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Colour-permuting automorphisms of complete Cayley graphs*

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To Dragan Marušič on his 70th birthday.

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Abstract

Let G be a (finite or infinite) group, and let $K_G = \operatorname{Cay}\bigl(G; G \smallsetminus \{\mathbf{1}\}\bigr)$ be the complete graph with vertex set G, considered as a Cayley graph of G. Being a Cayley graph, it has a natural edge-colouring by sets of the form $\{s, s^{-1}\}$ for $s \in G \smallsetminus \{\mathbf{1}\}$. We prove that every colour-permuting automorphism of K_G is an affine map, unless $G \cong Q_8 \times B$, where Q_8 is the quaternion group of order 8, and B is an abelian group, such that b^2 is trivial for all $b \in B$.

We also prove (without any restriction on G) that every colour-permuting automorphism of K_G is the composition of a group automorphism and a colour-preserving graph automorphism. This was conjectured by D. P. Byrne, M. J. Donner, and T. Q. Sibley in 2013.

Keywords: Cayley graph, automorphism, colour-permuting, complete graph.

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0 Preliminaries

The statements of our main results use the following standard notation and terminology.

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Definition 0.1 ([8, pages 189–191]). Let S be an inverse-closed subset of a group G. (This means that $s \in S \Rightarrow s^{-1} \in S$.)

(1) The *Cayley graph* of G, with respect to S, is the graph $\mathrm{Cay}(G; S)$ whose vertices are the elements of G, and with an edge between g and gs, for each $g \in G$ and $s \in S$.

(2) $\mathrm{Cay}(G; S)$ has a natural edge-colouring. Namely, each edge of the form $g — gs$ is coloured with the set $\{s, s^{-1}\}$. (In order to make the colouring well-defined, it is necessary to include s^{-1}, because $g — gs$ is the same as the edge $gs — g$, which is of the form $h — hs^{-1}$, with $h = gs$.)

(3) A function $\varphi \colon G \to G$ is said to be *affine* if it is the composition of an automorphism of G with left translation by an element of G. This means $\varphi(x) = \alpha(gx)$, for some $\alpha \in \mathrm{Aut}\, G$ and $g \in G$.

(4) A Cayley graph $\mathrm{Cay}(G; S)$ is *CCA* if all of its colour-preserving automorphisms are affine functions on G. (CCA is an abbreviation for the *Cayley Colour Automorphism* property.)

(5) We say that G is *CCA* if every connected Cayley graph of G is CCA. (We recall that $\mathrm{Cay}(G; S)$ is connected if and only if the set S generates G.)

(6) An automorphism φ of a Cayley graph $\mathrm{Cay}(G; S)$ is *colour-permuting* if it respects the colour classes; this means that if two edges have the same colour, then their images under φ must also have the same colour. Hence, there is a permutation π of S, such that $\varphi(gs) \in \{\varphi(g)\, \pi(s)^{\pm 1}\}$ for all $g \in G$ and $s \in S$ (and $\pi(s^{-1}) = \pi(s)^{-1}$).

(7) $\mathrm{Cay}(G; S)$ is *strongly CCA* if all of its colour-permuting automorphisms are affine functions.

(8) We say that G is *strongly CCA* if every connected Cayley graph of G is strongly CCA.

(9) $\mathrm{Cay}(G; S)$ is *normal* if every automorphism of $\mathrm{Cay}(G; S)$ is an affine function (cf. [10, Definition 1] and [7, Lemma 2.1]). The motivation for this terminology is that $\mathrm{Cay}(G; S)$ is normal if and only if the left-translations form a *normal* subgroup of the automorphism group of $\mathrm{Cay}(G; S)$. (*Warning:* This definition is not completely standard, because some graph theorists use "normal" to mean that the connection set S is a union of conjugacy classes [1, page 2].)

1 Introduction

Many groups are CCA, but not strongly CCA. (The smallest example is the dihedral group of order 12 [8, Proposition 7.3].) This means there are groups G, such that every connected Cayley graph of G is CCA, but not every connected Cayley graph of G is strongly CCA. The following result shows that this is not true if we replace the universal quantifier with the existential quantifier; that is, it shows that if there exists a Cayley graph of G that is CCA, then there exists a Cayley graph of G that is strongly CCA. (Specifically, the complete graph on G is strongly CCA.)

Theorem 1.1. *If G is a (finite or infinite) group, then the following are equivalent:*

(1) *G has a Cayley graph that is strongly CCA.*

(2) *The complete graph* $\mathrm{Cay}\big(G; G \setminus \{\mathbf{1}\}\big)$ *is strongly CCA.*

(3) *G has a Cayley graph that is CCA.*

(4) *The complete graph* $\mathrm{Cay}\big(G; G \setminus \{\mathbf{1}\}\big)$ *is CCA.*

(5) *G is not of the form $Q_8 \times B$, where Q_8 is the quaternion group of order 8, and B is an abelian group, such that b^2 is trivial for all $b \in B$.*

Furthermore, if G is finite, then these are also equivalent to:

(6) *Either G has a Cayley graph that is normal, or $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.*

Remark 1.2. Condition (6) implies all of the other conditions listed in the statement of Theorem 1.1. (The implication (6) $\Rightarrow$ (1) is easy, as is explained in Lemma 2.4 below, and the theorem tells us that all of the conditions except (6) are equivalent.) However, we do not know whether the converse is true: does every infinite group with a Cayley graph that is CCA also have a Cayley graph that is normal?

A group is said to be *finitely generated* if it has a finite generating set. It is immediate from work of P.–H. Leemann and M. de la Salle [9, Thm. 1] that every finitely generated, infinite group has a Cayley graph that is normal (see Corollary 2.16). This establishes that (6) holds for these groups, so it follows from the first sentence of Remark 1.2 that all of the other conditions in the theorem must also hold:

Corollary 1.3. *If G is an infinite group that has a finite generating set, then Conditions (1)–(6) in the statement of* Theorem 1.1 *are all true.*

For finitely generated groups, one is usually most interested in the Cayley graphs that come from finite generating sets, and we have the following standard definition:

Definition 1.4. A Cayley graph $\mathrm{Cay}(G; S)$ is *locally finite* if S is finite.

The theorem of Leemann and de la Salle that was mentioned above provides a Cayley graph that is locally finite, in addition to being normal (see Corollary 2.16 again). Combining this with Theorem 1.1 (and Remark 1.2) yields the following result. (Note that the complete graph on an infinite group is never locally finite.)

Corollary 1.5. *If G is a* (*finite or infinite*) *group that has a finite generating set, then the following are equivalent:*

(1)′. *G has a locally finite Cayley graph that is strongly CCA.*

(2). *The complete graph* $\mathrm{Cay}\big(G; G \setminus \{\mathbf{1}\}\big)$ *is strongly CCA.*

(3)′. *G has a locally finite Cayley graph that is CCA.*

(4). *The complete graph* $\mathrm{Cay}\big(G; G \setminus \{\mathbf{1}\}\big)$ *is CCA.*

(5). *G is not of the form $Q_8 \times B$, where Q_8 is the quaternion group of order 8, and B is an abelian group, such that b^2 is trivial for all $b \in B$.*

(6)′. *Either G has a locally finite Cayley graph that is normal, or $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.*

Note 1.6.

