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On graphs with the smallest eigenvalue at least $-1-\sqrt{2}$, part III

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Abstract

There are many results on graphs with the smallest eigenvalue at least -2. In order to study graphs with the eigenvalues at least $-1-\sqrt{2}$, R. Woo and A. Neumaier introduced Hoffman graphs and $\mathcal{H}$-line graphs. They proved that a graph with the sufficiently large minimum degree and the smallest eigenvalue at least $-1-\sqrt{2}$ is a slim $\{[\mathfrak{h}_2],[\mathfrak{h}_5],[\mathfrak{h}_7],[\mathfrak{h}_9]\}$-line graph. After that, T. Taniguchi researched on slim $\{[\mathfrak{h}_2],[\mathfrak{h}_5]\}$-line graphs. As an analogue, we reveal the condition under which a strict $\{[\mathfrak{h}_1],[\mathfrak{h}_4],[\mathfrak{h}_7]\}$-cover of a slim $\{[\mathfrak{h}_7]\}$-line graph is unique, and completely determine the minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs.

Keywords: Hoffman graph, line graph, smallest eigenvalue.

Math. Subj. Class.: 05C50, 05C75

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1 Introduction

Throughout this paper, we will consider only undirected graphs without loops or multiple edges, and denote by $\lambda_{\min}(\Gamma)$ and $\delta(\Gamma)$ the minimum eigenvalue and the minimum degree of a graph Γ, respectively.

P. J. Cameron, J. M. Goethals, J. J. Seidel and E. E. Shult have characterized generalized line graphs as the graphs with the smallest eigenvalue at least -2 except for finitely many graphs with at most 36 vertices in [3]. After that, A. Hoffman proved the following theorem in [6].

Theorem 1.1. *There exists an integer valued function f defined on the intersection of the half-open interval $(-1-\sqrt{2}, -2]$ and the set of the smallest eigenvalues of graphs, such that if Γ is a connected graph with $\delta(\Gamma) \geq f(\lambda_{\min}(\Gamma))$ then*

(i) if $-1 \geq \lambda_{\min}(\Gamma) > -2$ then Γ is a complete graph and $\lambda_{\min}(\Gamma) = -1$.

(ii) if $-2 \geq \lambda_{\min}(\Gamma) > -1-\sqrt{2}$ then Γ is a generalized line graph and $\lambda_{\min}(\Gamma) = -2$.

In [12], R. Woo and A. Neumaier introduced Hoffman graphs and $\mathcal{H}$-line graphs, where $\mathcal{H}$ is a family of isomorphism classes of Hoffman graphs, to extend the result of A. Hoffman, and proved Theorem 1.2. Moreover, they raised the problem [12, Open problem 3] to reveal the list of minimal forbidden graphs for the slim $\{[\mathfrak{h}_2], [\mathfrak{h}_5], [\mathfrak{h}_7], [\mathfrak{h}_9]\}$-line graphs. These Hoffman graphs and some ones that appear in this paper and [12] are listed in Figure 1 (here, the names $\mathfrak{h}_1, \mathfrak{h}_2, \ldots$ depend on [12]).

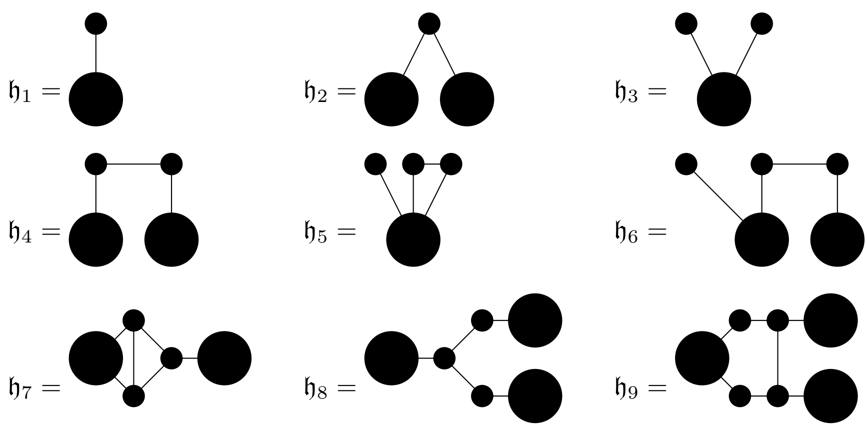


Figure 1: Hoffman graphs with slim (resp. fat) vertices depicted as small (resp. large) black dots.

Theorem 1.2. *Let α_4 (≈ -2.4812) be the smallest root of the polynomial $x^3 + 2x^2 - 2x - 2$. There exists an integer valued function f defined on the intersection of the half-open interval $(\alpha_4, -1-\sqrt{2}]$ and the set of the smallest eigenvalues of graphs, such that if Γ is a graph with $\lambda_{\min}(\Gamma) \in (\alpha_4, -1-\sqrt{2}]$ and $\delta(\Gamma) \geq f(\lambda_{\min}(\Gamma))$, then Γ is an $\{[\mathfrak{h}_2], [\mathfrak{h}_5], [\mathfrak{h}_7], [\mathfrak{h}_9]\}$-line graph.*

Since it is difficult to solve the open problem, T. Taniguchi considered a partial problem. In [11], he completely determined the 38 minimal forbidden graphs for the slim $\{[\mathfrak{h}_2], [\mathfrak{h}_5]\}$-line graphs by using Theorem 1.3 [10].

Theorem 1.3. *A slim $\{[\mathfrak{h}_2], [\mathfrak{h}_5]\}$-line graph with at least 8 vertices has a unique strict $\{[\mathfrak{h}_2], [\mathfrak{h}_3], [\mathfrak{h}_5]\}$-cover up to equivalence.*

As an analogue of his result, we reveal the minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graph. In Section 2, we introduce a part of the basic theory of Hoffman graphs summarized in detail in [7]. In Section 3, we introduce minimal forbidden graphs. In Section 4, we aim to prove Theorem 4.11 which reveals the necessary and sufficient condition that a strict $\{[\mathfrak{h}_1], [\mathfrak{h}_4], [\mathfrak{h}_7]\}$-cover of a slim $\{[\mathfrak{h}_7]\}$-line graph becomes unique up to equivalence. Furthermore, when the condition is not satisfied, the theorem shows the shape of the slim $\{[\mathfrak{h}_7]\}$-line graph and indicates its strict $\{[\mathfrak{h}_1], [\mathfrak{h}_4], [\mathfrak{h}_7]\}$-covers are exactly two up to equivalence. This helps us to examine the minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs. In order to prove our main result Theorem 5.1, in which we determine the minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs, we computed the minimal forbidden graphs with at most 9 vertices by the software MAMGA [2]. In Section 5, we determine the minimal forbidden graphs apart from those with at most 9 vertices.

2 Hoffman graphs

We introduce definitions related to Hoffman graphs. Details are in [7].

Definition 2.1. A *Hoffman graph* $\mathfrak{h}$ is a pair (H, μ) of a graph $H = (V, E)$ and a labelling map $\mu\colon V \to \{f, s\}$, satisfying the following conditions:

(i) every vertex with label f is adjacent to at least one vertex with label s;

(ii) vertices with label f are pairwise non-adjacent.

We call a vertex with label s a *slim vertex*, and one with label f a *fat vertex*. We denote by $V_s(\mathfrak{h})$ (resp. $V_f(\mathfrak{h})$) the set of slim (resp. fat) vertices of $\mathfrak{h}$.

For a vertex x of a Hoffman graph $\mathfrak{h}$, we denote by $N^f_{\mathfrak{h}}(x)$ (resp. $N^s_{\mathfrak{h}}(x)$) the set of neighbors labelled f (resp. s) of x, and set $N_{\mathfrak{h}}(x) = N^f_{\mathfrak{h}}(x) \cup N^s_{\mathfrak{h}}(x)$. For a set X of vertices of $\mathfrak{h}$, we let $N^f_{\mathfrak{h}}(X) := \bigcup_{x\in X} N^f_{\mathfrak{h}}(x)$ and $N^s_{\mathfrak{h}}(X) := \bigcup_{x\in X} N^s_{\mathfrak{h}}(x)$. We regard an ordinary graph H without labelling as a Hoffman graph (H, μ) without fat vertices, that is, $\mu(x) = s$ for any vertex x of H. Such a graph is called a *slim graph*.

Definition 2.2. A Hoffman graph $\mathfrak{h}' = (H', \mu')$ is called an *induced Hoffman subgraph* of a Hoffman graph $\mathfrak{h} = (H, \mu)$, if H' is an induced subgraph of H and $\mu|_{V(H')} = \mu'$. For a subset S of $V_s(\mathfrak{h})$, we denote by $\langle\langle S\rangle\rangle_{\mathfrak{h}}$ the induced Hoffman subgraph of $\mathfrak{h}$ by $S \cup N^f_{\mathfrak{h}}(S)$.

We denote by $\langle S\rangle_\Gamma$ the ordinary induced subgraph by S of a graph Γ for a subset S of $V(\Gamma)$. For a Hoffman graph $\mathfrak{h}$, $\langle V_s(\mathfrak{h})\rangle_{\mathfrak{h}}$ is called the *slim subgraph* of $\mathfrak{h}$. The *diameter* of a graph is the maximum distance between two distinct vertices. Let Γ be a graph and C be a subset of $V(\Gamma)$. Then, C is a *clique* in Γ if the induced subgraph $\langle C\rangle_\Gamma$ is a complete graph. The size of the largest clique in Γ is called the *clique number*. A partition $\pi = \{C_1, C_2, \ldots, C_t\}$ of $V(\Gamma)$ is called a *clique partition* if all cells C_i are cliques. Focusing on cliques is useful for discovering the structure of line graphs. Also in this paper, we may focus on clique numbers and clique partitions.

Definition 2.3. Let $\mathfrak{h}$ be a Hoffman graph, and let $\mathfrak{h}^1$ and $\mathfrak{h}^2$ be two induced Hoffman subgraphs of $\mathfrak{h}$. The Hoffman graph $\mathfrak{h}$ is said to be the *sum* of $\mathfrak{h}^1$ and $\mathfrak{h}^2$, written as $\mathfrak{h} = \mathfrak{h}^1 \oplus \mathfrak{h}^2$, if the following conditions are satisfied:

(i) $V(\mathfrak{h}) = V(\mathfrak{h}^1) \cup V(\mathfrak{h}^2)$;

(ii) $V_s(\mathfrak{h}) = V_s(\mathfrak{h}^1) \cup V_s(\mathfrak{h}^2)$ and $V_s(\mathfrak{h}^1) \cap V_s(\mathfrak{h}^2) = \emptyset$;

(iii) if $x \in V_s(\mathfrak{h}^i)$, $y \in V_f(\mathfrak{h})$ for $i = 1$ or 2, and $x \sim y$, then $y \in V_f(\mathfrak{h}^i)$;

(iv) if $x \in V_s(\mathfrak{h}^1)$ and $y \in V_s(\mathfrak{h}^2)$, then x and y have at most one common fat neighbor, and they have one if and only if they are adjacent.

If $\mathfrak{h}$ is the sum of some two nonempty Hoffman graphs, then it is said to be *decomposable*. Otherwise, $\mathfrak{h}$ is said to be *indecomposable*.

Remark that the sum of Hoffman graphs satisfies commutative and associative laws.

Definition 2.4. Let $\mathfrak{h}$ and $\mathfrak{m}$ be Hoffman graphs, and let ϕ be a graph morphism from the underlying graph of $\mathfrak{h}$ to that of $\mathfrak{m}$. Mapping $\phi\colon \mathfrak{h} \to \mathfrak{m}$ is called a *morphism* if it preserves the labelling, that is, $\phi(V_s(\mathfrak{h})) \subset V_s(\mathfrak{m})$ and $\phi(V_f(\mathfrak{h})) \subset V_f(\mathfrak{m})$. If ϕ is a morphism and a graph isomorphism, then it is called an *isomorphism*, and $\mathfrak{h}$ and $\mathfrak{m}$ are said to be *isomorphic*, written as $\mathfrak{h} \simeq \mathfrak{m}$. Let $[\mathfrak{h}]$ denote the isomorphism class of $\mathfrak{h}$.

