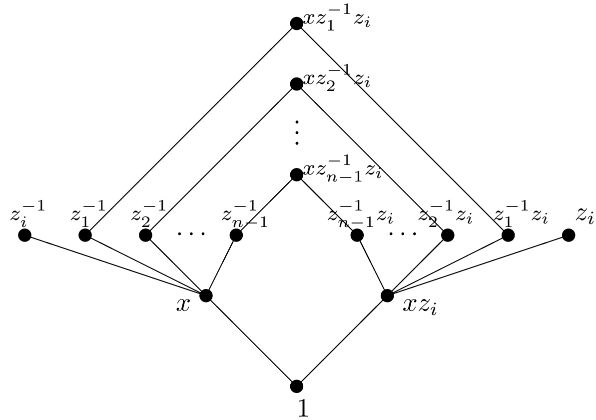

Figure 3: 6-cycles passing through $(x, 1, xz_i)$

Now we are ready to complete the proof. Let $A = \mathrm{Aut}(X_n)$ and let A_1 be the stabilizer of the identity 1 in A. Let A_1^* be the kernel of A_1 acting on Ω. Then A_1^* fixes every element in Ω. For any xz_i ($1 \le i \le n-1$), by Claim 2, there are exactly $(n-2)$ 6-cycles in X_n passing through the 2-arc $(x, 1, xz_i)$, namely, $C^{i,j} = (1, x, z_j^{-1}, xz_iz_j^{-1}, z_j^{-1}z_i, xz_i, 1)$ with $j \ne i$ and $1 \le j \le n-1$ (see Fig. (3)). Note that the neighborhood of x is $\{1, z_i^{-1} \mid 1 \le i \le n-1\}$ and the neighborhood of xz_i is $\{1, z_i, z_j^{-1}z_i \mid 1 \le i, j \le n-1, j \ne i\}$. Since there are no 6-cycles passing through $z_i^{-1}, x, 1, xz_i$ and z_i, it follows that A_1^* fixes z_i^{-1} and z_i ($1 \le i \le n-1$).

By [3, Lemma 4.2], X_n has girth 6, and so $C^{i,j}$ is the unique 6-cycle passing through $z_j^{-1}, x, 1, xz_i, z_j^{-1}z_i$. As A_1^* fixes $z_j^{-1}, x, 1$ and xz_i, A_1^* must fix $z_j^{-1}z_i$. By the arbitrariness of i, j, we obtain that A_1^* fixes every vertex of the set $\{z_i^{-1}, z_i, z_j^{-1}z_i \mid 1 \le i, j \le n-1, j \ne i\}$ which is just the set of vertices at distance 2 from the identity 1. By the vertex-transitivity and connectivity of X_n, A_1^* fixes all vertices of X_n. It follows that $A_1^* = 1$, and so A_1 acts faithfully on Ω. Therefore, $A_1 \lesssim S_n$. By Claim 1, $\mathrm{Aut}(T_n, \Omega) \cong S_n$, and since $\mathrm{Aut}(T_n, \Omega) \le A_1$, one has $\mathrm{Aut}(T_n, \Omega) = A_1$. By Proposition 2.3, X_n is normal, and so $A = R(T_n) \rtimes \mathrm{Aut}(T_n, \Omega)$. □

Now we are ready to determine the automorphism group of BH_n.

Theorem 3.2. *For* $n \ge 1$, $\mathrm{Aut}(BH_n) = \mathbb{Z}_2 \wr (T_n \rtimes S_n)$.

Proof. By Proposition 1.5, $BH_n \cong X_n[2K_1]$. By Proposition 2.1, $\mathrm{Aut}(BH_n) \cong \mathbb{Z}_2 \wr \mathrm{Aut}(X_n)$. From Theorem 3.1 we obtain that $\mathrm{Aut}(X_n) = R(T_n) \rtimes \mathrm{Aut}(T_n, \Omega) \cong T_n \rtimes S_n$. It follows that $\mathrm{Aut}(BH_n) = \mathbb{Z}_2 \wr (T_n \rtimes S_n)$. □

4 Related results

As the automorphism group of BH_n is known, we can obtain more information about the symmetry properties of BH_n. By Proposition 1.3, BH_n is arc-transitive, and by Theorem 3.1, X_n is 2-arc-transitive. It is natural to ask: whether BH_n has much stronger symmetry property? We answer this in negative by showing that BH_n is not 2-arc-transitive.

