The Art of Discrete and Applied Mathematics

Non-commutative structures 2018

This special issue of The Art of Discrete and Applied Mathematics is dedicated to the proceedings of *Non-commutative structures 2018: A workshop in honor of Jonathan Leech*, which took place at the University of Primorska, in the period 23 – 27 May 2018. It is therefore devoted to original research in the field of noncommutative structures.

The study of noncommutative lattices began in 1949 with Ernst Pascual Jordan's paper *Über nichtkommutative Verbände* [1]. Jordan was a theoretical and mathematical physicist, a co-worker of Werner Karl Heisenberg, and he made a significant contribution to the development of quantum mechanics and, in particular, quantum field theory. Jordan introduced noncommutative lattices as an algebraic structure potentially suitable to encompass the logic of the quantum world.

The modern theory of noncommutative lattices began 40 years later, with Jonathan Leech's 1989 paper *Skew lattices in rings* [2]. Recently, noncommutative generalizations of lattices and related structures have seen an upsurge in interest, with new ideas and applications emerging, from quasilattices to skew Heyting algebras. Much of this activity derives in some way from the initiation, thirty years ago, by Jonathan Leech, of a research program into structures based on Pascual Jordan's notion of a noncommutative lattice. The present volume contains nine papers on noncommutative lattices, beginning with Leech's tutorial which provides the essential definitions and main structural results of the theory, thus enabling a potentially uninitiated reader to follow the papers of this volume. It ends with a list of open problems that were posed during the NCS 2018 conference. As such, this volume aims to present the breadth of contemporary research in the area, with contributions from international and Slovenian mathematicians. Many of the papers connect noncommutative lattices to other mathematical structures, like (dual) discriminator varieties, graphs, partitions and groupoids.

Karin Cvetko-Vah, Michael Kinyon, Tomaž Pisanski

Guest Editors

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THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P2.01
https://doi.org/10.26493/2590-9770.1282.e7a
(Also available at http://adam-journal.eu)

My journey into noncommutative lattices and their theory

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Received 30 January 2019, accepted 28 May 2019, published online 7 August 2019

Abstract

This paper describes the motivations leading to a renewed interest in the study of noncommutative lattices, and especially skew lattices, beginning with the initial work of the author. Not only are primary concepts and results recalled, but recognition is given to the individuals involved and their particular contributions. It is the written version of a talk given at the NCS2018 workshop in May, 2018 in Portorož, Slovenia.

Keywords: Noncommutative, skew lattice, skew Boolean algebra, partial function algebra, distributivity.

Math. Subj. Class.: 06B75, 06E75

I started thinking about skew lattices in 1983, while visiting Case Western Reserve University as a guest of Charles Wells. My connection with Charles was a common interest in the cohomology of monoids. I had published a paper that presented a new type of cohomology for monoids in the *Memoirs of the American Mathematical Society* in 1975 and Charles had published a follow-up paper in the *Semigroup Forum* in 1978 that connected my work to a general approach to cohomology theories due to Jonathan Beck. In my office at Case-Western I was studying the Wells-Beck approach for specific classes of monoids. In the case where the underlying monoid was a semilattice, I was led to consider bands whose maximal semilattice image was the given semilattice. Now, every band that arose within the Wells-Beck confines was *regular* in that it satisfied the identity, $xyxzx = xyzx$. This led me to look at the occurrence of regular bands in other mathematical contexts, and in particular, to their occurrence *and behavior* as multiplicative subsemigroups of a ring. This in turn led me straight to skew lattices.

Suppose first that we are given a multiplicative semilattice of idempotents L in a ring R. (L is thus closed and commutative under multiplication.) It is well known that L will generate a lattice L' of idempotents with the meet and join given by

$$x \wedge y = xy \quad \text{and} \quad x \vee y = x + y - xy \text{ (the \textit{quadratic join})}.$$

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Indeed if we include 0 in L and throw in the relative complement $x \setminus y = x - xy$, L generates a generalized Boolean algebra of idempotents. It will be fully Boolean if a top element is generated from L, and in particular if R has an identity 1 that is thus generated.

The obvious question: what occurs for a multiplicative band S of idempotents in a ring R, be S regular or otherwise? Well the following occur:

(1) In general, these two operations need not generate a larger class of idempotents that is closed under both operations . . . even if S is known to be regular.

(2) But, if S is known to be *left regular* ($xyx = xy$) or *right regular* ($xyx = yx$), then S generates a set of idempotents S' that is closed under both operations above.