(1) The following implications of Theorem 1.1 are fairly obvious (see Lemma 2.4):

$$\begin{array}{c} \phantom{(2) \Rightarrow{}} (6) \phantom{{}\Rightarrow (3) \Leftrightarrow (4) \Rightarrow (5)} \\ \phantom{(2) \Rightarrow{}} \Downarrow \phantom{{}\Rightarrow (3) \Leftrightarrow (4) \Rightarrow (5)} \\ (2) \Rightarrow (1) \Rightarrow (3) \Leftrightarrow (4) \Rightarrow (5) \end{array}$$

(2) The implication (5) $\Rightarrow$ (4) is also known (but it is not obvious). It was proved by D. P. Byrne, M. J. Donner, and T. Q. Sibley [2, cf. page 324] for finite groups, and the general case was established by E. Dobson, A. Hujdurović, K. Kutnar, and J. Morris [3, Theorem 2.3].

(3) For groups that are *finite,* the implication (5) $\Rightarrow$ (6) was established by C. Wang, D. Wang, and M. Xu [10]. The proof relies on the classification of finite groups that do not have a Graphical Regular Representation (GRR), which is a difficult theorem that was proved in a series of papers by several authors (see [6]).

Therefore, the implication (1) $\Rightarrow$ (2) is the only new content of the theorem for finite groups, but (3) $\Rightarrow$ (1) is new for infinite groups (that are not finitely generated). Actually, we prove (5) $\Rightarrow$ (2) (see Corollary 4.8), which contains both of these.

We also prove the following result (see Section 5), which was conjectured by Byrne, Donner, and Sibley in [2, cf. page 332].

Theorem 1.7. *If G is any group, then every colour-permuting automorphism of the complete graph* $\operatorname{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)$ *is the composition of a group automorphism and a colour-preserving graph automorphism.*

As an offshoot of our proof of Theorem 1.7, we also establish the following result (see Corollary 4.7), which shows that placing a natural restriction on the colour-preserving automorphisms of a Cayley graph can also restrict the colour-permuting automorphisms:

Proposition 1.8. *If every colour-preserving automorphism of* $\operatorname{Cay}(G; S)$ *is a translation, then every colour-permuting automorphism of* $\operatorname{Cay}(G; S)$ *is an affine map* (*i.e.,* $\operatorname{Cay}(G; S)$ *is strongly CCA*).

2 Preliminaries

Graphs in this paper are simple and undirected, but they may be infinite.

Notation 2.1.

(1) G always denotes a group (which may be infinite).

(2) $\mathbf{1}$ is the identity element of G.

(3) S (and S') always denotes an inverse-closed subset of G.

Remark 2.2. In the statements of our results, it does not matter whether graphs are allowed to have loops. This is because all of our results concern automorphisms of Cayley graphs, and adding a loop to every vertex of a graph does not affect its automorphism group. (For a loopless Cayley graph $\operatorname{Cay}(G; S)$, adding a loop at all vertices is achieved by replacing S with $S \cup \{\mathbf{1}\}$.)

2A Colour-permuting automorphisms

The following result collects several elementary observations about colour-permuting automorphisms. All are trivial and/or well known.

Lemma 2.3.

(1) *A permutation φ of G is a colour-preserving automorphism of* $\mathrm{Cay}(G;S)$ *if and only if, for all $g \in G$ and $s \in S$, we have $\varphi(gs) \in \{\varphi(g)\, s^{\pm 1}\}$.* [8, page 190]

(2) *For each $g \in G$, the left translation $L_g \colon x \mapsto gx$ is a colour-preserving automorphism of* $\mathrm{Cay}(G;S)$. [8, page 190]

(3) *If α is an automorphism of the group G, such that $\alpha(s) \in \{s^{\pm 1}\}$ for all $s \in S$, then α is a colour-preserving automorphism of* $\mathrm{Cay}(G;S)$. [8, page 190]

(4) *Every strongly CCA Cayley graph is CCA. So every strongly CCA group is CCA.* [8, page 191]

(5) *Every normal Cayley graph is strongly CCA.* (cf. [8, Remark 6.2])

(6) *Assume $S \subseteq S' \subseteq G$. If* $\mathrm{Cay}(G;S)$ *is CCA, then* $\mathrm{Cay}(G;S')$ *is CCA.* (cf. [5, Lemma. 3.1])

We now establish the easy parts of Theorem 1.1:

Lemma 2.4. *The implications stated in* Note 1.6(1) *are all true.*

Proof. The implications (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3) are completely trivial. See Lemma 2.3(4) for (1) $\Rightarrow$ (3), and see [3, 1st paragraph of proof of Theorem 2.3] (or Example 3.1(c)) for (4) $\Rightarrow$ (5).

It is easy to see that $\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_2; S)$ is strongly CCA for $S = \{(\pm 1, 0), (0, 1)\}$. The other part of (6) $\Rightarrow$ (1) is Lemma 2.3((5)).

To establish the remaining implication (3) $\Rightarrow$ (4), note that if $\mathrm{Cay}(G;S)$ is CCA, then we see from Lemma 2.3(6) that $\mathrm{Cay}(G;G)$ is CCA, because $S \subseteq G$. Then the graph obtained by removing the loop at every vertex is also CCA (see Remark 2.2), so $\mathrm{Cay}\big(G; G \setminus \{\mathbf{1}\}\big)$ is CCA. □

For completeness, we now establish a few additional well-known, elementary facts.

Lemma 2.5. *Assume φ is a colour-permuting automorphism of* $\mathrm{Cay}(G;S)$*, $g \in G$, and $s, t \in S$. If $\varphi(gs) = \varphi(g)\, t$, then $\varphi(gs^n) = \varphi(g)\, t^n$ for all $n \in \mathbb{Z}$.*

Proof. Since φ is colour-permuting, we see that it is an isomorphism from $\mathrm{Cay}\big(G; \{s^{\pm 1}\}\big)$ to $\mathrm{Cay}\big(G; \{t^{\pm 1}\}\big)$. So it must restrict to an isomorphism from the connected component of $\mathrm{Cay}\big(G; \{s^{\pm 1}\}\big)$ that contains g to the connected component of $\mathrm{Cay}\big(G; \{t^{\pm 1}\}\big)$ that contains $\varphi(g)$. Since these components are cycles or two-way-infinite paths (except in the trivial cases where they have only one or two vertices), the restriction of φ to this component is determined by what it does to a single edge. □

