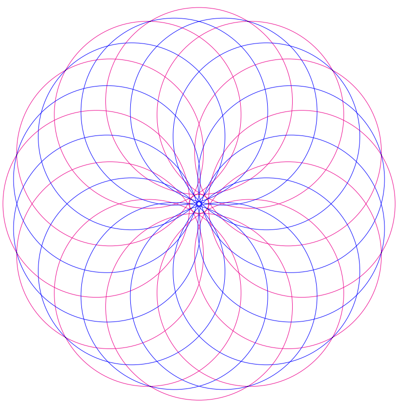
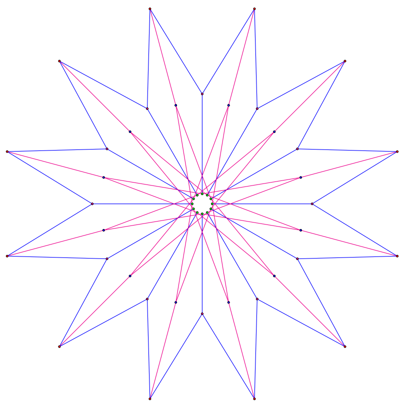
THE ART OF DISCRETE AND
APPLIED MATHEMATICS

The ADAM graph and its configuration[1]

It is well-known that exactly seven of the generalised Petersen graphs are symmetric (= arc-transitive), namely the following:

- $G(4,1)$ – the cube graph,
- $G(5,2)$ – the Petersen graph,
- $G(8,3)$ – the Möbius-Kantor graph,
- $G(10,2)$ – the dodecahedron graph,
- $G(10,3)$ – the Desargues graph,
- $G(12,5)$ – the Nauru graph, and
- $G(24,5)$ – the graph that we hereby name *the ADAM graph*.

Both $G(8,3)$ and $G(10,3)$ are associated with point-line configurations: $G(8,3)$ is the Levi graph (= incidence graph) of the Möbius-Kantor (8_3) configuration, while $G(10,3)$ is the Levi graph if the Desargues (10_3) configuration. A point-circle configuration is called an isometric configuration if all circles have the same radius, and a graph drawn in the plane is called unit-distance graph if all straight edges have the same length.

The above figures depict an isometric point-circle configuration (24_3) on the left, whose Levi graph is the generalised Petersen graph $G(24,5)$ drawn as the unit-distance graph on the right. The central detail has been adopted as the logo of our new journal, *The Art of Discrete and Applied Mathematics*, and because its abbreviation is ADAM, we propose that the generalised Petersen graph $G(24,5)$ and the corresponding (24_3) configuration be called respectively the *ADAM graph* and the *ADAM configuration*.

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The k-independence number of graph products[*]

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Abstract

The concept of k-independent number is a natural generalization of classical independence number. A *k-independent set* is a set of vertices whose induced subgraph has maximum degree at most k. The *k-independence number* of G, denoted by $\alpha_k(G)$, is defined as the maximum cardinality of a k-independent set of G. In this paper, we study the k-independence number on the lexicographical, strong, Cartesian and direct product and present several upper and lower bounds for these products of graphs.

Keywords: Independence number, k-independent set, k-independence number, lexicographical product, strong product, Cartesian product, direct product.

Math. Subj. Class.: 05C69, 05C76

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1 Introduction

Graphs considered in this paper are undirected, finite and simple. We refer to [1] for undefined notations and terminology. In particular, we use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and minimum degree of a graph G, respectively. If $X \subseteq V(G)$ or $X \subseteq E(G)$, then $G[X]$ is the subgraph of G induced by X. For two subsets X and Y of $V(G)$ we denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other end in Y.

Independence number is one of the most basic concepts in graph theory. A subset $S \subseteq V(G)$ is said to be *independent* if $E(G[S]) = \emptyset$. The *independence number* of G denoted by $\alpha(G)$ is the size of a maximum independent set in G. In [6, 7], Fink and Jacobson generalized the concept of independent set. In this paper, k will be an integer. We say that a subset S of V is *k-independent* if $\Delta(G[S]) \leq k$, that is, the maximum degree of the subgraph induced by the vertices of S is less or equal to k. The *k-independence number*, denoted $\alpha_k(G)$, as the maximum cardinality of a k-independent set. Thus for $k = 0$, the 0-independent is the classical independent set. Every k-independent set is $(k+1)$-independent; so $\alpha_{k+1}(G) \geq \alpha_k(G)$ for a graph G. Moreover, the vertex set V is the only maximal Δ-independent but is not a $(\Delta - 1)$-independent set. Thus every graph G satisfies

$$\alpha(G) = \alpha_0(G) \leq \alpha_1(G) \leq \alpha_2(G) \leq \cdots \leq \alpha_{\Delta-1}(G) < \alpha_\Delta(G) = n.$$

For k-independent set and k-independence number, Chellali, Favaron, Hansberg, and Volkmann published a survey paper on this subject; see [3]. We must mention that the k-independence number of G is defined as the size of a largest k-colorable subgraph of G in [17].

In graph theory, Cartesian product, strong product, lexicographical product, and direct product are four of main products, each with its own set of applications and theoretical interpretations. Product networks were proposed based upon the idea of using the cross product as a tool for "combining" two known graphs with established properties to obtain a new one that inherits properties from both [5]. For more details on graph products, we refer to the book [10].

- The *Cartesian product* of two graphs G and H, written as $G \,\square\, H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$.
- The *lexicographic product* $G \circ H$ of graphs G and H has the vertex set $V(G \circ H) = V(G) \times V(H)$. Two vertices $(u, v), (u', v')$ are adjacent if $uu' \in E(G)$, or if $u = u'$ and $vv' \in E(H)$.
- The *strong product* $G \boxtimes H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (u, v) and (u', v') are adjacent whenever $uu' \in E(G)$ and $v = v'$, or $u = u'$ and $vv' \in E(H)$, or $uu' \in E(G)$ and $vv' \in E(H)$.
- The *direct product* $G \times H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (u, v) and (u', v') are adjacent if the projections on both coordinates are adjacent, i.e., $uu' \in E(G)$ and $vv' \in E(H)$.

Note that unlike the other three products, the lexicographic product is a non-commutative product since $G \circ H$ is usually not isomorphic to $H \circ G$.

For the independence number of Cartesian product graphs, Vizing [16] observed:

Theorem 1.1 ([10, 16]). *For any graphs G and H,*

(i) $\alpha(G\Box H) \leq \min\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$;

(ii) $\alpha(G\Box H) \geq \alpha(G)\alpha(H) + \min\{|V(G)| - \alpha(G), |V(H)| - \alpha(H)\}$.

Geller and Stahl [9] obtained the following result for the independence number of lexicographical product graphs.

Theorem 1.2 ([9]). *For any graphs G and H, $\alpha(G \circ H) = \alpha(G)\alpha(H)$.*

The following result is immediate, since $G \boxtimes H$ is a subgraph of $G \circ H$.

Corollary 1.3 ([10]). *For any graphs G and H, $\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H)$.*

In 2011, Špacapan [17] proved the following theorem.

Theorem 1.4 ([17]). *For any graph G and H,*

(i) $\alpha(G \times H) \geq \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$;

(ii) $\alpha(G \times H) \leq \alpha(H)|V(G)| + \alpha(G)|V(H)| - \alpha(H)\alpha(G)$.

For the independence number of four graph products, Jha and Slutzki obtained the following relation in 1994.

Theorem 1.5 ([12]). *For any graphs G and H,*

$$\alpha(G \circ H) \leq \alpha(G \boxtimes H) \leq \alpha(G\Box H) \leq \alpha(G \times H).$$

In this paper, we consider four standard products: the lexicographic, the strong, the Cartesian and the direct with respect to the k-independence number. Every of these four products will be treated in one of the forthcoming subsections in Section 2. Our results can be seen as extensions of Theorems 1.1, 1.2, 1.4, 1.5 and Corollary 1.3.

2 Main results

In this section, let G and H be two connected graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Then $V(G*H) = \{(u_i, v_j) \,|\, 1 \leq i \leq n,\ 1 \leq j \leq m\}$, where $*$ denotes lexicographic product operation, strong product operation, Cartesian product operation or direct product operation. For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u_i, v) \,|\, 1 \leq i \leq n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u, v_j) \,|\, 1 \leq j \leq m\}$.

2.1 The lexicographic product

In this subsection, we give upper and lower bounds of $\alpha_k(G \circ H)$.

Theorem 2.1. *(i) Let $k \geq 0$ be an integer. For graphs G and H,*

$$\alpha_k(G \circ H) \leq \alpha_k(H)|V(G)|.$$

(ii) Let $k, r \geq 0$ be two integers. Let H be a graph of order m. For graphs G and H,

$$\alpha_k(G \circ H) \geq \alpha_r(G)\alpha_{k-rm}(H)$$

where $\alpha_{k-rm}(H) = 0$ if $k \leq rm$.
Moreover, the bounds are sharp.

Proof. (i) Let I be a maximum k-independent set of $G \circ H$. We claim that $|I \cap V(H(u_i))| \leq \alpha_k(H(u_i))$ for each $u_i \in V(G)$. To see this, we observe that $H(u_i)[I \cap V(H(u_i))]$ is a subgraph of $G \circ H[I]$.

(ii) Let I be a maximum r-independent set of G, and J be a maximum $(k - rm)$-independent set of H. Set

$$I = \{u_i \,|\, 1 \leq i \leq s\} \text{ and } J = \{v_j \,|\, 1 \leq j \leq t\}.$$

For any $(u_i, v_j) \in I \times J$, we show that the degree of (u_i, v_j) in $G \circ H[I \times J]$ is at most k. Since I is a maximum r-independent set of G, it follows that $d_{G[I]}(u_i) \leq r$, where $u_i \in V(G[I])$. Similarly, since J is a maximum $(k - mr)$-independent set of H, it follows that $d_{H[J]}(v_j) \leq k - mr$, where $v_j \in V(H[J])$. Then

$$d_{G \circ H[I \times J]}(u_i, v_j) \leq d_{H[J]}(v_j) + m d_{G[I]}(u_i) \leq k - mr + mr = k,$$

and hence $I \times J$ is a k-independent set of $G \circ H$. So $\alpha_k(G \circ H) \geq \alpha_r(G)\alpha_{k-rm}(H)$.

See Remarks 2.4 and 2.5 for the sharpness. □

2.2 The strong product

In this subsection, we derive upper and lower bounds of $\alpha_k(G \boxtimes H)$.

Theorem 2.2. *(i) Let $k \geq 0$ be an integer. For graphs G and H,*

$$\alpha_k(G \boxtimes H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}.$$

(ii) Let $k, r \geq 0$ be two integers. For graphs G and H,

$$\alpha_k(G \boxtimes H) \geq \alpha_r(G)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(H)$$

and

$$\alpha_k(G \boxtimes H) \geq \alpha_r(H)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(G).$$

Moreover, the bounds are sharp.

Proof. (i) Let I be a maximum k-independent set of $G \boxtimes H$. If $|G(v_j) \cap I| > \alpha_k(G)$ for some $j \leq m$, then I is not a k-independent set in $G \boxtimes H$. It follows $\alpha_k(G \boxtimes H) \leq \alpha_k(G)|V(H)|$. From the symmetry, we have

$$\alpha_k(G \boxtimes H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}.$$

(ii) Let I be a maximum r-independent set of G, and J be a maximum $(\frac{k}{2r+1})$-independent set of H. Set

$$I = \{u_i \,|\, 1 \leq i \leq s\} \text{ and } J = \{v_j \,|\, 1 \leq j \leq t\}.$$

For any $(u_i, v_j) \in I \times J$, we show that the degree of (u_i, v_j) in $G \circ H[I \times J]$ is at most k. Since I is a maximum r-independent set of G, it follows that $d_{G[I]}(u_i) \leq r$, where $u_i \in V(G[I])$. Similarly, since J is a maximum $(\frac{k}{2r+1})$-independent set of H, it follows that $d_{H[J]}(v_j) \leq \frac{k}{2r+1}$, where $v_j \in V(H[J])$. Then

$$d_{G\boxtimes H[I\times J]}(u_i, v_j) \leq d_{H[J]}(v_j) + \frac{2k}{2r+1} d_{G[I]}(u_i) \leq \frac{k}{2r+1} + \frac{2rk}{2r+1} = k,$$

and hence $I \times J$ is a k-independent set of $G \boxtimes H$. So $\alpha_k(G \boxtimes H) \geq \alpha_r(G)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(H)$.

See Remarks 2.4 and 2.5 for the sharpness. □

2.3 The Cartesian product

Upper and lower bounds of $\alpha_k(G \,\square\, H)$ are derived in this subsection.

Theorem 2.3. *Let $k, r \geq 0$ be two integers. For graphs G and H,*

(i) $\alpha_k(G \,\square\, H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}$;

(ii) $\alpha_k(G \,\square\, H) \geq \alpha_r(G)\alpha_{k-r}(H)$

$$+ \begin{cases} st, & \text{if } k \geq s+t-2; \\ t(k-t+2), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor, t < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{and } k \leq s+t-3; \\ s(k-s+2), & \text{if } t \geq \lfloor \frac{k+3}{2} \rfloor, s < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{and } k \leq s+t-3; \\ \min\{p,q\}\left(\lceil \frac{k}{2} \rceil + 1\right)\left(\lfloor \frac{k}{2} \rfloor + 1\right), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor \text{ and } t \geq \lfloor \frac{k+3}{2} \rfloor, \end{cases}$$

where $0 \leq r \leq k$, $s = |V(G)| - \alpha_r(G)$, $t = |V(H)| - \alpha_{k-r}(H)$, $s = (\lceil \frac{k}{2} \rceil + 1)p + s'$, $t = (\lfloor \frac{k}{2} \rfloor + 1)q + t'$, $0 \leq s' < \lceil \frac{k}{2} \rceil + 1$ *and* $0 \leq t' < \lfloor \frac{k}{2} \rfloor + 1$.

Moreover, the bounds are sharp.

Proof. *(i)* The proof is similar to the proof of *(i)* of Theorem 2.1.

(ii) Suppose I is a r-independent set in G and J is a $(k-r)$-independent set in H, respectively. We will prove that $I \times J$ is a k-independent set of $G \,\square\, H$. By commutativity, we may assume $|V(G)| - \alpha_r(G) \leq |V(H)| - \alpha_{k-r}(H)$. Say $V(H) \setminus J = \{y_1, y_2, \cdots, y_t\}$, and take a subset $\{x_1, x_2, \cdots, x_s\} \subseteq V(G) \setminus I$. Then $s \leq t$. Set

$$K = \{(x_i, y_j) \,|\, 1 \leq i \leq s,\ 1 \leq j \leq t\}.$$

Let $F = G \,\square\, H$. Since $F[K]$ is a spanning subgraph of $K_s \,\square\, K_t$, it follows that $\alpha_k(F[K]) \geq \alpha_k(K_s \,\square\, K_t)$, and hence there is a $\alpha_k(K_s \,\square\, K_t)$-independent set of $F[K]$, say K'.

Claim 1: $(I \times J) \cup K'$ is a k-independent set of $G \square H$.

Proof of Claim 1. For any $(u_i, v_j) \in I \times J$ where $u_i \in V(G)$ and $v_j \in V(H)$, we have

$$d_{G\square H[I\times J]}(u_i, v_j) = d_{G[I]}(u_i) + d_{H[J]}(v_j) \leq r + (k-r) = k.$$

Therefore, $I \times J$ is a k-independent set of $G \square H$. From the structure of Cartesian product graphs, we have $E_{G\square H}[I \times J, K'] = \emptyset$. Then $(I \times J) \cup K'$ is a k-independent set of $G \square H$. □

From Claim 1, we have $\alpha_k(G \,\square\, H) \geq |(I \times J) \cup K'| = \alpha_r(G)\alpha_{k-r}(H) + \alpha_k(K_s \,\square\, K_t)$ for graphs G and H.

If $k \geq s+t-2$, then $(V(G)-I) \times (V(H)-J) = K_s \times K_t$ is a k-independent set of $K_s \,\square\, K_t$, and hence $\alpha_k(K_s \,\square\, K_t) \geq st$. If $s \geq \lfloor \frac{k+3}{2} \rfloor$, $t < \lfloor \frac{k+3}{2} \rfloor$, and $k \leq s+t-3$, then $\alpha_k(K_s \,\square\, K_t) \geq \alpha_k(K_{k-t+2} \,\square\, K_t) \geq t(k-t+2)$. Similarly, if $t \geq \lfloor \frac{k+3}{2} \rfloor$, $s < \lfloor \frac{k+3}{2} \rfloor$, and $k \leq s+t-3$, then $\alpha_k(K_s \,\square\, K_t) \geq \alpha_k(K_s \,\square\, K_{k-s+2}) \geq s(k-s+2)$. If $s \geq \lfloor \frac{k+3}{2} \rfloor$, $t \geq \lfloor \frac{k+3}{2} \rfloor$, then

$$
\begin{aligned}
\alpha_k(K_s \,\square\, K_t) &\geq \alpha_{\lceil \frac{k}{2} \rceil}(K_s)\alpha_{\lfloor \frac{k}{2} \rfloor}(K_t) + \alpha_k(K_{s-\lceil \frac{k}{2} \rceil -1} \,\square\, K_{t-\lfloor \frac{k}{2} \rfloor -1}) \\
&\geq \left(\left\lceil \frac{k}{2} \right\rceil + 1\right)\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + \alpha_k(K_{s-\lceil \frac{k}{2} \rceil -1} \,\square\, K_{t-\lfloor \frac{k}{2} \rfloor -1}) \\
&\geq \left(\left\lceil \frac{k}{2} \right\rceil + 1\right)\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + \alpha_{\lceil \frac{k}{2} \rceil}(K_{s-\lceil \frac{k}{2} \rceil -1})\alpha_{\lfloor \frac{k}{2} \rfloor}(K_{t-\lfloor \frac{k}{2} \rfloor -1}) \\
&\quad + \alpha_k(K_{s-2\lceil \frac{k}{2} \rceil -2} \,\square\, K_{t-2\lfloor \frac{k}{2} \rfloor -2}) \\
&= 2\left(\left\lceil \frac{k}{2} \right\rceil + 1\right)\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + \alpha_k(K_{s-2\lceil \frac{k}{2} \rceil -2} \,\square\, K_{t-2\lfloor \frac{k}{2} \rfloor -2}) \\
&= \cdots \\
&= \min\{p,q\}\left(\left\lceil \frac{k}{2} \right\rceil + 1\right)\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \\
&\quad + \alpha_k(K_{s-\min\{p,q\}(\lceil \frac{k}{2} \rceil +1)} \,\square\, K_{t-\min\{p,q\}(\lfloor \frac{k}{2} \rfloor +1)}) \\
&\geq \min\{p,q\}\left(\left\lceil \frac{k}{2} \right\rceil + 1\right)\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right),
\end{aligned}
$$

where $s = (\lceil \frac{k}{2} \rceil + 1)p + s'$, $t = (\lfloor \frac{k}{2} \rfloor + 1)q + t'$, $0 \leq s' < \lceil \frac{k}{2} \rceil + 1$ and $0 \leq t' < \lfloor \frac{k}{2} \rfloor + 1$. So the result follows.

See Remarks 2.4 and 2.5 for the sharpness. □

Remark 2.4. From Theorems 2.1, 2.2 and 2.3, we have the following upper bounds for k-independent number.

- $\alpha_k(G \circ H) \leq \alpha_k(H)|V(G)|$;
- $\alpha_k(G \boxtimes H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}$;
- $\alpha_k(G \square H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}$.

To show the sharpness of these upper bounds, we consider the following example. Let $G = nK_1$ and $|V(H)| = m$. Then $G * H$ consists of n copies of H, where $*$ denotes the lexicographical or Cartesian or strong product operation. It is clear that $\alpha_k(G * H) = \alpha_k(H)n = \alpha_k(H)|V(G)|$. So all these upper bounds are sharp.

Remark 2.5. From Theorems 2.1, 2.2 and 2.3, we have the following lower bounds for k-independent number.

- $\alpha_k(G \circ H) \geq \alpha_r(G)\alpha_{k-rm}(H)$, where $m = |V(H)|$;
- $\alpha_k(G \boxtimes H) \geq \alpha_r(G)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(H)$;

- $\alpha_k(G\Box H) \geq \alpha_r(G)\alpha_{k-r}(H) + X$, where $s = |V(G)| - \alpha_r(G)$, $t = |V(H)| - \alpha_{k-r}(H)$, and

$$X = \begin{cases} st, & \text{if } k \geq s+t-2; \\ t(k-t+1), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor, t < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{ and } k \leq s+t-3; \\ s(k-s+1), & \text{if } t \geq \lfloor \frac{k+3}{2} \rfloor, s < \lfloor \frac{k+3}{2} \rfloor, \\ & \text{ and } k \leq s+t-3; \\ \min\{p,q\}\left(\lceil \frac{k}{2} \rceil + 1\right)\left(\lfloor \frac{k}{2} \rfloor + 1\right), & \text{if } s \geq \lfloor \frac{k+3}{2} \rfloor \text{ and } t \geq \lfloor \frac{k+3}{2} \rfloor. \end{cases}$$

To show the sharpness of these lower bounds, we first consider the following example. Let $G = K_2$ and $H = K_2$. Then $G \circ H = G \boxtimes H = K_4$, and $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_k(K_4)$. For $k = 0$, $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_0(K_4) = 1$; for $k = 1$, $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_1(K_4) = 2$. From Theorems 2.1 and 2.2, $\alpha_k(G \circ H) \geq \alpha_r(G)\alpha_{k-rm}(H)$ and $\alpha_k(G \boxtimes H) \geq \alpha_r(G)\alpha_{\lfloor \frac{k}{2r+1} \rfloor}(H)$. Set $r = 0$. Then $\alpha_k(G \circ H) \geq \alpha_0(K_2)\alpha_k(K_2) = \alpha_k(K_2)$ and $\alpha_k(G \boxtimes H) \geq \alpha_0(K_2)\alpha_k(K_2) = \alpha_k(K_2)$. For $k = 0$, $\alpha_0(K_2) = 1$; for $k = 1$, $\alpha_1(K_2) = 2$. For $k = 0$, $\alpha_k(G \circ H) = \alpha_k(G \boxtimes H) = \alpha_0(G)\alpha_k(H)$. This implies that the first two lower bounds are sharp.

Next, we consider the examples for Cartesian product. Let $G = K_2$ and $H = K_2$. Clearly, $G\Box H = C_4$, and $\alpha_k(G\Box H) = \alpha_k(C_4)$. If $k = 0$, then $r = 0$, $s = t = p = q = 1$, and $\alpha_k(G\Box H) = \alpha_0(C_4) = 2 = \alpha_0(K_2)\alpha_0(K_2) + st = \alpha_0(K_2)\alpha_0(K_2) + \min\{p,q\}\left(\lceil \frac{0}{2} \rceil + 1\right)\left(\lfloor \frac{0}{2} \rfloor + 1\right)$. So the bound for the case $k \geq s+t-2$ or $s \geq \lfloor \frac{k+3}{2} \rfloor, t \geq \lfloor \frac{k+3}{2} \rfloor$ is sharp. For the case $s \geq \lfloor \frac{k+3}{2} \rfloor$, $t < \lfloor \frac{k+3}{2} \rfloor$, and $k \leq s+t-3$, we let $G = K_7$ and $H = K_4$. If $k = 3$, $r = 2$, $s = 4$, and $t = 2$, then $\alpha_3(G\Box H) \geq \alpha_2(K_7)\alpha_1(K_4) + t(k-t+2) = 12$. It suffices to show that $\alpha_3(G\Box H) \leq 12$. Assume, to the contrary, that $\alpha_3(G\Box H) \geq 13$. Let $V(G) = V(K_7) = \{u_i \,|\, 1 \leq i \leq 7\}$ and $V(H) = V(K_4) = \{v_i \,|\, 1 \leq i \leq 4\}$. Then $\bigcup_{i=1}^4 V(G(v_i)) = V(G\Box H)$. Let I be a maximum 3-independent set in $G\Box H$. Then $|I| \geq 13$. Since $k = 3$, it follows that $|I \cap V(G(v_i))| \leq 4$ for each i $(1 \leq i \leq 4)$. Then there exists some $G(v_i)$ such that $|I \cap V(G(v_i))| = 4$. Without loss of generality, let $I \cap V(G(v_1)) = \{(u_j, v_1) \,|\, 1 \leq j \leq 4\}$. Since $k = 3$ and $|I| \geq 13$, it follows that $|I \cap V(G(v_i))| = 3$ for each i $(2 \leq i \leq 4)$. Since $k = 3$, it follows that $I \cap V(G(v_i)) = \{(u_j, v_i) \,|\, 5 \leq j \leq 7\}$ for each i $(2 \leq i \leq 4)$. Then the degree of the subgraph induced by I is at least 4, a contradiction. So $\alpha_3(G\Box H) = 12$, and hence the lower bound is also sharp.

2.4 The direct product

We give upper and lower bounds for $\alpha_k(G \times H)$ in this section.

Theorem 2.6. *Let $k \geq 0$ be an integers. For graphs G and H,*

(i) $\alpha_k(G \times H) \geq \max\left\{\alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)|V(H)|, \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)|\right\}$;

(ii) $\alpha_k(G \times H)$

$$\leq \min\left\{\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G)|V(H)| - \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G), \right.$$
$$\left. \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(G)|V(H)| + \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(H)|V(G)| - \alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)\alpha_{\lfloor \frac{k}{\delta(G)} \rfloor}(H)\right\}.$$

Moreover, the bounds are sharp.

Proof. *(i)* If I is a $\lfloor \frac{k}{\Delta(H)} \rfloor$-independent set of G, then $I \times V(H)$ is a k-independent set of $G \times H$. Therefore, $\alpha_k(G \times H) \geq \alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)|V(H)|$. By symmetry of direct product graphs, we have

$$\alpha_k(G \times H) \geq \max\left\{\alpha_{\lfloor \frac{k}{\Delta(H)} \rfloor}(G)|V(H)|, \alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H)|V(G)|\right\}.$$

(ii) Let I be a k-independent set of $G \times H$. Partition I into two vertex subsets J, K such that

$$J = \left\{(u, v) \in I \,|\, (u, v_j) \in I, v_j \in S_{(u,v)}, \text{and } |S_{(u,v)}| \leq \left\lfloor \frac{k}{\Delta(G)} \right\rfloor \right\}$$

and $K = I \setminus J$, where $S_{(u,v)} = \{v_j \in N_H(v) \,|\, (u, v_j) \in I\}$.

Set

$$J^{u_i} = J \cap H(u_i) \text{ and } K^{v_j} = K \cap G(v_j).$$

Let I_H be a maximum $\lfloor \frac{k}{\Delta(G)} \rfloor$-independent set of H. Set

$$Y = (V(G) \times I_H) \cap K$$

and

$$Y^{u_i} = Y \cap H(u_i)$$

Note that $J^{u_i} \cap Y^{u_i} = \varnothing$. From the definition of J, $J^{u_i} \cup Y^{u_i}$ is a $\lfloor \frac{k}{\Delta(G)} \rfloor$-independent set of H, and hence

$$\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H) \geq |J^{u_i}| + |Y^{u_i}|. \tag{2.1}$$

Claim 1: For $v_j \in V(H)$, $\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) \geq |K^{v_j}|$.

Proof of Claim 1. For $(u, v_j) \in K^{v_j}$ where $u \in V(G)$, from the definition of K^{v_j}, we have $d_H(v_j) > \lfloor \frac{k}{\Delta(G)} \rfloor$. Since $d_G(u) \cdot d_H(v_j) \leq k$, it follows that

$$d_G(u) \leq \frac{k}{d_H(v_j)} \leq \frac{k}{\delta(H)}.$$

Note that K^{v_j} is a $\lfloor \frac{k}{\delta(H)} \rfloor$-independent set of $G(v_j)$. Therefore, $\alpha_{\frac{k}{\delta(H)}}(G) \geq |K^{v_j}|$. □

Since $\sum_{u_i \in V(G)} |Y^{u_i}| = \sum_{v_j \in I(H)} |K^{v_j}|$, it follows from (2.1) and Claim 1 that

$$\begin{aligned}
&\sum_{u_i \in V(G)} (\alpha_{\lfloor \frac{k}{\Delta(G)} \rfloor}(H) - |J^{u_i}|) + \sum_{v_j \in V(H)} (\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) - |K^{v_j}|) \\
&\geq \sum_{u_i \in V(G)} |Y^{u_i}| + \sum_{v_j \in V(H)} (\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) - |K^{v_j}|) \\
&\geq \sum_{v_j \in I(H)} |K^{v_j}| + \sum_{v_j \in I(H)} (\alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G) - |K^{v_j}|) \\
&\geq \sum_{v_j \in I(H)} \alpha_{\lfloor \frac{k}{\delta(H)} \rfloor}(G)
\end{aligned}$$

and hence

$$\sum_{u_i\in V(G)} \alpha_{\lfloor \frac{k}{\Delta(G)}\rfloor}(H) + \sum_{v_j\in V(H)} \alpha_{\lfloor \frac{k}{\delta(H)}\rfloor}(G) - \sum_{v_j\in I(H)} \alpha_{\lfloor \frac{k}{\delta(H)}\rfloor}(G) \geq \sum_{u_i\in V(G)} |J^{u_i}| + \sum_{v_j\in V(H)} |K^{v_j}|.$$

Then

$$\alpha_{\lfloor \frac{k}{\Delta(G)}\rfloor}(H)|V(G)| + \alpha_{\lfloor \frac{k}{\delta(H)}\rfloor}(G)|V(H)| - \alpha_{\lfloor \frac{k}{\Delta(G)}\rfloor}(H)\alpha_{\lfloor \frac{k}{\delta(H)}\rfloor}(G) \geq |I|.$$

From the symmetry of direct product, we have

$$\begin{aligned}&\alpha_k(G\times H)\\ &\leq \min\Big\{\alpha_{\lfloor \frac{k}{\Delta(G)}\rfloor}(H)|V(G)| + \alpha_{\lfloor \frac{k}{\delta(H)}\rfloor}(G)|V(H)| - \alpha_{\lfloor \frac{k}{\Delta(G)}\rfloor}(H)\alpha_{\lfloor \frac{k}{\delta(H)}\rfloor}(G),\\ &\quad \alpha_{\lfloor \frac{k}{\Delta(G)}\rfloor}(G)|V(H)| + \alpha_{\lfloor \frac{k}{\delta(H)}\rfloor}(H)|V(G)| - \alpha_{\lfloor \frac{k}{\Delta(H)}\rfloor}(G)\alpha_{\lfloor \frac{k}{\delta(G)}\rfloor}(H)\Big\}.\end{aligned}$$

The proof is now complete. See Remark 2.7 for the sharpness. □

Remark 2.7. To show the sharpness of the lower and upper bounds in Theorem 2.6, we let $G = K_2$ and $H = K_2$. Then

- $\alpha_k(G\times H) \geq \max\{\alpha_k(K_2)|V(K_2)|, \alpha_k(K_2)|V(K_2)|\} = 2\alpha_k(K_2)$;
- $$\begin{aligned}\alpha_k(G\times H) &\leq \min\{\alpha_k(H)|V(G)| + \alpha_k(G)|V(H)| - \alpha_k(H)\alpha_k(G),\\ &\qquad \alpha_k(G)|V(H)| + \alpha_k(H)|V(G)| - \alpha_k(G)\alpha_k(H)\}\\ &= \alpha_k(K_2)|V(K_2)| + \alpha_k(K_2)|V(K_2)| - \alpha_k(K_2)\alpha_k(K_2)\\ &= (4-\alpha_k(K_2))\alpha_k(K_2).\end{aligned}$$

For $k \geq 1$, we have $\alpha_k(G\times H) = 2$, which implies that the upper and lower bounds in Theorem 2.6 are sharp.

2.5 Relation of four graph products

For the k-independence number of four graph products, we have the following relation.

Proposition 2.8. *For any graphs G and H,*

$$\alpha_k(G\circ H) \leq \alpha_k(G\boxtimes H) \leq \alpha_k(G\,\Box\, H) \leq \min\{\alpha_{k\Delta(H)}(G\times H), \alpha_{k\Delta(G)}(G\times H)\}.$$

Proof. Since $G\boxtimes H$ is a subgraph of $G\circ H$, it follows that $\alpha_k(G\circ H) \leq \alpha_k(G\boxtimes H)$. Similarly, since $G\,\Box\, H$ is a subgraph of $G\boxtimes H$, it follows that $\alpha_k(G\boxtimes H) \leq \alpha_k(G\,\Box\, H)$. From Theorem 2.3, $\alpha_k(G\,\Box\, H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}$. From Theorem 2.6, we have

$$\begin{aligned}\alpha_{k\Delta(H)}(G\times H) &\geq \max\{\alpha_k(G)|V(H)|, \alpha_{\frac{k\Delta(H)}{\Delta(G)}}(H)|V(G)|\}\\ &\geq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\}\\ &\geq \alpha_k(G\,\Box\, H).\end{aligned}$$

Similarly, we have $\alpha_{k\Delta(G)}(G\times H) \geq \alpha_k(G\,\Box\, H)$, and hence

$$\alpha_k(G\,\Box\, H) \leq \min\{\alpha_{k\Delta(H)}(G\times H), \alpha_{k\Delta(G)}(G\times H)\}.$$

The proof is now complete. □

3 Applications

In this section, we demonstrate the usefulness of the proposed constructions by applying them to some instances of Cartesian and lexicographical product networks.

The following results are immediate.

Proposition 3.1. *Let $k \geq 0, n \geq 2$ be two integers and $\{\frac{n}{3}\}$ be the integer such that $n \equiv \{\frac{n}{3}\}(mod\, 3)$.*

(i) For a complete graph K_n,

$$\alpha_k(K_n) = \begin{cases} k+1, & \text{if } 0 \leq k \leq n-1; \\ n, & \text{if } k \geq n. \end{cases}$$

(ii) For a path P_n,

$$\alpha_k(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } k = 0; \\ 2\lfloor \frac{n}{3} \rfloor + \{\frac{n}{3}\}, & \text{if } k = 1; \\ n, & \text{if } k \geq 2. \end{cases}$$

(iii) For a cycle C_n,

$$\alpha_k(C_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } k = 0; \\ 2\lfloor \frac{n}{3} \rfloor, & \text{if } k = 1 \text{ and } n \equiv 0,1\ (mod\ 3); \\ 2\lfloor \frac{n}{3} \rfloor + 1, & \text{if } k = 1 \text{ and } n \equiv 2\ (mod\ 3); \\ n, & \text{if } k \geq 2. \end{cases}$$

3.1 n-dimensional generalized hypercube

Let K_m be a clique of m vertices, $m \geq 2$. An *n-dimensional generalized hypercube* [5, 8] is the product of n cliques.

We first focus our attention on 2-dimensional generalized hypercube.

Proposition 3.2. *For network $K_{m_1} \Box K_{m_2}$,*

$$\begin{aligned} &\min\{m_1, \lceil k/2 \rceil + 1\} \min\{m_2, \lfloor k/2 \rfloor + 1\} \\ &\quad \leq \alpha_k(K_{m_1} \Box K_{m_2}) \\ &\quad \leq \begin{cases} \min\{m_2, m_1\}(k+1), & \text{if } k \leq m_i - 1\ (i = 1,2); \\ (k+1)m_1, & \text{if } k \leq m_2 - 1,\ k \geq m_1; \\ (k+1)m_2, & \text{if } k \leq m_1 - 1,\ k \geq m_2; \\ m_1 m_2, & \text{if } k \geq m_1,\ k \geq m_2. \end{cases} \end{aligned}$$

Proof. We first investigate the upper bound of $\alpha_k(K_{m_1} \Box K_{m_2})$. If $k \geq m_i$ $(i = 1, 2)$, then $\alpha_k(K_{m_i}) = m_i$ and $\alpha_k(K_{m_1} \Box K_{m_2}) \leq \min\{\alpha_k(K_{m_1})|V(K_{m_2})|, \alpha_k(K_{m_2})|V(K_{m_1})|\} = m_1 m_2$ by Theorem 2.3. If $k \leq m_2 - 1$ and $k \geq m_1$, then $\alpha_k(K_{m_1}) = m_1$ and $\alpha_k(K_{m_2}) = k+1$ and $\alpha_k(K_{m_1} \Box K_{m_2}) \leq \min\{\alpha_k(K_{m_1})|V(K_{m_2})|, \alpha_k(K_{m_2})|V(K_{m_1})|\} = \min\{m_1 m_2, (k+1)m_1\} = (k+1)m_1$. Similarly, if $k \leq m_1 - 1$ and $k \geq m_2$, then $\alpha_k(K_{m_1} \Box K_{m_2}) \leq (k+1)m_2$. If $k \leq m_i - 1$ $(i = 1, 2)$, then $\alpha_k(K_{m_i}) = k + 1$, and hence $\alpha_k(K_{m_1} \Box K_{m_2}) \leq \min\{(k+1)m_2, (k+1)m_1\} = \min\{m_2, m_1\}(k+1)$.

Next, we consider the lower bound of $\alpha_k(K_{m_1}\Box K_{m_2})$. From Theorem 2.3, we have $\alpha_k(K_{m_1}\Box K_{m_2}) \geq \alpha_r(K_{m_1})\alpha_{k-r}(K_{m_2})$, where $0 \leq r \leq k$. If $r = \lceil k/2 \rceil$, then $k - r = \lfloor k/2 \rfloor$, $\alpha_r(K_{m_1}) = \min\{m_1, \lceil k/2 \rceil + 1\}$, and $\alpha_{k-r}(K_{m_2}) = \min\{m_2, \lfloor k/2 \rfloor + 1\}$. Furthermore, we have $\alpha_k(K_{m_1}\Box K_{m_2}) \geq \alpha_r(K_{m_1})\alpha_{k-r}(K_{m_2}) = \min\{m_1, \lceil k/2 \rceil + 1\}\min\{m_2, \lfloor k/2 \rfloor + 1\}$, as desired. □

Next, we consider n-dimensional generalized hypercube.

Proposition 3.3. *For network $K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}$, we have the following.*

(i) If $m_i \leq k$ $(1 \leq i \leq n)$, then

$$m_1 \leq \alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) \leq \prod_{i=1}^{n} m_i.$$

(ii) If $k \leq m_j - 1$ $(1 \leq j \leq n)$, then

$$k+1 \leq \alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) \leq (k+1)\prod_{i=2}^{n} m_i.$$

Proof. *(i)* Since $m_i \leq k$ $(1 \leq i \leq n)$, it follows that $\alpha_k(K_{m_i}) = m_i$, where $1 \leq i \leq n$. From Theorem 2.3, we have $\alpha_k(G\Box H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq \alpha_k(G)|V(H)|$ for any two graphs G and H, and hence

$$\begin{aligned}
\alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) &= \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})\Box K_{m_n}) \\
&\leq \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})m_n \\
&\leq \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-2}})m_{n-1}m_n \\
&\leq \cdots \\
&\leq \alpha_k(K_{m_1})m_2\cdots m_{n-1}m_n \\
&= \prod_{i=1}^{n} m_i.
\end{aligned}$$

From Theorem 2.3, we have $\alpha_k(G\Box H) \geq \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H. Set $r = k$. Then $\alpha_k(G\Box H) \geq \alpha_k(G)\alpha_0(H)$ for any two graphs G and H, and hence

$$\begin{aligned}
\alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_n}) &= \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})\Box K_{m_n}) \\
&\geq \alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}})\alpha_0(K_{m_n}) \\
&\geq \alpha_k(K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-1}}) \\
&\geq \alpha_k((K_{m_1}\Box K_{m_2}\Box\cdots\Box K_{m_{n-2}}) \\
&\geq \cdots \\
&\geq \alpha_k(K_{m_1}) \\
&= m_1.
\end{aligned}$$

(ii) Since $k \leq m_j - 1$ $(1 \leq j \leq n)$, it follows that $\alpha_k(K_{m_j}) = k+1$, where $1 \leq j \leq n$. From Theorem 2.3, we have $\alpha_k(G\Box H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq$

$\alpha_k(G)|V(H)|$ for any two graphs G and H, and hence

$$\begin{aligned}\alpha_k(K_{m_1}\square K_{m_2}\square\cdots\square K_{m_n}) &= \alpha_k((K_{m_1}\square K_{m_2}\square\cdots\square K_{m_{n-1}})\square K_{m_n})\\ &\le \alpha_k((K_{m_1}\square K_{m_2}\square\cdots\square K_{m_{n-1}})m_n\\ &\le \alpha_k((K_{m_1}\square K_{m_2}\square\cdots\square K_{m_{n-2}})m_{n-1}m_n\\ &\le \cdots\\ &\le \alpha_k((K_{m_1})m_2\cdots m_{n-1}m_n\\ &= (k+1)\prod_{i=2}^{n} m_i.\end{aligned}$$

From Theorem 2.3, we have $\alpha_k(G\square H) \ge \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H. Set $r = k$. Then $\alpha_k(G\square H) \ge \alpha_k(G)\alpha_0(H)$ for any two graphs G and H, and hence

$$\begin{aligned}\alpha_k(K_{m_1}\square K_{m_2}\square\cdots\square K_{m_n}) &= \alpha_k((K_{m_1}\square K_{m_2}\square\cdots\square K_{m_{n-1}})\square K_{m_n})\\ &\ge \alpha_k(K_{m_1}\square K_{m_2}\square\cdots\square K_{m_{n-1}})\alpha_0(K_{m_n})\\ &\ge \alpha_k(K_{m_1}\square K_{m_2}\square\cdots\square K_{m_{n-1}})\\ &\ge \alpha_k((K_{m_1}\square K_{m_2}\square\cdots\square K_{m_{n-2}})\\ &\ge \cdots\\ &\ge \alpha_k(K_{m_1}) = k+1,\end{aligned}$$

as desired. □

Proposition 3.4. *For network $K_{m_1}\circ K_{m_2}\circ\cdots\circ K_{m_n}$,*

$$\alpha_k(K_{m_1}\circ K_{m_2}\circ\cdots\circ K_{m_n}) = \begin{cases} k+1, & \text{if } 0\le k\le \prod_{i=1}^n m_i - 1;\\ \prod_{i=1}^n m_i, & \text{if } k\ge \sum_{i=1}^n m_i.\end{cases}$$

Proof. From the definition of lexicographical product, $K_{m_1}\circ K_{m_2}\circ\cdots\circ K_{m_n}$ is a complete graph. From Proposition 3.1, if $0\le k\le \prod_{i=1}^n m_i - 1$, then $\alpha_k(K_{m_1}\circ K_{m_2}\circ\cdots\circ K_{m_n}) = k+1$; if $k+1k\ge \sum_{i=1}^n m_i$, then $\alpha_k(K_{m_1}\circ K_{m_2}\circ\cdots\circ K_{m_n}) = \prod_{i=1}^n m_i$. □

3.2 Two-dimensional grid graph

A *two-dimensional grid graph* is the Cartesian product $P_n\square P_m$ of path graphs on m and n vertices. For more details on grid graph, we refer to [2, 11]. The network $P_n\circ P_m$ is the lexicographical product $P_n\circ P_m$ of path graphs on m and n vertices; see [15]. Let $\{m/3\}$ be the integer such that $m\equiv \{m/3\}(mod\, 3)$.

Proposition 3.5. *For network $P_n\square P_m$ $(n\ge 3, m\ge 3)$, we have the following.*

(i) If $k\ge 4$, then $\alpha_k(P_n\square P_m) = mn$.

(ii) If $k = 2,3$, then $\min\{m\lceil n/2\rceil, n\lceil m/2\rceil\}\le \alpha_k(P_n\square P_m)\le mn$.

(iii) If $k = 1$, then

$$\begin{aligned}\lceil n/2\rceil(2\lfloor m/3\rfloor + \{m/3\}) &\le \alpha_k(P_n\square P_m)\\ &\le \min\{(2\lfloor n/3\rfloor + \{n/3\})m, (2\lfloor m/3\rfloor + \{m/3\})n\}.\end{aligned}$$

(iv) If $k = 0$*, then* $\lceil n/2\rceil\lceil m/2\rceil \le \alpha_k(P_n\Box P_m) \le \min\{\lceil n/2\rceil m, \lceil m/2\rceil n\}$.

Proof. *(i)* Choose all vertices in $P_n \Box P_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most 4, it follows that $\alpha_k(P_n \Box P_m) = mn$.

(ii) From Theorem 2.3, $\alpha_2(P_n \Box P_m) \le \min\{\alpha_2(P_n)|V(P_m)|, \alpha_2(P_m)|V(P_n)|\} = \min\{nm, mn\} = mn$ and $\alpha_2(P_n \Box P_m) \ge \alpha_r(P_n)\alpha_{2-r}(P_m)$. If $r = 0$, then we have $\alpha_2(P_n \Box P_m) \ge \alpha_0(P_n)\alpha_2(P_m) = \lceil n/2\rceil m$. If $r = 2$, then $\alpha_2(P_n \Box P_m) \ge \alpha_2(P_n)\alpha_0(P_m) = \lceil m/2\rceil n$. So, we have $\alpha_2(P_n \Box P_m) \ge \min\{m\lceil n/2\rceil, n\lceil m/2\rceil\}$. Similarly, if $k = 3$, then $\min\{m\lceil n/2\rceil, n\lceil m/2\rceil\} \le \alpha_3(P_n \Box P_m) \le mn$.

(iii) From Theorem 2.3, $\alpha_1(P_n \Box P_m) \le \min\{\alpha_1(P_n)|V(P_m)|, \alpha_1(P_m)|V(P_n)|\} = \min\{(2\lfloor n/3\rfloor + \{n/3\})m, (2\lfloor m/3\rfloor + \{m/3\})n\}$. From Theorem 2.3, $\alpha_1(P_n \Box P_m) \ge \alpha_r(P_n)\alpha_{1-r}(P_m)$. If $r = 0$, then $\alpha_1(P_n \Box P_m) \ge \alpha_0(P_n)\alpha_1(P_m) = \lceil n/2\rceil(2\lfloor m/3\rfloor + \{m/3\})$.

(iv) From Theorem 2.3, $\alpha_0(P_n \Box P_m) \le \min\{\alpha_0(P_n)|V(P_m)|, \alpha_0(P_m)|V(P_n)|\} = \min\{\lceil n/2\rceil m, \lceil m/2\rceil n\}$, and $\alpha_0(P_n \Box P_m) \ge \alpha_0(P_n)\alpha_0(P_m) = \lceil n/2\rceil\lceil m/2\rceil$. □

Proposition 3.6. *For network* $P_n \circ P_m$ $(n \ge 4, m \ge 3)$*, we have the following.*

(i) If $k \ge 2m + 2$*, then* $\alpha_k(P_n \circ P_m) = mn$.

(ii) If $2 \le k < 2m + 2$*, then* $\lceil n/2\rceil m \le \alpha_k(P_n \circ P_m) \le mn$.

(iii) If $k = 1$*, then*

$$\lceil n/2\rceil(2\lfloor m/3\rfloor + \{m/3\}) \le \alpha_1(P_n \circ P_m) \le n(2\lfloor m/3\rfloor + \{m/3\}).$$

(iv) If $k = 0$*, then*

$$\lceil n/2\rceil\lceil m/2\rceil \le \alpha_k(P_n \circ P_m) \le n\lceil m/2\rceil.$$

Proof. From Theorem 2.1, we have $\alpha_k(P_n \circ P_m) \le \alpha_k(P_m)|V(P_n)| = n\alpha_k(P_m)$ and $\alpha_k(P_n \circ P_m) \ge \alpha_r(P_n)\alpha_{k-rm}(P_m)$. Let $r = 0$. Then $\alpha_k(P_n \circ P_m) \ge \alpha_0(P_n)\alpha_k(P_m)$, and hence

$$\lceil n/2\rceil\alpha_k(P_m) \le \alpha_k(P_n \circ P_m) \le n\alpha_k(P_m). \tag{3.1}$$

(i) For $k \ge 2m + 2$, we choose all vertices in $P_n \circ P_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most $2m + 2$, it follows that $\alpha_k(P_n \circ P_m) = mn$.

(ii) Since $2 \le k < 2m + 2$, it follows that $\alpha_k(P_m) = m$. From (3.1), $\lceil n/2\rceil m \le \alpha_k(P_n \circ P_m) \le mn$.

(iii) For $k = 1$, $\alpha_k(P_m) = 2\lfloor m/3\rfloor + \{m/3\}$. From (3.1), $\lceil n/2\rceil(2\lfloor m/3\rfloor + \{m/3\}) \le \alpha_1(P_n \circ P_m) \le n(2\lfloor m/3\rfloor + \{m/3\})$.

(iv) For $k = 0$, $\alpha_k(P_m) = \lfloor m/2\rfloor$. From (3.1), we have $\lceil n/2\rceil\lceil m/2\rceil \le \alpha_k(P_n \circ P_m) \le n\lceil m/2\rceil$. □

3.3 n-dimensional mesh

An *n-dimensional mesh* is the Cartesian product of n paths. By this definition, two-dimensional grid graph is a 2-dimensional mesh. An n-dimensional hypercube is a special case of an n-dimensional mesh, in which the n linear arrays are all of size 2; see [13].

Proposition 3.7. *For n-dimensional mesh $P_{m_1} \square P_{m_2} \square \cdots \square P_{m_n}$,*

$$\alpha_k(P_{m_1} \square P_{m_2} \square \cdots \square P_{m_n}) \leq \begin{cases} \lceil \frac{m_1}{2} \rceil \prod_{i=2}^n m_i, & \text{if } k = 0; \\ (2\lfloor \frac{m_1}{3} \rfloor + \{\frac{m_1}{3}\}) \prod_{i=2}^n m_i, & \text{if } k = 1; \\ \prod_{i=1}^n m_i, & \text{if } k \geq 2, \end{cases}$$

and

$$\alpha_k(P_{m_1} \square P_{m_2} \square \cdots \square P_{m_n}) \geq \begin{cases} \lceil \frac{m_1}{2} \rceil \prod_{i=2}^n \lceil m_i/2 \rceil, & \text{if } k = 0; \\ (2\lfloor \frac{m_1}{3} \rfloor + \{\frac{m_1}{3}\}) \prod_{i=2}^n \lceil m_i/2 \rceil, & \text{if } k = 1; \\ m_1 \prod_{i=2}^n \lceil m_i/2 \rceil, & \text{if } k \geq 2. \end{cases}$$

Proof. From Theorem 2.3, we have $\alpha_k(G \square H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq \alpha_k(G)|V(H)|$ for any two graphs G and H, and hence

$$\begin{aligned} \alpha_k(P_{m_1} \square P_{m_2} \square \cdots \square P_{m_n}) &= \alpha_k((P_{m_1} \square P_{m_2} \square \cdots \square P_{m_{n-1}}) \square P_{m_n}) \\ &\leq \alpha_k((P_{m_1} \square P_{m_2} \square \cdots \square P_{m_{n-1}})m_n \\ &\leq \alpha_k((P_{m_1} \square P_{m_2} \square \cdots \square P_{m_{n-2}})m_{n-1}m_n \\ &\leq \cdots \\ &\leq \alpha_k(P_{m_1})m_2 \cdots m_{n-1}m_n. \end{aligned}$$

So, the result follows.

From Theorem 2.3, we have $\alpha_k(G \square H) \geq \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H. Set $r = k$. Then $\alpha_k(G \square H) \geq \alpha_k(G)\alpha_0(H)$ for any two graphs G and H, and hence

$$\begin{aligned} \alpha_k(P_{m_1} \square P_{m_2} \square \cdots \square P_{m_n}) &= \alpha_k((P_{m_1} \square P_{m_2} \square \cdots \square P_{m_{n-1}}) \square P_{m_n}) \\ &\geq \alpha_k(P_{m_1} \square P_{m_2} \square \cdots \square P_{m_{n-1}})\alpha_0(P_{m_n}) \\ &\geq \alpha_k(P_{m_1} \square P_{m_2} \square \cdots \square P_{m_{n-1}})\lceil m_n/2 \rceil \\ &\geq \alpha_k((P_{m_1} \square P_{m_2} \square \cdots \square P_{m_{n-2}})\lceil m_{n-1}/2 \rceil \lceil m_n/2 \rceil \\ &\geq \cdots \\ &\geq \alpha_k(P_{m_1}) \prod_{i=2}^{n} \lceil m_i/2 \rceil, \end{aligned}$$

and hence the result holds. □

Similarly to the proof of Proposition 3.7, we can obtain the following result.

Proposition 3.8. *For n-dimensional mesh $P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_n}$,*

$$\begin{cases} \lceil \frac{m_1}{2} \rceil \leq \alpha_k(P_{m_1} \circ \cdots \circ P_{m_n}) \leq \lceil \frac{m_1}{2} \rceil \prod_{i=2}^n m_i, & \text{if } k = 0; \\ 2\lfloor \frac{m_1}{3} \rfloor + \{\frac{m_1}{3}\} \leq \alpha_k(P_{m_1} \circ \cdots \circ P_{m_n}) \leq (2\lfloor \frac{m_1}{3} \rfloor + \{\frac{m_1}{3}\}) \prod_{i=2}^n m_i, & \text{if } k = 1; \\ m_1 \leq \alpha_k(P_{m_1} \circ \cdots \circ P_{m_n}) \leq \prod_{i=1}^n m_i, & \text{if } k \geq 2. \end{cases}$$

3.4 n-dimensional torus

An *n-dimensional torus* is the Cartesian product of n cycles $C_{m_1}, C_{m_2}, \cdots, C_{m_n}$ of size at least three. The cycles C_{m_i} are not necessary to have the same size. Ku et al. [14] showed that there are n edge-disjoint spanning trees in an n-dimensional torus. The network $C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_n}$ is investigated in [15]. Here, we consider the networks constructed by $C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_n}$ and $C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_n}$, respectively.

Proposition 3.9. *For network $C_n \Box C_m$ $(n \geq 3, m \geq 3)$, we have the following.*

(i) *If $k \geq 4$, then $\alpha_k(C_n \Box C_m) = mn$.*

(ii) *If $k = 3$ or $k = 2$, then $\min\{m\lfloor n/2 \rfloor, n\lfloor m/2 \rfloor\} \leq \alpha_k(C_n \Box C_m) \leq mn$.*

(iii) *If $k = 1$, then $2\lfloor n/2 \rfloor \lfloor \frac{m}{3} \rfloor \leq \alpha_k(C_n \Box C_m) \leq \min\{m(2\lfloor \frac{n}{3} \rfloor + 1), n(2\lfloor \frac{m}{3} \rfloor + 1)\}$.*

(iv) *If $k = 0$, then $\lfloor n/2 \rfloor \lfloor m/2 \rfloor \leq \alpha_k(C_n \Box C_m) \leq \min\{\lfloor n/2 \rfloor m, \lfloor m/2 \rfloor n\}$.*

Proof. *(i)* Choose all vertices in $C_n \Box C_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most 4, it follows that $\alpha_k(C_n \Box C_m) = mn$.

(ii) From Theorem 2.3, $\alpha_3(C_n \Box C_m) \leq \min\{\alpha_3(C_n)|V(C_m)|, \alpha_3(C_m)|V(C_n)|\} = \min\{nm, mn\} = mn$, and $\alpha_3(P_n \Box P_m) \geq \alpha_r(C_n)\alpha_{3-r}(C_m)$. If $r = 0$, then we have $\alpha_3(C_n \Box C_m) \geq \alpha_0(C_n)\alpha_3(C_m) = \lfloor n/2 \rfloor m$. If $r = 3$, then $\alpha_3(C_n \Box C_m) \geq \alpha_3(C_n)\alpha_0(C_m) = \lfloor m/2 \rfloor n = \lfloor m/2 \rfloor n$. So, $\alpha_3(C_n \Box C_m) \geq \min\{m\lfloor n/2 \rfloor, n\lfloor m/2 \rfloor\}$. The case $k = 2$ can be similarly proved.

(iii) From Theorem 2.3, $\alpha_1(C_n \Box C_m) \geq \alpha_r(C_n)\alpha_{1-r}(C_m)$. If $r = 0$, then we have $\alpha_1(C_n \Box C_m) \geq \alpha_0(C_n)\alpha_1(C_m) = \lfloor n/2 \rfloor(2\lfloor \frac{m}{3} \rfloor)$, and $\alpha_1(C_n \Box C_m) \leq \min\{\alpha_1(C_n) |V(C_m)|, \alpha_1(C_m)|V(C_n)|\} = \min\{m(2\lfloor \frac{n}{3} \rfloor + 1), n(2\lfloor \frac{m}{3} \rfloor + 1)\}$.

(iv) From Theorem 2.3, $\alpha_0(C_n \Box C_m) \leq \min\{\lfloor n/2 \rfloor m, \lfloor m/2 \rfloor n\}$, and $\alpha_0(C_n \Box C_m) \geq \alpha_0(C_n)\alpha_0(C_m) = \lfloor n/2 \rfloor \lfloor m/2 \rfloor$. $\Box$

For network $C_n \circ C_m$, we have the following result.

