

Cayley Graphs and Symmetric 4-Polytopes

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Abstract

Previously we have investigated the medial layer graph \mathcal{G} for a finite, self-dual, regular or chiral abstract 4-polytope \mathcal{P} . Here we study the Cayley graph \mathcal{C} on a natural group generated by polarities of \mathcal{P} , show that \mathcal{C} covers \mathcal{G} in a readily computable way and construct \mathcal{C} as a voltage graph over \mathcal{G} . We then examine such symmetric graphs for several interesting families of polytopes of type $\{p, q, p\}$, $p = 3, 4, 5$.

Keywords: Abstract regular or chiral polytopes, symmetric graphs.

Math. Subj. Class.: 05C25, 51M20

1 Introduction

In [7] Coxeter studied the graph on the edges and polygonal faces of each of the two self-dual, regular convex 4-polytopes, then with the second author of this paper, widened the investigation to include certain families of twisted honeycombs [9, 29]. Much later, in [23] and [18], we revisited these objects in the even wider setting of abstract polytopes.

To be specific, we focussed there on finite, self-dual, abstract regular (or chiral) 4-polytopes \mathcal{P} . In fact, self-duality served mainly to help guarantee vertex-transitivity in the associated graph \mathcal{G} . Here we take a closer look at the full group D of dualities and automorphisms for \mathcal{P} . We are particularly interested in the Cayley graph \mathcal{C} on the subgroup of D generated by a natural set of polarities (dualities of period 2); see Definition 3 and the remarks after it. In Proposition 11 we show that \mathcal{C} is symmetric, indeed transitive on t -arcs for some $t \geq 1$. Proposition 12 then describes how \mathcal{C} covers the edge-face graph \mathcal{G} mentioned earlier; also, in many cases \mathcal{C} can be naturally constructed as a voltage graph over \mathcal{G} (Proposition 14). Having established this foundation, we then take a detailed look at several interesting classes of polytopes. Before proceeding, however, we want to thank Marston Conder, Tomo Pisanski, Egon Schulte and the referees for several helpful comments.

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2 Polytopes and graphs

Rather than repeat in detail what can be found elsewhere, we refer the reader to [23] and [18] for notation and general background, and to [2] and [17] for a deeper discussion of symmetric graphs and polytopes. Here let us simply recall that an abstract n -polytope \mathcal{P} is a partially ordered set having some of the key combinatorial properties of the face lattice of a convex n -polytope; in general, however, \mathcal{P} need not be a lattice, need not be finite, need not have any familiar geometric realization.

Beginning with any abstract 4-polytope \mathcal{P} , one can construct a simple, bipartite graph $\mathcal{G} = \mathcal{G}(\mathcal{P})$, whose *nodes* are all 1-faces (edges) and 2-faces (polygons) in \mathcal{P} , where two such nodes are joined by a *branch* when incident as faces of \mathcal{P} . (We employ the lesser-used terms ‘node’ and ‘branch’ since we occasionally discuss the edges and vertices (0-faces) of \mathcal{P} itself.) Following [23, 18] we say that \mathcal{G} is the *medial layer graph* of \mathcal{P} . Viewing \mathcal{P} as a ranked, partially ordered set, then \mathcal{G} is quite literally the graph on the elements of rank 1 or 2 in a Hasse diagram for \mathcal{P} .

Usually \mathcal{P} will be *equivelar*, say with Schläfli type $\{p, q, r\}$. This encodes a sort of local symmetry in which every 2-face of \mathcal{P} is a p -gon, with r such p -gons around every edge; furthermore, in each facet (3-face) of \mathcal{P} there are q edges (and hence q polygons) around each vertex. Consequently, the graph \mathcal{G} contains cycles of length $2q$ and has bipartition classes consisting, respectively, of p -valent and r -valent nodes.

In fact, we shall enforce equivelarity by demanding that the group $A = A(\mathcal{P})$ of order preserving bijections (*automorphisms*) on \mathcal{P} be quite large. To frame our discussion we fix a *base flag* $\Phi = \{F_0, F_1, F_2, F_3\}$ (suppressing the improper faces F_{-1}, F_4). For $0 \leq j \leq 3$, there is a unique j -*adjacent* flag Φ^j which differs from Φ in just the proper face F_j .

Now recall from [17] that \mathcal{P} is *regular* if A is transitive on flags. For $0 \leq j \leq 3$, let ρ_j be the unique automorphism mapping Φ to Φ^j . Then A is generated by the involutions ρ_j , which we usefully think of as ‘reflections’ (see [17, 2B]); moreover, \mathcal{P} is equivelar of type $\{p, q, r\}$, where p, q, r are the periods of the ‘rotations’ $\sigma_1 := \rho_0\rho_1$, $\sigma_2 := \rho_1\rho_2$ and $\sigma_3 := \rho_2\rho_3$, respectively. The *rotation group*

$$A^+ := \langle \sigma_1, \sigma_2, \sigma_3 \rangle \tag{1}$$

has index at most 2 in A . If this index equals 2 we say that \mathcal{P} is *directly regular*. Otherwise, $A^+ = A$, and we say that \mathcal{P} is *non-orientably regular* (since the order complex for \mathcal{P} must be non-orientable; see [17, 2C] and [26, p. 501]).

Relaxing symmetry a little, we say that \mathcal{P} is *chiral* if A has just two flag orbits, with adjacent flags always in different orbits [26]. Here the automorphism group is also generated by ‘rotations’ behaving much like the σ_j in the directly regular case. In this case, too, we have $A = A^+$; and again \mathcal{P} is equivelar of type $\{p, q, r\}$.

If \mathcal{P} is regular or chiral, the generating rotations σ_i satisfy the *standard relations*

$$\sigma_1^p = \sigma_2^q = \sigma_3^r = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1, \tag{2}$$

(and usually other independent relations). Moreover, apart from one special case, we have the following *intersection condition* on special subgroups:

$$\begin{aligned} \langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle &= \{1\} \\ \langle \sigma_2 \rangle \cap \langle \sigma_3 \rangle &= \{1\} \\ \langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle &= \langle \sigma_2 \rangle. \end{aligned} \tag{3}$$

However, if \mathcal{P} is non-orientably regular, the very last intersection sometimes becomes

$$\langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle = \langle \rho_1, \rho_2 \rangle.$$

(For regular polytopes, the subgroup $\langle \sigma_2 \rangle$ always has index 2 in $\langle \rho_1, \rho_2 \rangle$.) One particularly nice instance of this is the 11-cell described in Example 17 below.

Remarks. When \mathcal{P} is a regular 4-polytope, so that $A = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$, the equations in (3) are related to the perhaps more familiar result that

$$\langle \rho_i : i \in I \rangle \cap \langle \rho_i : i \in J \rangle = \langle \rho_i : i \in I \cap J \rangle, \tag{4}$$

for any subsets $I, J \subseteq \{0, 1, 2, 3\}$. This in turn recalls a well-known fact concerning parabolic subgroups of general Coxeter groups [15, Theorem 5.5(c)]. Now in either regular or chiral cases, it is possible to reconstruct \mathcal{P} in a natural way as a coset geometry over A ([17, 2E] and [26, §5]). Conditions (3) or (4) translate in this construction into essential properties of \mathcal{P} ; for example, in a 4-polytope every 2-face will be incident with exactly two 3-faces.

Lemma 1. $\sigma_2^2 = \sigma_1^{-1} \sigma_3 \sigma_1 \sigma_3^{-1} \in \langle \sigma_1, \sigma_3 \rangle$.

Proof. From (2) we have

$$\begin{aligned} \sigma_2^2 &= \sigma_1^{-1} \sigma_1 \sigma_2^2 \\ &= \sigma_1^{-1} \sigma_1 \sigma_2 (\sigma_1 \sigma_2 \sigma_3)^2 \sigma_2 \\ &= \sigma_1^{-1} (\sigma_1 \sigma_2)^2 \sigma_3 \sigma_1 (\sigma_2 \sigma_3 \sigma_2) \\ &= \sigma_1^{-1} \sigma_3 \sigma_1 \sigma_3^{-1}. \end{aligned}$$

□

Corollary 2. *If q is odd, then $\sigma_2 = (\sigma_3 \sigma_1^{-1} \sigma_3^{-1} \sigma_1)^{\frac{q-1}{2}} \in \langle \sigma_1, \sigma_3 \rangle$.*

Clearly the graph \mathcal{G} inherits considerable symmetry from the polytope \mathcal{P} . If \mathcal{P} is regular or chiral, then A acts transitively on each of the two sets of 1-arcs in \mathcal{G} , namely those with initial node a 1-face or 2-face of \mathcal{P} .

Of course, no automorphism of \mathcal{P} maps a 1-face to a 2-face, so that A cannot be transitive on the set of all nodes of \mathcal{G} . As a remedy, we may also suppose \mathcal{P} to be *self-dual*. In this case, $A = A(\mathcal{P})$ has index 2 in the group $D = D(\mathcal{P})$ of all dualities and automorphisms. Furthermore, the graph \mathcal{G} is *symmetric*, that is, $\text{Aut}(\mathcal{G})$ is transitive on the set of all 1-arcs [2, p. 118]. We refer to [23, 18] for a discussion of such graphs in the case that $p = r = 3$.