Lemma 2.6. *Assume φ is a colour-permuting automorphism of* $\mathrm{Cay}(G;S)$*, such that $\varphi(\mathbf{1}) = \mathbf{1}$. Then, for all $g \in G$, $s \in S$, and $n \in \mathbb{Z}$, we have*

$$\varphi(gs^n) \in \{\varphi(g)\, \varphi(s)^{\pm n}\}.$$

Proof. Fix $s \in S$. Since φ is colour-permuting, there is some $t \in S$, such that

$$\varphi(gs) \in \{\varphi(g)\, t^{\pm 1}\} \quad \text{for all } g \in G.$$

By taking $g = \mathbf{1}$, we see that $\varphi(s) \in \{t^{\pm 1}\}$. Therefore $\varphi(gs) \in \{\varphi(g)\, \varphi(s)^{\pm 1}\}$ for all g. The desired conclusion now follows from Lemma 2.5. □

Lemma 2.7. *Assume φ is a colour-permuting automorphism of* $\operatorname{Cay}(G; S)$*, such that $\varphi(\mathbf{1}) = \mathbf{1}$, and let $H = \langle\, s \in S \mid s^2 = \mathbf{1}\,\rangle$. Then $\varphi(gh) = \varphi(g)\,\varphi(h)$ for all $g \in G$ and $h \in H$.*

Proof. Let S_2 be the set of all elements of order 2 in S. Let $h \in H$, so $h = s_1 \cdots s_n$ for some $s_1, s_2, \ldots, s_n \in S_2$.

Note that the elements of S_2 are precisely the elements of S that correspond to colour classes whose edges form a matching (i.e., all edges of this colour are vertex-disjoint from each other). Since φ is a colour-permuting automorphism (and $\varphi(\mathbf{1}) = \mathbf{1}$), this implies $\varphi(S_2) = S_2$, so $\varphi(s)^{-1} = \varphi(s)$ for all $s \in S_2$. Hence, we see from Lemma 2.6 (with $n = 1$) that $\varphi(gs_i) = \varphi(g)\,\varphi(s_i)$ for all i.

So

$$\begin{aligned}
\varphi(gh) &= \varphi(gs_1 s_2 \cdots s_n) \\
&= \varphi(gs_1 s_2 \cdots s_{n-1})\,\varphi(s_n) \\
&= \varphi(gs_1 s_2 \cdots s_{n-2})\,\varphi(s_{n-1})\,\varphi(s_n) \\
&\qquad \vdots \\
&= \varphi(g)\,\varphi(s_1)\,\varphi(s_2) \cdots \varphi(s_n).
\end{aligned}$$

By taking $g = \mathbf{1}$, we see that

$$\varphi(h) = \varphi(s_1)\,\varphi(s_2) \cdots \varphi(s_n).$$

So $\varphi(gh) = \varphi(g)\,\varphi(h)$, as desired. □

Lemma 2.8. *Assume φ is a colour-preserving automorphism of* $\operatorname{Cay}(G; S)$*, and let $H = \langle\, s \in S \mid s^2 = \mathbf{1}\,\rangle$. Then:*

(1) *$\varphi(gh) = \varphi(g)h$, for all $g \in G$ and $h \in H$.*

(2) *In particular, if $\varphi(\mathbf{1}) = \mathbf{1}$, then $\varphi(h) = h$ for all $h \in H$.*

Proof. Let $\psi(g) = \varphi(\mathbf{1})^{-1}\,\varphi(g)$, so ψ is colour-preserving and $\psi(\mathbf{1}) = \mathbf{1}$. This implies $\psi(s) = s$ for all elements of order 2 in S. Therefore, the main calculation in the proof of Lemma 2.7 establishes that $\psi(gh) = \psi(g)h$, for all $g \in G$ and $h \in H$. This immediately implies $\varphi(gh) = \varphi(g)h$. □

2B A few facts from group theory

We recall some well known facts.

Definition 2.9 (cf. [8, Definition 2.6])**.** A group G is of *dicyclic type* if it has an abelian subgroup A of index 2, an element z of order 2 that is in A, and an element $q \notin A$, such that $q^2 = z$, and every element of A is inverted by q (in the action by conjugation). We write $G = \operatorname{Dic}(A, z, q)$.

Notation 2.10 ([4, §1.5, page 36])**.** We let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8, so $i^2 = j^2 = k^2 = -1$ and $ij = k$.

Definition 2.11 (cf. [12])**.** A group G is *hamiltonian* if it is nonabelian, and every subgroup of G is normal. It can be shown that G is hamiltonian if and only if it is isomorphic to $Q_8 \times A \times B$, for some abelian groups A and B, such that every element of A has odd order, and every element of B has order 1 or 2.

Lemma 2.12. *If H is a proper subgroup of a group G, then $\langle G \setminus H \rangle = G$, where $G \setminus H = \{\, g \in G \mid g \notin H \,\}$.*

Proof. Since $G \setminus H$ is obviously contained in $\langle G \setminus H \rangle$, it suffices to show that $H \subseteq \langle G \setminus H \rangle$. Since H is proper, there is some $g \in G \setminus H$. Then, for any $h \in H$, we have $g^{-1}h \in G \setminus H$, so

$$h = g\,(g^{-1}h) \in (G \setminus H)(G \setminus H) \subseteq \langle G \setminus H \rangle. \qquad \square$$

Lemma 2.13 (internal direct product [4, Theorem 9, page 171])**.** *If H and K are normal subgroups of G, such that $HK = G$ and $H \cap K = \{\mathbf{1}\}$, then G is isomorphic to the direct product of H and K, and we write $G = H \times K$.*

Lemma 2.14 (cf. [4, Proposition 14, page 94])**.** *Assume H and K are subgroups of G. If $G = HK$, then $G = KH$.*

2C A theorem of Leeman and de la Salle

The following result was mentioned after the statement of Theorem 1.1:

Theorem 2.15 (Leeman and de la Salle [9, Theorem 1])**.** *If G is an infinite group that has a finite generating set, then there is a finite, inverse-closed generating set S of G, such that the translations form a subgroup of index at most 2 in the automorphism group of* $\operatorname{Cay}(G; S)$.

Since every subgroup of index 1 or 2 is normal, this has the following immediate consequence:

Corollary 2.16 (Leeman and de la Salle)**.** *If G is an infinite group that has a finite generating set, then G has a locally finite Cayley graph that is normal.*

3 Colour-preserving automorphisms of complete Cayley graphs

In this section, we recall a result that provides an explicit description of all the colour-preserving automorphisms of any complete Cayley graph $\operatorname{Cay}\big(G; G \setminus \{\mathbf{1}\}\big)$ (see Theorem 3.2). It was proved by D. P. Byrne, M. J. Donner, and T. Q. Sibley [2, page 324] when G is finite, and it was generalized to infinite groups by P.-H. Leemann and M. de la Salle [9, Theorem 2]. This theorem will be needed in Section 4, so, for completeness, we provide a proof.

