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4 Understanding the Second Quantization of Fermions in Clifford and in Grassmann Space — New Way of Second Quantization of Fermions — Part I ⋆

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Abstract. Both algebras, Clifford and Grassmann, offer the second quantized fermions [1–3] without postulating the second quantization conditions of Dirac [13]. But while fermions with the internal degrees of freedom described by the Clifford algebras manifest the half integer spins — in agreement with the observed properties of quarks and leptons and antiquarks and antileptons — the Grassmann ”fermions” manifest integer spins. In Part I properties of the second quantized integer spins ”fermions” in Grassmann space are presented. In Part II the conditions are discussed under which the Clifford algebra offers the appearance of families of the second quantized fermions.

Povzetek. Avtorja sta v članku [3] pokazala, da ponudita obe algebri — Cliffordova in Grassmannova — razlago za Diracove postulate druge kvantizacije fermionov [1–3], saj imajo vektorji v obeh prostorih vse lastnosti, ki jih zahteva Diracov pogoj za drugo kvantizacijo [13]. Clanek razloži v prvem delu tega prispevka drugo kvantizacijo v Grassmannovem prostoru. Pri tem opisu nosijo“fermioni” celoštevilčni spin in naboje, kadar je prostor šest razsežen ali več, v adjungirani upodobitvi. Avtorja demonstrirata lastnosti teh “fermionov” na primeru šest razsežnega prostora. Spin v peti in šesti dimenziji (se po zlomitvi simetrije) “vidi” v $(3 + 1)$-razsežnem prostoru kot naboj “fermiona”. V drugem delu obravnavata lastnosti fermionov s polštevilčnimi spini v Cliffordovi algebri.

Keywords: Second quantization of fermion fields in Clifford and in Grassmann space, Spinor representations in Clifford and in Grassmann space, Kaluza-Klein-like theories, Higher dimensional spaces, Beyond the standard model

4.1 Introduction

It is demonstrated in this paper how does the Grassmann algebra — in Part I — and the two kinds of the Clifford algebras — in Part II — take care of the second quantization of fermions without postulating anticommutation relations [13].

⋆ Talk presented by N.S. Mankoč Borštnik

In d-dimensional Grassmann space of anticommuting coordinates θ^a's, $i = (0, 1, 2, 3, 5, \cdots, d)$, there are 2^d operators ("vectors"), which are superposition of products of θ^a. One can arrange them into irreducible representations with respect to the Lorentz group. There are as well derivatives with respect to θ^a's, $\frac{\partial}{\partial\theta_a}$'s, which are Hermitian conjugated to θ^a's [3], ($\theta^{a\dagger} = \eta^{aa}\frac{\partial}{\partial\theta_a}$, $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$, which again form 2^d operators ("vectors"). Grassmann space offers correspondingly $2 \cdot 2^d$ degrees of freedom.

There are two kinds of the Clifford operators ("vectors"), which are expressible with θ^a and $\frac{\partial}{\partial\theta_a}$ — $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})$ [2,4,5]. Each of these two kinds of the Clifford algebra objects has 2^d operators ("vectors"), together again $2 \cdot 2^d$ degrees of freedom. The Grassmann and each of the two Clifford algebras split into odd and even part with respect to the odd and even number of θ^a's, $\frac{\partial}{\partial\theta_a}$'s, γ^a's, $\tilde{\gamma}^a$'s. There is the odd algebra in all three cases which fulfills the second quantized anticommutation relations without postulating them [13].

We present in Sect. 4.2 properties of the Grassmann odd anticommuting algebra and even commuting algebra of the corresponding creation and annihilation operators representing the second quantized "fermion" fields, manifesting in the Grassmann case an integer spin, and offering in d-dimensional space, $d > (3 + 1)$, the description of the corresponding charges in adjoint representations. We follow in this paper to some extent the Ref. [3].

In Part II we present in equivalent section properties of the two kinds of the Clifford algebras and discuss conditions under which operators of the two Clifford algebras demonstrate the anticommutation relations required for the second quantized fermion fields, this way with the half integer spin, offering in d-dimensional space, $d \geq (3 + 1)$, the description of charges, as well as the appearance of families of fermions [3], both needed to describe the properties of the observed quarks and leptons and antiquarks and antileptons, explaining the appearance of families.