Theorem 4.1. *For $n \geq 2$, BH_n is arc-transitive but not 2-arc-transitive.*

Proof. Suppose, by way of contradiction, that BH_n is 2-arc-transitive. Recall that $BH_n = \text{Cay}(G_n, S)$. Then the vertex-stabilizer $\text{Aut}(BH_n)_1$ of the identity 1 of G_n in $\text{Aut}(BH_n)$ is 2-transitive on S. That is, for any two distinct ordered pairs from $S \times S$, say (u_1, v_1) and (u_2, v_2), there exists $\alpha \in \text{Aut}(BH_n)_1$ such that $(u_1, v_1)^\alpha = (u_2, v_2)$. In particular, there exists $\alpha \in \text{Aut}(BH_n)_1$ such that $(x, xy)^\alpha = (x, xz_1)$. This implies that x and xz_1 have the same neighborhood because x and xy have the same neighborhood. However, from [22, Lemma 3.8], we see that xy is the unique vertex which has the same neighborhood as x, a contradiction. □

By Proposition 1.5, $BH_n \cong X_n[2K_1]$. As a consequence of Theorem 3.1, we can also prove that X_n is a $\mathbb{Z}_2^{n-1}$-regular cover of the hypercube Q_n. This reveals the relationship between the balanced hypercube BH_n and the hypercube Q_n.

Lemma 4.2. *For $n \geq 1$, let $N = \mathbb{Z}_2^n$. Let $G = \text{Cay}(N, S)$ be a connected n-valent Cayley graph. Then G is isomorphic to the n-dimensional hypercube Q_n.*

Proof. It is well-known that Q_n is a Cayley graph on N with respect to the subset

$$T = \{(1, 0, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, 0, 0, \cdots, 1)\}.$$

Viewing N as an n-dimensional vector space on the field $\mathbb{Z}_2$, one may see that T is a basis of N. Since G is an n-valent Cayley graph, one has $|S| = n$, and since G is connected, one has $N = \langle S \rangle$. This means that S is also a basis of N. So, there is an automorphism of N which maps S to T. By Proposition 2.2, $G \cong Q_n$, as desired. □

Theorem 4.3. *For $n \geq 3$, X_n is a $\mathbb{Z}_2^{n-1}$-regular cover of Q_n.*

Proof. By Theorem 3.1, $R(T_n)$ is normal in $\text{Aut}(X_n)$. Remember that $T_n = \text{Dih}(L_n) = L_n \rtimes \langle x \rangle$, where

$$L_n = \langle z_1 \rangle \times \ldots \times \langle z_{n-1} \rangle \cong \underbrace{\mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4}_{n-1 \text{ times}},$$

and x is an involution inverting every element in L_n. Set $Z = \langle R(z_1^2) \rangle \times \ldots \times \langle R(z_{n-1}^2) \rangle$. Then

$$Z \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{n-1 \text{ times}},$$

and Z is just the center of $R(T_n)$. It follows that Z is characteristic in $R(T_n)$. Since $R(T_n) \trianglelefteq \text{Aut}(X_n)$, one has $Z \trianglelefteq \text{Aut}(X_n)$. Consider the quotient graph Y_n of X_n relative to Z. Clearly, Z is semiregular on the vertex-set of X_n, and so it has more than 2 orbits on $V(X)$. Since X_n is 2-arc-transitive, by Proposition 2.4, Y_n is also an n-valent graph with

$\mathrm{Aut}(X_n)/Z$ as a 2-arc-transitive automorphism group, and X_n is a Z-regular cover of Y_n. To complete the proof, it suffices to prove that $Y_n \cong Q_n$.

Noting that $Z \trianglelefteq R(T_n)$ and $R(T_n)$ is regular on $V(X_n)$, $R(T_n)/Z$ is regular on $V(Y_n)$. It follows that Y_n is a Cayley graph on $R(T_n)/Z$. As $R(T_n) = \mathrm{Dih}(L_n)$, one has $R(T_n)/Z \cong \mathbb{Z}_2^n$. Since Y_n has valency n, by Lemma 4.2, one has $Y_n \cong Q_n$. □

Conclusion

In [14], the authors introduced the balanced hypercube to enhance some properties of the hypercube. Graph symmetry is an important factor in the design of an interconnection network. In 1997, it has been shown that the balanced hypercube is vertex-transitive. Recently, it was shown that the balanced hypercube is also arc-transitive. However, the full automorphism group of the balanced hypercube remained unknown. In this paper, we solve this problem. As applications, we first analyze the symmetry properties of the balanced hypercube and show that the balanced hypercube is not 2-arc-transitive. Then, we give a theoretical explanation of the relationship between the balanced hypercube and the hypercube.

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