Example 3.1. Here are some important (known) examples of colour-preserving automorphisms of complete Cayley graphs.

(a) Assume G is abelian, and define $\iota\colon G \to G$ by $\iota(g) = g^{-1}$. Then ι a colour-preserving automorphism of the complete graph $\operatorname{Cay}\big(G; G \setminus \{\mathbf{1}\}\big)$ that fixes $\mathbf{1}$. It is a group automorphism, and it is nontrivial if there exists $g \in G$, such that $g^2 \neq \mathbf{1}$.

(b) Let $G = \mathrm{Dic}(A, z, q)$ be a (finite or infinite) group of dicyclic type, and define $\varphi\colon G \to G$ by $\varphi(q^i a) = q^{-i} a$ for $i \in \mathbb{Z}$ and $a \in A$. Then φ is nontrivial colour-preserving automorphism of the complete graph $\mathrm{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)$ that fixes $\mathbf{1}$, and it is a group automorphism.

(c) Let G be a (finite or infinite) hamiltonian 2-group, so $G = Q_8 \times B$, for some group B of exponent dividing 2. For every $I \subseteq \{i, j, k\}$, there is a unique colour-preserving automorphism φ_I of the complete graph $\mathrm{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)$ that fixes $\mathbf{1}$, and satisfies

$$\text{for } \ell \in \{i, j, k\}, \ \varphi_I(\ell) = \ell \Leftrightarrow \ell \in I.$$

Furthermore, φ_I is a group automorphism if and only if $|I|$ is odd.

There is some overlap between (b) and (c), because any hamiltonian 2-group is also a dicyclic group. For example, if we write $Q_8 \times B = \mathrm{Dic}\big(\langle k, B\rangle, i^2, i\big)$, then the automorphism described in (b) is equal to $\varphi_{\{k\}}$.

Proof. All of these examples are included in the main theorem of [2], but the notation and general presentation there are very different from ours, so we provide brief justifications for the reader's convenience.

It is easy to see (and well known) that if φ is a group automorphism, such that $\varphi(s) \in \{s^{\pm 1}\}$ for all $s \in S$, then φ is a colour-preserving automorphism of $\mathrm{Cay}(G; S)$ that fixes $\mathbf{1}$.

(a) Note that ι is a group automorphism (because G is abelian). Since $\iota(g) = g^{-1}$ is obviously in $\{g^{\pm 1}\}$ for all $g \in G$, this implies that ι is a colour-preserving automorphism of $\mathrm{Cay}(G; S)$.

(b) For $a \in A$, we have $\varphi(a) = a$ and

$$\varphi(qa) = q^{-1} a = a^{-1} q^{-1} = (qa)^{-1}.$$

Therefore $\varphi(g) \in \{g^{\pm 1}\}$ for all $g \in G$.

Hence, all that remains is to show that φ is a group automorphism. For $b \in A$, we have

$$\varphi\big((q^i a) b\big) = \varphi\big(q^i (ab)\big) = q^{-i} ab = \varphi(q^i a)\, \varphi(b)$$

and

$$\varphi\big((q^i a)(qb)\big) = \varphi\big(q^{i+1} (a^{-1} b)\big) = q^{-i-1} a^{-1} b = q^{-i} a \cdot q^{-i} b = \varphi(q^i a)\, \varphi(qb).$$

(c) The uniqueness follows from Lemma 2.8. So we only prove existence.

(0) Let $\varphi_\emptyset = \iota$, where $\iota(x) = x^{-1}$, as in (a). This is not a group automorphism (because G is not abelian).

However, we can easily show that it is a colour-preserving graph automorphism (and it obviously fixes $\mathbf{1}$), by using an argument that is taken from the first paragraph of the proof of [3, Theorem 2.3]. Let $g, s \in G$. Then $\varphi_\emptyset(gs) = (gs)^{-1} = s^{-1} g^{-1}$. In $G = Q_8 \times B$, every element either inverts or commutes with every other element, so $s^{-1} g^{-1} \in \{g^{-1} s^{\pm 1}\} = \{\varphi(g) s^{\pm 1}\}$.

(1) For $\ell \in \{i, j, k\}$, we may let $\varphi_{\{\ell\}}$ be the conjugation by ℓ. This is a group automorphism.

(2) If $|I| = 2$, we have $I = \{i, j, k\} \smallsetminus \{\ell\}$, for some $\ell \in \{i, j, k\}$. Then $\varphi_I = \varphi_{\{\ell\}} \circ \varphi_\emptyset$. This is not a group automorphism, because it is the composition of an automorphism with a non-automorphism.

(3) The automorphism $\varphi_{\{i,j,k\}}$ is the trivial permutation (i.e., the identity map). This is a group automorphism. □

Theorem 3.2 (Byrne, Donner, and Sibley [2], Leemann and de la Salle [9, Theorem 2])**.** *Let G be a (finite or infinite) group. If the complete Cayley graph* $\mathrm{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)$ *has a nontrivial colour-preserving automorphism that fixes* $\mathbf{1}$*, then G is either abelian or dicyclic type. Furthermore, any such colour-preserving automorphism is one of the automorphisms in* Example 3.1.

Proof. This is similar to the proof of [3, Theorem 2.3].

By assumption, we may let φ be a nontrivial colour-preserving automorphism of the complete graph $\mathrm{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)$ that fixes $\mathbf{1}$. Let

$$F = \{\, f \in G \mid \varphi(f) = f \,\},$$

and let F^c be the complement of F. Note that F is a proper subset of G, because φ is nontrivial, and that

$$\varphi(g) = g^{-1} \text{ for all } g \in F^c$$

(because φ is colour-preserving and fixes $\mathbf{1}$).

For all $f \in F$ and $h \in G$, such that $fh \in F^c$, we have

$$\varphi(fh) = (fh)^{-1} = h^{-1} f^{-1}$$

and

$$\varphi(fh) \in \{\varphi(f) h^{\pm 1}\} = \{f h^{\pm 1}\}$$

However, since $fh \notin F$, we know that $\varphi(fh) \neq fh$. So we must have

$$fh^{-1} = \varphi(fh) = h^{-1} f^{-1},$$

which means that h inverts f. Therefore fh also inverts f. Thus, we have shown:

$$\text{every element of } F^c \text{ inverts every element of } F. \tag{3a}$$

Case 1. Assume $\varphi(x) = x^{-1}$ for all $x \in G$.

For all $g, h \in G$, we have

$$h^{-1} g^{-1} = (gh)^{-1} = \varphi(gh) \in \{\varphi(g) h^{\pm 1}\} = \{g^{-1} h^{\pm 1}\} \subseteq g^{-1}\langle h \rangle, \tag{3b}$$

so g normalizes $\langle h \rangle$. Since g and h are arbitrary elements of G, this implies that every cyclic subgroup of G is normal, from which it follows that every subgroup of G is normal. Therefore, G is either abelian or hamiltonian.