Elaborating on [14, p. 131], we say that \mathcal{P} is *properly* self-dual if there exists a duality preserving each flag orbit (under the natural action of A). This is forced when \mathcal{P} is regular but can fail in chiral cases. Indeed, an *improperly* self-dual chiral polytope admits dualities which swap the two flag orbits [14, Lemma 3.1]. Whenever \mathcal{P} is self-dual and regular or properly self-dual and chiral, there must exist a *polarity* δ which flips the base flag, that is, transposes F_0, F_3 and F_1, F_2 . This implies that

$$\begin{aligned} \delta^2 &= 1 \\ \delta^{-1} \sigma_1 \delta &= \sigma_3^{-1} \\ \delta^{-1} \sigma_2 \delta &= \sigma_2^{-1} \end{aligned} \tag{5}$$

and that $p = r$.

Conventions. We henceforth assume that the polytope \mathcal{P} is self-dual and regular or properly self-dual and chiral, with Schläfli type $\{p, q, p\}$, where $p < \infty$. (We allow $q = \infty$.) For convenience below, we take ‘self-dual’ to mean ‘properly self-dual’ in chiral cases.

3 A group generated by polarities

Now let us consider the group $D^+ := \langle \sigma_1, \sigma_2, \sigma_3, \delta \rangle$ of all rotations and dualities, which by (5) splits as

$$D^+ \simeq A^+ \rtimes C_2, \tag{6}$$

with $C_2 = \langle \delta \rangle$. Also, we define, in symmetrical fashion,

$$\delta_j = \sigma_1^{1-j} \delta \sigma_1^{j-1}, \text{ for } 1 \leq j \leq p,$$

so that $\delta_1 = \delta$. Now we may define the group of most interest to us here:

Definition 3. Let $G = \langle \delta_1, \dots, \delta_p \rangle$ be the subgroup of D^+ defined by the above set of distinguished polarities.

Observe that these generating polarities are distinct: if say $\delta_1 = \delta_j$, then

$$F_2 \delta = F_2 \sigma_1^{1-j} \delta \sigma_1^{j-1},$$

so that $F_1 = F_1 \sigma_1^{j-1}$, which forces $j \equiv 1 \pmod p$.

Remarks. By working within the group D^+ (rather than all of D), we can readily tackle both regular and chiral polytopes in one go. Our choice of the polarities δ_j is natural for the study of local properties of the medial layer graph \mathcal{G} , since the δ_j 's map a base vertex of \mathcal{G} to its neighbours. Looking ahead to Proposition 9, we note as well that the group G can be as large as D^+ , so it is hard to see how another set of generators could be more versatile. See also the comments after Definition 10.

Anticipating the subgroups described in (7) and (8) below, we now have the following arrangement of groups:

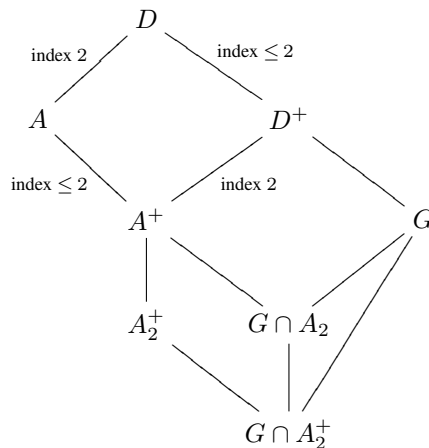


Figure 1: Duality and automorphism groups for self-dual regular or chiral polytopes.

Our ultimate goal is to understand a Cayley graph on the group G (see Definition 10); but first we must examine G more carefully.

Lemma 4. Taking subscripts modulo p , we have

- (a) $\delta_{j+k} = \sigma_1^{-k} \delta_j \sigma_1^k$;
- (b) $\delta_j \delta_k = \sigma_1^{1-j} \sigma_3^{k-j} \sigma_1^{k-1}$; in particular, $\delta_1 \delta_2 = \sigma_3 \sigma_1$ and $\delta_p \delta_1 = \sigma_1 \sigma_3$;
- (c) $\sigma_2^2 = \delta_2 \delta_3 \delta_2 \delta_1$;
- (d) $\sigma_3 \delta_j \sigma_3^{-1} = \delta_1 \delta_2 \delta_{j+1} \delta_2 \delta_1$;
- (e) $(\sigma_2 \sigma_3)^{-1} \delta_j \sigma_2 \sigma_3 = \delta_{3-j}$;
- (f) $\sigma_2^{-1} \delta_j \sigma_2 = \delta_1 \delta_2 \delta_{4-j} \delta_2 \delta_1$ and $\sigma_2 \delta_j \sigma_2^{-1} = \delta_2 \delta_3 \delta_{4-j} \delta_3 \delta_2$.

Proof. Part (a) is routine. For (b) note that

$$\delta_j \delta_k = \sigma_1^{1-j} \delta \sigma_1^{j-1} \sigma_1^{1-k} \delta \sigma_1^{k-1} = \sigma_1^{1-j} \sigma_3^{k-j} \sigma_1^{k-1} .$$

Part (c) follows from (b) and Lemma 1. From (b) we also have $\sigma_3 = \delta_1 \delta_2 \sigma_1^{-1}$, so that

$$\sigma_3 \delta_j \sigma_3^{-1} = \delta_1 \delta_2 \sigma_1^{-1} \delta_j \sigma_1 \delta_2 \delta_1 = \delta_1 \delta_2 \delta_{j+1} \delta_2 \delta_1 .$$

For part (e) we use (2), which gives $\sigma_1^{-1} = \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_3$, so that $(\sigma_2 \sigma_3) \sigma_1^j = \sigma_1^{-j} (\sigma_2 \sigma_3)$, for $1 \leq j \leq p$. Thus

$$\begin{aligned} (\sigma_2 \sigma_3)^{-1} \delta_j (\sigma_2 \sigma_3) &= \sigma_2 \sigma_3 \sigma_1^{1-j} \delta \sigma_1^{j-1} \sigma_2 \sigma_3 \\ &= \sigma_1^{j-1} \sigma_2 \sigma_3 \delta \sigma_2 \sigma_3 \delta \sigma_1^{1-j} \\ &= \sigma_1^{j-1} \sigma_2 \sigma_3 \sigma_2^{-1} \sigma_1^{-1} \delta \sigma_1^{1-j} \\ &= \sigma_1^{j-1} \sigma_2 \sigma_3 \sigma_1 \sigma_2 \delta \sigma_1^{1-j} \\ &= \sigma_1^{j-1} \sigma_1^{-1} \sigma_3^{-1} \delta \sigma_1^{1-j} \\ &= \sigma_1^{j-2} \delta \sigma_1^{2-j} \\ &= \delta_{3-j} . \end{aligned}$$

Finally, part (f) follows at once from (c), (d) and (e). □

Remarks. When $p = 2$, the polytope $\mathcal{P} = \{2, q, 2\}$ is uniquely specified by its Schläfli symbol. Both ρ_0 and ρ_3 act trivially on the graph \mathcal{G} , which is then a $2q$ -cycle.

However, as we observed in [23, §2], D does act faithfully on \mathcal{G} when $p (= r) \geq 3$. In this case, we can regard D as a subgroup of $\text{Aut}(\mathcal{G})$. A fragment of a polytope of type $\{3, q, 3\}$ is displayed in Figure 2. The nodes of the graph \mathcal{G} are indicated by black and white discs; and $\delta_1, \delta_2, \delta_3$ swap the (black and white) nodes incident with the nearby branches.

It is useful to recall here that, in chiral and directly regular cases, the stabilizer in A^+ of the node $v_2 = F_2$ (the base 2-face of \mathcal{P}) is the subgroup

$$A_2^+ := \langle \sigma_1, \sigma_2 \sigma_3 \rangle , \tag{7}$$

which is dihedral of order $2p$. In the non-orientably regular case, $A = A^+$ and the stabilizer of F_2 is now

$$A_2 := \langle \rho_0, \rho_1, \rho_3 \rangle , \tag{8}$$

which has order $4p$. Note that the index $[A_2 : A_2^+] = 2$. In Figure 2, σ_1 acts on F_2 (here a triangle) like a ‘rotation’ ($v_1 \rightarrow v_3 \rightarrow w_2 \rightarrow v_1$), whereas $\sigma_2 \sigma_3$ acts like a ‘reflection’ ($v_1 \leftrightarrow v_3, w_2 \leftrightarrow w_2$).

