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The $L_2(11)$-subalgebra of the Monster algebra

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Abstract

We study a subalgebra V of the Monster algebra, $V_{\mathbb{M}}$, generated by three Majorana axes $a_x,\ a_y$ and a_z indexed by the $2A$-involutions $x,\ y$ and z of $\mathbb{M}$, the Monster simple group. We use the notation $V = \langle\langle a_x, a_y, a_z\rangle\rangle$. We assume that xy is another $2A$-involution and that each of $xz,\ yz$ and xyz has order 5. Thus a subgroup G of $\mathbb{M}$ generated by $\{x,\ y,\ z\}$ is a non-trivial quotient of the group $G^{(5,5,5)} = \langle x,\ y,\ z \mid x^2,\ y^2,\ (xy)^2,\ z^2,\ (xz)^5,\ (yz)^5,\ (xyz)^5\rangle$. It is known that $G^{(5,5,5)}$ is isomorphic to the projective special linear group $L_2(11)$ which is simple, so that G is isomorphic to $L_2(11)$. It was proved by S. Norton that (up to conjugacy) G is the unique $2A$-generated $L_2(11)$-subgroup of $\mathbb{M}$ and that $K = C_{\mathbb{M}}(G)$ is isomorphic to the Mathieu group M_{12}. For any pair $\{t,\ s\}$ of $2A$-involutions, the pair of Majorana axes $\{a_t,\ a_s\}$ generates the dihedral subalgebra $\langle\langle a_t, a_s\rangle\rangle$ of $V_{\mathbb{M}}$, whose structure has been described in [16]. In particular, the subalgebra $\langle\langle a_t, a_s\rangle\rangle$ contains the Majorana axis a_{tst} by the conjugacy property of dihedral subalgebras. Hence from the structure of its dihedral subalgebras, V coincides with the subalgebra of $V_{\mathbb{M}}$ generated by the set of Majorana axes $\{a_t \mid t \in T\}$, indexed by the 55 elements of the unique conjugacy class T of involutions of $G \cong L_2(11)$. We prove that V is 101-dimensional, linearly spanned by the set $\{\ a_t \cdot a_s \mid s,\ t \in T\ \}$, and with $C_{V_{\mathbb{M}}}(K) = V \oplus \iota_{\mathbb{M}}$, where $\iota_{\mathbb{M}}$ is the identity of $V_{\mathbb{M}}$. Lastly we present a recent result of Á. Seress proving that V is equal to the algebra of the unique Majorana representation of $L_2(11)$.

Keywords: Majorana representation, Monster group, Conway-Griess-Norton algebra.

Math. Subj. Class.: 20C99, 20F05, 20C34, 20D05

1 Main result

We let $\big(V_{\mathbb{M}},\ \cdot\ ,(\ ,\ )\big)$ be the Monster algebra, a commutative non-associative algebra of dimension $196,884$ over $\mathbb{R}$, as described in [2]. As an $\mathbb{R}\mathbb{M}$-module, $V_{\mathbb{M}} = V'_{\mathbb{M}} \oplus \mathbb{1}_{\mathbb{M}}$, where $V'_{\mathbb{M}}$ is the minimal faithful irreducible $\mathbb{R}\mathbb{M}$-module of dimension $196,883$ and $\mathbb{1}_{\mathbb{M}}$

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is the trivial $\mathbb{RM}$-module which is the $\mathbb{R}$-span of the identity $\iota_{\mathbb{M}}$ of the algebra $V_{\mathbb{M}}$. The automorphism group of $\big(V_{\mathbb{M}},\ \cdot\ ,(\ ,\ )\big)$ is $\mathbb{M}$ the Monster simple group ([2], [7]). By $2A$ we denote the conjugacy class of involutions in $\mathbb{M}$ with the largest centraliser as in the Atlas [3]. For each $2A$ involution t of $\mathbb{M}$, the centraliser $C_{\mathbb{M}}(t) \cong 2.BM$ stabilises a 2-subspace W of $V_{\mathbb{M}}$ which has two non-trivial idempotents a_t and $\iota_{\mathbb{M}} - a_t$. In [2], J. Conway constructed an $\mathbb{M}$-invariant bijection ψ sending each $2A$ involution t to the non-trivial idempotent a_t of W with eigenvalue 1 and multiplicity 1 . We denote by $a_t := \psi(t)$ the image of t. In [8] A. A. Ivanov axiomatises some of the properties of the idempotents a_t into the definition of a Majorana axis.

A Majorana axis a of a real commutative non-associative algebra $\big(V,\ \cdot\ ,(\ ,\ )\big)$, where $\cdot$ associates with $(\ ,\ )$ in the sense that $(u \cdot v,\ w) = (u,\ v \cdot w)$ for all $u,\ v,\ w \in V$, is an idempotent of length 1, whose adjoint operator ad_a is semi-simple on V with spectrum $\{1, 0, \frac{1}{2^2}, \frac{1}{2^5}\}$. The eigenspaces of ad_a are denoted by $V_\mu^{(a)}$, with μ an eigenvalue, and satisfy the following conditions. The 1-eigenvectors of ad_a are exactly the scalar multiples of a. There exists a linear transformation $\tau(a)$ of V, called a Majorana involution, negating the $\frac{1}{2^5}$-eigenvectors, fixing the other eigenvectors and preserving both the algebra and inner products. Lastly there exists a linear transformation $\sigma(a)$ of $V_+^{(a)} = V_1^{(a)} \oplus V_0^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)}$ negating the $\frac{1}{2^2}$-eigenvectors, fixing the 0- and 1-eigenvectors, and preserving both products on $V_+^{(a)}$. From [8], this definition is equivalent to the 'Fusion Rules'. For two eigenvectors $u \in V_\lambda^{(a)}$ and $v \in V_\mu^{(a)}$ of a fixed Majorana axis a, the Fusion Rules specify in which part of the spectrum of ad_a the product $u \cdot v$ lies.

Sp	1	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
1	1	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
0	0	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
$\frac{1}{2^2}$	$\frac{1}{2^2}$	$\frac{1}{2^2}$	$1, 0$	$\frac{1}{2^5}$
$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$1, 0, \frac{1}{2^2}$

Table 1: Fusion rules

Definition 1.1. We denote by $\langle\langle A \rangle\rangle$ the subalgebra of $V_{\mathbb{M}}$ generated by a set A of Majorana axes.

The classification of subalgebras $\langle\langle a_t, a_s \rangle\rangle$ of $V_{\mathbb{M}}$, where $\{a_s,\ a_t\}$ is a pair of Majorana axes, was started in [2] and completed in [16]. We call them dihedral subalgebras as the corresponding pair of $2A$-involutions $\{t,\ s\}$ generates a dihedral subgroup of $\mathbb{M}$. We say the dihedral subalgebra has type C if the product of involutions ts belongs to the conjugacy class C of $\mathbb{M}$.

Some subalgebras of $V_{\mathbb{M}}$ generated by triples of Majorana axes are described by A. A. Ivanov et al in [11], [12], [13], [10], and [9].

In this paper, we investigate a subalgebra $V = \langle\langle a_x, a_y, a_z\rangle\rangle$ of $V_{\mathbb{M}}$ such that the dihedral subalgebra $\langle\langle a_x, a_y\rangle\rangle$ has type $2A$ and each of the dihedral subalgebras $\langle\langle a_x, a_z\rangle\rangle$, $\langle\langle a_y, a_z\rangle\rangle$, and $\langle\langle a_{xy}, a_z\rangle\rangle$ has type $5A$. The vector a_{xy} is the Majorana axis $\psi(xy)$ (since a dihedral subalgebra $\langle\langle a_s, a_t\rangle\rangle$ of type $2A$ contains the axis a_{st}).

Keeping in mind the bijection ψ we might ask whether there exists a subgroup of $\mathbb{M}$ generated by a triple of $2A$ involutions $\{x,\ y,\ z\}$ satisfying the relations:

$$x^2 = y^2 = z^2 = (xy)^2 = (xz)^5 = (yz)^5 = (xyz)^5 = 1.$$

A group affording the presentation

$$\langle x,\ y,\ z \mid x^2,\ y^2,\ (xy)^2,\ z^2,\ (xz)^5,\ (yz)^5,\ (xyz)^5\rangle$$

defines the Coxeter group $G^{(5,5,5)}$ and from [4] it is isomorphic to the projective special linear group $L_2(11)$. From classical results on $L_n(p^k)$, [5], $L_2(11)$ is a simple group of order $660 = 2^2\cdot 3\cdot 5\cdot 11$ and it has a single conjugacy class of involutions which we denote by T, and whose size is 55.