We may assume G is hamiltonian, for otherwise G and φ are described in Example 3.1(a). So $G = Q_8 \times A \times B$, for some torsion abelian group A in which every element has odd order, and some group B of exponent dividing 2 (see Definition 2.11).

We may assume A is nontrivial, for otherwise G and φ are described in Example 3.1(c) (with $\varphi = \iota = \varphi_\emptyset$). Now, let a be a nontrivial element of A. For $g, h \in G$, the calculation in (3b) shows that $h^{-1}g^{-1} \in \{g^{-1}h^{\pm 1}\}$. By taking inverses, we conclude that $gh \in \{h^{\pm 1}g\}$. Therefore, if $gh \neq hg$, then $gh = h^{-1}g$. Letting $g = j$ and $h = ia$, we conclude that

$$jia = gh = h^{-1}g = (i^{-1}a^{-1})j = jia^{-1},$$

so $a = a^{-1}$. This contradicts the fact that every element of A has odd order.

Note. Henceforth, we may assume that Case 1 does not apply, so there is some $x \in G$, such that $\varphi(x) \neq x^{-1}$. Then $\varphi(x) = x$, so $x \in F$. Since $\varphi(x) \neq x^{-1}$, we also see that $x \neq x^{-1}$. Hence, inverting the elements of F is not the same as centralizing the elements of F. So we conclude from (3a) that F contains its centralizer $C_G(F)$.

For any $q \in F^c$, we know that q^2 centralizes F (because q inverts F), so $q^2 \in C_G(F) \subseteq F$. This implies that q inverts q^2. However, q obviously centralizes q^2. So it must be the case that $q^{-2} = q^2$, which means $|q^2| \in \{1, 2\}$. However, we also know from Lemma 2.8(2) that $|q| > 2$, since $\varphi(q) \neq q$. So we conclude that $|q^2| = 2$. Thus:

$$|q| = 4 \text{ for all } q \in F^c. \tag{3c}$$

Furthermore, for all $g, h \in F^c$, we know that g and h both invert F, so $gh^{-1} \in C_G(F)$. This implies that all elements of F^c are in the same coset of $C_G(F)$:

$$F^c \subseteq C_G(F)\, q \quad \text{for all } q \in F^c.$$

Case 2. Assume F is a subgroup of G.

We will show that G is of dicyclic type, and φ is the automorphism in Example 3.1(b). Fix some $q \in F^c$.

- Since $F^c \subseteq C_G(F)\, q \subseteq Fq$, we see that all elements of F^c are in the same coset of F. This means $|G : F| = 2$.
- Since q inverts F, and conjugation by q is an automorphism, we see that inversion is an automorphism of F. This implies that F is abelian.
- Let $z = q^2$. We know from the above note that $|q| = 4$, so $|z| = 2$.

We have now established that $G = \mathrm{Dic}(F, z, q)$.

So it only remains to show that $\varphi(q^i f) = q^{-i} f$ for all $i \in \mathbb{Z}$ and $f \in F$.

- If i is even, then $q^i f \in F$, so $\varphi(q^i f) = q^i f$. However, since i is even and $|q| = 4$, we also have $q^i = q^{-i}$.
- If i is odd, then $q^i f \in F^c$, so

  $$\varphi(q^i f) = (q^i f)^{-1} = f^{-1} q^{-i} = q^{-i} f,$$

  since q inverts every element of f (and i is odd).

Case 3. Assume F is not a subgroup of G.

For convenience, let $C = C_G(F)$ be the centralizer of F. As mentioned in the above note, we have $C \subsetneq F$, and we have $F^c \subseteq Cq$ for every $q \in F^c$.

Since $F^c \subseteq Cq$, we know that F contains all of the cosets of C except Cq. Also, since F is not a subgroup, we know that $\langle F \rangle$ is strictly larger than F, which means it contains some element of F^c, and therefore contains some element of Cq. Since $C \subsetneq F$, this implies that $\langle F \rangle$ contains Cq. It also contains F, so it contains all of the other cosets of C, as well. Therefore $\langle F \rangle = G$. This implies that C is the centre of G.

Thus, all elements of F^c are in the same coset of the centre, so

the elements of F^c commute with each other.

Also recall (see (3a)) that

every element of F^c inverts every element of F.

Now, let $f_1, f_2 \in F$. If $f_1 f_2 \notin F$, then it inverts both f_1 and f_2, which implies that f_1 inverts f_2 and f_2 inverts f_1. On the other hand, if $f_1 f_2 \in F$, then the conjugation action of any element of F^c respects multiplication, but inverts f_1, f_2, and $f_1 f_2$, so we must have $f_1 f_2 = f_2 f_1$. Therefore, if f_1 and f_2 are any elements of F, then

either f_1 and f_2 commute, or they invert each other.

Now consider any $f \in F$, such that $f \notin C$. Then $qf \notin qC$, so $qf \notin F^c$. This means $qf \in F$, so

$$qf = \varphi(qf) \in \{\varphi(q)\, f^{\pm 1}\} = \{q^{-1}\, f^{\pm 1}\},$$

so $q^2 \in \{\mathbf{1}, f^{-2}\}$. However, we know from (3c) that $|q| = 4$, so $q^2 \neq \mathbf{1}$. So we conclude that

$$q^2 = f^{-2}. \tag{3d}$$

Therefore (recalling that q inverts f) we have

$$f^{-1} q f = f^{-2} q = q^2\, q = q^3 \in \langle q \rangle.$$

So f normalizes $\langle q \rangle$. Since this is also clearly true for the elements of F that are in C, we conclude that

if $f \in F$ and $q \in F^c$, then f normalizes $\langle q \rangle$.

We have now shown that g normalizes $\langle h \rangle$ for all $g, h \in G$. This means that every cyclic subgroup of G is normal. It follows easily that every subgroup of G is normal. (Also, G is not abelian, because elements of F^c invert elements of F.) Therefore G is a hamiltonian group (see Definition 2.11).

We claim that G is a 2-group. To see this, write $G = Q_8 \times A \times B$, as in Definition 2.11. We wish to show that A is trivial. Suppose not. Then, since F contains all but one coset of the centre, there is some $f \in F$, such that $f \notin C$, and the A-component of f is nontrivial. From (3d) we know that $|f^2| = 2$, which contradicts the fact that the A-component of f^2 is nontrivial.