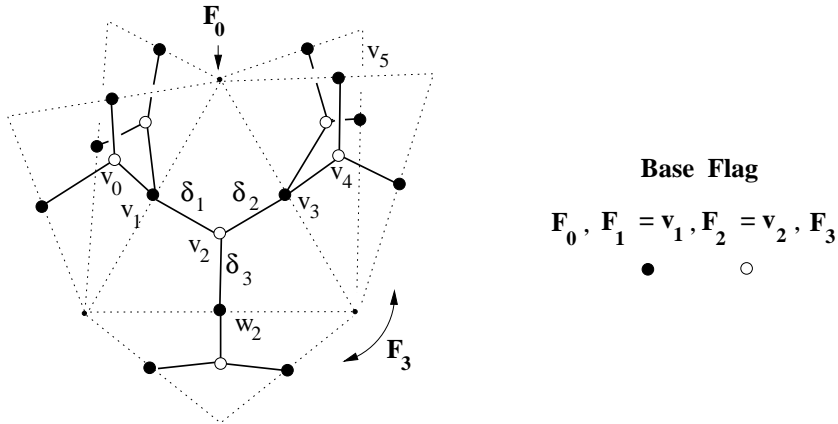


Figure 2: A fragment of a polytope of type $\{3, q, 3\}$.

Corollary 5. G is a normal subgroup of D^+ .

Proof. This follows at once from Lemma 4 (a),(d),(f). □

We can now enumerate the cosets of G in D^+ , though perhaps with some redundancy. Indeed, we define cosets

$$a_j := G\sigma_1^{j-1}; \quad b_j := a_j\sigma_2, \text{ for } 1 \leq j \leq p.$$

(Recall that we assume $p < \infty$.)

Lemma 6. *Again taking subscripts modulo p , we have*

$a_j\sigma_1 = a_{j+1}$	$a_j\sigma_2 = b_j$	$a_j\sigma_2^{-1} = b_j$	$a_j\sigma_3 = a_{j-1}$	$a_j\delta = a_j$
$b_j\sigma_1 = b_{j-1}$	$b_j\sigma_2 = a_j$	$b_j\sigma_2^{-1} = a_j$	$b_j\sigma_3 = b_{j+1}$	$b_j\delta = b_j$

Moreover, all cosets of G in D^+ occur among the $a_j, b_j, 1 \leq j \leq p$.

Proof. These identities follow easily from Lemma 4 and Corollary 5. For example,

$$b_j\sigma_2 = G\sigma_1^{j-1}\sigma_2^2 = \sigma_1^{j-1}\sigma_2^2G = \sigma_1^{j-1}G = G\sigma_1^{j-1} = a_j.$$

Since $a_1 = G$, every coset of G in D^+ is counted among the a_j, b_j . □

Proposition 7. G acts transitively on the nodes of \mathcal{G} .

Proof. D^+ certainly acts transitively on nodes of \mathcal{G} . But from Lemma 6 we find that every coset of G in D^+ has a representative in A_2^+ (see (7)). Indeed, $\sigma_1^{j-1} \in A_2^+$, so we need only observe that $G\sigma_1^{j-2}\sigma_2\sigma_3 = a_{j-1}\sigma_2\sigma_3 = b_{j-1}\sigma_3 = b_j$. It follows that G acts transitively on nodes of \mathcal{G} . □

Keeping in place a fixed regular or chiral, but self-dual polytope \mathcal{P} with Schläfli symbol $\{p, q, p\}$, we now introduce the *universal* polytope \mathcal{U} of type $\{p, \infty, p\}$. Certainly, \mathcal{U} is directly regular and self-dual; and its automorphism group $A(\mathcal{U})$ is the Coxeter group $[p, \infty, p]$ with diagram

$$\bullet \overset{p}{\text{---}} \bullet \overset{\infty}{\text{---}} \bullet \overset{p}{\text{---}} \bullet .$$

(In fact, we could replace the ‘ ∞ ’ by any even multiple of q .) For ease of notation, we let \bar{A}, \bar{D}^+ and so on denote objects attached to \mathcal{U} . In particular,

$$\bar{D}^+ = \langle \bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3, \bar{\delta} \rangle .$$

We note that defining relations for \bar{D}^+ are obtained by setting $p = r, q = \infty$ in (2) and (5) (and replacing σ_j by $\bar{\sigma}_j, \delta$ by $\bar{\delta}$).

Clearly the mapping $\bar{\sigma}_j \mapsto \sigma_j, \bar{\delta} \mapsto \delta$ induces an epimorphism $g : \bar{D}^+ \rightarrow D^+$. Thus $\bar{K} := \ker(g)$ is generated, as a normal subgroup of \bar{D}^+ , by any relations R which we must adjoin to (2) in order to establish a presentation for D^+ .

Even if the cosets in Lemma 6 are redundant, the action described there does suggest that we also investigate the dihedral group

$$I_2(p) = \langle \alpha_0, \alpha_1 \mid \alpha_0^2 = \alpha_1^2 = (\alpha_0\alpha_1)^p = 1 \rangle ,$$

of order $2p$. It is easy to check that the mapping

$$\bar{\sigma}_1 \mapsto \alpha_0\alpha_1, \bar{\sigma}_2 \mapsto \alpha_0, \bar{\sigma}_3 \mapsto \alpha_1\alpha_0, \bar{\delta} \mapsto 1 \tag{9}$$

induces another epimorphism $f : \bar{D}^+ \rightarrow I_2(p)$. Clearly, $\bar{G} \subseteq \ker(f)$. But \bar{G} has index at most $2p$ in \bar{D}^+ , by Lemma 6. Thus $\bar{G} = \ker(f)$, and the index equals $2p$ in the universal setting. The situation is summarized in Figure 3.

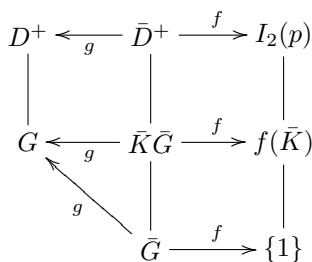


Figure 3: The index of G in D^+ .

Returning to the polytope \mathcal{P} , we can now discuss a useful *covering parameter*. We let $k = k(\mathcal{P})$ denote the order of the subgroup $G \cap A_2^+$ (see Figure 1).

Lemma 8. *Let $f(\bar{K})$ be the normal subgroup of $I_2(p)$ generated by the images $f(R)$ of the ‘additional’ relations required in a presentation of D^+ . Then*

$$I_2(p)/f(\bar{K}) \simeq D^+/G \simeq A_2^+/(G \cap A_2^+) .$$

Furthermore, $k = |f(\bar{K})|$ and the index of G in D^+ equals $2p/k$. Also, in non-orientably regular cases, $|G \cap A_2| = 2k$.

Proof. Since $\bar{K}\bar{G} \triangleleft \bar{D}^+$, we immediately get $I_2(p)/f(\bar{K}) \simeq D^+/G$. We easily check that $GA_2^+ = D^+$, so $D^+/G \simeq A_2^+/(G \cap A_2^+)$ follows from the second isomorphism theorem. But $A_2^+ \simeq I_2(p)$, so that $k = |G \cap A_2^+| = |f(\bar{K})|$.

When \mathcal{P} is non-orientably regular, we have $A = A^+$, so that $D = D^+ = GA_2^+ = GA_2$. Thus, $G \cap A_2$ also has index $2p/k$ in A_2 . Since $|A_2| = 4p$, we have $|G \cap A_2| = 2k$. \square

Now suppose, for example, that q is odd, so that $f(\bar{K})$ contains $f(\bar{\sigma}_2^q) = \alpha_0^q = \alpha_0$. Thus $f(\bar{K})$ contains the dihedral subgroup generated by the conjugacy class of the reflection α_0 in $I_2(p)$. Hence $f(\bar{K})$ is the full group $I_2(p)$ if p is odd as well, but could have index 2 when p is even. We have verified

Proposition 9. *Suppose \mathcal{P} is self-dual and regular or chiral of type $\{p, q, p\}$, with q odd. If p is odd, then $G = D^+$. If p is even, then G has index at most 2 in D^+ .*

At last we are ready for the following

Definition 10. Let $\mathcal{C} = \mathcal{C}(\mathcal{P})$ be the Cayley graph constructed on the group G , with specified (involutory) generators $\delta_1, \dots, \delta_p$.

Remarks. Recall from [2, ch. 16] that \mathcal{C} has node set G , where $\mu, \nu \in G$ are adjacent if and only if $\mu\nu^{-1} = \delta_j$ for some $j \in \{1, \dots, p\}$. Since all $\delta_j^2 = 1$, this unambiguously defines \mathcal{C} as a simple p -valent graph.

Since $\mu\tau(\nu\tau)^{-1} = \mu\nu^{-1}$, for all $\tau \in G$, the right regular action of G on itself serves to embed G as a subgroup of $\text{Aut}(\mathcal{C})$. Similarly, if $\varphi \in \text{Aut}(G)$ globally fixes the set $\{\delta_1, \dots, \delta_p\}$, then φ induces an automorphism of \mathcal{C} which stabilizes the vertex 1.

Such considerations also point out the unsuitability of other polarities, such as the $\sigma_2^{1-j}\delta\sigma_2^{j-1}$, $1 \leq j \leq q$, as possible generators. For one thing, the corresponding Cayley graph will have no obvious local symmetry, because of Lemma 4(f).