(3) The resulting algebra $(S', \wedge, \vee)$ is a *skew lattice* in that $\wedge$ and $\vee$ are associative, idempotent binary operations that together satisfy the absorption identities:

$$x \wedge (x \vee y) = x = (y \vee x) \wedge x \quad \text{and} \quad x \vee (x \wedge y) = x = (y \wedge x) \vee x.$$

Given that $\wedge$ and $\vee$ are associative and idempotent, these identities are equivalent to the *basic dualities*:

$$u \wedge v = u \text{ iff } u \vee v = v \quad \text{and} \quad u \wedge v = v \text{ iff } u \vee v = u.$$

(4) If S is left regular, then so is $(S', \wedge)$ with $(S', \vee)$ being right regular. Dual remarks hold when S is right regular.

(5) Conversely, given any skew lattice $(S, \wedge, \vee)$ both reducts $(S, \wedge)$ and $(S, \vee)$ are regular with one operation being left regular iff the other is right regular.

Skew lattices in general had a number of other discernable properties:

(1) A *natural partial order*: $x \leq y \Leftrightarrow x \wedge y = x = y \wedge x \Leftrightarrow x \vee y = y = y \vee x$.

(2) A *natural quasiorder*: $x \preceq y \Leftrightarrow x \wedge y \wedge x = x \Leftrightarrow y \vee x \vee y = y$.

(3) A *natural congruence* $\mathcal{D}$:

$$\begin{aligned} x \,\mathcal{D}\, y \quad &\text{iff} \quad x \wedge y \wedge x = x \;\&\; y \wedge x \wedge y = y \\ &\text{iff} \quad x \vee y \vee x = x \;\&\; y \vee x \vee y = y. \end{aligned}$$

(4) **Clifford-McLean Theorem:** *given a skew lattice $(S, \wedge, \vee)$, $(S/\mathcal{D}, \wedge, \vee)$ is its maximal lattice image and the $\mathcal{D}$-classes are its maximal* anti-commutative *subalgebras in that:*

$$x \wedge y = y \wedge x \Leftrightarrow x = y \Leftrightarrow x \vee y = y \vee x$$

holds.

Also $x \wedge y = y \vee x$ holds in every $\mathcal{D}$-class. Thus there are two basic subvarieties of skew lattices:

- Lattices (everybody commutes).
- *Anti-lattices*, also called *rectangular* skew lattices (no nontrivial commutation).

The Clifford-McLean Theorem thus states that *every skew lattice is a lattice of anti-lattices*. See Figure 1 below.

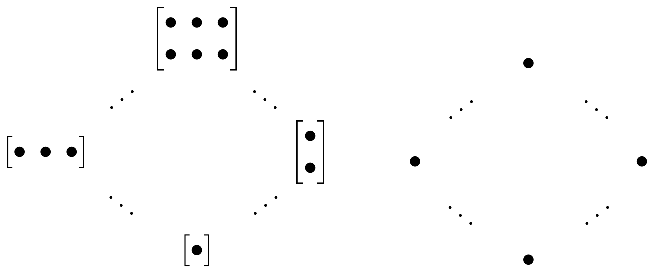


Figure 1: A lattice of anti-lattices.

Here are two more basic subvarieties:

- *Left-handed skew lattices:* $x \wedge y \wedge x = x \wedge y$ & $x \vee y \vee x = y \vee x$. $(S, \wedge)$ is left regular and thus $(S, \vee)$ is right regular.
- *Right-handed skew lattices:* $x \wedge y \wedge x = y \wedge x$ & $x \vee y \vee x = x \vee y$. $(S, \wedge)$ is right regular and thus $(S, \vee)$ is left regular. Their intersection is, of course, the variety of lattices.

(5) **Kimura's Theorem:** *If S_L and S_R are the maximal left- and right-handed images of a skew lattice S, the induced commuting diagram of epimorphisms is a pullback.*

$$\begin{array}{ccc} S & \longrightarrow & S_L \\ \downarrow & & \downarrow \\ S_R & \longrightarrow & S/\mathcal{D} \end{array}$$

Thus S is isomorphic to the fibered product: $S_R \times_{S/\mathcal{D}} S_L$.

Since both subvarieties are term equivalent, to the extent that one understands one, one understands the other, and thus to a large extent skew lattices in general. Both theorems above are so-named after two similar theorems about bands and regular bands respectively.

Here are *possible* properties that *do* occur in any skew lattice of idempotents in a ring:

(1) *Symmetry*: $x \wedge y = y \wedge x$ iff $x \vee y = y \vee x$. (A very nice condition.)