In Sect. 4.3 we comment what we have learned from the second quantized "fermion" fields with integer spin when internal degrees of freedom is described in Grassmann space and compare these recognitions with the recognitions, which the Clifford algebra is offering, discussions on which appears in Part II. We discuss as well a possible action for such an integer spin "fermions" and the corresponding equations of motion, both taken from [3], which are needed that the theory would have any prediction power.

The Clifford algebra offers in even d-dimensional spaces, $d \geq (13 + 1)$ indeed, the description of the internal degrees of freedom for the second quantized fermions with the half integer spins, explaining all the assumptions of the *standard model*: The appearance of charges of the observed quarks and leptons and their families, as well as the appearance of the dark matter, of the matter/antimatter asymmetry, offering several predictions [1,2,6–12].

4.2 Second quantized "fermions" in Grassmann space

In Grassmann d-dimensional space there are d anticommuting operators θ^{a_i}, $\{\theta^a, \theta^b\}_+ = 0$, $a = (0, 1, 2, 3, 5, .., d)$, and d anticommuting derivatives with respect to θ^a, $\frac{\partial}{\partial\theta_a}$, $\{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0$, offering together $2\cdot 2^d$ operators, the half of which are superposition of products of θ^a and another half corresponding superposition of $\frac{\partial}{\partial\theta_a}$.

$$\{\theta^a, \theta^b\}_+ = 0\,, \quad \{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0\,,$$
$$\{\theta_a, \frac{\partial}{\partial\theta_b}\}_+ = \delta_{ab}\,, (a,b) = (0,1,2,3,5,\cdots,d)\,. \tag{4.1}$$

Defining [3]

$$(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}\,,$$
it follows
$$(\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a\,, \tag{4.2}$$

The signature $\eta^{ab} = diag\{1,-1,-1,\cdots,-1\}$ is assumed.

One can arrange products of θ^a into 2^d irreducible representations with respect to the Lorentz group with the generators [2]

$$\mathbf{S}^{ab} = i\,(\theta^a\frac{\partial}{\partial\theta_b} - \theta^b\frac{\partial}{\partial\theta_a})\,, \quad (\mathbf{S}^{ab})^\dagger = \eta^{aa}\eta^{ab}\mathbf{S}^{ab}\,. \tag{4.3}$$

Half of the representations have an odd Grassmann character, those which are superposition of odd products of θ^a and half have an even Grassmann character, those which are superposition of even products of θ^a.

Since $\mathbf{S}^{ab}$ do not change the character of operators (”vectors”), that is the oddness and evenness of operators, all the members of one irreducible representation have the same Grassmann character. Different representations, either even or odd, are not reachable by $\mathbf{S}^{ab}$.

The Hermitian conjugated 2^d representations are reachable, due to Eq. (4.2), from the 2^d representations of θ^a’s.

It is useful to make a choice of the Cartan subalgebra of the commuting operators of the Lorentz algebra. We make the ordinary choice

$$\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\;d}\,, \tag{4.4}$$

and choose the irreducible representations of the Lorentz group to be the ”eigenvectors” of the Cartan subalgebra.

$$\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b) = k\,\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\,,$$
$$\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\theta^a\theta^b) = 0\,. \tag{4.5}$$

Let us point out that the Grassmann ”vectors” have an integer spin. Making a choice of $\eta^{aa} = 1,-1,-1,\ldots,-1$, the eigenvectors of $\mathbf{S}^{03}$, $\frac{1}{\sqrt{2}}\,(\theta^0 \mp \theta^3)$, have $k = \pm i$, respectively, all the others have $k = \pm 1$.

"Vectors" are normalized, up to a phase, in accordance with Eq. (4.21) of App. 4.4. Lorentz transformations change the Cartan subalgebra, correspondingly also the "eigenvectors" of the Cartan subalgebra change, since the choice of the Cartan subalgebra depends on the Lorentz frame.