Let us look more closely at our graph \mathcal{C} . It is clear on geometrical grounds, and follows easily from Lemma 4(a), (e), that the dihedral group $A_2^+ = \langle \sigma_1, \sigma_2\sigma_3 \rangle$ normalizes $\{\delta_1, \dots, \delta_p\}$, and, in fact, acts transitively and faithfully on that set, just as it would act on the vertices of a p -gon. This gives

Proposition 11. *The Cayley graph \mathcal{C} is a connected, symmetric, p -valent graph. The node stabilizer in $\text{Aut}(\mathcal{C})$ contains a dihedral subgroup of order $2p$.*

Remarks. From Proposition 9 we note that $A_2^+ \subset G$ is possible. Thus we cannot generally hope that $\text{Aut}(\mathcal{C})$ contains a subgroup isomorphic to $G \rtimes A_2^+$.

We also note that in non-orientably regular cases, it can never happen that the full F_2 -stabilizer $A_2 = \langle \rho_0, \rho_1, \rho_3 \rangle$ normalizes $\{\delta_1, \dots, \delta_p\}$. For example, suppose $\rho_0\delta_1\rho_0 = \delta_j$. Then

$$F_1 = F_2\rho_0\delta_1\rho_0 = F_2\delta_j = F_1\sigma_1^{j-1},$$

so that $j = 1$. But then $F_0 = F_3\rho_0\delta_1 = F_3\delta_1\rho_0 = F_0\rho_0$, a contradiction.

Proposition 12. *There is a surjective graph homomorphism*

$$h : \mathcal{C} \rightarrow \mathcal{G},$$

which is injective on any neighbourhood (of nodes distant at most 1 from a given node). Moreover, if \mathcal{P} is chiral or directly regular, then \mathcal{C} is a k -fold covering of \mathcal{G} , where $k = |G \cap A_2^+| = |f(\bar{K})|$. If \mathcal{P} is non-orientably regular, then \mathcal{C} is a $2k$ -fold covering of \mathcal{G} .

Proof. We let $h : \mu \rightarrow (F_2)\mu$, for all $\mu \in G$. Note that μ is a typical node of \mathcal{C} ; and h is surjective by Proposition 7. Now suppose that $\mu, \nu \in G$ are adjacent nodes with $\mu\nu^{-1} = \delta_j$. Then

$$F_2\mu = F_2\delta_j\nu = F_1\sigma_1^{j-1}\nu$$

is incident with $F_2\nu$, since $F_1\sigma_1^{j-1}$ is a 1-face of F_2 . Thus h is a graph homomorphism. The p neighbours of μ in \mathcal{C} are the nodes $\delta_1\mu, \dots, \delta_p\mu$. Their images under h are distinct since μ acts bijectively on \mathcal{P} .

Finally, we note that a typical node $F_2\mu$ of \mathcal{G} is covered by $\lambda \in G$ if and only if $\lambda\mu^{-1} \in \text{Stab}_G(F_2) = G \cap \text{Stab}_{A^+}(F_2)$. But in chiral or directly regular cases, the latter group is just $G \cap A_2^+$ and has order k . The non-orientably regular case also follows from Lemma 8. \square

At least in chiral and directly regular cases, we can exploit the foregoing proof to describe \mathcal{C} as a *derived voltage graph* over the base graph \mathcal{G} , with voltage group $S := \text{Stab}_G(F_2) = G \cap A_2^+$ (see [12, Ch. 2] or [2, Ch. 19]). Recall that we must first assign to each 1-arc $[u, v]$ in \mathcal{G} a voltage $\varphi[u, v] \in S$, satisfying $\varphi[v, u] = (\varphi[u, v])^{-1}$.

Lemma 13. *Suppose $\alpha\delta_i = \beta\delta_j$, for $\alpha, \beta \in S$ and $1 \leq i, j \leq p$. Then $i = j$ and $\alpha = \beta$.*

Proof. Since $F_2\sigma_1 = F_2$, we have $F_2\alpha\sigma_1^{1-i}\delta\sigma_1^{i-1} = F_2\beta\sigma_1^{1-j}\delta\sigma_1^{j-1}$, so $F_1\sigma_1^{i-1} = F_1\sigma_1^{j-1}$. Thus $i = j$. \square

By Proposition 7, the nodes of \mathcal{G} correspond to a (right) transversal $\{\mu_1, \dots, \mu_e\}$ for the subgroup S in G . Thus, for a typical 1-arc $[u, v]$ in \mathcal{G} we may take $u = F_2\mu_x$ and $v = F_2\mu_y = F_2\delta_j\mu_x$ for uniquely determined $x, y \in \{1, \dots, e\}$ and $j \in \{1, \dots, p\}$. We define

$$\varphi[u, v] := \delta_j\mu_x\mu_y^{-1}.$$

Note that $\varphi[u, v] \in S$, and

$$\mu_y\mu_x^{-1} = (\varphi[u, v])^{-1}\delta_j\varphi[u, v](\varphi[u, v])^{-1} = \delta_i(\varphi[u, v])^{-1},$$

say, since S normalizes $\{\delta_1, \dots, \delta_p\}$. (See the remarks after Proposition 11.)

Since $\delta_i\mu_y\mu_x^{-1} \in S$, we conclude from Lemma 13 that $i = j$ and $\varphi[v, u] = (\varphi[u, v])^{-1}$.

Proposition 14. *Suppose \mathcal{P} is chiral or directly regular. Then the Cayley graph \mathcal{C} is isomorphic to the derived voltage graph \mathcal{G}^φ .*

Proof. Recall ([2, Ch. 19]) that the nodes of \mathcal{G}^φ are all pairs (γ, u) , where u is a node of \mathcal{G} and $\gamma \in S$, with $(\gamma, u) \sim (\lambda, v)$ if and only if $u \sim v$ in \mathcal{G} and $\lambda = \gamma \cdot \varphi[u, v]$. But each node of \mathcal{C} can be uniquely written as $\gamma\mu_x$, where $1 \leq x \leq e$ and $\gamma \in S$. We may therefore define a map

$$\begin{aligned} F : \mathcal{C} &\rightarrow \mathcal{G}^\varphi \\ \gamma\mu_x &\mapsto (\gamma, F_2\mu_x) \end{aligned}$$

Clearly F is a bijection. Consider two adjacent nodes in \mathcal{C} , say $\mu = \gamma\mu_x, \nu = \lambda\mu_y$, where $\gamma, \lambda \in S$ and $\mu\nu^{-1} = \delta_j$. Again since S normalizes $\{\delta_1, \dots, \delta_p\}$, we have $\gamma^{-1}\delta_j\gamma = \delta_i$ and

$$\mu_y = \lambda^{-1}\delta_j\gamma\mu_x = (\lambda^{-1}\gamma)\delta_i\mu_x.$$

Thus $F_2\mu_y = F_2\delta_i\mu_x$, so $F_2\mu_y$ and $F_2\mu_x$ are adjacent in \mathcal{G} and

$$\gamma \cdot \varphi[F_2\mu_x, F_2\mu_y] = \gamma\delta_i\mu_x\mu_y^{-1} = \lambda .$$

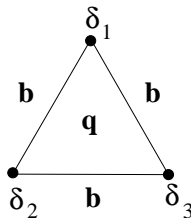
Thus F preserves adjacency (as does its inverse). □

4 Finite polytopes of type $\{3, q, 3\}$

Taking $p = 3$ in the more general discussion of the previous section, we find that both \mathcal{C} and \mathcal{G} are symmetric *trivalent* graphs. Much more is known about this very interesting family of graphs, both in general ([1, 2, 3, 5, 6]) and in the specific case of medial layer graphs for regular or chiral polytopes ([23, 18]). Although we could generalize a little, we shall assume that \mathcal{P} is finite, so that the graphs \mathcal{C} and \mathcal{G} are finite, too.

From Propositions 9, 11 and 12, we immediately get

Proposition 15. *Suppose \mathcal{P} is finite, self-dual and regular or chiral of type $\{3, q, 3\}$, with q odd. Then $D^+ = G = \langle \delta_1, \delta_2, \delta_3 \rangle$, whose generators satisfy at least the relations implicit in the diagram*



in which b is the order of the (left) Petrie motion $\sigma_1\sigma_3$ for \mathcal{P} , and q is the order of σ_2 (and of $\delta_1\delta_2\delta_3\delta_2$).

In all cases, the Cayley graph \mathcal{C} is t -transitive, for some $t \geq 2$ (and $t \leq 5$).

If \mathcal{P} is directly regular (chiral), then \mathcal{C} is a 6-fold cover of the 3-transitive (2-transitive) graph \mathcal{G} . If \mathcal{P} is non-orientably regular, then \mathcal{C} is a 12-fold cover of the 3-transitive graph \mathcal{G} .

Proof. From [2, Ch. 18] we recall that a finite, symmetric trivalent graph like \mathcal{C} is t -transitive for some t satisfying $1 \leq t \leq 5$, and that the node stabilizer has order $3 \cdot 2^{t-1}$, which is at least $2p = 6$ for \mathcal{C} (Proposition 11). Thus $t \geq 2$.

By [23, Thms. 2 and 5], \mathcal{G} is 3-transitive (resp. 2-transitive) when \mathcal{P} is regular (resp. chiral). □

Remark. It is very unlikely that \mathcal{C} could have transitivity $t = 4$ or 5 , but we have no proof of this.