Proposition 3.10. *For network $C_n \circ C_m$ $(n \geq 4, m \geq 3)$, we have the following.*

(i) *If $k \geq 2m + 2$, then $\alpha_k(C_n \circ C_m) = mn$.*

(ii) *If $2 \leq k < 2m + 2$, then $\lfloor n/2 \rfloor m \leq \alpha_k(C_n \circ C_m) \leq mn$.*

(iii) *If $k = 1$ and $n \equiv 0, 1 \pmod 3$, then $2\lfloor n/2 \rfloor \lfloor n/3 \rfloor \leq \alpha_k(C_n \circ C_m) \leq 2n\lfloor n/3 \rfloor$.*

(iv) *If $k = 1$ and $n \equiv 2 \pmod 3$, then*

$$\lfloor n/2 \rfloor (2\lfloor n/3 \rfloor + 1) \leq \alpha_k(C_n \circ C_m) \leq n(2\lfloor n/3 \rfloor + 1).$$

(v) *If $k = 0$, then $\lfloor m/2 \rfloor \lfloor n/2 \rfloor \leq \alpha_0(C_n \circ C_m) \leq n\lfloor m/2 \rfloor$.*

Proof. From Theorem 2.1, we have $\alpha_k(C_n \circ C_m) \leq \alpha_k(C_m)|V(C_n)| = n\alpha_k(C_m)$ and $\alpha_k(C_n \circ C_m) \geq \alpha_r(C_n)\alpha_{k-rm}(C_m)$. Let $r = 0$. Then $\alpha_k(C_n \circ C_m) \geq \alpha_0(C_n)\alpha_k(C_m)$, and hence

$$\lfloor n/2 \rfloor \alpha_k(C_m) \leq \alpha_k(C_n \circ C_m) \leq n\alpha_k(C_m). \tag{3.2}$$

(i) For $k \geq 2m + 2$, we choose all vertices in $C_n \circ C_m$. Since the degree of each vertex in the induced subgraph induced by these vertices is at most $2m + 2$, it follows that $\alpha_k(C_n \circ C_m) = mn$.

(ii) Since $2 \leq k < 2m+2$, it follows that $\alpha_k(C_m) = m$, and hence $\lfloor n/2 \rfloor m \leq \alpha_k(C_n \circ C_m) \leq mn$ by (3.2).

(iii) Since $k = 1$ and $n \equiv 0, 1 \ (mod\ 3)$, we have $\alpha_k(C_m) = 2\lfloor \frac{n}{3} \rfloor$. From (3.2), $2\lfloor n/2 \rfloor \lfloor n/3 \rfloor \leq \alpha_k(C_n \circ C_m) \leq 2n\lfloor n/3 \rfloor$.

(iv) For $k = 1$ and $n \equiv 2 \ (mod\ 3)$, $\alpha_k(C_m) = 2\lfloor \frac{n}{3} \rfloor + 1$. From (3.2), $\lfloor n/2 \rfloor (2\lfloor n/3 \rfloor + 1) \leq \alpha_k(C_n \circ C_m) \leq n(2\lfloor n/3 \rfloor + 1)$.

(v) For $k = 0$, $\alpha_k(C_m) = \lfloor m/2 \rfloor$. From (3.2), $\lfloor m/2 \rfloor \lfloor n/2 \rfloor \leq \alpha_k(C_n \circ C_m) \leq n\lfloor m/2 \rfloor$. □

For general case, we have the following two results.

Proposition 3.11. *For network $C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_n}$,*

$$\alpha_k(C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_n}) \leq \begin{cases} \lfloor \frac{m_1}{2} \rfloor \prod_{i=2}^n m_i, & \text{if } k = 0; \\ 2\lfloor \frac{m_1}{3} \rfloor \prod_{i=2}^n m_i, & \text{if } k = 1, m_1 \equiv 0, 1 \ (mod\ 3); \\ (2\lfloor \frac{m_1}{3} \rfloor + 1) \prod_{i=2}^n m_i, & \text{if } k = 1, m_1 \equiv 2 \ (mod\ 3); \\ \prod_{i=1}^n m_i, & \text{if } k \geq 2, \end{cases}$$

and

$$\alpha_k(C_{m_1} \Box \cdots \Box C_{m_n}) \geq \begin{cases} \lfloor \frac{m_1}{2} \rfloor \prod_{i=2}^n \lfloor m_i/2 \rfloor, & \text{if } k = 0; \\ 2\lfloor \frac{m_1}{3} \rfloor \prod_{i=2}^n \lfloor m_i/2 \rfloor, & \text{if } k = 1, m_1 \equiv 0, 1 \ (mod\ 3); \\ (2\lfloor \frac{m_1}{3} \rfloor + 1) \prod_{i=2}^n \lfloor m_i/2 \rfloor, & \text{if } k = 1, m_1 \equiv 2 \ (mod\ 3); \\ m_1 \prod_{i=2}^n \lfloor m_i/2 \rfloor, & \text{if } k \geq 2, \end{cases}$$

where m_i is the order of C_{m_i} and $1 \leq i \leq n$.

Proof. From Theorem 2.3, we have $\alpha_k(G \Box H) \leq \min\{\alpha_k(G)|V(H)|, \alpha_k(H)|V(G)|\} \leq \alpha_k(G)|V(H)|$ for any two graphs G and H, and hence

$$\begin{aligned} \alpha_k(C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_n}) &= \alpha_k((C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_{n-1}}) \Box C_{m_n}) \\ &\leq \alpha_k((C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_{n-1}})m_n \\ &\leq \alpha_k((C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_{n-2}})m_{n-1}m_n \\ &\leq \cdots \\ &\leq \alpha_k(C_{m_1})m_2 \cdots m_{n-1}m_n. \end{aligned}$$

From (iii) of Proposition 3.1, the result follows.

From Theorem 2.3, we have $\alpha_k(G \Box H) \geq \alpha_r(G)\alpha_{k-r}(H)$ for any two graphs G and H. Set $r = k$. Then $\alpha_k(G \Box H) \geq \alpha_k(G)\alpha_0(H)$ for any two graphs G and H, and hence

$$\begin{aligned} \alpha_k(C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_n}) &= \alpha_k((C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_{n-1}}) \Box C_{m_n}) \\ &\geq \alpha_k(C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_{n-1}})\alpha_0(C_{m_n}) \\ &\geq \alpha_k(C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_{n-1}})\lfloor m_n/2 \rfloor \\ &\geq \alpha_k((C_{m_1} \Box C_{m_2} \Box \cdots \Box C_{m_{n-2}})\lfloor m_{n-1}/2 \rfloor \lfloor m_n/2 \rfloor \\ &\geq \cdots \\ &\geq \alpha_k(C_{m_1}) \prod_{i=2}^n \lfloor m_i/2 \rfloor. \end{aligned}$$

From (3.2) of Proposition 3.1, the result holds. □

Similarly to the proof of Proposition 3.11, we can prove the following result.

Proposition 3.12. *For network $C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_n}$,*

$$\begin{cases} \lfloor \frac{m_1}{2} \rfloor \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq \lfloor \frac{m_1}{2} \rfloor \prod_{i=2}^{n} m_i, & \text{if } k = 0; \\ 2\lfloor \frac{m_1}{3} \rfloor \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq 2\lfloor \frac{m_1}{3} \rfloor \prod_{i=2}^{n} m_i, & \text{if } k = 1 \\ & \text{and } m_1 \equiv 0, 1 \,(mod\, 3); \\ 2\lfloor \frac{m_1}{3} \rfloor + 1 \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq (2\lfloor \frac{m_1}{3} \rfloor + 1) \prod_{i=2}^{n} m_i, & \text{if } k = 1 \\ & \text{and } m_1 \equiv 2 \,(mod\, 3); \\ m_1 \leq \alpha_k(C_{m_1} \circ \cdots \circ C_{m_n}) \leq \prod_{i=1}^{n} m_i, & \text{if } k \geq 2, \end{cases}$$

where m_i is the order of C_{m_i} and $1 \leq i \leq n$.

3.5 n-dimensional hyper Petersen network

An *n-dimensional hyper Petersen network* HP_n is the product of the well-known Petersen graph and Q_{n-3} [4], where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$-dimensional hypercube. Note that HP_3 is just the Petersen graph.

The network HL_n is the lexicographical product of the Petersen graph and Q_{n-3}, where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$-dimensional hypercube; see [15]. Note that HL_3 is just the Petersen graph, and HL_4 is a graph obtained from two copies of the Petersen graph by adding the edges between all the vertices from different copies of the Petersen graph.

Proposition 3.13. *(i) For network HP_3 and HL_3,*

$$\alpha_k(HP_3) = \alpha_k(HL_3) = \begin{cases} 4, & \text{if } k = 0; \\ 5, & \text{if } k = 1; \\ 5, & \text{if } k = 2; \\ 10, & \text{if } k \geq 3. \end{cases}$$

(ii) For network HP_4,

$$\begin{cases} 5 \leq \alpha_k(HP_4) \leq 8, & \text{if } k = 0; \\ 6 \leq \alpha_k(HP_4) \leq 10, & \text{if } k = 1; \\ 6 \leq \alpha_k(HP_4) \leq 10, & \text{if } k = 2; \\ 11 \leq \alpha_k(HP_4) \leq 20, & \text{if } k = 3; \\ \alpha_k(HP_4) = 30, & \text{if } k \geq 4. \end{cases}$$

(iii) For network HL_4,

$$\begin{cases} 4 \leq \alpha_k(HP_4) \leq 15, & \text{if } k = 0; \\ 8 \leq \alpha_k(HP_4) \leq 30, & \text{if } k \geq 1. \end{cases}$$

Proof. *(i)* Note that HL_3 or HP_3 is just the Petersen graph, and its maximum degree is 3. Since $|V(HP_3)| = 10$, it follows that $\alpha_k(HP_3) = 10$ for $k \geq 3$. One can also check that

$$\alpha_k(HP_3) = \alpha_k(HL_3) = \begin{cases} 4, & \text{if } k = 0; \\ 5, & \text{if } k = 1; \\ 5, & \text{if } k = 2. \end{cases}$$

(ii) For network HP_4, $HP_4 = HP_3 \Box K_2$. From Theorem 2.3, we have $\alpha_k(HP_4) = \alpha_k(HP_3 \Box K_2) \leq \min\{2\alpha_k(HP_3), 10\alpha_k(K_2)\}$. Note that $\alpha_k(K_2) = 1$ for $k = 0$; $\alpha_k(K_2) = 2$ for $k \geq 1$. Combining this with (i) of this proposition, we have

$$\alpha_k(HP_4) \leq \begin{cases} 8, & \text{if } k = 0; \\ 10, & \text{if } k = 1; \\ 10, & \text{if } k = 2; \\ 20, & \text{if } k \geq 3. \end{cases}$$

From Theorem 2.3, $\alpha_k(HP_4) \geq \alpha_r(HP_3)\alpha_{k-r}(K_2) + \alpha_k(K_s \Box K_t)$, where $s = |V(HP_3)| - \alpha_r(HP_3)$ and $t = |V(K_2)| - \alpha_{k-r}(K_2)$. Set $r = k$. Then $t = 1$ and $\alpha_k(HP_4) \geq \alpha_k(HP_3)\alpha_0(K_2) + \alpha_k(K_s \Box K_1) \geq \alpha_k(HP_3) + 1$, and hence

$$\alpha_k(HP_4) \geq \begin{cases} 5, & \text{if } k = 0; \\ 6, & \text{if } k = 1; \\ 6, & \text{if } k = 2; \\ 11, & \text{if } k \geq 3. \end{cases}$$

(iii) For network HL_4, $HL_4 = K_2 \circ HL_3$. From Theorem 2.3, we have $\alpha_k(HL_4) = \alpha_k(K_2 \circ HL_3) \leq |V(HL_3)|\alpha_k(K_2) = 10\alpha_k(K_2)$. Note that $\alpha_k(K_2) = 1$ for $k = 0$; $\alpha_k(K_2) = 2$ for $k \geq 1$. Combining this with (i) of this proposition, we have

$$\alpha_k(HL_4) \leq \begin{cases} 15, & \text{if } k = 0; \\ 20, & \text{if } k \geq 1. \end{cases}$$

From Theorem 2.3, $\alpha_k(HL_4) \geq \alpha_r(HL_3)\alpha_{k-2r}(K_2)$. Set $r = 0$. Then $\alpha_k(HL_4) \geq \alpha_0(HL_3)\alpha_k(K_2) = 4\alpha_k(K_2)$, and hence

$$\alpha_k(HL_4) \geq \begin{cases} 4, & \text{if } k = 0; \\ 8, & \text{if } k \geq 1. \end{cases}$$

$\square$

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Hurwitz's regular map $(3, 7)$ of genus 7: A polyhedral realization

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Dedicated to Prof. Dr. Dr. h.c. Jörg M. Wills on the Occation of his 80th Birthday.

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Abstract

A Hurwitz surface, named after Adolf Hurwitz, is a compact Riemann surface with precisely $84(g-1)$ automorphisms, where g is the genus of the surface. The Hurwitz surface of least genus is the Klein quartic of genus 3. A polyhedral realization without self-intersections of Klein's quartic of genus 3 was found by E. Schulte and J. M. Wills in 1985. For the next possible genus of a Hurwitz surface, i.e., for the genus 7 case with 72 vertices, we provide a polyhedral realization without self-intersections. We also show a topological representation for which we have a corresponding model.

Keywords: Hurwitz surface, regular map, polyhedral manifold.

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1 Introduction

In Riemann surface theory and hyperbolic geometry, a *Hurwitz surface*, named after Adolf Hurwitz, is a compact Riemann surface with precisely $84(g-1)$ automorphisms, where g is the genus of the surface. The Hurwitz surface of least genus is the Klein quartic of genus 3. The next possible genus is 7 with automorphism group $\mathrm{PSL}(2,8)$, which is the simple group of order $84 \times (7-1) = 504$; if one includes orientation-reversing isometries, the group is of order 1 008. Our paper is devoted to this genus 7 surface of Adolf Hurwitz from 1893, compare [12], which provides us in modern terminology with a regular map of type $(3,7)_{18}$. We have a closed triangular 2-manifold in which each vertex is incident with seven triangles. The Petrie polygon length is 18 and the automorphism group is flag transitive. In general a *regular map* is a decomposition of a two dimensional manifold into topological discs, such that every flag can be transformed into any other flag by a symmetry of the decomposition. When we describe a topological disc d via a circular sequence of its vertices $d = (v_1, v_2, \dots, v_k)$, a *flag* will be in this context a tripel $(v_i, (v_i, v_{i+1}), d)$ consisting of a vertex v_i, an edge (v_i, v_{i+1}), and the disc d itself. For the Hurwitz surface of genus 7, the name Macbeath surface is used as well, although the corresponding article of Macbeath is from 1965, [13].

We find under Wikipedia for *regular map*: *"regular maps are typically defined and studied in three ways: topologically, group-theoretically, and graph-theoretically"*. However, there are also results in which polyhedral realizations of regular maps have been studied, see e.g. the corresponding articles of Jörg M. Wills and of his co-authors or other colleagues in [2, 3, 4, 5, 6, 7, 8, 16, 17], and [18]. This article is devoted to such a question that was studied by Jörg M. Wills for some time. When only some abstract combinatorial data of a geometric object is given and when we are looking for a corresponding geometric realization or try to prove that no such realization exists, we are facing in general a hard problem that has been called a problem of *computational synthetic geometry* in [4].

Our main result of this article provides a polyhedral realization of Hurwitz's regular map $(3,7)_{18}$ of genus seven. We also show a topological representation for which we have a corresponding 3D-model. We refer the reader for additional aspects with respect to this paper to the homepage of the second author: http://www.iazd.uni-hannover.de/cuntz.html.

1.1 Previous polyhedral realizations of regular maps

Regular maps generalize on a combinatorial level Platonic solids with their geometric flag transitive automorphism groups. Mani's result [14] asserts that for each combinatorial automorphism of the boundary structure of a convex polyhedron, there does exist a convex polyhedron with a corresponding geometric symmetry. A corresponding result for general regular maps does not hold, the notion hidden symmetries has been used.

The polyhedral realization of Hurwitz's surface of least genus, i.e., a polyhedral realization of Klein's quartic of genus 3, with 24 vertices has been published by E. Schulte and J. M. Wills in [16]. In Figure 1 we have depicted two truncated tetrahedra the vertices of which are the vertices of this symmetric realization.

A first polyhedral realization of a regular map of Walther Dyck $(3,8)_6$ with 12 vertices was provided in Antibes in 1987 by Bokowski, see [1] and [2], thus disproving a conjecture of Schulte and Wills that it did not exist. A symmetrical version of this map was found later by Ulrich Brehm, [6]. U. Brehm and U. Leopold have found another polyhedral realization

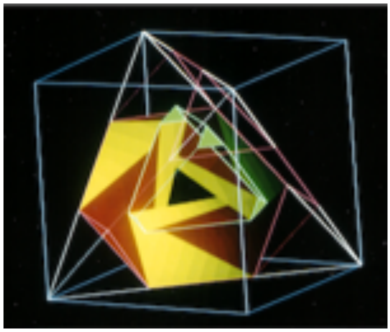

Figure 1: The points of a realization of Hurwitz's regular map of genus 3 by E. Schulte and J. M. Wills are the vertices of two truncated tetrahedra, [16].

Figure 2: First realization of Dyck's regular map of genus 3 presented by Bokowski (in the middle) in Antibes 1987, [1]. The other two men in the photo are R. Connelly, Cornell University (on the left) and J. M. Wills, University Siegen (on the right).

of a regular map $(3, 10)$ of genus 6 of W. Dyck with 15 vertices, [7]. See also a survey article of U. Brehm and E. Schulte in [8] and the papers cited there.

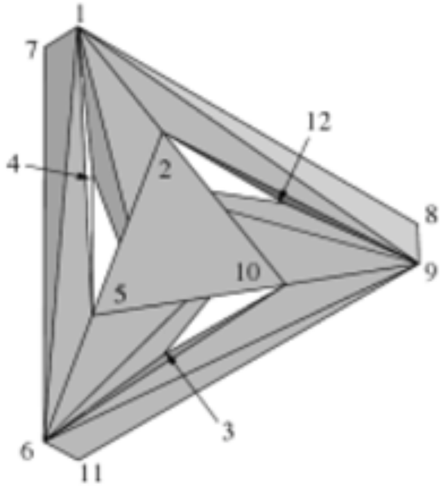

Figure 3: Symmetric realization of Dyck's regular map of genus 3 by U. Brehm [6].

2 Combinatorial description

For general descriptions of combinatorial regular maps we refer the reader to [9, 11], and [23]. Hurwitz's regular map of genus 7 consists of the following 168 triangles in Table 1. It has 252 edges and 72 vertices labeled $1, \ldots, 72$.

Compared with previous polyhedral realizations of regular maps we are faced with additional complexity. From the automorphism group of this manifold of order $1\,008$, we have used a dihedral subgroup of order 14 for sorting the triangles. The cyclic subgroup has the following generator.

$$(1)\,(72)\,(2,\ldots,8)\,(9,\ldots,15)\,(16,\ldots,22)\,(23,\ldots,29)\,(30,\ldots,36)\\ (37,\ldots,43)\,(44,\ldots,50)\,(51,\ldots,57)\,(58,\ldots,64)\,(65,\ldots,71)$$

When we assume the vertices from 2 to 71 to form ten regular seven-gons in horizontal equidistant planes with heights sorted according to labels belonging to the same orbit of the cyclic group, we have an additional up-side-down symmetry that maps vertex 1 to vertex 72 which is an automorphism of the map.

The combinatorial description of a complete list of small regular maps has been given in [10], an even more extended list is available from the first author of the same article.

3 Topological visualization

The study of topological visualizations of regular maps has recently been done by J. J. van Wijk, [21, 22], by C. H. Séquin, [20], and by Razafindrazaka and Polthier, [15]. From J. J. van Wijk we have a nice topological visualization as a computer film of our Hur-

Table 1: Triangles of Hurwitz's surface of genus seven.

(01, 02, 03),	(01, 03, 04),	(01, 04, 05),	(01, 05, 06),	(01, 06, 07),	(01, 07, 08),
(01, 08, 02),	(02, 08, 09),	(02, 10, 03),	(03, 11, 04),	(12, 05, 04),	(13, 06, 05),
(14, 07, 06),	(08, 07, 15),	(02, 09, 23),	(03, 10, 24),	(11, 25, 04),	(12, 26, 05),
(13, 27, 06),	(14, 28, 07),	(08, 15, 29),	(02, 16, 10),	(03, 17, 11),	(12, 04, 18),
(13, 05, 19),	(20, 14, 06),	(21, 15, 07),	(22, 09, 08),	(02, 23, 16),	(03, 24, 17),
(04, 25, 18),	(19, 05, 26),	(20, 06, 27),	(21, 07, 28],	(22, 08, 29),	(22, 51, 09),
(16, 52, 10),	(11, 17, 53),	(12, 18, 54),	(55, 13, 19),	(56, 14, 20),	(21, 57, 15),
(46, 23, 09),	(47, 24, 10),	(48, 25, 11),	(49, 26, 12),	(50, 27, 13),	(44, 28, 14),
(45, 29, 15),	(46, 09, 32),	(47, 10, 33),	(48, 11, 34),	(49, 12, 35),	(50, 13, 36),
(44, 14, 30),	(45, 15, 31),	(32, 09, 51),	(10, 52, 33),	(11, 53, 34),	(12, 54, 35),
(55, 36, 13),	(30, 14, 56),	(31, 15, 57),	(36, 16, 23),	(30, 17, 24),	(31, 18, 25),
(32, 19, 26),	(20, 27, 33),	(21, 28, 34),	(22, 29, 35),	(36, 42, 16),	(30, 43, 17),
(31, 37, 18),	(38, 19, 32),	(39, 20, 33),	(21, 34, 40),	(22, 35, 41),	(58, 16, 42),
(59, 17, 43),	(37, 60, 18),	(38, 61, 19),	(39, 62, 20),	(21, 40, 63),	(64, 22, 41),
(58, 52, 16),	(59, 53, 17),	(54, 18, 60),	(55, 19, 61),	(56, 20, 62),	(21, 63, 57),
(64, 51, 22),	(36, 23, 37),	(30, 24, 38),	(39, 31, 25),	(40, 32, 26),	(41, 33, 27),
(34, 28, 42),	(35, 29, 43),	(37, 23, 60),	(38, 24, 61),	(39, 25, 62),	(40, 26, 63),
(64, 41, 27),	(58, 42, 28),	(59, 43, 29),	(46, 60, 23),	(47, 61, 24),	(48, 62, 25),
(49, 63, 26),	(50, 64, 27),	(44, 58, 28),	(45, 59, 29),	(44, 30, 38),	(45, 31, 39),
(46, 32, 40),	(47, 33, 41),	(48, 34, 42),	(49, 35, 43),	(50, 36, 37),	(30, 56, 43),
(31, 57, 37),	(38, 32, 51),	(39, 33, 52),	(40, 34, 53),	(54, 41, 35),	(55, 42 ,36),
(50, 37, 57),	(44, 38, 51),	(45, 39, 52),	(46, 40, 53),	(47, 41, 54),	(48, 42, 55),
(49, 43, 56),	(44, 51, 65),	(45, 52, 66),	(46, 53, 67),	(47, 54, 68),	(48, 55, 69),
(49, 56, 70),	(50, 57, 71),	(44, 65, 58),	(45, 66, 59),	(46, 67, 60),	(47, 68, 61),
(48, 69, 62),	(49, 70, 63),	(50, 71, 64),	(64, 65, 51),	(58, 66, 52),	(67, 53, 59),
(68, 54, 60),	(55, 61, 69),	(56, 62, 70),	(63, 71, 57),	(58, 65, 66),	(67, 59, 66),
(67, 68, 60),	(68, 69, 61),	(69, 70, 62),	(63, 70, 71),	(64, 71, 65),	(66, 65, 72),
(67, 66, 72),	(67, 72, 68),	(68, 72, 69),	(69, 72, 70),	(71, 70, 72),	(65, 71, 72).

witz surface of genus 7 in [22]. We show corresponding pictures with a dihedral symmetry D_7 of order 14 in Figure 4 and Figure 5.

Figure 4: Topological visualization of Hurwitz's regular map $(3, 7)$ of genus 7 of J. J. van Wijk, [21].

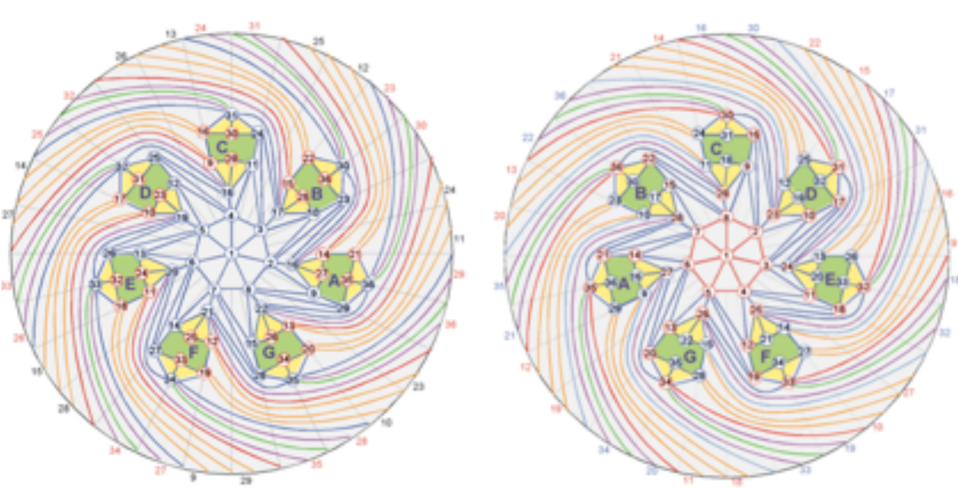

Figure 5: Topological visualization of Hurwitz's regular map $(3, 7)$ of genus 7 of C. H. Séquin, [20], see also [19].

Unfortunately, the method of Razafindrazaka and Polthier did not work in the case of Hurwitz's surface of genus 7 to provide an additional topological visualization. However, we have an additional different topological visualization as a 3D-Model that was helpful during our investigation for finding a polyhedral realization, see Figure 6.

This model of Figure 6 shows seven six-gons (marked by white connections inside the outer torus) around the axis that is fixed under the dihedral symmetry.

Figure 6: Topological visualization of Hurwitz's regular map $(3, 7)$ of genus 7 as a 3D-model. (This model was presented at the Jörgshop at the Technical University Berlin in June 2017.)

When we cut the model along those six-gons, we see that we can split the surface in two parts having 84 triangles each. On the one hand we obtain a topological torus with these seven six-gons as holes and on the other hand we have a topological 2-sphere with these seven holes.

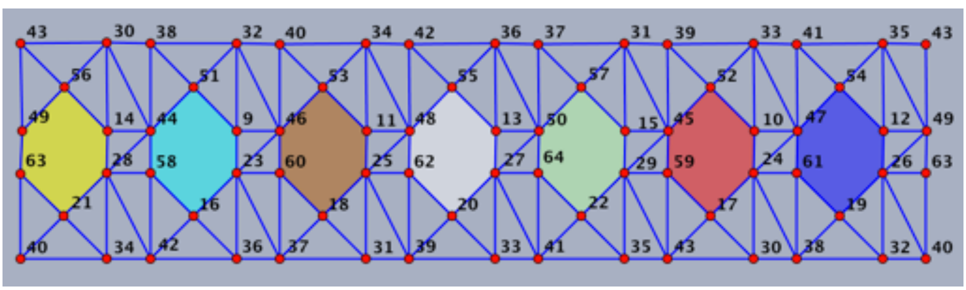

Figure 7: Triangles of the torus with seven holes each bounded by a polygon of length 6.

In Figure 7 we have depicted the combinatorial torus structure and in Figure 8 we see the corresponding 84 triangles of the sphere. Whereas both of these parts of the Hurwitz surface of genus 7 can easily be represented with planar triangles and without self-intersections, we see that the cyclic sequences of the holes in both cases do not coincide. However, they have to match. This is a clear indication that we probably cannot hope for a corresponding symmetric realization of order 7.

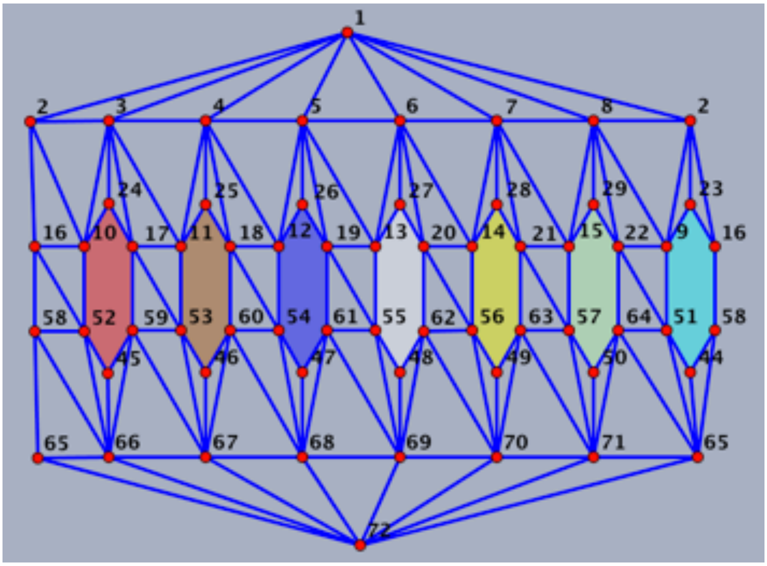

Figure 8: Triangles of the sphere with seven holes each bounded by a polygon of length 6.

4 Polyhedral realization

4.1 An algorithm

We use the following simple algorithm to obtain realizations within a few minutes (depending on the choice of distances and parameters):

1. Choose randomly a set of 72 distinct points $P = \{P_1, \dots, P_{72}\} \subseteq \mathbb{Q}^3$ with rational coordinates.
2. Count the number w_0 of pairs of labels of a triangle and labels of an edge of a triangle in Table 1 for which the corresponding points in P produce an intersection of a triangle and an edge.
3. Remember the points involved in these w_0 intersections in a set I.
4. While $w_0 > 0$, do:
   (a) Move a randomly chosen point of I into a random direction in such a way that it does not go too far away and not too close to the other points.
   (b) As above, count the number w of intersections and remember the points involved in intersections in a new set I.
   (c) If $w > w_0$ then move the point back to its place, else: $w_0 \leftarrow w$.
5. Output the solution.

An implementation in C produces for example the solution displayed in Table 2. To be completely sure that this output is correct one may check it using the code in Figure 9 or Figure 10.

Table 2: Coordinates of a polyhedral realization without self-intersections of Hurwitz's surface of genus seven.

no.	x	y	z	no.	x	y	z	no.	x	y	z
1	430	-270	-1000	2	959	-237	-213	3	434	-984	-70
4	-418	-861	-677	5	-988	98	-665	6	-272	-139	-814
7	299	577	-988	8	999	399	-854	9	981	727	-246
10	475	-498	408	11	361	-806	840	12	-509	115	609
13	-541,	-105	26	14	-299	434	-801	15	456	-230	-780
16	819	353	803	17	841	-663	868	18	-941	982	856
19	-928	694	-73	20	21	-294	158	21	-132	450	-319
22	526	305	-430	23	782	-550	996	24	172	-288	93
25	-859	-989	528	26	-679	983	697	27	-95	-239	-217
28	764	665	653	29	563	490	169	30	-872	507	-510
31	-413	-817	-561	32	136	921	30	33	432	-176	-157
34	522	778	359	35	489	-85	120	36	-470	84	709
37	-520	-823	679	38	-383	876	-325	39	365	-758	-25
40	114	900	838	41	240	176	-191	42	234	26	700
43	4	-150	345	44	261	843	-15	45	850	19	-196
46	902	679	797	47	17	36	114	48	-331	-763	720
49	-523	632	368	50	-254	-694	-243	51	367	659	-796
52	791	-11	367	53	194	442	411	54	34	376	304
55	-132	-413	773	56	103	-743	654	57	-240	-160	-832
58	940	632	175	59	567	-43	515	60	224	233	981
61	-254	268	182	62	-271	-721	265	63	-60	540	192
64	280	-119	-630	65	644	565	-266	66	516	538	265
67	-117	524	443	68	210	227	-110	69	-275	-204	444
70	-157	44	359	71	199	402	-282	72	90	510	140

4.2 Explanations

It is already very difficult to describe the geometric shape of any of the two parts of the Hurwitz surface of genus 7 that we have described in the last section. The Blender software is a powerful tool for 3D objects. In Figure 11, Figure 12, Figure 13, and Figure 14 you see some pictures of our realization. The reader can get a better understanding by using our corresponding Blender files for rotating the objects. Please write an e-mail to the authors. Glueing properly both parts along their boundaries leads to our polyhedral realization without self-intersections.

How did we check that the polyhedron has no self-intersections? We first confirmed that all vertices are in general position. This is equivalent to the fact that all determinants of any 4 points (by using homogeneous coordinates) are non-zero. Afterwards we have checked all pairs $(\text{edge}, \text{triangle})$ for intersections. Edges that have a vertex in common with a triangle cause no problem, because the points are in general position.

The other cases $(\text{edge}, \text{triangle})$ depend on the signs of the five determinants obtained from the five 4-element subsets of the set formed by the vertices of the edge and the triangle (again using homogeneous coordinates). When the two vertices of an edge lie on the same side of the plane determined by the triangle, we have no intersection. Otherwise we pick a vertex of the edge as the apex of a cone generated by the triangle. Precisely, when all three planes determined by the faces of this cone have the other vertex of the edge on the same side as the remaining vertex of the triangle, we have an intersection. In other words the other vertex of the edge lies within the convex cone, however beyond the triangle seen from the apex. We have double checked this result with two different programming methods, Haskell and Magma. When using exploded views, corresponding films, a symmetric realization, or even a geometric model, the reader might gain additional insight. Our attempts to find a symmetric realization were not successful. For a cyclic symmetry of order 7 we have even seen an argument that tells us how unlikely the existence of such a realization might be.

```
coordinates:= . . .
triangles:= . . .
edges:=[Sort(SetToSequence(k)) :
        k in &join [{{a[1],a[2]},{a[1],a[3]},{a[2],a[3]}} : a in triangles]];

for e in edges, t in triangles do
  if #(SequenceToSet(e) meet SequenceToSet(t)) eq 0 then
    x:=t[1]; y:=t[2]; z:=t[3]; a:=e[1]; b:=e[2];

Y:=<[x,y,z,a],[x,y,z,b],[a,x,y,b],[a,x,y,z],[a,y,z,b],[a,y,z,x],[a,x,z,b],[a,x,z,y]>;

    D:=[Determinant(Matrix(4,&cat [coordinates[i] : i in u])) : u in Y];
    D:=[d eq 0 select 0 else (d gt 0 select 1 else -1) : d in D];

    if D[1] ne D[2] and D[3] eq D[4] and D[5] eq D[6] and D[7] eq D[8] then
      printf "edge␣%o␣and␣triangle␣%o:␣",e,t;
      error "intersection!";
    end if;

    if 0 in D then
      error "zero␣determinant!";
    end if;
  end if;
end for;
```

Figure 9: Magma code.

```
module Hurwitz where
import Data.List
type MA =[[Integer]]         -- matrix
type OB =(Tu,Or)             -- oriented base
type Tu = [Int]              -- tuple of elements
type Or =  Int               -- orientation

-- ch=[([a,b,c,x],s),([a,b,c,y],s),([a,b,x,y],s),([a,c,x,y],s),([b,c,x,y],s)]
check::[Int]->[Int]->[[Integer]]->Bool
check triangle edge matrix | (length (nub(triangle++edge)) < 5) = True
                           | snd(ch!!0) == snd(ch!!1)           = True
                           | snd(ch!!0) == snd(ch!!2)           = True
                           | snd(ch!!0) /= snd(ch!!3)           = True
                           | snd(ch!!0) == snd(ch!!4)           = True
                           | otherwise                          = False
                     where ma = subMA (triangle++edge) matrix
                                ch  = m2Chi ma
edges::[[Int]]
edges = nub(  [[el!!0]++[el!!1]|el<-triangles]++[[el!!1]++[el!!2]|el<-triangles]
                                 ++[[el!!2]++[el!!0]|el<-triangles])

tuples::Int->Int->[[Int]] -- r -> n -> all r-tuples of [1..n]
tuples 0 n = [[]]
tuples r n = tuplesL r [1..n]

tuplesL::Int->[Int]->[[Int]]-- r -> list -> all r-tuples of list
tuplesL  r list@(x:xs)
  | length list <  r = []
  | length list == r = [list]
  | r == 1           = [[el]|el<-list]
  | otherwise = [[x]++el| el<-tuplesL (r-1) xs]++tuplesL r xs

det::MA -> Integer -- matrix -> determinant of matrix
det  m  |n == 1   = head (head m)
             |otherwise=sum(map (\i->((-1)^(i+1))*(head(m!!i))
                                *(det [(map tail m)!!l|l<-[0..n-1],l/=i]))[0..n-1])
             where n = length m

dets::[[Int]]->MA-> [Integer]-- rsets -> matrix -> (r x r)-sub-determinants
dets sets matrix = [det[matrix!!(i-1)|i<-set]|set<-sets]

m2Chi::MA->[OB]  -- matrix -> chirotope of matrix
m2Chi m =zip trn (map fromInteger (map signum(dets trn m)))
         where n  = length m
               r  = length(head m)
               trn= tuples r n

subMA::[Int]->MA->MA -- indices -> matrix -> submatrix
subMA t m = map(\i->m!!(i-1))t

coord::MA
coord =         list of homogeneous coordinates
triangles::[[Int]]
triangles=     list of triangles
```

Figure 10: Haskell code with some explanations.

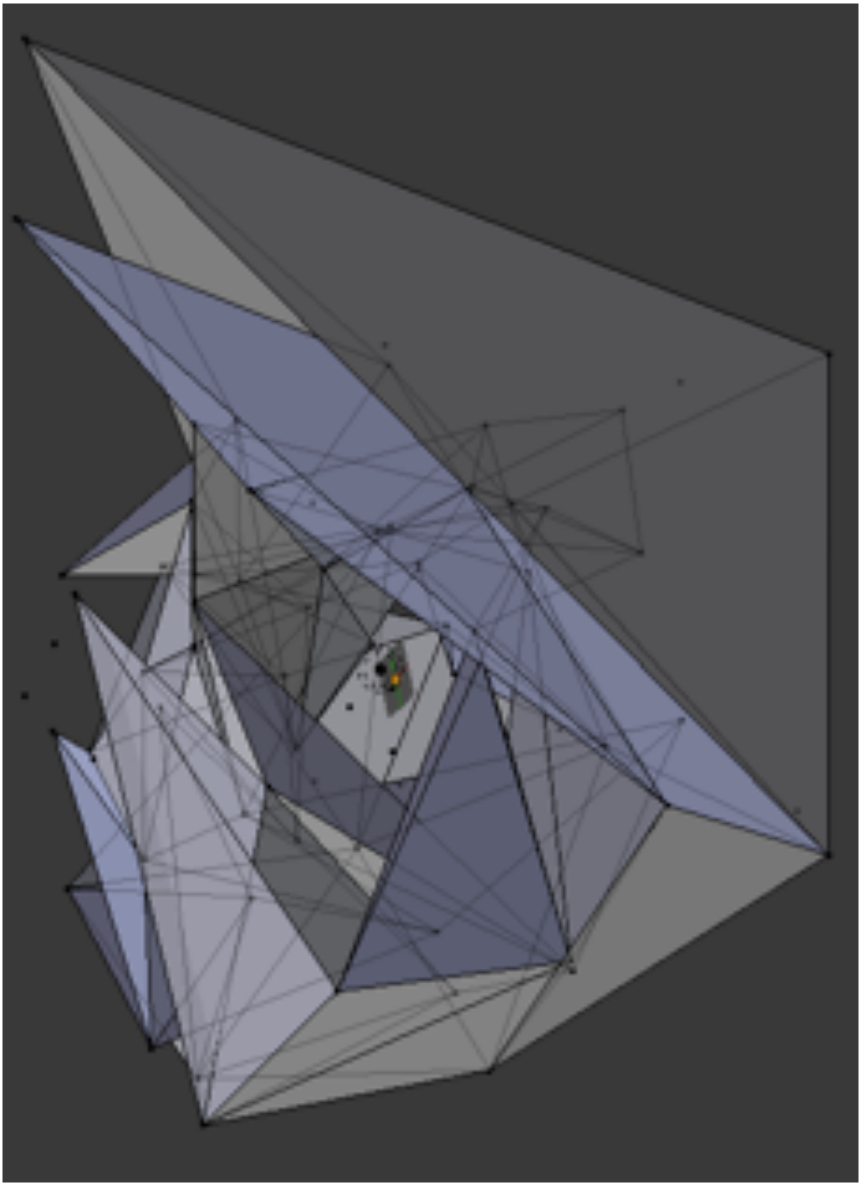

Figure 11: Polyhedral realization of the sphere with seven holes each bounded by a polygon of length 6.

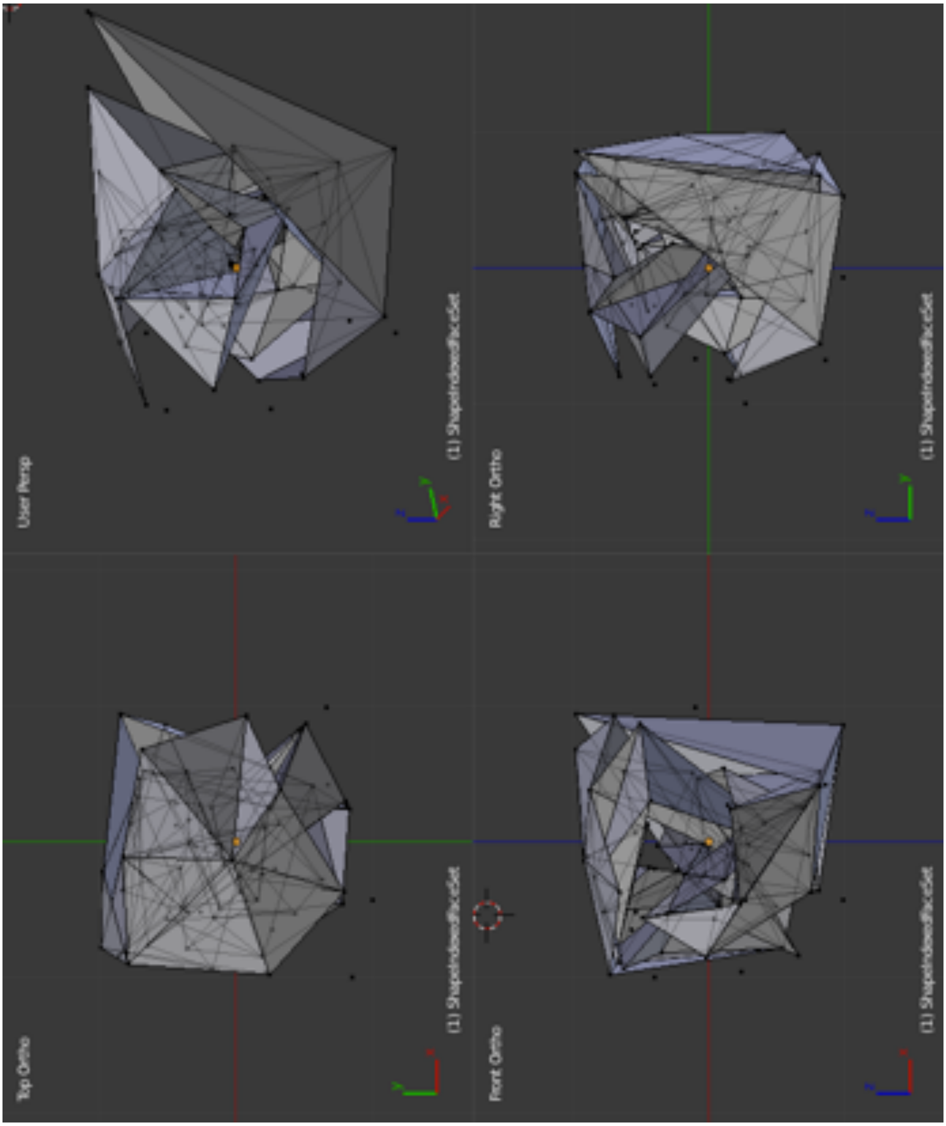

Figure 12: Polyhedral realization of the torus with seven holes each bounded by a polygon of length 6. Three orthogonal projections and a perspective view. Even half of the complete polyhedral realization is difficult to understand.

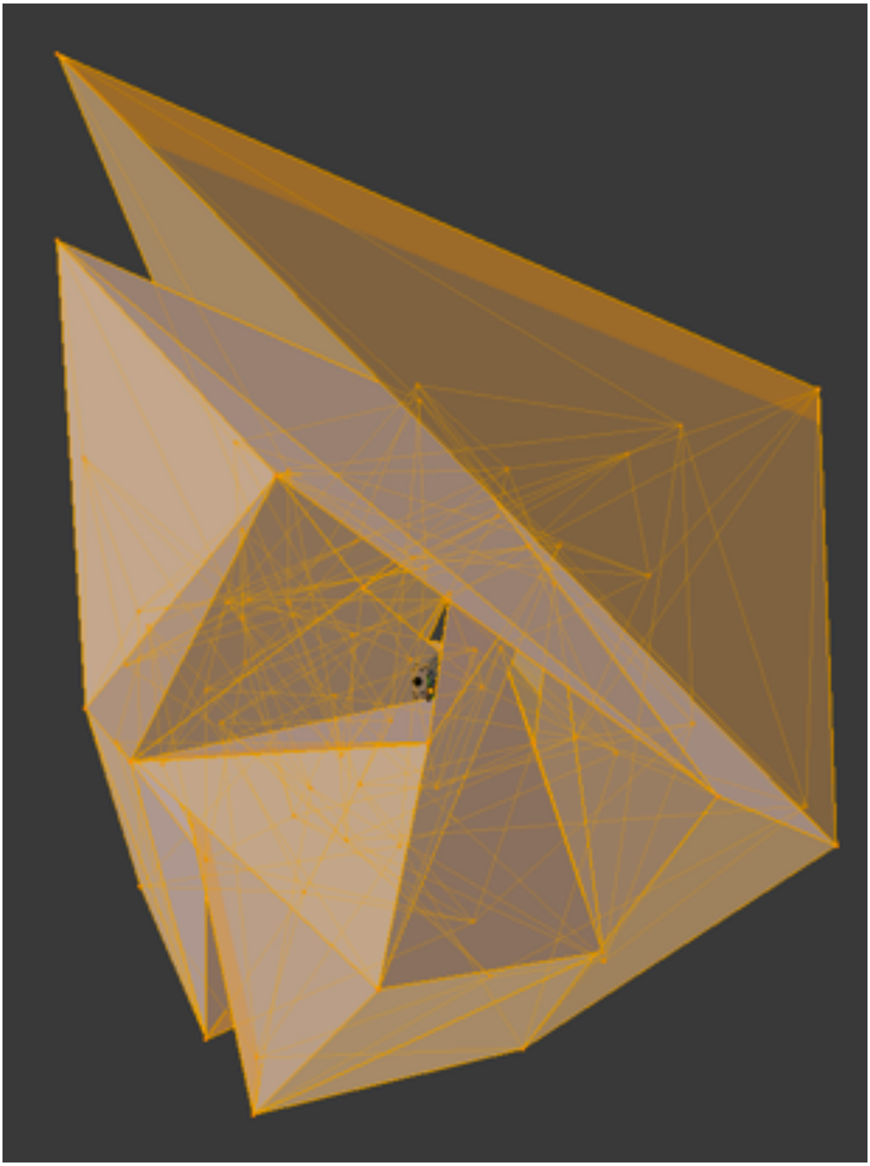

Figure 13: Polyhedral realization Hurwitz's surface of genus 7.

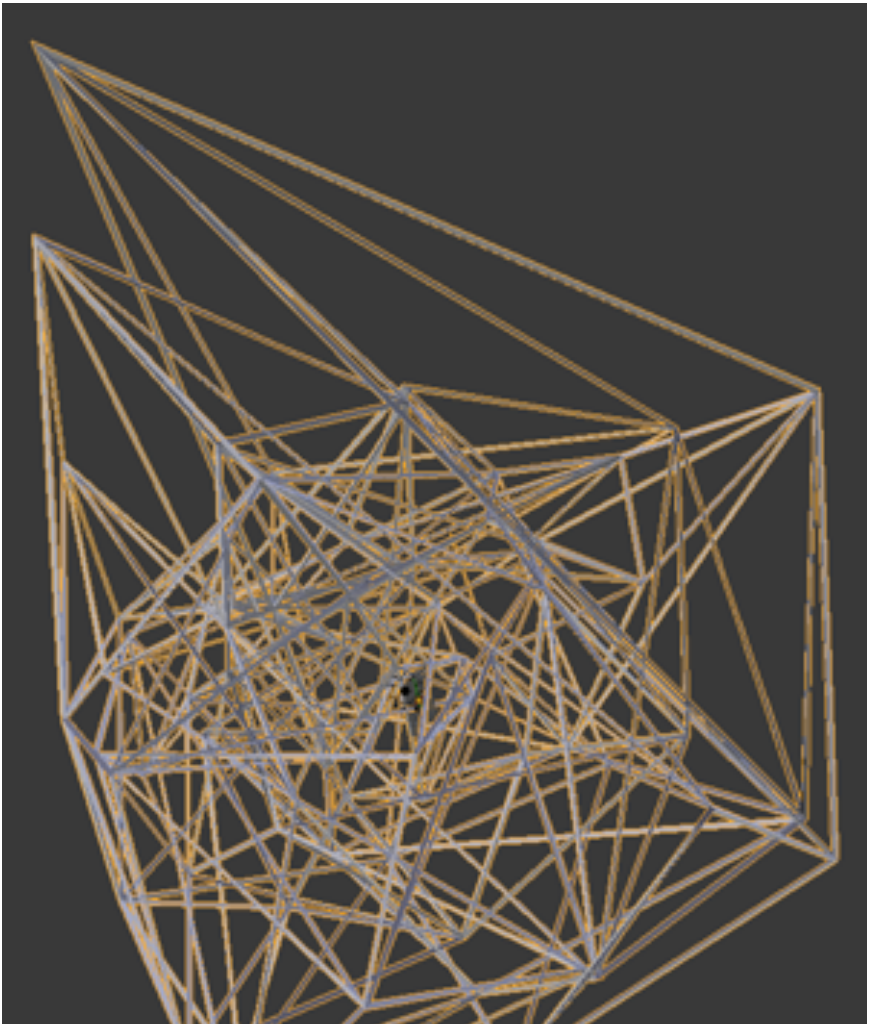

Figure 14: Polyhedral realization Hurwitz's surface of genus 7, complete wireframe.

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Classification of robust cycle bases and relations to fundamental cycle bases

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Abstract

The construction of a cycle in a graph can be realized by iteratively adding cycles of a cycle basis. The construction of each elementary cycle is only possible if this cycle basis is robust. In the last years, different classes of robust cycle bases have been established. We compare these classes and show that they are completely unrelated. More precisely, we draw a Venn diagram which displays the obvious containedness relations and show that each of its regions is not empty. In addition, we continue the comparison with fundamental cycle bases.

Keywords: Minimum cycle basis, robust cycle basis, quasi-robust cycle basis, fundamental cycle basis.

Math. Subj. Class.: 05C10, 05C38, 05C50

1 Introduction

Cycle bases of graphs have numerous applications, e.g. in the fields of periodic timetable optimization [9], coordination of traffic signals [15], or chemistry [4]. The first reference [9] additionally provides a useful classification of several types of cycle bases utilized for computations in the mentioned areas. The author considered the seven classes of directed, undirected, integral, totally unimodular, planar, as well as weakly and strictly fundamental cycle bases and compared them to each other.

Another line of research has been initiated by Kainen [6] who investigated robust cycle bases. Strengthening and weakening the concept of robust cycle bases led to four different types of robust cycle bases, which were further studied in [8] and [12], and recently in [7]. The latter paper provides an application of robust cycle bases to the analysis of commutative diagrams in groupoids.

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Similarly to the work of Liebchen [9], we show in our paper that no two of these four classes coincide and give a separating example for each pair of the classes. All of our examples provide a graph with its uniquely minimum cycle basis. This indicates that each class of robust cycle bases admits its own minimization problem.

A further focus in this paper is the relationship of robust and fundamental cycle bases, The investigation on this topic has been initiated in [8]. We continue this research by providing more examples of cycle bases which are even minimum in almost all cases. We are able to eliminate one of two question marks in a map given there, where the authors conjectured the existence of examples.

The results in this paper appeared also in the thesis [13].

2 Preliminaries

Throughout the paper, we consider only simple undirected weighted graphs $G = (V, E)$ with finite node set $V(G) = V$, finite edge set $E(G) = E$, and weight function $w\colon E \to \mathbb{R}_{>0}$. The *degree* of a node $v \in V$ is denoted by $\deg(v)$. A *path* P of length ℓ in a graph is a sequence $P = (v_0, v_1, \ldots, v_\ell)$ of pairwise disjoint nodes with $v_{i-1}v_i \in E$ for $1 \leq i \leq \ell$. The length of a shortest path between two nodes u and v in G is called the *distance* $\mathrm{dist}_G(u, v)$. A path from node u to node v is referred to as u-v-path.

A *circuit* C in G is a non-empty connected subgraph of G with $\deg(v) = 2$ for all $v \in V(C)$. We define $|C| := \sum_{e\in E(C)} w(e)$ as the *length* of a circuit C. A *cycle* Z in G is a subgraph of G where $\deg(v)$ is even for all $v \in V(Z)$.

For a spanning tree $T = (V, E(T))$ of G and an edge $e \in E \setminus E(T)$ define the *fundamental circuit* $C_T(e)$ as the unique circuit in $(V, E(T) \cup \{e\})$. The non-tree edges are also called *chords* of the spanning tree T. We usually identify circuits, cycles, and trees with their edge sets.

The *cycle space* $\mathcal{C}(G)$ of a graph $G = (V, E)$ is the vector subspace of $\mathrm{GF}(2)^E$ that is generated by the incidence vectors of the circuits in G. The sum of two cycles Z_1 and Z_2 in this vector space is their symmetric difference $(Z_1 \cup Z_2) \setminus (Z_1 \cap Z_2)$. A *cycle basis* B of G is a set of $\nu = m - n + 1$ circuits whose incidence vectors form a basis of $\mathcal{C}(G)$. The *size* $\Phi(B)$ of a cycle basis B is defined as $\Phi(B) := \sum_{C\in B} |C|$. A cycle basis B of $G = (V, E)$ is designated *strictly fundamental* iff there is a spanning tree $T = (V, E(T))$ with $B = \{C_T(e) \mid e \in E \setminus E(T)\}$.

A cycle basis $B = \{Z_1, \ldots, Z_\nu\}$ is *weakly fundamental* if there exists a permutation $\pi \in S_\nu$ such that

$$Z_{\pi(i)} \setminus \bigcup_{j=1}^{i-1} Z_{\pi(j)} \neq \emptyset \quad \text{for all } i = 2, \ldots, \nu. \tag{2.1}$$

Weakly fundamental cycle were also the matter of [5] where the authors characterized graphs for which every cycle basis is weakly fundamental.

If B is a cycle basis then every cycle Z has a unique representation $Z = \sum_{C\in B} \lambda_C C$ with $\lambda_C \in \{0, 1\}$. The subset $\{C \in B \mid \lambda_C = 1 \text{ and } Z = \sum_{C\in B} \lambda_C C\}$ is called the *support* $\mathrm{supp}(Z)$.

The following simple lemma is needed to justify the minimality of some of our cycle bases.

Lemma 2.1. *For a given strictly fundamental cycle basis B of an undirected graph $G = (V, E)$ one can always find a weight function w such that B is the unique minimum cycle* *basis of G.*

Proof. Let T be a fundamental spanning tree which induces B. For every edge $e \in T$ set $w(e) = 1$. Define $d := \max\{\mathrm{dist}_T(u,v) \mid uv \in E \backslash T\}$ and assign $w(e) = 2d - \mathrm{dist}_T(u,v)$ for the remaining edges $e = uv$. Observe that the minimum of w restricted to the chords is d. Now, every circuit in B has a weight of $2d$ while all other cycles of G have a greater weight since they contain at least two chords and at least one tree edge or at least three chords. □

For Example 5.5, the following enhancement of Lemma 2.1 is necessary.

Lemma 2.2. *For a given strictly fundamental cycle basis B of an undirected graph $G = (V, E)$ one can always find a weight function w such that B is the unique minimum cycle basis of G and such that there is a chord $e = uv$ with $w(e) < \mathrm{dist}_T(u, v)$.*

Proof. The proof has essentially the same structure as the proof of Lemma 2.1. Thus, set $w(e) = 1$ for all tree edges of a given fundamental spanning tree T which induces B. And again, let $d := \max\{\mathrm{dist}_T(u,v) \mid uv \in E \setminus T\}$. For the edges $e = uv$ in $E \setminus T$, we now assign the weight $w(e) = 2d - \mathrm{dist}_T(u,v) - \varepsilon$, for an $\varepsilon > 0$ whose value is determined later. The minimum of w restricted to the chords is $d - \varepsilon$, and each circuit $C \in B$ has the weight $w(C) = 2d - \varepsilon$.