The Hermitian conjugated representations of (odd and even) products of θ^a are obtainable according to Eq. (4.2).

$$\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)^\dagger = \eta^{aa}\,\frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta_a} + \frac{\eta^{aa}}{i(-k)}\frac{\partial}{\partial\theta_b})\,,$$
$$\frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\,\theta^a\theta^b)^\dagger = \frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\,\frac{\partial}{\partial\theta_a}\,\frac{\partial}{\partial\theta_b})\,. \tag{4.6}$$

4.2.1 Properties of Grassmann "vectors"

2^{d-1} odd and 2^{d-1} even Grassmann operators, which are superposition of odd and even products of θ^a's are well separated from their 2^{d-1} odd and 2^{d-1} even Hermitian conjugated operators, which are superposition of odd and even products of $\frac{\partial}{\partial\theta_a}$'s, Eq. (4.6) [1].

To make discussions concrete let us start with illustrating properties of the representations in Grassmann space in $d = (5+1)$-dimensional space. Table 4.1 represents two decuplets, which are "egenvectors" of the Cartan subalgbra ($\mathbf{S}^{03}$, $\mathbf{S}^{12}$, $\mathbf{S}^{5,6}$), Eq. (4.4), of the Lorentz algebra $\mathbf{S}^{ab}$. The two decouplets represent two Grassmann odd irreducible representations of $SO(5,1)$.

One can read on the same table, from the first to the third and from the fourth to the sixth line of both decuplets, two Grassmann even triplet representations of $SO(3,1)$, if paying attention on the "eigenvectors" of $\mathbf{S}^{03}$ and $\mathbf{S}^{12}$ alone, while the "eigenvactor" of $\mathbf{S}^{56}$ has, as a "spectator", the "eigenvalue" either $+1$ (the first triplet in both decouplets) or -1 (the second triplet in both decuplets). Each of the two decuplets contains also one fourplet $((7^{th}, 8^{th}, 9^{th}, 10^{th})$ lines in each of the two decuplets (Table II in Ref. [2])).

Paying attention on the eigenvectors of $\mathbf{S}^{03}$ alone one recognizes as well even and odd representations of $SO(1,1)$: $\theta^0\theta^3$ (Table II in Ref. [2] includes instead $1 \pm \theta^0\theta^3$) and $\theta^0 \pm \theta^3$, respectively.

The Hermitian conjugated "vectors" follow by using Eq. (4.6) and is for the first "vector" of Table 4.1 equal to $(-)^2(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})\,(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})\,(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$. One correspondingly finds that when $(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})\,(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})\,(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$ applies on $(\frac{1}{\sqrt{2}})^3(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)$ the result is identity. Application of $(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})\,(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})\,(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$ on all the rest of "vectors" of the decuplet I as well as on all the "vectors" of the decuplet II gives zero. "Vectors" are orthonormalized with respect to Eq. (4.21). Let us notice that $\frac{\partial}{\partial\theta_a}$ on a "state"

[1] Relations among operators and their Hermitian conjugated partners in both kinds of the Clifford algebra objects are more complicated than in the Grassmann case. In the Grassmann case Hermitian conjugated operators follow by taking into account Eq. (4.2). In the Clifford case $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)^\dagger$ is proportional to $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{i(-k)}\gamma^b)$, while $\frac{1}{\sqrt{2}}(1 + \frac{i}{k}\gamma^a\gamma^b)$ are self adjoint. This is the case also for representations in the sector of $\tilde{\gamma}^a$'s.

I	i	decuplet of "eigenvectors"	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	$\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	i	1	1	1	1
	2	$\frac{1}{\sqrt{2}}(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1	1	1
	3	$\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	$-i$	-1	1	1	1
	4	$\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	i	-1	-1	1	-1
	5	$\frac{1}{\sqrt{2}}(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1	1	-1
	6	$\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	$-i$	1	-1	1	-1
	7	$\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	i	0	0	1	0
	8	$\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	$-i$	0	0	1	0
	9	$\frac{1}{\sqrt{2}}(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0	1	0
	10	$\frac{1}{\sqrt{2}}(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0	1	0
II	i	decuplet of "eigenvectors"	S^{03}	S^{12}	S^{56}	$\gamma^{(5+1)}$	$\gamma^{(3+1)}$
	1	$\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	$-i$	1	1	-1	-1
	2	$\frac{1}{\sqrt{2}}(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1	-1	-1
	3	$\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	i	-1	1	-1	-1
	4	$\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	$-i$	-1	-1	-1	1
	5	$\frac{1}{\sqrt{2}}(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1	-1	1
	6	$\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	i	1	-1	-1	1
	7	$\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	$-i$	0	0	-1	0
	8	$\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	i	0	0	-1	0
	9	$\frac{1}{\sqrt{2}}(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0	-1	0
	10	$\frac{1}{\sqrt{2}}(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0	-1	0