Proposition 1.2. *There exists a monomorphism $\iota : L_2(11) \hookrightarrow \mathbb{M}$ such that $\iota(T) \subseteq 2A$ and ι is unique up to conjugacy in $\mathbb{M}$.*

Proof. In Table 5 of [17] S. Norton gives the list of simple subgroups of $\mathbb{M}$ having their elements of order 5 in the $\mathbb{M}$-conjugacy class $5A$. For $\iota(T)$ it is a requirement since if a product of two $2A$ involutions has order 5 it belongs to the conjugacy class $5A$ of $\mathbb{M}$ [2]. By Norton's list there is only one conjugacy class of groups isomorphic to $L_2(11)$ containing $5A$ elements and their involutions belong to class $2A$. □

Throughout the paper ι denotes the monomorphism as in Proposition 1.2, $G \cong L_2(11)$ denotes the image of ι, and T denotes the conjugacy class of involutions in G.

By the conjugacy property of dihedral subalgebras[1], the axis a_{tst} is contained in $\langle\langle a_t, a_s\rangle\rangle$. Hence from the dihedral subalgebras of V, we can restate our aim to be the study of the subalgebra V of $V_{\mathbb{M}}$ generated by the set of 55 Majorana axes $\{a_t \mid t \in T\}$. We determine the dimension of V and find a spanning set for V. In the next section we prove the following theorem.

Theorem. *Let V be the subalgebra of $V_{\mathbb{M}}$ generated by the set of 55 Majorana axes $\{a_t \mid t \in T\}$, where T is the class of involutions of the unique $2A$-generated $L_2(11)$-subgroup G of $\mathbb{M}$. Then*

(1) $dim(V) = 101$,

(2) *V is linearly spanned by the set $\{a_t \cdot a_s \mid t,\ s\ \in T\}$.*

(3) *If $K = C_{\mathbb{M}}(G)$ then $C_{V_{\mathbb{M}}}(K) = V \oplus \iota_{\mathbb{M}}$.*

[1] Let t and s be two $2A$ involutions, then tst is an involution conjugate to s. Hence tst is a $2A$ involution with corresponding Majorana axis $a_{tst} := \psi(tst)$.

In the last section we give some evidence towards the uniqueness of the map $\psi : t \to a_t$, where $t \in T$, within the class of Majorana representations of $L_2(11)$ satisfying conditions $(2A)$ and $(3A)$ (the terminology is explained in the last section)[2]. Lastly we state a recent result of Á. Seress proving that V is equal to the algebra of the unique Majorana representation of $L_2(11)$.

2 Some properties of $L_2(11)$

We present some of the standard properties of $G \cong L_2(11)$ used when calculating inner product values for V.

The group G is the automorphism group of the $(11, 5, 2)$-biplane, which we denote $\mathcal{B}$ (see [19]).
$\mathcal{B}$ is a 2-symmetric design with 11 points, $\{p_1, \ldots, p_{11}\}$, and 11 lines, $\{l_1, \ldots, l_{11}\}$, such that each line contains 5 points, each point lies on 5 lines, two lines intersect in exactly 2 points, and two points share exactly 2 lines. We call the incidence relation $p_i \in l_j$ a flag, which we denote $\alpha_{i,j}$, and the relation $p_i \notin l_j$ an anti-flag, which we denote by $w_{i,j}$.

From [14], the lines of $\mathcal{B}$ can be obtained by finding a difference set l_1 of size 5, with elements from $\mathbb{Z}_{11}$, such that every integer modulo 11 appears exactly twice as a difference $i - j$ mod 11 for i and j in l_1. We have that $l_1 = \{1, 3, 4, 5, 9\}$, which is the set of non-zero perfect squares in $\mathbb{Z}_{11}$, and all other lines l_k can be defined by $l_k = \{1 + k,\ 3 + k,\ 4 + k,\ 5 + k,\ 9 + k\}$, where $k \in \mathbb{Z}_{11}^*$ and addition is modulo 11.
The incidence matrix $\mathcal{N}$ of $\mathcal{B}$ is given below with the rows indexed by the points of $\mathcal{B}$, the columns indexed by the lines, and each flag is represented by a $'1'$ and each anti-flag by a $'0'$.

1	0	1	1	1	0	0	0	1	0	0
0	1	0	1	1	1	0	0	0	1	0
0	0	1	0	1	1	1	0	0	0	1
1	0	0	1	0	1	1	1	0	0	0
0	1	0	0	1	0	1	1	1	0	0
0	0	1	0	0	1	0	1	1	1	0
0	0	0	1	0	0	1	0	1	1	1
1	0	0	0	1	0	0	1	0	1	1
1	1	0	0	0	1	0	0	1	0	1
1	1	1	0	0	0	1	0	0	1	0
0	1	1	1	0	0	0	1	0	0	1

We can represent G as a permutation group on 11 letters, so that $G \subset Sym(11)$, by letting G act on the indices of the points or lines such that the incidence structure of $\mathcal{B}$ is preserved.

The stabiliser $G(\alpha_{i,j})$ of a flag $\alpha_{i,j}$ is isomorphic to A_4, the stabiliser $G(w_{k,l})$ of an anti-flag $w_{k,l}$ is isomorphic to D_{10}, and the stabiliser of a line (or a point) is isomorphic to A_5. We can associate to a flag $\alpha_{i,j}$ a unique subgroup $S(\alpha_{i,j}) \cong C_2 \times C_2$ and to an anti-flag $w_{k,l}$ a unique subgroup $S(w_{k,l}) \cong C_5$ such that $N_G\big(S(\alpha_{i,j})\big) = G(\alpha_{i,j})$ and $N_G\big(S(w_{k,l})\big) = G(w_{k,l})$. It is easy to see that each involution t stabilises 3 flags and to deduce that $C_G(t) \cong$

[2]When the first draft of this article was written, the author has learned that Ákos Seress has written a GAP program, [6], capable of checking this uniqueness conjecture.

D_{12}. Similarly for $\langle h\rangle$ a subgroup of order 3 we can deduce $N_G(\langle h\rangle) \cong D_{12}$. There are only one class of involutions and one class of elements of order 3 in G, so we can let d be the G-invariant bijection between subgroups of order 2 and 3 sending each involution t to the unique subgroup of order 3 commuting with t. Furthermore, by [14], G contains one class of subgroups isomorphic to the Frobenius group of order 55, which we denote F_{55}. These are the four conjugacy classes of maximal subgroups of G; two non-conjugate classes of subgroups isomorphic to A_5 each of size 11 and each stabilising a point or a line, one class of subgroups isomorphic to D_{12}, and one class of subgroups isomorphic to the Frobenius group F_{55}.

3 The algebra V

We start this section by finding an upper bound for $dim(V)$ based on the work of S. Norton ([15], [16], and [17]). We then calculate the Gram matrix of a particular subset of V which provides a lower bound for $dim(V)$.

3.1 S. Norton's observations

The upper bound on $dim(V)$ stems from the following inclusion.

Lemma 3.1. $V \subseteq C_{V_{\mathbb{M}}}\big(C_{\mathbb{M}}(G)\big)$

Proof. By the definition of a Majorana axis, a_t is fixed by $C_{\mathbb{M}}(t) \cong 2.BM$. Therefore $C_{\mathbb{M}}(G) = C_{\mathbb{M}}\big(\langle x, y, z\rangle\big) = \bigcap_{t=x,y,z} C_{\mathbb{M}}(t)$ fixes $V = \langle\langle a_x, a_y, a_z\rangle\rangle$ by $\mathbb{M}$-invariance of the algebra $V_{\mathbb{M}}$. □

We denote by K the group $C_{\mathbb{M}}(G)$. The dimension of the fixed space of K in $V_{\mathbb{M}}$ can be obtained by calculating the fusion of the character table of K in that of $\mathbb{M}$ (since the character of $V'_{\mathbb{M}}$ is known [3]). It is equal to the inner product of characters $\langle \chi_{V_{\mathbb{M}}} \downarrow_K, \mathbb{1}_K\rangle_{\mathbb{R}K}$, where $\mathbb{1}_K$ is the trivial character of K, and $\chi_{V_{\mathbb{M}}} \downarrow_K$ is the character of $V_{\mathbb{M}}$ restricted to K. We thus need to determine the isomorphism type of K and the inclusions of the conjugacy classes of G and K into those of $\mathbb{M}$.

We call an A_5-subgroup H of $\mathbb{M}$ an A_5 of type $(2A, 3A, 5A)$ if the elements of order 2, 3 and 5 of H are in the $\mathbb{M}$-conjugacy classes $2A$, $3A$ and $5A$ respectively. Clearly all A_5-subgroups of G are of type $(2A, 3A, 5A)$.