Definition 2.5. Let $\mathcal{H}$ be a family of isomorphism classes of Hoffman graphs. A Hoffman graph $\mathfrak{m}$ is called a *$\mathcal{H}$-line graph* if it is an induced subgraph of some Hoffman graph $\mathfrak{h} = \bigoplus_{i=1}^{n} \mathfrak{h}^i$, where $[\mathfrak{h}^i] \in \mathcal{H}$ for every i. In this case, $\mathfrak{m}$ is called a *slim $\mathcal{H}$-line graph* if $\mathfrak{m}$ is a slim graph, and $\mathfrak{h}$ is called a *strict $\mathcal{H}$-cover* of a graph Γ if $V_s(\mathfrak{h}) = V(\Gamma)$. Two strict $\mathcal{H}$-covers $\mathfrak{h}$ and $\mathfrak{h}'$ of a graph Γ are said to be *equivalent*, if there exists an isomorphism $\phi\colon \mathfrak{h} \to \mathfrak{h}'$ such that $\phi|_\Gamma$ is the identity automorphism of Γ.

Lemma 2.6. *Let $\mathcal{H}$ be a family of isomorphism classes of Hoffman graphs, and let $\mathcal{H}'$ be the family of the isomorphism classes of indecomposable induced Hoffman subgraphs by a nonempty set of slim vertices in a member of $\mathcal{H}$. Then, every slim $\mathcal{H}$-line graph has a strict $\mathcal{H}'$-cover.*

Proof. Let $\mathfrak{h} = \bigoplus_{i=1}^{n} \mathfrak{h}^i$, where $[\mathfrak{h}^i] \in \mathcal{H}$ for every i. Then, it holds that

$$\langle\langle S \rangle\rangle_{\mathfrak{h}} = \bigoplus_{i=1}^{n} \langle\langle S \cap V_s(\mathfrak{h}^i) \rangle\rangle_{\mathfrak{h}}$$

for a subset S of $V_s(\mathfrak{h})$. Therefore $\langle\langle S \rangle\rangle_{\mathfrak{h}}$ is a strict $\mathcal{H}'$-cover of the induced subgraph by S since every addend is the sum of some indecomposable induced Hoffman graphs of $\mathfrak{h}$. $\square$

3 Minimal forbidden graphs

In graph theory, various important families of graphs can be described by a set of graphs that do not belong to that family. This is the concept of so-called minimal forbidden graphs. First, we give the definition. Suppose that a family $\mathcal{G}$ of graphs is closed under the operation to take induced subgraphs, that is, $\mathcal{G}$ satisfies the condition that for a graph G in $\mathcal{G}$, any induced subgraph of G is also in $\mathcal{G}$. Then, we say that a graph F is a *minimal forbidden graph* for $\mathcal{G}$ if both of the following are satisfied:

(i) F is not in $\mathcal{G}$;

(ii) Every proper induced subgraph of F is in $\mathcal{G}$.

On the family of ordinary line graphs [1] and the family of slim $\{[\mathfrak{h}_2],[\mathfrak{h}_5]\}$-line graphs [11], their minimal forbidden graphs are revealed. Besides this, characterizations of forests, perfect graphs [4] and Threshold graphs [5] by minimal forbidden graphs are also known. In addition, Sumner [9] claimed that if Γ is a connected $K_{1,3}$-free graph of even order, then Γ has a 1-factor. As such, there are also known results that properties of a family of graphs in the case that "forbidden graphs" are specified in advance. As we can see from these results, revealing the minimal forbidden graphs is one way to understand families of graphs. Unfortunately not being finite, but we are able to reveal the minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs.

4 The condition that an $\{[\mathfrak{h}_1],[\mathfrak{h}_4],[\mathfrak{h}_7]\}$-strict cover of a slim $\{[\mathfrak{h}_7]\}$-line graph is unique up to equivalence

In this section, set $\mathcal{H}=\{[\mathfrak{h}_1],[\mathfrak{h}_4],[\mathfrak{h}_7]\}$. Note that every slim $\{[\mathfrak{h}_7]\}$-line graph has a strict $\mathcal{H}$-cover by Lemma 2.6. For exmaple, the graph Γ in Figure 2 is a slim $\{[\mathfrak{h}_7]\}$-line graph. Indeed, considering the sum $\mathfrak{h}=\mathfrak{h}_7\oplus\mathfrak{h}_1\oplus\mathfrak{h}_4$ of Hoffman graphs in Figure 9, we see that

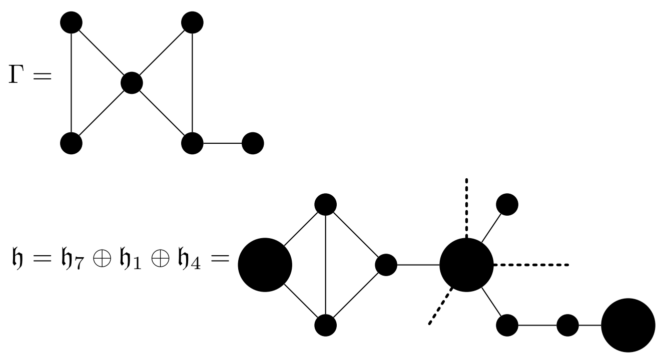


Figure 2: A slim $\{[\mathfrak{h}_7]\}$-line graph and its strict $\{[\mathfrak{h}_1],[\mathfrak{h}_4],[\mathfrak{h}_7]\}$-cover.

the slim subgraph of $\mathfrak{h}$ is the graph Γ. (In Figure 2, the dotted lines are used for convenience to show what kind of small Hoffman graphs the graph Γ is decomposed by. In addition, for two vertices x and y which belong to distinct addends, we omit the edge between x and y if they have a common fat neighbor since the existence of edge between x and y depends only on that of their common fat neighbor by Definition 2.3 (iv).) In addition, $\mathfrak{h}_1$ and $\mathfrak{h}_4$ are induced subgraphs of $\mathfrak{h}_7$, so the graph Γ is certainly a slim $\{[\mathfrak{h}_7]\}$-line graph. On the other hand, since $V_s(\mathfrak{h})=V(\Gamma)$ holds, the Hoffman graph $\mathfrak{h}$ is a strict $\{[\mathfrak{h}_1],[\mathfrak{h}_4],[\mathfrak{h}_7]\}$-cover of Γ.

Let $\mathfrak{h}=\bigoplus_{i=1}^{n}\mathfrak{h}^i$ where $[\mathfrak{h}^i]\in\mathcal{H}$ for every i. Then, we can regard $N_{\mathfrak{h}}^f$ as a mapping from $V_s(\mathfrak{h})$ to $V_f(\mathfrak{h})$ since every slim vertex is adjacent to exactly one fat vertex. For a slim vertex x of $\mathfrak{h}$, let $\mathfrak{h}(x)$ denote the addend $\mathfrak{h}^i$ containing x, and let $C_{\mathfrak{h}}(x)=N_{\mathfrak{h}}^s(N_{\mathfrak{h}}^f(x))$

and

$$\operatorname{cov}\mathfrak{h} := \{N^s_{\mathfrak{h}}(u) \mid u \in V_f(\mathfrak{h})\} = \{C_{\mathfrak{h}}(x) \mid x \in V_s(\mathfrak{h})\}.$$

Let x be a slim vertex of $\mathfrak{h}$. We show that $C_{\mathfrak{h}}(x)$ is a clique. First, we take $u \in V_f(\mathfrak{h})$ such that $N^f_{\mathfrak{h}}(x) = \{u\}$. We arbitrarily take two slim vertices y and z in $N^s_{\mathfrak{h}}(u)$ $(= C_{\mathfrak{h}}(x))$. It suffices to show that y and z are adjacent. If y and z are contained in the same indecomposable addend of $\mathfrak{h}$, then they are adjacent. Otherwise, so are they by Definition 2.3 (iv). Hence, the desired result follows. Note that $\operatorname{cov}\mathfrak{h}$ is a clique partition of $V_s(\mathfrak{h})$. Moreover, it holds clearly that $N^f_{\mathfrak{h}}|_\Delta = N^f_{\langle\langle\Delta\rangle\rangle_{\mathfrak{h}}}$ for any subset $\Delta \subset V_s(\mathfrak{h})$.

Lemma 4.1. *Let $\mathfrak{h} = \bigoplus_{i=1}^n \mathfrak{h}^i$, where $[\mathfrak{h}^i] \in \mathcal{H}$ for every i, and let C be a clique of the slim subgraph of $\mathfrak{h}$. Then, the following hold:*

(i) two distinct slim vertices x and y are adjacent if and only if $\mathfrak{h}(x) = \mathfrak{h}(y)$ or $N^f_{\mathfrak{h}}(x) = N^f_{\mathfrak{h}}(y)$;

(ii) $C \subset C_{\mathfrak{h}}(x)$ for any $x \in C$, or $C \subset V_s(\mathfrak{h}(y))$ for any $y \in C$;

(iii) If $C \subset C_{\mathfrak{h}}(x) \cap V_s(\mathfrak{h}(y))$ for some $x, y \in C$, then $|C| \leq 2$.

Proof. Statements (i) and (iii) hold clearly. Assume that $C \not\subset C_{\mathfrak{h}}(x)$ for some $x \in C$. There exists $y \in C$ such that $N^f_{\mathfrak{h}}(x) \neq N^f_{\mathfrak{h}}(y)$. Thus, $\mathfrak{h}(x) = \mathfrak{h}(y)$ holds by (i). Statement (ii) follows since $N^f_{\mathfrak{h}}(x) \neq N^f_{\mathfrak{h}}(z)$ or $N^f_{\mathfrak{h}}(x) \neq N^f_{\mathfrak{h}}(z)$ for each $z \in C$. □

We introduce some definitions to determine the strict $\mathcal{H}$-covers of a graph.

Definition 4.2. Let Γ be a graph, and let $\{C_i\}_{i \in I}$ be a partition of the vertex set of Γ. Then, define $n(x) = N_\Gamma(x) - C_i$ for $x \in C_i$. In addition, define $n^0(x) = \{x\}$ and

$$n^k(x) = n^{k-1}(x) \cup \bigcup_{y \in n^{k-1}(x)} n(y)$$

for a positive integer k. A vertex x of Γ is said to be *good* for the given partition $\{C_i\}_{i \in I}$ if x satisfies one of the following conditions:

(i) $n(x) = \emptyset$;

(ii) $n(x) = \{y\}$ for some y, and $n(y) = \{x\}$;

(iii) $n(x) = \{y, z\}$ for some y and z, $n(y) = \{x\}$, $n(z) = \{x\}$, and $y \sim z$;

(iv) $n(x) = \{y\}$ for some y, $n(y) = \{x, z\}$ for some z, $n(z) = \{y\}$, and $x \sim z$.

Furthermore, a set of vertices is said to be *good* if every element is good. Let $\mathcal{O}_\Gamma$ be the set of clique partitions for which every vertex is good.

We can regard cov as a mapping from the set of equivalent classes of strict $\mathcal{H}$-covers of Γ to $\mathcal{O}_\Gamma$, and Proposition 4.3 holds. It is clear that if $n(u)$ has a good vertex then u is good, and if u is good then $n(u)$ is good.