Table 4.1. The two decouplets, the largest odd "eigenvectors" of the Cartan subalgebra, Eq. (4.4), ($\mathbf{S}^{03}, \mathbf{S}^{12}$, $\mathbf{S}^{56}$, for $SO(5,1)$) of the Lorentz algebra in Grassmann $(5+1)$-dimensional space, forming two irreducible representations, are presented. Table is partly taken from Ref. [3]. "Vectors" within each decuplet are reachable from any member by $\mathbf{S}^{ab}$'s and are decoupled from another decouplet. The two operators of handedness, $\Gamma^{(d-1)+1}$ for $d=(5,4)$ are invariants of the Lorentz algebra, Eq. (4.23).

which is just an identity, $|\, I >$, gives zero, $\frac{\partial}{\partial\theta_a} |\, I >= 0$, while $\theta^a |\, I >$, or any superposition of products of θ^a's applied on $|\, I >$, gives the "vector" back.

The two by $\mathbf{S}^{ab}$ decoupled Grassmann decouplets of Table 4.1 are the largest two irreducible representations of odd products of θ^a's. There are 12 additional Grassmann odd "vectors", arranged into irreducible representation, $(\frac{1}{2}\,(\theta^0 \mp \theta^3)(1 \pm \theta^1\theta^2\theta^5\theta^6)$, $\frac{1}{2}\,(\theta^1 \pm i\theta^2)(1 \pm \theta^0\theta^3\theta^5\theta^6)$, $\frac{1}{2}\,(\theta^5 \pm i\theta^6)(1 \pm \theta^0\theta^3\theta^1\theta^2))$.

And there are 32 Grassmann "vectors" arranged into irreducible representations, which are superposition of even products of θ^a's.

4.2.2 Second quantized "Grassmann fermions" and bosons

It is not difficult to see that Grassmann "vectors" of an odd Grassmann character — odd products of superposition of θ^a's — anticommute among themselves and so do odd products of superposition of $\frac{\partial}{\partial\theta^a}$'s, while equivalent even products commute.

Defining the vacuum state in the Grassmann case as $|1>$ [3] [2], one easily sees that application of products of superposition of θ^a's on $|1>$ gives nonzero

[2] We shall see in Part II that the vacuum states are for both kinds of the Clifford algebra objects, γ^a's and $\tilde{\gamma}^a$'s, the sums of products of projectors.

contribution, while application of products of superposition of $\frac{\partial}{\partial\theta^a}$'s on $|1>$ gives zero.

Application of products of superposition of $\frac{\partial}{\partial\theta^a}$'s on the corresponding Hermitian conjugated partners, which are products of superposition of θ^a's, leads to identity for either even or odd Grassmann character [3].

All these algebras of an odd character, the Grassmann one and the Clifford two, offer the description of the anticommuting second quantized fields, as postulated by Dirac. But the Grassmann "fermions" carry the integer spins, while the observed fermions — quarks and leptons — carry half integer spin.

a. *Grassmann anticommuting "vectors" with integer spins*

Let us first study properties of Grassmann odd "vectors".

Let us use in $d = 2(2n+1)$, n is a positive integer, for the starting Grassmann odd "vector" — in $d = (5+1)$ this is the first "vector" on Table 4.1 — the notation

$$\hat{b}^{\theta 1\dagger}_1 := (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^0 \pm \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)\,,$$
$$(\hat{b}^{\theta 1\dagger}_1)^\dagger = \hat{b}^{\theta 1}_1 = (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^d})\cdots(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3})\,. \tag{4.7}$$

$\hat{b}^{\theta 1}_1$ is the Hermitian conjugate $(\hat{b}^{\theta 1\dagger}_1)^\dagger$.