Proposition 3.2. (i) *For K as above, $K \cong M_{12}$.*

(ii) *All A_5-subgroups H as above are conjugate in $\mathbb{M}$ and $C_{\mathbb{M}}(H) \cong A_{12}$.*

(iii) *The conjugacy classes of G fuse into those $\mathbb{M}$ as follows:*

Class in G	$1a$	$2a = T$	$3a$	$5a$	$5b$	$6a$	$11a$	$11b$
Class in $\mathbb{M}$	$1A$	$2A$	$3A$	$5A$	$5A$	$6A$	$11A$	$11A$

Proof. The result from part (i) can be read from the entry 31 of Table 3 of [15]. Part (ii) is proved in Lemma 4 of [15]. To prove (iii) we carry on from the proof of Proposition 1.1. From Table 5 of [17] we deduce the inclusion $3a \subset 3A$. In the character table of $\mathbb{M}$, given in [3], the information on p-powers[3] of elements $g \in 6A$ gives $g^2 \in 3A$ and $g^3 \in 2A$, and $6A$ is the unique conjugacy class of elements of order 6 with those p-powers, hence to avoid a contradiction we must have $6a \subset 6A$. Since $\mathbb{M}$ has a unique class $11A$ of elements of order 11 the classes $11a$ and $11b$ are subsets of $11A$. □

Proposition 3.3. *For the algebra $C_{V_{\mathbb{M}}}(K)$ we have $dim(C_{V_{\mathbb{M}}}(K)) = 102$.*

Within the proof of Proposition 3.3 we determine the fusion of the conjugacy classes of K into those of $\mathbb{M}$. We follow the Atlas's notation, [3], by writing the conjugacy of elements of order N in $\mathbb{M}$: $NA,\ NB, \dots (etc)$ in increasing order of the size of the class. Similarly for K we use the notation $NA_K,\ NB_K, \cdots (etc)$. The character tables used are those of the Atlas [3].

Proof. By part (i) of the previous proposition we have the inclusion of groups $H \subset G$, where $H \cong A_5$ is of type $(2A, 3A, 5A)$, which implies $K \subset C_{\mathbb{M}}(H) \cong A_{12}$. In A_{12}, the elements with cycle decompositions $2^2 1^8$, $2^4 1^4$, and 2^6 have 8, 4, and no fixed points respectively in the natural action of A_{12} on 12 points, and so by Lemma 6 in [15] they are mapped to the $\mathbb{M}$-conjugacy classes $2A,\ 2B$ and $2A$ respectively. There is a doubly transitive action of M_{12} on 12 points with character $\chi_1 + \chi_{11a}$ where χ_1 is the trivial character of M_{12} and χ_{11a} is the first irreducible character of degree 11 (as in the Atlas, [3]). This character takes the value 0 for the elements in the class $2A_{M_{12}}$, and the value 4 for the elements in the class $2B_{M_{12}}$, hence $2A_{M_{12}} \subset 2A$ and $2B_{M_{12}} \subset 2B$.

The structure class constants[4] for any pair of $2A$ involutions in $\mathbb{M}$ give the number of elements in each conjugacy class of $\mathbb{M}$ expressible as a product of two $2A$ involutions. The constants can be calculated directly from the character table. For $\mathbb{M}$ the product of two $2A$ involutions lies in either of the $\mathbb{M}$ classes : $1A,\ 2A,\ 2B,\ 3A,\ 3C,\ 4A,\ 4B,\ 5A$ or $6A$ (see [2] or [15]). Similarly for $K \cong M_{12}$ we obtain that the product of two $2A_K$ involutions lies in either of the K classes $1A_K,\ 2A_K,\ 2B_K,\ 3B_K,\ 4A_K,\ 4B_K,\ 5A_K$ or $6A_K$. To avoid a contradiction on the monomorphism ι we have $5A_K \subset 5A$, and $6A_K \subset 6A$ and $3A_K$ is a subset of either $3A$ or $3C$. The class $6A_K$ has p-powers $3B_K$, $2A_K$ in K, and the class $6A$ has p-powers $3A$, $2A$ in $\mathbb{M}$. Hence $3B_K \subset 3A$. From lemma 6 of [15] no elements of order 3 in $C_{\mathbb{M}}(H) \cong A_{12}$ belongs to class $3C$ of $\mathbb{M}$, hence $3A_K$ belongs to either $3A$ or $3B$. If $3A_K \subset 3A$ then $6B_K \subset 6C$ and if $3B_K \subset 3B$ then $6B_K$ is in either $6B$ or $6E$ according to the relevant p-powers in K and $\mathbb{M}$. We determine the fusion in $\mathbb{M}$ of $3A_K$ and $6B_K$ at the end of the proof. The classes $4A_K,\ 4B_K$ contain products of $2A$ involutions and their squares lie in class $2B_K \subset 2B$ hence $4A_K,\ 4B_K \subset 4A$ as $4A$ is the unique class of elements of order 4 squaring to $2B$. The classes $8A_K,\ 8B_K$ have their squares in classes $4A_K,\ 4B_K$ respectively, and in $\mathbb{M}$ the unique conjugacy class of elements of order 8 squaring to $4A$ is $8B$. Hence $8A_K,\ 8B_K \subset 8A$. The class $10A_K$ in M_{12} has p-powers $5A_K \subset 5A$ and $2A_K \subset 2A$ and in $\mathbb{M}$ the class $10A$ is the unique class of elements of

[3]For a finite group L, the p-power line in the character table of L records for each conjugacy class C of L, and for each prime p dividing the order of the elements of C, to which conjugacy class the p^{th}-power of the elements of C belongs to.

[4]For a finite group L, the structure class constants give the number of solutions $s_{1,2,3}$ to equations in the group of the type $x_1.x_2 = x_3$, where each x_i belongs to a conjugacy class C_i of L. From the table of complex characters of L:

$$s_{1,2,3} = \frac{|L|}{|C_L(x_1)| \cdot |C_L(x_2)|} \sum_{\chi \in \mathrm{Irr(L)}} \frac{\chi(x_1)\chi(x_2)\overline{\chi(x_3)}}{\chi(\mathbb{1})}$$

order 10 with such p-powers so that $10A_K \subset 10A$. There is a unique class of elements of order 11 in $\mathbb{M}$ so $11A_K,\ 11B_K \subset 11A$. If $3A_K \subset 3A$ then $6B_K \subset 6C$ and the completed fusion of conjugacy classes of K in those of $\mathbb{M}$ gives a value of $\langle \chi_{V_\mathbb{M}} \downarrow_K, \mathbb{1}_K \rangle_{\mathbb{R}K}$ which is not integral, a contradiction. Hence $3A_K \subset 3B$ and $6B_K$ is in $6B$ or $6E$. To determine which, we look at the fusion of the conjugacy classes of $A := C_\mathbb{M}(H) \cong A_{12}$ in $\mathbb{M}$. Apart from the conjugacy classes $6G_A,\ 9A_A,\ 9B_A$ and $9C_A$, the fusion of the classes of A in $\mathbb{M}$ is straightforward using the information on p-powers and the fusion of the classes of K in $\mathbb{M}$ already obtained. From a calculation of S. Shpectorov in [12] we know that $\langle \chi_{V_\mathbb{M}} \downarrow_A, \mathbb{1}_A \rangle_{\mathbb{R}A} = 26$. This can only happen if $6G_A \subset 6B$ and $9A_A,\ 9B_A,\ 9C_A \subset 9A$. In particular elements of order 6 in A_{12} cannot be subsets of $6E$, hence neither can the elements of order 6 in K. Hence $6B_K \subset 6B$. We have obtained the fusion of K in $\mathbb{M}$

Class in K	$1A_K$	$2A_K$	$2B_K$	$3A_K$	$3B_K$	$4A_K$	$4B_K$
p-powers	A	A	A	A	A	B	B
Class in $\mathbb{M}$	$1A$	$2A$	$2B$	$3B$	$3A$	$4A$	$4A$

	$5A_K$	$6A_K$	$6B_K$	$8A_K$	$8B_K$	$10A_K$	$11A_K$
	A	BA	AB	A	B	AA	A
	$5A$	$6B$	$6A$	$8B$	$8B$	$10A$	$11A$

and we can now compute the inner product of real characters of K

$$\langle \chi_{V_\mathbb{M}} \downarrow_K, \mathbb{1}_K \rangle_{\mathbb{R}K} = 102.$$

□

The following useful observation was made by S. Norton (in a more general context).