Example 16. The regular 4-simplex $\mathcal{P} = \{3, 3, 3\}$.

The automorphism group for this familiar self-dual, regular convex polytope is, of course, the Coxeter group $[3, 3, 3]$, so that $A \simeq S_5$ and $A^+ \simeq A_5$. Recall that the longest element in A is $\tau := \sigma_1\sigma_3\sigma_1\sigma_2^{-1}\sigma_1 \in A^+$. Then $\tau\delta$ is a central polarity in D^+ , so that

$$D^+ \simeq A_5 \times C_2 .$$

This group of order 120 is the icosahedral group $[3, 5]$. With the new generators $\delta_1, \delta_2, \delta_3$ we get a Cayley graph \mathcal{C} for the group $[3, 5]$, namely the 2-transitive graph **120B** in [3]. Evidently

we can think of $\delta_1, \delta_2, \delta_3$ as reflections in the sides of a triangular face of the icosahedron $\{3, 5\}$.

The graph \mathcal{C} is indeed a 6-fold cover of the medial layer graph \mathcal{G} , which in turn is 3-transitive and is listed as **20B** in [3]. We recall that \mathcal{G} is the *Levi graph* for the Desargues configuration 10_3 .

Example 17. The 11-cell $\mathcal{P} = \{\{3, 5\}_5, \{5, 3\}_5\}$ and its relatives.

This very interesting polytope with 11 facets and (dually) 11 vertices, was independently discovered by Coxeter [8] and Grünbaum [13]. The enhanced Schläfli symbol indicates that \mathcal{P} is the universal regular 4-polytope with hemi-icosahedral facets and hemi-dodecahedral vertex-figures. In fact, \mathcal{P} is non-orientably regular, with $A \simeq PSL_2(11)$ of order 660. Also, $G = D^+ \simeq PGL_2(11)$.

The Cayley graph \mathcal{C} is 2-transitive and has been described as graph C1320.3 in Conder’s recently expanded census of symmetric trivalent graphs [4]. It is a 12-fold cover of \mathcal{G} , which appears as the 3-transitive graph **110** in Foster’s *Census*. (Here, and often elsewhere, our supporting calculations were done using GAP and the subsidiary packages *GRAPE* and *nauty* [10, 28, 16].)

For a somewhat different description of the 11-cell we refer to [21], where the polytope appears as a singular offshoot from a family of finite, self-dual (directly) regular polytopes \mathcal{P}^π , all with icosahedral facets and dodecahedral vertex-figures. The parameter π is any prime in the ring $\mathbb{Z}[\tau]$ of algebraic integers in $\mathbb{Q}(\sqrt{5})$. In fact, the 11-cell is a natural quotient of \mathcal{P}^π , when $\pi = -(2 + 5\tau)$. On the other hand, if $\pi = 2$, then $A \simeq O(4, 2^2, -1)$, the 4-dimensional orthogonal group over $GF(2^2)$ with Witt index 1. The polytope \mathcal{P}^2 has 68 facets (and 68 vertices); its medial layer graph \mathcal{G} must be C1360.5, the only 3-transitive graph with 1360 nodes listed in [4].

Generally, if π is an odd rational prime $\equiv \pm 2 \pmod{5}$, then the number of nodes in \mathcal{C} is about π^{12} .

Remark. Before proceeding, let us comment on the notation used in Example 17. A *Petrie polygon* in an abstract n -polytope \mathcal{Q} is defined (inductively) as an edge-path along which any $n - 1$ consecutive edges, but no n , belong to a Petrie polygon of a facet of \mathcal{Q} ; the Petrie polygon of a polygon (2-polytope) is the polygon itself [17, p. 163].

For example, if \mathcal{Q} is a regular polyhedron, say of type $\{p, q\}$, then all Petrie polygons have the same length l , being the period of $\rho_0\rho_1\rho_2$ in the automorphism group of \mathcal{Q} . If these data determine the combinatorial type of \mathcal{Q} , then we may denote \mathcal{Q} by $\{p, q\}_l$. In particular, the hemi-icosahedron $\{3, 5\}_5$ has pentagonal Petrie polygons, obtained by antipodal identification along the decagonal Petrie polygons of the icosahedron $\{3, 5\}$.

In many cases, the regular polyhedron \mathcal{Q} has a *Petrie dual* \mathcal{Q}^π , which shares the vertices and edges of \mathcal{Q} but whose faces are the Petrie polygons of \mathcal{Q} . In particular, we have $\{p, q\}_l^\pi \simeq \{l, q\}_p$ [17, p. 192].

Example 18. The chiral polytopes $\{3, 5, 3\}_l$, for $l = 7, 9$.

In [9] we find a discussion of ‘honeycombs’ of type

$$\{3, 5, 3\}_l,$$

for which the group A^+ is obtained by adjoining the relation $(\sigma_1\sigma_3)^l = 1$ to the standard relations listed in (2) (with $p = r = 3, q = 5$, of course). We recall that l can be interpreted

as the length of a ‘right-handed’ Petrie polygon; the ‘left-handed’ Petrie polygons have length l' equal the period of $\sigma_2\sigma_1\sigma_2\sigma_3$.

For $l \leq 6$ and $l = 8$ the intersection condition (3) fails and the corresponding group is non-polytopal. (Even so, the resulting structures still have interesting medial layer graphs.) For $l \geq 10$, it seems, though we have no proof, that A is infinite. Thus, we are left with $l = 7$ or 9, which somewhat surprisingly give finite, properly self-dual chiral polytopes.

The polytope $\{3, 5, 3\}_9$ has $l' = 10$ and group $A^+ \simeq PSL(2, 19)$, of order 3420. The medial layer graph \mathcal{G} appears as C1140.5 in [4]; and the Cayley graph \mathcal{C} for $G = D^+ \simeq PGL(2, 19)$ has 6840 nodes.

The polytope $\{3, 5, 3\}_7$ has $l' = 29$ and group $A^+ \simeq PSL(2, 29)$. However, this polytope admits a central polarity and so $D^+ \simeq PSL(2, 29) \times C_2$.

Let us turn now to the case that q is even. We recall from Lemma 8 that the index in D^+ of G equals the index in $I_2(3)$ of the normal subgroup $f(\bar{K})$ generated by the images under (9) of all ‘additional’ relations. Various examples show that all possible quotients do occur: D^+/G could be any one of $I_2(3)$, C_2 or $\{1\}$. Therefore, we shall make do with organizing various subcases of interest.

Proposition 19. *Suppose \mathcal{P} is finite, self-dual and directly regular or chiral of type $\{3, q, 3\}$, with q even. If $|G| = |D^+|/6$, then G acts regularly on the node set of \mathcal{G} and $\mathcal{G} \simeq \mathcal{C}$.*

Proof. This follows at once from Proposition 12 and the fact that both \mathcal{G} and \mathcal{C} have the same number of nodes. □

Remark. We cannot take \mathcal{P} to be non-orientably regular in Proposition 19. Indeed, from Proposition 12 we see that \mathcal{C} is then at least a 2-fold cover of \mathcal{G} .

Example 20. The 24-cell and other classical examples.

The medial layer graph for the 24-cell $\{3, 4, 3\}$ appears as the 3-transitive graph **192A** in [3]. Since $A^+ \simeq [3, 4, 3]^+$ is the rotation subgroup of a Coxeter group, there are no additional relations to adjoin to those in (2) (with $p = r = 3, q = 4$). Thus in Lemma 8 we have $f(\bar{K}) = \{1\}$, so $k = 1$ and G has index 6 in D^+ . By Proposition 19, $\mathcal{G} \simeq \mathcal{C}$ really is a Cayley graph for the group G of order 192.

Now consider the hemi-24-cell $\{3, 4, 3\}/\pm 1$, which is also isomorphic to $\{3, 4, 3\}_6$, through imposition of the extra relation $(\sigma_1\sigma_3)^6 = 1$. Consulting (9), we still have $f(\bar{K}) = \{1\}$, so that $\mathcal{G} \simeq \mathcal{C}$ is the 3-transitive graph **96** in [3].

The medial layer graph for the spherical polytope $\{3, 2, 3\}$ is $K_{3,3}$, which is indeed a Cayley graph for the group $I_2(3) \simeq S_3$.

Example 21. The universal locally toroidal polytopes

$$\{ \{3, 6\}_s, \{6, 3\}_s \}, \tag{10}$$

for $s = (1, 1), (2, 0), (3, 0)$.