All elements of the centre C of G have order 2. Therefore, since $\varphi(q) = q^{-1}$ (and φ is colour-preserving), we have $\varphi(qc) = \varphi(q) c^{\pm 1} = q^{-1} c = (qc)^{-1}$ for all $c \in C$. So F^c is the entire coset qC. Hence, F^c contains some $\ell \in \{i, j, k\}$. Then it is easy to see that φ is the permutation φ_I defined in Example 3.1(c), where $I = \{i, j, k\} \setminus \{\ell\}$. $\square$

The above theorem provides another proof of the (known) implication (5) $\Rightarrow$ (4) of Theorem 1.1, which was discussed in Note 1.6(2) of the introduction:

Corollary 3.3 (Byrne et al. [2, cf. page 324], Dobson et al. [3, Theorem 2.3])**.** *If the complete graph* $\operatorname{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)$ *is not CCA, then G is a hamiltonian 2-group.*

Proof. Since $\operatorname{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)$ is not CCA, there exists a colour-preserving automorphism φ of this complete Cayley graph, such that φ is not affine. After composing with a translation, we may assume $\varphi(\mathbf{1}) = \mathbf{1}$. Since φ is not affine, it is obviously not trivial, so Theorem 3.2 tells us that G is abelian or dicyclic, and that φ is one of the automorphisms in Example 3.1. However, since φ is not affine, it is not a group automorphism, so it is not listed in part (a) or (b) of that example. Hence, it must be part (c) that applies, so G is a hamiltonian 2-group. $\square$

4 Complete graphs that are not strongly CCA

Notation 4.1. Let $X = \operatorname{Cay}(G; S)$ be a Cayley graph of G.

1. $\mathcal{A}(X)$ is the group of all colour-permuting automorphisms of X.
2. $\mathcal{A}^0(X)$ is the group of all colour-preserving automorphisms of X.
3. $\mathcal{A}_{\mathbf{1}}(X)$ and $\mathcal{A}^0_{\mathbf{1}}(X)$ are the stabilizers of the vertex $\mathbf{1}$ in $\mathcal{A}(X)$ and $\mathcal{A}^0(X)$, respectively.
4. For $\varphi \in \mathcal{A}(X)$, and $g \in G$, let π^φ_g be the permutation of S that is induced by φ at the vertex g. That is, for all $s \in S$, we have
$$\varphi(gs) = \varphi(g)\, \pi^\varphi_g(s).$$
5. For $\varphi \in \mathcal{A}(X)$, let $\overline{\pi}^\varphi$ be the permutation of the colours that is induced by φ. Thus, for all $g \in G$ and $s \in S$, we have
$$\overline{\pi}^\varphi(\{s^{\pm 1}\}) = \{\pi^\varphi_g(s)^{\pm 1}\}.$$
6. For $g \in G$, let L_g be the left-translation by g, so $L_g(x) = gx$ for all $x \in G$. This is a colour-preserving automorphism of every Cayley graph of G.

We have the following elementary (and well-known) observation:

Lemma 4.2. *For $\varphi \in \mathcal{A}_{\mathbf{1}}(X)$, we have $\varphi \in \operatorname{Aut} G$ if and only if $\pi^\varphi_g = \pi^\varphi_h$ for all $g, h \in G$.*

Lemma 4.3. *Given $\varphi \in \mathcal{A}_{\mathbf{1}}(X)$, $g \in G$, and $s \in S$, let*
$$\psi = L_g^{-1}\, \varphi^{-1}\, L_{\varphi(g)}\, L_{\varphi(gs)}^{-1}\, \varphi\, L_{gs}.$$
Then $\psi \in \mathcal{A}^0_{\mathbf{1}}(X)$ and $\psi(s) = s$.

Proof. It is clear that ψ is colour-permuting, since it is a composition of colour-permuting automorphisms. Furthermore, letting id denote a trivial permutation, we have

$$\begin{aligned}
\overline{\pi}^{\psi} &= \overline{\pi}^{L_g^{-1}} \circ \overline{\pi}^{\varphi^{-1}} \circ \overline{\pi}^{L_{\varphi(g)}} \circ \overline{\pi}^{L_{\varphi(gs)}^{-1}} \circ \overline{\pi}^{\varphi} \circ \overline{\pi}^{L_{gs}} \\
&= \mathrm{id} \circ \overline{\pi}^{\varphi^{-1}} \circ \mathrm{id} \circ \mathrm{id} \circ \overline{\pi}^{\varphi} \circ \mathrm{id} \qquad \begin{pmatrix}\text{translations are} \\ \text{colour-preserving}\end{pmatrix} \\
&= \overline{\pi}^{\varphi^{-1}\varphi} \\
&= \overline{\pi}^{\mathrm{id}} \\
&= \mathrm{id}\,.
\end{aligned}$$

So the colour-permuting automorphism ψ is actually colour-preserving.

Also, a straightforward calculation shows that $\psi(\mathbf{1}) = \mathbf{1}$. Therefore $\psi \in \mathcal{A}_{\mathbf{1}}^0(X)$. Furthermore,

$$\begin{aligned}
\psi(s^{-1}) &= L_g^{-1}\, \varphi^{-1}\, L_{\varphi(g)}\, L_{\varphi(gs)}^{-1}\, \varphi\, L_{gs}(s^{-1}) \\
&= L_g^{-1}\, \varphi^{-1}\, L_{\varphi(g)}\, L_{\varphi(gs)}^{-1}\, \varphi(g) \\
&= L_g^{-1}\, \varphi^{-1}\, L_{\varphi(g)}\big(\varphi(gs)^{-1}\, \varphi(g)\big) \\
&= L_g^{-1}\, \varphi^{-1}\, L_{\varphi(g)}\big(\pi_g^{\varphi}(s)^{-1}\big) \\
&= L_g^{-1}\, \varphi^{-1}\, L_{\varphi(g)}\big(\pi_g^{\varphi}(s^{-1})\big) \qquad \text{(Lemma 2.5 with } g = \mathbf{1} \text{ and } n = -1) \\
&= L_g^{-1}\, \varphi^{-1}\big(\varphi(gs^{-1})\big) \\
&= L_g^{-1}(gs^{-1}) \\
&= s^{-1}.
\end{aligned}$$

Now, it follows from Lemma 2.5 (with $g = \mathbf{1}$ and $n = -1$) that $\psi(s) = s$. □

Lemma 4.4. *Let $X = \mathrm{Cay}(G; S)$ be a Cayley graph of G, and let*

$$S^* = \{ s \in S \mid \textit{no nontrivial element of } \mathcal{A}_{\mathbf{1}}^0(X) \textit{ fixes } s \}.$$

If S^* generates G, then X is strongly CCA.

Proof. Given $\varphi \in \mathcal{A}_{\mathbf{1}}(X)$, we wish to show that $\varphi \in \operatorname{Aut} G$. Fix some $g \in G$ and $s \in S^*$, and, as in Lemma 4.3, let

$$\psi = L_g^{-1}\, \varphi^{-1}\, L_{\varphi(g)}\, L_{\varphi(gs)}^{-1}\, \varphi\, L_{gs}.$$

so $\psi \in \mathcal{A}_{\mathbf{1}}^0(X)$ and $\psi(s) = s$. Since $s \in S^*$, it follows from the definition of S^* that ψ is the identity map. Thus, for all $t \in S$, we have $\psi(t) = t$, so, by the definition of ψ, we have

$$L_{\varphi(g)}^{-1} \varphi L_g(t) = L_{\varphi(gs)}^{-1} \varphi L_{gs}(t),$$

which means $\pi_g^{\varphi}(t) = \pi_{gs}^{\varphi}(t)$. Therefore $\pi_g^{\varphi} = \pi_{gs}^{\varphi}$. Since s is an arbitrary element of S^* and S^* generates G, repeated application of this fact implies that $\pi_g^{\varphi} = \pi_h^{\varphi}$ for all $g, h \in G$. Therefore $\varphi \in \mathrm{Aut}(G)$ (see Lemma 4.2). □

Definition 4.5 ([11, page 8]). A permutation group is *semiregular* if no nonidentity element of the group has any fixed points.