Lemma 3.4. *The identity $\iota_\mathbb{M}$ of $V_\mathbb{M}$ cannot be contained in V.*

Proof. The groups G and K centralise each other in $\mathbb{M}$ and $G \times K$ is a subgroup of $\mathbb{M}$. The unique conjugacy class of involutions T of G is in class $2A$ of $\mathbb{M}$, and we have proved that the classes $2A_K$ and $2B_K$ are in $2A$ and $2B$ respectively.

Claim : there exist elements $s \in T$ and $t \in 2A_K$ such that the element ts of $G \times K$ is in class $2B$ of $\mathbb{M}$.

From Proposition 3.2, part (iii), the element s can be taken from a A_5-subgroup X of G of type $(2A, 3A, 5A)$. From the proof of Proposition 3.2, the elements of $2A_K$ act fixed-point freely on 12 points, and the centraliser in $\mathbb{M}$ of an A_5-subgroup of type $(2A, 3A, 5A)$ is isomorphic to A_{12}. By Table 4, line 7 of [17], there exists a subgroup $Y \subseteq A_{12},\ Y \cong A_5$, that acts transitively on 12 points. Let $t \in Y$. Then, by Table 3, line 8 of [17], the involutions in the diagonal subgroups of $X \times Y$ are in $2B$. In particular $ts \in 2B$ and the claim is proved.
Since $ts \in 2B$, the axes a_t and a_s generate a dihedral algebra of type $2B$ and $a_t \cdot a_s = 0$ (see [2] or [11]). For all $z \in T$ there is an element $g \in G$ such that $z = s^g$, so by invariance of the algebra product $(a_t \cdot a_s)^g = 0 = a_t \cdot a_z$ since G normalises K. Now, V is generated by the 55 Majorana axes a_z for $z \in T$ and the 0-eigenspace of a_t is closed under the algebra product, so if the identiy $\iota_\mathbb{M}$ were in V we would get the contradiction $a_t \cdot \iota_\mathbb{M} = 0$. □

The identity of a commutative algebra being unique and therefore stable under the automorphism group we have $\iota_{\mathbb{M}} \in C_{V_{\mathbb{M}}}(K)$. And since $\iota_{\mathbb{M}}$ is not in V we obtain the main result of this subsection.

Proposition 3.5. *For the algebra V we have $dim(V) \leq 101$.* □

3.2 Inner product values for V

In this subsection we calculate all inner products on a well-chosen subset of V and compute the rank of the corresponding Gram matrix to bound below the dimension of V. We do so using the information on some subalgebras of V which have already been classified.

3.2.1 Dihedral subalgebras of V

The algebra V contains the dihedral subalgebras of types $2A$, $3A$, $5A$, and $6A$ (obtained by calculating the relevant structure class constants in the character table of G). From [16], for each type of dihedral algebra, we know the dimension of the algebra, and a basis for which all algebra and inner products are known. We follow the exposition given in [8] which is now accepted as standard in the Majorana Theory and where a different scaling to [16] is used. Table 2 is taken from [8], which notation we explain below.

Each dihedral subalgebra corresponds to a dihedral subgroup D of $\mathbb{M}$ generated by two $2A$ involutions t and s, whose product we denote by $\rho := ts$. We denote by a_0, a_1 and a_i the Majorana axes a_t, a_s and $a_{t\rho^i}$ in Table 2.
In the subalgebra of type $2A$, we have $a_\rho = \psi(\rho)$ which is also a Majorana axis, and in the types $3A$ and $5A$ the vectors u_ρ and w_ρ are introduced to close the algebra product. They correspond to elements of order 3 or 5 in D respectively. From [2], the 1-dimensional subspace linearly spanned by the vector u_ρ or w_ρ is invariant under the normaliser $N_{\mathbb{M}}(\langle\rho\rangle)$ which is isomorphic to $3.F_{24}$ or $(D_{10} \times F_5).2$ respectively. Also, in the type $3A$ the vector itself is stable under $N_{\mathbb{M}}(\langle\rho\rangle)$, so that $u_\rho = u_{\rho^{-1}}$, and in the type $5A$ the vector is stabilised up to negation $w_\rho = -w_{\rho^2} = -w_{\rho^3} = w_{\rho^4}$.

Any element of order 3 or 5 in G can be expressed as a product of two involutions, and any two involutions correspond to a dihedral subalgebra of V. Hence to study the algebra V we can consider the span of the vectors corresponding to the cyclic subgroups of order 2, 3 and 5 of G.
We let $G^{(i)}$ be a set of non-trivial representatives of each cyclic subgroup of order i for $i = 2, 3, 5$, of size 55, 55 and 66 respectively, where for $i = 5$ the representatives are taken from the same conjugacy class of G. We use the notation $A := \{a_t \mid t \in G^{(2)}\}$, $U := \{u_h \mid h \in G^{(3)}\}$, and $W := \{w_f \mid f \in G^{(5)}\}$, and we let $S := A \cup U \cup W$.

3.2.2 A_5 subalgebras of V

The algebra V also contains 22 A_5-subalgebras of type $(2A, 3A, 5A)$ as G contains two conjugacy classes of A_5-subgroups of type $(2A, 3A, 5A)$, of size 11 each. The structure of a subalgebra V_H generated by the Majorana axes indexed by the involutions of an A_5-subgroup H of type $(2A, 3A, 5A)$ follows from [12]. We reformulate it so as to present V_H as a subalgebra of $V_{\mathbb{M}}$ generated by a triple of Majorana axes.

Type	Basis	Products and angles
2A	a_0, a_1, a_ρ	$a_0 \cdot a_1 = \frac{1}{2^3}(a_0 + a_1 - a_\rho),\ a_0 \cdot a_\rho = \frac{1}{2^3}(a_0 + a_\rho - a_1)$ $(a_0, a_1) = (a_0, a_\rho) = (a_1, a_\rho) = \frac{1}{2^3}$
3A	$a_{-1}, a_0, a_1,$ u_ρ	$a_0 \cdot a_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_{-1}) - \frac{3^3 \cdot 5}{2^{11}} u_\rho$ $a_0 \cdot u_\rho = \frac{1}{3^2}(2a_0 - a_1 - a_{-1}) + \frac{5}{2^5} u_\rho$ $u_\rho \cdot u_\rho = u_\rho$ $(a_0, a_1) = \frac{13}{2^8}, (a_0, u_\rho) = \frac{1}{2^2}, (u_\rho, u_\rho) = \frac{2^3}{5}$
5A	$a_{-2}, a_{-1}, a_0,$ a_1, a_2, w_ρ	$a_0 \cdot a_1 = \frac{1}{2^7}(3a_0 + 3a_1 - a_2 - a_{-1} - a_{-2}) + w_\rho$ $a_0 \cdot a_2 = \frac{1}{2^7}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$ $a_0 \cdot w_\rho = \frac{7}{2^{12}}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2^5} w_\rho$ $w_\rho \cdot w_\rho = \frac{5^2 \cdot 7}{2^{19}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$ $(a_0, a_1) = \frac{3}{2^7},\ (a_0, w_\rho) = 0, (w_\rho, w_\rho) = \frac{5^3 \cdot 7}{2^{19}}$
6A	$a_{-2}, a_{-1}, a_0,$ a_1, a_2, a_3 a_{ρ^3}, u_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-2} - a_{-1} - a_2 - a_3 + a_{\rho^3}) + \frac{3^2 \cdot 5}{2^{11}} u_{\rho^2}$ $a_0 \cdot a_2 = \frac{1}{2^5}(2a_0 + 2a_2 + a_{-2}) - \frac{3^3 \cdot 5}{2^{11}} u_{\rho^2}$ $a_0 \cdot u_{\rho^2} = \frac{1}{3^2}(2a_0 - a_2 - a_{-2}) + \frac{5}{2^5} u_{\rho^2}$ $a_0 \cdot a_3 = \frac{1}{2^3}(a_0 + a_3 - a_{\rho^3}), a_{\rho^3} \cdot u_{\rho^2} = 0, (a_{\rho^3}, u_{\rho^2}) = 0$ $(a_0, a_1) = \frac{5}{2^8}, (a_0, a_2) = \frac{13}{2^8}, (a_0, a_3) = \frac{1}{2^3}$

Table 2: Dihedral subalgebras

Proposition 3.6. *Let $V_H = \langle\langle a_x, a_y, a_z\rangle\rangle$ be a subalgebra of $V_{\mathbb{M}}$ where the dihedral subalgebra $\langle\langle a_x, a_y\rangle\rangle$ has type $2A$ and where the dihedral subalgebras $\langle\langle a_x, a_z\rangle\rangle$, $\langle\langle a_y, a_z\rangle\rangle$, and $\langle\langle a_{xy}, a_z\rangle\rangle$ have types $5A$, $5A$ and $3A$ respectively. Then V_H has dimension 26 and it is linearly spanned by the products of all pairs of Majorana axes indexed by the involutions of H.* □

For explicit formulas for the algebra product in V_H or a list of all inner product values, we refer the reader to [12]. In the rest of the paper we will simply refer to an A_5-subgroup H to mean an A_5-subgroup of type $(2A, 3A, 5A)$.