In the case of $d = 4n$, n is a positive integer, the starting Grassmann odd "vectors" of one Lorentz irreducible representation, and correspondingly the creation operator must be of the kind

$$b^{\theta 1\dagger}_1 := (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^d \tag{4.8}$$

All the rest of "vectors" belonging to the same irreducible representation follow by the application of $\mathbf{S}^{ab}$. We denote them by $\hat{b}^{\theta k\dagger}_1$ and their Hermitian conjugated partners by $\hat{b}^{\theta k}_1$.

Let those "vectors" belonging to different irreducible representations be denoted by $\hat{b}^{\theta k\dagger}_j$ and their Hermitian conjugated partners by $\hat{b}^{\theta k}_j = (\hat{b}^{\theta k\dagger}_j)^\dagger$.

From Sect. 4.2.1 we derive

$$\begin{aligned}
\{\hat{b}^{\theta k}_i, \hat{b}^{\theta l\dagger}_j\}_+ |\,1> &= \delta_{ij}\,\delta^{kl}\,|\,1>\,,\\
\{\hat{b}^{\theta k}_i, \hat{b}^{\theta l}_j\}_+ |\,1> &= 0\,|\,1>\,,\\
\{\hat{b}^{\theta k\dagger}_i, \hat{b}^{\theta l\dagger}_j\}_+\,|\,1> &= 0\,|\,1>\,,\\
\hat{b}^{\theta k}_j\,|\,1> &= 0\,|\,1>\;.
\end{aligned} \tag{4.9}$$

[3] The Clifford case requires more detailed analyses, as we shall see in Part II: Clifford odd "vectors" of each of the two Clifford algebras anticommute with all the members of the same irreducible representation and so do anticommute among themselves their Hermitian conjugated partners. One must, however, introduce the family quantum numbers in order that anticommutator of a "vector" only with its Hermitian conjugated parter gives a nonzero contribution.

These anticommutation relations are just the relations among creation and annihilation operators required by Dirac [13] for fermions. Fermion states correspondingly follow by the application off creation operators on the vacuum state $\mid 1>$.

$$|\phi^{k}_{ib}> = \hat{b}^{\theta k\dagger}_{i} \mid 1> \tag{4.10}$$

But Grassmann "fermions" have an integer spin — this follows from Eq. (4.5), and is demonstrated on Table 4.1 — and not half integer spin as it is the case for the so far observed fermions.

b. *Grassmann commuting "vectors" with integer spins*

Grassmann even "vectors" commute, and not anticommute as it is the case for the Grassmann odd "vectors". Let us use in the Grassmann even case, that is in the case of even number of θ^{a}'s, and correspondingly of the commuting "vectors", in $d = 2(2n+1)$ the notation

$$\hat{a}^{\theta 1\dagger}_{j} = (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^{0}-\theta^{3})(\theta^{1}+i\theta^{2})(\theta^{5}+i\theta^{6})\cdots(\theta^{d-3}+i\theta^{d-2})\theta^{d-1}\theta^{d} \tag{4.11}$$

Again the rest of "vectors", belonging to the same Lorentz irreducible representation, follow by the application of $\mathbf{S}^{ab}$. The Hermitian conjugated partner of $\hat{a}^{\theta 1\dagger}_{1}$ is $\hat{a}^{\theta 1}_{1} = (\hat{a}^{\theta 1\dagger}_{1})^{\dagger}$

$$\hat{a}^{\theta 1}_{1} = (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,\frac{\partial}{\partial\theta^{d}}\frac{\partial}{\partial\theta^{d-1}}(\frac{\partial}{\partial\theta^{d-3}} - i\frac{\partial}{\partial\theta^{d-2}})\cdots(\frac{\partial}{\partial\theta^{0}} - \frac{\partial}{\partial\theta^{3}})\,. \tag{4.12}$$

Let us noticed, that the "vector" identity, 1*, is not allowed, since the Hermitian conjugated "vector" of the identity is the identity back.* Then the last requirement of Eq.(4.9) for the commutation relations in the case of Grassmann even "vectors", instead of the anticommutation relations in the case of Grassmann odd "vectors", presented in Eq. (4.9), could not be fulfilled.