Now, look at a circuit which is not in B. It consists of $c \geq 2$ chords and $t \geq 0$ tree edges. Furthermore, $c = 2$ implies $t \geq 1$. The length of the circuit is at least $c(d - \varepsilon) + t$. For all $\varepsilon \in (0, \frac{(c-2)d+t}{c-1})$, this value is greater than $2d - \varepsilon$, i.e. greater than the weight of a basic circuit. Because $c \geq 2$, the denominator of the upper endpoint of the interval is not zero, and since $c + t \geq 3$, also the numerator is not zero. Hence, this interval is not empty and we can take any ε from this interval.

Finally, for a chord $e = uv$ with $\mathrm{dist}_T(u,v) = d$, the weight $w(e)$ has the value $d - \varepsilon < \mathrm{dist}_T(u,v)$. □

3 Classes of robust cycle bases

In order to define the four different types of robust cycle bases, we essentially follow the exposition in [12]. Similarly as there, we need at first the concept of (strictly) well-arranged sequences of circuits. Afterwards, we deduce several simple inclusions and present a map of the relationship between the different classes of robust cycle bases.

Definition 3.1 (*(Strictly) well-arranged sequence*)**.** A sequence $S = (C_1, \ldots, C_k)$ of circuits in an undirected graph is called *well-arranged* if for all $j = 1, \ldots, k$ the GF(2)-sum $\sum_{i=1}^{j} C_i$ is also a circuit. A well-arranged sequence of circuits is *strictly well-arranged* if for all $j = 2, \ldots, k$ the intersection $C_j \cap \sum_{i=1}^{j-1} C_i$ is a single path.

The path in Definition 3.1 contains at least one edge. Otherwise, the sum $C_j + \sum_{i=1}^{j-1} C_i$ was not a circuit and thus, the sequence was not even well-arranged, at all. It is clear that every strictly well-arranged sequence is also well-arranged. Furthermore, it is known that there are well-arranged sequences that are not strictly well-arranged. The authors of [8] provide such an example in which the sum of two basic circuits is again a circuit, but they intersect in three paths. Note that it is not forbidden that a circuit appears more than once in a (strictly) well-arranged sequence.

With this in mind, we are now able to define the four different types of robust cycle bases which were developed in [12].

Definition 3.2 (*Cyclically/strictly robust and (strictly) quasi-robust cycle basis*)**.** A cycle basis B of a graph G is *(strictly) quasi-robust* if for each circuit C in G there is a (strictly) well-arranged sequence $S_C = (C_1, \dots, C_{k-1}, C_k)$ such that $C = \sum_{i=1}^{k} C_i$ and $C_i \in B$ for $i = 1, \dots, k$. A strictly quasi-robust cycle basis is *strictly robust* if the circuits in the strictly well-arranged sequence are pairwise disjoint. Analogously, a quasi-robust cycle basis is *cyclically robust* if the according well-arranged sequence does not contain a circuit twice. If we do not want to specify the particular type of robustness, we simply speak about a *robust cycle basis*.

It can be concluded that for strictly and for cyclically robust cycle bases the well-arranged sequence of a circuit C must not contain basic circuits which are not in the support of C. Also, directly from these definitions, we can immediately derive the following facts:

- every strictly quasi-robust cycle basis is quasi-robust,
- every strictly robust cycle basis is strictly quasi-robust,
- every strictly robust cycle basis is cyclically robust, and
- every cyclically robust cycle basis is quasi-robust.

These inclusions hold since in each case, we require additional properties for the more specific class. The inclusions give rise to the diagram in Figure 1.

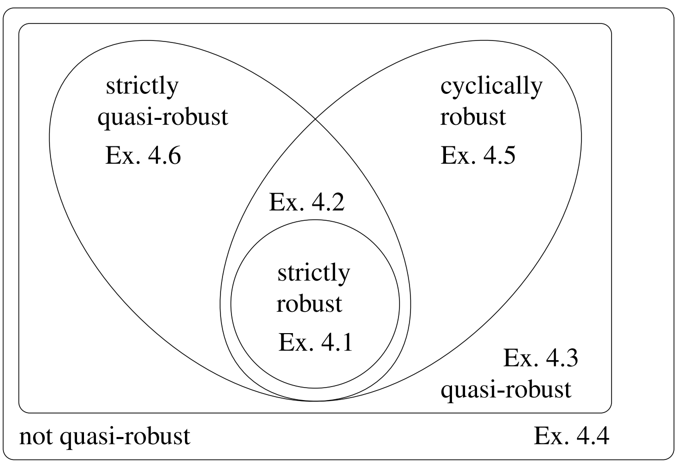

Figure 1: Map of robust cycle bases.

Not much is known about which graph classes can have which type of robust cycle bases. Furthermore, it is unknown whether each graph admits a robust cycle basis of any of the four types. Table 1 summarizes the related results. To the best of our knowledge, these are the only known ones.

Table 1: Summary of known graph classes for which the stated type of cycle bases is guaranteed.

Graph class	Robustness	Reference
planar graphs	strictly robust	[2]
complete graphs	strictly robust	[6]
complete bipartite graphs $K_{m,n}$ with $m \leq 4$ and $n \leq 5$	strictly robust	[12]
general complete bipartite graphs	quasi-robust	[12]
wheels	cyclically robust	[8]

4 Examples of robust cycle bases

In this section, we show that the inclusions derived in the last section are valid only in the given direction. Thus, no two of the classes are equivalent. To point this out, we give an example of a graph and a cycle basis for each region in the map in Figure 1 and thus show that it is not empty.

Except in Example 4.2, all cycle bases are strictly fundamental. According to Lemma 2.1, we can choose a weight function such that this cycle basis is the unique minimum cycle basis on this graph. However, the given cycle basis in Example 4.2 is also the unique minimum one. The existence of a graph with a minimum cycle basis in each region of the map indicates that each class—actually even each non-empty difference of two classes—of robust cycle bases admits its own minimization problem.

Remember that we do not know an efficient algorithm for the computation or for the recognition of any type of robust cycle bases on general graphs. Thus, to prove a cycle basis of a graph G as (strictly) quasi-robust, we have to indicate a (strictly) well-arranged sequence of basic circuits for every circuit in G. Analogous sequences have to be found for (strictly) robust cycle bases. In the latter case, a basic circuit is allowed to occur at most one time in each of these sequences.

On the other hand, a cycle basis B of a graph G is not quasi-robust if there exists a circuit C' in G such that for each $C \in B$ the sum $C' + C$ is not a circuit. To show that the cycle basis is not strictly quasi-robust, one has to verify that the cut $C' \cap C$ does not form a path for one circuit C' in the graph and for all $C \in B$. Finally, to show that a cycle basis is not a cyclically robust or a strictly robust cycle basis it suffices to check only the circuits of the support of such a circuit C'.

We now start with the description of the examples.

Example 4.1 (*Strictly robust cycle basis*)**.** The first example is the simple graph C_3 that consists of exactly one circuit of length 3. Clearly, its unique cycle basis is strictly robust—and strictly fundamental and minimum, as well.

Example 4.2 (*Cyclically robust and strictly quasi-robust cycle basis—not strictly robust*)**.** Our second example is the complete bipartite graph $K_{3,3}$, see Figure 2 (a). The cycle basis $B = \{C_1, C_2, C_3, C_4\}$ is highlighted in Figure 2 (b). It is not strictly fundamental, thus, we suggest the indicated weights to make the cycle basis minimum. All other circuits have a greater weight. The weights of all circuits are denoted below the graph in Figures 2 (b), (c) and (d). We show that B is cyclically robust and strictly quasi-robust, but not strictly robust. For $2 \leq k \leq \nu$ we denote $C_{i_1,\ldots,i_k} := \sum_{j=1}^{k} C_{i_j}$.

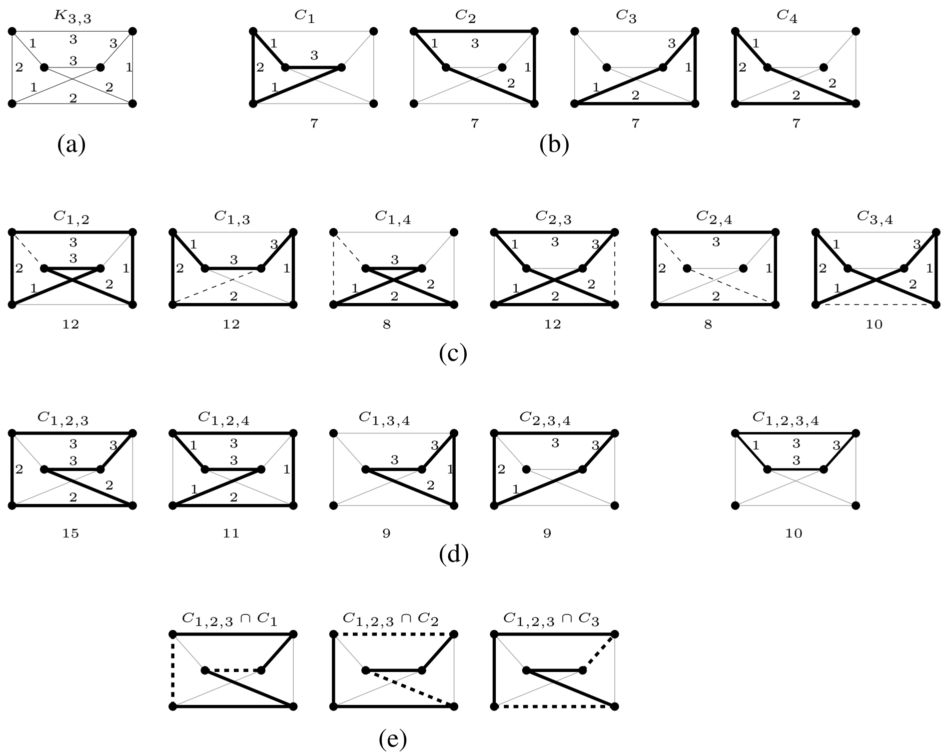

Figure 2: The $K_{3,3}$ with weights on the edges (a). The four basic circuits and their weights below (b). All other circuits and their weights (c) and (d). The intersections (dashed edges) of $C_{1,2,3}$ with the circuits of its support (e).

Cyclically robust. The $K_{3,3}$ is cubic, hence it contains no cycle with vertices of degree 4 or more. Furthermore, it has only six vertices, but it is triangle free. Thus, there is no cycle consisting of two triangles. This means that every cycle is a circuit and therefore, each cycle basis of $K_{3,3}$ is cyclically robust.

Not strictly robust. The given basis is not strictly robust, since there is no strictly well-arranged sequence for $C_{1,2,3}$, in which every basic circuit occurs only once. Observe this by looking at Figure 2 (e). It is indicated that $C_{1,2,3}$ has an intersection consisting of two path (dashed edges) with each circuit from its support.

Strictly quasi-robust. For the circuits which have exactly two basic circuits in their supports, these two circuits intersect in a single path, illustrated by the dashed edges in Figure 2 (c). For the circuits depicted in Figure 2 (d) we provide the sequences $S_{C_{1,2,3}} = (C_1, C_3, C_4, C_2, C_4)$, $S_{C_{1,2,4}} = (C_1, C_4, C_2)$, $S_{C_{1,3,4}} = (C_1, C_3, C_4)$, $S_{C_{2,3,4}} = (C_2, C_4, C_3)$, and $S_{C_{1,2,3,4}} = (C_1, C_3, C_4, C_2)$, which are all strictly well-arranged. Hence, the cycle basis is strictly quasi-robust.

Example 4.3 (*Quasi-robust cycle basis—neither cyclically robust nor strictly quasi-robust*)**.** This example is borrowed from [6]. We consider the complete bipartite graph $K_{5,5}$, the strictly fundamental cycle basis B induced by the spanning tree T shown in Figure 3 (a), and the circuit C aside in Figure 3 (b). The sixteen basic circuits themselves are also depicted in Figure 4 as black edges. Assigning weights according to Lemma 2.1, B becomes the unique minimum cycle basis.

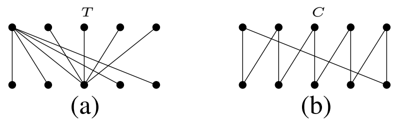

Figure 3: Spanning tree T of $K_{5,5}$ (a). The circuit C considered in the text (b).

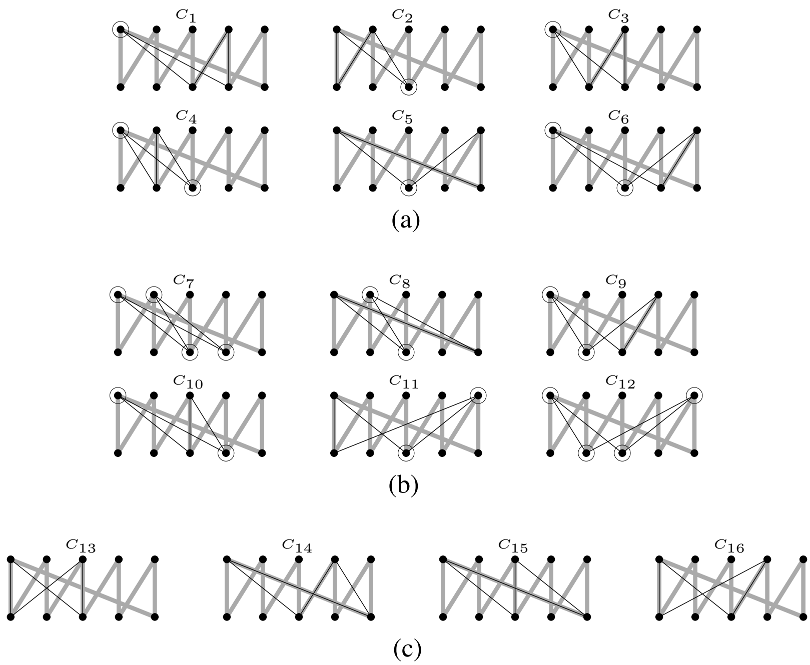

Figure 4: The sixteen basic circuits of B (black edges) and the circuit C (grey edges). For the sake of clearness we dropped the edges which are neither in the basic circuit nor in C.

Quasi-robust. The described basis had been shown to be quasi-robust in [12] in an elaborate manner.

Not cyclically robust. The circuit C can be written as $C = \sum_{i=1}^{6} C_i$ and all the sums $C + C_i$ for $i = 1, \ldots, 6$ are cycles with node degrees greater than 2 (marked by a circle), see Figure 4 (a). Hence, this cycle basis is not cyclically robust.

Not strictly quasi-robust. Looking at the remaining basic circuits, we observe that also C_7 to C_{12} yield cycles with node degrees of 4, Figure 4 (b), again marked by a circle. The intersection of C_{13} to C_{16} with C is not a single path in each case, as can be seen in Figure 4 (c). In addition, $C + C_{13}$ and $C + C_{14}$ are disconnected. All in all, the cycle basis is not strictly quasi-robust.

Example 4.4 (*Cycle basis, not even quasi-robust*)**.** The example of a cycle basis which is not even quasi-robust presented here had been inspired by a talk of Ostermeier [11].[1]

The cycle basis is strictly fundamental and it is induced by the fat drawn tree in Figure 5 (a).

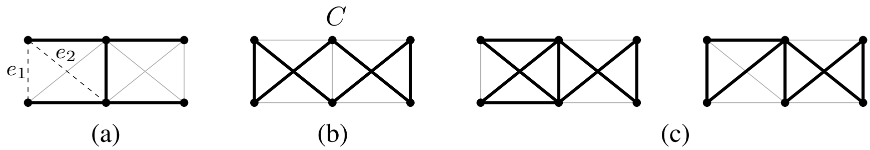

Figure 5: Graph with an inducing fundamental spanning tree (fat edges) and dashed chords e_1 and e_2 (a), a circuit C (b), and sums of C with two basic circuits generated by the chords e_1 and e_2 (c).

Not quasi-robust. Due to symmetry, we have to consider only the basic circuits induced by the dashed edges e_1 and e_2. In both cases, they add up with C to a cycle that is not a circuit, see Figure 5 (c).

Example 4.5 (*Cyclically robust cycle basis—not strictly quasi-robust*)**.** This example is a cycle basis on Wagner's graph V_8 which is cyclically robust, but not strictly quasi-robust. The strictly fundamental basis is indicated by the spanning tree which is highlighted in Figure 6 (a). The basic circuits are denoted at the chords. We use the notation from Example 4.2, i.e. $C_{i_1,\ldots,i_k} := \sum_{j=1}^{k} C_{i_j}$.

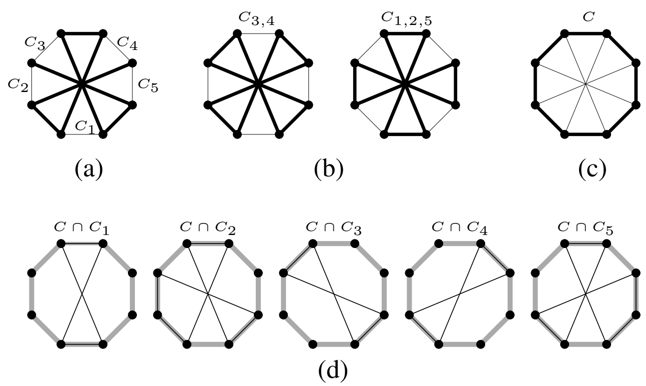

Figure 6: Wagner's graph V_8 with a fundamental spanning tree (a). The only two non-circuits in V_8 (b). The circuit C (c). The intersections of C (grey) with the five basic circuits do not form a single path (d), edges which are not in a basic circuit or in C are dropped.

Cyclically robust. Wagner's graph V_8 is cubic which implies that every cycle is 2-regular. The only critical cycles in V_8 are thus the two non-circuit pictured in Figure 6 (b). We provide the well-arranged sequences $S_{C_{3,4}+C_1} = S_{C_{1,3,4}} = (C_1, C_3, C_4)$, $S_{C_{3,4}+C_2} = S_{C_{2,3,4}} = (C_2, C_3, C_4)$, and $S_{C_{3,4}+C_5} = S_{C_{3,4,5}} = (C_4, C_5, C_3)$ for the circuits which arise by adding a remaining basic circuit to $C_{3,4}$. For the cycle $C_{1,2,5}$ we give the sequences

[1] A similar example already appeared in [14].

$S_{C_{1,2,5}+C_3} = S_{C_{1,2,3,5}} = (C_1, C_2, C_3, C_5)$ and $S_{C_{1,2,5}+C_4} = S_{C_{1,2,4,5}} = (C_1, C_2, C_4, C_5)$. In each of these sequences, every basic circuit appears at most once. This shows that the basis B is cyclically robust.

Not strictly quasi-robust. To see that the basis is not strictly quasi-robust, consider the circuit C in Figure 6 (c). Its intersection with each basic circuit does not form a single path. This is illustrated in Figure 6 (d).

Example 4.6 (*Strictly quasi-robust cycle basis—not cyclically robust*)**.** The last example provides a graph with a cycle basis $B = \{C_1, \ldots, C_6\}$ which is strictly quasi-robust but not cyclically robust. As in Example 4.2 denote $C_{i_1,\ldots,i_k} := \sum_{j=1}^{k} C_{i_j}$ for $2 \le k \le \nu$.

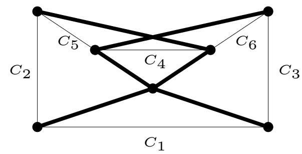

Figure 7: Graph with a fundamental spanning tree which induces a cycle basis that is strictly quasi-robust but not cyclically robust.

Strictly quasi-robust. Since $\nu = 6$ we have to investigate $2^6 - 6 - 1 = 57$ cycles; the six basic circuits and the zero vector are not interesting. The 22 cycles listed below are not circuits.

$$\begin{array}{llllllll} C_{1,4}, & C_{2,3}, & C_{1,5,6}, & C_{4,5,6}, & C_{1,4,5,6}, & C_{2,3,5,6}, & C_{1,2,3,4,5}, & C_{1,2,4,5,6}, \\ C_{1,5}, & C_{1,4,5}, & C_{2,4,6}, & C_{1,2,4,6}, & C_{2,3,4,5}, & C_{2,4,5,6}, & C_{1,2,3,4,6}, & C_{1,3,4,5,6}, \\ C_{1,6}, & C_{1,4,6}, & C_{3,4,5}, & C_{1,3,4,5}, & C_{2,3,4,6}, & C_{3,4,5,6} & & \end{array}$$

For the remaining eleven circuits $C_{i,j}$ with $|\operatorname{supp}(C_{i,j})| = 2$ we may ignore the order of the basic circuits. The intersection of the two basic circuits is a path in each case, and thus, the sequences are strictly well-arranged. For the 24 circuits with at least three elements in their supports, we provide the following strictly well-arranged sequences.

$$\begin{array}{llll} (C_1, C_2, C_3), & (C_1, C_3, C_6), & (C_4, C_6, C_3), & (C_1, C_2, C_5, C_6), \\ (C_1, C_2, C_4), & (C_3, C_4, C_2), & (C_5, C_6, C_3), & (C_4, C_6, C_3, C_1), \\ (C_1, C_2, C_5), & (C_3, C_5, C_2), & (C_1, C_2, C_3, C_4), & (C_5, C_6, C_3, C_1), \\ (C_1, C_2, C_6), & (C_3, C_6, C_2), & (C_1, C_2, C_3, C_5), & (C_1, C_2, C_3, C_5, C_6), \\ (C_1, C_3, C_4), & (C_4, C_5, C_2), & (C_1, C_2, C_3, C_6), & (C_1, C_2, C_3, C_6, C_5, C_4, C_1), \\ (C_1, C_3, C_5), & (C_2, C_5, C_6), & (C_4, C_5, C_2, C_1), & (C_1, C_2, C_3, C_5, C_6, C_4). \end{array}$$

Not cyclically robust. Figure 8 illustrates that the treated cycle basis is not cyclically robust. More precisely, look at the circuit $C_{2,3,4,5,6}$. For $i = 2, \ldots, 6$, the cycles $C_{2,3,4,5,6} + C_i$ have nodes with degree greater than 2, marked by circles in Figure 8. Hence, this circuit does not admit a well-arranged sequence in which the circuits are pairwise disjoint.

5 Relationship with fundamental bases

One approach for a better understanding of strictly robust and cyclically robust cycle bases had been presented in [8]. Therein, the authors investigated the relationship between strictly

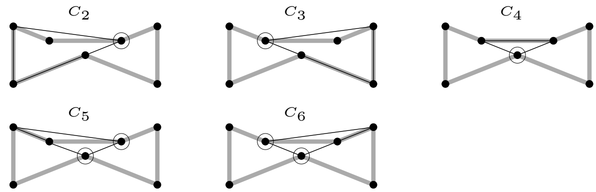

Figure 8: The circuit $C_{2,3,4,5,6}$ (grey) and the five basic circuits of its support (black edges).

robust, cyclically robust, and non-robust cycle bases on one hand, and strictly fundamental, weakly fundamental, and non-fundamental cycle bases on the other hand. Their motivation was the detailed exploration of strictly and weakly fundamental cycle bases which had been done in the years before. They concluded that robustness and fundamentality of cycle bases "are essentially unrelated concepts".

In more detail, they considered the combination (`robustness`, `fundamentality`), where `robustness` $\in$ {"strictly robust", "robust", "non-robust"} and `fundamentality` $\in$ {"strictly fundamental", "weakly fundamental", "non-fundamental"}. This immediately led to nine possibilities, and an example of a graph with an according cycle basis was presented in seven of these cases.

In this section, we follow up this line of research and provide for eight cases a graph with an appropriate cycle basis which is additionally minimum. For the ninth case, we are able to retire to a strictly quasi-robust cycle basis instead of a strictly robust one. However, this basis is not the minimum basis of the presented graph.

At the end of this section, we summarize our results in Table 2.

We start with three examples of strictly fundamental bases, that is the first column in Table 2. Two of them are taken from [8], the third one correlates to the basis in Example 4.5. Due to Lemma 2.1, all bases can be made minimum.

Example 5.1 (*Strictly fundamental—strictly robust*)**.** This example is directly taken from [8]. To be more accurate, we deal with the complete graph K_n and the cycle basis B_n which is induced by the complete bipartite graph $K_{1,n-1}$ as *fundamental spanning tree*. It is *strictly robust* as shown in [6]. With a weighting assigned according to Lemma 2.1 it is also the *unique minimum* cycle basis.

We decided to present this example here because it constitutes a whole class of graphs and cycle bases with the required properties. On the other hand, also the triangle graph in Example 4.1 could have served as an example at this place.

Example 5.2 (*Strictly fundamental—not strictly robust—cyclically robust*)**.** Wagner's graph V_8 and the cycle basis which had already been presented in Example 4.5 provide the necessary properties for this example. We remark that this example eliminates one of the two question marks in [8] where the authors conjectured the existence of such an example.

Example 5.3 (*Strictly fundamental—not cyclically robust*)**.** Again, we borrow the example given in [8] which is called there "Ostrowski's basis". It is simply the K_5 with a path consisting of four edges as *fundamental spanning tree*. This spanning tree induces a basis consisting of three triangles, two quadrangles, and one pentagon. To verify that the basis is *non-robust*, take a look at the circuit C which is the sum of the three triangles and the two quadrangles. The sum of C with each of these basic circuits constitutes a non-circuit.

Similarly to Example 5.2, we could have borrowed the graph with a non-robust cycle basis from Example 4.4. Anyway, we used Ostrowski's basis at this place because there is an easy way to construct an infinite class of graphs and cycle bases with the required properties. More precisely, we speak about the family of complete graphs with an odd number of vertices. For such a graph $G_k = (V_k, E_k)$ with $V_k = \{v_0, v_1, \ldots, v_{2k}\}$ we choose the path $(v_0, v_1, \ldots, v_{2k})$ as inducing spanning tree for the strictly fundamental cycle basis. As a certificate for the non-robustness, we provide the circuit $C_k = \bigcup_{i=0}^{2k} \{v_i v_{i+2}\} = \sum_{i=0}^{2k} \{v_i v_{i+1}, v_{i+1} v_{i+2}, v_{i+2} v_i\}$, where the indices are taken modulo $2k+1$. Adding one basic circuit $C_k^i = \{v_i v_{i+1}, v_{i+1} v_{i+2}, v_{i+2} v_i\}$ to C_k results in a cycle C'_k with $\deg_{C'_k}(v_{i+1}) = 4$. In Figure 9, the graph G_3 is given as an example.

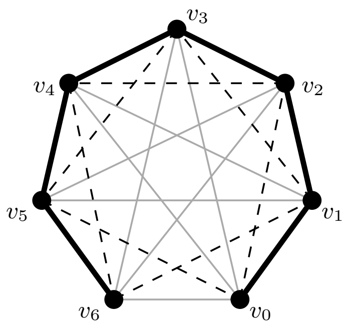

Figure 9: The graph G_3, the inducing spanning tree (fat edges), and the circuit C_k (dashed edges).

We continue with three examples which are weakly fundamental but not strictly fundamental. In Table 2, these examples appear in the second column. One example is taken from [9]. For the other two, we destroy the strictly fundamentality of the according examples above by gluing suitable graphs together. In doing so, we keep in mind that we want the bases to stay minimum.

Example 5.4 (*Not strictly fundamental—weakly fundamental—strictly robust*)**.** The additional demand for a minimum cycle basis prevents us from simply copying the according example in [8]. Instead, we copy Example 11.2 from [9], which deals with the sunflower graph SF(3), depicted in Figure 10. Therein, it served as an example for a 2-basis which is not strictly fundamental, where for a 2-basis, each edge is contained in at most two basic cycles.

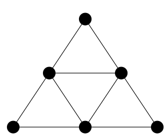

Figure 10: The sunflower graph SF(3).

The cycle basis B consisting of the only four triangles is the unique minimum cycle basis. Each edge of the middle triangle is contained in another circuit of B, thus, B is *not strictly* *fundamental*. But since the basis is a 2-basis, it is *weakly fundamental*, see e.g. [9], and *strictly robust* due to [2].

Example 5.5 (*Not strictly fundamental—weakly fundamental—not strictly robust—cyclically robust*)**.** The idea in this example is to adapt Wagner's graph and its cycle basis presented in Examples 4.5 resp. 5.2 such that it is not strictly fundamental anymore. To do this, we append a further path (v_1, v_2, v_3) at the two adjacent vertices v_1 and v_3 at the right hand side of the graph, see Figure 11.

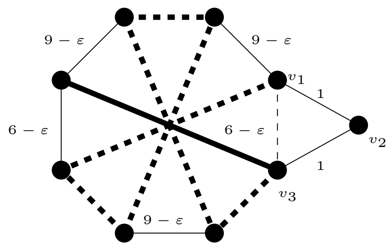

Figure 11: The modified Wagner's graph with a partial spanning tree (fat edges) and a circuit without private edge (dashed).

The weights of the graph are assigned according to Lemma 2.2. The second statement of this lemma does hold for v_1v_3, i.e. $w(v_1v_3) < \text{dist}_T(v_1, v_3)$. For the new edges set $w(v_1v_2) = w(v_2v_3) = 1$. To yield the cycle basis, inherit the basic circuits from the original example and append the circuit $C_6 = (v_1v_2, v_2v_3, v_3v_1)$. Remark that the weights of the old edges were chosen according to Lemma 2.2 and that C_6 is the shortest circuit which contains the new vertex v_2. Hence, the obtained cycle basis is minimum.

The basis is *not strictly robust* for the same reasons as in Example 4.5. On the other hand, assume that a circuit C in this graph contains the vertex v_2. A well-arranged sequence for C can be achieved by concatenating C_6 with the well-arranged sequence of $C + C_6$, hence, the basis is *cyclically robust*. Finally, the cycle basis is *not strictly fundamental*, since the dashed basic circuit does not have a private edge. But it is *weakly fundamental* because Inequality (2.1) holds for each permutation π with $C_{\pi(6)} = C_6$.

Example 5.6 (*Not strictly fundamental—weakly fundamental—not cyclically robust*)**.** Similarly to the example above, we destroy the strictly fundamentality of Ostrowski's basis of the K_5. We also could have used Lemma 2.2 and could have constructed a graph by simply appending a path of length 2 as in Example 5.5. Anyway, we decided to provide a larger example in favor of an integer weight function.

Remember that the basis of this graph was induced by a path of four edges as fundamental spanning tree. There is one edge between the end nodes of the path, denote it e_P. Now take three copies of K_5 and assemble them in a way such that the three copies of e_P constitute a triangle, add a vertex and connect it to the three corners of the triangle. See Figure 12 for the construction. The edge weights in the three copies of K_5 are assigned according to Lemma 2.1, the three new edges get the weight 2. Again, the fat edges get weight 1.

To get a cycle basis for the merged graph, combine the cycle bases of the three copies and append the three new triangles with weight 8, i.e. the triangles constituted by two new edges and one copy of e_P. The $\nu = 21$ shortest circuits have weight 8, hence, the combined

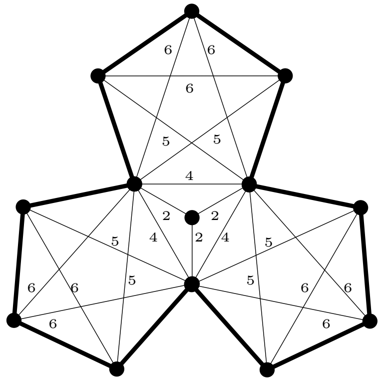

Figure 12: Three merged copies of K_5 with Ostrowski's bases.

cycle basis is minimum. It is *not robust* because Ostrowski's basis is not. It is not *strictly fundamental* since the circuits induced by e_P in each K_5 do not have private edges, as well as the three new triangles. In the end, it is *weakly fundamental*. Permute the basis such that the three new triangles appear at first, followed by the three circuits induced by the copies of e_P.

The last three examples present non-fundamental cycle bases, listed in the third column of Table 2. Two of them are again borrowed from [9].

Example 5.7 (*Not weakly fundamental—strictly quasi-robust*)**.** Unfortunately, we were not able to give an example of a minimum non-fundamental cycle basis which is strictly robust. But we provide a strictly quasi-robust one, at least. Therefore, look at the graph depicted in Figure 13 and the indicated cycle basis.

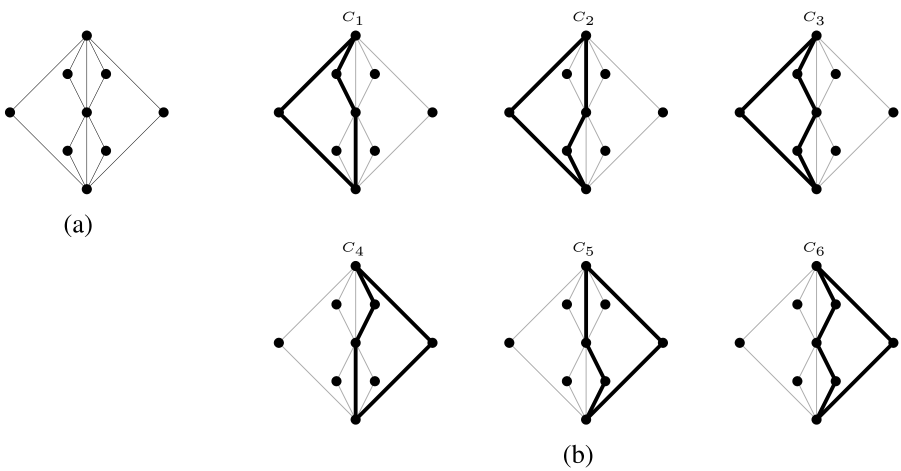

Figure 13: A graph (a) and a non-fundamental cycle basis which is strictly quasi-robust, but not strictly robust (b).

The basis is *non-fundamental* since each edge is contained in at least two basic circuits.

To see that it is *strictly quasi-robust*, we take a look at $2^6 - 6 - 1 = 57$ cycles, analogous to Example 4.6. Among these cycles, there are 38 which do not constitute circuits. For the other 19 circuits, we provide the strictly well-arranged sequences below.

(C_1, C_3),	(C_2, C_3, C_5),	$(C_1, C_3, C_4, C_6, C_2)$,
(C_2, C_3),	(C_6, C_5, C_2),	$\boldsymbol{(C_3, C_1, C_4, C_6, C_5, C_2, C_3)}$,
(C_4, C_6),	(C_6, C_5, C_4),	$(C_6, C_5, C_2, C_3, C_1)$,
(C_5, C_6),	(C_1, C_3, C_4, C_6),	$(C_1, C_3, C_4, C_6, C_5)$,
(C_3, C_2, C_1),	(C_2, C_3, C_5, C_6),	$(C_2, C_3, C_5, C_6, C_4)$,
(C_1, C_3, C_4),	$\boldsymbol{(C_6, C_4, C_1, C_3, C_2, C_5, C_6)}$,	$(C_6, C_4, C_1, C_3, C_2, C_5)$
(C_6, C_4, C_1),		

For the circuits which belong to the bold written sequences, there are no strictly well-arranged sequences in which all circuits are pairwise disjoint. Thus, the cycle basis is strictly quasi-robust, but not strictly robust.

Example 5.8 (*Not weakly fundamental—not strictly robust—cyclically robust*)**.** The cycle basis in this example is borrowed from [9] where it serves as an example of a minimum cycle basis which is not integral[2]. It is a basis of the generalized Petersen graph $P_{11,4}$, see Figure 14.

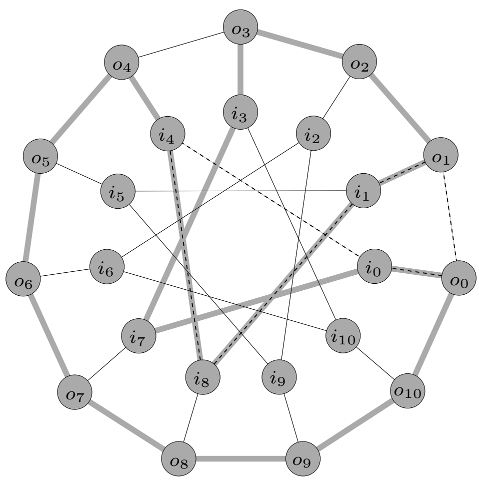

Figure 14: Generalized Petersen graph $P_{11,4}$ with the basic circuit C_1 (dashed) and the circuit $C_{1,4,12} = C_1 + C_4 + C_{12}$ (grey).

The discussed basis B contains the circuits $C_{j+1} = (o_j i_j, i_j i_{j+4}, i_{j+4} i_{j+8}, i_{j+8} i_{j+1}, i_{j+1} o_{j+1}, o_{j+1} o_j)$ for $j = 0, \ldots, 10$ where the indices are taken modulo 11, and the circuit $C_{12} = \{o_0 o_1, \ldots, o_9 o_{10}, o_{10} o_0\}$. In the figure above we emphasized the circuit C_1 with dashed edges. With the weights $w(o_j o_{j+1}) = 4$, $w(i_j i_{j+4}) = 5$, and $w(o_j i_j) = 12$, again for $j = 0, \ldots, 10$ and again modulo 11, this basis becomes the unique minimum one, see [9].

[2]For the definition of integral cycle bases we refer to [9].

Each edge $i_j i_{j+4}$ is contained in three basic circuits, all other edges in exactly two basic circuits. This shows the *non-fundamentality* of the basis. To see that it is *not strictly robust*, consider for example the circuit $C_{1,4,12} = C_1 + C_4 + C_{12}$ whose cuts with C_1, C_4, and C_{12} do not form a single path in each case.

It remains to show that the basis is cyclically robust. This was done by a small program implemented in C++ using LEDA ([10]). The program tested for each of the 2^{12} linear combinations if it constitutes a circuit C, and if so, if there is a circuit $C_j \in \text{supp}(C)$ such that $C + C_j$ is a circuit. This applied to each circuit and thus, the cycle basis is *cyclically robust*.

Example 5.9 (*Not weakly fundamental—not cyclically robust*)**.** To construct cycle bases of a biconnected graph which are neither robust nor fundamental, the authors in [8] suggest the following operation. Given a 2-connected graph G' with a non-robust cycle basis B' and a 2-connected graph G'' with a non-fundamental cycle basis B'', construct a graph G by identifying two arbitrary edges of G' and G''. The basis $B = B' \cup B''$ is a basis of G. However, even if B' and B'' are the minimum cycle bases of G' and G'', respectively, it is not guaranteed that B is a minimum cycle basis of G. In contrast to this construction, we propose Champetier's graph with its minimum cycle basis as a representative for a minimum non-robust and non-fundamental cycle basis.

Also this graph and the cycle basis are taken from [9]. In his Example 11.7, Liebchen considered Champetier's graph whose unique minimum cycle basis is integral but neither weakly fundamental nor totally unimodular. In Champetier's original paper [1], it served as a counter-example of a conjecture expressed in [3]: "If G is null-homotopic (i.e., if every cycle of G is the modulo 2 edge sum of triangles), there is an edge e of G such that $G \setminus e$ is still null-homotopic." Champetier's graph is visualized in Figure 15.

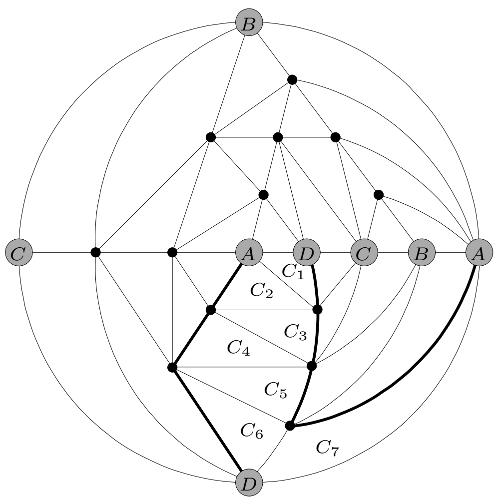

Figure 15: Champetier's graph and a certificate for the non-robustness of the minimum cycle basis.

Champetier's graph arises from the embedding by identifying the vertices A, B, C and D with their copies. The cycle basis we deal with is formed by the 36 triangles in the embedded version. This basis is minimum since there is neither a further triangle which is not the boundary of a face in the embedding in Figure 15 nor a path of length 3 between two copies of one of the vertices A to D. After the vertex identifications, such a path would also compose to a triangle. Hence, the basic circuits are the only triangles.

Since each edge is contained in two triangles at least, the basis is non-fundamental. As a proof for the non-robustness, we take the same certificate as in Example 11.7 in [9], i.e. the circuit $C = \sum_{i=1}^{7} C_i$, indicated in Figure 15 by two paths. In fact, $C + C_i$ does not form a circuit for $i = 1, \ldots, 7$. This shows that the basis is non-robust.

Table 2 summarizes the results of this section. It has been inspired by the Venn diagram in [8] which also illustrates the relationship between fundamental and robust cycle bases. In the table, we contrast our results with the results listed there. New examples and improvements are emphasized in italic.

Table 2: Overview of the results in this section.

	strictly fundamental	**weakly fundamental**	**non-fundamental**	
strictly robust	K_n with $K_{1,n-1}$ as fund. sp. tree minimum basis	Fig. 2 in [8] basis not minimum	?	[8]
strictly robust	as above minimum basis	sunflower graph $SF(3)$ *minimum basis*	Ex. 5.7, basis only strictly quasi-robust basis not minimum	this paper
cyclically robust	?	Kainen's basis of K_4 basis not minimum	non-fundamental basis of the 4-wheel basis not minimum	[8]
cyclically robust	*Wagner's graph with a P_7 as fund. sp. tree* *minimum basis*	*Wagner's graph joined up with a triangle* *minimum basis*	*Petersen graph $P_{11,4}$* *minimum basis*	this paper
non-robust	K_5 with P_4 as fund. sp. tree basis not minimum	Vogt's example basis not minimum	merging non-rob. basis with non-fund. basis basis not minimum	[8]
non-robust	as above *basis minimum with a suitable weighting*	*three merged K_5* *minimum basis*	*Champetier's graph* *minimum basis*	this paper

6 Concluding remarks

In this paper, we considered robust cycle bases and isolated strictly and cyclically robust cycle bases, as well as the newer concepts of quasi-robust and strictly quasi-robust cycle bases from each other. We did this by giving suitable examples. Since each of our cycle bases is the uniquely minimum one of its graph, and hence each type of robust cycle basis comes along with its own minimization problem, we can view the classification of robust cycle bases as completed.

A second focus was the continuation of the comparison between robust and fundamental types of cycle bases. We were able to further fill the Venn diagram of robust and fundamental cycle bases given in [8], where we demanded in addition that the provided cycle basis is minimum. Our results were summarized in a table which has only one missing item. We could not present a minimum cycle basis which is non-fundamental and strictly robust, but could provide an example of a cycle basis which is strictly quasi-robust, at least.

Despite all, there is still plenty of work to do in the field of robust cycle bases. For example, it is still unknown whether each graph provides a strictly robust cycle basis, or a cycle basis of any other robust type, at least. Furthermore, there is nothing known about the complexity of recognition and construction of robust cycle bases.

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Cayley graphs on groups with commutator subgroup of order $2p$ are hamiltonian

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Abstract

We show that if G is a finite group whose commutator subgroup $[G,G]$ has order $2p$, where p is an odd prime, then every connected Cayley graph on G has a hamiltonian cycle.

Keywords: Cayley graph, hamiltonian cycle, commutator subgroup.

Math. Subj. Class.: 05C25, 05C45

1 Introduction

Let G be a finite group. It is easy to show that if G is abelian (and $|G| > 2$), then every connected Cayley graph on G has a hamiltonian cycle. (See Definition 2.1 for the definition of the term *Cayley graph*.) To generalize this observation, one can try to prove the same conclusion for groups that are close to being abelian. Since a group is abelian precisely when its commutator subgroup is trivial, it is therefore natural to try to find a hamiltonian cycle when the commutator subgroup of G is close to being trivial. The following theorem, which was proved in a series of papers, is a well-known result along these lines.

Theorem 1.1 (Marušič [13], Durnberger [4, 5], 1983–1985)**.** *If the commutator subgroup $[G,G]$ of G has prime order, then every connected Cayley graph on G has a hamiltonian cycle.*

D. Marušič (personal communication) suggested more than thirty years ago that it should be possible to replace the prime with a product pq of two distinct primes:

Problem 1.2 (D. Marušič, personal communication, 1985)**.** Show that if the commutator subgroup of G has order pq, where p and q are two distinct primes, then every connected Cayley graph on G has a hamiltonian cycle.

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This has recently been accomplished when G is either nilpotent [8] or of odd order [16]. As another step toward the solution of this problem, we establish the special case where $q = 2$:

Theorem 1.3. *If the commutator subgroup of G has order $2p$, where p is an odd prime, then every connected Cayley graph on G has a hamiltonian cycle.*

See the bibliography of [12] for references to other results on hamiltonian cycles in Cayley graphs.

The proof of Theorem 1.3 is a lengthy case-by-case analysis, based on the choice of certain elements a and b of the Cayley graph's connection set (see Notation 3.3). Here is an outline of the paper:

2 Some known results

We recall a few results that provide hamiltonian cycles in various Cayley graphs.

Definition 2.1 (cf. [9, p. 34])**.** For any subset S of a finite group G, $\mathrm{Cay}(G; S)$ is the graph whose vertex set is G, with an edge joining g to gs, for each $g \in G$ and $s \in S$. This is called the *Cayley graph* of the connection set S on the group G.

Remark 2.2. Unlike most authors (including [9]), we do not require the connection set S to be symmetric in the definition of a Cayley graph; that is, we do not assume S is closed under inverses. This does not change the set of graphs that are considered to be Cayley graphs, because, in our notation, $\mathrm{Cay}(G; S) = \mathrm{Cay}(G; S \cup S^{-1})$, where $S^{-1} = \{s^{-1} \mid s \in S\}$.

Theorem 2.3 ([3, 6, 7, 12])**.** *Every connected Cayley graph on G has a hamiltonian cycle if $|G| = kp$ for some prime p and some $k \in \mathbb{N}$ with $1 \leq k < 32$ and $k \neq 24$.*

Notation 2.4.

- The symbol G always represents a finite group.
- For $g \in G$ and $s_1, \ldots, s_n \in S \cup S^{-1}$, we use $[g](s_1, \ldots, s_n)$ to denote the walk in $\mathrm{Cay}(G; S)$ that visits (in order), the vertices

  $$g,\ gs_1,\ gs_1s_2,\ gs_1s_2s_3,\ \ldots,\ gs_1s_2\cdots s_n.$$

  We may write $(s_1, \ldots, s_n)$ for $[e](s_1, \ldots, s_n)$.
- We use $(s_1, \ldots, s_n)^k$ to denote the concatenation of k copies of the sequence $(s_i)_{i=1}^{n}$.

- Appending # to a sequence deletes the last term; that is, $(s_i)_{i=1}^{n}\# = (s_i)_{i=1}^{n-1}$.
- If $W = [g](s_1, \ldots, s_n)$ is a walk in $\mathrm{Cay}(G;S)$, and $h \in G$, we use hW to denote the translate $[hg](s_1, \ldots, s_n)$.
- When C is an oriented cycle, we use $-C$ to denote the same cycle as C, but with the opposite orientation.
- For $g, h \in G$:

$$[g,h] = g^{-1}h^{-1}gh, \quad g^h = h^{-1}gh, \quad \text{and} \quad {}^h g = hgh^{-1} \, (= g^{h^{-1}}).$$

- We use G' to denote the commutator subgroup $[G,G]$ of G.
- For convenience, we let $\overline{G} = G/G'$.
- For $g \in G$, we let $\overline{g} = gG'$ be the image of g in $\overline{G}$.
- We use $Z(G)$ to denote the center of G.

Definition 2.5 (cf. [10, §2.1.3, p. 61])**.** Suppose

- N is an abelian, normal subgroup of G, and
- $C = [Nv](s_i)_{i=1}^{n}$ is an (oriented) cycle in $\mathrm{Cay}(G/N;S)$.

The *voltage* of C is ${}^v(\prod_{i=1}^n s_i)$. This is an element of N, and it may be denoted $\Pi\, C$.

We have the following straightforward observations:

Lemma 2.6. *Assume the notation of Definition 2.5. Then:*

1. *$\Pi\, C$ is determined by the oriented cycle C: it is independent of the choice of the vertex Nv of C, and of the choice of the representative v of Nv.*
2. *$\Pi\, gC = {}^g(\Pi\, C)$ for all $g \in G$.*
3. *$\Pi(-C) = (\Pi\, C)^{-1}$.*

Definition 2.7. A subset S of G is an *irredundant* generating set of G if S generates G, but no proper subset of S generates G.

Lemma 2.8 ("Factor Group Lemma" [15, §2.2])**.** *Suppose*

- *N is a cyclic, normal subgroup of G,*
- *$(s_i)_{i=1}^{m}$ is a hamiltonian cycle in $\mathrm{Cay}(G/N;S)$, and*
- *the voltage $\Pi(s_i)_{i=1}^{m}$ generates N.*

Then $(s_1, s_2, \ldots, s_m)^{|N|}$ is a hamiltonian cycle in $\mathrm{Cay}(G;S)$.

Corollary 2.9 ([12, Cor. 2.11])**.** *Suppose*

- *N is a normal subgroup of G, such that $|N|$ is prime,*
- *the image of S in G/N is an irredundant generating set of G/N,*
- *there is a hamiltonian cycle in $\mathrm{Cay}(G/N;S)$, and*
- *$s \equiv t \pmod{N}$ for some $s, t \in S \cup S^{-1}$ with $s \neq t$.*

Then there is a hamiltonian cycle in $\operatorname{Cay}(G;S)$.

Lemma 2.10 ([2, Lem. 1 on p. 24])**.** *Let $P_k \mathbin{\square} P_\ell$ be the Cartesian product of a path of length k with a path of length ℓ. If $k\ell$ is even, and $k,\ell \geq 2$, then $P_k \mathbin{\square} P_\ell$ has a hamiltonian path from any corner vertex v to any vertex that is at odd distance from v.*

Corollary 2.11. *Suppose N is a subgroup of an abelian group H, and $\{x,y\} \cup S_0$ is a subset of H that generates H/N. Let $k = |\langle x, N\rangle : N|$ and $\ell = |\langle x,y,N\rangle : \langle x, N\rangle|$. If $k\ell$ is even, $k,\ell \geq 2$, $0 \leq p < k$, $0 \leq q < \ell$, and $p+q$ is odd, then* $\operatorname{Cay}(H/N;\{x,y\}\cup S_0)$ *has a hamiltonian path $(s_i)_{i=1}^r$, such that $s_1 s_2 \cdots s_r = x^p y^q$.*

Proof. If we identify the vertices of $P_k \mathbin{\square} P_\ell$ with $\{(i,j) \mid 0 \leq i < k,\ 0 \leq j < \ell\}$ in the natural way, then the map $(i,j) \mapsto x^i y^j$ is an isomorphism from $P_k \mathbin{\square} P_\ell$ to a subgraph X of $\operatorname{Cay}\big(\langle x,y\rangle; x,y\big)$. So Lemma 2.10 provides a hamiltonian path $(t_i)_{i=1}^{k\ell-1}$ in X from e to $x^p y^q$. So $t_1 t_2 \cdots t_{k\ell-1} = x^p y^q$.

Let $L = (u_j)_{j=1}^n$ be a hamiltonian path in $\operatorname{Cay}\big(H/\langle x,y,N\rangle\big)$, and let

$$(s_i)_{i=1}^r = (L, t_{2i-1}, L^{-1}, t_{2i})_{i=1}^{k\ell/2}\#.$$

From the definition of k and ℓ, we see that the natural map from X to the Cayley graph $\operatorname{Cay}\big(\langle x,y,N\rangle/N; x,y\big)$ is an isomorphism onto a spanning subgraph. Therefore, $(s_i)_{i=1}^r$ is a hamiltonian path in $\operatorname{Cay}(H/N;S)$. Since H is abelian, it is easy to see that $s_1 s_2 \cdots s_r = x^p y^q$. □

Given a hamiltonian cycle C_0 in $\operatorname{Cay}(\overline{G};S)$, the following result often provides a second hamiltonian cycle C_1, such that the voltage of at least one of these two cycles generates G'. (Then the Factor Group Lemma (2.8) provides a hamiltonian cycle in $\operatorname{Cay}(G;S)$.)

Lemma 2.12 (cf. Marušič [13] and Durnberger [4], or see [16, Lem. 3.1])**.** *Assume:*

- *N is an abelian normal subgroup of G, such that G/N is abelian,*
- *C_0 is an oriented hamiltonian cycle in* $\operatorname{Cay}(G/N;S)$,
- *$s,t,u \in S^{\pm 1}$ and $h \in G$,*
- *C_0 contains:*
  - *the oriented path $[\overline{hs^{-1}u^{-1}}](s,t,s^{-1})$, and*
  - *either the oriented edge $[\overline{h}](t)$ or the oriented edge $[\overline{ht}](t^{-1})$.*

Then there is a hamiltonian cycle C_1 in $\operatorname{Cay}(G/N;S)$*, such that*

$$\Big(\big(\Pi\, C_0\big)^{-1}\big(\Pi\, C_1\big)\Big)^h = \begin{cases} [u,t^{-1}]\,[s,t^{-1}]^u & \text{if } C_0 \text{ contains } [\overline{h}](t), \\ [t^{-1},u]\,[s,t^{-1}]^u & \text{if } C_0 \text{ contains } [\overline{ht}](t^{-1}). \end{cases}$$

Furthermore, C_0 and C_1 have exactly the same oriented edges, except for some of the edges in the subgraph induced by $\{\overline{h}, \overline{hu^{-1}}, \overline{hs^{-1}u^{-1}}, \overline{ht}, \overline{htu^{-1}}, \overline{hts^{-1}u^{-1}}\}$.

Lemma 2.13 ([4, Lem. 2.8])**.** *Assume*

- *S is an irredundant generating set of G,*
- *$s,t \in S$, with $s \neq t$,*

- *s commutes with t,*
- *$\langle S \setminus \{s\}\rangle \triangleleft G$, and*
- *there is a hamiltonian cycle in $\mathrm{Cay}\big(\langle S \setminus \{s\}\rangle; S \setminus \{s\}\big)$.*

Then there is a hamiltonian cycle in $\mathrm{Cay}(G;S)$.

We do not need the general theory of nilpotent groups, but we will make use of the following two facts. (The first is essentially the definition of a nilpotent group, which can be found in any graduate-level textbook on group theory.)

Lemma 2.14 ([14, (iii) on p. 175 and Prop. VI.1.h on page 176])**.**

1. *Every abelian group is nilpotent.*
2. *If $G/Z(G)$ is nilpotent, then G is nilpotent.*

Therefore, if $G' \subseteq Z(G)$ (in other words, if $G/Z(G)$ is abelian), then G is nilpotent.

Theorem 2.15 ([8])**.** *If G is a nontrivial, nilpotent, finite group, and the commutator subgroup of G is cyclic, then every connected Cayley graph on G has a hamiltonian cycle.*

The following observation is well known (and easy to prove).

Lemma 2.16 ([12, Lem. 2.27])**.** *Let S generate a finite group G and let $s \in S$, such that $\langle s\rangle \triangleleft G$. If*

- *$\mathrm{Cay}\big(G/\langle s\rangle; S\big)$ has a hamiltonian cycle, and*
- *either*
  1. *$s \in Z(G)$, or*
  2. *$Z(G) \cap \langle s\rangle = \{e\}$, or*
  3. *$|s|$ is prime,*

then $\mathrm{Cay}(G;S)$ has a hamiltonian cycle.

Corollary 2.17. *Suppose*

- *G' is cyclic of order pq, where p and q are distinct primes,*
- *S is an irredundant generating set of G, and*
- *some nontrivial element s of S is in G'.*

Then $\mathrm{Cay}(G;S)$ has a hamiltonian cycle.

Proof. We may assume $G' = \mathbb{Z}_p \times \mathbb{Z}_q$. Since every subgroup of a cyclic, normal subgroup is also normal, we know that $\langle s\rangle \triangleleft G$. Also, there are hamiltonian cycles in $\mathrm{Cay}(G/\mathbb{Z}_p;S)$, $\mathrm{Cay}(G/\mathbb{Z}_q;S)$, and $\mathrm{Cay}(G/G';S)$ (by Theorem 1.1 and the elementary fact that Cayley graphs on abelian groups have hamiltonian cycles). Hence, we may assume $\langle s\rangle = G'$ and $G' \cap Z(G) = \mathbb{Z}_q$ (perhaps after interchanging p and q), for otherwise Lemma 2.16 applies.