For an A_5-subgroup H, we denote by $H^{(2)}$, $H^{(3)}$ and $H^{(5)}$ the sets of non-trivial conjugate representatives of cyclic subgroups of order 2, 3 and 5 and in the corresponding algebra V_H we denote by A_H, U_H and W_H the sets of vectors $\{a_t \mid t \in H^{(2)}\}$, $\{u_h \mid h \in H^{(3)}\}$, and $\{w_f \mid f \in H^{(5)}\}$. Let w_H be the sum of all vectors in W_H. By [12], the set $S_H := A_H \cup U_H \cup W_H$ is a spanning set of size 31 for V_H, and V_H is 26-dimensional with a basis $A_H \cup U_H \cup \{w_H\}$. The five independent linear relations on S_H, which can be found in [12] or [16], are called the Norton Relations.

Proposition 3.7. ***The Norton Relations***
In the algebra V_H corresponding to an A_5-subgroup H of type $(2A, 3A, 5A)$, all vectors $w_f \in W_H$ satisfy:

$$w_f = \frac{1}{6} w_H + \frac{1}{2^7}\left(\sum_{t\in H_5^{(2)}(f)} a_t - \sum_{t\in H_3^{(2)}(f)} a_t\right) + \frac{3^2.5}{2^{12}}\left(\sum_{\substack{h\in H^{(3)},\\ o([h,f])=3}} u_h - \sum_{\substack{h\in H^{(3)},\\ o([h,f])=5}} u_h\right)$$

where

$$H_5^{(2)}(f) := \{t \in H^{(2)} \mid o(tf) = 5\},$$
$$H_3^{(2)}(f) := \{t \in H^{(2)} \mid o(tf) = 3\}.$$

□

We denote by $\mathcal{H}_1 = \{H_1, \ldots, H_{11}\}$ and $\mathcal{H}_2 = \{H_1', \ldots, H_{11}'\}$ the two classes of A_5-subgroups in G. One class corresponds to the rows of $\mathcal{N}$ and the other to the columns, so the intersection between A_5's taken from different classes can be read directly from the entries of $\mathcal{N}$.

For a given A_5-subgroup H_i in G let W^i be the sum of all vectors in W_{H_i}. For a vector $w_f \in W_{H_i}$ we rewrite the Norton relation for w_f as

$$w_f = \frac{1}{6} W^i + \lambda\, A_i(f) + \mu\, U_i(f) \tag{3.1}$$

where the meaning of λ, μ, $A_i(f)$ and $U_i(f)$ is clear from Proposition 3.7.

Corollary 3.8. *Let w be the sum of all vectors w_f in $W \subseteq V$. Then $S' := A \cup U \cup \{w\}$ is a spanning set of size 111 for S.*

Proof. Consider an A_5-subgroup $H_1 \in \mathcal{H}_1$. From $\mathcal{N}$, any subgroup $H_{i'} \in \mathcal{H}_2$ intersect H_1 in a D_{10} or an A_4. If $H_1 \cap H_{i'} \cong D_{10}$ there exists a representative f of $H_1^{(5)}$ in $H_1 \cap H_{i'}$ for which the Norton relations give

$$w_f = \begin{cases} \frac{1}{6}W^1 + \lambda\, A_1(f) + \mu\, U_1(f) \\ \\ \frac{1}{6}W^{i'} + \lambda\, A_{i'}(f) + \mu\, U_{i'}(f) \end{cases}$$

and so $W^{i'}$ is in $Sp(A \cup U \cup \{W^1\})$, the $\mathbb{R}$-linear span of $A \cup U \cup \{W^1\}$.
If $H_1 \cap H_{i'} \cong A_4$ then the situation can be visualized as the following submatrix of $\mathcal{N}$, where each anti-flag has been replaced by the unique element of $G^{(5)}$ stabilising it, and the rows and columns are indexed with the copy of A_5 stabilising the corresponding line or point of $\mathcal{B}$.

$$\begin{array}{cc} & \begin{array}{cc} H_1 & H_i \end{array} \\ \begin{array}{c} H_i' \\ H_j' \end{array} & \begin{pmatrix} 1 & w_g \\ w_f & w_k \end{pmatrix} \end{array}$$

From the Norton relations for $g \in H_i \cap H_{i'},\ k \in H_i \cap H_{j'}$ and $\ f \in H_1 \cap H_{j'}$, we also get $W^{i'} \in Sp\big(A \cup U \cup \{W^1\}\big)$. From $\mathcal{N}$, for any subgroup $H_i \in \mathcal{H}_1$ there are 3 elements of $\mathcal{H}_2$ intersecting both H_1 and H_i in a D_{10}, with say $H_{l'}$ being one of them:

$$\begin{array}{cc} & \begin{array}{cc} H_1 & H_i \end{array} \\ H_{l'} & \begin{pmatrix} w_f & w_l \end{pmatrix} \end{array}$$

so the Norton relations for f and l give $W^i \in Sp\big(A \cup U \cup \{W^1\}\big)$. Hence there exists $v \in Sp(A \cup U)$ such that

$$22\, W^1 = \textstyle\sum_{H_i \in \mathcal{H}_1} W^i + \sum_{H_{i'} \in \mathcal{H}_2} W^{i'} + v,$$

and from $\mathcal{N}$ every element of $G^{(5)}$ is contained in exactly one element of $\mathcal{H}_1$ and one of $\mathcal{H}_2$ so that

$$\textstyle\sum_{H_i \in \mathcal{H}_1} W^i + \sum_{H_{i'} \in \mathcal{H}_2} W^{i'} = 2\, w,$$

where w is the sum of all vectors w_f in W, and hence

$$W^1 = \tfrac{1}{11} w + v' \quad \text{for some } v' \in Sp(A \cup U).$$

□

4 Inner product values

Definition 4.1. For each pair $(G^{(i)}, G^{(j)})$ with $i,\ j \in \{2, 3, 5\}$ we call the inner product values on $G^{(i)} \times G^{(j)}$ inner products of type **(i, j)**.

If we let $E^{(i)}$ be the equivalence class of elements of order i in G belonging to the same cyclic subgroup, then the orbits of G acting by conjugation on $E^{(i)} \times E^{(j)}$ form a subpartition of the distinct inner products values of type (i, j) (these orbits were calculated using [1]).

We will only explain the inner products values (u_k, v_l) for which the subgroup $\langle k, l\rangle$ is isomorphic to F_{55} or to the whole of G. They arise as the solutions of equations of intersecting subalgebras inside V, or equivalently as particular configurations of subgroups inside G, which can be read from the incidence matrix $\mathcal{N}$ or found using a code written in [1].

4.1 Inner products of type $(2,2)$

From the dihedral subalgebras of V we know all possible inner product values of any two Majorana axes in V, see [11].

Case	$o(ts)$	$\langle\langle a_t, a_s\rangle\rangle$	$\langle t, s\rangle$	(a_t, a_s)
1	1	$1A$	1	1
2	2	$2A$	D_4	$\frac{1}{2^3}$
3	3	$3A$	D_6	$\frac{13}{2^8}$
4	5	$5A$	D_{10}	$\frac{3}{2^7}$
5	6	$6A$	D_{12}	$\frac{5}{2^8}$

Table 3: Inner Products of type $(2,2)$, with $t, s \in G^{(2)}$

4.2 Inner products of type $(2,3)$

The value for case 5) of the inner product of type $(2,3)$ was computed using the following lemma.