If $\hat{a}^{\theta k}_{j}$ represents a Grassmann even operator, then one obtains, with the index j denoting different irreducible representations and the index k denoting a particular member of the j^{th} irreducible representations, taking into account Sect. 4.2.1, the relations

$$\begin{aligned}
\{\hat{a}^{\theta k}_{i}, \hat{a}^{\theta k'\dagger}_{j}\}_{-} \mid 1> &= \delta_{ij}\;\delta_{kl} \mid 1>\,,\\
\{\hat{a}^{\theta k}_{i}, \hat{a}^{\theta l}_{j}\}_{-} \mid 1> &= 0 \mid 1>\,,\\
\{\hat{a}^{\theta k\dagger}_{i}, \hat{a}^{\theta k'\dagger}_{j}\}_{-} \mid 1> &= 0 \mid 1>\,,\\
\hat{a}^{\theta k}_{j} \mid 1> &= 0 \mid 1>\,,\\
\hat{a}^{\theta k\dagger}_{i} \mid 1> &= |\phi^{k}_{ia}>\;.
\end{aligned} \tag{4.13}$$

c. *Action for free massless Grassmann "fermions" with integer spin* [3]

To obtain the equations of motion for at least noninteracting Grassmann massless "fermions" the corresponding Lorentz invariant action for a free massless "fermions" must be proposed. We follow here the suggestion from Ref. [3].

$$\mathcal{A}_{G} = \int d^{d}x\; d^{d}\theta\;\omega\{\phi^{\dagger}(1-2\theta^{0}\frac{\partial}{\partial\theta^{0}})\,\frac{1}{2}\,\theta^{a}p_{a}\phi\} + h.c.\,. \tag{4.14}$$

We use the integral over θ^a coordinates with the weight function ω from Eq. (4.21, 4.22). Requiring the Lorentz invariance we add after $\phi^\dagger$ the operator γ^0_G (γ^a_G $= (1 - 2\theta^a \frac{\partial}{\partial\theta^a})$), which takes care of the Lorentz invariance. Namely

$$
\begin{aligned}
\mathbf{S}^{ab\dagger}\,(1 - 2\theta^0 \frac{\partial}{\partial\theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial\theta^0})\,\mathbf{S}^{ab}\,,\\
\mathbf{S}^{\dagger}\,(1 - 2\theta^0 \frac{\partial}{\partial\theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial\theta^0})\,\mathbf{S}^{-1}\,,\\
\mathbf{S} &= e^{-\frac{i}{2}\omega_{ab}(L^{ab}+\mathbf{S}^{ab})}\,,
\end{aligned}
\tag{4.15}
$$

while θ^a, $\frac{\partial}{\partial\theta_a}$ and p^a transform as Lorentz vectors. The equations of motion follow from the action, Eq. (4.14),

$$
\frac{1}{2}\gamma^0_G\,(\theta^a - \frac{\partial}{\partial\theta^a})\,p_a\,|\phi> \,= 0\,,
\tag{4.16}
$$

as well as the Klein-Gordon equation, $\gamma^0_G\,(\theta^a - \frac{\partial}{\partial\theta^a})\,p_a\,\gamma^0_G\,(\theta^b - \frac{\partial}{\partial\theta^a})\,p_b\,|\phi>= 0$, leading to

$$
\{\theta^a p_a, \frac{\partial}{\partial\theta_b}\,p_b\}_+ \,= p^a p_a = 0\,.
\tag{4.17}
$$

From the Lagrange density, presented in Eq. (4.14), using Eq. (4.2), and the relations $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})$, $\gamma^0_G = -i\eta^{aa}\gamma^a\tilde{\gamma}^a$, it follows, up to the surface term,

$$
\begin{aligned}
\mathcal{L}_G &= -i\frac{1}{2}\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,(\hat{p}_a\phi)\\
&= -i\frac{1}{4}\{\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,\hat{p}_a\phi\, - \hat{p}_a\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,\phi\,\}.
\end{aligned}
\tag{4.18}
$$

One correspondingly finds equations of motion

$$
\begin{aligned}
\frac{\partial\mathcal{L}_G}{\partial\phi^\dagger} - \hat{p}_a\,\frac{\partial\mathcal{L}_G}{\partial\hat{p}_a\phi^\dagger} &= 0 = \frac{-i}{2}\gamma^0_G\,\tilde{\gamma}^a\,\hat{p}_a\,\phi\,,\\
\frac{\partial\mathcal{L}_G}{\partial\phi} - \hat{p}_a\,\frac{\partial\mathcal{L}_G}{\partial(\hat{p}_a\phi)} &= 0 = \frac{i}{2}\hat{p}_a\,\phi^\dagger\gamma^0_G\,\tilde{\gamma}^a\,,
\end{aligned}
\tag{4.19}
$$