Let $\widehat{G} = G/\mathbb{Z}_p$. We may assume $|\widehat{G}| \neq 27$, for otherwise $|G| = 27p$ so Theorem 2.3 applies. Then, since $\widehat{G}$ is nilpotent (see Lemma 2.14) and its commutator subgroup is $\mathbb{Z}_q$, the proof in [11, §4] implies there is a hamiltonian cycle $(t_i)_{i=1}^n$ in $\mathrm{Cay}\big(\widehat{G}/\widehat{G}'; S'\big)$ whose voltage generates $\widehat{G}'$. Then, since $\mathbb{Z}_p \cap Z(G) = \{e\}$, the proof of Lemma 2.16(2) in [12, Lem. 2.27(2)] tells us that $(t_i, s^{p-1})_{i=1}^{n}$ is a hamiltonian cycle in $\mathrm{Cay}\big(G/\mathbb{Z}_q; S\big)$.

Note that, since $\widehat{G}$ is a nilpotent group whose commutator subgroup is in the center and has prime order q, the order of $|\widehat{G}/\widehat{G}'|$ must be a multiple of q; that is, n is a multiple of q (cf. Lemma 3.6 below). Calculating modulo $\mathbb{Z}_p$, we have

$$\begin{aligned} \Pi(t_i, s^{p-1})_{i=1}^{n} &\equiv s^{(p-1)n}\ \Pi(t_i)_{i=1}^{n} && (\widehat{s} \in \widehat{G}' = \widehat{\mathbb{Z}_q} \subseteq Z(\widehat{G})) \\ &\equiv \Pi(t_i)_{i=1}^{n} && (n \text{ is a multiple of } q) \\ &\not\equiv e && (\Pi(t_i)_{i=1}^{n} \text{ generates } \widehat{G}'). \end{aligned}$$

Therefore $\Pi(t_i, s^{p-1})_{i=1}^{n}$ generates $\mathbb{Z}_q$. So the Factor Group Lemma (2.8) tells us that $\big((t_i, s^{p-1})_{i=1}^{n}\big)^q$ is a hamiltonian cycle in $\mathrm{Cay}(G; S)$. □

3 Assumptions, group theory, and connected sums

Assumptions 3.1. The remainder of this paper provides a proof of Theorem 1.3, so

- p is an odd prime,
- G is a finite group whose commutator subgroup has order $2p$, and
- S is an irredundant generating set of G.

We wish to show that the Cayley graph $\mathrm{Cay}(G; S)$ has a hamiltonian cycle.

3A Basic group theory

Assumption 3.2. Because of Corollary 2.17, we may assume $S \cap G' = \emptyset$.

Notation 3.3. The assumption that the commutator subgroup has order $2p$ implies that G' is cyclic (cf. [16, §2E, proof of Cor. 1.4]), so we may write

$$G' = \mathbb{Z}_2 \times \mathbb{Z}_p.$$

From Theorem 2.15, we may assume that G is not nilpotent, so $G' \not\subseteq Z(G)$ (see Lemma 2.14). This implies $\mathbb{Z}_p \cap Z(G) = \{e\}$. Hence there exists $a \in S$, such that

$$a \text{ does not centralize } \mathbb{Z}_p. \tag{3.3A}$$

Then there exists $b \in S$, such that

$$\mathbb{Z}_p \subseteq \langle [a, b] \rangle. \tag{3.3B}$$

The assumptions (3.3A) and (3.3B) are the basis of most of the arguments in the later sections of the paper.

For ease of reference, we now collect a few well-known facts from group theory (specialized to our setting).

Lemma 3.4. *If $S_0 \subseteq G$, such that $\langle S_0, \mathbb{Z}_2 \rangle = G$, then $\langle S_0 \rangle = G$.*

Proof. Since $\mathbb{Z}_2 \subseteq Z(G)$, we have

$$\langle S_0 \rangle' = \big\langle S_0, Z(G) \big\rangle' \supseteq \langle S_0, \mathbb{Z}_2 \rangle' = G'.$$

Therefore

$$\langle S_0 \rangle = \big\langle S_0, \langle S_0 \rangle' \big\rangle = \langle S_0, G' \rangle \supseteq \langle S_0, \mathbb{Z}_2 \rangle = G. \qquad \square$$

Corollary 3.5. *Suppose S_0 is a proper subset of S, such that $\mathbb{Z}_p \subseteq \langle S_0 \rangle$. (In particular, this will be the case if $\{a, b\} \subseteq S_0$.) Then $\langle \overline{S_0} \rangle \neq \overline{G}$.*

Proof. Suppose $\langle \overline{S_0} \rangle = \overline{G}$. This means $\langle S_0, G' \rangle = G$. Since $G' = \mathbb{Z}_2 \times \mathbb{Z}_p$ and $\mathbb{Z}_p \subseteq \langle S_0 \rangle$, this implies $\langle S_0, \mathbb{Z}_2 \rangle = G$. So Lemma 3.4 tells us that $\langle S_0 \rangle = G$. This contradicts the fact that the generating set S is irredundant. $\square$

Lemma 3.6. *Let H be a group. If $x, y, z \in H$, and y centralizes H', then $[xy, z] = [x, z]\,[y, z]$. Therefore $[y^k, z] = [y, z]^k$ for all $k \in \mathbb{Z}$.*

Corollary 3.7. *If $x, y \in G$, such that y centralizes G', and $\mathbb{Z}_p \subseteq \langle [x, y] \rangle$, then $|y|$ is divisible by p.*

Corollary 3.8. *Let $S_0 \subseteq G$, such that $\mathbb{Z}_2 \not\subseteq \langle S_0 \rangle'$. If $g \in G$, such that $\mathbb{Z}_2 \subseteq \langle g, S_0 \rangle'$, then $|\langle \overline{g}, \overline{S_0} \rangle : \langle \overline{S_0} \rangle|$ is even.*

In particular, if $\mathbb{Z}_2 \subseteq \langle [g, h] \rangle$, then, by taking $S_0 = \{h\}$, we see that $|\langle \overline{g}, \overline{h} \rangle : \langle \overline{h} \rangle|$ is even, so $|\overline{g}|$ is even (and, similarly, $|\overline{h}|$ must also be even).

Corollary 3.9. *$|\overline{G}|$ is divisible by 4.*

3B Connected sums

Definition 3.10 ([8, Defn. 5.1])**.** Assume C_1 and C_2 are two vertex-disjoint oriented cycles in $\mathrm{Cay}(\overline{G}; S)$, and let $g \in G$, and $s, t \in S \cup S^{-1}$. If

- C_1 contains the oriented edge $[\overline{g}](t)$, and
- C_2 contains the oriented edge $[\overline{gst}](t^{-1})$,

then we use $C_1 \#_t^s C_2$ to denote the oriented cycle obtained from $C_1 \cup C_2$ by

- removing the oriented edges $[\overline{g}](t)$ and $[\overline{gst}](t^{-1})$, and
- inserting the oriented edges $[\overline{g}](s)$ and $[\overline{gst}](s^{-1})$.

This is called the *connected sum* of C_1 and C_2.

If $[g](t)$ is any oriented edge of an oriented cycle C, and $s \in S$, such that sC is vertex disjoint from C, then we can form the connected sum $C \#_t^s -sC$. This construction can be iterated:

Definition 3.11. Suppose

- $[g_1](t_1), \ldots, [g_k](t_k)$ are oriented edges of an oriented cycle C in $\mathrm{Cay}(\overline{G}; S)$, such that $g_i \neq g_{i+1}$ for all i, and

- $s_1, s_2, \ldots, s_k \in S \cup S^{-1}$, such that the cycles C, s_1C, s_2s_1C, …, $s_ks_{k-1}\cdots s_1C$ are pairwise vertex-disjoint.

Then we can form the connected sum

$$C \ \#_{t_1}^{s_1} \ -s_1C \ \#_{t_2}^{s_2} \ s_2s_1C \ \#_{t_3}^{s_3} \ \cdots \ \#_{t_k}^{s_k} \ \pm s_ks_{k-1}\cdots s_1C.$$

We call this a *connected sum of signed translates of* C.

Lemma 3.12 (cf. [8, Lem. 5.2]). *If C_1, C_2, g, s, and t are as in Definition 3.10, then*

$$\Pi(C_1 \#_t^s C_2) = \Pi\, C_1 \cdot {}^{g}[s^{-1}, t^{-1}] \cdot \Pi\, C_2.$$

Proof. We may assume $g = t^{-1}$ (or, in other words, $gt = e$), after translating the cycles by $(gt)^{-1}$ (cf. Lemma 2.6(2)). Write $C_1 = (s_i)_{i=1}^m$ and $C_2 = [st^{-1}](t_j)_{j=1}^n$, so

$$(C_1 \#_t^s C_2) = \big((s_i)_{i=1}^{m-1}, s, (t_j)_{j=1}^{n-1}, s^{-1}\big).$$

By assumption, C_1 contains the edge $\overline{t^{-1}} \to \overline{e}$ and C_2 contains the edge $\overline{s} \to \overline{st^{-1}}$, so $s_m = t$ and $t_n = t^{-1}$. Therefore

$$\begin{aligned}
\Pi(C_1 \#_t^s C_2) &= \prod_{i=1}^{m-1}(s_i) \cdot s \cdot \prod_{j=1}^{n-1}(t_j) \cdot s^{-1} \\
&= \prod_{i=1}^{m}(s_i) \cdot t^{-1}s \cdot \prod_{j=1}^{n}(t_j) \cdot ts^{-1} \\
&= \Pi\, C_1 \cdot t^{-1}s \cdot (\Pi\, C_2)^{st^{-1}} \cdot ts^{-1} \\
&= \Pi\, C_1 \cdot t^{-1}sts^{-1} \cdot \Pi\, C_2 \\
&= \Pi\, C_1 \cdot {}^{t^{-1}}[s^{-1}, t^{-1}] \cdot \Pi\, C_2 \\
&= \Pi\, C_1 \cdot {}^{g}[s^{-1}, t^{-1}] \cdot \Pi\, C_2.
\end{aligned}$$

□

Corollary 3.13. *Assume that C_1, C_2, g, s, and t are as in Definition 3.10. If C_0 is another oriented cycle that is vertex-disjoint from C_2 and contains the oriented edge $\overline{g}(t)$, then*

$$\big(\Pi(C_0 \#_t^s C_2)\big)\big(\Pi(C_1 \#_t^s C_2)\big)^{-1} = (\Pi\, C_0)(\Pi\, C_1)^{-1}.$$

Corollary 3.14 ([8, Lem. 5.2]). *If C_1, C_2, g, s, and t are as in Definition 3.10, then*

$$\Pi(C_1 \#_t^s C_2) \equiv \Pi\, C_1 \cdot \Pi\, C_2 \cdot [s, t] \pmod{\mathbb{Z}_p}.$$

The following result describes a fairly common situation in which the connected sum provides hamiltonian cycles in $\mathrm{Cay}(G; S)$:

Lemma 3.15. *Let S_0 be a nonempty subset of S, $g \in G$, $c \in S \smallsetminus S_0$, and $s, t \in S \smallsetminus \{c\}$. Assume C_0 and C_1 are oriented hamiltonian cycles in $\mathrm{Cay}\big(\langle\overline{S_0}\rangle; S_0\big)$, such that*

- *$(\Pi\, C_0)^{-1}(\Pi\, C_1)$ is a nontrivial element of $\mathbb{Z}_p$,*
- *C_0 and C_1 both contain the oriented edge $[\overline{g}](s)$,*

- *for every $x \in S_0$, C_0 contains at least two edges that are labelled either x or x^{-1},*
- *$\mathbb{Z}_2 \subseteq \langle [c,t] \rangle$, and*
- *either $|\overline{G} : \langle \overline{S_0} \rangle| > 2$ or $s = t$.*

If either

1. *there exists $u \in S \smallsetminus \{c\}$, such that $\mathbb{Z}_2 \not\subseteq \langle [u,c] \rangle$, or*
2. *$|\overline{G} : \langle \overline{S_0}, \overline{t} \rangle|$ is even,*

then there is a hamiltonian cycle C in $\mathrm{Cay}(\overline{G}; S)$, such that $\langle \Pi\, C \rangle = G'$, so the Factor Group Lemma (2.8) yields a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Proof. Let $r = |\overline{G} : \langle \overline{S_0} \rangle|$. We have $\mathbb{Z}_p \subseteq \langle (\Pi\, C_0)^{-1}(\Pi\, C_1) \rangle \subseteq \langle S_0 \rangle$, so Corollary 3.5 implies $r \neq 1$.

Suppose $r = 2$. By assumption, this implies $s = t$, which means that C_0 and C_1 both contain the oriented edge $[\overline{g}](t)$. Then the translate $c\, C_0$ contains the oriented edge $[\overline{gc}](t)$. The connected sums $C = C_0 \#_t^c -c\, C_0$ and $C' = C_1 \#_t^c -c\, C_0$ are hamiltonian cycles in $\mathrm{Cay}(\overline{G}; S)$. From Corollary 3.14, we have

$$\Pi\, C \equiv \Pi\, C_0 \cdot \Pi\, C_0 \cdot [c,t] \equiv [c,t] \not\equiv 0 \pmod{\mathbb{Z}_p},$$

so $\Pi\, C$ projects nontrivially to $\mathbb{Z}_2$. Corollary 3.13 says $(\Pi\, C)(\Pi\, C')^{-1} = (\Pi\, C_0)(\Pi\, C_1)^{-1}$, which generates $\mathbb{Z}_p$ (because it is conjugate to the inverse of $(\Pi\, C_0)^{-1}(\Pi\, C_1)$, which is assumed to be a nontrivial element of $\mathbb{Z}_p$). Therefore, we see that either $\Pi\, C$ or $\Pi\, C'$ generates G', as desired. So we may assume henceforth that $r > 2$.

We now show that we may assume $t \in S_0$. To this end, suppose it is not the case that $t \in S_0$. Let $n = |\langle \overline{S_0}, \overline{t} \rangle : \langle \overline{S_0} \rangle|$. Then, by choosing a sequence $\{[g_i](s_i)\}_{i=1}^{n-1}$ of oriented edges of C_0, we can form a connected sum C_0' of signed translates of C_0:

$$C_0' = C_0 \#_{s_1}^t -tC_0 \#_{s_2}^t \cdots \#_{s_{n-1}}^t \pm t^{n-1} C_0.$$

This is a hamiltonian cycle in $\mathrm{Cay}\big(\langle \overline{S_0}, \overline{t} \rangle; S_0 \cup \{t\}\big)$. We may assume $s_1 = s$. Then another hamiltonian cycle C_1' can be constructed by replacing the leftmost occurrence of C_0 with C_1, and Lemma 3.12 tells us that $(\Pi\, C_0')(\Pi\, C_1')^{-1} = (\Pi\, C_0)(\Pi\, C_1)^{-1}$, which is a nontrivial element of $\mathbb{Z}_p$ (and $(\Pi\, C_0)^{-1}(\Pi\, C_1)$ is conjugate to the inverse of this). From the definition of connected sum, it is obvious that C_0' contains at least two edges labelled $t^{\pm 1}$. So the hamiltonian cycles C_0' and C_1' satisfy the hypotheses of the lemma with $S_0 \cup \{t\}$ in the role of S_0 and with t in the role of s.

Case 1. *Assume there exists $u \in S \smallsetminus \{c\}$, such that $\mathbb{Z}_2 \not\subseteq \langle [u,c] \rangle$.*

Subcase 1.1. *Assume $u \in S_0$.* Fix a hamiltonian path $(s_i)_{i=1}^n$ in $\mathrm{Cay}(\overline{G}/\langle \overline{S_0} \rangle; S \smallsetminus S_0)$ with $s_1 = c$, and let $\pi_i = \prod_{j=1}^i s_j$. Any connected sum $C_0 \#_{t_1}^{s_1} (-\pi_1 C_0) \#_{t_2}^{s_2} \cdots \#_{t_n}^{s_n} (\pm \pi_n C_0)$ is a hamiltonian cycle C in $\mathrm{Cay}(\overline{G}; S)$.

Since $[t,c]$ and $[u,c]$ do not have the same projection to $\mathbb{Z}_2$, the voltages of $C_0 \#_t^c -\pi_1 C_0$ and $C_0 \#_u^c -\pi_1 C_0$ do not have the same projection to $\mathbb{Z}_2$. Therefore, by choosing t_1 to be the appropriate element of $\{t, u\}$, we may assume the projection of $\Pi\, C$ to $\mathbb{Z}_2$ is nontrivial (see Corollary 3.14). Note also that if $|\overline{G} : \langle \overline{S_0} \rangle| = 2$, then we must have $t_1 = t$.

We may assume that $t_n = s$, and that the connected sum $(-1)^{n-1}\pi_{n-1}C_0 \#_s^{s_n} (-1)^n \pi_n C_0$ is relative to the oriented edge $[\overline{\pi_n g}](s)$ of $\pi_n C_0$ that is also in $\pi_n C_1$. Therefore, another hamiltonian cycle C' can be constructed by replacing $\pi_n C_0$ with $\pi_n C_1$ in the connected sum. Then Lemma 3.12 (together with Lemma 2.6(2)) implies that $(\prod C)^{-1}(\prod C')$ is conjugate to $(\prod C_0)^{-1}(\prod C_1)$, which is a generator of $\mathbb{Z}_p$. Therefore, either $\prod C$ or $\prod C'$ generates G', as desired.

Subcase 1.2. *Assume $u \notin S_0$.* Let $S_u = \{u\} \cup S_0$, let $n = |\langle \overline{S_u} \rangle : \langle \overline{S_0} \rangle| - 1$, let $(s_i)_{i=1}^m$ be a hamiltonian path in $\mathrm{Cay}\big(\overline{G}/\langle \overline{S_u} \rangle; S \smallsetminus S_u\big)$ with $s_1 = c$, and let $\pi_i = \prod_{j=1}^i s_j$. (Since $S \smallsetminus S_0$ is an irredundant generating set for $\overline{G}/\langle \overline{S_0} \rangle$, we have $m, n \geq 1$.) Any connected sum

$$C_u = C_0 \#_{t_1}^u -uC_0 \#_{t_2}^u \cdots \#_{t_n}^u \pm u^n C_0$$

is a hamiltonian cycle in $\mathrm{Cay}\big(\langle \overline{S_u} \rangle; S_u\big)$, so any connected sum

$$C = C_u \#_{t_1'}^{s_1} -\pi_1 C_u \#_{t_2'}^{s_2} \cdots \#_{t_m'}^{s_m} \pm \pi_m C_u$$

is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$.

Since $t \in S_0$, we know that C_0 contains more than one edge labeled $t^{\pm 1}$, so $-uC_0$ has an edge labeled $t^{\pm 1}$ that was not removed in the construction of the connected sum $C_0 \#_{t_1}^u -\pi_1 C_0$. Furthermore, the definition of the connected sum implies that $C_0 \#_{t_1}^u -\pi_1 C_0$ also contains an edge labeled u. Therefore, we may form connected sums

$$C_u \#_{t^{\pm 1}}^c -\pi_1 C_u \quad \text{and} \quad C_u \#_u^c -\pi_1 C_u$$

without removing any of the edges of C_u. Since $[c, t]$ and $[c, u]$ do not have the same projection to $\mathbb{Z}_2$, the voltages of these two connected sums do not have the same projection to $\mathbb{Z}_2$ (see Corollary 3.14). Therefore, by choosing t_1' to be the appropriate element of $\{t^{\pm 1}, u\}$, we may assume the projection of $\prod C$ to $\mathbb{Z}_2$ is nontrivial.

We have

$$C = C_u \#_{t_1'}^{s_1} -\pi_1 C_u \#_{t_2'}^{s_2} \cdots \#_{t_{m-1}'}^{s_{m-1}} \pm \pi_{m-1} C_u \#_{t_m'}^{s_m} \big(\pm \pi_m C_0 \#_{t_1}^u \pm \pi_m u C_0 \#_{t_2}^u \cdots \#_{t_n}^u \pm \pi_m u^n C_0\big),$$

so the proof can be completed almost exactly as in the final paragraph of Subcase 1.1 (by constructing another connected sum in which $\pi_m u^n C_0$ is replaced with $\pi_m u^n C_1$).

Case 2. *Assume $[u, c]$ projects nontrivially to $\mathbb{Z}_2$, for every $u \in S \smallsetminus \{c\}$.* In particular, $[d, c]$ projects nontrivially to $\mathbb{Z}_2$, for every $d \in S \smallsetminus \big(S_0 \cup \{c\}\big)$. Since we may assume that Case 1 does not apply with d in the place of c, we conclude that we may assume

$$[u, d] \text{ projects nontrivially to } \mathbb{Z}_2, \text{ for all } d \in S \smallsetminus S_0 \text{ and } u \in S \smallsetminus \{d\}. \tag{3.15A}$$

Choose a hamiltonian path $(s_i)_{i=1}^n$ in $\mathrm{Cay}(\overline{G}/\langle \overline{S_0} \rangle; S \smallsetminus S_0)$. Any connected sum

$$C = C_0 \#_{t_1}^{s_1} -\pi_1 C_0 \#_{t_2}^{s_2} \cdots \#_{t_n}^{s_n} \pm \pi_n C_0$$

is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$. Calculating modulo $\mathbb{Z}_p$, and letting z be the nontrivial element of $\mathbb{Z}_2$, we have

$$\begin{aligned} \Pi\, C &\equiv \Pi\, C_0 \cdot [s_1, t_1] \cdot \Pi(-\pi_1 C_0) \ \cdots \ [s_n, t_n] \cdot \Pi(\pm \pi_n C_0) && \text{(Corollary 3.14)} \\ &\equiv \Pi\, C_0 \cdot z \cdot \Pi\, C_0 \cdots z \cdot \Pi\, C_0 && \text{(Lemma 2.6(2) \& (3.15A))} \\ &= (\Pi\, C_0)^{n+1} \cdot z^n \\ &\equiv z && (n \text{ is odd}). \end{aligned}$$

The proof is now completed exactly as in the final paragraph of Subcase 1.1. □

Corollary 3.16. *Let $S_0 \subseteq S$, $g \in G$, and $s \in S_0$. Assume C_0 and C_1 are oriented hamiltonian cycles in $\mathrm{Cay}\big(\langle \overline{S_0} \rangle; S_0\big)$, such that*

- *$(\Pi\, C_0)^{-1}(\Pi\, C_1)$ is a nontrivial element of $\mathbb{Z}_p$,*
- *C_0 and C_1 both contain the oriented edge $[\overline{g}](s)$,*
- *for every $x \in S_0$, C_0 contains at least two edges that are labelled either x or x^{-1}, and*
- *$\mathbb{Z}_2 \not\subseteq \langle S_0 \rangle'$.*

Then there is a hamiltonian cycle C in $\mathrm{Cay}(\overline{G}; S)$, such that $\langle \Pi\, C \rangle = G'$, so the Factor Group Lemma (2.8) yields a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Proof. We may assume $[c, t] \in \mathbb{Z}_p$, for all $c \in S$ and $t \in S_0$. (Otherwise, we see from Corollary 3.8 that Lemma 3.15(2) applies.) Choose $c, d \in S$, such that $[c, d] \notin \mathbb{Z}_p$, let $S_0^+ = S_0 \cup \{d\}$, and let $r = |\langle S_0^+ \rangle : \langle S_0 \rangle|$. Any connected sum of the following form is a hamiltonian cycle in $\mathrm{Cay}\big(\langle S_0^+ \rangle; S_0^+\big)$:

$$C = C_0 \#^d_{s_1} -dC_0 \#^d_{s_2} \cdots \#^d_{s_{r-1}} \pm d^{r-1} C_0 .$$

We may assume $s_1 = s$, and that the connected sum $C_0 \#^d_{s_1} -dC_0$ is formed by using the oriented edge $[\overline{g}](s)$ that is also in C_1. Therefore, a second hamiltonian cycle C' can be constructed by replacing the leftmost occurrence of C_0 with C_1. Then Corollary 3.8 implies that Lemma 3.15(2) applies (with S_0^+, d, d, C, and C' in the roles of S_0, s, t, C_0, and C_1, respectively). □

4 Case with $\overline{s} = \overline{t}$

Case 4.1. *Assume there exist $s, t \in S \cup S^{-1}$ with $\overline{s} = \overline{t}$ and $s \neq t$.*

Proof. Write $t = s\gamma$ with $\gamma \in G'$. We may assume $\langle \gamma \rangle = G'$, for otherwise $|\gamma|$ is prime, so Corollary 2.9 applies with $N = \langle \gamma \rangle$. Note that the irredundance of S implies $\langle S \smallsetminus \{s\} \rangle$ and $\langle S \smallsetminus \{t\} \rangle$ do not contain $\mathbb{Z}_p$. This implies that every element of $S \smallsetminus \{s, t\}$ centralizes $\mathbb{Z}_p$. So s and t do not centralize $\mathbb{Z}_p$.

Let $m = |\overline{t}|$ and $n = |\overline{G}|/m$.

Subcase 4.1.1. *Assume* $|\overline{t}| > 2$. Since $\overline{G}$ is abelian, it is easy to find a hamiltonian cycle $C = (t_i)_{i=1}^{mn}$ in $\mathrm{Cay}\big(\overline{G}; S \smallsetminus \{s\}\big)$, such that $t_1 = t_2 = \cdots = t_{m-1} = t$. Since $\langle \Pi\, C \rangle \subseteq \langle S \smallsetminus \{s\} \rangle$, and $\mathbb{Z}_p \not\subseteq \langle S \smallsetminus \{s\} \rangle$, we must have $\Pi\, C \in \mathbb{Z}_2$.

For each subset I of $\{1, \ldots, m-1\}$, we define C_I to be the hamiltonian cycle constructed from C by changing t_i to s for all $i \in I$. The proof is completed by noting that I may be chosen such that $\Pi\, C_I$ generates G', so the Factor Group Lemma (2.8) applies:

- If $\prod C = e$, let $I = \{1\}$.
- If $\prod C$ is the nontrivial element of $\mathbb{Z}_2$, and t does not invert $\mathbb{Z}_p$, then we may let $I = \{1, 2\}$.
- If $\prod C$ is the nontrivial element of $\mathbb{Z}_2$, and t inverts $\mathbb{Z}_p$, then $|\bar{t}|$ is even, so we must have $|\bar{t}| \geq 4$. We may let $I = \{1, 3\}$.

Subcase 4.1.2. *Assume* $|\bar{t}| = 2$. (Since t does not centralize $\mathbb{Z}_p$, this implies that t inverts $\mathbb{Z}_p$.) Choose a hamiltonian cycle $(s_i)_{i=1}^n$ in $\mathrm{Cay}\big(\overline{G}/\langle\bar{t}\rangle; S \setminus \{s, t\}\big)$, and let

$$C_0 = (t, s_i)_{i=1}^n = (t_j)_{j=1}^{2n}.$$

Since $n = |\overline{G}|/2$ is even (see Corollary 3.9) and $S \setminus \{s\}$ is an irredundant generating set of $\overline{G}$, it is easy to see that C_0 is a hamiltonian cycle in $\mathrm{Cay}\big(\overline{G}; S \setminus \{s\}\big)$. Note that $t_i = t$ whenever i is odd, and that $\prod C_0 \in \mathbb{Z}_2$ (because $\mathbb{Z}_p \not\subseteq \langle S \setminus \{s\}\rangle$).

We may assume $n \geq 6$ (for otherwise $|G| = 4np \leq 20p$, so Theorem 2.3 applies). We construct a hamiltonian cycle C_1 from C_0:

- If $\prod C_0 = e$, construct C_1 by changing t_1 to s.
- If $\prod C_0 \neq e$, construct C_1 by changing both t_1 and t_5 to s.

In each case, $\prod C_1$ generates G'. (To see this in the second case, note that $t_2 t_3 t_4 t_5 = s_1 t s_2 t$ centralizes G', because t inverts G', and each s_i centralizes G'.) Therefore, the Factor Group Lemma (2.8) applies. □

5 Cases with $|\bar{a}| > 2$ and $\bar{b} \notin \langle\bar{a}\rangle$

Recall that the elements a and b of S satisfy (3.3A) and (3.3B).

Case 5.1. *Assume* $|\bar{a}| > 2$, $\bar{b} \notin \langle\bar{a}\rangle$, *and there exists* $c \in S$, *such that* $\mathbb{Z}_2 \subseteq \langle[a, c]\rangle$. *(It may be the case that* $b = c$.*)*

Proof. Let $m = |\bar{a}|$ and $n = |\overline{G} : \langle\bar{a}\rangle|$. Since $\bar{b}, \bar{c} \notin \langle\bar{a}\rangle$ (and $\overline{G}/\langle\bar{a}\rangle$ is abelian), it is easy to find a hamiltonian cycle $(s_i)_{i=1}^n$ in $\mathrm{Cay}\big(\overline{G}/\langle\bar{a}\rangle; S \setminus \{a\}\big)$, such that $s_n \in \{c^{\pm 1}\}$, and $s_k = b$ for some $k < n$. Since $\mathbb{Z}_2 \subseteq \langle[a, c]\rangle$, we know m and n are both even (see Corollary 3.8). Since n is even, we have the following (well-known) hamiltonian cycle C_0 in $\mathrm{Cay}(\overline{G}; \overline{S})$:

$$C_0 = \big(a, (a^{m-2}, s_{2i-1}, a^{-(m-2)}, s_{2i})_{i=1}^{n/2}\#, a^{-1}, (s_{n-j}^{-1})_{j=1}^{n-1}\big). \tag{5.1A}$$

Letting $\widehat{G} = G/\mathbb{Z}_p$, we have $\widehat{G}' = \mathbb{Z}_2$, so $\widehat{a}^{m-2} \in Z(\widehat{G}\,)$ (because m is even). Therefore

$$a^{m-2} s_{2i-1} a^{-(m-2)} \equiv s_{2i-1} \pmod{\mathbb{Z}_p},$$

so, calculating modulo $\mathbb{Z}_p$, we have

$$\prod C_0 \equiv a \cdot \left(\prod_{i=1}^{n-1} s_j\right) \cdot a^{-1} \cdot \left(\prod_{i=1}^{n-1} s_j\right)^{-1} \equiv a \cdot s_n^{-1} \cdot a^{-1} \cdot s_n = [a^{-1}, s_n] = [a^{-1}, c^{\pm 1}],$$

which is nontrivial (mod $\mathbb{Z}_p$).

Recall that $s_k = b$. Let $g = \prod_{i=1}^{k-1} s_i$ and $\delta = (-1)^{k+1}$. Then C_0 contains both the oriented edge $[\overline{gb}](b^{-1})$ and the oriented path $[\overline{ga^{-2\delta}}](a^\delta, b, a^{-\delta})$. So Lemma 2.12 (with $s = a^\delta$, $t = b$, $u = a^\delta$ and $h = g$) provides a hamiltonian cycle C_1, such that $(\Pi\, C_0)^{-1}(\Pi\, C_1)$ is conjugate to $[b^{-1}, a^\delta][a^\delta, b^{-1}]^{a^\delta}$. Since a centralizes $\mathbb{Z}_2$, but not $\mathbb{Z}_p$, this voltage is a generator of $\mathbb{Z}_p$.

Thus, either $\Pi\, C_0$ or $\Pi\, C_1$ generates $\mathbb{Z}_2 \times \mathbb{Z}_p = G'$, so the Factor Group Lemma (2.8) provides a hamiltonian cycle in $\mathrm{Cay}(G; S)$. □

Case 5.2. *Assume $|\overline{a}| > 2$, $\overline{b} \notin \langle \overline{a} \rangle$, and there does not exist $c \in S$, such that $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$.*

Proof. Choose $c, d \in S$ with $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$. Let

$$m = |\overline{a}|,\ n = |\langle \overline{S} \setminus \{\overline{d}\} \rangle|/m, \text{ and } r = |\overline{G}|/(mn).$$

By assumption, we know $a \notin \{c, d\}$. Also, we may assume $d \neq b$ (after interchanging c and d if necessary). Then Corollary 3.5 tells us $r > 1$. Furthermore, we see from Corollary 3.8 that the image of c in $\overline{G}/\langle \overline{a} \rangle$ has even order, so n is even.

Subcase 5.2.1. *Assume $n > 2$.* It is not difficult to construct a hamiltonian cycle $(s_i)_{i=1}^n$ in $\mathrm{Cay}\big(\langle \overline{S} \setminus \{\overline{d}\} \rangle / \langle \overline{a} \rangle; \overline{S} \setminus \{\overline{a}, \overline{d}\}\big)$, such that $s_1 = b$ and $s_k = c^{\pm 1}$ for some $k \notin \{1, n\}$. Then, since n is even, we may define C_0 as in (5.1A), so C_0 is a hamiltonian cycle in $\mathrm{Cay}\big(\langle \overline{S} \setminus \{\overline{d}\} \rangle; S \setminus \{d\}\big)$.

Let $g = s_1 s_2 \cdots s_k$, and note that C_0 contains the oriented edges $[\overline{e}](a)$ and $[\overline{g}](c^{\mp 1})$. Since $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$, but $\mathbb{Z}_2 \nsubseteq \langle [a, d] \rangle$, we see from Lemma 3.12 that there is a connected sum

$$C = C_0 \#_{t_1}^d -dC_0 \#_{t_2}^d \cdots \#_{t_{r-1}}^d \pm d^{r-1} C_0,$$

with $t_1 \in \{a, c^{\pm 1}\}$, such that $\mathbb{Z}_2 \subseteq \langle \Pi\, C \rangle$. Note that C is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$.

The cycle C_0 contains both $[\overline{b}](b^{-1})$ and $[\overline{a}^{-2}](a, b, a^{-1})$, and neither of these paths contains either the edge $[\overline{e}](a)$ or the edge $[\overline{g}](c^{\mp 1})$. Therefore, C also contains both of these paths, so Lemma 2.12 (with $s = a$, $t = b$, $u = a$, and $h = e$) provides a hamiltonian cycle C' in $\mathrm{Cay}(\overline{G}; S)$, such that $\big(\Pi\, C\big)^{-1}\big(\Pi\, C'\big)$ is a conjugate of $[b^{-1}, a]\,[a, b^{-1}]^a$, which is a generator of $\mathbb{Z}_p$ (since a centralizes $\mathbb{Z}_2$, but not $\mathbb{Z}_p$). Then either $\Pi\, C$ or $\Pi\, C'$ generates G', so the Factor Group Lemma (2.8) applies.

Subcase 5.2.2. *Assume $n = 2$ and $r > 2$.* Since $n = 2$ (and $\overline{b} \notin \langle \overline{a} \rangle$), we have $\langle \overline{a}, \overline{b}, \overline{d} \rangle = \overline{G}$, so Corollary 3.5 implies $S = \{a, b, d\}$. (Therefore $b = c$, which means $\mathbb{Z}_2 \subseteq \langle [b, d] \rangle$.) We have the following hamiltonian cycle in $\mathrm{Cay}\big(\langle \overline{a}, \overline{b} \rangle; \overline{a}, \overline{b}\big)$:

$$C_0 = [\overline{e}](a^{m-1}, b, a^{-(m-1)}, b^{-1}).$$

Using the oriented edge $[\overline{e}](a)$, we can form the connected sum $C_0 \#_a^d -dC_0$. Then, since dC_0 contains both $[\overline{db}](b^{-1})$ and $[\overline{dab}](a^{-1})$, we can extend this to a connected sum

$$C = C_0 \#_a^d -dC_0 \#_{t_2}^d \cdots \#_{t_{r-1}}^d \pm d^{r-1} C_0,$$

with $t_2 \in \{a, b\}$, such that $\mathbb{Z}_2 \subseteq \langle \Pi\, C \rangle$ (see Corollary 3.14). Since C contains both $[\overline{b}](b^{-1})$ and $[\overline{a}^{-2}](a, b, a^{-1})$, we may argue as in the last paragraph of Subcase 5.2.1. Namely, Lemma 2.12 (with $s = a$, $t = b$, $u = a$, and $h = e$) provides a hamiltonian cycle C' in $\mathrm{Cay}(\overline{G}; S)$, such that $\big(\Pi\, C\big)^{-1}\big(\Pi\, C'\big)$ is a conjugate of $[b^{-1}, a]\,[a, b^{-1}]^a$, which is a generator of $\mathbb{Z}_p$. Then either $\prod C$ or $\prod C'$ generates G', so the Factor Group Lemma (2.8) applies.

Subcase 5.2.3. *Assume* $n = r = 2$. As in Subcase 5.2.2, we must have $S = \{a, b, d\}$ and $b = c$ (so $\mathbb{Z}_2 \subseteq \langle [b, d] \rangle$).

Subsubcase 5.2.3.1. *Assume* $m \neq 3$. Since $m = |\overline{a}| > 2$ (by an assumption of this case), we have $m \geq 4$. We have the following hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$:

$$C_0 = \big(d, b, a, b^{-1}, d^{-1}, a^{m-2}, d, a^{-(m-3)}, b, a^{m-3}, d^{-1}, a^{-(m-1)}, b^{-1}\big).$$

Since a is central in $G/\mathbb{Z}_p$ (by an assumption of this case), we know that

$$\prod C_0 \equiv dbb^{-1}d^{-1}dbd^{-1}b^{-1} = dbd^{-1}b^{-1} = [d^{-1}, b^{-1}] \equiv [d, b] = [d, c] \pmod{\mathbb{Z}_p},$$

so $\mathbb{Z}_2 \subseteq \langle \prod C_0 \rangle$.

Note that C_0 contains both $[\overline{dab}](b^{-1})$ and $[\overline{d}\overline{a}^3](a^{-1}, b, a)$ (because $m \geq 4$), so applying Lemma 2.12 (with $s = a^{-1}$, $t = b$, $u = a^{-1}$ and $h = da$) yields a hamiltonian cycle C_1 in $\mathrm{Cay}(G; S)$, such that $\big(\prod C_0\big)^{-1}\big(\prod C_1\big)$ is a conjugate of $[b^{-1}, a^{-1}]\,[a^{-1}, b^{-1}]^{a^{-1}}$, which is a generator of $\mathbb{Z}_p$. Then either $\prod C$ or $\prod C'$ generates G', so the Factor Group Lemma (2.8) applies.

Subsubcase 5.2.3.2. *Assume* $m = 3$ *and* d *does not centralize* G'. Since the walk (a^{-2}, b^{-1}, a^2) is a hamiltonian path in $\mathrm{Cay}\big(\langle \overline{a}, \overline{b} \rangle; a, b\big)$, we have the following hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$:

$$C = (a^{-2}, b^{-1}, a^2, d^{-1}, a^{-2}, b, a^2, d).$$

Note that

$$\prod C = (a^{-2}b^{-1}a^2)\, d^{-1}(a^{-2}ba^2)\, d = (b^{a^2})^{-1}\, d^{-1}(b^{a^2})\, d = [b^{a^2}, d].$$

Since a^2 does not invert G', we know that $b^{a^2} \not\equiv b^{a^{-2}} \pmod{\mathbb{Z}_2}$. Therefore, since d does not centralize G', we may assume $[b^{a^2}, d] \not\equiv e \pmod{\mathbb{Z}_2}$ (by replacing a with its inverse if necessary). Also, since G' is central modulo $\mathbb{Z}_p$, we have $[b^{a^2}, d] \equiv [b, d] \not\equiv e \pmod{\mathbb{Z}_p}$. Therefore, $\prod C$ generates G', so the Factor Group Lemma (2.8) applies.

Subsubcase 5.2.3.3. *Assume* $m = 3$ *and* d *centralizes* G'. Suppose $[b, d] \in \mathbb{Z}_2$. Let $\widehat{G} = G/\mathbb{Z}_2$ and $\widehat{H} = \langle \widehat{a}, \widehat{b} \rangle$. From Theorem 1.1, we know there is a hamiltonian cycle in $\mathrm{Cay}\big(\widehat{H}; a, b\big)$. Deleting an edge labeled $b^{\pm 1}$ (and passing to the reverse and/or a translate if necessary) yields a hamiltonian path $L = (t^i)_{i=1}^{2mp-1}$ in $\mathrm{Cay}\big(\widehat{H}; a, b\big)$ from $\widehat{e}$ to $\widehat{b}$. Let

$$C = (L^{-1}, d^{-1}, L, d).$$

Then

$$\prod C = \Big[\prod_{i=1}^{2mp-1} t_i, d\Big] \in [b\mathbb{Z}_2, d] = \{[b, d]\},$$

because $\mathbb{Z}_2$ is in the center of G. Since $[b, d] \in \mathbb{Z}_2$, this calculation implies that C is a closed walk in $G/\mathbb{Z}_2 = \widehat{G}$. So C is a hamiltonian cycle in $\mathrm{Cay}\big(\widehat{G}; S\big)$. The calculation also implies that the Factor Group Lemma (2.8) applies, because $\langle [b, d] \rangle = \mathbb{Z}_2$.

We may now assume $[b, d] \notin \mathbb{Z}_2$. Therefore, since d centralizes G', and $p^2 \nmid 12 = |\overline{G}|$, we see from Lemma 3.6 that b does not centralize G'. Also, we may assume $[a, d] \neq e$, for otherwise Lemma 2.13 applies with $s = d$ and $t = a$. However, we know $\mathbb{Z}_2 \not\subseteq \langle [a, d] \rangle$ (by an assumption of this case). Therefore $\langle [a, d] \rangle = \mathbb{Z}_p$. So Subsubcase 5.2.3.2 applies after interchanging b and d. □

6 Cases with $\overline{b} \in \langle \overline{a} \rangle$

Case 6.1. *Assume $\overline{b} \in \langle \overline{a} \rangle$ and a does not invert G'.*

Proof. Let $m = |\overline{a}|$. We may assume (perhaps after replacing b with its inverse) that we may write $b = a^k \gamma$ with $1 \leq k \leq m/2$ and $\gamma \in G'$. Assume $k \geq 2$, for otherwise Case 4.1 applies. This implies $m - 1 \geq k + 1$ (since $m = |\overline{a}| \geq 2k \geq k + 2$).

Subcase 6.1.1. *Assume there exists $c \in S$, such that $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$.* Let $n = |\overline{G} : \langle \overline{a} \rangle|$. Note that Corollary 3.8 implies m and n are even, and $c \notin \langle \overline{a} \rangle$ (so $c \neq b$).

Choose a hamiltonian cycle $(s_i)_{i=1}^{n}$ in $\mathrm{Cay}\big(\overline{G}/\langle \overline{a} \rangle; S \smallsetminus \{a, b\}\big)$, such that $s_n = c$, and define C_0 as in (5.1A). Then $\langle \prod C_0 \rangle$ contains $\mathbb{Z}_2$ by the same calculation as in Case 5.1.

Since $m - 1 \geq k + 1$, we may construct a hamiltonian cycle C_1 in $\mathrm{Cay}(\overline{G}; S)$ by replacing the path (a^{k+1}) at the start of C_0 with $(b, a^{-(k-1)}, b)$. Then

$$\big(\prod C_1\big)\big(\prod C_0\big)^{-1} = ba^{-(k-1)}ba^{-(k+1)} = (a^k\gamma)a^{-(k-1)}(a^k\gamma)a^{-(k+1)} = a^{k+1}\gamma^a \, \gamma a^{-(k+1)}.$$

This is a generator of $\mathbb{Z}_p$, since a inverts $\mathbb{Z}_2$, but not $\mathbb{Z}_p$. Hence, either $\prod C_0$ or $\prod C_1$ generates G', so the Factor Group Lemma (2.8) provides a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Subcase 6.1.2. *Assume there does not exist $c \in S$, such that $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$.* Choose $c, d \in S$, such that $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$. (It is possible that $b \in \{c, d\}$, but we know, by the assumption of this subcase, that $a \notin \{c, d\}$.) Let $n = |\langle \overline{a}, \overline{d} \rangle : \langle \overline{a} \rangle|$ and $r = |\overline{G}|/(mn)$. From Corollary 3.8 (and the assumption of this subcase), we know n and r are even.

We have the following hamiltonian cycle in $\mathrm{Cay}\big(\langle \overline{a}, \overline{d} \rangle; a, d\big)$:

$$C_0 = \big((a, (a^{m-2}, d, a^{-(m-2)}, d)^{n/2} \#, a^{-1}, d^{-(n-1)}\big).$$

As in the final paragraph of Subcase 6.1.1, another hamiltonian cycle C_1 can be constructed by replacing the path (a^{k+1}) at the start of C_0 with $(b, a^{-(k-1)}, b)$, and the calculation in Subcase 6.1.1 shows that $(\prod C_1)(\prod C_0)^{-1}$ generates $\mathbb{Z}_p$. Therefore, since $[c, d] \notin \mathbb{Z}_p$, but $[c, a] \in \mathbb{Z}_p$, we see that Lemma 3.15(1) applies (with $S_0 = \{a, b, d\}$, $g = a^{-1}$, $s = t = d$, and $u = a$). □

Case 6.2. *Assume $\overline{b} \in \langle \overline{a} \rangle$ and a inverts G'.*

Proof. As in Case 6.1, we let $m = |\overline{a}|$ and write $b = a^k \gamma$ with $2 \leq k \leq m/2$ and $\gamma \in G'$. We now consider the same five subcases as in [4, pp. 60–62].

Subcase 6.2.1. *Assume $2 < k < m/2$ and k is even.* Let $C_1 = (a^m)$. The proof in the last paragraph of [4, p. 60] provides a hamiltonian cycle

$$C_0 = \big(b, a^{-(k-4)}, b, a^{m-2k-2}, b, a^{-1}, b, a^2, b^{-2}, a^{k-3}\big)$$

in $\mathrm{Cay}\big(\langle \overline{a} \rangle; a, b\big)$, such that $(\prod C_0)^{-1}(\prod C_1)$ is a generator of $\mathbb{Z}_p$. Therefore, Corollary 3.16 applies (with $S_0 = \{a, b\}$), because C_0 and C_1 both contain the oriented edge $[\overline{a}^{-1}](a)$.

Subcase 6.2.2. *Assume $2 < k < m/2$ and k is odd.* Let

$$C_0 = \big((b, a, b^{-1}, a)^{(k-1)/2}, b, a^{m-2k+1}\big)$$

and

$$C_1 = \big((b, a^{-1}, b^{-1}, a^{-1})^{(k-1)/2}, b^2, a^{m-2k-1}, b\big).$$

Calculations in [4, p. 61] show that $(\Pi\, C_0)^{-1}(\Pi\, C_1)$ is a generator of $\mathbb{Z}_p$. Therefore, Corollary 3.16 applies (with $S_0 = \{a, b\}$), because C_0 and C_1 both contain the oriented edge $[\overline{e}](b)$.

Subcase 6.2.3. *Assume $k = m/2$ and k is even.* We follow the argument of [11, Subcase iii, p. 97]. Since k is even, we know a^k centralizes G', so

$$b^2 = (a^k\gamma)^2 = a^{2k}\gamma^2 = a^m\gamma^2 \in \mathbb{Z}_2 \cdot \gamma^2 \not\ni e.$$

Therefore Corollary 2.9 applies (with $s = b$ and $t = b^{-1}$).

Subcase 6.2.4. *Assume $k = m/2$ and k is odd.* Choose $c \in S$ so that $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$, if such c exists. Otherwise, choose c so that there exists $d \in S$, such that $\mathbb{Z}_2 \subseteq \langle [c, d] \rangle$. In either case, Corollary 3.8 implies $c \in S \setminus \{a, b\}$, and $|\langle \overline{a}, \overline{c} \rangle : \langle \overline{a} \rangle|$ is even.

We may assume $b^2 = e$, for otherwise Corollary 2.9 applies (with $s = b$ and $t = b^{-1}$). Therefore, noting that a^k inverts G' (since k is odd), we have

$$e = b^2 = (a^k\gamma)(a^k\gamma) = a^{2k} \cdot \gamma^{-1}\gamma = a^m.$$

Subsubcase 6.2.4.1. *Assume $|\overline{G} : \langle \overline{a} \rangle| > 2$.* It suffices to find a hamiltonian cycle C_* in $\mathrm{Cay}(\overline{G}; S)$, such that $\Pi\, C_*$ projects nontrivially to $\mathbb{Z}_2$, and C_* contains the paths $[\overline{a^{k-3}}](a, b, a^{-1})$ and $[\overline{a^{k-1}b}](b^{-1})$. For then Lemma 2.12 (with $s = a$, $t = b$, $u = a$, and $h = a^{k-1}$) provides a hamiltonian cycle C'_*, such that $\langle (\Pi\, C_*)^{-1}(\Pi\, C'_*) \rangle = \mathbb{Z}_p$. Therefore, either $\Pi\, C_*$ or $\Pi\, C'_*$ generates G', so the Factor Group Lemma (2.8) applies.

Note that

$$C = (a^{k-2}, b, a^{-(k-2)}, c, a^{k-1}, c^{-1}, b^{-1}, c, a^{-(k-1)}, c^{-1})$$

is a cycle through the vertices of $\mathrm{Cay}(\overline{G}; \{a, b, c\})$ in $\langle \overline{a} \rangle \cup c\langle \overline{a} \rangle$. A connected sum of translates of C yields a hamiltonian cycle C_0 in $\mathrm{Cay}(\overline{G}; S)$.

If $\mathbb{Z}_2 \not\subseteq \langle [a, c] \rangle$, then the connected sum defining C_0 can be chosen so that $\mathbb{Z}_2 \subseteq \langle \Pi\, C_0 \rangle$ (see the proof of Lemma 3.15). So we may let $C_* = C_0$.

We may now assume $\mathbb{Z}_2 \subseteq \langle [a, c] \rangle$. Construct a hamiltonian cycle C_1 in $\mathrm{Cay}(\overline{G}; S)$ by replacing the rightmost translate of C in the connected sum with

$$C' = (a^{k-1}, b, a^{-(k-1)}, c, a^{k-1}, b^{-1}, a^{-(k-1)}, c^{-1}).$$

A straightforward calculation shows that $(\Pi\, C)^{-1}(\Pi\, C') \notin \mathbb{Z}_p$, so we have $\mathbb{Z}_2 \subseteq \langle \Pi\, C_i \rangle$ for some $i \in \{0, 1\}$. Let $C_* = C_i$.

Assumptions 6.2.4.2. We may now assume $|\overline{G} : \langle \overline{a} \rangle| = 2$, so the irredundance of S implies $S = \{a, b, c\}$. Since $\overline{b} \in \langle \overline{a} \rangle$, the irredundance of S also implies $\langle [a, c] \rangle = \mathbb{Z}_2$. Furthermore, we may also assume that c either centralizes G' or inverts G'. (Otherwise, a preceding case applies after interchanging a with c.)

Subsubcase 6.2.4.3. *Assume c inverts G'.* Let

$$L = \begin{cases} (a, b)^k\# & \text{if } p \mid k \\ (b, a)^k\# & \text{if } p \nmid k \end{cases} \qquad \text{and} \qquad C = (L^{-1}, c^{-1}, L, c).$$

Then L is a hamiltonian path in $\operatorname{Cay}\big(\langle\overline{a}\rangle; a, b\big)$, so C is a hamiltonian cycle in $\operatorname{Cay}(\overline{G}; S)$. Since $(ab)^k = \gamma^k$, we have

$$\prod L = \begin{cases} \gamma^k b^{-1} = \gamma^{k-1} a^{-k} & \text{if } p \mid k, \\ \gamma^{-k} a^{-1} & \text{if } p \nmid k. \end{cases}$$

Thus, in either case, we have $\prod L = \gamma^y a^z$, where $p \nmid y$ and z is odd, so

$$\begin{aligned} \prod C &= (\prod L)^{-1} c^{-1} (\prod L) c = [\prod L, c] = [\gamma^y a^z, c] \\ &= [\gamma^y, c]^{a^z} \cdot [a^z, c] = (\gamma^{-2y})^{a^z} \cdot [a, c]^z = \gamma^{2y} \cdot [a, c]. \end{aligned}$$

This generates G', so the Factor Group Lemma (2.8) applies.

Subsubcase 6.2.4.4. *Assume c centralizes G' and $k \geq 5$.* Let

$$C_0 = (L, c, L^{-1}, c^{-1}),$$

where $L = (b, a)^k\#$. Since C_0 contains both $[\overline{e}](b, a, b)$ and $[\overline{a}\overline{b}\overline{c}](a^{-1})$, and also contains both $[\overline{a}^2](b, a, b)$ and $[\overline{a}^3\overline{b}\overline{c}](a^{-1})$ we can apply Lemma 2.12 twice (first with $s = b$, $t = a$, $u = c$, and $h = bc$, and then with $s = b$, $t = a$, $u = c$, and $h = a^2bc$), to obtain a hamiltonian cycle C_2, such that

$$(\prod C_0)^{-1}(\prod C_2) = [a^{-1}, b]^2,$$

which generates $\mathbb{Z}_p$. Then, since

$$\prod C_0 = \big((ba)^k a^{-1}\big) c \big((ba)^k a^{-1}\big)^{-1} c^{-1} = [a, c]$$

is a generator of $\mathbb{Z}_2$, we conclude that $\prod C_2$ generates G', so the Factor Group Lemma (2.8) applies.

Subsubcase 6.2.4.5. *Assume c centralizes G' and $k = 3$.* Assume, for the moment, that $\gamma \notin \mathbb{Z}_p$. Let

$$C = (c, b, c^{-1}, a, b^{-1}, c, b, a, b^{-1}, c^{-1}, b, a).$$

Then C is a hamiltonian cycle in $\operatorname{Cay}(\overline{G}; S)$, and a straightforward calculation shows that $\prod C = ba^3 = \gamma^{-1}$ generates G', so the Factor Group Lemma (2.8) applies.

Now, suppose that $p \geq 5$, and, because of the preceding paragraph, that $\gamma \in \mathbb{Z}_p$. Let

$$C = (b, a, b^{-1}, a, b, c, a^{-5}, c^{-1}).$$

Then C is a hamiltonian cycle in $\operatorname{Cay}(\overline{G}; S)$ and

$$\prod C = bab^{-1}abcac^{-1} = bab^{-1}aba[a, c^{-1}] = \gamma^{-3}[a, c].$$

Therefore $\langle \prod C \rangle = G'$ (since $p \neq 3$ and γ projects trivially to $\mathbb{Z}_2$), so the Factor Group Lemma (2.8) applies.

We may now assume $p = 3$ (so $|G| = 72$), and that $\gamma \in \mathbb{Z}_p$. Let $\widehat{G} = G/\mathbb{Z}_p$. We have the following hamiltonian cycle in $\operatorname{Cay}(\widehat{G}; S)$:

$$C = (a^2, c, a^5, c^{-1}, a^{-2}, b, a^2, c, a^{-5}, c^{-1}, a^{-2}, b).$$

Calculating modulo $\mathbb{Z}_2$ (so c is in the center), we have

$$\Pi C = a^2ca^5c^{-1}a^{-2}ba^2ca^{-5}c^{-1}a^{-2}b \equiv a^2a^5a^{-2}ba^2a^{-5}a^{-2}b = a^{-1}bab = [a,b] = \gamma^2.$$

This is nontrivial (mod $\mathbb{Z}_2$), so ΠC must be nontrivial. Therefore ΠC generates $\mathbb{Z}_p$, so the Factor Group Lemma (2.8) applies.

Subcase 6.2.5. *Assume* $k = 2 < m/2$.

Subsubcase 6.2.5.1. *Assume* $|\overline{G} : \langle \overline{a} \rangle| > 2$. Note that

$$C = \big(b, a, b^{-1}, c, b, a^{-1}, b, c^{-1}, (a, c, a, c^{-1})^{(m-4)/2}\big)$$

is a cycle through the vertices of $\mathrm{Cay}(\overline{G}; \{a, b, c\})$ in $\langle \overline{a} \rangle \cup c\langle \overline{a} \rangle$. A connected sum of translates of C yields a hamiltonian cycle C_0 in $\mathrm{Cay}(\overline{G}; S)$. Since k is even, we know that $\mathbb{Z}_2 \not\subseteq \langle [b, c] \rangle$, so it is easy to choose the connected sum in such a way that $\mathbb{Z}_2 \subseteq \langle \Pi C_0 \rangle$ (see the proof of Lemma 3.15).

The cycle C contains the paths $[\overline{e}](b, a, b^{-1})$ and $[\overline{b}^2](a)$. By taking just a bit of care in the creation of C_0 (namely, not using any of these edges for the first connected sum), we may assume that C_0 also contains these paths. Then Lemma 2.12 (with $s = b$, $t = a$, $u = b$, and $h = b^2$) provides a hamiltonian cycle C_1, such that $(\Pi C_0)^{-1}(\Pi C_1) = [a, b]^2$ (because b centralizes G'). This is a generator of $\mathbb{Z}_p$, so either ΠC_0 or ΠC_1 generates G'. Therefore, the Factor Group Lemma (2.8) applies.