Corollary 4.6. *Let $X = \mathrm{Cay}(G;S)$ be a connected Cayley graph of G. If the action of $\mathcal{A}^0_{\mathbf{1}}(X)$ on S is semiregular, then X is strongly CCA.*

Proof. In this case, we have $S^* = S$. Since X is connected, we know that S generates G. □

Corollary 4.7. *Let $X = \mathrm{Cay}(G;S)$ be a Cayley graph of G. If $\mathcal{A}^0_{\mathbf{1}}(X)$ is trivial, then X is strongly CCA.*

Proof. Any trivial permutation group is obviously semiregular, so this is almost immediate from Corollary 4.6. The only issue (which is minor) is that the corollary requires X to be connected. However, it is easy to see that if $\mathcal{A}^0_{\mathbf{1}}(X)$ is trivial, then either X is connected, or X has only two vertices. □

We can now establish the implication (5) $\Rightarrow$ (2) of Theorem 1.1.

Corollary 4.8. *If G is not a hamiltonian 2-group, then $\mathrm{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)$ is strongly CCA.*

Proof. In the notation of Lemma 4.4, it suffices to show that $\langle S^* \rangle = G$.

To this end, let φ be a nontrivial element of $\mathcal{A}^0_{\mathbf{1}}(X)$. By Theorem 3.2 (and the assumption that G is not a hamiltonian 2-group), φ must be described in (a) or (b) of Example 3.1. Since each of these parts describes a unique automorphism, this implies that φ is the only nontrivial element of $\mathcal{A}^0_{\mathbf{1}}(X)$. (Also note that $S = G \smallsetminus \{\mathbf{1}\}$ in the complete graph $\mathrm{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)$.) Therefore, we have

$$S^* = G \smallsetminus F, \text{ where } F = \{\, g \in G \mid \varphi(g) = g \,\}.$$

Since φ is nontrivial, we know that F is a proper subset of G. Also, in both (a) and (b) of Example 3.1, it is easy to see that F is a subgroup of G. (In (a), it is the subgroup $\{\, x \in G \mid x^2 = \mathbf{1} \,\}$; in (b), it is the subgroup A.) Therefore, it is immediate from Lemma 2.12 that $\langle S^* \rangle = G$. □

5 Colour-permuting automorphisms of complete Cayley graphs

In this section, we prove Theorem 1.7. For the reader's convenience, we state the result again here:

Theorem 1.7. *If G is any (finite or infinite) group, then every colour-permuting automorphism of the complete graph $\mathrm{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)$ is the composition of a group automorphism and a colour-preserving graph automorphism.*

In other words (and using the symbols defined in Notation 4.1), we prove:

$$\mathcal{A}\big(\mathrm{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)\big) = \mathrm{Aut}(G) \cdot \mathcal{A}^0\big(\mathrm{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)\big).$$

(Note that the order of the factors on the right-hand side does not matter (see Lemma 2.14).)

Proof of Theorem 1.7. Let $K_G = \mathrm{Cay}\big(G; G \smallsetminus \{\mathbf{1}\}\big)$, and let φ be a colour-permuting automorphism of K_G. We wish to show that φ is the composition of a group automorphism and a colour-preserving graph automorphism. By composing φ with a translation, we may assume, without loss of generality, that $\varphi(\mathbf{1}) = \mathbf{1}$.

If K_G is strongly CCA, then φ must be a group automorphism. So the desired conclusion is true in this case.

We may now assume that K_G is not strongly CCA. By the contrapositive of Corollary 4.8, this implies that G is a hamiltonian 2-group:

$$G = Q_8 \times B,$$

for some abelian group B of exponent ≤ 2. Let $G_2 = \langle i^2, B \rangle$ be the subgroup generated by the elements of order 2 in G. (Recall that $i^2 = j^2 = k^2$ is the unique element of order 2 in Q_8.) This is the centre of G, and it is an abelian group of exponent 2 that has index 4 in G. By Lemma 2.7, we have

$$\varphi(gx) = \varphi(g)\,\varphi(x) \text{ for all } g \in G \text{ and } x \in G_2.$$

Therefore

$$\varphi(G_2) = G_2, \text{ and the restriction of } \varphi \text{ to } G_2 \text{ is a group automorphism.} \tag{5a}$$

Note that $\mathrm{Cay}\big(G; \{i^{\pm 1}\}\big)$ is a disjoint union of 4-cycles (because i has order 4). Hence, the isomorphic graph $\mathrm{Cay}\big(G; \{\varphi(i)^{\pm 1}\}\big)$ is also a disjoint union of 4-cycles, which means that $\varphi(i)$ has order 4. Therefore $\varphi(i)^{\pm 2} = i^2$, because $g^2 = i^2$ for all $g \in G$ of order 4. Also, we have $\varphi(i^2) \in \{\varphi(i)^{\pm 2}\}$ (see Lemma 2.5). Combining these observations yields

$$\varphi(i^2) = i^2.$$

Now, let $\widetilde{Q}_8 = \langle \varphi(i), \varphi(j) \rangle$, and define $\alpha \colon Q_8 \to G$ by

$$\alpha(i^m j^n) = \varphi(i)^m\, \varphi(j)^n.$$

(We will soon see that α is well-defined.) Since i and j have order 4 (and $\varphi \in \mathcal{A}_{\mathbf{1}}(K_G)$), we know that $\varphi(i)$ and $\varphi(j)$ have order 4 (by the argument in the preceding paragraph). Also note that $i \notin j\,G_2$, so (5a) implies that $\varphi(i) \notin \varphi(j)\,G_2$; hence, $\varphi(i)$ does not centralize $\varphi(j)$. However, any two elements of G either commute or invert each other; therefore $\varphi(i)$ and $\varphi(j)$ invert each other, so $\varphi(i)$ and $\varphi(j)$ satisfy exactly the same relations as i and j. We conclude that

$$\alpha \text{ is a (well-defined) isomorphism from } Q_8 \text{ to } \widetilde{Q}_8. \tag{5b}$$