We recall from [17, 1D] how the toroidal polyhedron $\{3, 6\}_s$, with $s = (b, c)$, is obtained from $\mathcal{T} = \{3, 6\}$, the regular tiling of the Euclidean plane by equilateral triangles. The rotation group for \mathcal{T} is $[3, 6]^+ = \langle \sigma_1, \sigma_2 \rangle$, whose defining relations are a suitable subset of (2):

$$\sigma_1^3 = \sigma_2^6 = (\sigma_1\sigma_2)^2 = 1. \tag{11}$$

One obtains the toroid by identifying faces of \mathcal{T} through the action of the normal subgroup of translations in $[3, 6]^+$ generated by $x^b y^c$, where $x := \sigma_2^2 \sigma_1^{-1}$ and $y := \sigma_2^{-2} \sigma_1$ are unit translations along adjacent edges in the tiling. In brief, we adjoin to (11) the extra relation

$$(\sigma_2^2 \sigma_1^{-1})^b (\sigma_2^{-2} \sigma_1)^c = 1 . \tag{12}$$

The resulting toroidal map is actually a 3-polytope precisely when $b^2 + bc + c^2 \geq 3$ and is regular if and only if $bc(b - c) = 0$. Otherwise, the polyhedron is chiral. We denote the dual of $\{3, 6\}_s$ by $\{6, 3\}_s$. Clearly this dual is obtained in like manner from the hexagonal tiling $\{6, 3\}$. (Later we will need the remaining family of toroidal polyhedra $\{4, 4\}_s$, for which $s = (b, c)$ satisfies $b^2 + c^2 \geq 2$.)

A polytope \mathcal{P} is *locally toroidal* if its facets and vertex-figures are spherical or toroidal, with at least one kind toroidal. The regular locally toroidal polytopes have not yet been fully classified; see [17, Chs. 10–12].

Now let us consider the universal regular 4-polytope described by (10). This locally toroidal 4-polytope is known to be finite only when $s = (1, 1), (2, 0)$ or $(3, 0)$ [17, 11E]. In each of these cases, the polytope is self-dual. Thus D^+ is obtained from (2) and (5), with $p = r = 3, q = 6$, by adjoining just the relation (12); duality then forces the defining relation $(\sigma_2^{-2} \sigma_3)^b (\sigma_2^2 \sigma_3^{-1})^c = 1$ for the vertex-figure.

From (12) and (9), we conclude that $f(\bar{K})$ is generated by

$$(\alpha_0^2 \alpha_1 \alpha_0)^b (\alpha_0^{-2} \alpha_0 \alpha_1)^c = (\alpha_0 \alpha_1)^{c-b} \in I_2(3) . \tag{13}$$

Thus it is trivial to apply Lemma 8. Drawing on [17, Table 11E1], we can summarize the results in the following table:

$s = (b, c)$	$ D^+ $	$k = f(\bar{K}) $	$ \mathcal{C} $	$ \mathcal{G} $
(1, 1)	108	1	18	18
(2, 0)	240	3	120	40
(3, 0)	2916	1	486	486

For $s = (1, 1)$, the medial layer graph $\mathcal{G} \simeq \mathcal{C}$ is listed as **18** in [3] and, in fact, is the Levi graph for the Pappus configuration 9_3 . The corresponding group G of order 18 is actually isomorphic to the rotation subgroup $[3, 2, 3]^+ = \langle \eta_1, \eta_2, \eta_3 \rangle$ for $\{3, 2, 3\}$, if identify δ_1 with η_2, δ_2 with $\eta_1 \eta_2 \eta_3$ and δ_3 with $\eta_1 \eta_2$.

When $s = (3, 0)$, we again have $\mathcal{C} \simeq \mathcal{G}$, listed as the 3-regular graph **486C** in [3]. We note from [30] that the polytope \mathcal{P} can also be described as $\{3, 6, 3\}_6$.

Finally, when $s = (2, 0)$, we find that \mathcal{C} is again the graph **120B** in [3]; and once more G is the icosahedral group $[3, 5]$. However, now \mathcal{C} is a 3-fold cover of \mathcal{G} , which appears as **40** in [3].

Example 22. The universal polytope $\{\{3, 2b\}_6, \{2b, 3\}_6\}, b \geq 2$.

It is easy to check that

$$\{6, 3\}_{(b,0)} \simeq \{6, 3\}_{2b} ,$$

indicating that the combinatorial type of this toroidal polyhedron is forced by Petrie polygons having length $2b$. Hence, for the dual of the Petrie dual, namely $\{3, 2b\}_6$, the crucial extra defining relation is

$$(\rho_0\rho_1\rho_2)^6 = (\sigma_1^2\sigma_2^2)^3 = 1. \tag{14}$$

The polyhedron $\{3, 2b\}_6$ has the same group of order $12b^2$ as the original toroid (with different specified generators, of course).

In [31, §5], the second author of this paper proved that the universal, self-dual regular polytope

$$\{\{3, 2b\}_6, \{2b, 3\}_6\}$$

has automorphism group of order

$$|A| = \begin{cases} 36b^4, & b \text{ odd,} \\ 72b^4, & b \text{ even.} \end{cases}$$

(We require $b \geq 2$ for the original toroidal map to be polyhedral.) From (14) and (9) we conclude that $f(\bar{K})$ is generated by $(\alpha_0\alpha_1)^6 = 1$, so that $k = 1$ in all cases. Thus $\mathcal{G} \simeq \mathcal{C}$ is a 3-transitive trivalent graph with $6b^4$ (resp. $12b^4$) nodes for b odd (resp. even).

Example 23. Modular polytopes in the class $\langle \{3, 6\}_{(b,c)}, \{6, 3\}_{(b,c)} \rangle$

We begin with the infinite Coxeter group $W = [3, 6, 3]$ and exploit its action on the conformal model of hyperbolic space \mathbb{H}^3 . Referring to [27, §10] or [22, §6], we therefore consider the unimodular group $L_2^{(-1)}(\mathbb{Z}[\omega])$, consisting of all 2×2 matrices of determinant ± 1 over the domain of Eisenstein integers $\mathbb{Z}[\omega]$ (where $\omega = e^{2\pi i/3}$). Next let H be the subgroup generated by

$$S_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} \omega^2 & 0 \\ -\omega^2 & -\omega \end{bmatrix}, S_3 = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}, \tag{15}$$

noting that $\det(S_1) = \det(S_3) = 1$, $\det(S_2) = -1$ and $S_1^3 = -I$ [22, Eq. 15]. We recall that H has index 4 in $L_2^{(-1)}(\mathbb{Z}[\omega])$ and that the rotation group

$$W^+ = [3, 6, 3]^+ \simeq H/\{\pm I\}.$$

Of course, $[3, 6, 3]^+$ is infinite, as is the corresponding polytope $\{3, 6, 3\}$, which is realized as a self-dual, regular tessellation of \mathbb{H}^3 . However, if we reduce H modulo $m \in \mathbb{Z}[\omega]$, then almost always we obtain a finite regular or chiral 4-polytope. More precisely, we first reduce H to H_m , then examine

$$X := \{x \in \mathbb{Z}[\omega]/(m) : xI \in H_m\},$$

which is a group of units in $\mathbb{Z}[\omega]/(m)$ arising from the centre of H_m . For any admissible subgroup S of such units, with $\{\pm 1\} \subseteq S \subseteq X$, we set

$$H_m^S := H_m/\{sI : s \in S\},$$

and let σ_j be the natural image of S_j in this quotient. By [22, Th. 6.1], if the non-zero modulus $m \nmid (1 - \omega)$, then $A^+ := H_m^S = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is the rotation group for a finite 4-polytope \mathcal{P}_m^S . Briefly, \mathcal{P}_m^S is directly regular if $m \mid \bar{m}$ and $S = \bar{S}$, but is chiral otherwise.

For example, if $m = 3$ then $\mathcal{P}_3^{\{\pm 1\}}$ is the universal regular polytope

$$\{ \{3, 6\}_{(3,0)}, \{3, 6\}_{(1,1)} \},$$

which is clearly not self-dual. By [18, Th. 2], the medial layer graph \mathcal{G} cannot then be symmetric. Indeed, \mathcal{G} is the *Gray graph*, the smallest trivalent, semisymmetric graph [18, §3.4].

In fact \mathcal{P}_m^S is self-dual if and only if $\gcd(m, 1 - \omega) = 1$, which we now assume. We also suppose $m \nmid 2$, since $\mathcal{P}_2^{\{1\}} \simeq \{3, 3, 3\}$ collapses to the regular 4-simplex. Restricting $m = b - c\omega$ in this way (where $b, c \in \mathbb{Z}$), we have that \mathcal{P}_m^S is a self-dual polytope in the class

$$\langle \{3, 6\}_{(b,c)}, \{6, 3\}_{(b,c)} \rangle.$$

From [22] we note that

$$|A^+| = |H_m^S| = \frac{2(m\bar{m})^3}{|S|} \prod_{\pi|m} (1 - (\pi\bar{\pi})^{-2}), \tag{16}$$

where the product is over all non-associated prime divisors π of m in $\mathbb{Z}[\omega]$. Thus we can readily compute the the number of nodes in the medial layer graph \mathcal{G} . Indeed, a typical node in \mathcal{G} is the basic 2-face F_2 , whose stabilizer in A^+ is the group of order 6 described in (7). Thus \mathcal{G} has $|H_m^S|/3$ nodes.

However, recalling our comments in Example 21, it is very unlikely that \mathcal{P}_m^S will be universal for its facets and vertex-figures. Thus, in order to understand the corresponding Cayley graph \mathcal{C} , we must be a little more devious.