Lemma 4.2. *For $t \in G^{(2)}$ and $h \in G^{(3)}$ such that $\langle t, h\rangle \cong L_2(11)$ we have*

$$(a_t, u_h) = \tfrac{1}{2.3.5}.$$

Proof. We fix an element $h \in G^{(3)}$ and we let $t \in G^{(2)}$ such that $\langle t, h\rangle = G$. Since $N_G(\langle h\rangle) \cong D_{12}$ then $\langle h\rangle$ is contained in exactly two distinct dihedral groups of order 6. Let S_h be one of the two sets of 3 involutions, $S_h := \{s, sh, sh^2\}$, such that $\langle S, h\rangle \cong D_6$, and up to permutation of the set S_h we have

$$\langle t,\ s\rangle \cong D_6,$$
$$\langle t,\ sh\rangle \cong D_{12},$$
$$\langle t,\ sh^2\rangle \cong D_{10}.$$

In the $3A$-dihedral subalgebra $\langle\langle a_s, u_h\rangle\rangle$ we have the equality

$$u_h = -\tfrac{2^{11}}{3^3.5}\Big[a_s \cdot a_{sh} - \tfrac{1}{2^5}(2a_s + 2a_{sh} - a_{sh^2})\Big],$$

so taking the inner product with a_t gives

$$(a_t\ ,\ u_h) = -\tfrac{2^{11}}{3^3.5}\Big[(a_t\ ,\ a_s \cdot a_{sh}) - \tfrac{1}{2^5}(a_t\ ,\ 2a_s + 2a_{sh} - a_{sh^2})\Big],$$

Case	$o(ht)$	$\langle t, h\rangle$	(a_t, u_h)
1	2	D_6	$\frac{1}{2^2}$
2	3	A_4	$\frac{1}{3^2}$
3	5	A_5	$\frac{1}{2.3^2}$
4	6	C_6	0
5	11	$L_2(11)$	$\frac{1}{2.3.5}$

Table 4: Inner Products of type $(2, 3)$, with $t \in G^{(2)}$ and $h \in G^{(3)}$

Case	$o(tf) = o(tf^{-1})$	$o([t, f])$	$\langle t, f\rangle$	(a_t, w_f)
1	2	5	D_{10}	0
2	3	5	A_5	$-\frac{7^2}{2^{14}}$
3	5	3	A_5	$\frac{7^2}{2^{14}}$
4	5	5	$L_2(11)$	$-\frac{1}{2^{14}}$
5	6	6	$L_2(11)$	$-\frac{3}{2^{12}}$
6	11	5	$L_2(11)$	$\frac{19}{2^{14}}$

Table 5: Inner Products of type $(2, 5)$, with $t \in G^{(2)}$ and $f \in G^{(5)}$

and by associativity of the algebra product with the inner product

$$(a_t,\ a_s \cdot a_{sh}) = (a_s,\ a_t \cdot a_{sh}).$$

Since $\langle t, sh\rangle \cong D_{12}$, the element $\rho = tsh$ has order 6 so the algebra product $a_t \cdot a_{sh}$ is contained in the dihedral algebra $\langle\langle a_t, a_{sh}\rangle\rangle$ of type $6A$, and so

$$(a_s,\ a_t \cdot a_{sh}) = \frac{1}{2^6}(a_s,\ a_t + a_{sh} - a_{t\rho^2} - a_{t\rho^3} - a_{t\rho^4} - a_{t\rho^5} + a_{\rho^3}) + \frac{3^2.5}{2^{11}}(a_s, u_{\rho^2}),$$

where $\langle s, \rho^2\rangle \cong A_4$, so the value of $(a_s\ ,\ u_{\rho^3})$ is known to be $\frac{1}{9}$ from [11]. Since all the required inner products are now known, one can compute $(a_t, u_h) = \frac{1}{2.3.5}$.

□

4.3 Inner products of type $(2, 5)$

The next lemma justifies the values found in cases $4)$, $5)$ and $6)$. We omit its proof which is similar to the proof of Lemma 4.2.

Lemma 4.3. *Let $t \in G^{(2)}$ and $f \in G^{(5)}$ such that $\langle t,\ f\rangle \cong L_2(11)$. Then exactly one of the following holds.*

(i) *There exists $s \in G^{(2)}$ commuting with t and inverting f, and*

$$(a_t, w_f) = -\tfrac{1}{2^{11}} + \tfrac{1}{2^6}p - \tfrac{1}{2^3}q, \text{ where}$$

$$p = 7(a_t, a_{sf}) + (a_t, a_{sf^2}) \text{ and } q = (a_{ts}, a_{sf}).$$

(ii) *There exists $s \in G^{(2)}$ inverting f and generating with t a dihedral group of order 6, and*

$$(a_t, w_f) = \tfrac{3^2}{2^{14}} + \tfrac{1}{2^7}p - \tfrac{3^3.5}{2^{11}}q, \text{ where}$$

$$p = (a_t,\ 5a_{sf} + a_{sf^2} + a_{sf^3} + a_{sf^4}) \text{ and } q = (u_{st}, a_{sf}).$$

(iii) *There exists $s \in G^{(2)}$ inverting f and generating with t a dihedral group of order 12, and*

$$(a_t, w_f) = -\tfrac{3}{2^{15}} + \tfrac{1}{2^6}p + \tfrac{1}{2^7}q - \tfrac{3^2.5}{2^{11}}r, \text{ where}$$

$$p = (a_{sf},\ a_{\rho^3} - 2a_t - a_{t\rho^2} - a_{t\rho^3} - a_{t\rho^4} - a_{t\rho^5}),\ q = (a_t,\ a_{sf^2} + a_{sf^3} + a_{sf^4}) \text{ and } r = (u_{\rho^2}, a_{sf}),$$

$$\text{for } \rho := ts \text{ of order } 6 \text{ in } \langle t,\ s\rangle \cong D_{12}.$$

□

4.4 Inner products of type $(3, 3)$

In the next lemma part (i) addresses cases $4)$ and $6)$ and part (ii) addresses case $5)$. The lemma assumes all products of type $(2, 3)$ are known.

Lemma 4.4. *Let $h,\ k \in G^{(3)}$ with $\langle h, k\rangle \cong L_2(11)$. Then exactly one of the following holds.*

(i) *There exists an involution t inverting both h and k, and*

$$(u_h, u_k) = \tfrac{2^4}{3^5.5}(5 - 2^3.3^2p + 2^6q), \text{ where}$$

$$p = (u_h, a_{tk^2}) = (u_k, a_{th^2}) \text{ and}$$
$$q = (a_{th}, a_{tk^2}) + (a_{th}, a_{tk}) + (a_{th^2}, a_{tk}) + (a_{th^2}, a_{tk^2}).$$

(ii) *There exists an involution t inverting h and generating with k an alternating group A_4, and*

$$(u_h, u_k) = \tfrac{2^5}{3^3.5}(\tfrac{1}{2^2.3} - p - 2q - r), \text{ where}$$

$$p = (a_{th},\ 3u_{tk} - 4u_{tk} - 4u_{tk^2}),\ q = (a_{th^2}, u_k) \text{ and } r = (a_{th}, u_k).$$

□

Case	$\{o(hk), o(hk^{-1})\}$	$\langle h, k\rangle$	(u_h, u_k)
1	$\{1, 3\}$	C_3	$\frac{2^3}{5}$
2	$\{2, 3\}$	A_4	$\frac{2^3.17}{3^4.5}$
3	$\{5, 5\}$	A_5	$\frac{2^4}{3^4.5}$
4	$\{5, 6\}$	$L_2(11)$	$\frac{2^3}{3.5^2}$
5	$\{5, 11\}$	$L_2(11)$	$\frac{2^3.7}{3^3.5^2}$
6	$\{6, 6\}$	$L_2(11)$	$\frac{2^5}{3^4.5}$

Table 6: Inner Products of type $(3, 3)$, with $h, k \in G^{(3)}$

4.5 Inner products of type $(3, 5)$

Part (i) of the next lemma addresses case 3), and part (ii) addresses cases 5) and 6).

Lemma 4.5. *Let $h \in G^{(3)}$ and $f \in G^{(5)}$ such that $\langle h,\ f\rangle \cong L_2(11)$. Then exactly one of the following holds.*

(i) *There exists an involution t inverting f and h, and*

$$(u_h, w_f) = -\tfrac{2^{11}}{3^3.5}(p - \tfrac{1}{2^5}q), \text{ where}$$

$$p = \tfrac{7}{2^{12}}(a_{tf} + a_{tf^4} - a_{tf^2} - a_{tf^3},\ a_{th}) + \tfrac{7}{2^5}(w_f, a_{th}) \text{ and}$$
$$q = (2a_t + 2a_{th} + a_{th^2},\ w_f).$$

(ii) *There exists an involution t inverting f and generating with h a subgroup isomorphic to A_5, and*

$$(u_h, w_f) = \tfrac{7}{2^7}p + \tfrac{1}{2^7}q - \tfrac{1}{2^4}r, \text{ where}$$

$$p = (a_{sf} + a_{sf^4},\ u_h),\ \ q = (a_{sf^2} + a_{sf^3},\ u_h) \text{ and } r = (a_{sf},\ u_{sh} + u_{sh^2}).$$

□

The inner product value for case 4) of the inner product of type $(3, 5)$ can be found using the Norton relations inside some A_5-subalgebras of V, see equation (1). The proof of the following lemma uses similar arguments to the proof of Corollary 3.8. We use the notation of (1).