The eigenstates of Eq. (4.16, 4.19) for free massless "fermions" are superposition of states $|\phi^k_i>$, describing their internal degrees of freedom, with coefficients depending on momentum p^a, $a = (0, 1, 2, 3, 5, \ldots, d)$ of the plane wave solution $e^{-ip_a x^a}$

$$
|\phi^k_{sp}> = \sum_i c^k{}_{spi}\,\hat{b}^{\theta k\dagger}_i\,|\,1>e^{-ip_a x^a}\,,
\tag{4.20}
$$

with s representing different solutions of the equations of motion, and, since they are orthogonalized, they fulfill the relation $<\phi^k_{sp}|\phi^{k'}_{s'p'}>= \delta_{kk'}\,\delta_{ss'}\,\delta^{pp'}$, where we assumed the discretization of momenta.

One of the plane wave massless solutions of these equations, in $d = (5+1)$, for $p^a = (p^0, p^1, p^2, p^3, 0, 0)$, the positive energy $p^0 = |p^0|$, the spin $\frac{1}{2}$ and the charge $\frac{1}{2}$, from the point of view of $d = (3+1)$, for example, is $\hat{b}^{\theta\dagger}_{\frac{1}{2},\frac{1}{2}}(\vec{p}) = \beta\{(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1+i\theta^2) - \frac{2(|p^0|-|p^3|)}{p^1-ip^2}(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^1\theta^2)\}\overset{56}{(+)})\, e^{-i(|p^0|x^0-\vec{p}\cdot\vec{x})}$, β is the normalization factor.

The corresponding state follows by the application of the creation operator $\hat{b}^{\theta\dagger}_{\frac{1}{2},\frac{1}{2}}(\vec{p})$ on the vacuum state $|\,1>$, $|\phi_{\frac{1}{2},\frac{1}{2}}> = \hat{b}^{\theta\dagger}_{\frac{1}{2},\frac{1}{2}}(\vec{p})\,|\,1>$. More solutions can be found in [3] and the references therein.

4.3 Conclusions

We learn in this paper, in Part I, that products of superposition of θ^a's, Eqs. (4.7, 4.5), exist, which together with their Hermitian conjugated partners, Eqs. (4.7, 4.6), fulfill all the requirements for the anticommutation relations for Dirac fermions. No postulation of anticommutation relations is needed. If using products of superposition of θ^a's as creation operators to describe the internal degrees of freedom of "Grassmann fermions", these "fermions" carry the integer spin, and in spaces $d \geq 5$ the corresponding charges belong to adjoint representations. No families appear in this case, that means that there is no available operators, which would connect different irreducible representations of the Lorentz group (without breaking symmetries).

The presented Lorentz invariant action leads to the equations of motion for free massless "Grassmann fermions" [3].

No elementary fermions with these properties have been observed. The interaction of such "Grassmann fermions" [3] with the corresponding gauge fields could tell more about the possibility whether or not these "Grassmann fermions" exist in nature, not yet observed.

In Part II two kinds of operators are studied; There are namely two kinds of the Clifford algebra objects, $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})$, which anticommute, $\{\gamma^a, \tilde{\gamma}^a\}_+ = 0$, and correspondingly form two kinds of independent representations.

Each of these two kinds of independent representations can be arranged into irreducible representations with respect to the two Lorentz generators — $S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$. All the Clifford irreducible representations of any of the two kinds of algebras are independent and completely disconnected.

The Dirac action in d-dimensional space for free massless fermions — $\mathcal{A} = \int d^dx\, \frac{1}{2}(\psi^\dagger\gamma^0\,\gamma^a\,p_a\psi) + h.c.$ (or $\mathcal{A} = \int d^dx\, \frac{1}{2}(\psi^\dagger\tilde{\gamma}^0\,\tilde{\gamma}^a\,p_a\psi) + h.c.$) — leads to equations of motions, which have the solutions in both kinds of algebras for either even or odd Clifford character, that is for an even or odd products of the superposition of γ^a in one kind and $\tilde{\gamma}^a$ in another kind of the Clifford algebra objects.