Subsubcase 6.2.5.2. *Assume* $|\overline{G} : \langle \overline{a} \rangle| = 2$. The irredundance of S implies that $S = \{a, b, c\}$ (see Corollary 3.5). We have the following hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$:

$$C = (b^2, a^{m-5}, c, a^{-(m-4)}, c^{-1}, b^{-1}, c, a, b^{-1}, c^{-1}).$$

Since $\overline{b} \in \langle \overline{a} \rangle$, the irredundance of S implies $\langle [a, c] \rangle = \mathbb{Z}_2$. So m is even (see Corollary 3.8). However, $\mathbb{Z}_2 \not\subseteq \langle [b, c] \rangle$, because $k = 2$ is even. So

$$\begin{aligned} \Pi C &= b^2(a^{m-5}ca^{-(m-4)}c^{-1})(b^{-1}cab^{-1}c^{-1}) \\ &\equiv b^2(a^{-1})(b^{-2}a[a, c]) \equiv [a, c] \pmod{\mathbb{Z}_p}, \end{aligned}$$

which generates $\mathbb{Z}_2$. We may also assume that c either centralizes G' or inverts G' (for otherwise a preceding case applies after interchanging a with c). Therefore

$$\begin{aligned} \Pi C &= b^2(a^{m-5}ca^{-(m-4)}c^{-1})(b^{-1}cab^{-1}c^{-1}) \equiv a^4\gamma^2(a^{-1})(\gamma^{-1}a^{-2}ca\gamma^{-1}a^{-2}c^{-1}) \\ &= \gamma^3 \cdot (\gamma^{-1})^c = \gamma^3 \cdot \gamma^{\pm 1} \in \{\gamma^2, \gamma^4\} \pmod{\mathbb{Z}_2}, \end{aligned}$$

which generates $\mathbb{Z}_p$. We now know that ΠC projects nontrivially to both $\mathbb{Z}_2$ and $\mathbb{Z}_p$, so it generates G'. Therefore, the Factor Group Lemma (2.8) applies. □

7 Cases with $|\overline{a}| = 2$ and $\#S = 2$

Assumption 7.1. In this section, we assume

- $|\overline{a}| = 2$, for all $a \in S$, such that a does not centralize G', and
- $\#S = 2$.

We may assume $|a| = 2$, for otherwise Case 4.1 applies with $s = a$ and $t = a^{-1}$.

We may also assume that b centralizes G', for otherwise we must have $|\bar{b}| = 2$, so $|G| = 8p$, so Theorem 2.3 applies. Since a does not centralize G', this implies $\overline{a} \notin \langle \bar{b} \rangle$. Let

$$n = |\overline{G} : \langle \overline{a} \rangle| = |\overline{G}|/2 = |\bar{b}|.$$

Case 7.2. *Assume* $n \not\equiv 1 \pmod{p}$.

Proof. Let $C = (a^{-1}, b^{-(n-1)}, a, b^{n-1})$, so C is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$ with $\prod C = [a, b^{n-1}] = [a, b]^{n-1}$, since b centralizes G'. Note that n is even (see Corollary 3.8), and, by assumption, $n \not\equiv 1 \pmod{p}$. Therefore, $n - 1$ is relatively prime to $2p$, so $\prod C$ generates G', so the Factor Group Lemma (2.8) applies. □

Case 7.3. *Assume* $n \equiv 1 \pmod{p}$.

Proof. We claim that $\mathbb{Z}_p \subseteq \langle b \rangle$. Suppose not. Then $|\langle b, \mathbb{Z}_2 \rangle| = 2n$. Since $\gcd(2n, p) = 1$, the abelian group $\langle b, G' \rangle$ has a unique subgroup of order $2n$, so we conclude that $\langle b, \mathbb{Z}_2 \rangle$ is normal in G. This implies that

$$\langle a \rangle \langle b, \mathbb{Z}_2 \rangle = \langle a, b, \mathbb{Z}_2 \rangle \supseteq \langle a, b \rangle = G,$$

so

$$|G| \leq |a| \cdot |\langle b, \mathbb{Z}_2 \rangle| = 2 \cdot 2n = 4n.$$

This contradicts the fact that $|G| = 4np$.

Subcase 7.3.1. *Assume* $\mathbb{Z}_2 \subseteq \langle b \rangle$. Combining this assumption with the above claim, we see that $G' \subseteq \langle b \rangle$. This implies $\langle b \rangle \triangleleft G$, so $G = \langle a \rangle \ltimes \langle b \rangle$. Since $|a| = 2$, this implies that $\mathrm{Cay}(G; a, b)$ is a generalized Petersen graph. Then the main result of [1] tells us that $\mathrm{Cay}(G; a, b)$ has a hamiltonian cycle.

Subcase 7.3.2. *Assume* $\mathbb{Z}_2 \not\subseteq \langle b \rangle$. Since $\langle b, G' \rangle$ is abelian, $\gcd(n, p) = 1$, and $\mathbb{Z}_2 \not\subseteq \langle b \rangle$, we may write

$$\langle b, G' \rangle = \mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_n.$$

Then $G = \langle a \rangle \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_n)$, and we may assume $b = (0, 1, 1)$ and $[a, b] = (1, 2, 0)$. For $\underline{G} = G/\langle b^2 \rangle = G/(\mathbb{Z}_p \times 2\mathbb{Z}_n)$, it is straightforward to check that $\big((a, b)^4\#, b^{-1}\big)$ is a hamiltonian cycle in $\mathrm{Cay}(\underline{G}; a, b)$ whose voltage is $(0, -2, 2)$. (This hamiltonian cycle is taken from the final paragraph of Case 1 of the proof of [3, Prop. 6.1].) This voltage generates $\mathbb{Z}_p \times 2\mathbb{Z}_n$ (since $\gcd(p, n) = 1$), so the Factor Group Lemma (2.8) applies. □

8 Cases with $|\overline{a}| = 2$ and $\#S = 3$

Assumption 8.1. In this section, we assume

$$S = \{a, b, c\},$$

and

$$|\overline{s}| = 2, \text{ for all } s \in S, \text{ such that } s \text{ does not centralize } G'.$$

We also assume Case 4.1 does not apply. (So $|s| = 2$.) In particular, we have $|a| = 2$.

Note that $\overline{a} \notin \langle \bar{b} \rangle$. (If $\overline{a} \in \langle \bar{b} \rangle$, then b, like a, does not centralize G', so our assumption implies $|\bar{b}| = 2$. Then $\overline{a} = \bar{b}$, contradicting the fact that Case 4.1 does not apply.)

Notation 8.2. Let

$$n = |\overline{b}| = |\langle \overline{a}, \overline{b} \rangle : \langle \overline{a} \rangle| \geq 2 \quad \text{and} \quad \ell = |\overline{G} : \langle \overline{a}, \overline{b} \rangle| = |\overline{G}|/(2n) \geq 2.$$

The last inequality is because the irredundance of S implies $\overline{c} \notin \langle \overline{a}, \overline{b} \rangle$ (see Corollary 3.5).

Case 8.3. *Assume* $|\overline{b}| = 3$.

Proof. Since $|\overline{b}| \neq 2$, Assumption 8.1 implies that b centralizes G'. Also, since $|\overline{b}|$ is odd, Corollary 3.8 implies that $[a, b]$ and $[b, c]$ project trivially to $\mathbb{Z}_2$, so $[a, c]$ must project nontrivially (and ℓ must be even). We have the following hamiltonian path in $\mathrm{Cay}\big(\overline{G}/\langle \overline{a} \rangle; S\big)$:

$$L = (c^{\ell-1}, b, c^{-(\ell-1)}, b, c^{\ell-1}).$$

Then $C = (L, a, L^{-1}, a)$ is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$. Since $\ell - 1$ is odd, it is easy to see that $\mathbb{Z}_2 \subseteq \langle \Pi\, C \rangle$.

Since C contains both $[\overline{c}^{\ell-2}](c, b, c^{-1})$ and $[\overline{c}^{\ell-1}\overline{a}\overline{b}](b^{-1})$, Lemma 2.12 (with $s = c$, $t = b$, $u = a$, and $h = c^{\ell-1}a$) provides a hamiltonian cycle C', such that $(\Pi\, C)^{-1}(\Pi\, C')$ is conjugate to $[t^{-1}, u]\,[s, t^{-1}]^u = [b^{-1}, a]\,[c, b^{-1}]^a = [a, b]\,[c, b]$. This is an element of $\mathbb{Z}_p$. If it generates $\mathbb{Z}_p$, then either $\Pi\, C$ or $\Pi\, C'$ generates G', so the Factor Group Lemma (2.8) applies.

Thus, we may assume $[a, b]\,[c, b]$ is trivial. Since $\mathbb{Z}_p \subseteq \langle [a, b] \rangle$ (see (3.3B)), this implies that $[c, b]$ is nontrivial. So we may assume that c does not centralize $\mathbb{Z}_p$ (for otherwise replacing c with c^{-1} would replace $[c, b]$ with $[c, b]^{-1}$, which would not cancel $[a, b]$).

Now, Assumption 8.1 implies $|\overline{c}| = 2$, so we have the hamiltonian cycle

$$C_0 = (b^2, a, b^2, c, a, b, a, b, a, c),$$

in $\mathrm{Cay}(\overline{G}; S)$. This contains both the path $[bac](a, b, a)$ and the edge $[b](b)$, so applying Lemma 2.12 (with $s = a$, $t = b$, $u = c$, and $h = b$) provides a hamiltonian cycle C_1, such that $\big(\Pi\, C_0\big)^{-1}\big(\Pi\, C_1\big)$ is conjugate to $[u, t^{-1}]\,[s, t^{-1}]^u = [c, b^{-1}]\,[a, b^{-1}]^c$. This is not equal to $[a, b]\,[c, b]$ (which is trivial), because $[a, b^{-1}]^c = [a, b]$, but $[c, b^{-1}] = [c, b]^{-1} \neq [c, b]$. So $\big(\Pi\, C_0\big)^{-1}\big(\Pi\, C_1\big)$ is nontrivial, and therefore generates $\mathbb{Z}_p$. Since a straightforward calculation shows that $\mathbb{Z}_2$ is contained in $\langle \Pi\, C_0 \rangle$, this implies that either $\Pi\, C_0$ or $\Pi\, C_1$ generates G', so the Factor Group Lemma (2.8) applies. □

Case 8.4. *Assume* $\ell = 2$.

Proof. We may assume $|\overline{b}| \geq 4$, for otherwise either $|\overline{b}| = 2$, so Theorem 2.3 applies (because $|G| = 16p$), or $|\overline{b}| = 3$, so Case 8.3 applies. Let

$$L = (a, b, a, b^{n-2}, a, b^{-(n-3)}) \qquad \text{and} \qquad C = (L, c, L^{-1}, c^{-1}),$$

so L is a hamiltonian path in $\mathrm{Cay}\big(\langle \overline{a}, \overline{b} \rangle; a, b\big)$ and C is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$.

Subcase 8.4.1. *Assume* $[a, c]$ *and* $[a, b][b, c]$ *are not both in* $\mathbb{Z}_p$. A straightforward calculation (using Lemma 3.6) shows that $\Pi\, C \equiv [a, c] \pmod{\mathbb{Z}_p}$. If this is in $\mathbb{Z}_p$, then, by assumption, $[a, b][b, c] \notin \mathbb{Z}_p$, so applying Lemma 2.12 to the paths $[\overline{e}](a, b, a)$ and $[\overline{abc}](b^{-1})$ in C (so $s = a$, $t = b$, $u = c$, and $h = ac$) yields a hamiltonian cycle C', such that $\Pi\, C'$ projects nontrivially to $\mathbb{Z}_2$. Therefore, we have a hamiltonian cycle (either C or C') whose voltage is not in $\mathbb{Z}_p$.

Now, since $|\overline{b}| \geq 4$, we see that C (and also C') contains the path $[\overline{b^{-2}ac}](b, a, b^{-1})$ and $[\overline{ac}](a)$. Furthermore, we know that $[b, a][b, a]^b$ is a nontrivial element of $\mathbb{Z}_p$ (because b does not invert $[a, b]$). Therefore, Lemma 2.12 (with $s = b$, $t = a$, $u = b$, and $h = ac$) yields a hamiltonian cycle C_1 (or C_1') whose voltage generates G', so the Factor Group Lemma (2.8) applies.

Subcase 8.4.2. *Assume* $[a, c]$ *and* $[a, b][b, c]$ *are both in* $\mathbb{Z}_p$. Since $[a, c]$, $[a, b]$, and $[b, c]$ generate G', they cannot all be in $\mathbb{Z}_p$, so this assumption implies that neither $[a, b]$ nor $[b, c]$ is in $\mathbb{Z}_p$. Also, we may assume $\langle [a, c] \rangle = \mathbb{Z}_p$, for otherwise $[a, c] = e$, so we could apply Lemma 2.13 with $s = c$.

We have the following hamiltonian cycle in $\operatorname{Cay}(\overline{G}; S)$:

$$C_0 = (b^{n-1}, c, b^{-(n-2)}, a, b^{n-2}, c^{-1}, b^{-(n-1)}, c, a, c^{-1}).$$

Then

$$\begin{aligned}
\Pi\, C_0 &= b^{n-1}c\big(b^{-(n-2)}ab^{n-2}\big)c^{-1}b^{-(n-1)}cac^{-1} \\
&= b^{n-1}c\big(a[a, b]^{n-2}\big)c^{-1}b^{-(n-1)}cac^{-1} \\
&= ([a, b]^{-(n-2)})^c \cdot b^{n-1}(cac^{-1})b^{-(n-1)}(cac^{-1}) \\
&= ([a, b]^{-(n-2)})^c \cdot [b, cac^{-1}]^{-(n-1)} \\
&= ([a, b]^{-(n-2)})^c \cdot [b, a]^{-(n-1)} && (cac^{-1} \in aG' \text{ and } G' \subseteq C_G(b)) \\
&= ([a, b]^{-(n-2)})^c \cdot [a, b]^{n-1}.
\end{aligned}$$

If c centralizes $\mathbb{Z}_p$, then $\Pi\, C_0 = [a, b]$ generates G', so the Factor Group Lemma (2.8) applies.

We may now assume c does not centralize $\mathbb{Z}_p$. Then Assumption 8.1 tells us that c inverts $\mathbb{Z}_p$, so $\Pi\, C_0 = [a, b]^{2n-3}$ (and $|c| = 2$). Hence, we may assume $2n \equiv 3 \pmod{p}$, for otherwise $\Pi\, C_0$ generates G', so the Factor Group Lemma (2.8) applies. We now consider the following hamiltonian cycle in $\operatorname{Cay}(\overline{G}; S)$:

$$C_* = (b^{n-3}, c, b^{-(n-4)}, a, b^{n-4}, c^{-1}, b^{-(n-3)}, c, (b^{-1}, c)^2, a, (c, b)^2, c^{-1}).$$

We have

$$\Pi\, C_* = b^{n-3}c\big(b^{-(n-4)}ab^{n-4}\big)c^{-1}b^{-(n-3)}c\big((b^{-1}c)^2a(cb)^2\big)c^{-1}.$$

Since cb inverts G', we know that $(b^{-1}c)^2a(cb)^2 = a$, so $\Pi\, C_*$ is exactly the same as the voltage of C_0, but with n replaced by $n - 2$; that is,

$$\Pi\, C_* = [a, b]^{2(n-2)-3} = [a, b]^{2n-7}.$$

Since $2n \equiv 3 \pmod{p}$, we have

$$2n - 7 \equiv 3 - 7 = -4 \not\equiv 0 \pmod{p},$$

so $\Pi\, C_*$ generates G', so the Factor Group Lemma (2.8) applies. □

Case 8.5. *Assume* $|\overline{b}| \neq 3$ *and* $\ell \neq 2$.

Proof. Since $\ell \neq 2$, we know $|\overline{c}| > 2$, so c must centralize G' (by Assumption 8.1). Also, Corollary 3.8 implies that $|\overline{b}|$ and ℓ cannot both be odd.

- If $|\overline{b}|$ is odd (so ℓ is even), let

$$L = \big(c^{\ell-1}, b, c^{-1}, b, c, b, (b^{n-4}, c^{-1}, b^{-(n-4)}, c^{-1})^{\ell/2}\#, b^{-1}, c^{\ell-3}, b^{-1}, c^{-(\ell-3)}\big).$$

- If $|\overline{b}|$ is even, let

$$L = \big(c^{\ell-1}, b^{n-1}, c^{-1}, (c^{-(\ell-2)}, b^{-1}, c^{\ell-2}, b^{-1})^{(n-2)/2}, c^{-(\ell-2)}\big).$$

In either case, L is a hamiltonian path in $\mathrm{Cay}\big(\overline{G}/\langle\overline{a}\rangle; \{b, c\}\big)$ from $\overline{e}$ to $\overline{b}$. Now, let

$$C = (L, a, L^{-1}, a) \quad \text{and} \quad (g, \epsilon) = \begin{cases} (c^{\ell-1}, -1) & \text{if } |\overline{b}| = 2 \text{ or } |\overline{b}| \text{ is odd,} \\ (ab^2, 1) & \text{if } |\overline{b}| > 2 \text{ and } |\overline{b}| \text{ is even,} \end{cases}$$

so C is a hamiltonian cycle in $\mathrm{Cay}(\overline{G}; S)$ that contains the paths

$$[\overline{bc}](c^{-1}, a, c), \quad [\overline{ca}](c^{-1}, a, c), \quad [\overline{g}](b), \quad \text{and} \quad [\overline{gbac^{\epsilon}}](c^{-\epsilon}, b^{-1}, c^{\epsilon}).$$

Note that $[\overline{bc}](c^{-1}, a, c)$ contains $[\overline{b}](a)$ and that $[\overline{ca}](c^{-1}, a, c)$ contains $[\overline{a}](a)$. Also note that all of these paths are vertex-disjoint (except for the vertices $\overline{ac}$ and $\{abc\}$ when $|\overline{b}| = 2$ and $\ell = 3$). We introduce some terminology:

- Applying Lemma 2.12 to the oriented paths $[\overline{ca}](c^{-1}, a, c)$ and $[\overline{b}](a)$ (so $s = c^{-1}$, $t = a$, $u = b$, and $h = ab$) will be called the "a-transform." This multiplies the voltage by γ_a, where $\gamma_a = [a, b^{-1}][c, a]$.
- Applying Lemma 2.12 to the oriented paths $[\overline{g}](b)$ and $[\overline{gbac^{\epsilon}}](c^{-\epsilon}, b^{-1}, c^{\epsilon})$ (so $s = c^{-\epsilon}$, $t = b^{-1}$, $u = a$, and $h = gb$) will be called the "b-transform." This multiplies the voltage by a conjugate of γ_b, where $\gamma_b = [b, a][b, c^{-\epsilon}]$.

Subcase 8.5.1. *Assume precisely one of γ_a and γ_b is in $\mathbb{Z}_p$.* Write $\{a, b\} = \{x, y\}$, such that $\gamma_x \in \mathbb{Z}_p$ and $\gamma_y \notin \mathbb{Z}_p$. We may assume $\langle\gamma_x\rangle = \mathbb{Z}_p$ (by replacing c with its inverse, if necessary). Choose C' to be either C or the y-transform of C, such that $\Pi\, C'$ projects nontrivially to $\mathbb{Z}_2$. Then choose C'' to be either C' or the x-transform of C', such that $\Pi\, C''$ generates G', so the Factor Group Lemma (2.8) applies.

Subcase 8.5.2. *Assume γ_a and γ_b are both in $\mathbb{Z}_p$.* Since $[a, b]$, $[a, c]$, and $[b, c]$ cannot all be in $\mathbb{Z}_p$, this assumption implies that none of them are in $\mathbb{Z}_p$. Therefore, since the path L has odd length, we see that $\Pi\, C$ has nontrivial projection to $\mathbb{Z}_2$.

We may assume (by replacing c with its inverse, if necessary), that γ_a has nontrivial projection to $\mathbb{Z}_p$, so $\langle\gamma_a\rangle = \mathbb{Z}_p$. Therefore, by choosing C' to be either C or the a-transform of C, such that $\Pi\, C'$ generates G', we may apply the Factor Group Lemma (2.8).

Subcase 8.5.3. *Assume neither γ_a nor γ_b is in $\mathbb{Z}_p$, and b centralizes G'.* Note that the sum of the exponents of the occurrences of b in L is 1, and the sum of the exponents of the occurrences of c is 0. Therefore, since b and c centralize G', Lemma 3.6 implies that $\Pi\, C = [a, b]$. Hence, we may assume $[a, b] \in \mathbb{Z}_p$ (for otherwise $\langle\Pi\, C\rangle = G'$, so the Factor Group Lemma (2.8) applies). Then, by the assumption of this subcase, we conclude that

$[a,c] \notin \mathbb{Z}_p$. So we may assume $\langle [a,c] \rangle = \mathbb{Z}_2$, for otherwise b and c could be interchanged, resulting in a situation in which $[a,b] \notin \mathbb{Z}_p$, and which has therefore already been covered. Also, since $[a,b] \in \mathbb{Z}_p$ and $[a,c] \notin \mathbb{Z}_p$, Corollary 3.8 tells us that ℓ is even (and recall that $\ell \neq 2$).

Since $[a,b]$ is a nontrivial element of $\mathbb{Z}_p$, and b centralizes G', we see from Corollary 3.7 that $|b|$ is divisible by p. Therefore, $|b| \neq 2$, so we may assume $|\overline{b}| > 2$ (for otherwise Case 4.1 applies with $s = b$ and $t = b^{-1}$). Since $|\overline{b}| \neq 3$ (by the assumption of this case), this implies $n = |\overline{b}| \geq 4$, so we may let

$$L_0 = \bigl(c^{\ell-1}, b, c^{-(\ell-1)}, b^2, (b^{n-4}, c, b^{-(n-4)}, c)^{\ell/2}\#, b^{-1}, c^{-(\ell-2)}\bigr),$$

so L_0 is a hamiltonian path from $\overline{e}$ to $\overline{b^2 c}$ in $\operatorname{Cay}\bigl(\overline{G}/\langle \overline{a} \rangle; \{b,c\}\bigr)$. Note that the sum of the exponents of the occurrences of b in L is 2, and the sum of the exponents of the occurrences of c is 1. Therefore, since b and c centralize G', Lemma 3.6 implies $\prod(L_0, a, L_0^{-1}, a) = [a,b]^2[a,c]$. This generates G', so the Factor Group Lemma (2.8) applies.

Subcase 8.5.4. *Assume neither γ_a nor γ_b is in $\mathbb{Z}_p$, and b does not centralize $\mathbb{Z}_p$.* From Assumption 8.1, we know $\overline{b} = 2$ (so b must invert G').

We may assume $[a,c] \in \mathbb{Z}_2$, for otherwise Case 8.4 could be applied by interchanging b and c. Then we may assume $[a,c]$ is the nontrivial element of $\mathbb{Z}_2$, for otherwise the assumption that $\gamma_a \notin \mathbb{Z}_p$ implies $\langle [a,b] \rangle = G'$, so $\langle a,b \rangle \lhd G$, and then Lemma 2.13 applies with $s = c$.

By applying the same argument, with a and b interchanged, we may assume $[b,c]$ is also the nontrivial element of $\mathbb{Z}_2$. This implies $[a,b] \in \mathbb{Z}_p$, since $\gamma_b \notin \mathbb{Z}_p$.

Note that, since a and b both have order 2 (and invert G'), the image of $\langle a,b \rangle$ in $G/\mathbb{Z}_2$ is the dihedral group of order $2p$. Also, the preceding two paragraphs imply that c is in the center of $G/\mathbb{Z}_2$. Therefore, we have the following hamiltonian cycle in $\operatorname{Cay}\bigl(G/\mathbb{Z}_2; S\bigr)$:

$$C = \bigl(c, (c^{\ell-2}, a, c^{-(\ell-2)}, b)^p\#, c^{-1}, (a^{-1}, b^{-1})^p\#\bigr).$$

Since $[a,b]$ projects trivially to $\mathbb{Z}_2$, Corollary 3.8 implies that ℓ is even, so, calculating modulo $\mathbb{Z}_p$, we have

$$\begin{aligned}
\prod C &= c(c^{\ell-2}ac^{-(\ell-2)}b)^p b^{-1} c^{-1} (a^{-1}b^{-1})^p b \\
&\equiv c(ab)^p b^{-1} c^{-1} (a^{-1}b^{-1})^p b && \begin{pmatrix} \ell-2 \text{ is even, so } c^{\ell-2} \\ \text{is central modulo } \mathbb{Z}_p \end{pmatrix} \\
&\equiv z^{2p-1}(ab)^p b^{-1} (a^{-1}b^{-1})^p b && \begin{pmatrix} \text{letting } z = [a,c] = [b,c] \text{ be} \\ \text{the nontrivial element of } \mathbb{Z}_2 \end{pmatrix} \\
&\equiv z && (z^2 = e \text{ and } [a,b] \in \mathbb{Z}_p).
\end{aligned}$$

Since this generates $\mathbb{Z}_2$, the Factor Group Lemma (2.8) applies. □

9 Cases with $|\overline{a}| = 2$ and $\#S \geq 4$

Assumption 9.1. In this section, we assume

- $\#S \geq 4$, and
- $|\overline{s}| = 2$, for all $s \in S$, such that s does not centralize G'.

We also assume Case 4.1 does not apply. (So $|s| = 2$.)

Furthermore, we assume $\overline{b} \notin \langle \overline{a} \rangle$ (otherwise, Case 4.1 applies). Then it is easy to see that we also have $\overline{a} \notin \langle \overline{b} \rangle$.

Outline. This final section of the proof is longer than the others, so here is an outline of the cases and subcases that it considers.

9.4: *Assume no element of S centralizes G'.*

- 9.4.1: *Assume $\#S \geq 5$.*
- 9.4.2: *Assume $\#S = 4$.*

9.5: *Assume there exists $s \in S$, such that $[a, s] \notin \mathbb{Z}_p$, and, in addition, either $s = b$, or b centralizes G', or $\mathbb{Z}_p \subseteq \langle S \smallsetminus \{a\} \rangle'$.*

- 9.5.1: *Assume $\mathbb{Z}_p \nsubseteq \langle S \smallsetminus \{a\} \rangle'$.*
- 9.5.2: *Assume $\mathbb{Z}_p \subseteq \langle S \smallsetminus \{a\} \rangle'$.*

9.6: *Assume b centralizes G'.*

- 9.6.1: *Assume there exists $c \in S$, such that $[c, b] \notin \mathbb{Z}_p$.*
- 9.6.2: *Assume $[c, b] \in \mathbb{Z}_p$ for all $c \in S$.*

9.7: *Assume that none of the preceding cases apply.*
Since Case 9.4 does not apply, some element c of S centralizes G'.

- 9.7.1: *Assume $\langle [s, c] \rangle \neq \mathbb{Z}_2$, for some $s \in S \smallsetminus \{c\}$.*
- 9.7.2: *Assume $\langle [s, c] \rangle = \mathbb{Z}_2$, for all $s \in S \smallsetminus \{c\}$.*

Notation 9.2. Let $n = |\overline{b}|$ and $\ell = |\overline{G} : \langle \overline{a}, \overline{b} \rangle| = |\overline{G}|/(2n)$.

Note 9.3. The irredundance of S implies $S \smallsetminus \{a, b\}$ is an irredundant generating set for $\overline{G}/\langle \overline{a}, \overline{b} \rangle$ (see Corollary 3.5), so $\ell \geq 4$.

Case 9.4. *Assume no element of S centralizes G'.*

Proof. From Assumption 9.1, we see that every element of S inverts G' (and has order 2). We may assume no two elements of S commute, for otherwise it is not difficult to see that Lemma 2.13 applies.

Let $c, d \in S \smallsetminus \{a, b\}$, and let $\gamma = [a, b]\,[a, c]$. We claim that we may assume $\gamma \notin \mathbb{Z}_2$, by permuting b, c, d. To this end, first note that if $\gamma \in \mathbb{Z}_2$, then $\mathbb{Z}_p \subseteq \langle [a, c] \rangle$, so there is no harm in putting c into the role of b. Now, let us suppose $[a, b][a, c]$, $[a, c][a, d]$, and $[a, d][a, b]$ are all in $\mathbb{Z}_2$. Then

$$[a, b] \equiv [a, c]^{-1} \equiv [a, d] \equiv [a, b]^{-1} \pmod{\mathbb{Z}_2},$$

which contradicts the fact that $[a, b] \notin \mathbb{Z}_2$ (and p is odd).

Let

$$C = \big((c, a, c, b)^2 \#, d\big)^2,$$

so C is a hamiltonian cycle in $\mathrm{Cay}\big(\langle \overline{a}, \overline{b}, \overline{c}, \overline{d} \rangle; \{a, b, c, d\}\big)$ that contains the vertex-disjoint paths $[\overline{e}](c, a, c)$, $[\overline{abc}](a)$, $[\overline{bd}](c, a, c)$, and $[\overline{acd}](a)$. Applying Lemma 2.12 to the paths

$[\overline{e}](c,a,c)$ and $[\overline{abc}](a)$ (so $s = c$, $t = a$, $u = b$, and $h = bc$) will multiply the voltage by γ. Applying Lemma 2.12 to the other two paths $[\overline{bd}](c,a,c)$ and $[\overline{acd}](a)$ (so $s = c$, $t = a$, $u = b$, and $h = cd$) will also multiply the voltage by γ (because bc and cd both centralize G'). Therefore, applying Lemma 2.12 twice yields a hamiltonian cycle C'', such that $(\Pi\, C)^{-1}(\Pi\, C'') = \gamma^2$, which is a generator of $\mathbb{Z}_p$.

Subcase 9.4.1. *Assume* $\#S \geq 5$. If there exist $s,t \in S$, such that $s \notin \{a,b,c\}$, and $[s,t] \notin \mathbb{Z}_p$, then the preceding paragraph implies that Lemma 3.15(2) applies.

Thus, we may assume that the preceding condition does not apply (for any legitimate choice of a, b, and c). Fix two elements $x,y \in S \setminus \{a,b,c\}$. The failure of the condition implies $[x,S] \subseteq \mathbb{Z}_p$. In particular, $[x,y]$ must be a generator of $\mathbb{Z}_p$ (because no two elements of S commute), so we may let $\{x,y\}$ play the role of $\{a,b\}$. So we may let $\{x,y,b,c\}$ play the role of $\{a,b,c,d\}$. Then, since $a \notin \{x,y,b,c\}$, the failure of the condition implies $[a,S] \subseteq \mathbb{Z}_p$. Similarly, $[b,S]$ and $[c,S]$ are also in $\mathbb{Z}_p$. So $[s,t] \subseteq \mathbb{Z}_p$ for all $s,t \in S$. This contradicts the fact that $\langle [S,S] \rangle = G' \nsubseteq \mathbb{Z}_p$.

Subcase 9.4.2. *Assume* $\#S = 4$. For convenience, in this subcase (and only in this subcase), we drop our standing assumption that $\langle [a,b] \rangle$ contains $\mathbb{Z}_p$. Instead, choose $b,d \in S$, such that $[b,d]$ projects nontrivially to $\mathbb{Z}_2$. A straightforward calculation (using the fact that a, b, c, and d all invert G') shows that

$$\Pi\, C = [c,d]^4 [d,a]^2 [d,b].$$

Since $[d,b]$ projects nontrivially to $\mathbb{Z}_2$, but $[c,d]^4$ and $[d,a]^2$ have even exponents, so they obviously do not, we see that $\mathbb{Z}_2 \subseteq \langle \Pi\, C \rangle$. Therefore, we may assume $\Pi\, C \in \mathbb{Z}_2$, for otherwise the Factor Group Lemma (2.8) applies.

We may assume $\gamma \in \mathbb{Z}_2$, for otherwise applying Lemma 2.12 twice (as in the paragraph immediately before Subcase 9.4.1) yields a hamiltonian cycle whose voltage generates G', so the Factor Group Lemma (2.8) applies. By the definition of γ, this means $[a,b][a,c] \in \mathbb{Z}_2$. And we may assume the same is true when b and d are interchanged, which means $[a,d][a,c] \in \mathbb{Z}_2$. So

$$[a,b] \equiv [a,c]^{-1} \equiv [a,d] \pmod{\mathbb{Z}_2}.$$

By interchanging a and c, we conclude that we may also assume

$$[c,b] \equiv [c,a]^{-1} \equiv [c,d] \pmod{\mathbb{Z}_2}.$$

So

$$[c,d] \equiv [c,a]^{-1} = [a,c] \equiv [a,d]^{-1} = [d,a] \pmod{\mathbb{Z}_2}.$$

Therefore

$$[d,a]^6[d,b] = [d,a]^4[d,a]^2[d,b] \equiv [c,d]^4[d,a]^2[d,b] = \Pi\, C \equiv 0 \pmod{\mathbb{Z}_2}.$$

If $p \neq 3$, then, since we may assume the same is true when we interchange a and c, we conclude that $[d,c] \equiv [d,a] \pmod{\mathbb{Z}_2}$. Since we also have $[c,d] \equiv [d,a] \pmod{\mathbb{Z}_2}$, we conclude that $[c,d]$ and $[a,d]$ are in $\mathbb{Z}_2$. This implies $[b,d] \notin \mathbb{Z}_2$ (since d does not centralize $\mathbb{Z}_p$, and is therefore not in the center of $G/\mathbb{Z}_2$), so

$$\Pi\, C = [c,d]^4[d,a]^2[d,b] \equiv e^4 e^2 [d,b] = [d,b] \not\equiv 0 \pmod{\mathbb{Z}_2}.$$

This contradicts the fact that $\Pi\, C \in \mathbb{Z}_2$.

We now assume $p = 3$. Then the equation $[d,a]^6[d,b] \equiv 0 \pmod{\mathbb{Z}_2}$ implies $[d,b] \in \mathbb{Z}_2$. This conclusion came from assuming only that $[d,b] \notin \mathbb{Z}_p$. Therefore, for all $s,t \in S$, the commutator $[s,t]$ must be in either $\mathbb{Z}_2$ or $\mathbb{Z}_p$. However,

$$[a,b] \equiv [c,a] \equiv [a,d] \equiv [b,c] \equiv [d,c] \pmod{\mathbb{Z}_2},$$

and $[a,b] \notin \mathbb{Z}_2$. Therefore, we conclude all five of these other commutators are in $\mathbb{Z}_p$. (Therefore, the stated congruences between these commutators are actually equalities.)

Now, interchanging $a \leftrightarrow b$ and $c \leftrightarrow d$ in C yields a hamiltonian cycle C^*, such that

$$\Pi\, C^* = [d,c]^4[c,b]^2[c,a] = [d,c][b,c][c,a] = [c,a]^3 = e$$

(because $p = 3$). Let $\gamma^* = [b,a]\,[b,d]$, so γ^* is obtained from $\gamma = [a,b][a,c]$ by interchanging $a \leftrightarrow b$ and $c \leftrightarrow d$. Then, since applying Lemma 2.12 to C can multiply the voltage by $\gamma = [a,b]\,[a,c]$, we know that applying Lemma 2.12 to C^* can multiply the voltage by γ^*, which generates G'. So the Factor Group Lemma (2.8) applies. □

Case 9.5. *Assume there exists $s \in S$, such that $[a,s] \notin \mathbb{Z}_p$, and:*

$$\textit{either } \ s = b, \ \textit{ or } \ b \textit{ centralizes } G', \ \textit{ or } \ \mathbb{Z}_p \subseteq \langle S \smallsetminus \{a\}\rangle'.$$

Proof. Let $S_0 = S \smallsetminus \{a\}$. Note that the irredundance of S implies $a \notin \langle S_0\rangle \mathbb{Z}_2$ (see Lemma 3.4).

Subcase 9.5.1. *Assume $\mathbb{Z}_p \not\subseteq \langle S_0\rangle'$.* If $[a,b] \notin \mathbb{Z}_p$, we assume that $s = b$. Let

$$g = \begin{cases} s & \text{if } [s,a] \notin \mathbb{Z}_2, \\ sb^2 & \text{if } [s,a] \in \mathbb{Z}_2. \end{cases}$$

Note that $\langle [g,a]\rangle = G'$.

Let $H^* = \langle S_0\rangle \mathbb{Z}_2/\mathbb{Z}_2$. From the assumption of this subcase, we know that H^* is abelian. Therefore, Corollary 2.11 provides a hamiltonian path $L = (s_i)_{i=1}^r$ in $\operatorname{Cay}(\overline{H^*}; S_0)$, such that $s_1 s_2 \cdots s_r \in g\mathbb{Z}_2$. Then (L^{-1}, a, L, a) is a hamiltonian cycle in $\operatorname{Cay}(\overline{G}; S)$, and

$$\Pi\, C = [s_1 s_2 \cdots s_r, a] \in [g\mathbb{Z}_2, a] = \{[g,a]\}$$

(since $\mathbb{Z}_2$ is in the center of G). This voltage generates G', so the Factor Group Lemma (2.8) applies.

Subcase 9.5.2. *Assume $\mathbb{Z}_p \subseteq \langle S_0\rangle'$.* Suppose $w, x, y \in S^{\pm 1} \smallsetminus \{a\}$, such that

$$\langle \overline{w}\rangle \subsetneq \langle \overline{w}, \overline{x}\rangle \subsetneq \langle \overline{w}, \overline{x}, \overline{y}\rangle. \tag{9.5A}$$

It is easy to construct a hamiltonian cycle C_0 in $\operatorname{Cay}(\langle\overline{S_0}\rangle; S_0)$, such that C_0 contains the oriented paths $[\overline{hw^{-1}y^{-1}}](w, x, w^{-1})$ and $[\overline{hx}](x^{-1})$, for some $h \in G$. Furthermore, if

$$\text{either } \ x \notin \{s^{\pm 1}\} \ \text{ or } \ |\overline{G}| > 16, \tag{9.5B}$$

then, for some $\epsilon \in \{\pm 1\}$, it is not difficult to arrange that the hamiltonian cycle C_0 contains the oriented edge $[\overline{s^\epsilon}](s^{-\epsilon})$, and that this edge is not in either of the above-mentioned paths.

Applying Lemma 2.12 to the first two paths (so $s = w$, $t = x$, and $u = y$) yields a hamiltonian cycle C_1, such that $(\Pi\, C_0)^{-1}(\Pi\, C_1)$ is conjugate to $[x^{-1}, y]\,[w, x^{-1}]^y$. Removing the edge $[\overline{s^\epsilon}](s^{-\epsilon})$ yields hamiltonian paths $C_0\#$ and $C_1\#$ from $\overline{e}$ to $\overline{s}^\epsilon$.

From Lemma 3.4 and the assumption of this subcase, we see that $\langle \overline{S_0}\rangle \neq \overline{G}$. So

$$C_0^+ = \big(C_0\#, a, (C_0\#)^{-1}, a\big) \text{ and } C_1^+ = \big(C_1\#, a, (C_1\#)^{-1}, a\big)$$

are hamiltonian cycles in $\mathrm{Cay}(\overline{G}; S)$. For $k = 0, 1$, we have

$$\Pi\, C_k^+ = \big[\big((\Pi\, C_k) s^\epsilon\big)^{-1}, a\big].$$

Since $\Pi\, C_k \in G'$, and G' is central modulo $\mathbb{Z}_p$ (and from the choice of s), we have

$$\Pi\, C_k^+ \equiv [s^\epsilon, a] \not\equiv e \pmod{\mathbb{Z}_p}.$$

Furthermore, if $[x^{-1}, y]\,[w, x^{-1}]^y$ projects nontrivially to $\mathbb{Z}_p$, then $(\Pi\, C_0^+)^{-1}(\Pi\, C_1^+)$ does not centralize a modulo $\mathbb{Z}_2$, so $\Pi\, C_0^+$ and $\Pi\, C_1^+$ are not both in $\mathbb{Z}_2$. This implies that $\Pi\, C_k^+$ generates G' for some k, so the Factor Group Lemma (2.8) applies. Therefore (after replacing x^{-1} with x for simplicity), we may assume

$$[w, x]^y\,[x, y] \in \mathbb{Z}_2 \text{ for all } w, x, y \in S^{\pm 1} \smallsetminus \{a\} \text{ that satisfy (9.5A) and (9.5B).} \tag{9.5C}$$

We will show that this leads to a contradiction.

Assume, for the moment, that b centralizes G'. Then $n = |\overline{b}| > 2$ (because Corollary 3.7 implies that $|b| \neq 2$), so $|\overline{G}| = 2n\ell > 2 \cdot 2 \cdot 4 = 16$. Therefore (9.5B) is automatically satisfied. Let $x, y \in S_0 \smallsetminus \{b\}$, such that $x \neq y$. We see from Note 9.3 that (9.5A) is satisfied for $w = b^{\pm 1}$, so (9.5C) tells us

$$[b, x]^y\,[x, y] \text{ and } [b^{-1}, x]^y\,[x, y] \text{ are both in } \mathbb{Z}_2.$$

However, we also know that $[b^{-1}, x] = [b, x]^{-1}$ (because we are assuming in this paragraph that b centralizes G'). Therefore

$$[b, x]^y \equiv [x, y]^{-1} \equiv [b^{-1}, x]^y = \big([b, x]^{-1}\big)^y \pmod{\mathbb{Z}_2},$$

so $[b, x] \in \mathbb{Z}_2$ (for all $x \in S_0$). Then, since $[b, x]^y\,[x, y] \in \mathbb{Z}_2$, we conclude that $[x, y] \in \mathbb{Z}_2$, for all $x, y \in S_0$. This contradicts the assumption of this subcase.

Now assume b does not centralize G'. We may assume Case 9.4 does not apply, so G' is centralized by some $t \in S$ (and $t \neq b$). Let $w, x \in S_0 \smallsetminus \{t\}$ with $w \neq x$. Combining the irredundance of S with the fact that $t \neq b$ implies that (9.5A) is satisfied for $y = t^{\pm 1}$ (unless $\overline{w} = \overline{x}$, when Case 4.1 applies). We may assume $x \neq s$ (by interchanging w and x, if necessary), so (9.5B) is satisfied. Then (9.5C) tells us

$$[w, x]^t\,[x, t] \text{ and } [w, x]^{t^{-1}}\,[x, t^{-1}] \text{ are both in } \mathbb{Z}_2.$$

Since t centralizes G', this implies $[x, t] \equiv [x, t^{-1}] = [x, t]^{-1} \pmod{\mathbb{Z}_2}$, so $[x, t] \in \mathbb{Z}_2$ (for all $x \in S_0$). Since $[w, x]^t\,[x, t] \in \mathbb{Z}_2$, this implies $[w, x] \in \mathbb{Z}_2$ (for all $w, x \in S_0$). This contradicts the assumption of this subcase. □

Case 9.6. *Assume b centralizes G'.*

Proof. We consider two subcases.

Subcase 9.6.1. *Assume there exists $c \in S$, such that $[c, b] \notin \mathbb{Z}_p$.* We use some of the arguments of Case 8.5. We may assume $[a, s] \in \mathbb{Z}_p$ for all $s \in S$. (Otherwise, Case 9.5 applies, because b centralizes G'.) Therefore $c \neq a$. Let $L = (s_i)_{i=1}^{r}$ be a hamiltonian path from $\overline{e}$ to $\overline{b}$ in $\mathrm{Cay}\big(\overline{G}/\langle\overline{a}\rangle; S \setminus \{a\}\big)$, such that $s_1 = c = s_r^{-1}$, and L contains a path of the form $[\overline{gc^\epsilon}](c^{-\epsilon}, b^\delta, c^\epsilon)$ (for some $\delta, \epsilon \in \{\pm 1\}$) that is vertex-disjoint from $\{\overline{e}, \overline{c}, \overline{b}, \overline{bc}\}$. Now let $C = (L, a, L^{-1}, a)$. Then C contains vertex-disjoint paths of the form

$$[\overline{b}](a), \quad [\overline{ca}](c^{-1}, a, c), \quad [\overline{gc^\epsilon}](c^{-\epsilon}, b^\delta, c^\epsilon), \quad \text{and} \quad [\overline{gab^\delta}](b^{-\delta}).$$

- Applying Lemma 2.12 to $[\overline{b}](a)$ and $[\overline{ca}](c^{-1}, a, c)$ (so $s = c^{-1}$, $t = a$, $u = b$, and $h = ab$) will be called the "a-transform." It multiplies the voltage by
$$\gamma_a = [b, a][a, c^{-1}].$$
- Applying Lemma 2.12 to $[\overline{gc^\epsilon}](c^{-\epsilon}, b^\delta, c^\epsilon)$ and $[\overline{gab^\delta}](b^{-\delta})$ (so $s = c^{-\epsilon}$, $t = b^\delta$, $u = a$, and $h = ga$) will be called the "b-transform." It multiplies the voltage by a conjugate of
$$\gamma_b = [a, b][c^{-\epsilon}, b].$$

Since $[a, b], [a, c] \in \mathbb{Z}_p$ and $[b, c] \notin \mathbb{Z}_p$ we know $\gamma_a \in \mathbb{Z}_p$ and $\gamma_b \notin \mathbb{Z}_p$. Also, we may also assume γ_a is nontrivial (by replacing b with b^{-1} if necessary). Therefore, the argument of Subcase 8.5.1 applies. Namely, choose C' to be either C or the b-transform of C, such that $\prod C'$ projects nontrivially to $\mathbb{Z}_2$. Then choose C'' to be either C' or the a-transform of C', such that $\prod C''$ generates G', so the Factor Group Lemma (2.8) applies.

Subcase 9.6.2. *Assume $[c, b] \in \mathbb{Z}_p$ for all $c \in S$.* Choose $c, d \in S$, such that $[c, d] \notin \mathbb{Z}_p$. Assuming that Case 9.5 and Subcase 9.6.1 do not apply, we have

$$[s, t] \in \mathbb{Z}_p \text{ for all } s \in \{a, b\} \text{ and } t \in S.$$

Therefore, $c, d \notin \{a, b\}$, and the element $\gamma = [a, b][d^{-1}, a]$ is in $\mathbb{Z}_p$, and we may assume (by replacing b with its inverse, if necessary) that γ generates $\mathbb{Z}_p$.

Let $S_0 = \{a, b, d\}$, and choose a hamiltonian cycle C_0 in $\mathrm{Cay}\big(\langle\overline{S_0}\rangle; S_0\big)$ that contains the oriented paths $[\overline{d}](d^{-1}, a, d)$ and $[\overline{ab}](a)$, and has at least two edges labelled $x^{\pm 1}$, for every $x \in S_0$. Lemma 2.12 (with $s = d^{-1}$, $t = a$, $u = b$, and $h = b$) provides a hamiltonian cycle C_1, such that $(\prod C_0)^{-1}(\prod C_1)$ is conjugate to γ, and therefore generates $\mathbb{Z}_p$. Furthermore, C_1 contains all of the oriented edges of C_0 that are not in these two above-mentioned paths, so Lemma 3.15(2) applies (with $g = b$ and $t = d$). □

Case 9.7. *Assume that none of the preceding cases apply.*

Proof. This implies that:

#1. $[a, b] \in \mathbb{Z}_p$. (Otherwise, Case 9.5 applies.)

#2. *If $s \in S$, and there exists $t \in S$, such that t inverts G' and $\mathbb{Z}_p \subseteq \langle [s, t] \rangle$, then s inverts G'.* (If s does not invert G', then we see from Assumption 9.1 that s centralizes G', so Case 9.6 applies with s and t in the roles of b and a, respectively.)

#3. *There exists $c \in S$, such that c centralizes G'.* (Otherwise, Case 9.4 applies.) From (#2), we know $[a, c] \in \mathbb{Z}_2$.

Subcase 9.7.1. *Assume $\langle [s, c] \rangle \neq \mathbb{Z}_2$, for some $s \in S \setminus \{c\}$.* Suppose, for the moment, that s centralizes G'. Then Lemma 3.6 implies $\big[a, [s, c]\big] = \big[[a, s], [a, c]\big] = e$ (because G' is abelian), so $[s, c]$ projects trivially to $\mathbb{Z}_p$. Since $\langle [s, c] \rangle \neq \mathbb{Z}_2$, we conclude from this that $[s, c] = e$, so Lemma 2.16 applies.

We may now assume s does not centralize G', so there is no harm in assuming that $s = a$. Since (#2) implies that $[a, c] \in \mathbb{Z}_2$, we see that $[a, c]$ must be trivial. Let $H = \langle S \setminus \{c\} \rangle$. We may assume $\mathbb{Z}_2 \not\subseteq H$, for otherwise $H \triangleleft G$, so Lemma 2.13 applies with $s = c$ and $t = a$. Therefore, $[x, y] \in \mathbb{Z}_p$ for all $x, y \in S \setminus \{c\}$, but there is some $d \in S \setminus \{c\}$, such that $[c, d]$ projects nontrivially to $\mathbb{Z}_2$.

Similarly, we may assume $\mathbb{Z}_p \not\subseteq \langle S \setminus \{a\} \rangle$, for otherwise we have $\langle S \setminus \{a\} \rangle \triangleleft G$, so Lemma 2.13 applies with $s = a$ and $t = c$. This means $[x, y] \in \mathbb{Z}_2$ for all $x, y \in S \setminus \{a\}$. In particular, since b and d are in both $S \setminus \{a\}$ and $S \setminus \{c\}$, we must have $[b, d] \in \mathbb{Z}_2 \cap \mathbb{Z}_p = \{e\}$.

Choose a hamiltonian cycle C_0 in $\mathrm{Cay}\big(\overline{H}; S \setminus \{c\}\big)$ that contains the oriented paths $[\overline{d}](d^{-1}, b, d)$ and $[\overline{ab}](b)$. If we apply Lemma 2.12 to these paths (so $s = d^{-1}$, $t = b$, $u = a$, and $h = a$), then the voltage is multiplied by a conjugate of $[b, a]\,[b, d^{-1}]$, which is a generator of $\mathbb{Z}_p$ (since $[a, b]$ generates $\mathbb{Z}_p$ and $[b, d]$ is trivial). Therefore, Lemma 3.15(1) applies with $s = t = d$ and $u = a$.

Subcase 9.7.2. *Assume $\langle [s, c] \rangle = \mathbb{Z}_2$, for all $s \in S \setminus \{c\}$.* For convenience, let $\widehat{G} = G/\mathbb{Z}_2$ and $\widehat{H} = \langle \widehat{S} \setminus \{\widehat{c}\} \rangle$. Then $|\widehat{H}'| = p$ is prime, so Theorem 1.1 provides a hamiltonian path L in $\mathrm{Cay}\big(\widehat{H}; S \setminus \{c\}\big)$. Since $\widehat{c}$ is central in $\widehat{G}$, there is a spanning subgraph of $\mathrm{Cay}(\widehat{G}; S)$ that is isomorphic to the Cartesian product $L \,\square\, (\widehat{c}^{\ell-1})$, where $\ell = |\overline{G} : \langle \overline{S \setminus \{c\}} \rangle|$. Since $|\widehat{G}|$ is even, it is easy to find a hamiltonian cycle C in $L \,\square\, (\widehat{c}^{\ell-1})$ (see Lemma 2.10), and this yields a hamiltonian cycle $\widehat{C}$ in $\mathrm{Cay}(\widehat{G}; S)$.

To complete the proof, we carry out a straightforward (and well-known) calculation to verify that $\Pi\, \widehat{C}$ is nontrivial, so the Factor Group Lemma (2.8) applies.

If we view the Cartesian product $L \,\square\, (\widehat{c}^{\ell-1})$ as a grid of squares, then the interior of the hamiltonian cycle C is a union of squares of the grid. Graph theoretically, this means C is the connected sum of some number N of digons of the form $[g](t, t^{-1})$ (where $t \in S^{\pm 1}$). Note that if $\mathcal{C}$ is an r-cycle (with $r \geq 2$), then $\mathcal{C} \#_t^s (t, t^{-1})$ is an $(r+2)$-cycle. Therefore, since the length of C is $|\widehat{G}|$, we have $2N = |\widehat{G}| \equiv 0 \pmod 4$, so N is even.

Now, each 4-cycle in $L \,\square\, (\widehat{c}^{\ell-1})$ is of the form $[\widehat{g}](s^{-1}, t^{-1}, s, t)$, where one of s and t is in $\{c^{\pm 1}\}$, and the other is in $S^{\pm 1} \setminus \{c^{\pm 1}\}$. This means that in any connected sum $\mathcal{C} \#_t^s [g](t, t^{-1})$, one of s and t is in $\{c^{\pm 1}\}$, and the other is in $S^{\pm 1} \setminus \{c^{\pm 1}\}$. By the assumption of this subcase, we conclude that $[s, t] = z$, where z is the generator of $\mathbb{Z}_2$. Therefore

$$
\begin{aligned}
\Pi\, C &= \Pi\Big([\widehat{g_1}](t_1, t_1^{-1}) \;\#_{t_2}^{s_2}\; [\widehat{g_2}](t_2, t_2^{-1}) \;\#_{t_3}^{s_3}\; \cdots \;\#_{t_N}^{s_N}\; [\widehat{g_N}](t_N, t_N^{-1}) \Big) \\
&\equiv \prod_{i=2}^{N} [s_i, t_i] && \text{(Corollary 3.14 and } \Pi(t, t^{-1}) = e\text{)} \\
&= z^{N-1} \\
&\not\equiv e \pmod{\mathbb{Z}_p} && (N-1 \text{ is odd}). \qquad \square
\end{aligned}
$$

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Most rigid representations and Cayley index*

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Abstract

For any finite group G, a natural question to ask is the order of the smallest possible automorphism group for a Cayley graph on G. A particular Cayley graph whose automorphism group has this order is referred to as an MRR (Most Rigid Representation), and its Cayley index is a numerical indicator of this value. Study of GRRs showed that with the exception of two infinite families and thirteen individual groups, every group admits a Cayley graph whose MRR is a GRR, so that the Cayley index is 1. The full answer to the question of finding the smallest possible Cayley index for a Cayley graph on a fixed group was almost completed in previous work, but the precise answers for some finite groups and one infinite family of groups were left open. We fill in the remaining gaps to completely answer this question.

Keywords: Cayley graph, Cayley index, GRR, MRR, automorphisms.

Math. Subj. Class.: 05C25

1 Introduction

All groups and graphs in this paper are finite. All of our graphs are simple, undirected, and have no loops.

A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ where $S \subseteq G$ with $S = S^{-1}$ and $1 \notin S$, is the graph whose vertices are the elements of G, with $(g, gs) \in E(\Gamma)$ if and only if $g \in G$ and $s \in S$. We refer to S as the connection set for Γ. Let $A = \mathrm{Aut}(\Gamma)$. Observe that L_G, the left-regular representation of G, lies in A, so $|G|$ divides $|A|$.

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Definition 1.1. The *Cayley index* $c(\Gamma)$ of the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$, is $|A : L_G|$. The *Cayley index* $c(G)$ of the group G is $\min_{S \subseteq G, S=S^{-1}} c(\mathrm{Cay}(G, S))$; that is, the lowest Cayley index of any Cayley graph on the group G.

Definition 1.2. A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is a GRR (Graphical Regular Representation) for G if $c(\Gamma) = 1$.

Thus, groups that admit GRRs are precisely the groups whose Cayley index is 1. In order to completely characterise these groups, we require another definition.

Definition 1.3. Let A be an abelian group of even order, and y an involution in A. Then the generalised dicyclic group $\mathrm{Dic}(A, y)$ is $\langle A, x\rangle$ where $x \notin A$, $x^2 = y$, and $x^{-1}ax = a^{-1}$ for every $a \in A$.

Notice that under this definition, the generalised dicyclic group $\mathrm{Dic}(A, y)$ will be abelian if and only if A is an elementary abelian 2-group.

The study of GRRs involved many researchers and papers. Some of the most influential work along the way appeared in [6, 7, 9]. Watkins [16] observed that there are two infinite families of graphs that cannot admit GRRs: generalised dicyclic groups, and abelian groups that are not elementary abelian 2-groups. Imrich [7] resolved the problem for abelian groups by classifying the elementary abelian 2-groups, finding exactly three that admit no GRR. Watkins in a series of papers, some with coauthor Nowitz [11, 12, 16, 17, 18], discovered ten nonabelian groups that admit no GRRs. Hetzel [5] proved that aside from the two infinite families noted by Watkins, and the thirteen small solvable groups (of order at most 32) found by Imrich, Nowitz, and Watkins, every solvable group admits a GRR. Godsil [3] showed that every non-solvable group admits a GRR.

In the case where a group fails to admit a GRR, a natural question to ask is: what is the Cayley index of the group, and what is a Cayley graph on the group that has that Cayley index? The following terminology was coined in [10].

Definition 1.4. Let G be a group with $c(G) > 1$, and let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph on G with $c(\Gamma) = c(G)$. Then we say that Γ is an MRR (Most Rigid Representation) for G.

The bulk of this paper is divided into 4 sections. In Section 2, we describe the groups that do not admit a GRR but do not lie in either of the infinite families of groups that do not admit a GRR. For each of these groups, we find its Cayley index and an MRR. In Section 3, we find the Cayley index of every abelian group, and find MRRs for those groups whose Cayley index is greater than 2. In Section 4, we consider a subfamily of generalised dicyclic groups (specifically, the hamiltonian 2-groups), and show that the smallest two of these have Cayley index 16, while the rest have Cayley index 8. Finally, in Section 5, we find the Cayley index for every generalised dicyclic group that was not included in Section 4.

Much of the work that we summarise in this paper was done in [10], but the authors of [10] left some gaps. Our paper fills all of these gaps, thus completing their work. Specifically, we fill the following gaps. We examine the Cayley indices of the groups that do not lie in either of the infinite families; we give the Cayley indices for the four abelian groups for which they did not specify it (although they stated that these had been found by computer); we find the precise Cayley index for generalised dicyclic groups of order at most 96 (they bounded almost all of these by 4, but most in fact have Cayley index 2); and we find the

Cayley indices for all hamiltonian 2-groups (they bounded these by 16, but almost all have Cayley index 8). Table 1 summarises this work, providing the Cayley index for every finite group.