Proposition 4.3. *The mapping* cov *for a graph is bijective.*

Proof. We construct the inverse mapping of cov. Let $\{C_i\}_{i\in I} \in \mathcal{O}_\Gamma$. A Hoffman graph $\mathfrak{m}$ is defined as $V_s(\mathfrak{m}) := V(\Gamma)$, $V_f(\mathfrak{m}) := \{C_i\}_{i\in I}$ and

$$E(\mathfrak{m}) := E(\Gamma) \cup \{\{x, C\} \mid x \in V(\Gamma), \text{ and } C \in \{C_i\}_{i\in I} \text{ and } x \in C\}.$$

For $x \in V(\Gamma)$, define the induced Hoffman graph $\mathfrak{m}_x := \langle\langle n^2(x)\rangle\rangle_{\mathfrak{m}}$. It holds that

$$\mathfrak{m} = \bigoplus\{\mathfrak{m}_x \mid x \in V(\Gamma)\}, \qquad \text{and}$$
$$[\mathfrak{m}_x] = [\mathfrak{h}_1], [\mathfrak{h}_4] \text{ or } [\mathfrak{h}_7] \text{ for each vertex } x.$$

Hence, $\mathfrak{m}$ is a strict $\mathcal{H}$-cover of Γ. The mapping

$$\phi\colon \mathcal{O}_\Gamma \ni \{C_i\}_{i\in I} \mapsto \mathfrak{m} \in \text{ the set of strict } \mathcal{H}\text{-covers of } \Gamma$$

is the inverse mapping of the mapping cov. □

We have the following lemma:

Lemma 4.4. *Let* Γ *be a graph with a partition* $\{C_i\}_{i\in I}$ *of the vertex set. Then, a vertex* x *is good for* $\{C_i\}_{i\in I}$ *if and only if* x *is good for* $\{n^3(u) \cap C_i\}_{i\in I}$ *in* $\langle n^3(u)\rangle_\Gamma$.

Let Γ be a connected graph, and let K be a nonempty set of vertices. Then, let

$$\partial_{K,\Gamma}(x) = \partial_K(x) = \partial(x) := \min_{k\in K} d(x,k)$$

for $x \in V(\Gamma)$, where $d(x,y)$ is the distance between x and y. Define

$$\partial_{\max} = \max_{y\in V(\Gamma)} \partial_{K,\Gamma}(y),$$

and let $\Psi_\Gamma(K)$ denote the family

$$\big\{\{y \in \{x\} \cup N(x) \mid \partial_{K,\Gamma}(y) \geq \partial_{K,\Gamma}(x)\} \mid x \in V(\Gamma) \text{ and } \partial_{K,\Gamma}(x) \in 2\mathbb{N}+1\big\} \cup \{K\}$$

of sets of vertices. If $K \in \operatorname{cov}\mathfrak{h}$ then $\Psi_\Gamma(K) = \operatorname{cov}\mathfrak{h}$ for every strict $\mathcal{H}$-cover $\mathfrak{h}$ of Γ. This means that we can restore the clique partition if we find a member of a partition in $\mathcal{O}_\Gamma$. We have the following lemmas:

Lemma 4.5. *Let* Γ *be a graph with a clique* K*. If* $\Psi_\Gamma(K)$ *is a partition of* $V(\Gamma)$ *and* Γ *has no induced subgraph isomorphic to* $K_{1,3}$*, then* $\Psi_\Gamma(K)$ *is a clique partition.*

Lemma 4.6. *Let* Γ *be a connected slim* $\{[\mathfrak{h}_7]\}$*-line graph with a clique* C *of size* c*. Let* $\mathfrak{h}$ *be a strict* $\mathcal{H}$*-cover of* Γ*. If the following (i) or (ii) holds, then* $C_\mathfrak{h}(x)$ *is the maximal clique containing* C *for any* $x \in C$*, and a strict* $\mathcal{H}$*-cover of* Γ *is unique up to equivalence.*

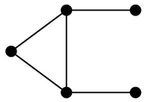
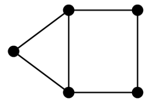
(i) $c \geq 4$,

(ii) $c = 3$*, and* $|N^f_\mathfrak{m}(C)| = 1$ *for any strict* $\mathcal{H}$*-cover* $\mathfrak{m}$ *of* Γ.

Proof. In the case of (i), for each clique D which contains C and any $x \in C$, $D \subset C_\mathfrak{h}(x)$ holds by Lemma 4.1 (ii) and $|D| \geq 4$. Hence, $C_\mathfrak{h}(x)$ is a unique maximal clique containing C for any $x \in C$. In the case of (ii), we can prove as well by Lemma 4.1 (ii) and (iii).

Next, we show the uniqueness of a strict $\mathcal{H}$-cover of Γ. The maximal clique D containing C is defined independently of a choice of a strict $\mathcal{H}$-cover. Hence, $\Psi_\Gamma(D)$ is also defined independently of one. By Proposition 4.3, a strict $\mathcal{H}$-cover of Γ is unique. □

We define

$S_1 =$  and $S_2 =$  .

Lemma 4.7. *Let $\mathfrak{h}$ be a strict $\mathcal{H}$-cover of a graph Γ. If Γ has an induced subgraph isomorphic to S_1 or S_2, then the vertices of the triangle of the induced subgraph are adjacent to the same fat vertex in $\mathfrak{h}$.*

Proof. Let Δ be the triangle in the induced subgraph $S \simeq S_1$ or S_2. Let $\mathfrak{m}$ be a strict $\mathcal{H}$-cover of Γ. We suppose that $|N^f_{\mathfrak{m}}(\Delta)| \geq 2$ to prove $|N^f_{\mathfrak{m}}(\Delta)| = 1$ by contradiction. Then, we have Δ is not contained in $C_{\mathfrak{m}}(x)$ for every $x \in V(\Gamma)$ since every slim vertex in $C_{\mathfrak{m}}(x)$ are adjacent to the same fat vertex. This together with Lemma 4.1 (ii) implies that

$$\Delta \subset V_s(\mathfrak{m}(y)) \text{ for any } y \in \Delta.$$

We take a vertex $y \in V(\Delta)$. Then, $\Delta \subset V_s(\mathfrak{m}(y))$, and hence $[\mathfrak{m}(y)] = [\mathfrak{h}_7]$. Moreover, $\langle\langle V(S)\rangle\rangle_{\mathfrak{m}}$ is a strict $\mathcal{H}$-cover of S. Hence, we have

$$\begin{aligned}\langle\langle V(S)\rangle\rangle_{\mathfrak{m}} &= \langle\langle \Delta\rangle\rangle_{\mathfrak{m}(y)} \oplus \langle\langle V(S) \setminus \Delta\rangle\rangle_{\mathfrak{m}'} \\ &\simeq \mathfrak{h}_7 \oplus \langle\langle V(S) \setminus \Delta\rangle\rangle_{\mathfrak{m}'},\end{aligned}$$

where $\mathfrak{m}'$ denotes the Hoffman graph so that $\mathfrak{m} = \mathfrak{m}(y) \oplus \mathfrak{m}'$. It is easy to verify that the slim subgraph of $\mathfrak{h}_7 \oplus \langle\langle V(S)\rangle\rangle_{\mathfrak{m}'}$ is distinct from S_1 and S_2. This is a contradiction to $S \simeq S_1$ or S_2. Therefore the desired result follows. □

The Lemma 4.6 gives conditions that a strict $\mathcal{H}$-cover is unique, and Lemma 4.7 gives a concrete situation satisfying one of the conditions.

Lemma 4.8. *If the slim subgraph of a Hoffman graph $\mathfrak{h} = \bigoplus_{i=1}^{N} \mathfrak{h}^i$ with $[\mathfrak{h}^i] \in \mathcal{H}$ for every i is connected, then that of $\bigoplus_{i=1, i\neq k}^{N} \mathfrak{h}^i$ is connected for some k.*

Proof. Note that an $\{[\mathfrak{h}_7]\}$-line graph is connected if and only if the slim subgraph is connected. Let Γ be the graph with the vertices $\{1, \ldots, N\}$ whose two distinct vertices x and y are adjacent if and only if $V(\mathfrak{h}^i) \cap V(\mathfrak{h}^j) \neq \emptyset$. Since Γ is connected, there exists integer k such that $\Gamma - k$ is also connected. Hence, $\bigoplus_{i=1, i\neq k}^{N} \mathfrak{h}^i$ is connected, and the slim subgraph is connected. □

Let $t = (t_i)_{i=1}^n$ be a finite sequence of positive integers. Then, define the graphs P_t and C_t by

$$\begin{aligned}V(P_t) = V(C_t) &:= \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq t_i\},\\ E(P_t) &:= \{\{(i, j), (i', j')\} \mid i - i' = 1, \text{ or } i = i' \text{ and } j \neq j'\},\\ E(C_t) &:= \{\{(i, j), (i', j')\} \mid i - i' \equiv 1 \pmod{n}, \text{ or } i = i' \text{ and } j \neq j'\},\end{aligned}$$

respectively (see Example 4.10). Let

$$[a_1, \ldots, a_k] := \{(a_i, j) \in V(\Gamma) \mid 1 \leq i \leq k, 1 \leq j \leq t_{a_i}\}$$

for $\{a_1, \dots, a_k\} \subset \{1, \dots, n\}$, where $\Gamma = P_t$ or C_t. In addition, let

$$TP := \{(t_i)_{i=1}^n \in \{1,2\}^n \mid n \in \mathbb{Z}_{\geq 2}, t_i + t_{i+1} \leq 3 \ (1 \leq \forall i \leq n-1)\} \quad \text{and} \tag{4.1}$$

$$TC := \{(t_i)_{i=1}^n \in \{1,2\}^n \mid n \in (2\mathbb{Z})_{\geq 4}, t_i + t_{(i+1 \bmod n)} \leq 3 \ (1 \leq \forall i \leq n)\}. \tag{4.2}$$

Furthermore, a vertex u of a graph is said to be *end* if the graph is isomorphic to P_t for some $t \in TP$ with the length n, and $u \in [1]$ or $[n]$. In the following lemma, we see that P_t and C_t are slim $\{[\mathfrak{h}_7]\}$-line graphs, and reveal their strict $\mathcal{H}$-covers.

Lemma 4.9. *For $t \in TP$ of length n, we have $\mathcal{O}_{P_t}$ is the set of*

$$\{[1], [2,3], [4,5], [6,7], \dots\} \qquad \text{and} \qquad \{[1,2], [3,4], [5,6], \dots\}. \tag{4.3}$$

For $t \in CP$ of length n, we have $\mathcal{O}_{C_t}$ is the set of

$$\{[1,2], [3,4], \dots, [n-1,n]\} \qquad \text{and} \qquad \{[n,1], [2,3], \dots, [n-2,n-1]\}. \tag{4.4}$$

Namely, for $t \in TP$ (resp. TC), the graph P_t (resp. C_t) has precisely two strict $\mathcal{H}$-covers up to equivalence.

Proof. Recall that $\Psi_\Gamma(C) = \pi$ holds for every $C \in \pi$ where Γ is a slim $\{[\mathfrak{h}_7]\}$-line graph and $\pi \in \mathcal{O}_\Gamma$. In order to reveal $\mathcal{O}_\Gamma$, it suffices to verify whether $\Psi_\Gamma(K)$ is in $\mathcal{O}_\Gamma$ for every clique K of Γ.

We fix a sequence $t \in TP$ of length n, and determine $\mathcal{O}_{P_t}$. Since if $n = 2$ then desired result holds, we may assume that $n \geq 3$. On the other hand, every clique of P_t is contained in $[i, i+1]$ for an integer $i \in \{1, \dots, n-1\}$. The clique partitions in (4.3) are obtained from cliques $[1]$, $[n]$ and $[i, i+1]$ for $i \in \{1, \dots, n-1\}$. Moreover we can verify that $\Psi_\Gamma(K)$ is not in $\mathcal{O}_{P_t}$ for one clique K of the other following cliques:

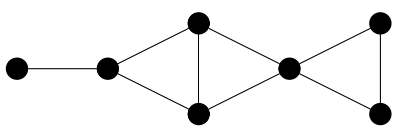
(i) non-empty subsets of $[i]$ for $i \in \{2, \dots, n-1\}$;

(ii) $\{(i,1), (i+1,1)\}$ and $\{(i,2), (i+1,1)\}$ for $i \in \{1, \dots, n-1\}$ with $t_i = 2$;

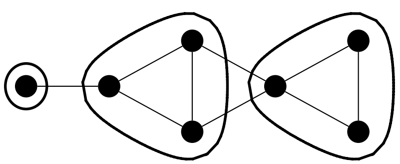
(iii) $\{(i,1), (i+1,1)\}$ and $\{(i,1), (i+1,2)\}$ for $i \in \{1, \dots, n-1\}$ with $t_{i+1} = 2$.