Proposition 24. *For any modulus $m = b - c\omega \in \mathbb{Z}[\omega]$, with $\gcd(m, 1 - \omega) = 1$ and $m \not\sim 2$, and for any admissible unit group S , let \mathcal{P}_m^S be the self-dual 4-polytope with rotation group H_m^S . Let \mathcal{G} be its (trivalent) medial layer graph. Then*

- (a) *if $m \mid \bar{m}$ and $S = \bar{S}$, then \mathcal{P}_m^S is regular and \mathcal{G} is 3-transitive.*
- (b) *if $m \nmid \bar{m}$ or $S \neq \bar{S}$, then \mathcal{P}_m^S is chiral and \mathcal{G} is 2-transitive.*

Furthermore, the associated (trivalent) Cayley graph \mathcal{C} is transitive on 2-arcs and is a k -fold cover of \mathcal{G} , where

$$k := \begin{cases} 3, & \text{if } s^2 = 1 \text{ for all } s \in S; \\ 6, & \text{if } s^2 = -1 \text{ for some } s \in S. \end{cases}$$

Proof. The transitivity properties of \mathcal{G} follow at once from [23, Ths. 2 and 5]. By Proposition 12 and Lemma 8 it remains only to compute $k = |f(\bar{K})|$. (Recall that $6 = k[D^+ : G]$ and that \bar{K} has nothing to do with complex conjugation.)

From (13) we have $(\alpha_1\alpha_0)^{b-c} \in f(\bar{K}) \subseteq I_2(3)$. Now $m\bar{m} = b^2 + bc + c^2 \equiv (b - c)^2 \pmod{3}$. Since, by hypothesis, m is relatively prime to $3 = -\omega^2(1 - \omega)^2$, we conclude that $b - c$ is relatively prime to 3, and so $\alpha_0\alpha_1 \in f(\bar{K})$. Thus $k = 3$ or 6.

Now suppose $s^2 = 1$ for all $s \in S$. Thus, the determinant map is well-defined on $A^+ = H_m^S$, and we have $\det(\sigma_j) = \det(\delta\sigma_j\delta)$ for $j = 1, 2, 3$. With (6) this gives a well-defined homomorphism

$$\begin{aligned} h : D^+ &\rightarrow \{\pm 1\} \\ \alpha\delta^i &\mapsto \det(\alpha), \text{ for } \alpha \in A^+, i = 0, 1. \end{aligned} \tag{17}$$

Clearly $G \subseteq \ker(h)$, which has index 2 in D^+ , since $\det(\sigma_2) = -1$. Thus $k = 3$ in all such cases.

Otherwise we have $s^2 = -1$ for some $s \in S$. But $sI = S_{i_1}S_{i_2} \cdots S_{i_l}$ for suitable $i_j \in \{1, 2, 3\}$. Since $\det(S_l) = -1$ only when $l = 2$, we have $i_j = 2$ for an odd number of j 's. But in H_m^S we have $1 = \sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_l}$. From (9) we therefore conclude that $f(\bar{K})$ contains a reflection, so that $k = 6$. □

Remarks. Any rational prime $p \equiv 1 \pmod{3}$ factors over $\mathbb{Z}[\omega]$ as $p = \pi\bar{\pi} = b^2 + bc + c^2$, where $\pi = b - c\omega$ is prime in $\mathbb{Z}[\omega]$. Let us take our modulus m to be π , with $S = \{\pm 1\}$. We conclude from Proposition 24(b) that $\mathcal{P}_\pi^{\{\pm 1\}}$ is self-dual and chiral of type $\{\{3, 6\}_{(b,c)}, \{6, 3\}_{(b,c)}\}$. From (16) we have

$$|A^+| = (\pi\bar{\pi})^3(1 - (\pi\bar{\pi})^{-2}) = p(p^2 - 1).$$

Thus the medial layer graph \mathcal{G} is 2-transitive and has $p(p^2 - 1)/3$ nodes. Since $k = 3$, the Cayley graph \mathcal{C} has $p(p^2 - 1)$ nodes and has an automorphism group which is transitive on 2-arcs. For example, when $p = 7$, we find that \mathcal{G} and \mathcal{C} are the 2-transitive graphs **112A** and **336B**, respectively, in [3].

Suppose now that the modulus m is an odd rational prime p . For a moment we put aside the conformal action of W^+ on \mathbb{H}^3 and turn instead to the standard faithful representation of $W = [3, 6, 3]$ as a *crystallographic linear Coxeter group* (see [15, Cor. 5.4] and [19, §4]). Abusing notation a little, we therefore now suppose $W \subset GL_4(\mathbb{Z})$, then reduce W modulo p and so obtain a finite representation W_p in some orthogonal space V over \mathbb{Z}_p . When $p > 3$, the reduced group W_p turns out to be the automorphism group of a finite, self-dual regular polytope \mathcal{Q}_p in the class

$$\langle \{3, 6\}_{(p,0)}, \{6, 3\}_{(p,0)} \rangle.$$

(See [20, Eqs. 22, 10]; this reference and [19] give a detailed description of the construction and applications to several natural families of polytopes.)

We can now establish a connection with the earlier family of polytopes \mathcal{P}_m^S . For a suitable matrix $B \in GL_4(\mathbb{Q}(\omega))$, we may define an epimorphism

$$\begin{aligned} H &\rightarrow W^+ \\ \text{with } A &\mapsto B(A \otimes \bar{A})B^{-1}. \end{aligned}$$

As long as $s\bar{s} = 1$ for all $s \in S$, this induces an epimorphism

$$\lambda : H_p^S \rightarrow W_p^+,$$

which in fact becomes an isomorphism when S is chosen carefully. We refer to [19, 20] for more notation and, without much explanation, summarize our calculations in

Corollary 25. *For the rational prime $p > 3$, let \mathcal{Q}_p be the self-dual, directly regular 4-polytope with automorphism group $W_p = [3, 6, 3]_p$. Then the (trivalent) medial layer graph \mathcal{G} for this polytope is 3-transitive with $|W_p|/6$ nodes. Furthermore, the associated (trivalent) Cayley graph \mathcal{C} is transitive on 2-arcs and is a k -fold cover of \mathcal{G} , as indicated below:*

$p \pmod{12}$	$S \subset \mathbb{Z}[\omega]/(p)$	$H_p^S \simeq W_p^+ \simeq$	$ W_p $	k
1	$\{\pm 1, \pm s\}$	$O_1(4, p, 1)$	$p^2(p^2 - 1)^2$	6
5	$\{\pm 1\}$	$O(4, p, -1)$	$2p^2(p^4 - 1)$	3
7	$\{\pm 1\}$	$O(4, p, 1)$	$2p^2(p^2 - 1)^2$	3
11	$\{\pm 1, \pm s\}$	$O_1(4, p, -1)$	$p^2(p^4 - 1)$	6

Proof. For $p \equiv \pm 1 \pmod{12}$ there exists $s \in \mathbb{Z}[\omega]$ with $s^2 \equiv -1$, $s\bar{s} \equiv 1 \pmod{p}$. By Proposition 24 we then have $k = 6$. \square

Note that \mathcal{C} has $p^2(p^2 - \epsilon)(p^2 - 1)$ nodes, where $\epsilon = +1$ for $p \equiv 1, 7 \pmod{12}$, $\epsilon = -1$ for $p \equiv 5, 11 \pmod{12}$. For example, when $p = 5$, the graph \mathcal{C} with 15600 nodes is actually 2-transitive and is a 3-fold cover of the 3-transitive graph \mathcal{G} .

5 Finite polytopes \mathcal{P} of type $\{4, q, 4\}$

Suppose \mathcal{P} is a finite, self-dual and regular or chiral polytope of type $\{4, q, 4\}$. Then our observations in Section 3 tell us that the corresponding medial layer graph \mathcal{G} and Cayley graph \mathcal{C} are symmetric *tetravalent* graphs. The general theory of these graphs is somewhat less developed than in the trivalent case, although considerable information can be found in [2, Ch. 17] and in [11, 24, 32]. We shall have correspondingly less to say here about such graphs as arise from polytopes. We summarize what we can easily say in the following

Proposition 26. *Suppose \mathcal{P} is finite, self-dual and regular or chiral of type $\{4, q, 4\}$. The corresponding Cayley graph \mathcal{C} and medial layer graph \mathcal{G} are symmetric, tetravalent graphs. Furthermore, \mathcal{C} is a l -fold cover of \mathcal{G} , as described in the following Table:*

\mathcal{P}	$ \mathcal{G} $	q odd	q even
		$l =$	$l =$
<i>directly regular</i>	$ D^+ /8$	8 or 4	8, 4, 2 or 1
<i>chiral</i>	$ D^+ /8$	8 or 4	8, 4, 2 or 1
<i>non-orientably regular</i>	$ D^+ /16$	16 or 8	16, 8, 4 or 2

In particular, if \mathcal{P} is directly regular or chiral and $|G| = |D^+|/8$, then $\mathcal{C} \simeq \mathcal{G}$. This isomorphism is impossible when q is odd and in all non-orientably regular cases.