For a number of the small individual groups, we found MRRs using Sage [15] and its GAP package [14]. The Cayley index of any of the graphs we present can be easily checked via computer, using this or other appropriate software.

Throughout this paper,

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k : ij = k = -ji, i^2 = j^2 = k^2 = -1\}$$

is the usual representation of the quaternion group of order 8. We use D_{2n} for $n \geq 3$ to represent the dihedral group of order $2n$. Four of the exceptional groups listed in Theorem 2.1 we denote by H_i for $i \in \{1, 2, 3, 4\}$; a precise representation of each of these groups is given in Theorem 2.1.

To represent some of our MRRs, we use cartesian products. For two graphs Γ_1 and Γ_2, the *cartesian product* of Γ_1 with Γ_2 is denoted by $\Gamma_1 \square \Gamma_2$. It is the graph whose vertices are the elements of $V(\Gamma_1) \times V(\Gamma_2)$, with (u_1, v_1) adjacent to (u_2, v_2) if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in Γ_2, or $v_1 = v_2$ and u_1 is adjacent to u_2 in Γ_1. We say that a graph Γ on more than one vertex is *prime* with respect to the cartesian product if $\Gamma \cong \Gamma_1 \square \Gamma_2$ implies that for some $i \in \{1, 2\}$, $\Gamma_i \cong \Gamma$ and Γ_{2-i} has just one vertex. It is well-known that every graph has a unique *prime factorisation* as the cartesian product of prime graphs. We say that two graphs are *relatively prime* with respect to the cartesian product if they have no common factors in their prime factorisations. We sometimes simply refer to the graphs as prime or relatively prime.

2 Exceptional groups

We begin by listing the 13 groups that do not admit a GRR but do not lie in either of the infinite families that do not admit GRRs.

The following theorem is the end result of considerable work by a number of researchers. Imrich [7] completed the abelian case (correcting an earlier error by Sabidussi [13] and Chao [2], who showed that no graph has a transitive abelian automorphism group, but overlooked the case of elementary abelian 2-groups). The construction given in [7] also has an error in the case of the elementary abelian 2-group of order 32; it is mentioned in [10] that this was pointed out and corrected by Alspach, Hell, Hetzel, and Lim, and a GRR for that group (due to Hetzel) appears in [10]. Watkins, alone and in joint work with Nowitz [11, 12, 16, 17, 18] found the other ten exceptional groups and proved in [12] that any nonabelian group whose order is coprime to 6 admits a GRR. Imrich [8] then showed that every nonabelian group whose order is odd and at least $3^7 \cdot 5^4$ admits a GRR. Hetzel [5, Satz 14.38] showed that the exceptions we have mentioned are the only solvable groups that fail to have GRRs, and Godsil [3] completed the result by showing that every nonsolvable group has a GRR. We therefore cite Godsil's work for the final result, but attribute it to all of the researchers who made major contributions.

Theorem 2.1 (Godsil, Hetzel, Imrich, Nowitz, and Watkins; see [3]). *The following are the only groups that are neither generalised dicyclic nor abelian of exponent greater than* 2*, yet admit no GRR:*

- $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$;

- D_6, D_8, D_{10} *where these represent the dihedral groups of orders* 6, 8, *and* 10 *(respectively);*
- A_4, *the alternating group of degree* 4*;*
- $H_1 := \langle a, b, c : a^2 = b^2 = c^2 = 1, abc = bca = cab\rangle$;
- $H_2 := \langle a, b : a^8 = b^2 = 1, bab = a^5\rangle$;
- $H_3 := \langle a, b, c : a^3 = b^3 = c^2 = 1, ab = ba, (ac)^2 = (bc)^2 = 1\rangle$;
- $H_4 := \langle a, b, c : a^3 = c^3 = 1, ac = ca, bc = cb, b^{-1}ab = ac\rangle$;
- $Q_8 \times \mathbb{Z}_3$, $Q_8 \times \mathbb{Z}_4$, *where* Q_8 *is the quaternion group of order* 8.

The groups listed in the first bullet are abelian, and their Cayley indices are given in Section 3.

All of the remaining groups have Cayley index 2. Their Cayley index must be at least 2 by Theorem 2.1, since they admit no GRR. This was shown explicitly in [16, Theorem 2]

Table 1: Cayley indices for all finite groups.

Group	Cayley index	See
Abelian groups		
$\mathbb{Z}_2, \mathbb{Z}_2^n, n \geq 5$	1	[7], 1.2 of [10]
$\mathbb{Z}_2^3, \mathbb{Z}_4 \times \mathbb{Z}_2$	6	Lemma 2.7 of [10]
$\mathbb{Z}_2^4$	8	Table 3
$\mathbb{Z}_4^2$	4	Table 3
$\mathbb{Z}_4 \times \mathbb{Z}_2^2$	8	Table 3
$\mathbb{Z}_3^2$	8	Lemma 2.4 of [10]
$\mathbb{Z}_3^3$	12	Table 3
all other abelian groups	2	Theorem 1 of [10]
Hamiltonian 2-groups		
Q_8	16	Lemma 2.6 of [10]
$Q_8 \times \mathbb{Z}_2$	16	Section 4
$Q_8 \times \mathbb{Z}_2^n, n \geq 2$	8	Proposition 4.8
Other generalised dicyclic groups		
$\text{Dic}(\mathbb{Z}_6, 3)$	4	Table 4
$\text{Dic}(\mathbb{Z}_8, 4)$	4	Table 4
$\text{Dic}(\mathbb{Z}_{10}, 5)$	4	Table 4
$\text{Dic}(\mathbb{Z}_4 \times \mathbb{Z}_2, (0, 1))$	4	Table 4
all other generalised dicyclic groups	2	Section 5, and Theorem 2 of [10]
Exceptional groups		
D_6, D_8, D_{10}	2	Section 2 of [10], or Table 2
A_4	2	Table 2
$Q_8 \times \mathbb{Z}_3$, $Q_8 \times \mathbb{Z}_4$	2	Table 2
H_1 of order 16	2	Table 2
H_2 of order 16	2	Table 2
H_3 of order 18	2	Table 2
H_4 of order 27	2	Table 2
Every group not listed above	1	[3]

for the dihedral groups in the second bullet. It was shown in [18, Proposition 3.7] for A_4. For the groups H_1 and H_3, it was shown in [18, Proposition 5.3 and Theorem 2]. The group H_2 was dealt with in [11, Theorem 2 or Proposition 3.1], and H_4 in [12, Theorem 3]. Finally, $Q_8 \times \mathbb{Z}_3$ and $Q_8 \times \mathbb{Z}_4$ were addressed in [17, Theorem].

To show that the Cayley index of each is precisely 2, we present Table 2. For each group, we give the connection set for a Cayley graph on that group that has Cayley index 2. The Cayley indices of these graphs can be verified by hand or by computer.

Table 2: MRRs for exceptional groups.

Group G	S such that $c(\mathrm{Cay}(G,S)) = 2$
$D_{2n} = \langle a, b : a^2 = b^n = 1, aba = b^{-1}\rangle$, $n \in \{3,4,5\}$	$\{a, ab\}$
A_4	$\{(1\ 2\ 3)^{\pm 1}, (1\ 2)(3\ 4)\}$
$H_1 = \langle a, b, c : a^2 = b^2 = c^2 = 1$, $abc = bca = cab\rangle$	$\{a, b, c, (ab)^{\pm 1}\}$
$H_2 = \langle a, b : a^8 = b^2 = 1, bab = a^5\rangle$	$\{a^{\pm 1}, a^{\pm 2}, b\}$
$H_3 = \langle a, b, c : a^3 = b^3 = c^2 = 1, ab = ba$, $(ac)^2 = (bc)^2 = e\rangle$	$\{a^{\pm 1}, c, ac, bc\}$
$H_4 = \langle a, b, c : a^3 = c^3 = 1, ac = ca, bc = cb$, $b^{-1}ab = ac\rangle$	$\{a^{\pm 1}, b^{\pm 1}, (a^{-1}b)^{\pm 1}, (bab^{-1})^{\pm 1}\}$
$Q_8 \times \mathbb{Z}_3 = \langle i, j, z : z^3 = 1, iz = zi, jz = zj\rangle$	$\{\pm i, (iz)^{\pm 1}, (jz)^{\pm 1}\}$
$Q_8 \times \mathbb{Z}_4 = \langle i, j, z : z^4 = 1, iz = zi, jz = zj\rangle$	$\{z^{\pm 1}, \pm i, \pm j, (iz)^{\pm 1}, (-kz)^{\pm 1}\}$

The MRRs listed in the first line of this table were also mentioned in [10].

3 Abelian groups

The Cayley index of every abelian group was determined in [10]. However, for a small number of these they stated only that the Cayley index had been found by Hetzel on computer, and cite a private communication. The known results on abelian groups are as follows.

Theorem 3.1 ([10, Theorem 1, Lemma 2.4, Lemma 2.7]). *The only finite abelian groups with a Cayley index greater than 2 are:*

- $\mathbb{Z}_2^3$ *and* $\mathbb{Z}_4 \times \mathbb{Z}_2$*, for which the Cayley index is* 6*, with MRR* $K_2 \,\square\, K_2 \,\square\, K_2$ *(the cube);*
- $\mathbb{Z}_3^2$*, for which the Cayley index is* 8*, with MRR* $K_3 \,\square\, K_3$*;*
- $\mathbb{Z}_2^4$*,* $\mathbb{Z}_4 \times \mathbb{Z}_2^2$*, and* $\mathbb{Z}_4^2$*; and*
- $\mathbb{Z}_3^3$.

In the rest of this section, we list the Cayley index for each of the last four groups together with an MRR for each group. The Cayley indices for these graphs and the fact that these are the Cayley indices for these groups can be verified by computer.

If A is an abelian group that we are presenting as being isomorphic to $\mathbb{Z}_{i_1} \times \cdots \times \mathbb{Z}_{i_k}$, then we let $\{z_1, \ldots, z_k\}$ be the canonical generating set for this group, so $|z_j| = i_j$. We present the Cayley index and an MRR for each group in Table 3.

Table 3: MRRs for abelian groups not given in [10].

Group	Cayley index	Connection set for an MRR
$\mathbb{Z}_2^4$	8	$\{z_1, z_2, z_3, z_4, z_1z_2, z_1z_3, z_2z_4\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2^2$	8	$\{z_1^{\pm 1}, z_2, z_3, (z_1z_2)^{\pm 1}, (z_1z_3)^{\pm 1}\}$
$\mathbb{Z}_4^2$	4	$\{z_1^{\pm 1}, z_2^{\pm 1}, z_1^2, (z_1z_2)^{\pm 1}\}$
$\mathbb{Z}_3^3$	12	$\{z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}, (z_1z_2)^{\pm 1}, (z_1z_3)^{\pm 1}, (z_2z_3)^{\pm 1}\}$

It may seem odd that $c(\mathbb{Z}_2^4) > c(\mathbb{Z}_2^3)$. However, Lemma 4.4 does not apply here, because neither MRR for $\mathbb{Z}_2^3$ ($K_2 \square K_2 \square K_2$ and its complement, $K_4 \square K_2$), is relatively prime to K_2, which is the unique connected MRR for $\mathbb{Z}_2$.

4 The groups $Q_8 \times \mathbb{Z}_2^n$

In this section we deal with a particular family of generalised dicyclic groups: groups of the form $Q_8 \times \mathbb{Z}_2^n$ for some nonnegative integer n.

Definition 4.1. A hamiltonian group is a nonabelian group all of whose subgroups are normal. A hamiltonian 2-group is a hamiltonian group whose order is a power of 2.

It is well-known (see, for example, [4, Theorem 12.5.4]) that the hamiltonian 2-groups are precisely the groups of the form $Q_8 \times \mathbb{Z}_2^n$ for some nonnegative integer n that we are considering in this section.

We begin with three important results from [10].

Lemma 4.2 ([10, Lemma 2.6])**.** *The group Q_8 has Cayley index 16, with $C_4 \square \overline{K_2}$ as an MRR.*

Lemma 4.3 ([10, Proposition 2.9])**.** *Every group other than $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, and $\mathbb{Z}_3^2$ admits a connected MRR that is prime with respect to the cartesian product.*

Lemma 4.4 ([10, Lemma 2.8])**.** *Let G_1 and G_2 be groups having connected MRRs that are relatively prime with respect to the cartesian product. Then $c(G_1 \times G_2) \le c(G_1)c(G_2)$.*

In fact, if Γ_1 and Γ_2 are connected MRRs for G_1 and G_2 (respectively) that are relatively prime with respect to the cartesian product, then $c(\Gamma_1 \square \Gamma_2) = c(G_1)c(G_2)$ and $\Gamma_1 \square \Gamma_2$ is a Cayley graph on $G_1 \times G_2$.

The following observation is made in [10] and is implicit in their Theorem 2(b), which states that $c(Q_8 \times \mathbb{Z}_2^n) \le 16$ for every integer $n \ge 0$. It can be deduced from Lemmas 4.2, 4.3, and 4.4, using the fact that $c(\mathbb{Z}_2) = 1$.

Corollary 4.5. *For every group $G \notin \{\mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_3^2\}$, $c(G \times \mathbb{Z}_2) \le c(G)$.*

The following result is key to providing a lower bound for the Cayley index of every group $Q_8 \times \mathbb{Z}_2^n$.

Proposition 4.6 ([1, Classification Theorem]). *There are 8 permutations φ of the elements of $G = Q_8 \times \mathbb{Z}_2^n$ that fix the identity, and have the property that for every $g, h \in G$, $\varphi(gh)$ is either $\varphi(g)h$, or $\varphi(g)h^{-1}$.*

Corollary 4.7. *The Cayley index of $Q_8 \times \mathbb{Z}_2^n$ is at least 8 for every integer $n \geq 0$.*

Proof. Fix n, and let $G = Q_8 \times \mathbb{Z}_2^n$. Let S be any inverse-closed subset of $G \setminus \{1_G\}$, and let $\Gamma = \mathrm{Cay}(G, S)$. Let φ be any of the 8 permutations given in Proposition 4.6. To prove this result, it will be sufficient to show that φ is an automorphism of Γ.

We know that for any $g \in G$, g is adjacent to gs if and only if $s \in S$. We also know that $\varphi(gs)$ is either $\varphi(g)s$, or $\varphi(g)s^{-1}$. Since S is inverse-closed, each of these is adjacent to $\varphi(g)$ if and only if $s \in S$. Thus, φ is indeed an automorphism of Γ. $\square$

To complete this section, we note that $\overline{C_4 \,\square\, \overline{K_2}} \,\square\, K_2$ is an MRR for $Q_8 \times \mathbb{Z}_2$ with Cayley index 16, verified by computer. However, for $Q_8 \times \mathbb{Z}_2^2$, the Cayley index is 8, with MRR $\mathrm{Cay}(Q_8 \times \mathbb{Z}_2^2, \{\pm i, \pm j, \pm k, \pm iz_1, \pm kz_1z_2, z_1, z_2\})$, where z_1 and z_2 are two distinct central involutions that do not lie in Q_8.

Thus, using Corollary 4.5 and Corollary 4.7 we are able to conclude the following.

Proposition 4.8. *For every integer $n \geq 2$, the Cayley index of $Q_8 \times \mathbb{Z}_2^n$ is 8.*

5 Other generalised dicyclic groups

Imrich and Watkins [10] showed that generalised dicyclic groups of order greater than 96 that are not of the form $Q_8 \times \mathbb{Z}_2^n$ have Cayley index 2. Many of the ideas from their proof in fact apply to generalised dicyclic groups of smaller orders. We reproduce these key ideas here, without their assumptions on order. We generally need to find two elements that satisfy a number of conditions. We note that the condition $a_1 \neq ya_2$ was not listed in [10] but is required; for this reason we provide a full proof of Lemma 5.4.

Definition 5.1. Let $\mathrm{Dic}(A, y)$ be a generalised dicyclic group. We say that the 2-set $\{a_1, a_2\}$ for $a_1, a_2 \in A$ is a *suitable pair* of elements of $\mathrm{Dic}(A, y)$ if for every $\{i, j\} = \{1, 2\}$ we have

(i) $a_1 \neq a_2, ya_2$;

(ii) $a_i^2 \neq 1, y$;

(iii) $a_i \neq a_j^2, ya_j^2$; and

(iv) $a_1a_2 \neq 1, y$.

Lemma 5.2. *Let $D = \mathrm{Dic}(\langle z \rangle, z^n)$ (the dicyclic group of order $4n$), where $|z| = 2n > 10$. Then $\{z, z^{-2}\}$ is a suitable pair for D. If $\Gamma = \mathrm{Cay}(D, \{z^{\pm 1}, x^{\pm 1}, (xz)^{\pm 1}, (xz^{-2})^{\pm 1}\})$, where $x^2 = z^n$, then $\langle z \rangle$ is invariant under $\mathrm{Aut}(\Gamma)_1$.*

Proof. We have $y = z^n$. We verify the conditions for $\{z, z^{-2}\}$ to be a suitable pair. Since $n > 5$, (i) and (ii) are satisfied; (iii) and (iv) are equally easy to check.

Let $\varphi \in \mathrm{Aut}(\Gamma)_1$ be arbitrary. It is straightforward to verify that when $n > 4$, z^n is the unique vertex that has 6 common neighbours with 1, so $\varphi(z^n) = z^n$. In fact, this shows that for any vertex v, vz^n is uniquely determined as the vertex that has 6 common neighbours with v. Since the neighbours of 1 can be partitioned into three pairs of this sort ($\{x, x^{-1} = xz^n\}$, $\{xz, xz^{n+1}\}$, and $\{xz^{-2}, xz^{n-2}\}$) and two elements (z and z^{-1}) whose match in this respect (z^{n+1}, and z^{n-1} respectively) is not a neighbour of 1, it must be the case that $\{z, z^{-1}\}$ and $\{x, x^{-1}, xz, xz^{n+1}, xz^{-2}, xz^{n-2}\}$ are fixed setwise by φ. Repeating this argument shows that $\varphi(z^i) \in \langle z\rangle$ for every i. Thus, $\varphi(\langle z\rangle) = \langle z\rangle$. □

Lemma 5.3. *Let $A = \langle z_1, z_2\rangle$ where $|z_1| = 2n \geq 6$, $|z_2| = 2$, and $z_1z_2 = z_2z_1$, so $A \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2$. Then $\{z_1, z_1^{-2}\}$ is a suitable pair for $D = \mathrm{Dic}(A, x^2)$.*

Also, if $\Gamma = \mathrm{Cay}(D, \{z_1^{\pm1}, z_2, x^{\pm1}, (xz_1)^{\pm1}, (xz_1^{-2})^{\pm1})$ then A is invariant under $\mathrm{Aut}(\Gamma)_1$.

Proof. Checking the conditions for $\{z_1, z_1^{-2}\}$ to be a suitable pair is straightforward.

Since $z_2 \in S$ is central in D and $x^{-1} = xz_2$, the following pairs of neighbours of 1 are adjacent in Γ: $\{x, x^{-1}\}$; $\{xz_1, x^{-1}z_1\}$; $\{xz_1^{-2}, x^{-1}z_1^{-2}\}$. However, z_1, z_1^{-1} and z_2 have no neighbours in S. Thus, we can distinguish the neighbours of 1 that lie in A from the neighbours of 1 that lie in xA. Repeating this argument shows that every element of A is invariant under $\mathrm{Aut}(\Gamma)_1$. □

Lemma 5.4. *Let $\Delta = \mathrm{Cay}(A, S)$ be an MRR of the abelian group A of Cayley index 2. Let $D = \mathrm{Dic}(A, y)$ be a generalised dicyclic group with suitable pair $\{a_1, a_2\}$. Let*

$$\Gamma = \mathrm{Cay}(D, S \cup \{x, x^{-1}, xa_1, x^{-1}a_1, xa_2, x^{-1}a_2\})$$

(where x is as in Definition 1.3) and suppose that for every $\varphi \in \mathrm{Aut}(\Gamma)_1$, we have $\varphi(A) = A$. If φ is not the identity automorphism, then $\varphi(a) = a$, and $\varphi(xa) = (xa)^{-1}$ for every $a \in A$.

Proof. Since $\varphi(A) = A$ and the induced subgraph on A is Δ which has Cayley index 2, we know that we either have $\varphi(a) = a$ for every $a \in A$, or $\varphi(a) = a^{-1}$ for every $a \in A$. (This is always the case in a Cayley graph of Cayley index 2 on an abelian group.)

Similarly, since $\varphi(A) = A$ we have $\varphi(xA) = xA$. Observe that the induced subgraph on xA is isomorphic to Δ, which has Cayley index 2. This means that there are exactly two graph automorphisms that fix xA and take x to any given vertex xa where $a \in A$. Clearly one of these automorphisms is given by left-multiplication by a^{-1}, and therefore maps each vertex of the form xa' to the vertex $a^{-1}xa' = xaa'$. The other graph automorphism that fixes x and xA (aside from the identity) is the automorphism that maps every vertex xa' to the vertex $x(a')^{-1}$. This implies that the other automorphism that maps x to xa must take each vertex of the form xa' to the vertex $a^{-1}x(a')^{-1} = xa(a')^{-1}$.

In the remainder of this proof, we use $N_X(v)$ to denote the set of neighbours of the vertex v that lie in the subset X of the vertices of Γ. First we will show that $\varphi(x) \in \{x, x^{-1}\}$.

We are assuming that $\varphi(A) = A$, and need to show that $\varphi(x) \notin \{xa_1, x^{-1}a_1, xa_2, x^{-1}a_2\}$. Suppose that $\varphi(x) \notin \{x, x^{-1}\}$. By symmetry, without loss of generality we may assume that $\varphi(x) = xa_1$.

Since $\varphi(x) = xa_1$ and $\varphi(xA) = xA$, as noted above we must have either $\varphi(xa) = xaa_1$ for every $a \in A$, or $\varphi(xa) = xa^{-1}a_1$ for every $a \in A$.

Suppose the first of these possibilities holds, so $\varphi(xa_1) = xa_1^2$, which must therefore be a neighbour of 1 in xA, and hence an element of

$$N_{xA}(1) = \{x, x^{-1}, xa_1, x^{-1}a_1, xa_2, x^{-1}a_2\}.$$

Each of these possibilities contradicts one of the properties of being a suitable pair: any of the first four would contradict (ii); either of the last two contradict (iii).

If on the other hand the second possibility holds, then $\varphi(xa_2) = xa_2^{-1}a_1 \in N_{xA}(1)$. Again, each possible equality contradicts one of the properties of being a suitable pair: either of the first two contradict (i); the third or fourth each contradicts (ii); and either of the last two contradict (iii). We therefore conclude that $\varphi(x) \in \{x, x^{-1}\}$, as claimed.

Next we show that $\varphi(a) = a$ for every $a \in A$.

Observe that
$$N_A(x^{-1}) = N_A(x) = \{1, y, a_1, ya_1, a_2, ya_2\}.$$
Thus, since $\varphi(x) \in \{x, x^{-1}\}$, we have $\varphi(N_A(x)) = N_A(x)$. If $\varphi(a) = a^{-1}$ for every $a \in A$, then this implies that $a_1^{-1} \in N_A(x)$, leading to a contradiction to the definition of a suitable pair, as above. (If a_1^{-1} is any of the first four elements, this contradicts (ii); if it is either of the last two, this contradicts (iv).) Thus, we must have $\varphi(a) = a$ for every $a \in A$.

Next we show that if $\varphi(x) = x$ then $\varphi = 1$.

Again as noted above, we must either have $\varphi(xa) = xa^{-1}$ for every $a \in A$, or $\varphi(xa) = xa$ for every $a \in A$. In the latter case, $\varphi = 1$ and we are done. In the former case, we must have $\varphi(N_A(xa_1^{-1})) = N_A(xa_1)$. Observe that $a_1 = xa_1^{-1}x^{-1} \in N_A(xa_1^{-1})$, so this would imply that
$$a_1 = \varphi(a_1) \in N_A(xa_1) = \{a_1^{-1}, ya_1^{-1}, 1, y, a_1^{-1}a_2, ya_1^{-1}a_2\}.$$
Similar to the arguments above, each of these possibilities contradicts some property of suitable pairs. If a_1 were any of the first four elements of $N_A(xa_1)$ this would contradict (i); if it were either of the last two, this would contradict (iii).

Finally, we show that if $\varphi(x) = x^{-1}$ then $\varphi(xa) = (xa)^{-1}$ for every $a \in A$.

Again as noted above, we must either have $\varphi(xa) = x^{-1}a = (xa)^{-1}$ for every $a \in A$, or $\varphi(xa) = x^{-1}a^{-1}$ for every $a \in A$. In the former case we are done. In the latter case, we must have $\varphi(N_A(x^{-1}a_1^{-1})) = N_A(xa_1)$. Observe that $a_1 = x^{-1}a_1^{-1}x \in N_A(x^{-1}a_1^{-1})$, so this would imply that $a_1 = \varphi(a_1) \in N_A(xa_1)$, yielding the same contradiction as in the previous paragraph. □

Proposition 5.5. *Let $A_1 = \langle z_1 \rangle$ be a cyclic group of order $2n \geq 6$, and $A_2 = \langle z_1, z_2 \rangle$ with $|z_2| = 2$ and $z_1z_2 = z_2z_1$. Let $S_1 = \{z_1, z_1^{-1}\}$ and $S_2 = \{z_1, z_1^{-1}, z_2\}$, and let $D_1 = \mathrm{Dic}(A_1, z_1^n)$, and $D_2 = \mathrm{Dic}(A_2, z_2)$. Then*
$$\Gamma_i = \mathrm{Cay}(D_i, S_i \cup \{x, x^{-1}, xz_1, xz_1^{n+1}, xz_1^{-2}, xz_1^{n-2}\})$$
for $i \in \{1, 2\}$ is connected and has Cayley index 2 when $n \geq 6$, and Γ_2 is connected and has Cayley index 2 when $n \geq 3$.

Proof. It is easy to see that S_1 is the connection set for a Cayley graph on A_1 with Cayley index 2. It is slightly less obvious that S_2 is the connection set for a Cayley graph on A_2 with Cayley index 2, but becomes clear upon noting that each z_1-edge lies in a unique 4-cycle, while each z_2-edge lies in two 4-cycles. Fix $i \in \{1, 2\}$, and if $i = 1$, ensure that $n \geq 5$.

By Lemma 5.2 or Lemma 5.3, we know that $\{z_1, z_1^{-2}\}$ is a suitable pair for D_i, and that for any $\varphi \in \mathrm{Aut}(\Gamma_i)_1$, $\varphi(A_i) = A_i$. By Lemma 5.4 with $S = S_i$ and this suitable pair, we see that there are only two possibilities for φ: $\varphi = 1$, or $\varphi(a) = a$ and $\varphi(xa) = (xa)^{-1}$ for every $a \in A$. Thus, Γ has Cayley index 2. □

Proposition 5.6. *Let A be an abelian group of even order that contains an involution y, and let $D = \mathrm{Dic}(A, y)$. Suppose that D has a connected MRR with Cayley index 2. Let $A' = A \times \mathbb{Z}_2$. Then $D' = \mathrm{Dic}(A', y)$ has Cayley index 2.*

Proof. Observe that $D' \cong D \times \mathbb{Z}_2$. The result is now immediate from Corollary 4.5. □

As an immediate consequence of Proposition 5.5 and Proposition 5.6, we obtain the following.

Corollary 5.7. *The following generalised dicyclic groups have Cayley index 2:*

- $\mathrm{Dic}(A \times \mathbb{Z}_2^k, z_1^n)$ *where* $A = \langle z_1 \rangle \cong \mathbb{Z}_{2n}$, $n \geq 6$, *and* $k \geq 0$; *and*
- $\mathrm{Dic}(A \times \mathbb{Z}_2^k, z_2)$ *where* $A = \langle z_1, z_2 \rangle \cong \mathbb{Z}_{2n} \times \mathbb{Z}_2$, $|z_1| = 2n$, $|z_2| = 2$, $n \geq 3$, *and* $k \geq 0$.

This leads us to the following theorem.

Theorem 5.8. *Every generalised dicyclic group that is neither abelian nor a hamiltonian 2-group has Cayley index 2, with the following four exceptions, each of which has Cayley index 4:* $\mathrm{Dic}(\mathbb{Z}_6, 3)$, $\mathrm{Dic}(\mathbb{Z}_8, 4)$, $\mathrm{Dic}(\mathbb{Z}_{10}, 5)$, *and* $\mathrm{Dic}(\mathbb{Z}_4 \times \mathbb{Z}_2, (0, 1))$.

Proof. All generalised dicyclic groups of order greater than 96 have Cayley index of 2 (see [10]). To deal with the remaining cases, we begin by considering all abelian groups of even order at most 48. For each group, we choose one representative for each automorphism class of elements of order 2 to be the distinguished element $y = x^2$.

By Corollary 5.7, the result holds for every dicyclic group of order at least 24; this deals with every cyclic group of even order at least 12, all of which have a unique element of order 2. Since $\mathbb{Z}_2$ produces an abelian dicyclic group and $\mathbb{Z}_4$ produces Q_8 which is a hamiltonian 2-group, we need only consider the dicyclic groups over the groups $\mathbb{Z}_6$, $\mathbb{Z}_8$, and $\mathbb{Z}_{10}$.

We note that if n is odd, then $\mathbb{Z}_{2n} \times \mathbb{Z}_2$ has only one automorphism class of elements of order 2, so that Corollary 5.7 provides an MRR for the unique generalised dicyclic group over any of these groups when $n \geq 3$, and in fact produces two MRRs when $n > 6$. Also, if n is even, $\mathbb{Z}_{2n} \times \mathbb{Z}_2$ has two automorphism classes of elements of order 2 (the element $(n, 1)$ lies in the same class as $(0, 1)$). Thus Corollary 5.7 produces an MRR for each of the two possible generalised dicyclic groups over these abelian groups whenever $n \geq 6$, and an MRR for one of them when $n \geq 3$. When $n = 1$ there is a unique generalised dicyclic group which is actually abelian; and when $n = 2$, one of the two generalised dicyclic groups is the hamiltonian 2-group $Q_8 \times \mathbb{Z}_2$. Thus we need only consider the two groups $\mathrm{Dic}(\mathbb{Z}_4 \times \mathbb{Z}_2, (0, 1))$ and $\mathrm{Dic}(\mathbb{Z}_8 \times \mathbb{Z}_2, (4, 0))$.

The generalised dicyclic group over $\mathbb{Z}_2^3$ is abelian, and the groups

$$\mathrm{Dic}(\mathbb{Z}_4 \times \mathbb{Z}_2^2, (2, 0, 0)) \cong Q_8 \times \mathbb{Z}_2^2 \quad \text{and} \quad \mathrm{Dic}(\mathbb{Z}_4 \times \mathbb{Z}_2^3, (2, 0, 0, 0))$$

are hamiltonian 2-groups, so these need not be considered.

Finally, if a group has the form $D \times \mathbb{Z}_2$ for some smaller generalised dicyclic group D with $c(D) = 2$, then Corollary 4.5 gives $c(D \times \mathbb{Z}_2) = 2$, so we do not have to consider these groups either. This eliminates all generalised dicyclic groups over $\mathbb{Z}_6 \times \mathbb{Z}_2^2$, $\mathbb{Z}_{10} \times \mathbb{Z}_2^2$, and $\mathbb{Z}_{12} \times \mathbb{Z}_2^2$, as well as $\mathrm{Dic}(\mathbb{Z}_8 \times \mathbb{Z}_2^2, (0, 1, 0))$.

With all of this in mind, there are 18 generalised dicyclic groups that remain to be considered. We conclude this section and the paper with Table 4, showing the Cayley index and the connection set for an MRR for each of these generalised dicyclic groups. For four of these groups that have the form $D \times \mathbb{Z}_2$ for some smaller generalised dicyclic group D, we use Corollary 4.5, but only after showing that $c(D) = 2$ in a previous line of the table. For these, instead of explicitly giving the connection set for an MRR, we present the group as $D \times \mathbb{Z}_2$. This table completes the proof, and its results are straightforward to verify by computer. □

Table 4: MRRs for generalised dicyclic groups.

Group	Cayley index	Connection set for an MRR
$\mathrm{Dic}(\mathbb{Z}_6, 3)$	4	$\{z_1^{\pm 1}, x^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_8, 4)$	4	$\{z_1^{\pm 1}, x^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_{10}, 5)$	4	$\{z_1^{\pm 1}, x^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_4 \times \mathbb{Z}_2, (0,1))$	4	$\{z_1^{\pm 1}, x^{\pm 1}, (z_1x)^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_8 \times \mathbb{Z}_2, (4,0))$	2	$\{z_1^{\pm 1}, z_2, x^{\pm 1}, (z_1x)^{\pm 1}, (z_2x)^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_4 \times \mathbb{Z}_4, (2,0))$	2	$\{z_1^{\pm 1}, z_2^{\pm 1}, (z_1z_2)^{\pm 1}, x^{\pm 1}, (z_1x)^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_4 \times \mathbb{Z}_2^2, (0,1,0))$	2	$\{z_1^{\pm 1}, z_3, x^{\pm 1}, (z_1x)^{\pm 1}, (z_3x)^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_6 \times \mathbb{Z}_3, (3,0))$	2	$\{z_1^{\pm 1}, z_2^{\pm 1}, x^{\pm 1}, (z_2x)^{\pm 1}, (z_1z_2x)^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_8 \times \mathbb{Z}_2^2, (4,0,0))$	2	$D \cong \mathrm{Dic}(\mathbb{Z}_8 \times \mathbb{Z}_2, (4,0)) \times \mathbb{Z}_2$
$\mathrm{Dic}(\mathbb{Z}_8 \times \mathbb{Z}_4, (4,0))$	2	$\{z_1^{\pm 1}, z_2^{\pm 1}, x^{\pm 1}, (z_1^6z_2^{-1}x)^{\pm 1}, (z_1^5z_2x)^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_8 \times \mathbb{Z}_4, (0,2))$	2	$\{z_1^{\pm 1}, z_2^{\pm 1}, x^{\pm 1}, (z_1^5x)^{\pm 1}, (z_1^3z_2x)^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_4^2 \times \mathbb{Z}_2, (2,0,0))$	2	$D \cong \mathrm{Dic}(\mathbb{Z}_4 \times \mathbb{Z}_4, (2,0)) \times \mathbb{Z}_2$
$\mathrm{Dic}(\mathbb{Z}_4^2 \times \mathbb{Z}_2, (0,0,1))$	2	$\{z_1^{\pm 1}, z_2^{\pm 1}, x^{\pm 1}, (z_2^3x)^{\pm 1}, (z_1^3z_2^2x)^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_4 \times \mathbb{Z}_2^3, (0,1,0,0))$	2	$D \cong \mathrm{Dic}(\mathbb{Z}_4 \times \mathbb{Z}_2^2, (0,1,0)) \times \mathbb{Z}_2$
$\mathrm{Dic}(\mathbb{Z}_{12} \times \mathbb{Z}_3, (6,0))$	2	$\{z_1^{\pm 1}, z_2^{\pm 1}, x^{\pm 1}, (z_1^7z_2x)^{\pm 1}, (z_1^3z_2x)^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_6 \times \mathbb{Z}_6, (3,0))$	2	$D \cong \mathrm{Dic}(\mathbb{Z}_6 \times \mathbb{Z}_3, (3,0)) \times \mathbb{Z}_2$
$\mathrm{Dic}(\mathbb{Z}_{12} \times \mathbb{Z}_4, (6,0))$	2	$\{z_1^{\pm 1}, z_2^{\pm 1}, x^{\pm 1}, (z_1^4z_2x)^{\pm 1}, (z_1^9z_2^3x)^{\pm 1}\}$
$\mathrm{Dic}(\mathbb{Z}_{12} \times \mathbb{Z}_4, (0,2))$	2	$\{z_1^{\pm 1}, (z_1^3z_2)^{\pm 1}, x^{\pm 1}, (z_1x)^{\pm 1}, (z_1z_2x)^{\pm 1}\}$

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Fault-Hamiltonicity of Cartesian products of directed cycles*

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Abstract

Although the Cartesian product of two Hamiltonian graphs is Hamiltonian, the corresponding statement for directed graphs is not true. Indeed, it is known that it does not always hold even for the Cartesian products of two directed cycles. In this paper, we study the Cartesian product and its generalization of a directed graph G and a directed cycle. We show that if G has "strong" fault-Hamiltonicity properties, then so does $G \,\square\, C_n$, that is, the Cartesian product of G and a cycle of length n. We also discuss some related problems.

Keywords: Digraphs, fault-Hamiltonicity, Cartesian product.

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1 Introduction

The interconnection network is one of the center pieces of a parallel architecture. The underlying topology of such a parallel machine is a graph, usually referred to as an interconnection network. Depending on the applications, the graph may be undirected or directed. A *Hamiltonian cycle* in a graph is a cycle that visits every vertex of the graph (exactly once). (If the underlying graph is directed, then a cycle means a directed cycle.) A graph is *Hamiltonian* if it has a Hamiltonian cycle. A *Hamiltonian path from u to v* in a graph is a path from u to v that visits every vertex of the graph. (Again, if the underlying graph is directed, then a path means a directed path.) A graph is *Hamiltonian connected* if there exists a Hamiltonian path from u to v for every distinct ordered pair of vertices u and v. Hamiltonicity is an important issue in the study of interconnection networks and there are many papers in this area. Paper [8] contains many references in this area and we refer the readers to [8] for an extensive list of references on Hamiltonicity related problems in interconnection networks. (A small partial list of such papers is [3, 4, 5, 7, 10, 13, 14].) However, most research has been done in the undirected setting as the analysis is, in general, more complicated in the directed case. A directed graph G is *k-regular* if the in-degree and out-degree of every vertex is k. So a connected 1-regular directed graph is a directed cycle. (We will simply refer to directed cycles as *cycles* if it is clear from the context.)

The Cartesian product of two directed graphs G_1 and G_2 is the directed graph $G_1 \square G_2 = (V, E)$ where $V = V_1 \times V_2$ and $((u_1, v_1), (u_2, v_2))$ is in E if either

(1) $u_1 = u_2$ and $(v_1, v_2) \in E_2$, or

(2) $v_1 = v_2$ and $(u_1, u_2) \in E_1$.

The Cartesian product of undirected graphs can be defined similarly. (One can check that given three directed graphs G_1, G_2 and G_3, $(G_1 \square G_2) \square G_3$ is isomorphic to $G_1 \square (G_2 \square G_3)$. Thus the Cartesian product of finitely many directed graphs can be naturally defined.) Cartesian product is an important topic in the study of interconnection networks. For example, the classical hypercube is K_2^n, that is, a Cartesian product of n complete graphs on two vertices. Although the Cartesian product of two Hamiltonian graphs is always Hamiltonian, this is false for directed graphs. Trotter and Erdős [12] gave a necessary and sufficient condition for the Cartesian product of two Hamiltonian directed cycles to be Hamiltonian. To be precise, let $\gcd(m, n)$ denote the greatest common divisor of two positive integers m and n. Then the Cartesian product of two directed cycles C_m and C_n is Hamiltonian if and only if $\gcd(m, n) \geq 2$ and there exists positive integers d_1 and d_2 such that $\gcd(m, n) = d_1 + d_2$, $\gcd(m, d_1) = 1$, and $\gcd(n, d_2) = 1$. So $C_2 \square C_3$ is not Hamiltonian since $\gcd(2, 3) = 1 < 2$.

Vertices in an interconnection network represent processors and edges represent links between processors. Since processors and links may fail, it is meaningful to study such faulty networks. A graph $G = (V, E)$ is *k-Hamiltonian* if $G - F$ is Hamiltonian for every $F \subseteq V \cup E$ and $|F| \leq k$. Similarly, a graph $G = (V, E)$ is *k-Hamiltonian-connected* if $G - F$ is Hamiltonian-connected for every $F \subseteq V \cup E$ and $|F| \leq k$. Here F is the set of *faults* that represent failed processors (vertices) and failed links (edges). We note that if $G = (V, E)$ is k-Hamiltonian-connected, then G is k-Hamiltonian whenever $|V| > k + 2$. For undirected graphs, many related results on k-Hamiltonicity and k-Hamiltonian connectedness with respect to the Cartesian product are known. See, for example, [1, 6, 11].

We have already mentioned the interesting result given in [12]. It is even more interesting if one considers the Cartesian product of three directed cycles. In particular, one can check by brute force that $C_2 \,\square\, C_3 \,\square\, C_4$ is a 3-regular, 2-Hamiltonian and 1-Hamiltonian-connected directed graph. In fact, $C_2 \,\square\, C_3 \,\square\, C_5$, $C_2 \,\square\, C_3 \,\square\, C_6$, $C_2 \,\square\, C_4 \,\square\, C_5$ and $C_2 \,\square\, C_5 \,\square\, C_5$ are also 3-regular, 2-Hamiltonian and 1-Hamiltonian-connected directed graphs. Results similar to the one given in [12] appeared in [2, 9]. This gives an indication that Hamiltonicity problems for directed graphs are more difficult than the undirected version. In addition, it is proved in [2] that every product of more than two directed cycles is Hamiltonian.

The ultimate goal is to obtain a result on k-Hamiltonicity and k-Hamiltonian connectedness with respect to the Cartesian product of directed graphs. Given the above example, we believe that this problem is difficult. Thus we study directed graphs of the form $G \,\square\, C_n$. We want to show that if G has "strong" Hamiltonicity property, then so does $G \,\square\, C_n$. In fact, we will generalize the concept of Cartesian product by considering the following. Let $\mathcal{G}$ be a set of directed graphs, each with the same fixed number of vertices. We say that $\mathcal{G}$ has a certain property if every directed graph in $\mathcal{G}$ has this property. Now we take n graphs $G_0, G_1, \ldots, G_{n-1}$ from $\mathcal{G}$ with repetitions allowed. Let $f_i \colon V(G_i) \to V(G_{i+1})$, for $i = 0, 1, \ldots, n-1$, be bijections where addition is taken modulo n. We construct the directed graph $H = (V, E)$ by letting $V = \cup_{i=0}^{n-1} V(G_i)$ and $E = \left(\cup_{i=0}^{n-1} E(G_i)\right) \cup \left(\cup_{i=0}^{n-1}\{(u, f_i(u)) : u \in V(G_i)\}\right)$. We call H an *n-$\mathcal{G}$-directed graph*. So $G \,\square\, C_n$ is an n-$\{G\}$-directed graph. For notational simplicity, we denote $EC_i = \{(u, f_i(u)) : u \in V(G_i)\}$ and we let CG_{ij} be the subgraph of H induced by $\cup_{r=i}^{j} V(G_r)$ (modulo n). Given $(u, v) \in EC_i$, we may refer to v as $f_i(u)$ and $u = f_i^{-1}(v)$. For the case $G \,\square\, C_n$, we may simply refer to f_i as f. Whenever we refer to a range $[i, j]$, it is considered modulo n.

In this paper, we consider deleting vertices and arcs. As mentioned before, these deleted elements correspond to failed processors and links in an interconnection network, and we refer them as faults. Let G be an r-regular directed graph. Clearly the best one can hope for is for G to be $(r-1)$-Hamiltonian and $(r-2)$-Hamiltonian connected. As pointed out earlier, there exist directed graphs that achieve such optimal properties when $r = 3$. In this paper, we show that if G has such optimal properties, then so does $G \,\square\, C_n$. In fact, our result covers the more general n-$\mathcal{G}$-directed graph. At first glance, one may wonder whether this is consistent with the necessary and sufficient condition given by Trotter and Erdős for $C_n \,\square\, C_m$ to be Hamiltonian. After all, C_m is 1-regular and Hamiltonian but $C_m \,\square\, C_n$ may not be Hamiltonian. One may argue that in this case, the condition "-1"-Hamiltonian connected is meaningless. As we shall see, our main result requires the regularity of G to be at least 3.

2 The main result

In this section, we present our main result. We want to show that if $\mathcal{G}$ has good Hamiltonian properties, then so does an n-$\mathcal{G}$-directed graph. We start with the following lemma.

Lemma 2.1. *Let $k \geq 2$ and $N \geq k + 5$. Let $\mathcal{G}$ be a class of $(k+1)$-regular and $(k-1)$-Hamiltonian-connected graphs on N vertices. Let H be an n-$\mathcal{G}$-directed graph obtained from $G_0, G_1, \ldots, G_{n-1}$ in $\mathcal{G}$ with the corresponding bijections $f_0, f_1, \ldots, f_{n-1}$. Let $[i, j]$ be a range. Let $F_r \subseteq V(G_r) \cup E(G_r)$ for every r in the range $[i, j]$. Let $F_{r,r+1} \subseteq EC_r$. Let s and t be vertices in $G_i - F_i$ and $G_j - F_j$, respectively. Suppose*

1. *$|F_r| \leq k - 1$ for every r in the range $[i, j]$ and*

2. $|F_r| + |F_{r+1}| + |F_{r,r+1}| \leq k + 2$ *for every* r *in the range* $[i, j - 1]$.

Then there is a Hamiltonian path from s *to* t *in* $CG_{i,j} - (\cup_{r=i}^{j} F_r) - (\cup_{r=i}^{j-1} F_{r,r+1})$.

Proof. If $i = j$, then there is nothing to prove as G_i is $(k-1)$-Hamiltonian-connected. For notational simplicity, we may assume that $i = 1$. We consider two cases.

Case 1: $j = 2$. We want to find an arc $(u_1, v_2) \in EC_1 - F_{1,2}$ where $u_1 \in V(G_1) - (F_1 \cup \{s\})$ and $v_2 \in V(G_2) - (F_2 \cup \{f_1(s)\})$ and $(u_1, v_2) \neq (f_1^{-1}(t), t)$. Such an arc exists if

$$N > |F_1| + |F_2| + |F_{1,2}| + |\{(s, f_1(s))\}| + |\{(f_1^{-1}(t), t)\}|.$$

But $|F_1| + |F_2| + |F_{1,2}| \leq k + 2$. Thus we are done as $N > k + 2 + 2 = k + 4$. We now obtain a desired Hamiltonian path by using a Hamiltonian path from s to u_1, the arc (u_1, v_2) and a Hamiltonian path from v_2 to t.

Case 2: $j \geq 3$. We first find an arc $(u_1, v_2) \in EC_1 - F_{1,2}$ where $u_1 \in V(G_1) - (F_1 \cup \{s\})$ and $v_2 \in V(G_2) - (F_2 \cup \{f_1(s)\})$. Such an arc exists if

$$N > |F_1| + |F_2| + |F_{1,2}| + |\{(s, f_1(s)\}|.$$

But $|F_1| + |F_2| + |F_{1,2}| \leq k + 2$. Thus we are done as $N > k + 2 + 1 = k + 3$.

Similarly, we can obtain an arc $(u_2, v_3) \in EC_2 - F_{2,3}$ where $u_2 \neq v_2$, and so on, via an inductive argument, in obtaining (u_i, v_{i+1})'s, until we obtain an arc

$$(u_{j-2}, v_{j-1}) \in EC_{j-2} - F_{j-2,j-1}$$

where $u_{j-2} \in V(G_{j-2}) - (F_{j-2} \cup \{v_{j-2}\})$. Now, we need to find an arc

$$(u_{j-1}, v_j) \in EC_{j-1} - F_{j-1,j}$$

where $u_{j-1} \in V(G_{j-1}) - (F_{j-1} \cup \{v_{j-1}\})$ and $v_j \in V(G_j) - (F_j \cup \{t\})$ which can be guaranteed since $N > |F_{j-1}| + |F_j| + |F_{j-1,j}| + 2$ (as $|F_{j-1}| + |F_j| + |F_{j-1,j}| \leq k + 2$ and $N \geq k + 5$). Now since G_r is $(k-1)$-Hamiltonian-connected, we have a Hamiltonian path from v_r to u_r in G_r for every r in $[i, j]$ with $v_1 = s$ and $u_j = t$. These paths together with the arcs (u_r, v_{r+1})'s give a desired Hamiltonian path. □

We remark that if we replace (2) by $|F_r| + |F_{r+1}| + |F_{r,r+1}| \leq k + 1$ for every r in the range $[i, j - 1]$ in Lemma 2.1, then the assumption that $N \geq k + 5$ can be replaced with the weaker assumption that $N \geq k + 4$.

Theorem 2.2. *Let* $k \geq 2$ *and* $n \geq 3$. *Let* $\mathcal{G}$ *be a class of* $(k+1)$*-regular,* k*-Hamiltonian and* $(k-1)$*-Hamiltonian connected graphs on* N *vertices. Let* H *be an* n*-*$\mathcal{G}$*-directed graph. Then* H *is* $(k+2)$*-regular. Moreover* H *is* $(k+1)$*-Hamiltonian if* $N \geq k + 4$ *and* k*-Hamiltonian connected if* $N \geq k + 5$ *and* $k \geq 3$.

Proof. We first prove that H is $(k+1)$-Hamiltonian. Let F be a set of faults with $|F| \leq k + 1$. We let F_i be the set of faults in G_i. We consider two cases.

Case 1: $|F_i| = k+1$ for some i. Without loss of generality, we may assume that $|F_0| = k + 1$. Let $x \in F_0$ and define $F'_0 = F_0 - \{x\}$. By assumption, there is a Hamiltonian cycle C'_0 in $G_0 - F'_0$. Regardless of whether x is a vertex or an arc, $C'_0 - \{x\}$ is a Hamiltonian path P'_0 from u to v for some u and v. Now let $y = f_0(v)$ and $z = f_{n-1}^{-1}(u)$. By Lemma 2.1, there is a Hamiltonian path from y to z in the $CG_{1,n-1} - (\cup_{r=1}^{n-1} F_r) = CG_{1,n-1}$. (Note that equality holds as $F_r = \emptyset$ for $r \in \{1, 2, \ldots, n-1\}$.) This together with P_0' gives a Hamiltonian cycle in $H - F$.

Case 2: $|F_i| \le k$ for every i. We first note that $2k > k+1$ as $k \ge 2$. Thus there is at most one i with $|F_i| = k$. Therefore we may assume that $|F_0|$ is the largest and $|F_i| \le k-1$ for $i \ne 0$. Now, by assumption, there is a Hamiltonian cycle C_0 in $G_0 - F_0$. We want to find an arc (v, u) in C_0 such that

$$(v, f_0(v)), (f_{n-1}^{-1}(u), u), f_0(v), f_{n-1}^{-1}(u) \notin F.$$

Here $|F_r| + |F_{r+1}| + |F_{r,r+1}| \le k+1$ for $r \in \{0, 1, 2, \ldots, n-2\}$ as $|F| \le k+1$. So we only require $N \ge k+4$ from the remark after Lemma 2.1. Now C_0 has at least $N - |F_0|$ arcs. Since $N - |F_0| > |F| - |F_0|$, such (v, u) exists. Now the argument in Case 1 applies, and we are done.

This completes the proof for H being $(k+1)$-Hamiltonian. The case for H being k-Hamiltonian connected is much more difficult. We assume $N \ge k+5$. (We will see later why $k+5$ is needed.) Let F be a set of faults with $|F| \le k$ and we define the F_i's as before. Let s and t be two fault-free vertices and our goal is to construct a Hamiltonian path from s to t in $H - F$. We consider two main cases. (Unfortunately, subcases are needed here.)

Case 1: $|F_i| = k$ for some i. Without loss of generality, we may assume that $|F_0| = k$. So all the faults are in F_0. We have to consider subcases depending on the locations of s and t.

Subcase 1.1: s and t are in $G_0 - F_0$. Let $x \in F_0$ and define $F_0' = F_0 - \{x\}$. By assumption, there is a Hamiltonian path P_0' from s to t in $G_0 - F_0'$. Regardless of whether x is a vertex or an arc, $P_0' - \{x\}$ contains the following two disjoint paths that span $G_0 - F_0$: Q_0 from s to u and Q_0' from v to t for some u and v. (It is possible that $s = u$ or $v = t$.) Moreover, Q_0 and Q_0' cover all the vertices in $G_0 - F_0$. Now let $y = f_0(u)$ and $z = f_{n-1}^{-1}(v)$. By Lemma 2.1, there is a Hamiltonian path from y to z in $CG_{1,n-1} - (\cup_{r=1}^{n-1} F_r) = CG_{1,n-1}$. This, together with the edges (u, y) and (z, v), Q_0 and Q_0', gives a Hamiltonian path from s to t in $H - F$.

Subcase 1.2: s is in $G_0 - F_0$ and t is in $G_i - F_i = G_i$ where $i \ne 0$. If $i = n-1$, then it is straightforward as $G_0 - F_0$ is a Hamiltonian. (Since $G_0 - F_0$ is Hamiltonian, there is a Hamiltonian path Q_0 from s to y in $G_0 - F_0$ for some y. Now apply Lemma 2.1 to obtain a Hamiltonian path from $f_0(y)$ to t in $CG_{1,n-1} - (\cup_{r=1}^{n-1} F_r) = CG_{1,n-1}$. This, together with the edge $(y, f_0(y))$ and Q_0, gives a Hamiltonian path from s to t in $H - F$.) Thus we may assume that $i \ne n-1$. By assumption, $G_0 - F_0$ has a Hamiltonian cycle C_0. Since $N \ge k+5$, there exists a vertex u_0 on C_0 such that (u_0, s) is not an arc in C_0 and $u_i \ne t$ where $u_1 = f_0(u_0), u_2 = f_1(u_1), \ldots, u_i = f_{i-1}(u_{i-1})$. (It is possible that $u_0 = s$.) Now C_0 contains the following two disjoint paths that span $G_0 - F_0$: Q_0 from s to u_0 and Q_0' from v to x for some v and x as determined by C_0 and Q_0. (It is possible that $v = x$.) We note that since (u_0, s) is not an arc, Q_0' is not empty. Now, apply Lemma 2.1 to obtain a Hamiltonian path P_1 from $f_i(u_i)$ to $f_{n-1}^{-1}(v)$ in $CG_{i+1,n-1} - (\cup_{r=i+1}^{n-1} F_r) = CG_{i+1,n-1}$. We apply Lemma 2.1 again, this time to obtain a Hamiltonian path P_2 from $f_0(x)$ to t in $CG_{1,i} - (\cup_{r=1}^{i} (F_r \cup \{u_r\})) = CG_{1,i} - (\cup_{r=1}^{i} \{u_r\})$. For the moment, assume that $f_0(x) \ne t$. Then

$$Q_0, (u_0, u_1, \ldots, u_i, f_i(u_i)), P_1, (f_{n-1}^{-1}(v), v), Q_0', (x, f_0(x)), P_2$$

is a desired Hamiltonian path from s to t in $H - F$. (See Figure 1.)

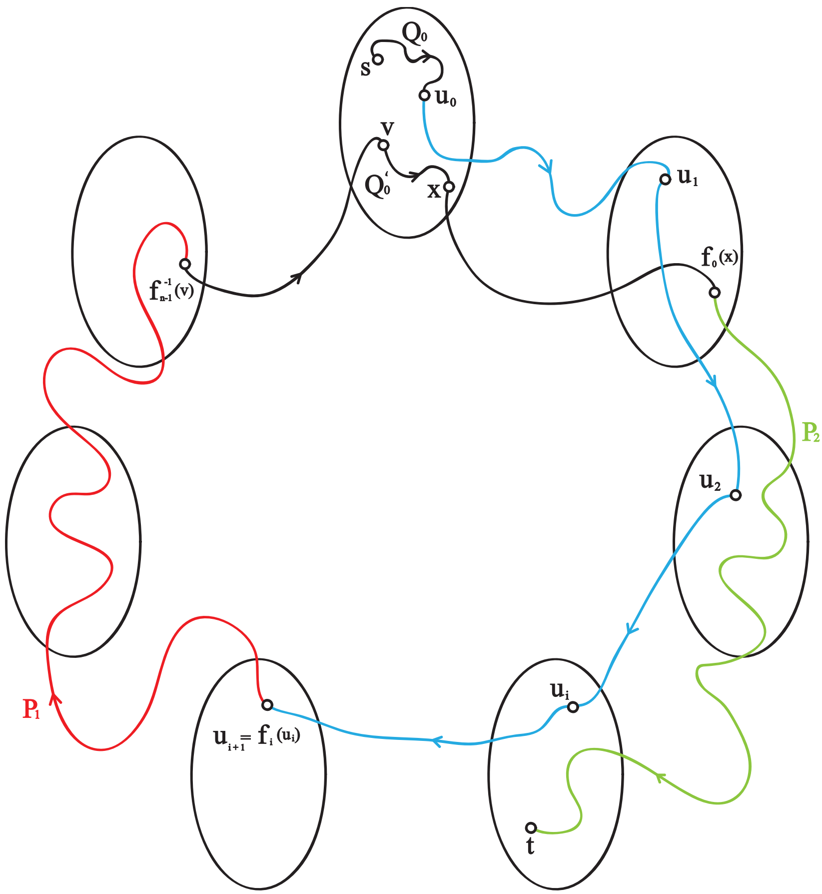

Figure 1: The Hamiltonian path of Subcase 1.2.