Although the "vectors" of one irreducible representation of an odd Clifford algebra character, anticommute among themselves and so do their Hermitian conjugated partners in each of the two kinds of the Clifford algebras, the anticommutation relations among creation and annihilation operators in each of the two

Clifford algebras separately, do not fulfill the requirement, that only the Hermitian conjugated partner of the creation operator gives nonzero contribution.

The decision, the postulate, that only one kind of the Clifford algebra objects — let say γ^a — is used to describe the internal space of fermions, while the second kind — $\tilde{\gamma}^a$ in this case — which does not contribute to description of the internal space of fermions, determines quantum numbers of the irreducible representations of the S^{ab}, solves both problems: **a.** Different irreducible representations with respect to S^{ab} carry now different "family" quantum numbers determined by $\frac{d}{2}$ commuting operators among $\tilde{S}^{ab}$. **b.** Creation operators and their Hermitian conjugated partners, which are odd products of superpositions of γ^a, fulfill all the requirements which Dirac postulated for fermions.

4.4 APPENDIX: Norms in Grassmann space and Clifford space

Let us define the integral over the Grassmann space [2] of two functions of the Grassmann coordinates $< \mathbf{B}|\theta >< \mathbf{C}|\theta >$, $< \mathbf{B}|\theta >=< \theta|\mathbf{B} >^{\dagger}$, $< \mathbf{b}|\theta >= \sum_{k=0}^{d} b_{a_1\dots a_k}\theta^{a_1}\cdots\theta^{a_k}$, by requiring

$$\{d\theta^a, \theta^b\}_+ = 0\,, \quad \int d\theta^a = 0\,, \quad \int d\theta^a \theta^a = 1\,, \quad \int d^d\theta \;\theta^0\theta^1\cdots\theta^d = 1\,,$$
$$d^d\theta = d\theta^d \dots d\theta^0\,, \quad \omega = \Pi_{k=0}^{d}(\frac{\partial}{\partial\theta_k} + \theta^k)\,, \tag{4.21}$$

with $\frac{\partial}{\partial\theta_a}\theta^c = \eta^{ac}$. We shall use the weight function [2] $\omega = \Pi_{k=0}^{d}(\frac{\partial}{\partial\theta_k} + \theta^k)$ to define the scalar product in Grassmann space $< \mathbf{B}|\mathbf{C} >$

$$< \mathbf{B}|\mathbf{C} > = \int d^d\theta^a \;\omega < \mathbf{B}|\theta >< \theta|\mathbf{C} >= \sum_{k=0}^{d}\int b^*_{b_1\dots b_k} c_{b_1\dots b_k}\,. \tag{4.22}$$

To define norms in Clifford space Eq. (4.21) can be used as well.

4.5 APPENDIX: Handedness in Grassmann and Clifford space

The handedness $\Gamma^{(d)}$ is one of the invariants of the group SOd, with the infinitesimal generators of the Lorentz group S^{ab}, defined as

$$\Gamma^{(d)} = \alpha\varepsilon_{a_1 a_2\dots a_{d-1}\, a_d}\, S^{a_1 a_2}\cdot S^{a_3 a_4}\cdots S^{a_{d-1} a_d}\,, \tag{4.23}$$

with α, which is chosen so that $\Gamma^{(d)} = \pm 1$.

In the Grassmann case S^{ab} is defind in Eq. (4.3), while in the Clifford case Eq. (4.23) simplifies, if we take into account that $S^{ab}|_{a\neq b} = \frac{i}{2}\gamma^a\gamma^b$ and $\tilde{S}^{ab}|_{a\neq b} = \frac{i}{2}\tilde{\gamma}^a\tilde{\gamma}^b$, as follows

$$\Gamma^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}}\gamma^a), \quad \text{if} \quad d = 2n,$$
$$\Gamma^{(d)} := (i)^{(d-1)/2} \prod_a (\sqrt{\eta^{aa}}\gamma^a), \quad \text{if} \quad d = 2n+1\,. \tag{4.24}$$

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