Similarly, we can determine $\mathcal{O}_{C_t}$ for every $t \in TC$. Finally, by Proposition 4.3, which claims that cov is a bijection from the set of strict $\mathcal{H}$-cover of a slim $\{[\mathfrak{h}_7]\}$-line graph Γ to $\mathcal{O}_\Gamma$, P_t and C_t have precisely two strict $\mathcal{H}$-covers up to equivalence. □

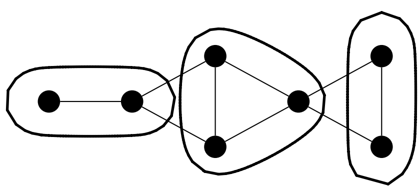
Example 4.10. We give examples of strict $\mathcal{H}$-covers of P_t and C_t. In the case of $t = (1,1,2,1,2)$, the graph P_t is



and $\mathcal{O}_{P_t}$ is the set consisting of the partitions corresponding to



and



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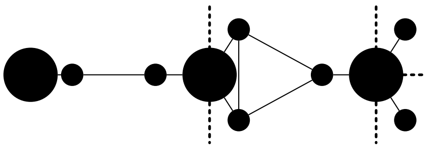
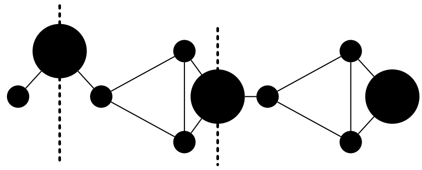
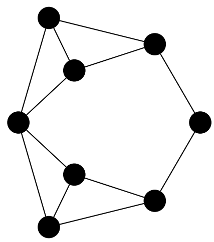
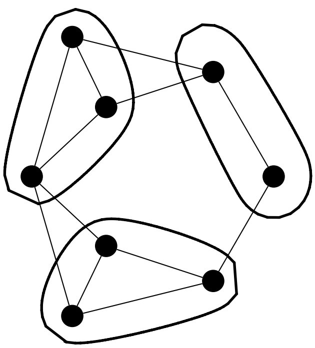
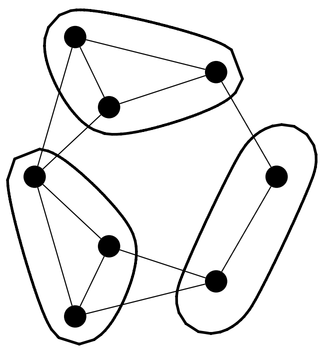
By Proposition 4.3, these clique partitions give strict $\mathcal{H}$-covers. They are

 and ,

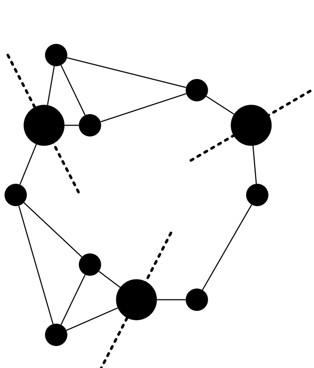
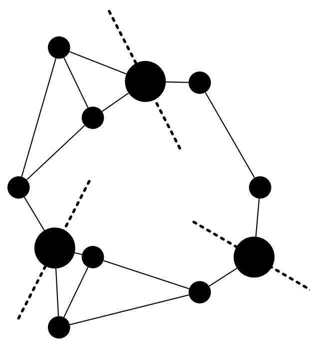
respectively. The similar consideration is applied to C_t for $t \in TC$. For example, we consider $t = (1, 1, 2, 1, 2, 1)$. Then the graph C_t is



and $\mathcal{O}_{C_t}$ is the set consisting of the partitions corresponding to

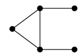
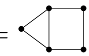
 and .

By Proposition 4.3, we have the following two strict $\mathcal{H}$-covers:

 and .

Theorem 4.11. *If a connected slim $\{[\mathfrak{h}_7]\}$-line graph Γ with the clique number c satisfies one of the following conditions:*

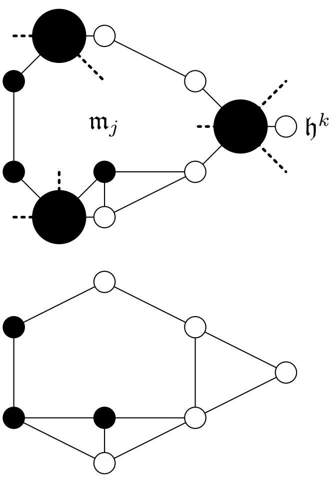
(a) $c = 1$ *or* $c \geq 4$*;*

(b) Γ *has an induced subgraph isomorphic to* $S_1 =$  *or* $S_2 =$ ,

then it has a unique strict $\{[\mathfrak{h}_1], [\mathfrak{h}_4], [\mathfrak{h}_7]\}$*-cover up to equivalence. Otherwise,* Γ *is isomorphic to* P_t *for some* $t \in TP$ *or* C_t *for some* $t \in TC$*, and it has precisely two strict* $\{[\mathfrak{h}_1], [\mathfrak{h}_4], [\mathfrak{h}_7]\}$*-covers up to equivalence.*

Proof. If (a) or (b) holds then a strict $\mathcal{H}$-cover is unique by Lemma 4.6 and Lemma 4.7 (see Example 4.12). Otherwise, it is proved that Γ is isomorphic to either P_t for some $t \in TP$ or C_t for some $t \in TC$ by induction on the number of addends of a strict $\mathcal{H}$-covers of Γ. Fix a strict $\mathcal{H}$-cover $\mathfrak{h} = \bigoplus_{i=1}^{N} \mathfrak{h}^i$, where $[\mathfrak{h}^i] \in \mathcal{H}$ for every i. If $N = 1$ then $\Gamma \simeq P_{\{1,1\}}$ or $P_{\{1,2\}}$. Otherwise, we can take an integer k such that the subgraph Γ' induced by the slim vertices of $\mathfrak{h}' = \bigoplus_{i=1, i\neq k}^{N} \mathfrak{h}^i$ is connected by Lemma 4.8. Each of S_1 and S_2 is not isomorphic to any induced subgraph in Γ'. Note that the clique number c' of Γ' is at most 3. Suppose $c' = 2$ or 3 since the result follows if $c' = 1$.

$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}^k = \mathfrak{m}_j \oplus \mathfrak{h}^k =$



The slim subgraph of $\mathfrak{h} =$

Figure 3: An example of the case that the slim subgraph of $\mathfrak{h}'$ is isomorphic to C_t for $t \in TC$.

If $\Gamma' \simeq C_{t'}$ for some $t' \in TC$, then $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}^k$ must have an induced subgraph isomorphic to either S_1 or S_2 (see Figure 3). Otherwise, $\Gamma' \simeq P_{t'}$ for some $t' \in TP$. Let

$$\mathfrak{m}_1 := \mathrm{cov}^{-1}(\{[1], [2,3], [4,5], [6,7], \ldots\}),$$
$$\mathfrak{m}_2 := \mathrm{cov}^{-1}(\{[1,2], [3,4], [5,6], \ldots\}),$$

and let n denote the length of t. By the induction hypothesis, we can take $j \in \{1, 2\}$ so that $\mathfrak{h}'$ and $\mathfrak{m}_j$ are equivalent. Then, we show that the following two conditions hold:

(A) $|N^s_{\mathfrak{m}_j}(u)| + |N^s_{\mathfrak{h}^k}(u)| \leq 3$ holds for every $u \in V_f(\mathfrak{m}_j) \cap V_f(\mathfrak{h}^k)$;

(B) $N^s_{\mathfrak{m}_j}(u) = [1], [1,2], [n]$ or $[n-1, n]$ holds for every $u \in V_f(\mathfrak{m}_j) \cap V_f(\mathfrak{h}^k)$.

First, if $|N^s_{\mathfrak{m}_j}(u)| + |N^s_{\mathfrak{h}^k}(u)| \geq 4$ holds for $u \in V_f(\mathfrak{m}_j) \cap V_f(\mathfrak{h}^k)$, then $N^s_{\mathfrak{m}_j}(u) \cup N^s_{\mathfrak{h}^k}(u)$ is a clique of size greater than 4 in Γ, a contradiction to the assumption that Γ does not satisfy the condition (a). Second, we suppose that the condition (B) does not hold. Then we can take a fat vertex $u \in V_f(\mathfrak{m}_j) \cap V_f(\mathfrak{h}^k)$ so that

$$N^s_{\mathfrak{m}_j}(u) = [i, i+1]$$

for $i \in \{2, \ldots, n-2\}$. Thus, Γ has an induced subgraph isomorphic to S_1 (see Figure 4), a contradiction to the assumption that Γ does not satisfy the condition (b). Therefore the two conditions (A) and (B) are proved.

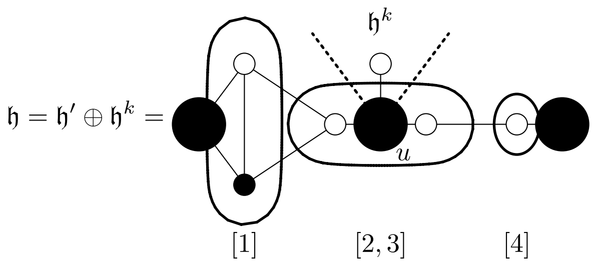


Figure 4: An example of the case that $\Gamma' \simeq P_{t'}$ and the condition (B) does not hold.

In the case of $[\mathfrak{h}^k] = [\mathfrak{h}_1]$, by the condition (B), the fat vertex u in $\mathfrak{h}^k$ equals

$$N^f_{\mathfrak{m}_j}([1]), \quad N^f_{\mathfrak{m}_j}([1,2]), \quad N^f_{\mathfrak{m}_j}([n]) \quad \text{or} \quad N^f_{\mathfrak{m}_j}([n-1,n])$$

for some $j = 1$ or 2. If $u = N^f_{\mathfrak{m}_j}([1])$ or $N^f_{\mathfrak{m}_j}([n])$ then Γ is isomorphic to P_y for some $y \in TP$. Otherwise, without loss of generality we can assume that

$$u = N^f_{\mathfrak{m}_j}([1,2]).$$

Then $t'_1 = t'_2 = 1$ by the condition (A). Hence, Γ is isomorphic to P_y for some $y \in TP$.

We consider the case of $[\mathfrak{h}^k] = [\mathfrak{h}_4]$ or $[\mathfrak{h}_7]$. If $n = 2$ then the desired result holds. Thus, we may assume that $n \geq 3$. Then

$$u \neq N^f_{\mathfrak{m}_j}([i, i+1]) \text{ for every fat vertex } u \in V_f(\mathfrak{h}^k) \text{ and } 1 \leq i \leq n-1 \tag{4.5}$$

since if (4.5) does not hold then $\mathfrak{h}$ has an induced subgraph isomorphic to either S_1 or S_2 (see Figure 5), a contradiction. Let u and v are distinct fat vertices of $\mathfrak{h}^k$. Then one of the following holds:

(i) $u = N^f_{\mathfrak{m}_j}([1])$ and $v \notin V_f(\mathfrak{h}')$, or $u = N^f_{\mathfrak{m}_j}([n])$ and $v \notin V_f(\mathfrak{h}')$;

(ii) $u = N^f_{\mathfrak{m}_j}([1]), v = N^f_{\mathfrak{m}_j}([n])$

by exchanging u and v if necessary. Hence, $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}^k$ is isomorphic to either P_y for some $y \in TP$ or C_y for some $y \in TC$. □

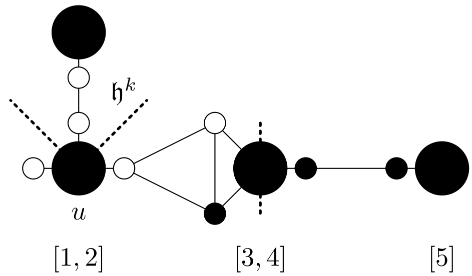


Figure 5: An example of the case that $\Gamma' = P_{t'}$, $\mathfrak{h}^k \simeq \mathfrak{h}_4$ and $u = N^f_{\mathfrak{m}_j}([1,2])$.