The remaining possibility is $f_0(x) = t$. Then $i = 1$. This case is actually simpler as we can obtain a desired Hamiltonian path by using Q_0, a Hamiltonian path from u_1 to $f_{n-1}^{-1}(v)$ in $CG_{1,n-1} - \{t\}$ (via Lemma 2.1), the edge $(f_{n-1}^{-1}(v), v)$, the path Q_0', and the edge (x, t).

Subcase 1.3: t is in $G_0 - F_0$ and s is in $G_i - F_i$ where $i \neq 0$. This is similar to Subcase 1.2 by observing instead of going from G_0 to G_i via $G_1, G_2, \ldots, G_{i-1}$ to obtain a directed path from s to t, we can "trace backward" from to t to s via $G_{n-1}, G_{n-2}, \ldots, G_{i+1}$. To be precise, we let G^R be the directed graph obtained from $G - F$ by reversing the direction on every arc. Then a directed path from s to t in $G - F$ can be obtained from a directed path from t to s in G^R, whose existence is proved in Subcase 1.2.

Subcase 1.4: s and t are in $G_i - F_i$ where $i \neq 0$. We have to consider several scenarios. We first assume that $i = n - 1$. By assumption, there is a Hamiltonian path P from s to t in $G_{n-1} - F_{n-1} = G_{n-1}$. Choose any (u, v) on P such that $f_{n-1}(u) \notin F_0 = F$. (Again such u exists as $N \geq k + 5$. Henceforth, we will not explicitly mention this when choosing an appropriate vertex.) Now P contains the following two disjoint paths that span $G_{n-1} - F_{n-1} = G_{n-1}$: Q from s to u and Q' from v to t. (It is possible that $s = u$ or $v = t$.) By assumption, there is a Hamiltonian cycle in $G_0 - F_0$, which implies that there is a Hamiltonian path P from $f_{n-1}(u)$ to w in $G_0 - F_0$ for some w. Now apply Lemma 2.1 to obtain a Hamiltonian path R from $f_0(w)$ to $f_{n-2}^{-1}(v)$ in $CG_{1,n-2} - (\cup_{r=1}^{n-2} F_r) = CG_{1,n-2}$. Now

$$Q, (u, f_{n-1}(u)), P, (w, f_0(w)), R, (f_{n-2}^{-1}(v), v), Q'$$

is a desired Hamiltonian path from s to t in $H - F$.

We now assume $i = 1$. By assumption, there is a Hamiltonian cycle in $G_0 - F_0$, which implies that there is a Hamiltonian path P from u to v in $G_0 - F_0$ for some u and v. We may choose v such that $f_0(v) \notin \{s, t\}$. By assumption, there is a Hamiltonian path Q from $f_0(v)$ to t in $G_1 - (F_1 \cup \{s\}) = G_1 - \{s\}$. Now by Lemma 2.1, there is a Hamiltonian path R from $f_1(s)$ to $f_{n-1}^{-1}(u)$ in $CG_{2,n-1} - (\cup_{r=2}^{n-1} F_r) = CG_{2,n-1}$. Now

$$(s, f_1(s)), R, (f_{n-1}^{-1}(u), u), P, (v, f_0(v)), Q$$

is a desired Hamiltonian path from s to t in $H - F$.

We may now assume that $2 \leq i \leq n - 2$. By assumption, there is a Hamiltonian cycle in $G_0 - F_0$, which implies that there is a Hamiltonian path P_0 from u to v in $G_0 - F_0$ for some u and v. By assumption, there is a Hamiltonian path P_i from s to t in $G_i - F_i = G_i$. Pick any (y, z) on P_i such that $f_i(y) \neq f_{n-1}^{-1}(u)$ and $f_{i-1}^{-1}(z) \neq f_0(v)$. Now P_i contains two disjoint paths that span $G_i - F_i = G_i$: Q_i from s to y and Q_i' from z to t. Apply Lemma 2.1 to get a Hamiltonian path R from $f_0(v)$ to $f_{i-1}^{-1}(z)$ in $CG_{1,i-1} - (\cup_{r=1}^{i-1} F_r) = CG_{1,i-1}$. Apply Lemma 2.1 to get a Hamiltonian path R' from $f_i(y)$ to $f_{n-1}^{-1}(u)$ in $CG_{i+1,n-1} - (\cup_{r=i+1}^{n-1} F_r) = CG_{i+1,n-1}$. Now

$$Q_i, (y, f_i(y)), R', (f_{n-1}^{-1}(u), u), P_0, (v, f_0(v)), R, (f_{i-1}^{-1}(z), z), Q_i'$$

is a desired Hamiltonian path from s to t in $H - F$. (See Figure 2.)

Subcase 1.5: s is in $G_i - F_i$ and t is in $G_j - F_j$ where $1 \leq i < j \leq n - 1$. By assumption, there is a Hamiltonian cycle in $G_0 - F_0$, which implies that there is a Hamiltonian path P_0 from u to v in $G_0 - F_0$ for some u and v. We first assume that $i \neq 1$. We may assume that $f_0(v) \neq s$. By Lemma 2.1, we obtain a Hamiltonian path P' from $f_0(v)$ to w

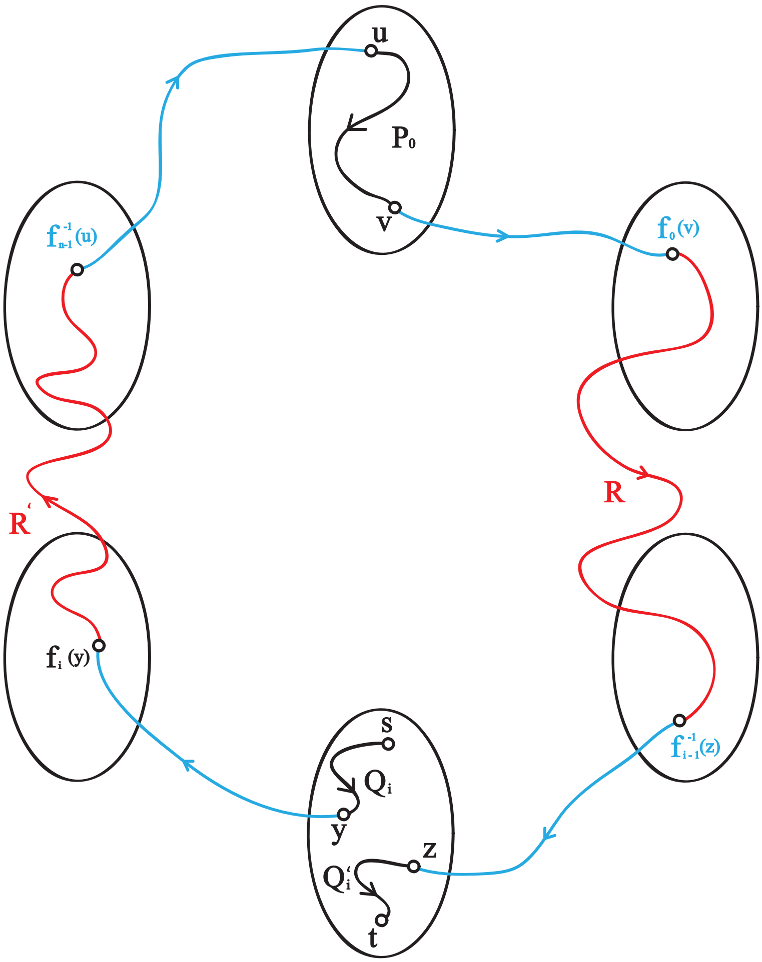

Figure 2: The Hamiltonian path of Subcase 1.4.

in $CG_{1,i-1}$ for some w in $G_{i-1} - F_{i-1} = G_{i-1}$ to be determined. By assumption, there is a Hamiltonian path P_i from s to y in $G_i - F_i = G_i$ for some y such that $y \neq s$ and $f_i(y) \neq t$. Let choose (x, z) on P_i such that

$$f_{j-1}(\cdots(f_{i+1}(f_i(x))) \neq t \text{ and } f_{i-1}^{-1}(z) \neq f_0(v).$$

(If $n-1 = j+1$, we further require that $f_j(f_{j-1}(\cdots(f_{i+1}(f_i(x)))) \neq f_{n-1}^{-1}(u)$.) Now we choose $w = f_{i-1}^{-1}(z)$. Now let $A = (x, f_i(x), \ldots, f_{j-1}(\cdots(f_{i+1}(f_i(x))))$. Let $y' = f_i(y)$ if $j = i+1$ and y' be any vertex in $G_j - \{t, f_{j-1}(\cdots(f_{i+1}(f_i(x)))\}$ otherwise. (If $j = i+2$, we further require $f_i(y) \neq f_{j-1}^{-1}(y')$.) For notational convenience, let $x' = f_{j-1}(\cdots(f_{i+1}(f_i(x)))$. Let A' be (y, y') if $j = i+1$ and let A' be a Hamiltonian path from $f_i(y)$ to $f_{j-1}^{-1}(y')$ in $CG_{i+1,j-1} - (\cup_{r=i+1}^{j-1}\{f_{r-1}(\cdots(f_{i+1}(f_i(x)))\})$. Now P_i contains two disjoint paths that span $G_i - F_i = G_i$: Q_i from s to x and Q'_i from $z = f_{i-1}(w)$ to y.

Now let R be a Hamiltonian path from y' to t in $G_j - \{x'\}$ and R' be a Hamiltonian path from $f_j(x')$ to $f_{n-1}^{-1}(u)$ in $CG_{j+1,n-1}$. Then

$$Q_i, A, (x', f_j(x')), R', (f_{n-1}^{-1}(u), u), P_0, (v, f_0(v)), P', (w, z), Q'_i, \\ (y, f_i(y)), A', (f_{j-1}^{-1}(y'), y'), R$$

is a desired Hamiltonian path from s to t in $H - F$. (See Figure 3.) If $i = 1$, then a small adjustment is needed for P'. We note that there is freedom in choosing v, y and z. However, once v is chosen, u is forced; similarly, once z is chosen, x is forced. Here we reduce the degree of freedom by one and choose v such that $z = f_0(v)$ in addition to the other restrictions. It is not difficult to see that such v can be chosen. So P' is the arc $(v, f_0(v))$.

Subcase 1.6: s is in $G_j - F_j$ and t is in $G_i - F_i$ where $1 \leq i < j \leq n-1$. We construct a desired Hamiltonian path in several steps. (This is similar to Subcase 1.5 but it is slightly more complicated.) By assumption, there is a Hamiltonian cycle C_0 in $G_0 - F_0$. We want to find two arcs (v, u) and (x, y) on C_0 to delete so that C_0 contains two disjoint paths that span $G_0 - F_0$: Q_0 from u to x and Q'_0 from y to v. It is possible that $u = x$ or $y = v$. (But both cannot occur at the same time.) There are only a few restrictions on the candidacies of (v, u) and (x, y). We call the path

$$(v, f_0(v), f_1(f_0(v)), \ldots, f_{i-1}(\cdots f_1(f_0(v))), f_i(f_{i-1}(\cdots f_1(f_0(v)))),$$

R_1; and the requirement is $f_{i-1}(\cdots f_1(f_0(v))) \neq t$. For convenience, let $w = f_i(f_{i-1}(\cdots f_1(f_0(v))))$. (For the case $i+1 = j$, then w is in G_j and we further require $w \neq s$.) We will call the path

$$(f_j^{-1}(\cdots f_{n-2}^{-1}(f_{n-1}^{-1}(u))), \ldots, f_{n-2}^{-1}(f_{n-1}^{-1}(u)), f_{n-1}^{-1}(u), u),$$

R_2; moreover, the requirement is $f_j^{-1}(\cdots f_{n-2}^{-1}(f_{n-1}^{-1}(u))) \neq s$. For convenience, let $w' = f_j^{-1}(\cdots f_{n-2}^{-1}(f_{n-1}^{-1}(u)))$. (For the case $i+1 = j$, then w is in G_j and we further require $w' \neq w$.) Now let R_3 be a Hamiltonian path from w to w' in $CG_{i+1,j} - \{s\}$. It turns out that the case $j = n-1$ and $f_{n-1}(s) \in F$ requires modification of our construction. So for now, we assume that this is not the case. (Note that $j = n-1$ and $f_{n-1}(s) \in F$ implies $(s, f_{n-1}(s))$ is fault-free as $F = F_0$.) We have more freedom for (x, y) in most

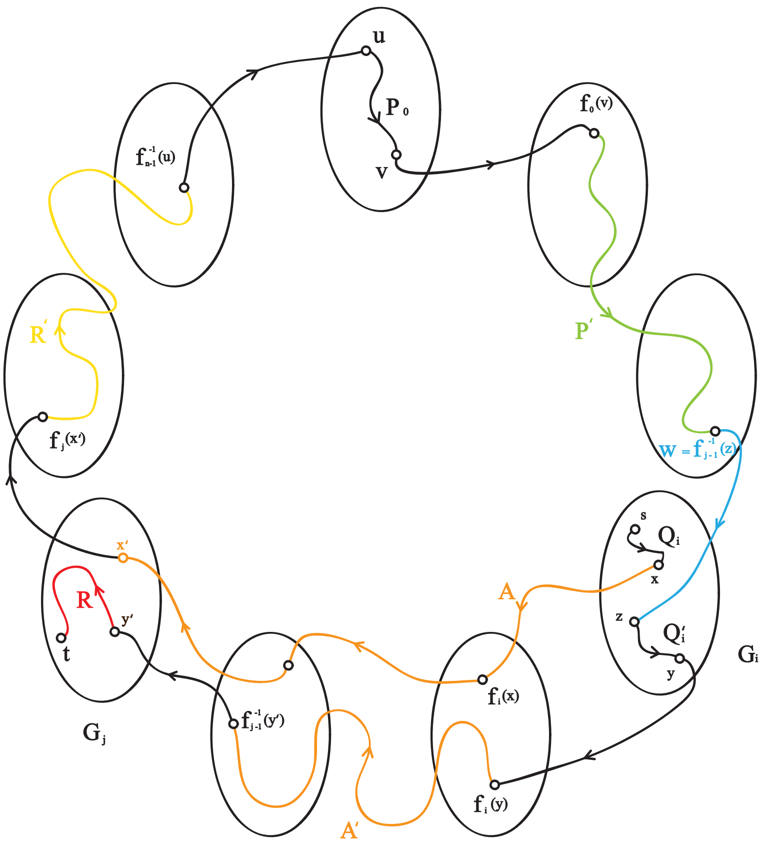

Figure 3: The Hamiltonian path of Subcase 1.5.

instances. If $i = 1$, we require $f_0(x) \neq t$. If $j = n-1$ and $f_{n-1}(s) \notin F$, then we let $y = f_{n-1}(s)$. (Recall that the case $j = n-1$ and $f_{n-1}(s) \in F$ is deferred.) We further note that if $j = n-1$, there is only one choice for y and hence there is only one choice of (x, y), so we should pick (x, y) first and then (v, u). If $j \neq n-1$, let A be the path consisting of $(s, f_j(s))$, the Hamiltonian path from $f_j(s)$ to $f_{n-1}^{-1}(y)$ in $CG_{j+1,n-1} - \cup_{r=j+1}^{n-1}\{f_{r-1}(f_{r-2}(\cdots f_j(w)))\}$ and $(f_{n-1}^{-1}(y), y)$; otherwise (that is, $j = n-1$ and $f_{n-1}(s) \notin F$), let $A = (s, y)$. (Note that if $j \neq n-1$, then $(s, f_j(s))$ is fault-free as $F = F_0$.) Let B be the path consisting of $(x, f_0(x))$ and the Hamiltonian path from $f_0(x)$ to t in $CG_{1,i} - \cup_{r=1}^{i}\{f_{r-1}(f_{r-2}(\cdots f_0(v)))\}$. Then

$$A, Q_0', R_1, R_3, R_2, Q_0, B$$

is a desired Hamiltonian path from s to t in $H - F$. (See Figure 4.) Now we consider the case when $j = n-1$ and $f_{n-1}(s) \in F$. Then we consider the $k+1$ arcs $(s, s^1), (s, s^2), \ldots, (s, s^{k+1})$ in G_j that start at s. Since $|F| = k$, we may assume, without loss of generality, that $f_{n-1}(s^1) \notin F$. So we let $y = f_{n-1}(s^1)$, and the path A will be (s, s^1, y). So we pick (x, y) first, then we pick (v, u) as before but we now need to include the restriction that $w' \neq s^1$. We note that then R_3 needs to be a Hamiltonian path from w to w' in $CG_{i+1,n-1} - \{s, s^1\}$. Since $k \geq 3$, such a path exists via the usual explanation. We remark that one can actually adjust the proof to give a construction for $k = 2$.

Case 2: $|F_i| \leq k-1$ for every i. We consider two subcases depending on the locations of s and t.

Subcase 2.1: s and t belong to the same $G_i - F_i$. Without loss of generality, we may assume that s and t belong to $G_0 - F_0$. By assumption, there is a Hamiltonian path P_0 from s to t in $G_0 - F_0$. Choose (u, v) on P_0 such that $f_0(u), (u, f_0(u)) \notin F$ and $f_{n-1}^{-1}(v), (f_{n-1}^{-1}(v), v) \notin F$. Now apply Lemma 2.1 to obtain a Hamiltonian path from $f_0(u)$ to $f_{n-1}^{-1}(v)$ in $CG_{1,n-1} - (\cup_{r=1}^{n-1} F_r)$. Now, the usual argument gives a desired Hamiltonian path.

Subcase 2.2: s and t belong to different $G_i - F_i$'s. We may assume that s belong to $G_0 - F_0$ and t belong to $G_j - F_j$ where $j \neq 0$. If $j = n-1$, then this result follows directly from Lemma 2.1. So we may assume that $j \leq n-2$.

Subsubcase 2.2.1: $|F_i| \leq k-2$ for every i. Find x in $G_0 - F_0$ such that $f_0(x) \neq t$ (if $j = 1$). We remark that the argument in this subcase requires only $|F_1|, |F_2|, \ldots, |F_j| \leq k-2$. By assumption, there is a Hamiltonian path P_0 from s to x in $G_0 - F_0$. We want to find (u, v) on P_0 to delete so that P_0 contains two disjoint paths that span $G_0 - F_0$: Q_0 from s to u and Q_0' from v to x. There are only a few restrictions on the candidacy of (u, v). We want $f_{n-1}^{-1}(v), (f_{n-1}^{-1}(v), v) \notin F$, the path

$$(u, f_0(u), f_1(f_0(u)), \ldots, y = f_{j-1}(f_{j-2}(\cdots f_1(f_0(u)))), f_j(y))$$

be a fault-free path, and $y \neq t$. (We note that there is a definition embedded in the path. The penultimate vertex is $f_{j-1}(f_{j-2}(\cdots f_1(f_0(u))))$ which we call y. Thus the last vertex is $f_j(y)$.) It is easy to see that such an edge (u, v) exists. Let R_1 be $(f_0(u), f_1(f_0(u)), \ldots, y)$. Let R_2 be the Hamiltonian path from $f_0(x)$ to t in $CG_{1,j} - (\cup_{r=1}^{j} F_r) - \{f_0(u), f_1(f_0(u)), \ldots, y\}$. (Such a path exists by Lemma 2.1 since $|F_r| \leq k-2$ and we delete at most one additional vertex from each G_r so $|F_r|' + |F_{r+1'}| + |F_{r,r+1}| \leq k+2$ where F_r' is F_r. Here we need $N \geq k+5$.) Let R_3 be the Hamiltonian path from $f_j(y)$ to $f_{n-1}^{-1}(v)$

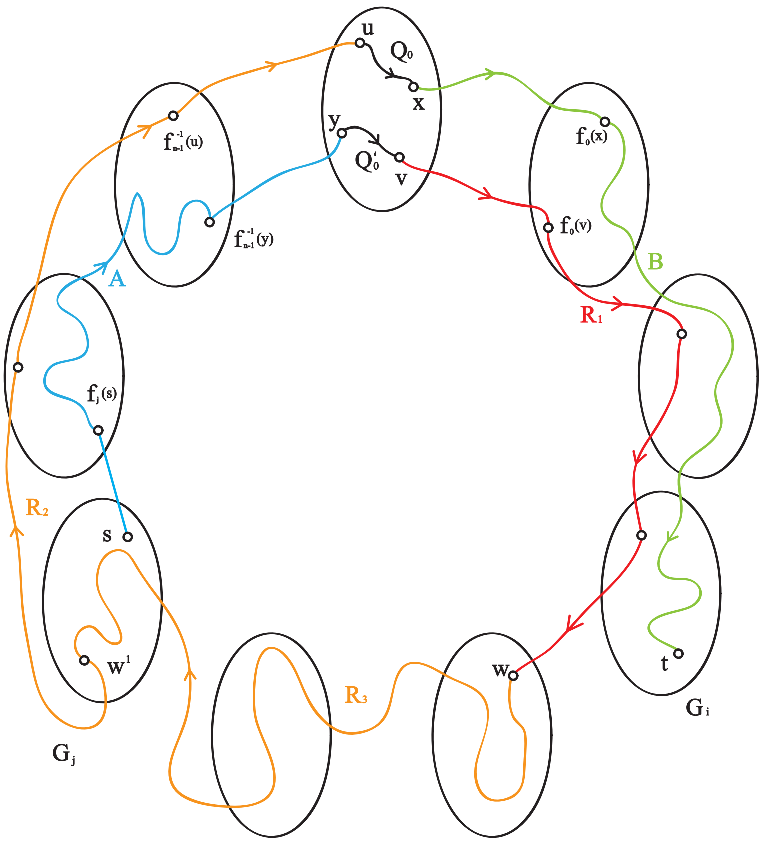

Figure 4: The Hamiltonian path of Subcase 1.6.

in $CG_{j+1,n-1} - (\cup_{r=j}^{n-1} F_r)$. Then

$$Q_0, (u, f_0(u)), R_1, (y, f_j(y)), R_3, (f_{n-1}^{-1}(v), v), Q_0', (x, f_0(x)), R_2$$

is a desired Hamiltonian path from s to t in $H - F$. (See Figure 5.)

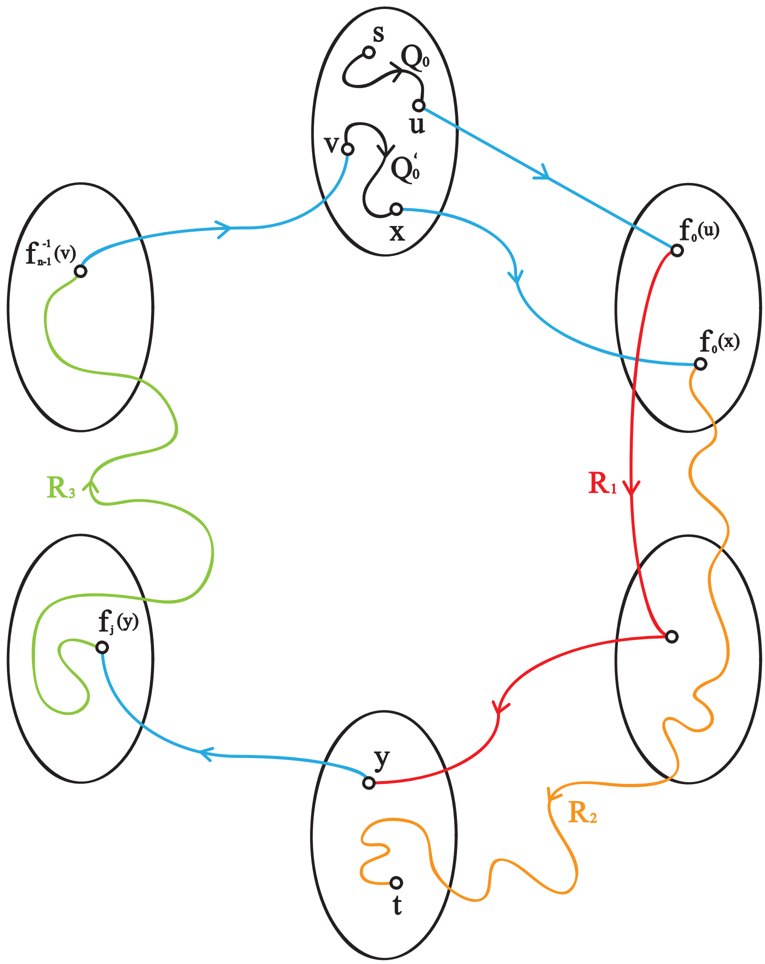

Figure 5: The Hamiltonian path of Subsubcase 2.2.1.

Henceforth, $|F_i| = k - 1$ for some i. Since $k \geq 3$, such an i is unique.

Subsubcase 2.2.2: $|F_0| = k - 1$ or $|F_q| = k - 1$ for some $q = j + 1, j + 2, \ldots, n - 1$. Then the argument of Subsubcase 2.2.1 applies by the remark given at the start of its argument.

Subsubcase 2.2.3: $|F_j| = k - 1$. We will adjust the construction given in Subsubcase 2.2.1. We note that there is at most one fault not in F_j. We find a vertex $y \neq t$ in $G_j - F_j$ such that $(f_0^{-1}(\cdots f_{j-1}^{-1}(y)), \ldots, f_{j-1}^{-1}(y), y)$ is fault-free. Now for this chosen y, let P_j be a Hamiltonian path from y to t in $G_j - F_j$. We find an arc (w, w') on P_j such that $(w, f_j(w))$ and $(f_{j-1}^{-1}(w'), w')$ are fault-free. (We allow $y = w$ or $w' = t$.) If

$j = 1$, we further require $f_0^{-1}(w') \neq s$. Let P_j^y and P_j^t be the subpaths of P_j from y to w and from w' to t, respectively. Let $u = f_0^{-1}(\cdots f_{j-1}^{-1}(y))$ and R_1 be $(u, \ldots, f_{j-1}^{-1}(y), y)$, followed by P_j^y and $(w, f_j(w))$. If $j = 1$, then let $x = f_0^{-1}(w')$. Otherwise, we pick x in $G_0 - (F_0 \cup \{s, u\})$ such that $(x, f_0(x))$ is fault-free. (If $j = 2$, then we further require $f_0(x) \neq f_1^{-1}(w')$.) We can now construct R_2 (similar to Subsubcase 2.2.1) by taking a fault-free Hamiltonian path from $f_0(x)$ to $f_{j-1}^{-1}(w')$ in

$$CG_{1,j-1} - (\cup_{r=1}^{j} F_r) - \{f_0(u), f_1(f_0(u)), \ldots, f_{j-1}^{-1}(y)\},$$

followed by P_j^t. Now consider $G_0 - F_0$. Recall that $|F_0| \leq 1$. If there is a z such that (u, z) is an arc in G_0 and $\{f_{n-1}^{-1}(z), (f_{n-1}^{-1}(z), z)\} \cap F \neq \emptyset$, then set $F_0' = F_0 \cup \{(u, z)\}$; otherwise, $F_0' = F_0$. Now we find a Hamiltonian path P_0 from s to x in $G_0 - F_0'$. Let (u, v) on P_0. By construction, $f_{n-1}^{-1}(v), (f_{n-1}^{-1}(v), v) \notin F$. We can now construct

$$R_1, R_3, Q_0, Q_0'$$

as in Subsubcase 2.2.1 with the following extra condition for choosing v when $j = n - 2$: $f_{n-2}(w) \neq f_{n-1}^{-1}(v)$. We also note that R_3 starts at $f_j(w)$ rather than $f_j(y)$. (See Figure 6.)

Subsubcase 2.2.4: $|F_q| = k - 1$ for some $q = 1, 2, \ldots, j - 1$. One can adapt the construction in Subsubcase 2.2.3 to cover this case. For completeness, we describe the procedure. We note that there is at most one fault not in F_q. We pick two (distinct) vertices a and b in $G_q - F_q$ such that $(f_0^{-1}(\cdots f_{q-1}^{-1}(a)), \ldots, f_{j-1}^{-1}(a), a)$ is fault-free and $(b, f_q(b))$ is fault-free. If $q = j - 1$, we further require that $f_q(b) \neq t$. Let P_q be a Hamiltonian path from a to b in $G_q - F_q$. We find an arc (w, w') on P_q such that $(w, f_q(w))$ and $(f_{q-1}^{-1}(w'), w')$ are fault-free. (We allow $a = w$ or $w' = b$.) If $j = 1$, we further require $f_0^{-1}(w') \neq s$. Let P_q^y and P_q^t be the subpaths of P_j from a to w and from w' to b, respectively. Let $u = f_0^{-1}(\cdots f_{q-1}^{-1}(a))$ and R_1 be $(u, \ldots, f_{q-1}^{-1}(a), a)$, followed by P_q^a and $(w, f_j(w))$. If $j = 1$, then let $x = f_0^{-1}(w')$. Otherwise, we pick x in $G_0 - (F_0 \cup \{s\})$ such that $(x, f_0(x))$ is fault-free. We can now construct R_2 by taking a fault-free Hamiltonian path from $f_0(x)$ to $f_{j-1}^{-1}(w')$ in

$$CG_{1,q-1} - (\cup_{r=1}^{j} F_r) - \{f_0(u), f_1(f_0(u))), \ldots, f_{j-1}^{-1}(a)\},$$

followed by P_q^t. The rest is the same as Subsubcase 2.2.3. □

We remark that the main reason that the argument given in Subsubcase 2.2.3 is not valid for $k = 2$ is because two vertices may be removed from a G_i and hence R_2 cannot be constructed as G_i is only 1-Hamiltonian connected. The same problem occurs for the other subcases. In fact, we did not notice this gap and gave this proof for $k \geq 2$ in an earlier draft. Fortunately for us, the anonymous referee noticed the error. We do not see an easy way to repair this gap. We note that for Subsubcase 2.2.3, the path R_1 and R_2 together span several G_i's, and in general, R_1 only covers one vertex of such G_i's. One idea is to be less restrictive in using the two paths covering such G_i's. (For example, the vertices in $G_j - F_j$ in Subsubcase 2.2.3 are spanned by two paths R_2 and R_3, with each path covering possibly more than one vertex of $G_i - F_i$.) Due to the distribution of s and t and the two vertices in F, there are 8 cases to consider with additional "boundary" subcases in each. We feel that a full discussion adds minimal value. So we choose to present the result for $k \geq 3$ only.

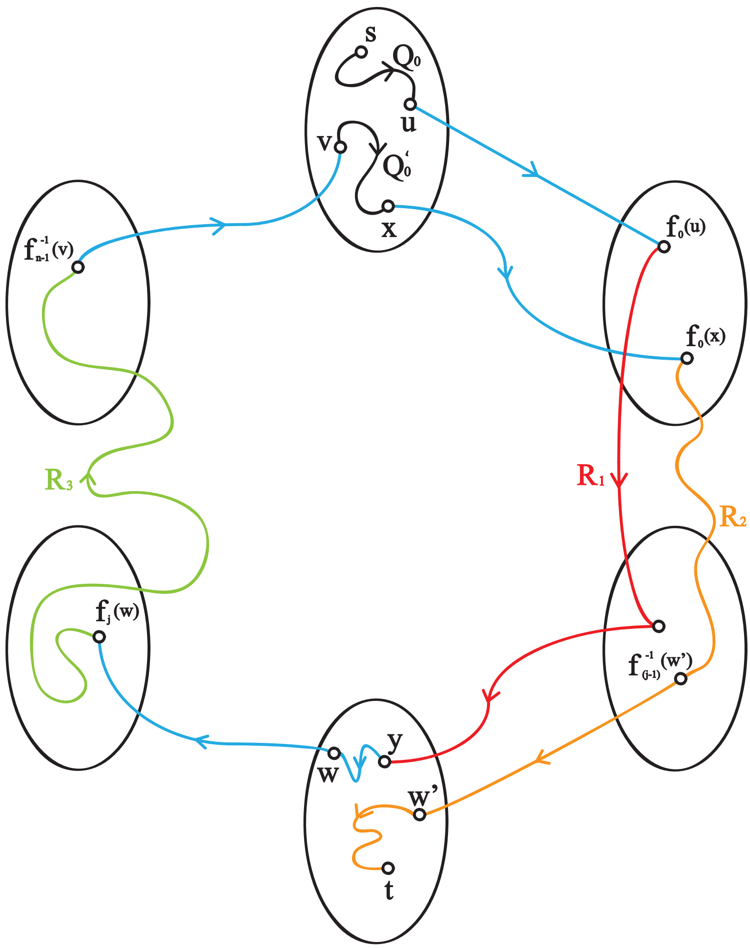

Figure 6: The Hamiltonian path of Subsubcase 2.2.3.

3 Analyzing the conditions in Theorem 2.2

In Theorem 2.2, one of the conditions is the requirement that $N \geq k+4$ for $(k+1)$-Hamiltonicity and $N \geq k+5$ for k-Hamiltonian connectedness. We are unsure whether this condition can be relaxed. However, we do know that the result for k-Hamiltonian connectedness does not hold if $N = k+2$. We choose $G = K_{k+2}$, the complete directed graph on $k+2$ vertices. Clearly it is $(k+1)$-regular and one can check that it is k-Hamiltonian and $(k-1)$-Hamiltonian connected. Then consider $H = G \square C_3$. Let F be k vertices in G_0, s be a vertex in $G_0 - F$ and $t = f(s)$. Then it is clear that there is no Hamiltonian path from s to t. One may wonder whether there is a counterexample $N = k+3$? An obvious choice is to let G be the directed graph obtained from the complete graph K_{k+3}, where k is even, by deleting a perfect matching (and treat the resulting graph as a directed graph). So G is $(k+1)$-regular. If G is k-Hamiltonian and $(k-1)$-Hamiltonian connected, then we have a counterexample. Unfortunately, this is not true as if $k = 2i-1$, then by deleting appropriate $k-1$ vertices from G, we have a 4-cycle which is not Hamiltonian connected.

We now consider the condition on k. As we pointed out earlier that $k = 0$ is not applicable, that is, G needs to be at least 2-regular. We have the condition $k \geq 2$ (that is, G is at least 3-regular) in the statement. In the proof, we did use this assumption; for example, we used it in Case 2 in proving that H is $(k+1)$-Hamiltonian. This is not to say that the result is not true for $k = 1$. On the other hand, we know of no 2-regular, 1-Hamiltonian and Hamiltonian-connected directed graphs. We have already commented on the condition of $k \geq 3$ for the k-Hamiltonian connectedness portion of the theorem.

Finally, there is the condition on n. In an undirected graph, a cycle must have at least three vertices. By the same convention, one usually requires a directed cycle in a directed graph to have at least three vertices; thus the condition $n \geq 3$. However, some authors do consider the two arcs (x, y) and (y, x) to form a directed cycle of length two. In any case, one may consider two directed graphs G and H with the same number of vertices and construct a new directed graph by two set of matchingss that match the vertices of G with the vertices of H and orient the edges in the first set from G to H and vice versa for the second set. One can ask if both G and H have "strong" Hamiltonian properties, does the resulting graph have "strong" Hamiltonian properties. One can apply similar analysis as in the proof of Theorem 2.2 for this problem.

We further remark that Theorem 2.2 seeks the strongest possible property, that is, for a $(k+1)$-regular graph G to be k-Hamiltonian and $(k-1)$-Hamiltonian connected, and then consider an n-$\mathcal{G}$-directed graph. The proof of Theorem 2.2 mainly relies on G being k-Hamiltonian and $(k-1)$-Hamiltonian connected, and not G being k-regular. So our proof is applicable to establish the following: Let $k \geq 2$ and $n \geq 3$. Let $1 \leq r \leq k$. Suppose the class of directed graphs $\mathcal{G}$ is $(k+1)$-regular, r-Hamiltonian, $(r-1)$-Hamiltonian connected and of order N. Let H be an n-$\mathcal{G}$-directed graph. Then H is $(k+2)$-regular, $(r+1)$-Hamiltonian if $N \geq k+4$ and r-Hamiltonian connected if $N \geq k+5$.

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Hamiltonicity of token graphs of fan graphs*

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Abstract

In this note we show that the token graphs of fan graphs are Hamiltonian. This result provides another proof of the Hamiltonicity of Johnson graphs and also extends previous results obtained by Mirajkar and Priyanka on the token graphs of wheel graphs.

Keywords: Token graphs, Johnson graphs, Hamiltonian graphs.

Math. Subj. Class.: 05C38, 05C45

1 Introduction

Let G be a simple graph of order n and let k be an integer such that $1 \leq k \leq n-1$. The k-*token graph*, or *symmetric kth power*, of G is the graph $G^{(k)}$ whose vertices are the k-subsets of $V(G)$ and two vertices are adjacent in $G^{(k)}$ if their symmetric difference is an edge of G. A classical example is the Johnson graph $J(n,k)$, which is isomorphic to the k-token graph of the complete graph K_n. This class of graphs is widely studied and has connections with coding theory [7, 9, 11, 13, 15] (another connection of token graphs with coding theory was showed in [18]).

The definition of k-token graphs (without a name) appeared in a work of Rudolph [17], in connection with problems in quantum mechanics and with the graph isomorphism problem. Rudolph presented examples of cospectral graphs G and H such that their corresponding 2-token graphs are not cospectral. Audenaert et al. [3], proved that the 2-token graphs

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of strongly regular graphs with the same parameters are cospectral, and suggested that for a given positive integer k there exists infinitely many pairs of non-isomorphic graphs with cospectral k-token graphs. This conjecture was proved by Barghi and Ponomarenko [16] and, independently, by Alzaga et al. [2]. Later, Fabila-Monroy et al. [8] reintroduced the k-token graphs as part of several models of swapping in the literature [19, 20], and studied some properties of these graphs: connectivity, diameter, cliques, chromatic number, Hamiltonian paths and Cartesian product of token graphs. This line of research was continued by Carballosa et al. [4] who studied regularity and planarity, de Alba et al. [6], who presented some results about independence and matching numbers, and Mirajkar et al. [14], who studied some covering properties of token graphs. Finally, Leaños and Trujillo-Negrete [12] proved a conjecture of Fabila-Monroy et al. [8] about the connectivity of token graphs and de Alba et al. [5] classified the triangular graphs (in other words, the 2-token graphs of complete graphs) that are Cohen-Macaulay.

A graph is Hamiltonian if it contains a Hamiltonian cycle. It is well known that $J(n, k)$ is Hamiltonian [10, 21], in fact, it is Hamiltonian connected [1]. As was noted in [8], the existence of a Hamiltonian cycle in G does not imply that $G^{(k)}$ contains a Hamiltonian cycle. For example, if k is even then $K_{m,m}^{(k)}$ is not Hamiltonian.

We are interested in graphs G such that its token graphs are Hamiltonian. The fan graph F_n is the join of graphs K_1 and P_{n-1}. In this note we show that the token graphs of fan graphs are Hamiltonian. Our result provides another proof that $J(n, k)$ is Hamiltonian, and also extends some of the results obtained by Mirajkar and Priyanka Y. B [14] about the Hamiltonicity of the token graphs of wheel graphs.

2 Main result

First we present some definitions and notations. For vertices u, v in graph G we write $u \sim v$ to mean that u and v are adjacent vertices in G. We write $G \simeq G'$ if G and G' are isomorphic graphs. A spanning subgraph of G is a subgraph H such that $V(H) = V(G)$. The following proposition is obvious.

Proposition 2.1. *If H is a spanning subgraph of G and H is Hamiltonian then G is Hamiltonian.*

One of the main properties of token graphs is that $G^{(1)}$ and G are isomorphic. Moreover, $G^{(k)} \simeq G^{(n-k)}$ for any $k \in \{1, \ldots, n-1\}$. Another known property of token graphs is the following.

Proposition 2.2. *If H is a subgraph of G then $H^{(k)}$ is a subgraph of $G^{(k)}$. Even more, if H is a spanning subgraph of G then $H^{(k)}$ is a spanning subgraph of $G^{(k)}$.*

For a fan graph F_n we assume that the vertices of P_{n-1} are $\{1, \ldots, n-1\}$ and the vertex in K_1 is labeled as n. For vertex $A = \{a_1, \ldots, a_k\}$ of $F_n^{(k)}$ we use the convention that $a_1 < \cdots < a_k$.

The main result of this note is the following.

Theorem 2.3. *Let n and k be positive integers with $n \geq 3$ and $1 \leq k \leq n-1$. Then $F_n^{\{k\}}$ is Hamiltonian.*

Proof. For $k = 1$, $F_n^{(1)} \simeq F_n$ which is Hamiltonian so in the rest of the proof we assume that $k \geq 2$. We will show that $F_n^{(k)}$ has a Hamiltonian cycle such that the vertices

$\{n-k, n-k+1, \dots, n-2, n\}$ and $\{n-k, n-k+1, \dots, n-2, n-1\}$ are adjacent in the cycle. The sequence of vertices $\{1,3\}\{1,2\}\{2,3\}\{1,3\}$ is a Hamiltonian cycle in $F_3^{(2)}$. The proof for $n \geq 4$ is by induction on k. First we show the case $k = 2$ and $n \geq 4$. The sequence of vertices

$$\begin{array}{l}
\{1,n-1\}\{1,n\}\{1,n-2\}\{1,n-3\}\dots\{1,3\}\{1,2\} \\
\{2,n\}\{2,n-1\}\{2,n-2\}\{2,n-3\}\dots\{2,4\}\{2,3\} \\
\vdots \\
\{n-3,n\}\{n-3,n-1\}\{n-3,n-2\} \\
\{n-2,n\}\{n-2,n-1\} \\
\{n-1,n\} \\
\{1,n-1\}
\end{array}$$

is a Hamiltonian cycle in $F_n^{(2)}$, where vertices $\{n-2, n-1\}$ and $\{n-2, n\}$ are adjacent in the cycle. We assume as induction hypothesis that $F_{n'}^{(k')}$ satisfies the conditions whenever $k' < k$ and $n' > k'$.

Claim. Let S_i be the subgraph of $F_n^{(k)}$ induced by the vertex set

$$V_i = \left\{ \{a_1, \dots, a_k\} \in V(F_n^{(k)}) : a_1 = i \right\},$$

with $1 \leq i \leq n-k$. Then $S_i \simeq F_{n-i}^{(k-1)}$.

Proof of Claim. Suppose that $V(F_{n-i}) = \{i+1, \dots, n\}$ with $V(P_{n-i-1}) = \{i+1, \dots, n-1\}$ and n the vertex of K_1. Then the function $A \mapsto A \setminus \{i\}$ is a graph isomorphism between S_i and $F_{n-i}^{(k-1)}$. □

We identify S_i with $F_{n-i}^{(k-1)}$ using the isomorphism given in the proof of the claim. By induction there exists a Hamiltonian cycle C_i in S_i, where vertices $X_i := \{i, n-k+1, \dots, n-2, n-1\}$ and $Y_i := \{i, n-k+1, \dots, n-2, n\}$ are adjacent in C_i, for $1 \leq i \leq n-k$. Let P_i be the Hamiltonian subpath of C_i from X_i to Y_i, for $1 \leq i \leq n-k$. Let Z denote the vertex $\{n-k+1, n-k+2, \dots, n-1, n\}$. Therefore $V_{n-k+1} = \{Z\}$.

Let $D_i = \{n-k, n-k+1, \dots, n-1, n\} \setminus \{i\}$, with $n-k+1 \leq i \leq n$. Then the vertex set V_{n-k} of S_{n-k} is $\{D_n, D_{n-1}, \dots, D_{n-k+1}\}$. Also, we have $X_{n-k} = D_n$ and $Y_{n-k} = D_{n-1}$. Let

$$Q = D_{n-2}D_{n-3}\dots D_{n-k+2}D_{n-k+1},$$

which, in fact, is a path in S_{n-k} because $D_i \triangle D_{i-1} = \{i-1, i\}$, for $n-k+2 \leq i \leq n-2$. Now,

$$\begin{aligned}
X_{n-k} \triangle D_{n-2} &= \{n-2, n\} \\
Y_{n-k} \triangle D_{n-2} &= \{n-2, n-1\} \\
Z \triangle D_{n-k+1} &= \{n-k, n-k+1\}
\end{aligned}$$

and hence

$$X_{n-k} \sim D_{n-2},$$
$$Y_{n-k} \sim D_{n-2},$$
$$D_{n-k+1} \sim Z,$$

in $F_n^{(k)}$. Notice that $X_i \triangle X_{i+1} = \{i, i+1\}$ and $Y_i \triangle Y_{i+1} = \{i, i+1\}$, for $1 \leq i \leq n-k-1$, and $X_1 \triangle Z = \{1, n\}$. Therefore we can define a Hamiltonian cycle $\mathcal{C}$ in $F_n^{(k)}$ as

$$X_1 \overrightarrow{_{P_1}} Y_1 Y_2 \overrightarrow{_{P_2}} X_2 \ldots X_{(n-k-1)} \overrightarrow{_{P_{n-k-1}}} Y_{(n-k-1)} Y_{(n-k)} X_{(n-k)} D_{n-2} \overrightarrow{_{Q}} D_{n-k+1} Z X_1,$$

if $n-k$ is even, and

$$X_1 \overrightarrow{_{P_1}} Y_1 Y_2 \overrightarrow{_{P_2}} X_2 \ldots Y_{(n-k-1)} \overrightarrow{_{P_{n-k-1}}} X_{(n-k-1)} X_{(n-k)} Y_{(n-k)} D_{n-2} \overrightarrow{_{Q}} D_{n-k+1} Z X_1,$$

if $n-k$ is odd. Furthermore

$$\{n-k, n-k+1, ..., n-2, n-1\} = X_{n-k} \sim Y_{n-k} = \{n-k, n-k+1, ..., n-2, n\},$$

in $\mathcal{C}$, as desired. □

The wheel graph W_n is the joint graph of K_1 and C_{n-1}. It is known that Johnson graphs [10, 21] and the k-token graphs of wheel graphs [14] are Hamiltonian, the following corollary provides another proof of this facts.

Corollary 2.4. *If F_n is a spanning subgraph of G then $G^{(k)}$ is Hamiltonian. In particular the Johnson graphs and the k-token graphs of wheel graphs are Hamiltonian.*

Proof. As F_n is a spanning subgraph of G then $F_n^{(k)}$ is a spanning subgraph of $G^{(k)}$ by Proposition 2.2. The k-token graph of F_n is Hamiltonian by Theorem 2.3 and hence $G^{(k)}$ is Hamiltonian by Proposition 2.1. In particular F_n is a spanning subgraph of W_n and K_n. □

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A short note on undirected Fitch graphs*

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Abstract

Fitch graphs have been introduced as a model of xenology relationships in phylogenomics. Directed Fitch graphs $G = (X, E)$ are di-graphs that are explained by $\{0, 1\}$-edge-labeled rooted trees with leaf set X: there is an arc $xy \in E$ if and only if the unique path in T that connects the least common ancestor $\operatorname{lca}(x, y)$ of x and y with y contains at least one edge with label 1. In this contribution, we consider the undirected version of Fitch's xenology relation, in which x and y are xenologs if and only if the unique path between x and y in T contains an edge with label 1. We show that symmetric Fitch relations coincide with class of complete multipartite graph and thus cannot convey any non-trivial phylogenetic information.

Keywords: Labeled trees, forbidden subgraphs, phylogenetics, xenology, Fitch graph.

Math. Subj. Class.: 05C75, 05C05, 92B10

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Fitch graphs [4] form a class of directed graphs that is derived from rooted, $\{0,1\}$-edge-labeled trees T in the following manner: The vertices of the Fitch graph are the leaves of T. Two distinct leaves x and y of T are connected by an arc (x,y) from x to y if and only if there is at least one edge with label 1 on the (unique) path in T that connects the least common ancestor $\operatorname{lca}(x,y)$ of x and y with y. Fitch graphs model "xenology", an important binary relation among genes, was introduced by Walter M. Fitch [2]. Interpreting T as a phylogenetic tree and 1-edges as horizontal gene transfer events, the arc (x,y) in the Fitch graph encodes the fact that y is xenologous with respect to x. A complete characterization of directed Fitch graphs is given in [4] in terms of the eight forbidden induced subgraphs shown in Figure 1.

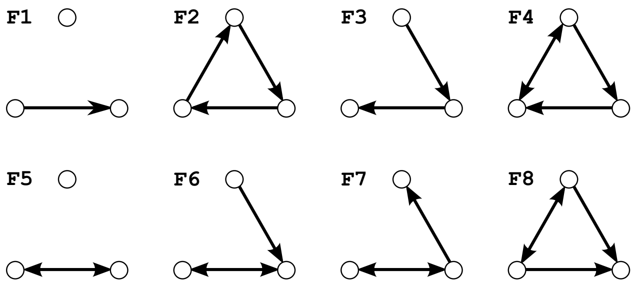

Figure 1: Shown are the eight forbidden induced subgraphs $F_1,\dots,F_8$ of Fitch graphs.

Theorem 0.1 ([4]). *A digraph $G=(X,E)$ is a directed Fitch graph if and only if it does not contain one the graphs $F_1,\dots,F_8$ in Figure 1 as an induced subgraph. It can be decided in $O(|X|+|E|)$ time whether G is a directed Fitch graph. In the positive case, there is a unique least-resolved tree (T,λ) explaining G, which also can be constructed in linear time.*

It is natural to consider also the symmetrized version of this relationship, i.e., to interpret $\{x,y\}$ as a xenologous pair whenever the evolutionary history separated x and y by at least one horizontal transfer event. In mathematical terms, this idea is captured by:

Definition 0.2. Let T be a rooted tree with leaf set X and let $\lambda\colon E(T)\to\{0,1\}$. Then the *undirected Fitch graph* F explained by (T,λ) has vertex set X and edges $\{x,y\}\in E(F)$ if and only if the (unique) path from x to y in T contains at least one edge e with $\lambda(e)=1$. A graph F is an undirected Fitch graph if and only if it is explained in this manner by some edge-labeled rooted tree (T,λ).

Undirected Fitch graphs are closely related to their directed counterparts. Since the path $\wp$ connecting two leaves x and y is unique and contains their least common ancestor $\operatorname{lca}(x,y)$, there is a 1-edge along $\wp$ if and only if there is a 1-edge on the path between x and $\operatorname{lca}(x,y)$ or between $\operatorname{lca}(x,y)$ and y. The undirected Fitch graph is therefore the underlying undirected graph of the directed Fitch graph, i.e., it is obtained from the directed version by ignoring the direction of the arcs.

The undirected Fitch graphs form a heritable family, i.e., if F is an undirected Fitch graph, so are all its induced subgraphs. This is an immediate consequence of the fact that

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directed Fitch graphs are a heritable family of digraphs [4]. The fact can also be obtained directly by considering the restriction of T to a subset of leaves. This obviously does not affect the paths or their labeling between the remaining vertices.

Clearly F does not depend on which of the non-leaf vertices in T is the root. Furthermore, a vertex v with only two neighbors and its two incident edges e' and e'' can be replaced by a single edge e. The new edge is labeled $\lambda(e) = 0$ if both $\lambda(e') = \lambda(e'') = 0$, and $\lambda(e) = 1$ otherwise. These operations do not affect the undirected Fitch graph. Hence, we can replace the rooted tree T by an unrooted tree in Definition 0.2 and assume that all non-leaf edges have at least degree 3. To avoid trivial cases we assume throughout that T has at least two leaves and hence a Fitch graph has at least two vertices.

Lemma 0.3. *If G is an undirected Fitch graph, then F does not contain $K_1 \mathbin{\dot\cup} K_2$ as an induced subgraph. In particular every undirected Fitch graph is a complete multipartite graph.*

Proof. There is a single unrooted tree with three leaves, namely the star S_3, which admits four non-isomorphic $\{0, 1\}$-edge labelings defined by the number N of 1-edges. The undirected Fitch graphs F_N are easily obtained. In the absence of 1-edges, $F_0 = \overline{K_3}$ is edge-less. For $N = 2$ and $N = 3$ there is a 1-edge along the path between any two leaves, i.e., $F_2 = F_3 = K_3$. For $N = 1$ one leaf is connected to the other two by a path in S_3 with an 1-edge; the path between the latter two leaves consists of two 0-edges, hence $F_1 = P_3$, the path of length two. Hence, only three of the four possible undirected graphs on three vertices can be realized, while $K_1 \mathbin{\dot\cup} K_2$ is not an undirected Fitch graph. By heredity, $K_1 \mathbin{\dot\cup} K_2$ is therefore a forbidden induced subgraph for the class of undirected Fitch graphs. Finally, it is well-known that the class of graphs that do not contain $K_1 \mathbin{\dot\cup} K_2$ as an induced subgraph are exactly the complete multipartite graphs, see e.g. [8]. □

We note in passing that the first part of Lemma 0.3 can also be obtained from the eight forbidden graphs on three vertices, using the fact that an undirected Fitch graph is the underlying (undirected) graph of a directed Fitch graph.

In order to show that forbidding $K_1 \mathbin{\dot\cup} K_2$ is also sufficient, we explicitly construct the edge-labeled trees necessary to explain complete multipartite graphs. We start by recalling that each complete multipartite graph $K_{n_1,\dots,n_k}$ is determined by its independent sets $V_1, \dots, V_k$ with $|V_i| = n_i$ for $1 \le i \le k$. By definition, $\{x, y\} \in E(K_{n_1,\dots,n_k})$ if and only if $x \in V_i$ and $y \in V_j$ with $i \ne j$. In particular, therefore, $K_{n_1,\dots,n_k}$ with at least two vertices is connected if and only if $k \ge 2$. The complete 1-partite graphs are the edge-less graphs $\overline{K_n}$.

Since $K_1 \mathbin{\dot\cup} K_2$ is an induced subgraph of the path on four vertices P_4, any graph G that does not contain $K_1 \mathbin{\dot\cup} K_2$ as an induced subgraph must be P_4-free, i.e., a cograph [1]. Cographs are associated with vertex-labeled trees known as cotrees, which in turn are a special case of modular decomposition trees [3]. The cotrees of connected multipartite graphs have a particularly simple shape, illustrated without the vertex labels in Figure 2. The cotree has a root labeled “1” and all inner vertices labeled “0”. Here we do not need the connection between cographs and their cotrees, however. Therefore, we introduce these trees together with an edge-labeling that is useful for our purposes in the following:

Definition 0.4. For $k = 1$, $T[n]$ is the star graph S_n with n leaves. For $k \ge 2$, the tree $T[n_1, \dots, n_k]$ has a root r with k children c_i, $1 \le i \le k$. The vertex c_i is a leaf if $|V_i| = n_i = 1$ and has exactly n_i children that are leaves if $|V_i| = n_i \ge 2$. For $k = 1$ all edges e of $T[n]$ are labeled $\lambda^*(e) = 0$. For $k \geq 2$ we set $\lambda^*(\{r, c_i\}) = 1$ for $1 \leq i \leq k$ and $\lambda^*(e) = 0$ for all edges not incident to the root.

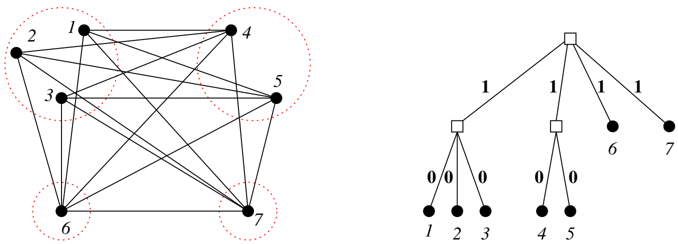

Figure 2: The complete multipartite graph $K_{3,2,1,1}$ is the Fitch graph explained by the tree $T[3, 2, 1, 1]$ with edge labeling λ^* shown with bold numbers **0** and **1**.

Now we can prove our main result:

Theorem 0.5. *A graph G is an undirected Fitch graph if and only if it is a complete multipartite graph. In particular, $K_{n_1,\dots,n_k}$ is explained by $(T[n_1, \dots, n_k], \lambda^*)$.*

Proof. Lemma 0.3 implies that an undirected Fitch graph is a complete multipartite graph. To show the converse, we fix an arbitrary complete multipartite graph $G = K_{n_1,\dots,n_k}$ and find an edge-labeled rooted tree (T, λ^*) that explains G.

For $k = 1$ it is trivial that $(T[n], \lambda^*)$ explains $\overline{K_n}$.

For $k \geq 2$ consider the tree $T[n_1, \dots n_k]$ with edge labeling λ^* and let F be the corresponding Fitch graph. The leaf set of $T[n_1, \dots n_k]$ is partitioned into exactly k subsets $L_1, \dots, L_k$ defined by (a) singletons adjacent to the root and (b) subsets comprising at least two leaves adjacent to the same child c_i of the root. Furthermore, we can order the leaf sets so that $|L_i| = n_i$. By construction, all vertices within a leaf set L_i are connected by a path that does not run through the root and thus, contains only 0-edges, if $|L_i| > 1$ and no edge, otherwise. The L_i are independent sets in F. On the other hand any two leaves $x \in L_i$ and $y \in L_j$ with $i \neq j$ are connected only by path through the root, which contains two 1-edges. Thus x and y are connected by an edge in F. Hence F is a complete multipartite graph of the form $K_{|L_1|,\dots,|L_k|} = K_{n_1,\dots,n_k}$. Since $K_{n_1,\dots,n_k}$ is explained by $(T[n_1, \dots, n_k], \lambda^*)$ for all $n_i \geq 1$ and all $k \geq 2$, and $\overline{K_n}$ is explained by $(T[n], \lambda^*)$, we conclude that every complete multipartite graph is a Fitch graph. □

The converse of Lemma 0.3 does not follow in a straightforward manner from the characterization of directed Fitch graphs in [4]. It is possible to make use of the connection between Fitch graphs and di-cographs [5, 6] to obtain the trees of Definition 0.4. This line of reasoning, however, is neither shorter nor simpler than the direct, elementary proof given above.