Lemma 4.6. *Let $f \in G^{(5)}$ and $h \in G^{(3)}$. If there exists an element $g \in G^{(5)}$ such that $A_1 := \langle f, g\rangle$ and $A_2 := \langle h, g\rangle$ are two non-conjugate A_5-subgroups of G, then*

$$(u_h, w_f) = \tfrac{1}{6}\big(u_h\ , W^2\big) + \tfrac{1}{2^7}\big(u_h\ , l_a\big) + \tfrac{3^2.5}{2^{12}}\big(u_h, l_u\big), \text{ where}$$
$$l_a = A_1(g) + A_1(f) - A_2(g) \text{ and } l_u = U_1(g) + U_1(f) - U_2(g).$$

□

Case	$o(hf)$	$o(hf^{-1})$	$\langle h, f\rangle$	(u_h, w_f)
1	2	5	A_5	$\frac{-5.7}{2^9.3^2}$
2	3	5	A_5	$\frac{5.7}{2^9.3^2}$
3	3	6	$L_2(11)$	$\frac{-67}{2^9.3^2.5}$
4	5	5	$L_2(11)$	$\frac{-1}{2^8.3^2.5}$
5	6	11	$L_2(11)$	$\frac{7}{2^6.3^2.5}$
6	11	11	$L_2(11)$	$\frac{-7}{2^7.3^2.5}$

Table 7: Inner Products of type $(3, 5)$, with $h \in G^{(3)}$ and $f \in G^{(5)}$

4.6 Inner products of type $(5, 5)$

In the next lemma, part (i) justifies the values of the inner product of type $(5, 5)$ for the cases 2), 3) and 4), and part (ii) justifies case 6). The proof is similar to that of Corollary 3.8 and the notation is the same as the one used in the previous lemma.

Lemma 4.7. *Let $f,\ g \in G^{(5)}$ not contained in a common A_5-subgroup, with f and g belonging to the pairs $\{H_i,\ H_{i'}\}$ and $\{H_j,\ H_{j'}\}$ respectively, of distinct non-conjugate A_5-subgroups of G. Then exactly one of the following holds.*

(i) *$H_i \cap H_{j'} \cong D_{10}$, or $H_j \cap H_{i'} \cong D_{10}$, with k an element of order 5 in $H_i \cap H_{j'}$, say, then*

$$(w_f, w_g) = \frac{1}{6^2}\big(W^j, W^j\big) + \frac{1}{2^8.3}\big(l_a\,, W^j\big) + \frac{3.5}{2^{13}}\big(l_u,\ W^j\big) + \frac{1}{2^{14}}\big(l_a\,, A_j(g)\big) + \frac{3^2.5}{2^{19}}\big(l_a\,,\ U_j(g)\big) + \frac{3^2.5}{2^{19}}\big(l_u\,, A_j(g)\big) + \frac{3^4.5^2}{2^{24}}\big(l_u\,, U_j(g)\big),$$

$$\textit{where}\ \ l_a = A_j(k) - A_{i'}(k) + A_{i'}(f)\ \ \textit{and}\ \ l_u = U_j(k) - U_{i'}(k) + U_{i'}(f).$$

(iii) *$H_i \cap H_{j'} \cong H_j \cap H_{i'} \cong A_4$, and there exist two elements $k \neq l \in G^{(5)}$ such that k belongs to H_i and $H_{m'}$ and l belongs to H_j and $H_{m'}$, so that*

$$(w_f, w_g) = \frac{1}{6^2}\big(W^j, W^j\big) + \frac{1}{2^8.3}\big(l_a\,, W^j\big) + \frac{3.5}{2^{13}}\big(l_u\,, W^j\big) + \frac{1}{2^{14}}\big(l_a\,, A_j(g)\big) + \frac{3^2.5}{2^{19}}\big(l_a\,, U_j(g)\big) + \frac{3^2.5}{2^{19}}\big(l_u\,, A_j(g)\big) + \frac{3^4.5^2}{2^{24}}\big(l_u\,, U_j(g)\big),$$

$$\textit{where}\ \ l_a = A_j(l) - A_{m'}(l) - A_i(k) + A_{m'}(k) + A_i(f),$$
$$\textit{and}\ \ l_u = U_j(l) - U_{m'}(l) - U_i(k) + U_{m'}(k) + U_i(f).$$

□

Case	$\{o(fg), o(fg^{-1})\}$	$o([f,g])$	$\langle f,g\rangle$	(u_h, w_f)
1	$\{1,5\}$	1	C_5	$\frac{5^3.7}{2^{19}}$
2	$\{3,5\}$	5	A_5	$\frac{7.29}{2^{19}}$
3	$\{5,11\}$	11	F_{55}	$\frac{-11}{2^{19}}$
4	$\{3,6\}$	5	$L_2(11)$	$\frac{3.151}{2^{21}}$
5	$\{2,6\}$	5	$L_2(11)$	$\frac{157}{2^{20}}$
6	$\{5,11\}$	2	$L_2(11)$	$\frac{59}{2^{20}}$
7	$\{5,5\}$	3	$L_2(11)$	$\frac{-3.41}{2^{20}}$

Table 8: Inner Products of type $(5,5)$, with $f, g \in G^{(5)}$

Corollary 4.8. *The inner product values between the vector w, and the vectors $a_t \in A$, $u_h \in U$, $w_f \in W$ and w itself are as follows*

(i) $(a_t, w) = \frac{3^2}{2^{11}}$;

(ii) $(u_h, w) = -\frac{3^2}{2^7.5}$;

(iii) $(w_f, w) = \frac{3^3.5.17}{2^{18}}$;

(iv) $(w, w) = \frac{3^4.5.11.17}{2^{17}}$.

□

4.7 Dependence relations in the algebra

We let $V_{S'}$ be the $\mathbb{R}$-vector space having the subset $S' = A \cup U \cup \{w\}$ of V as a basis. We turn $V_{S'}$ into a G-module by the natural action of G on S', and we let π be the natural projection

$$\pi : V_{S'} \to V.$$

Using [1] we find the rank of the Gram matrix of the set S' and give a description of the kernel of π.

We recall the bijection d introduced at the beginning of section 2 between subgroups of order 2 and 3 in G:

$$d : G^{(2)} \to G^{(3)}$$
$$\langle t\rangle \to \langle h\rangle$$

since $\forall t \in G^{(2)}\ \exists!\ h \in G^{(3)}$ where $[t,\ h] = 1$. For a fixed involution $t \in G^{(2)}$ its normaliser $N_G(t) \cong D_{12}$ has the following orbits on $G^{(2)}$ (the action is conjugation):

$$O_1,\ O_3^1,\ O_3^2,\ O_6^1,\ O_6^2,\ O_6^3,\ O_6^4,\ O_{12}^1 \text{ and } O_{12}^2,$$

where the subscript indicates the size of the orbit. If we write $N_G(t) = \langle\rho\rangle \rtimes \langle s\rangle$ then $\rho^3 = t$, so $O_1 = \{\rho^3\}$, $O_3^1 = \{s,\ s\rho^2,\ s\rho^4\}$ and $O_3^2 = \{s\rho,\ s\rho^3,\ s\rho^5\}$ wlog. Further we can describe the orbits as follows:

$$\begin{aligned} O_3^1 \cup O_3^2 =& \{s \in G^{(2)} | \langle s,\ t\rangle \cong 2^2\} \\ O_6^1 \cup O_6^2 =& \{s \in G^{(2)} | \langle s,\ t\rangle \cong D_{12}\} \\ O_6^3 \cup O_6^4 =& \{s \in G^{(2)} | \langle s,\ t\rangle \cong D_6\} \\ O_{12}^1 \cup O_{12}^2 =& \{s \in G^{(2)} | \langle s,\ t\rangle \cong D_{10}\}. \end{aligned}$$

For $(t_1,\ t_2)$ in $O_3^1 \times O_6^2$ or $O_3^2 \times O_6^1$ the subgroup $\langle t_1,\ t_2\rangle$ in G is isomorphic to either 2^2 or D_{10}. For $(t_1,\ t_2)$ in $O_3^1 \times O_6^1$ or $O_3^2 \times O_6^2$ the subgroup $\langle t_1,\ t_2\rangle$ is isomorphic to either D_6 or D_{12}.

Proposition 4.9. (i) *The rank of the Gram matrix for the set S' is* 101.