Proof. This follows immediately from Propositions 9, 11 and 12. \square

Example 27. The cubic toroids $\{4, 3, 4\}_s$, where $s = (b, 0, 0)$, $(b, b, 0)$ or (b, b, b) , for integers $b \geq 2$.

The cubic toroid $\mathcal{P}_s := \{4, 3, 4\}_s$ is a finite, self-dual and directly regular 4-polytope constructed in much the same way as the toroidal polyhedra described in Example 21. Thus \mathcal{P}_s is a quotient of $\{4, 3, 4\}$, the familiar tessellation of Euclidean 3-space by cubes; hence, we can regard \mathcal{P}_s as a tessellation of the 3-torus [17, 6D].

The toroids \mathcal{P}_s are parametrized by a type vector $s = (b^i, 0^{3-i})$, as indicated above. From [17, 6D5] we find that the group A^+ is defined by the relations in (2), with $p = r = 4$, $q = 3$, along with a single extra relation

$$(\sigma_1\sigma_3\beta)^b = 1,$$

where $\beta = \sigma_2^2, \sigma_1\sigma_3\sigma_2$ or $(\sigma_1\sigma_3)^2$, for $s = (b, 0, 0)$, $(b, b, 0)$ or (b, b, b) , respectively. Noting that $\sigma_2^3 = 1$, it is now easy to use (9) and Lemma 8 to show that the covering multiplicity $k = f(\bar{K}) = 4$ in every case. Drawing on [17, table 6D1], we summarize our calculations here:

s	$ A $	$ \mathcal{G} = A /8$	k	$ \mathcal{C} $
$(b, 0, 0)$	$48b^3$	$6b^3$	4	$24b^3$
$(b, b, 0)$	$96b^3$	$12b^3$	4	$48b^3$
(b, b, b)	$192b^3$	$24b^3$	4	$96b^3$

The smallest toroid in this class is in some sense the most interesting. The polytope $\mathcal{P}_{(2,0,0)}$ is *flat*, meaning that each of the 8 vertices is incident with each of the 8 facets. Of course, the 24 edges and 24 polygons comprise the 48 nodes of the medial layer graph \mathcal{G} , whose automorphism group is just the full duality group D of order 768. This graph is listed as $C4[48, 10]$ in [32] and is 1-transitive.

The Cayley graph \mathcal{C} has $192 = 4 \cdot 48$ vertices; and $\text{Aut}(\mathcal{C})$ has the unexpectedly large order $98304 = 192 \cdot 4 \cdot 2^7$. From [2, p. 134], we conclude that \mathcal{C} is 1-transitive, and that the stabilizer of a 1-arc has order 2^7 .

Example 28. The universal polytope $\{\{4, 2b\}_4, \{2b, 4\}_4\}$, $b \geq 2$.

This family of finite, self-dual regular polytopes is also described in [31, §5]. The analysis is very similar to that in Example 22, so we merely record some results. The toroidal map

$$\{4, 4\}_{(b,b)} \simeq \{4, 4\}_{2b}$$

has automorphism group of order $16b^2$. The dual of the Petrie dual is $\{4, 2b\}_4$ and the self-dual, universal regular polytope $\{\{4, 2b\}_4, \{2b, 4\}_4\}$ has $|A| = 64b^4$. Again we have $k = 1$ in all cases, so that $\mathcal{G} \simeq \mathcal{C}$ is a tetravalent graph on $8b^4$ nodes, with group transitive on 1-arcs.

Example 29. Modular polytopes in the class $\langle \{4, 4\}_{(p,0)}, \{4, 4\}_{(p,0)} \rangle$, for odd primes p .

The infinite Coxeter group $W = [4, 4, 4]$ is crystallographic. If, as in the previous section, we reduce the standard linear representation modulo any odd prime p , we find that $A = W_p$ is the automorphism group of a finite, self-dual and directly regular polytope \mathcal{P}_p in the class $\langle \{4, 4\}_{(p,0)}, \{4, 4\}_{(p,0)} \rangle$ [20, pp. 340-343]. This locally toroidal polytope is universal for its class when $p = 3$, but almost surely is not for larger primes [17, 10C]. Consulting [20, Eq. (6)], we note that

$$A = W_p = \begin{cases} O_1(4, p, 1), & \text{if } p \equiv 1 \pmod{8} \\ O_1(4, p, -1), & \text{if } p \equiv 7 \pmod{8} \\ O(4, p, 1), & \text{if } p \equiv 5 \pmod{8} \\ O(4, p, -1), & \text{if } p \equiv 3 \pmod{8}. \end{cases} \tag{18}$$

From here on the analysis is very similar to that for Corollary 25. We thus obtain

Corollary 30. For any odd prime p , let \mathcal{P}_p be the self-dual, directly regular 4-polytope with automorphism group $W_p = [4, 4, 4]_p$. Then the symmetric tetravalent Cayley graph \mathcal{C} is a k -fold cover of the symmetric, tetravalent medial layer graph \mathcal{G} , where

$$k := \begin{cases} 8 & \text{if } p \equiv \pm 1 \pmod{8} \\ 4 & \text{if } p \equiv \pm 3 \pmod{8} \end{cases} .$$

We conclude this section with a look at

Example 31. The universal, locally projective polytope $\{\{4, 3\}_3, \{3, 4\}_3\}$ [25, Table 1].

The facets of this self-dual, non-orientably regular polytope are hemicubes; dually its vertex-figures are hemi-octahedra. Along with the standard relations in (2) and (5), with $p = r = 4, q = 3$, we need only enforce $(\rho_0\rho_1\rho_2)^3 = 1$, or equivalently

$$(\sigma_2\sigma_1^2)^3 = 1 .$$

This gives $|A| = |A^+| = 96$. From (9), we conclude that $f(\bar{K})$ is the normal subgroup generated by $(\alpha_0(\alpha_0\alpha_1)^2)^3 = \alpha_1\alpha_0\alpha_1$ in $I_2(4)$, so that $k = 4$.

Note that \mathcal{G} has 12 nodes. From Proposition 12, we find that \mathcal{C} has 96 nodes and is a 8-fold cover of \mathcal{G} . Both \mathcal{C} and \mathcal{G} are 1-transitive.

6 Other examples and questions

We end our survey of particular examples with a look at

Example 32. The classical 4-dimensional star-polytopes.

Consulting [17, 7D], we find that just three of the 4-dimensional regular star-polytopes concern us here: $\{5, \frac{5}{2}, 5\}$, $\{\frac{5}{2}, 5, \frac{5}{2}\}$ and $\{5, 3, \frac{5}{2}\}$.

The first two of these are geometrically self-dual, and, as abstract regular polytopes, are each isomorphic to the universal polytope

$$\{ \{5, 5 | 3\}, \{5, 5 | 3\} \}$$

([17, Th. 7D16(b)]). The notation indicates, for instance, that the facet $\{5, 5 | 3\}$, with Schläfli type $\{5, 5\}$, is specified by its having triangular holes. This means that $(\sigma_1\sigma_2^{-1})^3 = 1$. Of course, the full automorphism group A is the Coxeter group H_4 of order 14400, so that \mathcal{G} is a symmetric pentavalent graph with 1440 nodes. By Propositions 9 and 12, we see that \mathcal{C} is a symmetric pentavalent graph on 14400 nodes and is a 10-fold cover of \mathcal{G} .

Now although the other polytope $\{5, 3, \frac{5}{2}\}$ is clearly not geometrically self-dual, it is nevertheless self-dual in the combinatorial sense used throughout this paper. In fact, it is isomorphic to

$$\{ 5, 3, 5 | 3 \} ,$$

whose deep triangular holes are specified by $(\sigma_1\sigma_3\sigma_2^{-1})^3 = 1$ [17, Th. 7D16(c)]. We obtain new pentavalent graphs \mathcal{G} and \mathcal{C} , again with 1440 and 14400 nodes, respectively.

We conclude by mentioning some open questions. First of all, suppose \mathcal{G} is t -transitive, while the cover \mathcal{C} is s -transitive. Invariably we have observed that $s \leq t$; however, we have no proof that this must be so. Of course, it would be very desirable to find examples with say $s = 4$ or 5 in the trivalent case (where $t = 2$ or 3); but this is most unlikely.

In Proposition 14, we have described \mathcal{C} as a derived voltage graph. If also A acts as a group of automorphisms on the stabilizer S , in a manner compatible with the voltage assignment φ , it is at least possible to lift t -transitivity on \mathcal{G} to \mathcal{C} , so that $s \geq t$ [2, Prop. 19.4]. However, we have no non-trivial examples of this very special situation. Also, we do not know if the voltage graph construction in Proposition 14 can be extended in a natural way to non-orientably regular cases (but see [12, Th. 2.2.2]).

Finally, we reconsider improperly self-dual chiral polytopes \mathcal{P} of type $\{p, q, p\}$. If q is even, then \mathcal{P} might not admit a polarity (duality of period 2); but when q is odd, \mathcal{P} does admit polarities [14, Th. 3.4]. Nevertheless, we have avoided all such polytopes in order to simplify the definition of the graph \mathcal{C} . Perhaps these complications can be overcome to yield interesting results in improperly self-dual cases.

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