Complete multipartite graphs $G = (V, E)$ obviously can be recognized in $O(|V|^2)$ time (e.g., by checking that its complement is a disjoint union of complete graphs), and even in $O(|V| + |E|)$ time by explicitly constructing its modular decomposition tree [7]. Given the tree $T[n_1, \dots, n_k]$, the canonical edge labeling λ^* is then assigned in $O(|V|)$ time.

A tree (T, λ) that explains a Fitch graph F is *minimum* if it has the smallest number of vertices among all trees that explain F. In this case, (T, λ) is also *least-resolved*, i.e., the contraction of any edge in (T, λ) results in a tree that does not explain F. Not surprisingly, the tree $T[n_1, \dots, n_k]$ is almost minimum in most, and minimum in some cases: Since the vertices of the Fitch graph must correspond to leaves of the tree, $T[n_1, \dots, n_k]$ is necessarily minimum whenever it is a star, i.e., for $T[n]$ and $T[1, \dots, 1]$. In all other cases, its only potentially "superfluous" part is its root. Indeed, exactly one of the edges connecting the root with a non-leaf neighbor can be contracted without changing the corresponding Fitch graph. It is clear that this graph is minimal: The leaf sets L_i must be leaves of an induced subtree without an intervening 1-edge. Having all vertices of L_i adjacent to the same vertex is obviously the minimal choice. Since the L_i must be separated from all other leaves by a 1-edge, at least one incident edge of c_i must be a 1-edge. Removing all leaves incident to a 0-edge results in a tree with at least k vertices that must contain at least $k-1$ 1-edges, since every path between leaves in this tree must contain a 1-edge. The contraction of exactly one of the k 1-edges incident to the root r in $T[n_1, \dots, n_k]$ indeed already yields a minimal tree. In general, the minimal trees are not unique, see Figure 3.

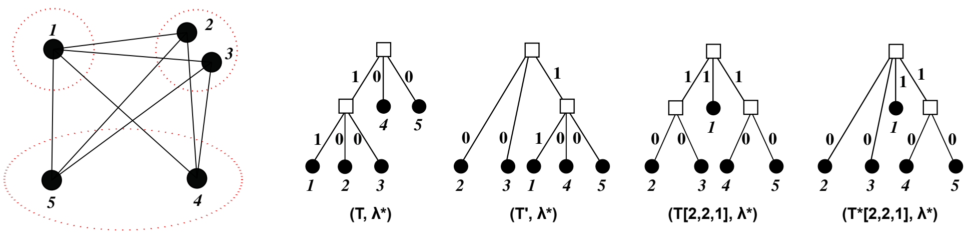

Figure 3: The non-isomorphic trees (T, λ^*), (T', λ^*) $(T[2,2,1], \lambda^*)$, and $(T^*[2,2,1], \lambda^*)$ all explain the same complete multipartite graph $K_{2,2,1}$. Three of these trees have the smallest possible number (7) of vertices and thus are minimal. These can be obtained from the tree $(T[2,2,1], \lambda^*)$ specified in Definition 0.2 by contraction of one of its inner 1-edges and possibly re-rooting the resulting tree.

It may be worth noting that $K_{n_1,\dots,n_k}$ can also be explained by binary trees. To see this, we convert a tree $(T[n_1, \dots, n_k], \lambda^*)$ into a binary tree in two simple steps. First, each group of $n_i > 1$ leaves with a common parent are replaced by an arbitrary binary tree with the same leaf set and all edges labeled 0. Second, the star consisting of the root and all its children C is replaced by an arbitrary rooted binary tree with leaf set C and all edges labeled 1. It is obvious that neither of the operations affects the graph that is explained.

The practical implication of Theorem 0.5 in the context of phylogenetic combinatorics is that the mutual xenology relation cannot convey any interesting phylogenetic information: Since the undirected Fitch graphs are exactly the complete multipartite graphs, which in turn are completely defined by their independent sets, the only insight we can gain by considering mutual xenology is the identification of the maximal subsets of taxa that have not experienced any horizontal transfer events among them.

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On a conjecture about the ratio of Wiener index in iterated line graphs*

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Abstract

Let G be a graph. Denote by $W(G)$ its Wiener index and denote by $L^i(G)$ its i-iterated line graph. Dobrynin and Mel'nikov proposed to estimate the extremal values for the ratio $R_k(G) = W(L^k(G))/W(G)$ for $k \geq 1$. Motivated by this we study the ratio for higher k's. We prove that among all trees on n vertices the path P_n has the smallest value of this ratio for $k \geq 3$. We conjecture that this holds also for $k = 2$, and even more, for the class of all connected graphs on n vertices. Moreover, we conjecture that the maximum value of the ratio is obtained for the complete graph.

Keywords: Wiener index, line graph, tree, iterated line graph.

Math. Subj. Class.: 05C12, 05C05, 05C76

1 Introduction

Let G be a graph. We denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. For any two vertices u, v let $d(u, v)$ be the distance from u to v. The *Wiener index* of G, $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

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where the sum is taken over all unordered pairs of vertices of G. Wiener index was introduced by Wiener in [17]. Since it is related to several properties of molecules (see [7]), it is widely studied by chemists. The interest of mathematicians was attracted in 1970's, when it was reintroduced as *the transmission* and *the distance of a graph*, see [16] and [5], respectively. For surveys and some up-to-date papers related to the Wiener index of trees and line graphs see [15, 18] and [2, 8, 13], respectively.

By definition, if G has a unique vertex, then $W(G) = 0$. In this case, we say that the graph G is *trivial*.

The line graph of G, $L(G)$, has vertex set identical with the set of edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are incident in G. Iterated line graphs are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0, \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

Observe that $W(P_n) = ((n-1) + \cdots + 1) + ((n-2) + \cdots + 1) + \cdots + 1 = \binom{n+1}{3}$. In the case when a tree contains a small number of branching vertices (i.e., vertices of degree at least three), then it is suitable to use the theorem of Doyle and Graver [4] for computing its Wiener index:

Theorem 1.1. *Let T be a tree on n vertices. Then*

$$W(T) = \binom{n+1}{3} - \sum_{v \in V(T)} \sum_{1 \le i < j < k \le p} n(T_i)\, n(T_j)\, n(T_k)\,,$$

where $T_1, T_2, \ldots, T_p$ are the components of $T - v$.

Wiener index of the line graph of a tree T can easily be computed from $W(T)$ by using the following result of Buckley [1]:

Theorem 1.2. *Let T be a tree on n vertices. Then $W(L(T)) = W(T) - \binom{n}{2}$.*

In [6] (see also [3]) Gutman proposed a problem to find an n-vertex graph G whose line graph $L(G)$ has the maximum Wiener index.

Dobrynin and Mel'nikov [3] proposed to estimate the extremal values of the ratio

$$R_k(G) = \frac{W(L^k(G))}{W(G)}. \tag{1.1}$$

Notice that

$$\frac{W(L(S_n))}{W(S_n)} = \frac{n-2}{2(n-1)}, \quad \frac{W(L(P_n))}{W(P_n)} = \frac{n-2}{n+1}, \quad \text{and} \quad \frac{W(L(K_n))}{W(K_n)} = \binom{n-1}{2}.$$

In [14], this problem was solved for the minimum in the case $k = 1$:

Theorem 1.3. *Among all connected graphs on n vertices, the fraction $R_1(G)$ is minimum for the star S_n.*

The problem for the maximum remains open:

Problem 1.4. Find n-vertex graph G with the maximum value of $R_1(G)$.

The line graph of K_n has the greatest number of edges and the smallest Wiener index, and henceforth, it may attain the maximum value. For higher iterations $k \geq 2$, we expect that the minimum should be at P_n, as it is the only graph whose line graph decreases in size. Thus, we believe the following holds:

Conjecture 1.5. *Let n be a large number and $k \geq 2$. Among all graphs G on n vertices, $W(L^k(G))/W(G)$ attains the maximum for K_n, and it attains the minimum for P_n.*

In what follows we support this conjecture for the minimum. In a series of papers [10, 9, 12, 11, 8] (see [11, Corollary 1.4]), where the equality $W(L^k(T)) = W(T)$ is solved for trees and $k \geq 3$, the following result was obtained:

Theorem 1.6. *Let T be a tree and $k \geq 4$. Then we have*

$$\begin{array}{ll} W(L^k(T)) = W(T) & \text{if } T \text{ is trivial,} \\ W(L^k(T)) < W(T) & \text{if } T \text{ is a nontrivial path or the claw } K_{1,3}, \\ W(L^k(T)) > W(T) & \text{otherwise.} \end{array}$$

The above result gives an immediate support to Conjecture 1.5:

Corollary 1.7. *Let $k \geq 4$. In the class of trees on n vertices, R_k attains the minimum value for P_n.*

In this paper we extend the above corollary to the case $k = 3$. Let H be a tree on six vertices, two of which have degree 3 and the other four have degree 1. Recall that two graphs G_1 and G_2 are homeomorphic if and only if there is a third graph F, such that both G_1 and G_2 can be obtained from F by means of edge subdivision. In the proof we will use the following result [9, Corollary 1.6]:

Theorem 1.8. *Let T be a tree which is not homeomorphic to a path, claw $K_{1,3}$ or H, and let $k \geq 3$. Then $W(L^k(T)) > W(T)$.*

By Theorem 1.8, to solve the case $k = 3$, it is sufficient to consider the ratios for paths and trees homeomorphic to the claw $K_{1,3}$ and H.

Note that $L^3(P_n) = P_{n-3}$ if $n \geq 4$, and we have

$$R_3(P_n) = \frac{\binom{n-2}{3}}{\binom{n+1}{3}} = \frac{(n-2)(n-3)(n-4)}{(n+1)n(n-1)}.$$

In Section 2 we prove the following two results:

Theorem 1.9. *Let T be a tree on n vertices homeomorphic to $K_{1,3}$. Then*

$$R_3(T) > R_3(P_n).$$

Theorem 1.10. *Let T be a tree on n vertices homeomorphic to H. Then*

$$R_3(T) > R_3(P_n).$$

These two results together with Theorem 1.8 and Corollary 1.7 give us the following:

Corollary 1.11. *Let $k \geq 3$. Then the path P_n attains the minimum value of R_k in the class of trees on n vertices.*

2 Proofs of Theorems 1.9 and 1.10

Proof of Theorem 1.9. Let $C_{a,b,c}$ be a tree homeomorphic to the claw $K_{1,3}$, such that the paths connecting the vertices of degree 1 with the vertex of degree 3 have lengths a, b and c, where $a \geq b \geq c \geq 1$. The tree $C_{a,b,c}$ has exactly $n = a+b+c+1$ vertices, see Figure 1 for $C_{4,3,2}$.

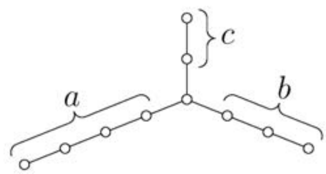

Figure 1: The graph $C_{4,3,2}$.

Further, for $i \in \{1, 2, 3\}$ let V_i be the set of vertices of $V(L(C_{a,b,c}))$ of degree i. This naturally splits the problem into four cases according to the size of V_1.

Denote

$$\Delta = W(L^3(C_{a,b,c})) - W(C_{a,b,c}). \tag{2.1}$$

In [8], the value of Δ for each of these four cases is evaluated. For the sake of simplicity, let $W_0 = W(C_{a,b,c})$ and $W_3 = W(L^3(C_{a,b,c}))$. Then $\Delta = W_3 - W_0$ and

$$R_3(C_{a,b,c}) = \frac{W_3}{W_0} = \frac{W_0 + \Delta}{W_0} = 1 + \frac{\Delta}{W_0}.$$

By Theorem 1.1 we have

$$W_0 = (a+b+c+2)(a+b+c+1)(a+b+c)/6 - abc. \tag{2.2}$$

We prove that when $|V(C_{a,b,c})| = |V(P_n)|$, that is when $n = a + b + c + 1$, then $R_3(C_{a,b,c}) > R_3(P_n)$. This inequality is equivalent to

$$1 + \frac{\Delta}{W_0} > \frac{(n-2)(n-3)(n-4)}{(n+1)n(n-1)}$$

and after multiplying by denominators also to

$$\Delta(n+1)n(n-1) + W_0\big((n+1)n(n-1) - (n-2)(n-3)(n-4)\big) > 0. \tag{2.3}$$

Since $3 \geq |V_i| \geq 0$, there are four cases to consider.

Case 1: $a, b, c \geq 2$. That is, $|V_1| = 3$. In [8] we have

$$\Delta = (a+b+c)^2 - 5(ab+ac+bc) + (a+b+c) + 21. \tag{2.4}$$

After substituting (2.4) and (2.2) into (2.3), we get that the left-hand side of (2.3) is equal to the following expression

$$1.5abc\big((a-b)^2+(a-c)^2+(b-c)^2\big)+44a+65a^2+25.5a^3+7a^4+2.5a^5+ 44b+130ab+66.5a^2b+13a^3b+7.5a^4b+65b^2+66.5ab^2+12a^2b^2+10a^3b^2+ 25.5b^3+13ab^3+10a^2b^3+7b^4+7.5ab^4+2.5b^5+44c+130ac+66.5a^2c+ 13a^3c+7.5a^4c+130bc+117abc+18a^2bc+3a^3bc+66.5b^2c+18ab^2c+ 13b^3c+3ab^3c+7.5b^4c+65c^2+66.5ac^2+12a^2c^2+10a^3c^2+66.5bc^2+18abc^2+ 12b^2c^2+10b^3c^2+25.5c^3+13ac^3+10a^2c^3+13bc^3+3abc^3+10b^2c^3+7c^4+ 7.5ac^4+7.5bc^4+2.5c^5.$$

Since $a,b,c \geq 2$, the expression $1.5abc\big((a-b)^2+(a-c)^2+(b-c)^2\big)$ and all the isolated terms are nonnegative. Moreover some of the terms, such as $44a$ for example, are strictly positive. Hence, (2.3) is satisfied, which means that $R_3(C_{a,b,c}) > R_3(P_{a+b+c+1})$.

Observe that the above long expression was obtained from the left-hand side of (2.3) by subtracting $1.5abc\big((a-b)^2+(a-c)^2+(b-c)^2\big)$, which is nonnegative, and then by expanding the difference. Since all the parameters a, b, c are nonnegative, all the coefficients in the expanded expression are positive and at least one of the terms is strictly positive, (2.3) is satisfied. We will use this way of reasoning especially in the proof of Theorem 1.10, where the expanded expressions are extremely long.

Case 2: $a,b \geq 2$, $c = 1$. That is, $|V_1| = 2$. In [8] we have

$$2\Delta = (a+b)^2 - 8ab - 5(a+b) + 30. \tag{2.5}$$

After substituting (2.5) and (2.2) into (2.3) and expanding the expression, we get that the left-hand side of (2.3) is equal to

$$96+170a+97a^2+32a^3+11a^4+2a^5+170b+164ab+43a^2b+11a^3b+6a^4b+ 97b^2+43ab^2+8a^3b^2+32b^3+11ab^3+8a^2b^3+11b^4+6ab^4+2b^5.$$

Hence (2.3) is satisfied and so $R_3(C_{a,b,1}) > R_3(P_{a+b+2})$.

Case 3: $a \geq 2$, $b = c = 1$. That is, $|V_1| = 1$. In [8] we have $\Delta = -6a+6$. After substituing this value of Δ and (2.2) into (2.3) and expanding the expression, we get that the left-hand side of (2.3) is equal to

$$1.5a^5+12a^4+26.5a^3+60a^2+300a+240.$$

Hence (2.3) is satisfied and so $R_3(C_{a,1,1}) > R_3(P_{a+3})$.

Case 4: $a = b = c = 1$. That is, $|V_1| = 0$. In this case $C_{a,b,c} = K_{1,3}$ has 4 vertices and $L^3(K_{1,3})$ is a cycle of length 3. Since $W(L^3(P_4)) = 0$, we have

$$R_3(C_{1,1,1}) > 0 = R_3(P_4),$$

which establishes this small case, and also the proof of the theorem. □

Proof of Theorem 1.10. Denote by $H_{a,b,c,d,e}$ a tree homeomorphic to H defined as follows: In $H_{a,b,c,d,e}$, the two vertices of degree 3 are joined by a path of length $e+1$, $e \geq 0$. Hence, this path has e vertices of degree 2. Further, at one vertex of degree 3 there start two pendant paths of lengths a and b, where $a, b \geq 1$, and at the other vertex of degree 3 there start another two pendant paths of lengths c and d, where $c, d \geq 1$. Thus $H_{a,b,c,d,e}$ has $n = a+b+c+d+e+2$ vertices, out of which two have degree 3, four have degree 1 and the remaining vertices have degree 2, see Figure 2 for $H_{3,3,4,2,2}$. By symmetry, we may assume that $a \geq b, c \geq d$, and $b \geq d$. That is, we assume that the shortest pendant path in $H_{a,b,c,d,e}$ has length d.

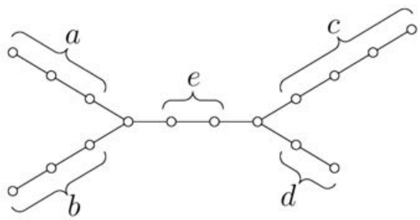

Figure 2: The graph $H_{3,3,4,2,2}$.

We proceed analogously as in the proof of Theorem 1.9. Denote

$$\Delta = W(L^3(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e}). \tag{2.6}$$

For the sake of simplicity, let $W_0 = W(H_{a,b,c,d,e})$ and $W_3 = W(L^3(H_{a,b,c,d,e}))$. Then $\Delta = W_3 - W_0$ and again

$$R_3(H_{a,b,c,d,e}) = 1 + \frac{\Delta}{W_0}.$$

By Theorem 1.1 we have

$$W_0 = \binom{a+b+c+d+e+3}{3} - ab(c+d+e+1) - cd(a+b+e+1). \tag{2.7}$$

If $e = 0$, then we have one vertex of degree 4 in $L(H_{a,b,c,d,e})$, while if $e \geq 1$, then the greatest degree of a vertex in $L(H_{a,b,c,d,e})$ is 3. Analogously as in [8], by symmetry we distinguish eleven cases. Five cases with at least one of a, b, c, d greater than or equal to 2 have $e \geq 1$, five cases with at least one of a, b, c, d greater than or equal to 2 have $e = 0$, and the last case has all a, b, c, d equal to 1. First we consider the cases with $\Delta > 0$.

Claim 1. *If $\Delta > 0$, then $R_3(H_{a,b,c,d,e}) > R_3(P_{a+b+c+d+e+2})$.*

Proof. Observe that $|V(H_{a,b,c,d,e})| = |V(P_{a+b+c+d+e+2})|$. If $\Delta > 0$, then $R_3(H_{a,b,c,d,e}) = 1 + \frac{\Delta}{W_0} > 1$. However, $R_3(P_n)$ is always smaller than 1. □

By [8], there are 8 cases (out of the 11) for which in [8] it was proved that $\Delta > 0$ (we remark that P is used instead of Δ in [8]). These are the cases:

1. (case 3 in [8]) $a, c \geq 2$, $b = d = 1$, $e \geq 1$;
2. (case 4 in [8]) $a, b \geq 2$, $c = d = 1$, $e \geq 1$;

3. (case 5 in [8]) $a \geq 2$, $b = c = d = 1$, $e \geq 1$;
4. (case 7 in [8]) $a, b, c \geq 2$, $d = 1$, $e = 0$;
5. (case 8 in [8]) $a, c \geq 2$, $b = d = 1$, $e = 0$;
6. (case 9 in [8]) $a, b \geq 2$, $c = d = 1$, $e = 0$;
7. (case 10 in [8]) $a \geq 2$, $b = c = d = 1$, $e = 0$;
8. (case 11 in [8]) $a = b = c = d = 1$, $e \geq 0$.

By Claim 1 it suffices to consider the remaining three cases.

We proceed analogously as in the proof of Theorem 1.9. Hence, we prove that when $|V(H_{a,b,c,d,e})| = |V(P_n)|$, that is when $n = a+b+c+d+e+2$, then $R_3(H_{a,b,c,d,e}) > R_3(P_n)$. This inequality is equivalent to

$$1 + \frac{\Delta}{W_0} > \frac{(n-2)(n-3)(n-4)}{(n+1)n(n-1)}$$

and after multiplying by denominators also to

$$\Delta(n+1)n(n-1) + W_0\big((n+1)n(n-1) - (n-2)(n-3)(n-4)\big) > 0. \tag{2.8}$$

Now we consider the remaining three cases.

Case 1: $a, b, c, d \geq 2, e \geq 1$. In [8] we have

$$\begin{aligned} 2\Delta = {} & 7(a{+}b{+}c{+}d{+}e)^2 - 20(ab{+}ac{+}ad{+}bc{+}bd{+}cd) - 10(ae{+}be{+}ce{+}de) \\ & + 5(a{+}b{+}c{+}d) + 65e + 234. \end{aligned} \tag{2.9}$$

Denote

$$\begin{aligned} D = 11\big(& cd(a-b)^2(a+b) + bd(a-c)^2(a+c) + ad(b-c)^2(b+c) \\ & + bc(a-d)^2(a+d) + ac(b-d)^2(b+d) + ab(c-d)^2(c+d)\big). \end{aligned}$$

Observe that $D \geq 0$. Now substitute (2.9) and (2.7) into the left-hand side of (2.8) and delete D. When we expand the resulting expression, all the coefficients will be positive. Since the constant term is 708, which is strictly positive, (2.8) is satisfied and so $R_3(H_{a,b,c,d,e}) > R_3(P_{a+b+c+d+e+2})$.

Case 2: $a, b, c \geq 2, d = 1, e \geq 1$. From [8] we have

$$\Delta = 3(a^2{+}b^2{+}c^2{+}e^2) - 3(ab{+}ac{+}bc) + (ae{+}be) + 2ce - 2(a+b) - c + 28e + 97.$$

In [8] it was shown that if $e \geq 2$ then $\Delta > 0$. By Claim 1,

$$R_3(H_{a,b,c,1,e}) > R_3(P_{a+b+c+e+3})$$

in this subcase, so it suffices to restrict ourselves to $e = 1$. For $e = 1$ we obtain

$$\Delta = 3(a^2{+}b^2{+}c^2) - 3(ab{+}ac{+}bc) - a - b + c + 128. \tag{2.10}$$

Now substitute (2.10) and (2.7) with $e = 1$ into the left-hand side of (2.8). When we expand the resulting expression, all the coefficients will be positive. Since the constant term is 8280, which is strictly positive, (2.8) is satisfied and so $R_3(H_{a,b,c,1,1}) > R_3(P_{a+b+c+4})$.

Case 3: $a, b, c, d \geq 2, e = 0$. In [8] we have

$$\Delta = 4(a+b+c+d)^2 - 11(ab+ac+ad+bc+bd+cd) + 3(a+b+c+d) + 137. \qquad (2.11)$$

Denote

$$D = 10\big(cd(a-b)^2(a+b) + bd(a-c)^2(a+c) + ad(b-c)^2(b+c) + bc(a-d)^2(a+d) + ac(b-d)^2(b+d) + ab(c-d)^2(c+d)\big).$$

Observe that $D \geq 0$. Now substitute (2.11) and (2.7) into the left-hand side of (2.8) and delete D. When we expand the resulting expression, all the coefficients will be positive. Since the constant term is 828, which is strictly positive, (2.8) is satisfied and so $R_3(H_{a,b,c,d,0}) > R_3(P_{a+b+c+d+2})$. This completes the proof. □

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The number of independent sets in a connected graph and its complement

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Abstract

For a connected graph G, the total number of independent vertex sets (including the empty vertex set) is denoted by $i(G)$. In this paper, we consider Nordhaus-Gaddum-type inequalities for the number of independent sets of a connected graph with connected complement. First we define a transformation on a graph that increases $i(G)$ and $i(\overline{G})$. Next, we obtain the minimum and maximum value of $i(G)+i(\overline{G})$, where graph G is a tree T with connected complement and a unicyclic graph G with connected complement, respectively. In each case, we characterize the extremal graphs. Finally, we establish an upper bound on the $i(G)$ in terms of the Wiener polarity index.

Keywords: Independent sets, connected complement, bounds, the Wiener polarity index, Nordhaus-Gaddum-type inequality.

Math. Subj. Class.: 05C69, 05C30

1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph of order n with vertex set $V(G)$ and edge set $E(G)$, denote by $N_G(u)$ the set of neighbors of a vertex u in G, and denote by $G[S]$ the graph which is induced by vertex set $S \subseteq V(G)$. A double star $S_{p,q}$ is obtained from S_p and S_q by connecting the center of S_p with that of S_q. A graph is unicyclic if and only if it is connected and has size equal to its order.

Given a graph G, a *k-independent set* is a set of k vertices, no two of which are adjacent. Denote by $i(G; k)$ the number of k-independent sets of G, $k \geq 1$. It is both consistent and convenient to define $i(G; 0) = 1$. The family of the independent sets in G which contains the vertex sets U and S is denoted by $I_{U,S}(G)$, and let $i_{U,S}(G; k)$ be its cardinality. The

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total number of independent vertex sets (including the empty vertex set) of a molecular graph $G = (V, E)$, denoted by $i(G)$, is defined as

$$i(G) = \sum_{k \geq 0} i(G; k).$$

In chemical literature, the number of the independent sets of graphs $i(G)$ is referred to as the *Merrifield-Simmons* index. It is a valuable topological index introduced by the American chemists Richard E. Merrifield and Howard E. Simmons [12] in 1989. It is one of the topological indices whose mathematical characteristics has been extensively studied in a monograph [11, 20]. Its applicability for QSPR and QSAR was also examined to a much lesser extent. In [12] it has been shown that $i(G)$ is correlated with boiling points. And, for the path P_n, $i(P_n)$ is equal to the Fibonacci number F_{n+1} [15].

The problem of counting the number of independent sets in a graph is NP-complete [16]. However, for certain types of graphs the problem of determining their number of independent subsets is polynomial. For instance, the number of independent sets in tree, uncyclic, and tricyclic graphs are calculated in [15, 14, 21], respectively. It is of significant interest to study the extremal graphs having maximal or minimal index. Zhu [20] characterized the extremal unicyclic graphs with a perfect matching which have maximal, second maximal Merrifield-Simmons index. In [17], S. Wagner and I. Gutman wrote a survey of results and techniques on the Hosoya index and Merrifield-Simmons index. Other recent results on the number of independent sets can be found in [2, 4, 3, 9].

The number of unordered vertices pairs that are at distance 3 in a graph G, denoted by $W_p(G)$, is

$$W_p(G) = |\{(u, v) \mid d_G(u, v) = 3, u, v \in V(G)\}|.$$

It is also referred as the Wiener polarity index ([5, 7, 8]). Motivated by the result of [7], Hua et al. gave an upper bound on the Wiener polarity index in terms of the Hosoya index. We can find that, in a graph G, every pair of vertices at distance 3 corresponds to some 2-independent sets. There are also some relationships between the number of independent sets and the Wiener polarity index.

The Nordhaus-Gaddum-type results are bounds of the sum or the product of a parameter for a graph and its complement. The name “Nordhaus-Gaddum-type” is given because Nordhaus and Gaddum [13] first found this type of inequality for the chromatic number of a graph and its complement in 1956. Since then, Nordhaus-Gaddum-type inequalities for many other graph invariants have been studied in a number of papers [1, 7, 10, 19]. We respectively research Nordhaus-Gaddum-type results for tree $i(T)$, unicyclic graph $i(G)$ and connected graph $i(G)$.

In this paper, we consider Nordhaus-Gaddum-type inequalities for the number of independent sets of a connected graph with a connected complement. Firstly, in Section 2 we establish a transformation on graphs that increases $i(G)$ and $i(\overline{G})$. Secondly, in Section 3 and 4, we obtain the minimum and maximum value of $i(G) + i(\overline{G})$, where graph G is a tree T with connected complement and a unicyclic graph G with connected complement, respectively. In each case, we characterize the extremal graphs. Finally, in Section 5 we establish a lower bound on $i(G)$ in terms of the Wiener polarity index. And, for a connected graph G with connected complement $\overline{G}$, we obtain the minimum of $i(G) + i(\overline{G})$. Also, we pose a conjecture about which graph obtains the maximum value of $i(G) + i(\overline{G})$.

Other notation and terminology not defined here will conform to those in [18].

2 Preliminary

Lemma 2.1 ([6]). *Let $G = (V, E)$ be a graph.*

(1) If $uv \in E(G)$, then $i(G) = i(G - uv) - i(G - (N[u] \cup N[v]))$;

(2) If $u \in V(G)$, then $i(G) = i(G - u) + i(G - N[u])$;

(3) If $G_1, G_2, \ldots, G_t$ are the components of the graph G, then $i(G) = \prod_{j=1}^{t} i(G_j)$.

Theorem 2.2. *Let G be a simple graph and uv an edge of G such that $N_G(u) \cap N_G(v) = \emptyset$ and $d(u), d(v) > 1$. Let $G_{u,v}$ denote the graph obtained from G by identifying vertex u and v (the new vertex is labeled as u) and attaching a pendent vertex v at u. Then*

(1) $i(\overline{G_{u,v}}) \geq i(\overline{G})$ with equality if and only if $\overline{G}[N_{G-u}(v) \cup N_G(u) \setminus \{v\}]$ is not an empty graph;

(2) $i(G_{u,v}) \geq i(G)$ with equality if and only if $G[N_{G-u}(v) \cup N_G(u) \setminus \{v\}]$ is not an empty graph.

Proof. For convenience, let $G' = G_{u,v}$. By Lemma 2.1(1), for all non-negative integers k, we have $i(G; k) = i(G - u; k) + i(G - u - N_G(u); k - 1)$.

(1) By $i(G; k) = i(G - u; k) + i(G - u - N_G(u); k - 1)$, we have

$$\begin{aligned} i(\overline{G}; k) &= i(\overline{G} - v; k) + i(\overline{G} - v - N_{\overline{G}}(v); k - 1) \\ &= i(\overline{G} - v - u; k) + i(\overline{G} - v - u - N_{\overline{G}-v}(u); k - 1) + i(\overline{G} - v - N_{\overline{G}}(v); k - 1) \end{aligned}$$

and

$$\begin{aligned} i(\overline{G'}; k) = i(\overline{G'} - v - u; k) + i(\overline{G'} - v - u - N_{\overline{G'}-v}(u); k - 1) \\ + i(\overline{G'} - v - N_{\overline{G'}}(v); k - 1). \end{aligned}$$

Obviously,

$$\begin{aligned} \overline{G} - v - u &= \overline{G'} - v - u, \\ N_{\overline{G}-v}(u) &= N_{\overline{G'}-v}(u) \cup (N_G(v) \setminus \{u\}), \\ N_{\overline{G'}}(v) &= N_{\overline{G}}(v) \cup (N_G(v) \setminus \{u\}), \\ N_{\overline{G}}(v) &= [N_{\overline{G}}(u) \setminus (N_G(v) \setminus \{u\})] \cup [N_G(u) \setminus \{v\}], \\ \overline{G'} - v - N_{\overline{G'}}(v) &= \overline{G} - v - N_{\overline{G}}(v) - (N_G(v) \setminus \{u\}), \\ N_{\overline{G}}(v) \cap (N_G(v) \setminus \{u\}) &= \emptyset, \text{ and} \\ N_{\overline{G'}-v}(u) \cap (N_G(v) \setminus \{u\}) &= \emptyset. \end{aligned}$$

So,

$$\begin{aligned} & i(\overline{G'}; k) - i(\overline{G}; k) \\ &= i(\overline{G} - v - u; k) + i(\overline{G} - v - N_{\overline{G}}(v) - (N_G(v) \setminus \{u\}); k - 1) \\ &\quad + i(\overline{G} - v - u - [N_{\overline{G}}(u) \setminus (N_G(v) \setminus \{u\})]; k - 1) - i(\overline{G} - v - u; k) \\ &\quad - i(\overline{G} - v - u - N_{\overline{G}-v}(u); k - 1) - i(\overline{G} - v - N_{\overline{G}}(v); k - 1) \\ &= i_{N_G(v) \setminus \{u\}}(\overline{G} - v - u - [N_{\overline{G}-v}(u) \setminus (N_G(v) \setminus \{u\})]; k - 1) \\ &\quad - i_{N_G(v) \setminus \{u\}}(\overline{G} - v - N_{\overline{G}}(v); k - 1) \end{aligned}$$

$$
\begin{aligned}
&= i_{N_G(v)\setminus\{u\}}(\overline{G} - v - u - [N_{\overline{G}-v}(u) \setminus (N_G(v) \setminus \{u\})]; k-1) \\
&\qquad - i_{N_G(v)\setminus\{u\}}(\overline{G} - v - u - N_{\overline{G}}(v); k-1) \\
&\qquad - i_{N_G(v)\setminus\{u\}}(\overline{G} - v - u - N_{\overline{G}}(v) - N_{\overline{G}-v-N_{\overline{G}}(v)}(u); k-2) \\
&= i_{N_G(v)\setminus\{u\}, N_G(u)\setminus\{v\}}(\overline{G} - v - u - [N_{\overline{G}-v}(u) \setminus (N_G(v) \setminus \{u\})]; k-1) \geq 0.
\end{aligned}
$$

Obviously, if

$$i_{N_G(v)\setminus\{u\}, N_G(u)\setminus\{v\}} \left(\overline{G} - v - u - [N_{\overline{G}-v}(u) \setminus (N_G(v) \setminus \{u\})]; k-1\right) = 0,$$

$\overline{G}[N_{G-u}(v) \cup N_G(u) \setminus \{v\}]$ is not an empty graph. Conversely, if graph $\overline{G}[N_{G-u}(v) \cup N_G(u) \setminus \{v\}]$ is not an empty graph,

$$i_{N_G(v)\setminus\{u\}, N_G(u)\setminus\{v\}} \left(\overline{G} - v - u - [N_{\overline{G}-v}(u) \setminus (N_G(v) \setminus \{u\})];\ k-1\right) = 0.$$

(2) By $i(G; k) = i(G - u; k) + i(G - u - N_G(u); k-1)$, we can similarly get:

$$
\begin{aligned}
i(G; k) &= i(G - u; k) + i(G - u - N_G(u); k-1) \\
&= i(G - u - v; k) + i(G - u - v - N_{G-u}(v); k-1) + i(G - u - N_G(u); k-1)
\end{aligned}
$$

and

$$
\begin{aligned}
i(G'; k) = i(G' - u - v; k) &+ i(G' - u - v - N_{G'-u}(v); k-1) \\
&+ i(G' - u - N_{G'}(u); k-1).
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
G - u - v &= G' - u - v, \\
N_{G'-u}(v) &= \emptyset, \\
N_G(u) \cap N_{G-u}(v) &= \emptyset, \text{ and} \\
G' - u - N_{G'}(u) &= G - u - N_G(u) - N_{G-u}(v).
\end{aligned}
$$

So,

$$
\begin{aligned}
i(G'; k) - i(G; k) &= i(G - u - v; k) + i(G - u - N_G(u) - N_{G-u}(v); k-1) \\
&\qquad + i(G - u - v; k-1) - i(G - u - v - N_{G-u}(v); k-1) \\
&\qquad - i(G - u - v; k) - i(G - u - N_G(u); k-1) \\
&= i_{N_{G-u}(v)}(G - u - v; k-1) - i_{N_{G-u}(v)}(G - u - N_G(u); k-1) \\
&= i_{N_{G-u}(v), N_G(u)\setminus\{v\}}(G - u - v; k-1) \geq 0.
\end{aligned}
$$

Obviously, if

$$i_{N_{G-u}(v), N_G(u)\setminus\{v\}}(G - u - v; k-1) = 0,$$

$G[N_{G-u}(v) \cup N_G(u) \setminus \{v\}]$ is not an empty graph. Conversely, if graph $G[N_{G-u}(v) \cup N_G(u) \setminus \{v\}]$ is not an empty graph,

$$i_{N_{G-u}(v), N_G(u)\setminus\{v\}}(G - u - v; k-1) = 0. \qquad \square$$

3 The Nordhaus-Gaddum-type inequality for trees

In this section, we consider a tree T with connected complement $\overline{T}$, then we obtain the minimum and maximum value of $i(T) + i(\overline{T})$ and characterize the extremal graph.

Lemma 3.1 ([15]). *The star S_n has the maximal Merrifield-Simmons index for all trees with n vertices. And, the path P_n has the minimal Merrifield-Simmons index for all trees with n vertices.*

For the proof, we give an equality involving $i(T) + i(\overline{T})$ as follows.

Lemma 3.2. *Let T be a tree of order n with connected complement $\overline{T}$. Then*

$$i(T) + i(\overline{T}) = 2n + i(T).$$

Proof. For connected complement $\overline{T}$ and all non-negative integers $k \geq 3$, it is easy to verify $i(\overline{T}; k) = 0$ and $i(\overline{T}; 2) = |E(T)| = n - 1$. Therefore

$$i(T) + i(\overline{T}) = i(T) + 1 + n + i(\overline{T}; 2) = 2n + i(T). \qquad \square$$

Now we give the Nordhaus-Gaddum-type inequality of a tree for $i(T)$.

Theorem 3.3. *Let T be a tree of order n with connected complement $\overline{T}$, then*

$$i(T) + i(\overline{T}) \geq 2n + F_{n+1}$$

with equality if and only if $T \cong P_n$, where F_{n+1} is the Fibonacci number.

Proof. By Lemma 3.1 and Lemma 3.2 graph which reaches the minimum value of $i(T) + i(\overline{T})$.

And, $i(P_n)$ is equal to the Fibonacci number F_{n+1}, then $i(T) + i(\overline{T}) \geq 2n + F_{n+1}$. $\square$

Theorem 3.4. *Let T be a tree of order n with connected complement $\overline{T}$, then*

$$i(T) + i(\overline{T}) \leq 2 + 2n + 2^{n-3} + 2^{n-2}$$

with equality if and only if $T \cong S_{2,n-2}$.

Proof. If T and $\overline{T}$ are connected graphs, then the star S_n is not the extremal graph which reaches the maximum value of $i(T) + i(\overline{T})$. So we assume $D(T) \geq 3$.

Let $P = v_0 v_1 \ldots v_{D(T)}$ be a diametrical path of tree T. By Theorem 2.2, we have

$$i(T_{D(T)-1,D(T)-2}) + i(\overline{T_{D(T)-1,D(T)-2}}) > i(T) + i(\overline{T}).$$

Obviously, graph $T_{D(T)-1,D(T)-2}$ is a tree of order n.

Therefore, for the tree of order n with connected complement, by shortening the diametrical path of a tree, we can get the extremal graph the double star $S_{p,q}$ which reaches the maximal value of $i(T) + i(\overline{T})$. For the double star $S_{p,q}$ of order n, we have

$$\begin{aligned} i(S_{p,q}) + i(\overline{S_{p,q}}) &= 2 + 2n + \sum_{i=2}^{n-2} \binom{n-2}{i} + \sum_{i=1}^{p-1} \binom{p-1}{i} + \sum_{i=1}^{q-1} \binom{q-1}{i} + p + q - 1 \\ &= 2n + 2^{n-2} + 2^{p-1} + 2^{q-1} \\ &\leq 2 + 2n + 2^{n-3} + 2^{n-2} \end{aligned}$$

with equality if and only if $p = n-2$ or $q = n-2$. So $i(T) + i(\overline{T}) \leq 2 + 2n + 2^{n-3} + 2^{n-2}$ with equality if and only if $T \cong S_{2,n-2}$. $\square$

4 The Nordhaus-Gaddum-type inequality for unicyclic graphs

In this section, we consider a unicyclic graph G of order n with connected complement $\overline{G}$, then we obtain the minimum and maximum value of $i(G) + i(\overline{G})$ and characterize the extremal graph. Obviously, if $n < 5$, any complement $\overline{G}$ is not connected. We need to consider the case when $n \geq 5$.

Lemma 4.1 ([14]). *If G is a unicyclic graph of order n, then*

(1) $i(G) \geq F_{n-1} + F_{n+1}$ and equality occurs if and only if $G \cong C_n$ or $G \cong L_{n,3}$, where $L_{n,3}$ is the unicyclic graph of order n obtained from the two vertex disjoint graphs C_3 and P_{n-3} by adding an edge joining a vertex of C_3 to an endvertex of P_{n-3}.

(2) $i(G) \leq 3 \times 2^{n-3} + 1$ and equality holds if and only if G is a 4-cycle or $G \cong H_{n,3}$, where $H_{n,3}$ is the unicyclic graph of order n constructed by attaching $n-3$ leaves to one vertex on a cycle of length 3.

For the proof, we give an equality about $i(G) + i(\overline{G})$ as follows.

Lemma 4.2. *Let G be a unicyclic graph of order $n \geq 5$ with connected complement $\overline{G}$, then*

$$i(G) + i(\overline{G}) = 1 + 2n + i(\overline{G}; 3) + i(G).$$

Proof. For connected complement $\overline{G}$ and all non-negative integers $k \geq 4$, it is easy to verify $i(\overline{G}; k) = 0$ and $i(\overline{G}; 2) = |E(G)| = n$. Therefore

$$i(G) + i(\overline{G}) = i(G) + 1 + n + i(\overline{G}; 2) + i(\overline{G}; 3) = 1 + 2n + i(\overline{G}; 3) + i(G). \qquad \square$$

Now we give the Nordhaus-Gaddum-type inequality of a unicyclic graph for $i(G)$.

Theorem 4.3. *Let G be a unicyclic graph of order $n \geq 5$ with connected complement $\overline{G}$, then*

$$i(G) + i(\overline{G}) \geq 1 + 2n + F_{n-1} + F_{n+1}$$

with equality if and only if $G \cong C_n$, where F_{n+1} is the Fibonacci number.

Proof. Obviously, $i(\overline{L_{n,3}}; 3) = 1 > i(\overline{C_n}; 3) = 0$, and the complement of graph C_n is a connected graph. Then by Lemma 4.1(1) and Lemma 4.2, we have $i(L_{n,3}) = i(C_n)$ and

$$\begin{aligned} i(G) + i(\overline{G}) &= 1 + 2n + i(\overline{G}; 3) + i(G) \\ &\geq 1 + 2n + i(\overline{C_n}; 3) + i(C_n) \\ &= 1 + 2n + F_{n-1} + F_{n+1}. \end{aligned} \qquad \square$$

In order to formulate our results, some graphs need to be defined. Let O_{x_1,x_2,x_3} denote a unicyclic graph on n vertices created from a cycle $C_3 = v_1v_2v_3$ by attaching x_i $(i = 1, 2, 3)$ pendent vertices to v_i such that $x_1 + x_2 + x_3 + 3 = n$ and $x_1 \geq x_2 \geq x_3$, $x_2 \geq 1$.

Let U_{y_1,y_2} denote a unicyclic graph on n vertices created from a cycle $C_3 = v_1v_2v_3$ by attaching y_1 pendent vertices $u_1, u_2, \ldots, u_{y_1}$ to v_1 and attaching y_2 pendent vertices to u_1 such that $y_1 + y_2 + 3 = n$ and $y_1 \geq 1, y_2 \geq 2$.

Theorem 4.4. *Let G be a unicyclic graph of order $n \geq 5$ with connected complement $\overline{G}$, then*

$$i(G) + i(\overline{G}) \leq 4 + 2n + 2^{n-4} + 2^{n-2}$$

with equality if and only if $G \cong O_{n-4,1,0}$.

Proof. If the cycle in G is of length greater than three, then by applying the transformation in Theorem 2.2 to the cycle, there is a unicyclic graph L_1 with a triangle such that $i(L_1) + i(\overline{L_1}) > i(G) + i(\overline{G})$.

Let $\mathcal{H}$ denote the set of all unicyclic graphs H with a triangle. Then for all $H \in \mathcal{H}$ by Lemma 4.2 we have $i(H) + i(\overline{H}) = 1 + 2n + i(\overline{H}; 3) + i(H) = 2 + 2n + i(H)$. The maximum value of $i(H)+i(\overline{H})$ is equal to the maximum value of $i(H)$. By Lemma 4.1(2), we know that the graph $H_{n,3}$ is the extremal graph which obtains the maximum value $i(H)$, but the graph $\overline{H_{n,3}}$ is not connected. So we calculate the second maximum value of $i(H)$.

Case 1: There is a vertex in H with distance at least two to the 3-cycle.

By the transformation in Theorem 2.2, we get there is a graph L_2 with $i(L_2) > i(H)$, where $L_2 \cong O_{x_1,x_2,x_3}$ or $L_2 \cong U_{y_1,y_2}$.

$$\begin{aligned}
i(U_{y_1,y_2}) &= i(U_{y_1,y_2} - v_3) + i(U_{y_1,y_2} - N_{U_{y_1,y_2}}[v_3]) \\
&= i(S_{y_1+1,y_2+1}) + i(K_{y_1-1} \cup S_{y_2+1}) \\
&= 3 \times 2^{n-4} + 2^{y_2} + 3 \times 2^{y_1-1} \\
&\leq 3 \times 2^{n-4} + 2 + 3 \times 2^{n-5} \\
&< 2 + 2^{n-4} + 2^{n-2} = i(O_{n-4,1,0})
\end{aligned}$$

Case 2: $G \cong O_{x_1,x_2,x_3}$

$$\begin{aligned}
i(O_{x_1,x_2,x_3}) &= i(O_{x_1,x_2,x_3} - v_3) + i(O_{x_1,x_2,x_3} - N_{O_{x_1,x_2,x_3}}[v_{v_3}]) \\
&= i(S_{x_1+1,x_2+2} \cup \overline{K_{x_3}}) + i(\overline{K_{x_1+x_2}}) \\
&= 2^{n-3} + 2^{x_1+x_3} + 2^{x_2+x_3} + 2^{n-3-x_3} \\
&\leq 2^{n-3} + 2^{x_1+x_3} + 2^{x_2} + 2^{n-3} = i(O_{x_1+x_3,x_2,0}) \\
&\leq 2^{n-3} + 2^{n-4} + 2 + 2^{n-3} = i(O_{n-4,1,0})
\end{aligned}$$

with equality if and only if $x_3 = 0$ and $x_2 = 1$. Obviously, the graph $\overline{O_{n-4,1,0}}$ is connected. So $i(G) + i(\overline{G}) \leq i(H) + i(\overline{H}) \leq 4 + 2n + 2^{n-4} + 2^{n-2}$ with equality if and only if $G \cong O_{n-4,1,0}$. □

5 The Nordhaus-Gaddum-type inequality for connected graphs

In this section, we obtain a lower bound on $i(G)$ in terms of the Wiener polarity index. And, for a connected graph G with connected complement $\overline{G}$, we obtain a minimum value of $i(G) + i(\overline{G})$ and characterize the extremal graph. Also, we pose a conjecture about which graph gets the maximum value of $i(G) + i(\overline{G})$.

Lemma 5.1 ([7]). *Let G be a connected graph with connected complement $\overline{G}$, then*

$$W_p(G) + W_p(\overline{G}) \geq D(G) + D(\overline{G}) - 4.$$

Moreover, equality holds if and only if $G \cong P_n$ or $G \cong G^{**}$ or $D(G) = D(\overline{G}) = 2$. The graph G^{**} of order $n \geq 5$ is obtained from a path P_4 by joining each vertex of H_{n-4} to each internal vertex of the path P_4 such that $V(G^{**}) \backslash V(P_4) = V(H_{n-4})$, where H_{n-4} is any graph of order $n - 4$.

In order to get the lower bound on the $i(G) + i(\overline{G})$, we give a lower bound on the $i(G)$ in terms of the Wiener polarity index.

Lemma 5.2. *Let G be a connected graph of order n and $D(G) \geq 2$. Then*

$$i(G) \geq 2 + n + 2W_p(G)$$

with equality if and only if $G \cong G_n$ or $G \cong B_{2,n-2}$, where $G_n = K_n - e, e \in E(K_n)$, $B_{2,n-2}$ is a graph on $n \geq 3$ vertices obtained from P_2 and K_{n-2} by coinciding any vertex of P_2 with that of K_{n-2}.

Proof. If $D(G) = 2$, then $W_p(G) = 0$. Let $P = uxy$ be a diametrical path, then $\{u, y\}$ is a 2-independent set of G. Therefore

$$i(G) \geq i(G; 0) + i(G; 1) + i(G; 2) \geq 1 + n + 1 = 2 + n + 2W_p(G)$$

follows readily. Suppose that equality is attained. Then G has only one 2-independent set and no k-independent set, where $k \geq 3$. Also, $D(G) = 2$. Then, we have $G \cong G_n$. Conversely, if $G \cong G_n$, then the equality is attained.

For the case $D(G) \geq 3$: Suppose that u and v are a pair of vertices in G such that $d_G(u, v) = 3$. Let $uxyv$ be a path of length 3 connecting u and v in G. Then $\{u, y\}$, $\{u, v\}$ and $\{x, v\}$ are 2-independent sets of G. Therefore, every pair of vertices at distance 3 corresponds to three 2-independent sets in G. Moreover, for any two paths connecting distinct pair vertices at distance 3, they correspond to two different 2-independent sets and one same 2-independent set, otherwise they correspond to three different 2-independent sets. From this it follows that

$$i(G; 2) \geq 2W_p(G) + 1.$$

Therefore, by the definition of Merrifield-Simmons index, $i(G) \geq i(G; 0) + i(G; 1) + i(G; 2) \geq 1 + n + 2W_p(G) + 1 = 2 + n + W_p(G)$, (2) follows readily.

Now, we check the equality condition. If $i(G; 2) = 2W_p(G) + 1$, by analysis, then any two paths of distinct pair vertices at distance 3 correspond to two different 2-independent sets and one same 2-independent set. If $i(G; 3) = 0$, $D(G) = 3$. So $G \cong B_{2,n-2}$.

Conversely, if $G \cong B_{2,n-2}$, then we clearly have $i(G; 2) = 2W_p(G) + 1$ and $i(G; 3) = 0$. So, the equality is attained if and only if $G \cong B_{2,n-2}$. □

Theorem 5.3. *Let G be a connected graph with connected complement $\overline{G}$, then*

$$i(G) + i(\overline{G}) \geq 2n + 2D(G) + 2D(\overline{G}) - 4$$

with equality if and only if $G \cong P_4$.

Proof. By Lemma 5.1 and Lemma 5.2, the result is obvious. □

Conjecture 5.4. *Let G be a connected graph with connected complement $\overline{G}$, then*

$$i(G) + i(\overline{G}) \leq 2 + 2n + 2^{n-3} + 2^{n-2}$$

with equality if and only if $G \cong S_{2,n-2}$.

For a connected graph G with connected complement $\overline{G}$, it is difficult to get the value of $\max\{i(G)+i(\overline{G})\}$. For $n \leq 5$, by enumeration and calculation, we can find $\max\{i(G)+i(\overline{G})\} = \max\{i(T)+i(\overline{T})\} = i(S_{2,n-2})+i(\overline{S_{2,n-2}})$. If we do not consider the connectivity of the graph, we can get:

Theorem 5.5. *Let G be a simple graph of order n. If we do not consider the connectivity of the graph, then*

$$i(G)+i(\overline{G}) \leq 1+n+2^n$$

with equality if and only if $G \cong K_n$ or $\overline{G} \cong K_n$.

Proof. Let $k, m \in \mathbb{N}$. Without loss of generality, we assume that $\alpha(G) > \alpha(\overline{G})$.

Every pair of vertices are not a 2-independent set of $\overline{G}$, which compose of a 2-independent set of G. Moreover, for any two vertices which do not compose of a 2-independent set of $\overline{G}$, they compose of a 2-independent set of G. Then, we have $i(G;2)+i(\overline{G};2) = C_n^2$. Suppose $i(G;3) = k$. Since every three vertices which are a 3-independent set of G are not a 3-independent set of $\overline{G}$, we have $i(\overline{G};3) \leq C_n^3 - k$. Therefore, we have

$$i(G)+i(\overline{G}) \leq 2+2n+C_n^2+C_n^3+\ldots+C_n^{\alpha(G)} = 1+n+2^n - \sum_{i=\alpha(G)+1}^{n} C_n^i. \quad (5.1)$$

Now, we check the equality condition in (1). If $i(\overline{G};m) = C_n^m - i(G;m)$, then for any m vertices which are not an independent set of $\overline{G}$, they are an m-independent set of G. Then, G is the empty graph, and $\overline{G} \cong K_n$. By the definition of G and $\overline{G}$, we have

$$i(G)+i(\overline{G}) \leq 1+n+2^n. \qquad \square$$

Obviously, for connected graph G with a connected complement, $1+n+2^n$ is an upper bound on the maximum value of $i(G)+i(\overline{G})$. And, the lower bound on the maximum value of $i(G)+i(\overline{G})$ is $i(S_{2,n-2})+i(\overline{S_{2,n-2}})$. The difference between the upper bound and the lower bound is $5 \cdot 2^{n-3} - n - 1$.

6 Conclusions

In this paper, we firstly establish a transformation on a simple graph that increases $i(G)$ and $i(\overline{G})$. Secondly, we prove the path P_n and the double star $S_{2,n-2}$ are the extremal graphs which respectively reach the minimum and maximum value of $i(T)+i(\overline{T})$. Then, for unicyclic graphs G, we get that the cycle C_n and the graph $O_{n-4,1,0}$ are the extremal graphs which respectively reach the minimum and maximum value of $i(G)+i(\overline{G})$. Finally, for connected graphs G, we find $i(G) \geq 2+n+2W_p(G)$ with equality if and only if $G \cong G_n$ or $G \cong B_{2,n-2}$. Then we obtain $i(G)+i(\overline{G}) \geq 2n+2D(G)+2D(\overline{G})-4$ with equality if and only if $G \cong P_4$. Also, we conjecture that the extremal graph which reaches the maximum value of $i(G)+i(\overline{G})$ is $S_{2,n-2}$. Which graph gives the maximum value on $i(G)+i(\overline{G})$ remains an open problem.

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THE ART OF DISCRETE AND APPLIED MATHEMATICS

Norman W. Johnson (12 November 1930 to 13 July 2017)

Norman W. Johnson was born on November 12, 1930 in Chicago, where his father had a bookstore and ran a local newspaper. He attended Carleton College, graduating in 1953. He did alternative service as a conscientious objector then went on to earn a Master's degree from the University of Pittsburgh. He then went to the University of Toronto to work with H. S. M. Coxeter in geometry. After receiving his PhD in 1966 he accepted a position in the Mathematics Department of Wheaton College in Massachusetts and taught there until his retirement in 1998. He is known for the "Johnson Solids," the ninety-two non-uniform convex solids with regular faces that he identified in a 1966 article [1] and speculated was the complete set. He also published a number of other articles on various aspects of polytopes. He died on July 13, 2017, but his completed book, *Geometries and Transformations* [2], is forthcoming from Cambridge University Press. His nearly-completed work on uniform polytopes, the subject of his dissertation, will be appearing.

Asia Ivić Weiss and Eva Marie Stehle

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Professor Wilfried Imrich awarded honorary doctorate at the University of Maribor

On January 31, 2018, Prof. Emer. Dr. Wilfried Imrich from the Montanuniversität Leoben, Austria, became a Honorary Doctor of the University of Maribor. The title was awarded to him for his scientific achievements and contributions to the development of the University of Maribor. The university awards this title since 1979, Wilfried Imrich is the first mathematician to receive this prestigious title. Moreover, he is the first foreign mathematician with the honorary doctor title at a Slovenian university.

The collaboration between Wilfried Imrich and the Slovenian graph theory school started when in the 1980s he established together with Tomo Pisanski the Leoben-Ljubljana seminar, which is still going on. The rest is then history. As a coincidence, the 30th Ljubljana-Leoben Graph Theory Seminar that happened in September 2017, took place for the first time in Maribor. In the last two decades, Wilfried was a frequent participant of the Seminar on discrete mathematics that is held at the Faculty of Natural Sciences and Mathematics in Maribor. He has written three books and close to fifty papers with a dozen co-authors from Maribor. The fact that at the present 16 academic descendants of Prof. Imrich have positions at the University of Maribor indicates that the award was more than deserved.

Boštjan Brešar and Sandi Klavžar

The Art of Discrete and Applied Mathematics

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This is a call for submission of papers for a special issue of the journal *The Art of Discrete and Applied Mathematics* (ADAM), on topics presented or related to talks given at the TSIMF workshop on 'Symmetries of Graphs and Networks' held at Sanya (China) in January 2018. The Sanya workshop added to the series of conferences and workshops on symmetries of graphs and networks initiated at BIRS (Canada) in 2008 and progressed in Slovenia every two years from 2010 to 2016.

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Marston Conder and Yan-Quan Feng

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