Example 4.12. In Theorem 4.11, there are the two conditions that a slim $\{[\mathfrak{h}_7]\}$-line graph has a unique strict $\mathcal{H}$-cover up to equivalence. For each condition, we give an example.

We let G and $\mathfrak{h}$ denote the slim $\{[\mathfrak{h}_7]\}$-line graph and its strict $\mathcal{H}$-cover in Figure 6, respectively. Then, the clique number c of G is equal to 4, and the set K of small circles of G is a maximal clique. Namely, G satisfies the condition (a) in Theorem 4.11. Take a vertex x in K. As shown in Lemma 4.6, $K = C_{\mathfrak{h}}(x)$ holds. Since $K = C_{\mathfrak{h}}(x) \in \operatorname{cov}\mathfrak{h}$, we have

$$\Psi_G(K) = \Psi_G(C_{\mathfrak{h}}(x)) = \operatorname{cov}\mathfrak{h}.$$

As with the proof of Proposition 4.3, we derive the Hoffman graph $\mathfrak{h}$ by adding fat vertices according to $\Psi_G(K)$.

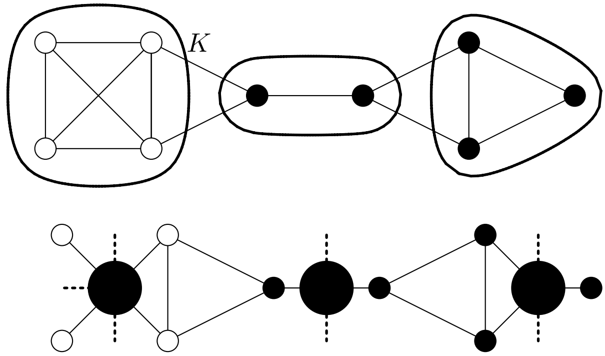


Figure 6: A slim $\{[\mathfrak{h}_7]\}$-line graph whose clique number c is 4 and its strict $\mathcal{H}$-cover corresponding to $\Psi_G(K)$.

Next, we let H and $\mathfrak{m}$ denote the slim $\{[\mathfrak{h}_7]\}$-line graph and its strict $\mathcal{H}$-cover in Figure 7, respectively. Let H' be the subgraph induced by the small circles in H. Then, the clique number c of H is equal to 3, and H' is isomorphic to S_1. Namely, H satisfies the condition (b) in Theorem 4.11. Let K be the triangle of H'. Take a vertex x in K. As shown in Lemma 4.7, the vertices in K are adjacent to the same fat vertex of $\mathfrak{m}$. In particular, $K = C_{\mathfrak{m}}(x)$ holds. Since $K = C_{\mathfrak{m}}(x) \in \operatorname{cov}\mathfrak{m}$, we have

$$\Psi_H(K) = \Psi_H(C_{\mathfrak{m}}(x)) = \operatorname{cov}\mathfrak{m}.$$

As with the proof of Proposition 4.3, we derive the Hoffman graph $\mathfrak{h}$ by adding fat vertices according to $\Psi_H(K)$. Similarly, for a slim $\{[\mathfrak{h}_7]\}$-line graph in Figure 8, we can construct its strict $\mathcal{H}$-cover.

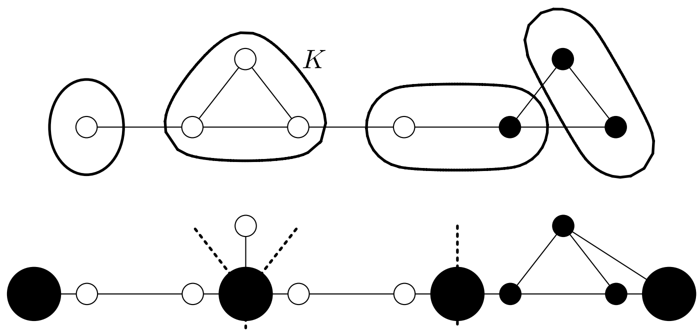


Figure 7: A slim $\{[\mathfrak{h}_7]\}$-line graph containing S_1 induced by the small circles and its strict $\mathcal{H}$-cover corresponding to $\Psi_G(K)$.

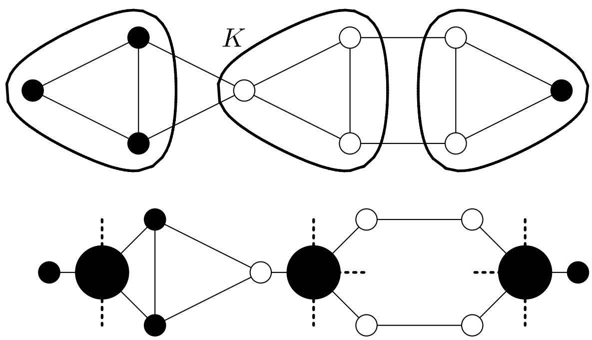


Figure 8: A $\{[\mathfrak{h}_7]\}$-line graph containing S_2 induced by the small circles and its strict $\mathcal{H}$-cover corresponding to $\Psi_H(K)$.

Remark 4.13. Let Γ be a connected slim $\{[\mathfrak{h}_7]\}$-line graph with the clique number at least 2. Let K be a maximal clique of Γ. We suppose that $|K| \geq 4$ when Γ satisfies the condition (a) in Theorem 4.11, and that K contains the triangle of S_1 or that of S_2 when Γ satisfies the condition (b). Then, as shown in Lemma 4.6, the strict $\{[\mathfrak{h}_1], [\mathfrak{h}_4], [\mathfrak{h}_7]\}$-cover is $\operatorname{cov}^{-1}(\Psi_\Gamma(K))$.

5 The minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs

The following theorem is the main result in this paper.

Theorem 5.1. *A graph is a minimal forbidden graph for the slim* $\{[\mathfrak{h}_7]\}$*-line graphs if and only if it is one of the following graphs:*

(i) M_i *(*$i = 1, 2, 3, 4, 6, 7, 11, 12, 19$*) in Figure 9;*

(ii) odd cycles with at least 5 *vertices;*

(iii) graphs in Figures 11 and 13.

We explain the reason that the graphs in Figure 9 are minimal forbidden graphs for the slim $\{\mathfrak{h}_7\}$-line graphs. They are obtained by enumeration by MAGMA. The following briefly describes the program.

The MAGMA program is available at [8]. It is also available at `https://doi.org/10.26493/1855-3974.1581.b47`.

Hoffman graphs can construct large new graphs little by little from small graphs by using the concept of sum. With this method, all possible $\{[\mathfrak{h}_7]\}$-line graphs with a small number of slim vertices can be obtained by considering all cases where fat vertices can be stuck together. Therefore, we can obtain all slim $\{[\mathfrak{h}_7]\}$-line graphs with a small number of vertices. On the other hand, the graphs up to 10 vertices have databases in MAGMA [2]. Using this, the list $\mathcal{F}$ of graphs with at most 10 vertices that are not slim $\{[\mathfrak{h}_7]\}$-line graphs is completely revealed. After that, the set of minimal elements of $\mathcal{F}$ can be calculated.

We will prove Theorem 5.1 separately.

(C1) Γ has an induced subgraph isomorphic to S_1, S_2 or the complete graph K_4;

(C2) For any maximal clique K containing the largest clique of some induced subgraph isomorphic to S_1, S_2 or K_4, $\Psi_\Gamma(K)$ is a partition of $V(\Gamma)$.

Proposition 5.2. *Let* Γ *be a minimal forbidden graph for the slim* $\{[\mathfrak{h}_7]\}$*-line graphs with at least* 10 *vertices. Then,* Γ *does not satisfy the condition (C1) if and only if* Γ *is an odd cycle.*

Proof. It is easy to verify that every odd cycle with at least 5 vertices is a minimal forbidden graph. Thus, the necessity is proved. Next, we prove the sufficiency. Pick two vertices x and y to determine the diameter of Γ. Then, $\Gamma - x$ and $\Gamma - y$ are connected and slim $\{[\mathfrak{h}_7]\}$-line graphs. By Theorem 4.11, $\Gamma - x$ is isomorphic to either P_t for some $t \in TP$ or C_t for some $t \in TC$. We have

$$|N_\Gamma(x)| \leq 4 \tag{5.1}$$

since $\Gamma - y$ is also isomorphic to either P_t for some $t \in TP$ or C_t for some $t \in TC$. By (4.1) and (4.2), which are the definition of TP and TC, we have the length l of t is at least 6 since

$$\sum_{i=1}^{l} t_i = |\Gamma| - 1 \geq 9.$$

In the rest of this proof, we will consider the decision on whether the vertex is end or non-end in $\Gamma - x$.

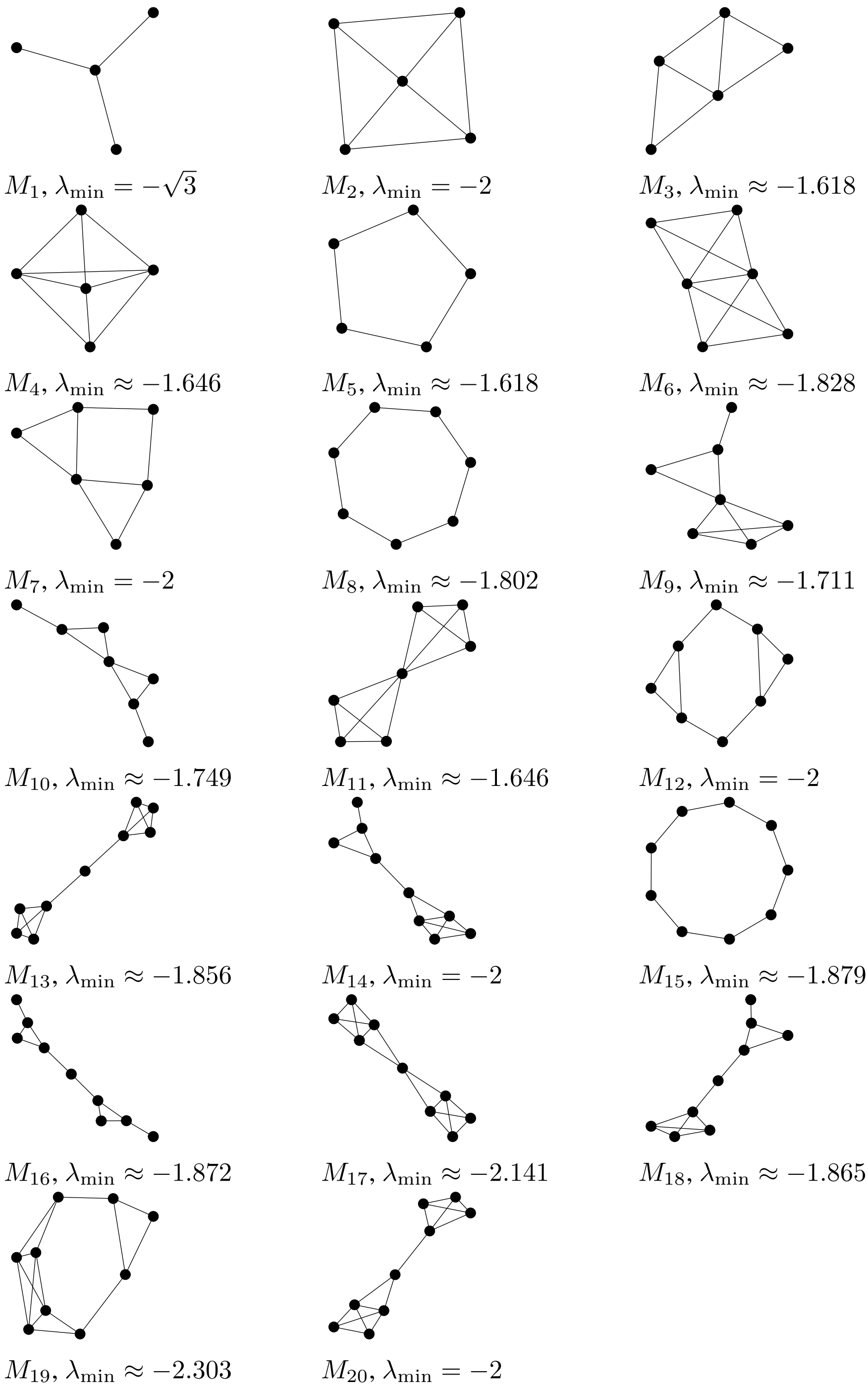


Figure 9: The minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs, with at most 9 vertices.