Also note that, for all $m, n \in \mathbb{Z}$, Lemma 2.6 implies

$$\varphi(i^m j^n) \in \{\varphi(i^m)\,\varphi(j)^{\pm n}\} \subseteq \{\varphi(i)^{\pm m}\,\varphi(j)^{\pm n}\} \subseteq \alpha(Q_8) = \widetilde{Q}_8,$$

so $\varphi(Q_8) = \widetilde{Q}_8$. Therefore, since $Q_8 \cap G_2 = \langle i^2 \rangle$, we see that

$$\widetilde{Q}_8 \cap \varphi(G_2) = \varphi(Q_8) \cap \varphi(G_2) = \varphi(Q_8 \cap G_2) = \varphi\big(\langle i^2 \rangle\big) = \langle i^2 \rangle.$$

Since $\varphi(B) \subseteq \varphi(G_2)$ and $i^2 \notin B$, this implies $\widetilde{Q}_8 \cap \varphi(B) = \{\mathbf{1}\}$. Also, since the restriction of φ to G_2 is a group automorphism, we know that $\varphi(B)$ is a subgroup of G and we have $|G : \varphi(B)| = |G : B| = 8$; from this, we can conclude that $\widetilde{Q}_8 \cdot \varphi(B) = G$. So we see from Lemma 2.13 that

$$G = \widetilde{Q}_8 \times \varphi(B). \tag{5c}$$

Define $\beta\colon G \to G$ by

$$\beta(qb) = \alpha(q)\,\varphi(b) \ \text{ for } q \in Q_8 \text{ and } b \in B.$$

We see from (5a), (5b), and (5c) that β is an automorphism of G.

Now, let $\psi = \beta^{-1} \circ \varphi$. To complete the proof, we will show that ψ is a colour-preserving automorphism of $\operatorname{Cay}\bigl(G; G \smallsetminus \{\mathbf{1}\}\bigr)$. (This implies that, as desired, $\varphi = \beta \circ \psi$ is the composition of a group automorphism, namely β, and a colour-preserving graph automorphism, namely ψ.) First, note that ψ is a colour-permuting automorphism (because it is the composition of two colour-permuting automorphisms) and that $\psi(\mathbf{1}) = \mathbf{1}$ (because β and φ both fix $\mathbf{1}$).

If $m, n \in \mathbb{Z}$, and n is odd, then conjugation by j^n inverts i^m, so

$$i^m j^{-n} = j^{-n} \cdot (j^n i^m j^{-n}) = j^{-n} \cdot i^{-m} = (i^m j^n)^{-1}.$$

On the other hand, if n is even, then $j^{-n} = j^n$, so $i^m j^{-n} = i^m j^n$. Thus, we have

$$i^m j^{-n} \in \{(i^m j^n)^{\pm 1}\} \text{ for all } m, n \in \mathbb{Z}. \tag{5d}$$

Hence, for all $q \in Q_8$, we have

$$\begin{aligned}
\varphi(q) &= \varphi(i^m j^n) && \text{(for some } m, n \in \mathbb{Z}) \\
&\in \{\varphi(i^m)\,\varphi(j)^{\pm n}\} && \text{(Lemma 2.6)} \\
&= \{\varphi(i)^m\,\varphi(j)^{\pm n}\} && \text{(Lemma 2.5 with } g = \mathbf{1}) \\
&= \{\beta(i)^m\,\beta(j)^{\pm n}\} && \text{(definitions of } \beta \text{ and } \alpha) \\
&= \{\beta(i^m j^{\pm n})\} && (\beta \text{ is a group homomorphism}) \\
&\subseteq \{\beta\bigl((i^m j^n)^{\pm 1}\bigr)\} && \text{((5d)} \\
&= \{\beta(q^{\pm 1})\} && \text{(definition of } m \text{ and } n) \\
&= \{\beta(q)^{\pm 1}\} && (\beta \text{ is a group homomorphism}).
\end{aligned}$$

Therefore

$$\beta(q) \in \{\varphi(q)^{\pm 1}\} \text{ for all } q \in Q_8. \tag{5e}$$

Now, for any $s \in G$, we have

$$\begin{aligned}
\beta(s) &= \beta(qb) && \text{(for some } q \in Q_8 \text{ and } b \in B) \\
&= \beta(q)\,\beta(b) && (\beta \text{ is a group automorphism}) \\
&\in \{\varphi(q)^{\pm 1}\,\beta(b)\} && \text{(5e)} \\
&= \{\bigl(\varphi(q)\,\beta(b)\bigr)^{\pm 1}\} && \begin{pmatrix} \beta(b) \in G_2 \text{ has order } 1 \text{ or } 2 \\ \text{and is in the centre of } G \end{pmatrix} \\
&= \{\bigl(\varphi(q)\,\varphi(b)\bigr)^{\pm 1}\} && \text{(definitions of } \beta \text{ and } \alpha) \\
&= \{\varphi(qb)^{\pm 1}\} && \text{(Lemma 2.7)} \\
&= \{\varphi(s)^{\pm 1}\} && \text{(definition of } q \text{ and } b).
\end{aligned}$$

Since $\varphi(s^{-1}) = \varphi(s)^{-1}$ (by Lemma 2.5 with $g = \mathbf{1}$) and $\beta(s^{-1}) = \beta(s)^{-1}$ (because β is a group homomorphism), this implies

$$\varphi(\{s^{\pm 1}\}) = \beta(\{s^{\pm 1}\}) \text{ for all } s \in G.$$

Therefore, we have

$$\psi(\{s^{\pm 1}\}) = \beta^{-1}(\varphi(\{s^{\pm 1}\})) = \beta^{-1}(\beta(\{s^{\pm 1}\})) = \{s^{\pm 1}\}.$$

Hence, for all $g, s \in G$, we see from Lemma 2.6 that

$$\psi(gs) \in \{\psi(g)\, \psi(s^{\pm 1})\} = \{\psi(g)\, s^{\pm 1}\}.$$

This means that ψ is colour-preserving, as desired. □

We will close by showing that this theorem yields a direct proof of the implication (4) $\Rightarrow$ (2) of Theorem 1.1. However, this observation does not shorten any of our arguments, because the logic is circular: the proof of Theorem 1.7 relies on the implication (5) $\Rightarrow$ (2) (i.e., it relies on Corollary 4.8), which is *a priori* stronger than (4) $\Rightarrow$ (2), because (4) $\Rightarrow$ (5) is trivial (see Note 1.6(1)).

Corollary 5.2. *If the complete graph* $\operatorname{Cay}(G; G \setminus \{\mathbf{1}\})$ *is CCA, then it is strongly CCA.*

Proof. Let φ be a colour-permuting automorphism of $\operatorname{Cay}(G; G \setminus \{\mathbf{1}\})$. By Theorem 1.7, we may write $\varphi = \psi \circ \alpha$, where ψ is a colour-preserving graph automorphism and $\alpha \in \operatorname{Aut} G$. However, since $\operatorname{Cay}(G; G \setminus \{\mathbf{1}\})$ is CCA, we know that ψ is an affine map. Hence, φ is the composition of two affine maps, and is therefore affine. □

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