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Unramified Brauer groups and isoclinism

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Abstract

We show that the Bogomolov multipliers of isoclinic groups are isomorphic.

Keywords: Unramified Brauer group, Bogomolov multiplier, isoclinism.

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1 Introduction

Let G be a finite group and V a faithful representation of G over an algebraically closed field k of characteristic zero. Suppose that the action of G upon V is generically free. A relaxed version of Noether's problem [11] asks as to whether the fixed field $k(V)^G$ is purely transcendental over k, i.e., whether the quotient space V/G is *rational*. A question related to the above mentioned is whether V/G is *stably rational*, that is, whether there exist independent variables $x_1, \ldots, x_r$ such that $k(V)^G(x_1, \ldots, x_r)$ becomes a pure transcendental extension of k. This problem has close connection with Lüroth's problem [12] and the inverse Galois problem [14, 13]. By the so-called *no-name lemma*, stable rationality of V/G does not depend upon the choice of V, but only on the group G, cf. [4, Theorem 3.3 and Corollary 3.4]. Saltman [13] found examples of groups G of order p^9 such that V/G is not stably rational over k. His main method was application of the unramified cohomology group $\mathrm{H}^2_{\mathrm{nr}}(k(V)^G, \mathbb{Q}/\mathbb{Z})$ as an obstruction. A version of this invariant had been used before by Artin and Mumford [1] who constructed unirational varieties over k that were not rational. Bogomolov [2] proved that $\mathrm{H}^2_{\mathrm{nr}}(k(V)^G, \mathbb{Q}/\mathbb{Z})$ is canonically isomorphic to

$$\mathrm{B}_0(G) = \bigcap_{\substack{A \leq G, \\ A \text{ abelian}}} \ker \mathrm{res}^G_A,$$

where $\mathrm{res}^G_A : \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z})$ is the usual cohomological restriction map. Following Kunyavskiĭ [7], we say that $\mathrm{B}_0(G)$ is the *Bogomolov multiplier* of G.

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We recently proved [9] that $\mathrm{B}_0(G)$ is naturally isomorphic to $\mathrm{Hom}(\tilde{\mathrm{B}}_0(G), \mathbb{Q}/\mathbb{Z})$, where $\tilde{\mathrm{B}}_0(G)$ is the kernel of the commutator map $G \curlywedge G \to [G,G]$, and $G \curlywedge G$ is a quotient of the *non-abelian exterior square* of G (see Section 2 for further details). This description of $\mathrm{B}_0(G)$ is purely combinatorial, and allows for efficient computations of $\mathrm{B}_0(G)$, and a Hopf formula for $\mathrm{B}_0(G)$. We also note here that the group $\tilde{\mathrm{B}}_0(G)$ can be defined for any (possibly infinite) group G.

Recently, Hoshi, Kang, and Kunyavskiĭ [6] classified all groups of order p^5 with non-trivial Bogomolov multiplier; the question was dealt with independently in [10]. It turns out that the only examples of such groups appear within the same isoclinism family, where isoclinism is the notion defined by P. Hall in his seminal paper [5]. The following question was posed in [6]:

Question 1.1 ([6]). *Let G_1 and G_2 be isoclinic p-groups. Is it true that the fields $k(V)^{G_1}$ and $k(V)^{G_2}$ are stably isomorphic, or at least, that $\mathrm{B}_0(G_1)$ is isomorphic to $\mathrm{B}_0(G_2)$?*

The purpose of this note is to answer the second part of the above question in the affirmative:

Theorem 1.1. *Let G_1 and G_2 be isoclinic groups. Then $\tilde{\mathrm{B}}_0(G_1) \cong \tilde{\mathrm{B}}_0(G_2)$. In particular, if G_1 and G_2 are finite, then $\mathrm{B}_0(G_1)$ is isomorphic to $\mathrm{B}_0(G_2)$.*

The proof relies on the theory developed in [9]. We note here that we have recently become aware of a paper by Bogomolov and Böhning [3] who fully answer the above question using different techniques. We point out that our approach here is purely combinatorial and does not require cohomological machinery.

2 Proof of Theorem 1.1

We first recall the definition of $G \curlywedge G$ from [9]. For $x, y \in G$ we write ${}^x y = xyx^{-1}$ and $[x,y] = xyx^{-1}y^{-1}$. Let G be any group. We form the group $G \curlywedge G$, generated by the symbols $g \curlywedge h$, where $g, h \in G$, subject to the following relations:

$$\begin{aligned} gg' \curlywedge h &= ({}^g g' \curlywedge {}^g h)(g \curlywedge h),\\ g \curlywedge hh' &= (g \curlywedge h)({}^h g \curlywedge {}^h h'),\\ x \curlywedge y &= 1, \end{aligned}$$

for all $g, g', h, h' \in G$, and all $x, y \in G$ with $[x,y] = 1$. The group $G \curlywedge G$ is a quotient of the non-abelian exterior square $G \wedge G$ of G defined by Miller [8]. There is a surjective homomorphism $\kappa : G \curlywedge G \to [G,G]$ defined by $\kappa(x \curlywedge y) = [x,y]$ for all $x, y \in G$. Denote $\tilde{\mathrm{B}}_0(G) = \ker \kappa$. By [9] we have the following:

Theorem 2.1 ([9]). *Let G be a finite group. Then $\mathrm{B}_0(G)$ is naturally isomorphic to $\mathrm{Hom}(\tilde{\mathrm{B}}_0(G), \mathbb{Q}/\mathbb{Z})$, and thus $\mathrm{B}_0(G) \cong \tilde{\mathrm{B}}_0(G)$.*

Let L be a group. A function $\phi : G \times G \to L$ is called a $\tilde{\mathrm{B}}_0$*-pairing* if for all $g, g', h, h' \in G$, and for all $x, y \in G$ with $[x,y] = 1$,

$$\begin{aligned} \phi(gg', h) &= \phi({}^g g', {}^g h)\phi(g,h),\\ \phi(g, hh') &= \phi(g,h)\phi({}^h g, {}^h h'),\\ \phi(x,y) &= 1. \end{aligned}$$

Clearly a $\tilde{\mathrm{B}}_0$-pairing ϕ determines a unique homomorphism of groups $\phi^* : G \curlywedge G \to L$ such that $\phi^*(g \curlywedge h) = \phi(g, h)$ for all $g, h \in G$.

We now turn to the proof of Theorem 1.1. Let G_1 and G_2 be isoclinic groups, and denote $Z_1 = Z(G_1)$, $Z_2 = Z(G_2)$. By definition [5], there exist isomorphisms $\alpha : G_1/Z_1 \to G_2/Z_2$ and $\beta : [G_1, G_1] \to [G_2, G_2]$ such that whenever $\alpha(a_1Z_1) = a_2Z_2$ and $\alpha(b_1Z_1) = b_2Z_2$, then $\beta([a_1, b_1]) = [a_2, b_2]$ for $a_1, b_1 \in G_1$. Define a map $\phi : G_1 \times G_1 \to G_2 \curlywedge G_2$ by $\phi(a_1, b_1) = a_2 \curlywedge b_2$, where a_i, b_i are as above. To see that this is well defined, suppose that $\alpha(a_1Z_1) = a_2Z_2 = \bar{a_2}Z_2$ and $\alpha(b_1Z_1) = b_2Z_2 = \bar{b_2}Z_2$. Then we can write $\bar{a_2} = a_2z$ and $\bar{b_2} = b_2w$ for some $w, z \in Z_2$. By the definition of $G_2 \curlywedge G_2$ we have that $\bar{a_2} \curlywedge \bar{b_2} = a_2z \curlywedge b_2w = a_2 \curlywedge b_2$, hence ϕ is well defined.

Suppose that $a_1, b_1 \in G_1$ commute, and let $a_2, b_2 \in G_2$ be as above. By definition, $[a_2, b_2] = \beta([a_1, b_1]) = 1$, hence $a_2 \curlywedge b_2 = 1$. This, and the relations of $G_2 \curlywedge G_2$, ensure that ϕ is a $\tilde{\mathrm{B}}_0$-pairing. Thus ϕ induces a homomorphism $\gamma : G_1 \curlywedge G_1 \to G_2 \curlywedge G_2$ such that $\gamma(a_1 \curlywedge b_1) = a_2 \curlywedge b_2$ for all $a_1, b_1 \in G_1$. By symmetry there exists a homomorphism $\delta : G_2 \curlywedge G_2 \to G_1 \curlywedge G_1$ defined via α^{-1}. It is straightforward to see that δ is the inverse of γ, hence γ is an isomorphism.

Let $\kappa_1 : G_1 \curlywedge G_1 \to [G_1, G_1]$ and $\kappa_2 : G_2 \curlywedge G_2 \to [G_2, G_2]$ be the commutator maps. Since $\beta\kappa_1(a_1 \curlywedge b_1) = \beta([a_1, b_1]) = [a_2, b_2] = \kappa_2\gamma(a_1 \curlywedge b_1)$, we have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\mathrm{B}}_0(G_1) & \longrightarrow & G_1 \curlywedge G_1 & \xrightarrow{\kappa_1} & [G_1, G_1] & \longrightarrow & 0\,. \\
 & & \downarrow{\scriptstyle \tilde{\gamma}} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \tilde{\mathrm{B}}_0(G_2) & \longrightarrow & G_2 \curlywedge G_2 & \xrightarrow{\kappa_2} & [G_2, G_2] & \longrightarrow & 0
\end{array}
$$

Here $\tilde{\gamma}$ is the restriction of γ to $\tilde{\mathrm{B}}_0(G_1)$. Since β and γ are isomorphisms, so is $\tilde{\gamma}$. This concludes the proof.

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