Step 1: Show that $N_\Gamma(x)\cap N_{\Gamma-x}(z)$ is non-empty if z is a non-end vertex in $N_\Gamma(x)$. Suppose that $N_\Gamma(x)\cap N_{\Gamma-x}(z)$ is empty. Then, there exists two distinct vertices u and v of $\Gamma-x$ such that $u\sim z\sim v$ and $u\not\sim v$. It is a contradiction that $M_1\simeq\langle\{x,z,u,v\}\rangle_\Gamma$.

Step 2: Show that $|N_\Gamma(x)\cap N_{\Gamma-x}(z)|\leq 1$ for $z\in N_\Gamma(x)$. Suppose $|N_\Gamma(x)\cap N_{\Gamma-x}(z)|\geq 2$, and let z_1 and z_2 be two distinct vertices in $N_\Gamma(x)\cap N_{\Gamma-x}(z)$. We have $z_1\not\sim z_2$ since $K_4\simeq\langle\{x,z,z_1,z_2\}\rangle_\Gamma$ if $z_1\sim z_2$. Thus, z is non-end. Let i be the integer such that $z\in[i]$. Then, the following hold:

(i) if $(t_{i-1},t_i,t_{i+1})=(1,1,2)$, then $M_3\simeq\langle\{x\}\cup[i-1,i,i+1]\rangle_\Gamma$;

(ii) if $(t_{i-1},t_i,t_{i+1})=(1,2,1)$, then $M_2\simeq\langle\{x\}\cup[i-1,i,i+1]\rangle_\Gamma$;

(iii) if $(t_{i-1},t_i,t_{i+1})=(2,1,2)$, then M_3 is isomorphic to an induced subgraph in $\{x\}\cup[i-1,i,i+1]$.

Then, Γ is isomorphic to either M_2 or M_3 by the minimality of Γ. This is a contradiction to $|V(\Gamma)|\geq 10$. Consider the case of $(t_{i-1},t_i,t_{i+1})=(1,1,1)$. If $|N_\Gamma(x)|=3$ then $\Gamma\simeq P_{t'}$ or $C_{t'}$ where $t'=(t_1,\ldots,t_{i-1},2,t_{i+1},\ldots,t_l)$. Otherwise, $|N_\Gamma(x)|=4$ holds by (5.1), and hence we let $\{z,z_1,z_2,z_3\}=N_\Gamma(x)$. Then, the following hold:

(i) if $z_3\in[i-2,i+2]$ then $M_3\simeq\langle\{x\}\cup N_\Gamma(x)\rangle_\Gamma$;

(ii) if $z_3\notin[i-2,i+2]$ then $M_1\simeq\langle\{x,z_1,z_2,z_3\}\rangle_\Gamma$

(see Figure 10). These are contradictions to $|V(\Gamma)|\geq 10$.

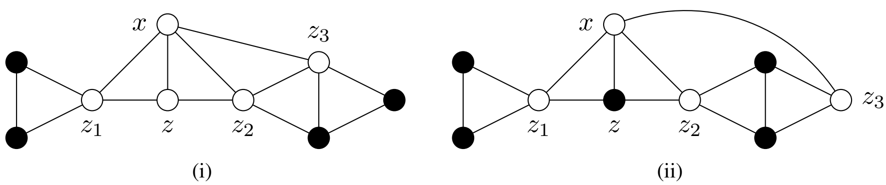


Figure 10: Examples of the case that $(t_{i-1},t_i,t_{i+1})=(1,1,1)$ and $|N_\Gamma(x)|=4$ in Step 2.

Step 3: Show that the vertices in $N_\Gamma(x)$ are end. Suppose that a vertex z is non-end in $N_\Gamma(x)$. By Step 1 and 2, $|N_\Gamma(x)\cap N_{\Gamma-x}(z)|=1$ holds. Thus, we can take a vertex z_1 so that

$$\{z_1\}=N_\Gamma(x)\cap N_{\Gamma-x}(z).$$

There are i and j such that $z\in[i]$ and $z_1\in[j]$. Let $I=[i,j,i\pm 1,j\pm 1]$. It follows that $N_\Gamma(x)\cap I=\{z,z_1\}$ by Step 2. If z_1 is non-end, then some induced subgraph of Γ is isomorphic to M_1 if $i=j$, S_1 otherwise, a contradiction. Otherwise, we may assume that $i=2$ and $j=1$ without loss of generality. Then, the following hold:

(i) if $(t_1,t_2)=(2,1)$, then M_1 is isomorphic to some induced subgraph of Γ;

(ii) if $(t_1,t_2)=(1,2)$, then M_3 is isomorphic to some induced subgraph of Γ;

(iii) if $(t_1, t_2) = (1, 1)$ and $|N_\Gamma(x)| \geq 3$, then Γ has an induced subgraph isomorphic to S_1 or S_2;

(iv) if $(t_1, t_2) = (1, 1)$ and $|N_\Gamma(x)| = 2$, then $\Gamma \simeq P_{(t_1+1,t_2,\ldots,t_l)}$.

The result follows. Moreover, $\Gamma - x$ is isomorphic to P_t.

Step 4: For $i = 1$ or l, if $N_\Gamma(x) \cap [i] \neq \emptyset$ then $N_\Gamma(x) \cap [i] = [i]$ since $N_\Gamma(x) \cap [i] \neq [i]$ implies that Γ has an induced subgraph isomorphic to S_1 by Step 3.

Step 5: If $N_\Gamma(x) = [1]$ then $\Gamma \simeq P_{(1,t_1,\ldots,t_l)}$ is an $\{[\mathfrak{h}_7]\}$-line graph, a contradiction. Hence, $\Gamma \simeq C_{(1,t_1,\ldots,t_l)}$. If l is odd then $C_{(1,t_1,\ldots,t_l)}$ is an $\{[\mathfrak{h}_7]\}$-line graph, a contradiction. Thus, Γ is an odd cycle. □

Proposition 5.3. *Let Γ be a minimal forbidden graph for the slim $\{[\mathfrak{h}_7]\}$-line graphs, with at least 10 vertices and the condition (C1). Then, Γ is isomorphic to one of the graphs in Figure 11 if and only if Γ does not satisfy the condition (C2).*

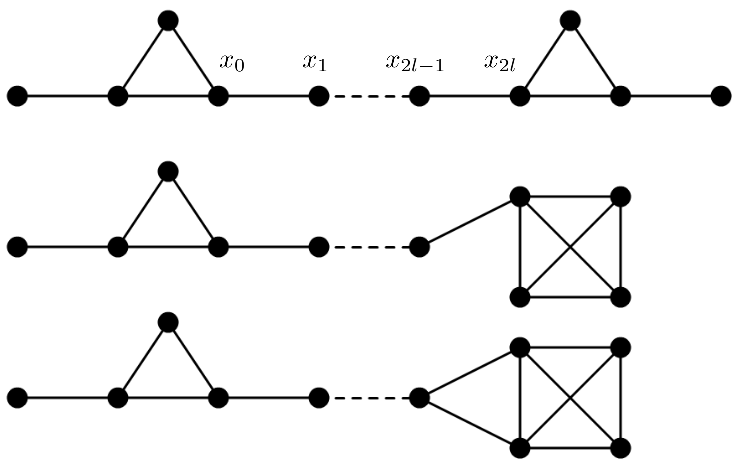


Figure 11: Minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs, with at least 10 vertices and the condition (C1), without (C2).

Proof. It is easy to verify that $\Psi_\Gamma(K)$ is not a partition, where Γ is one of the graphs of Figure 11, and K is the rightmost clique of size at least 3.

Next, we prove the necessity. By the condition (C1), Γ has an induced subgraph isomorphic to S_1, S_2 or K_4. Hence, we can take a maximal clique K containing the largest clique of some induced subgraph S isomorphic to S_1, S_2 or K_4. Since the condition (C2) is not satisfied, we may suppose that $\Psi_\Gamma(K)$ is not a partition of $V(\Gamma)$. If $|K| > 3$ then we replace S by $\langle K' \rangle_\Gamma$, where K' is a set of 4 vertices in K. Let $l = \lfloor (\partial_{\max} - 1)/2 \rfloor$ and $V' = \{x \in V(\Gamma) \mid \partial_K(x) \leq 2l\}$.

Step 1: There is a vertex g not in $V(S) \cup K$ with $\partial_K(g) = \partial_{\max}$. Then, $\{C \in \Psi_\Gamma(K) \mid \partial_K(c) \leq 2l \text{ for any } c \in C\} = \{C \in \Psi_{\Gamma-g}(K) \mid \partial_K(c) \leq 2l \text{ for any } c \in C\}$ is a clique partition of V' since $\Gamma - g$ is a slim $\{[\mathfrak{h}_7]\}$-line graph satisfying the condition (a) or (b) in

Theorem 4.11. Therefore, $\Psi_\Gamma(K)$ is not a partition if and only if there exists vertices x, y, p and q such that $\partial_K(x) = \partial_K(y) = 2l + 1$,

$$q \in ((N(x) \cup \{x\}) \cap (N(y) \cup \{y\})) - V',$$

and

$$p \in ((N(x) \cup \{x\}) - (N(y) \cup \{y\})) - V'.$$

If $z \in V(\Gamma) - (\{x, y, p, q\} \cup S)$ with $\partial_K(z) \geq 2l + 1$, then $\Gamma - z$ is an $\{[\mathfrak{h}_7]\}$-line graph and $p \sim y$ by Remark 4.13. This is a contradiction to $p \not\sim y$. It follows that

$$\{u \in V(\Gamma) \mid \partial_K(u) \geq 2l + 1\} - V(S) = \{x, y, p, q\}. \tag{5.2}$$

Step 2: Assume that $l = 0$. In the case of $|K| \geq 4$ (i.e., $S \simeq K_4$),

$$\{K - \{k\}, \{x, y, p, q\}\} \subset \Psi_{\Gamma - k}(K - \{k\})$$

is a clique partition for some $k \in K$ by Remark 4.13 and (5.2), i.e., $|K| \geq 6$. This is a contradiction to $p \not\sim y$. In the case of $|K| = 3$ (i.e., $S \simeq S_1$ or S_2), we obtain $|V(\Gamma)| \leq 9$ and a contradiction. Hence, l is a positive integer.