(ii) *The kernel of π is* 10*-dimensional and consists of* 10 *linearly independent relations, between the vectors of $A \cup U$, taken from a set of* 55 *G-invariant relations $R(t)$ indexed by the involutions of G.*

For a fixed involution t in $G^{(2)}$, $R(t)$ defines the following $N_G(t)$-invariant relation:

$$R(t) := \sum_{r\in T_1} a_r - \sum_{s\in T_2} a_s + \frac{3^2.5}{2^5}\Big(\sum_{h\in d(T_1)} u_h - \sum_{k\in d(T_2)} u_k\Big) = 0,$$

where T_1 and T_2 can taken to be $O_3^1 \cup O_6^1$ and $O_3^2 \cup O_6^2$ respectively (or vice versa). □

From the rank of the Gram matrix of the set S' we obtain the following proposition.

Proposition 4.10. *For the algebra V we have $dim(V) \geq 101$.* □

The above, together with Proposition 3.5, proves that $dim(V) = 101$. Hence the set S' spans V, so that $\{a_t \cdot a_s \mid t,\ s \in T\}$ also spans V. From Lemma 3.4, the identity $\iota_{\mathbb{M}}$ of $V_{\mathbb{M}}$ is not in V. The space $C_{V_{\mathbb{M}}}(K)$ is 102-dimensional, containing $\iota_{\mathbb{M}}$ and having V as a subspace. Hence $C_{V_{\mathbb{M}}}(K)$ decomposes as $V \oplus \iota_{\mathbb{M}}$, and we have proved our main theorem.

5 A Majorana representation of $L_2(11)$

The dihedral and A_5-subalgebras of V can be characterised under the axioms of Majorana theory; they are equal to the algebra of the Majorana representations of the dihedral groups D_4 of type $2A$, D_6 of type $3A$, D_{10} of type $5A$, and D_{12} of type $6A$, and of the alternating group A_5 of type $(2A, 3A, 5A)$.
Majorana theory was introduced by A. A. Ivanov in [8] to axiomatise some of the properties of $V_{\mathbb{M}}$ and its Majorana axes. We refer the reader to [8] and [11] for a full description.

Definition 5.1. A **Majorana representation** of a finite group G is a tuple

$$\mathcal{R} = \big(G,\ T,\ X,\ (\ ,\ ),\ \cdot,\ \varphi,\ \psi\big),$$

where T is a union of conjugacy classes of involutions generating G, and X is a commutative non-associative $\mathbb{R}$-algebra endowed with an inner product $(\ ,\ )$ associating with its algebra product $\cdot$ in the sense that $(u \cdot v,\ w) = (u,\ v \cdot w)$ for all $u,\ v,\ w \in X$ and satisfying the Norton Inequality

$$(u \cdot u,\ v \cdot v) \geq (u \cdot v,\ u \cdot v), \text{ for all } u, v \in X.$$

The image of the homomorphism $\varphi : G \hookrightarrow GL(X)$ is an automorphism of $\big(X,\ \cdot\ ,\ (\ ,\ )\big)$, and the map ψ is an injection sending each involution t of T to a Majorana axis a_t of X, as defined in the second paragraph of section 1 (the properties of the spectrum of ad_a and the Fusion Rules are assumed to hold), such that φ and ψ commute in the sense that :

$$a_{g^{-1}tg} = (a_t)^{\varphi(g)} \text{ for every } g \in G.$$

We require that the algebra X be generated by the set of Majorana axes $\psi(T)$ and that it must satisfy conditions $(2A)$ and $(3A)$ below.

Conditions $(2A)$ and $(3A)$ ensure that when constructing X in the above definition we get the right number of $3A$ vectors u_h from the Majorana axes.

(2A) Let $t_0, t_1 \in T$ and $\rho := t_0 t_1$ such that

- (a) if $\rho \in T$ and the vectors a_{t_0}, a_{t_1} generate a dihedral subalgebra of type $2A$ then $a_\rho = \psi(\rho)$,
- (b) if $\rho^i \in T$ for ρ of order 4 or 6 and the vectors a_{t_0} and a_{t_1} generate a subalgebra of type $4B$ or $6A$, then $\psi(\rho^i)$ coincides with the axis a_{ρ^i} ;

(3A) Let $t_0, t_1, t_2, t_3 \in T$ with $\langle t_0, t_1 \rangle \cong \langle t_2, t_3 \rangle \cong D_6$. We let $\rho_1 := t_0 t_1$ and $\rho_2 := t_2 t_3$ both of order 3.

If the following two conditions are satisfied:
(i) $\rho_1 = \rho_2$ or ρ_2^{-1}, and
(ii) the dihedral subalgebras generated by $\{a_{t_0}, a_{t_1}\}$ and $\{a_{t_2}, a_{t_3}\}$ have type $3A$,

then the corresponding $3A$-axial vectors u_{ρ_1} and u_{ρ_2} in the above subalgebras are equal in X.

We call $dim(X)$ the dimension of $\mathcal{R}$, and we say that $\mathcal{R}$ is based on an embedding of G into $\mathbb{M}$ if there exists a monomorphism $\iota : G \to \mathbb{M}$ with $\iota(T) \subset 2A$ and such that $\mathcal{R}$ is isomorphic to the subalgebra of $V_{\mathbb{M}}$ generated by the Majorana axes corresponding to $\iota(T)$.

Definition 5.2. The **shape** of a Majorana representation $\mathcal{R}$ of G specifies the types of dihedral subalgebras associated with all pairs of involutions on T.

Theorem 5.3. *A Majorana representation of* $G \cong L_2(11)$ *must have shape* $(2A, 3A, 5A, 6A)$.

Proof. Let $\mathcal{R}$ be a Majorana representation of G with associated algebra X. The group $L_2(11)$ has a single conjugacy class of involutions, $2a$, and a single class $3a$ of elements of order 3. From the structure class constants the product of any $2a$ involutions is in either of the $L_2(11)$ classes $1a,\ 2a,\ 3a,\ 5a,\ 5b$ or $6a$. Hence X contains dihedral subalgebras of type $5A$ and $6A$ since they are the only dihedral subalgebras associated with dihedral groups of order 10 and 12. By the inclusion of the dihedral subalgebras $3A \hookrightarrow 6A$ and $2A \hookrightarrow 6A$ the classes $3a$ and $2a$ are mapped to $3A$ and $2A$ under ψ. Hence X also contains dihedral subalgebras of type $3A$ and $2A$ and we have accounted for all possible dihedral subalgebras in X. □

Let $\mathcal{R}$ be a Majorana representation of G with associated algebra X. From the above theorem, $\mathcal{R}$ has the same shape as the subalgebra V of $V_{\mathbb{M}}$ and the same inner product values for the sets S' and S. Moreover the dihedral and A_5-subalgebras of X are equal to their Majorana representations from [11] and [12].

Proposition 5.4. (i) *The dihedral subalgebras of type* $2A$, $3A$, $5A$ *and* $6A$ *are equal to the unique Majorana representations of* D_4, D_6, D_{10}, *and* D_{12} *of shape* $2A$, $3A$, $5A$ *and* $6A$ *respectively.*

(ii) *The* A_5*-subalgebra of type* $(2A, 3A, 5A)$ *is equal to the unique Majorana representation of shape* $(2A, 3A, 5A)$ *of a group* A_5 *of type* $(2A, 3A, 5A)$, *which has dimension* 26. □

We would like to show that the shape of $\mathcal{R}$ uniquely determines the algebra product in X so that $X = V$. In particular it is necessary to find the closure of the algebra generated by S. This can be inspected computationally. We let $S^2 := \{u \cdot v \mid u,\ v \in S\}$ and $S^3 := \{(u \cdot v) \cdot w,\ \ u \cdot (v \cdot w) \mid u,\ v,\ \ \in S\}$ and for any positive integer n the set S^n is defined in a similar way. Already for the set of vectors $S \cup S^2$ the Majorana axioms yield a very large number of eigenvectors for each Majorana axis and the first computational step is to check whether or not the linear span of $S \cup S^2$ over $\mathbb{R}$ is contained in the closure of X. In fact during the reviewing stage of this paper, the author has learned that Á. Seress has proved in [18] that the system of linear equations in $S \cup S^2$, obtained from the eigenvectors of the axes $\{a_t \mid t \in T\}$, has a unique solution and that $dim_{\mathbb{R}}(X) = 101$. The result was obtained computationally with an algorithm written with [6].

Theorem 5.5. *The* $L_2(11)$*-subalgebra of the Monster algebra* $V_{\mathbb{M}}$ *is equal to* X, *the algebra corresponding to the unique Majorana representation of* $L_2(11)$. □

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