Step 3: Show that $x \sim y$. In addition, it holds that $x \neq p$ and $x = q$. Suppose that x and y are not adjacent. Then, $p = x$ and $q \notin \{x, y\}$ by Remark 4.13. If x and y are adjacent to a vertex r with $\partial_K(r) = 2l$, then Γ has an induced subgraph isomorphic to $K_{1,3}$, a contradiction. When $\partial_K(q) = 2l + 1$, there is a neighbor q' of q with $\partial_K(q') = 2l$ such that $q' \notin N(x) \cap N(y)$. Without loss of generality we assume that q' and y are not adjacent. Then, the induced subgraph Γ' obtained by deleting the neighbors of y expect q has a clique partition $\Psi_{\Gamma'}(K)$ which contains a set $\{x, y, q\}$. This is a contradiction, and $\partial_K(q) = 2l + 2$ follows. Then, Γ has an induced subgraph isomorphic to an odd cycle with at least 5 vertices. This is a contradiction since $\Psi_{\Gamma - q}(K)$ is a clique partition in $\mathcal{O}_{\Gamma - q}$.

Step 4: Let P denote a path $(x_0, \ldots, x_{2l+1})$ such that $x_0 \in K$ and $x = x_{2l+1}$. If $S \simeq K_4$, then replace S by $\langle K' \rangle_\Gamma$, where K' is a set of 4 vertices in K containing x_0 and a vertex not adjacent to x_1. The graph Γ' by deleting vertices other than $V(P) \cup V(S) \cup \{x, y, p\}$ is a slim $\{[\mathfrak{h}_7]\}$-line graph with a clique partition $\Psi_{\Gamma'}(K)$. Thus, it holds that $V(\Gamma) = V(P) \cup V(S) \cup \{x, y, p\}$ and $y \sim x_{2l}$.

Step 5: If $\partial_K(p) = 2l + 1$, then $p \sim x_{2l}$. Thus, $\langle \{y, p, x_{2l}, x_{2l-1}\} \rangle_\Gamma \simeq K_{1,3}$ holds, a contradiction. Thus, $\partial_K(p) = 2l + 2$.

Step 6: In the case of $|K| \geq 4$ (i.e., $S \simeq K_4$), $\deg(x_1) \leq 3$ since $\Gamma - p$ is an $\{[\mathfrak{h}_7]\}$-line graph. Obtain the second and third graphs in Figure 11.

Step 7: In the case of $|K| = 3$ (i.e., $S \simeq S_1$ or S_2), if $x_1 \notin S$ then $|N(x_1) \cap K| = 1$ since M_1 and M_3 are minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs. Then, we can replace S by the induced subgraph by $K \cup \{x_1, w\}$, where w is a vertex of S not adjacent to x_0. Hence, $x_1 \in S$ holds. We can draw Γ as Figure 12. The edge e exists if and only if the edge e' does in Figure 12. If the edges e and e' exist, then $\Gamma - x_0$ is not an $\{[\mathfrak{h}_7]\}$-line graph. Otherwise, we obtain the first graphs in Figure 11. □

Let Γ be a connected graph, and let K and D be nonempty subsets of $V(\Gamma)$. D is said to be *deletable* for K if $K - D \neq \emptyset$, $\Gamma - D$ is connected, and $\Psi_{\Gamma - D}(K - D) = \{C - D \mid$

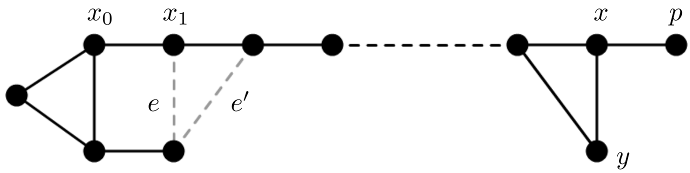


Figure 12: In the proof of Proposition 5.3.

$C \in \Psi_\Gamma(K)\} - \{\emptyset\}$. In addition, a vertex v is said to be *deletable* for K if $\{v\}$ is deletable for K.

Lemma 5.4. *Let Γ be a connected graph, and let K and D be nonempty subsets of $V(\Gamma)$. If $\Psi_\Gamma(K)$ is a partition of $V(\Gamma)$ and $\partial_{K,\Gamma}|_{\Gamma-D} = \partial_{K-D,\Gamma-D}$, then D is deletable for K.*

Proof. Write $\partial = \partial_{K,\Gamma}$ for short. Then, it holds that by $\partial_{K,\Gamma}|_{\Gamma-D} = \partial_{K-D,\Gamma-D}$,

$$\begin{aligned}&\Psi_{\Gamma-D}(K-D) - \{K-D\}\\ &\quad = \{\{y \in (\{x\} \cup N(x)) - D \mid \partial(y) \geq \partial(x)\} \mid x \in V(\Gamma - D), \partial(x) \in 2\mathbb{N}+1\},\end{aligned}$$

and

$$\begin{aligned}&\{C - D \mid C \in \Psi_\Gamma(K)\} - \{K - D\}\\ &\quad = \{\{y \in (\{x\} \cup N(x)) - D \mid \partial(y) \geq \partial(x)\} \mid x \in V(\Gamma), \partial(x) \in 2\mathbb{N}+1\}.\end{aligned}$$

Let $x \in D$ with odd $\partial(x)$, and take $C_x \in \Psi_\Gamma(K)$ containing x. Assuming that $C_x - D \neq \emptyset$, there exists $z \in \tilde{C}$ such that $\partial(z) = \partial(x)$ by the assumptions. It holds that

$$\begin{aligned}C_x - D &= \{y \in (\{x\} \cup N(x)) - D \mid \partial(y) \geq \partial(x)\}\\ &= \{y \in (\{z\} \cup N(z)) - D \mid \partial(y) \geq \partial(z)\} \in \Psi_{\Gamma-D}(K-D).\end{aligned}$$

□

Lemma 5.5. *Let Γ be a minimal forbidden graph for the slim $\{[\mathfrak{h}_7]\}$-line graphs, with the condition (C1) and (C2). Let S be an induced subgraph isomorphic to S_1, S_2 or K_4. Let K be a maximal clique of Γ contains the largest clique of S. Then, the following hold:*

(i) $\Psi_\Gamma(K)$ is a clique partition;

(ii) if u is a non good vertex for $\Psi_\Gamma(K)$, and $v \notin V(S) \cup K$ is a deletable vertex for K, then $v \in n^3(u)$ and v is non good for $\Psi_\Gamma(K)$, where $n^k(\cdot)$ is defined for $\Psi_\Gamma(K)$ and a non negative integer k.

Proof. By the condition (C2), $\Psi_\Gamma(K)$ is a partition of $V(\Gamma)$. Moreover, it is a clique partition of $V(\Gamma)$ by Lemma 4.5 since Γ has no induced subgraph isomorphic to $M_1 \simeq K_{1,3}$. Next, suppose that the vertex v is not in $n^3(u)$. Then,

$$\begin{aligned}\{C \cap n^3(u) \mid C \in \Psi_\Gamma(K)\} &= \{C \cap n^3(u) - \{v\} \mid C \in \Psi_\Gamma(K)\}\\ &= \{C \cap n^3(u) \mid C \in \Psi_{\Gamma-v}(K)\}\end{aligned}$$

holds. Thus, the vertex u is good for $\Psi_{\Gamma-v}(K)$ if and only if it is good for $\Psi_{\Gamma-v}(K)$ in $\Gamma - v$ by Lemma 4.4. On other hand, $v \notin V(S) \cup K$ and $\Gamma - v$ is connected. Hence, $\Psi_{\Gamma-v}(K) \in \mathcal{O}_{\Gamma-v}$, that is, every vertex of $\Gamma - v$ is good for $\Psi_\Gamma(K)$ by Remark 4.13 since Γ is a minimal forbidden graph for the slim $\{[\mathfrak{h}_7]\}$-line graphs. Therefore, u is good for $\Psi_{\Gamma-v}(K)$ and $\Psi_\Gamma(K)$. This is a contradiction that u is non good for $\Psi_\Gamma(K)$. □

Proposition 5.6. *Let* Γ *be a minimal forbidden graph for the slim* $\{[\mathfrak{h}_7]\}$*-line graphs, with at least* 10 *vertices and the condition (C1). Then,* Γ *is one of the graphs in Figure 13 if and only if* Γ *satisfies the condition (C2).*

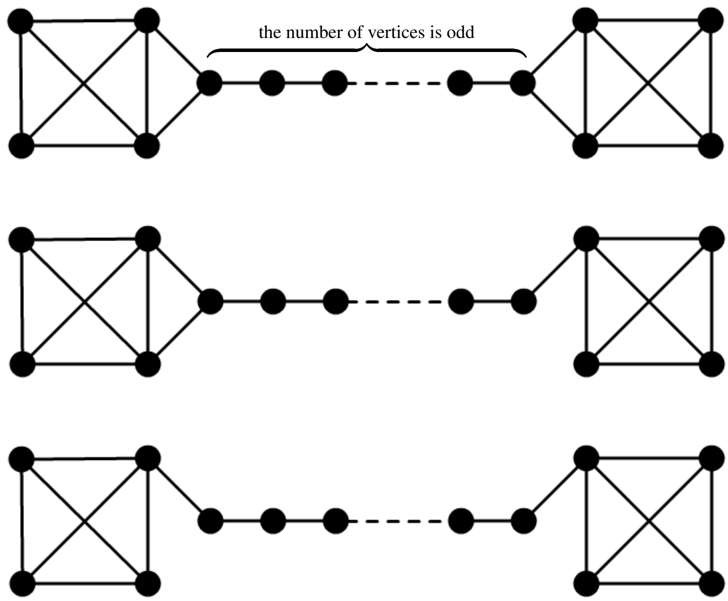


Figure 13: The minimal forbidden graphs for the slim $\{[\mathfrak{h}_7]\}$-line graphs, with at least 10 vertices and the conditions (C1) and (C2).

Proof. The sufficiency obviously holds. Prove the necessity. Fix an induced subgraph S isomorphic to S_1, S_2 or K_4, and let K be a maximal clique containing the largest clique of S. By Lemma 5.5 (i), $\Psi_\Gamma(K)$ is a clique partition. Then, $n^k(\cdot)$ is defined for $\Psi_\Gamma(K)$ and a non negative integer k.

If Γ has an induced subgraph isomorphic to K_4, then replace S by it. Let $l = \lfloor(\partial_{\max} - 1)/2\rfloor$. By the definition of $\Psi_\Gamma(K)$, we can pick the subset

$$\big\{\{y \in N(x) \cup \{x\} \mid \partial_K(y) \geq \partial_K(x)\} \mid x \in V(\Gamma), \partial_K(x) = 2l+1\big\}$$

of $\Psi_\Gamma(K)$. We denote by $\{C_i\}_{i=1}^{n}$ the subset. Note that C_i are pairwise distinct. If $l = 0$ then let $D_1 = K$ and $m = 1$. Otherwise we pick the subset

$$\big\{\{y \in N(x) \cup \{x\} \mid \partial_K(y) \geq \partial_K(x)\} \mid x \in V(\Gamma), \partial_K(x) = 2l-1\big\}$$

of $\Psi_\Gamma(K)$. We denote by $\{D_i\}_{i=1}^{n'}$ the subset. Note that D_i are pairwise distinct. Without loss of generality, we can take an integer m such that $D_i - V(S)$ is empty if and only if $i > m$.

In the case of $\partial_{\max} = 2l+1$, we show that Γ is isomorphic to one of the graphs in Figure 13.

Step 1: Show that $l \neq 0$. Suppose that $l = 0$ to prove by contradiction. Set $B = V(\Gamma) - (V(S) \cup K)$. Then, every vertex in B is deletable by Lemma 5.4. Moreover, the deletable vertex is non good for $\Psi_\Gamma(K)$ by applying Lemma 5.5 (ii) to a non good vertex for $\Psi_\Gamma(K)$ and the deletable vertex. We obtain a contradiction by checking the following:

(i) $|B| \leq 3$;

(ii) if $7 \leq |V(\Gamma) - K|$, then $5 \leq |V(\Gamma) - K| - 2 \leq |B|$;

(iii) if $4 \leq |V(\Gamma) - K| \leq 6$, then $S \simeq K_4$ and $4 \leq |B|$;

(iv) if $|V(\Gamma) - K| \leq 3$, then Γ is an $\{[\mathfrak{h}_7]\}$-line graph.

Assume that $|B| \geq 4$. If we find a vertex $k \in K$ with $|n(k)| \geq 3$, then we can take a vertex $b \in B$ such that $|n(k) - \{b\}| \geq 3$. This is a contradiction since the vertex b is deletable and $\Psi_{\Gamma-b}(K) \in \mathcal{O}_{\Gamma-b}$ by Remark 4.13. Thus,

$$|n(k)| \leq 2 \tag{5.3}$$

holds for every $k \in K$. Fix a vertex $b \in B$. Then, $|n(b)| \geq 3$ holds by (5.3) and applying Lemma 5.5 (ii) to the non good vertex b and each vertex in $B - \{b\}$. We obtain a contradiction as well and $|B| \leq 3$. Next, In the case of $|V(\Gamma) - K| \leq 3$, we have $|B| \leq 3$, $|K| \geq 7$ and hence $S \simeq K_4$. If $2 \leq |B| \leq 3$, then $|n(b)| \leq 2$ for every vertex $b \in B$. Hence, we can pick a vertex k in

$$K - \bigcup_{b \in B} n(b).$$

It is clear that k is deletable and $\Psi_{\Gamma-k}(K - \{k\}) \in \mathcal{O}_{\Gamma-k}$. Thus every vertex of Γ is good for $\Psi_\Gamma(K)$, a contradiction to Γ being a non $\{[\mathfrak{h}_7]\}$-line graph.

Step 2: Every vertex x with $\partial_K(x) = \partial_{\max}$ is not in $V(S) \cup K$ and deletable by $l \geq 1$ and Lemma 5.4. Moreover, such a vertex x is non good for $\Psi_\Gamma(K)$ by applying Lemma 5.5 (ii) to a non good vertex for $\Psi_\Gamma(K)$ and the deletable vertex x. Fix a vertex u with $\partial_K(u) = \partial_{\max}$. By applying Lemma 5.5 (ii) to u and each vertex x with $\partial_K(x) = \partial_{\max}$, we have

$$\{x \in V(\Gamma) \mid \partial_K(x) = \partial_{\max}\} \subset n^2(u)$$

since Γ has no induced subgraph isomorphic to $M_1 \simeq K_{1,3}$. For a vertex x with $\partial_K(x) = \partial_{\max} - 1 = 2l$, it holds that $x \in n^3(u)$ since if $n(x) = \emptyset$ then x is deletable by Lemma 5.4 and $x \in n^3(u)$ holds by applying Lemma 5.5 (ii) to the vertices u and x. Thus,

$$n^3(u) = \{x \in V(\Gamma) \mid \partial_K(x) \geq 2l\}. \tag{5.4}$$

Furthermore, D_i contains a vertex v_i with $\partial_K(v_i) = \partial_{\max} - 1$ for every $1 \leq i \leq m$, since every deletable vertex not in $V(S) \cup K$ is contained in $n^3(u)$ by Lemma 5.5 (ii).

Step 3: Show that $n = m = 1$. If $n \geq 2$ then Γ has an induced subgraph isomorphic to M_1 by (5.4), a contradiction. Thus, $n = 1$. Suppose that $m \geq 2$ to prove by contradiction. Without loss of generality, we can assume that $v_1 \sim u \sim v_2$ by (5.4). In the case of $m \geq 3$, the vertex v_3 is deletable clearly, and the vertex u is non good for $\Psi_{\Gamma - v_3}(K)$ in $\Gamma - v_3$. This is a contradiction to $\Psi_{\Gamma-d_3}(K) \in \mathcal{O}_{\Gamma-d_3}$ by Remark 4.13.

In the case of $m = 2$, we have $C_1 = \{u\}$ since if we find an vertex u' in C_1, then the vertex u' is deletable by Lemma 5.4 and u is non good for $\Psi_{\Gamma-u'}(K)$ in $\Gamma - u'$, a contradiction to $\Psi_{\Gamma-u'}(K) \in \mathcal{O}_{\Gamma-u'}$. Fix a vertex $v_i' \in D_i$ with $\partial_K(v_i') = 2l-1$ for $i = 1$ and 2, respectively. Then, the set

$$D_1 \cup D_2 - (V(S) \cup \{v_1, v_2, v_1', v_2'\})$$

is deletable and u is good. Hence, the set is empty. If v_1 and v_2 are not adjacent in Γ, then Γ has an induced subgraph isomorphic to an odd cycle with at least 5 vertices since u is deletable and $\Psi_\Gamma(K) - \{C_1\} = \Psi_{\Gamma-u}(K) \in \mathcal{O}_{\Gamma-u}$. This is a contradiction to the minimality of Γ. Hence, v_1 and v_2 are adjacent in Γ. First, consider the case of $l \geq 2$. Let d be the vertex adjacent to v_2' with $\partial_K(d) = 2l-2$. Note that d is not in S. Then, $\Gamma - d$

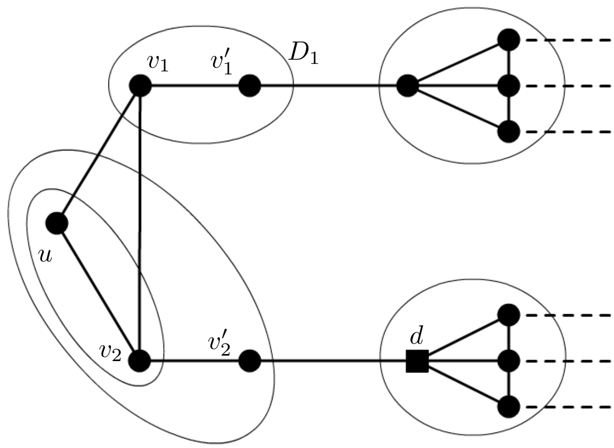


Figure 14: The cases of $n = 1, m = 2$ and $l \geq 2$ in the proof of Proposition 5.6.

is not an $\{[\mathfrak{h}_7]\}$-line graph since $\Psi_{\Gamma-d}(K) \notin \mathcal{O}_{\Gamma-d}$ (cf. Figure 14). Second, consider the case of $l = 1$. Suppose that $S \simeq K_4$. Note that $n' = m = 2$ holds. If $|K| \geq 5$, then we can take a deletable vertex k in K. By Remark 4.13, $\Psi_{\Gamma-k}(K - \{k\}) \in \mathcal{O}_{\Gamma-k}$ holds since $K - \{k\}$ is a maximal clique with at least 4 vertices. However, u is non good for $\Psi_{\Gamma-k}(K - \{k\})$, a contradiction to the minimality of Γ. We have $|V(\Gamma)| = |K| + |D_1| + |D_2| + |C_1| = 4+2+2+1 = 9$, a contradiction to $|V(\Gamma)| \geq 10$. Thus, Γ has no induced subgraph isomorphic to K_4. Suppose that $S \simeq S_1$ or S_2. We define the vertices w_i of S as Figure 15. Note that the vertex v_i' is not in $V(S)$ for $i = 1, 2$ since $|V(\Gamma)| \geq 10$. If both v_1' and v_2' are adjacent only to w_1, then Γ has an induced subgraph isomorphic to M_1, a contradiction. Hence, it is not so. Without loss of generality, we can assume that v_2' and w_2 are adjacent. Since Γ has no induced subgraph isomorphic to K_4, v_2' is not adjacent to some vertex $w \in \{w_1, w_3\}$. The vertices v_2' and w_5 are adjacent since $\langle v_2', w, w_2, w_5\rangle_\Gamma$ is not isomorphic to M_1. Furthermore, v_2 is also adjacent to w_5 since $\Psi_\Gamma(K)$ is a clique partition. Hence, v_2' is deletable for K, a contradiction to Ψ. We have $n = m = 1$.

Step 4: Let $p = |C_1|$ and $q = |\{x \in V(\Gamma) \mid \partial_K(x) = 2l\}|$. The induced subgraph $\langle C_1 \cup D_1\rangle_\Gamma$ is an $\{[\mathfrak{h}_7]\}$-line graph by the minimality of Γ. Hence, $\Psi_{\langle C_1 \cup D_1\rangle_\Gamma}(D_1) =$

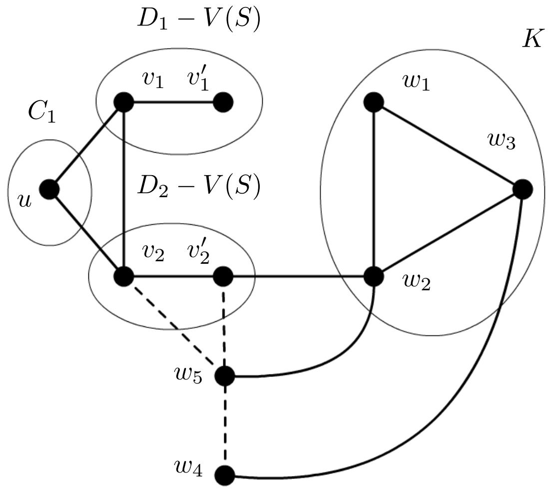


Figure 15: The cases of $n = 1, m = 2$ and $l = 1$ in the proof of Proposition 5.6.

$\{C_1, D_1\}$ holds by $|D_1| \geq 4$ and Remark 4.13. This is a contradiction since non good vertices for $\Psi_\Gamma(K)$ in $C_1 \cup D_1$ are also non good for $\Psi_{\langle C_1 \cup D_1 \rangle_\Gamma}(D_1)$. Thus, $q \leq 2$. When $q = 1, p = 3$ holds and we obtain the second and third graph in Figure 13 in the same way as the Step 4 in the proof of Proposition 5.3. Consider the case of $q = 2$. If $p = 1$ then Γ is an $\{[\mathfrak{h}_7]\}$-line graph. When $p = 2$, we obtain the first and second graph in Figure 13 since Γ has no induced subgraph isomorphic to M_3. If $p \geq 3$ then we can assume that $n(u) = \{x \in V(\Gamma) \mid \partial_K(x) = 2l\}$ by (5.4). Then, $\Gamma - w$ is not an $\{[\mathfrak{h}_7]\}$-line graph for some $w \in C_1 - \{v\}$, a contradiction.

Suppose that $\partial_{\max} = 2l + 2$. We have $l \geq 0$ and every vertex u with $\partial_K(u) = 2l + 2$ is deletable for K. By Lemma 5.5 (ii), the vertex u is non good for $\Psi_\Gamma(K)$. Moreover, every vertex v with $\partial_K(v) \leq 2l + 1$ is good for $\Psi_\Gamma(K)$ since $\langle n^3(v) \rangle_\Gamma$ is the induced subgraph of $\Gamma - u$, where $\partial_K(u) = 2l + 2$. Hence, a vertex u is good if and only if $\partial_K(u) \leq 2l + 1$. We have $n \neq 1$ since a vertex v with $\partial_K(v) = 2l + 2$ is non good. Let v be a vertex with $\partial_K(v) = 2l + 1$. If v is not a vertex of S, then we have a contradiction to Lemma 5.5 (ii) that v is deletable for K. Hence, v is a vertex of S. Thus, $S \simeq S_1$, $l = 0$ and $n = 2$. Then, $|V(\Gamma)| \leq 9$ since $|K| = 3$, $|C_1| \leq 3$ and $|C_2| \leq 3$, a contradiction. □

Proof of Theorem 5.1. The minimal forbidden graphs with at most 9 vertices are revealed in Figure 9. This theorem follows by Proposition 5.2, 5.3 and 5.6. □

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