(2) *Distributivity*:

$$x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x), \text{ and}$$
$$x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x).$$

(3) *Cancellation*:

$$x \wedge y = x \wedge z \text{ and } x \vee y = x \vee z \Longrightarrow y = z, \text{ and}$$
$$x \wedge z = y \wedge z \text{ and } x \vee z = y \vee z \Longrightarrow x = y.$$

Some general facts:

- Neither distributivity as defined nor cancellation implies the other.
- Cancellation does imply that a skew lattice is symmetric.
- Neither distributive identity implies the other.
- But for *symmetric* skew lattices, the two distributive identities *are* equivalent.
- In the *symmetric* case, every pairwise commuting subset generates a sublattice.
- A non-symmetric example exists with 3 commuting generators, that is not a lattice.
- Clearly, *maximal left (right) regular bands of idempotents in a ring form skew lattices that have all three properties.*

Another *possible* property: A band is *normal* if it is *mid-commutative*: $xyzw = xzyw$. Normal bands are easily seen to be regular. A skew lattice $(S, \wedge, \vee)$ is *normal* if its $\wedge$-reduct $(S, \wedge)$ is normal. (The $\vee - \wedge$ dual is *conormal*.) Clearly distributive skew lattices and normal skew lattices form subvarieties of skew lattices. So do symmetric skew lattices.

Some theorems:

- *A normal skew lattice S is distributive iff $S/\mathcal{D}$ is a distributive lattice.*
- *Normal, distributive skew lattices are characterized by the identity:*

$$x \wedge (y \vee z) \wedge w = (x \wedge y \wedge w) \vee (x \wedge z \wedge w).$$

- *Normal, symmetric and distributive (NSD) skew lattices are characterized by:*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \textit{and} \quad (y \vee z) \wedge w = (y \wedge w) \vee (z \wedge w).$$

- *A normal band of idempotents in a ring generates an NSD skew lattice.* (Normal skew lattices in rings need no longer be left or right-handed. An associative *cubic join* is given by $x \nabla y = x + y + yx - xyx - yxy$. In left- or right-handed contexts $x \nabla y$ reduces to the previous *quadratic join* $x + y - xy$.)
- *Maximal normal bands in a ring form skew Boolean algebras* (defined below).
- *If the idempotents of a ring are closed under multiplication, then they are normal as a band and thus form a skew Boolean algebra.*

A *skew Boolean algebra* (SBA) is an algebra $(S, \wedge, \vee, \setminus, 0)$ for which $(S, \wedge, \vee)$ is an NSD skew lattice, $\setminus$ is a binary operation and 0 is a constant such that for all x, y:

(i) $0 \wedge x = 0 = x \wedge 0$ and hence $0 \vee x = x = x \vee 0$;

(ii) $(x \wedge y \wedge x) \vee (x \setminus y) = x = (x \setminus y) \vee (x \wedge y \wedge x)$ and
$(x \wedge y \wedge x) \wedge (x \setminus y) = 0 = (x \setminus y) \wedge (x \wedge y \wedge x)$.

This brings us to a second class of motivating examples: *partial function algebras*.

If we are given sets A and B, let $\mathcal{P}(A, B)$ denote the set of all partial functions f from A to B. Special case: B is $\{1\}$. Here $\mathcal{P}(A, \{1\})$ may be identified with the power set $\mathcal{P}(A)$ of A under the map $f \to \mathrm{dom}(f)$. $\mathcal{P}(A)$ forms, of course, a typical example of a Boolean algebra.

Likewise $\mathcal{P}(A, B)$ forms a typical example of a skew Boolean algebra. One can do this in two ways: a left-handed way and a right-handed way. For both ways, $\mathcal{P}(A)$ forms the maximal Boolean algebra image. We consider the right-handed case, the left-handed version being dual. Given f, g in $\mathcal{P}(A, B)$ with respective domains F and G, set

$$f \wedge g = g|_{F\cap G}; \qquad f \vee g = f \cup g|_{G-F}; \qquad f \setminus g = f|_{F-G}; \quad \text{and} \qquad 0 = \emptyset.$$

Basic theorems:

(1) *Every left-handed skew Boolean algebra can be embedded in a left-handed partial function algebra.*

(2) *Every right-handed skew Boolean algebra can be embedded in a right-handed partial function algebra.*

(3) *Every skew Boolean algebra is the fibered product of a left-handed SBA and right-handed SBA over their common maximal generalized Boolean algebra image.*

(4) *A skew lattice can be embedded in a skew Boolean algebra iff it is normal, distributive and symmetric.*

Skew lattices in rings and partial function algebras formed the concrete bases of my first three full-length publications on skew lattices:

- "Skew lattices in rings" appeared in *Algebra Universalis* in 1989 [44];
- "Skew Boolean algebras" appeared in *Algebra Universalis* in 1990 [45];
- "Normal skew lattices" appeared in *Semigroup Forum* in 1992 [46].

The communicating editor for all three, Boris Schein, had once published a paper on a class of noncommutative lattices in a Russian journal that was later translated into English by the AMS.

What all was I doing between 1983 and the first publication in 1989?

(1) I gave a number of talks to various groups: seminars; AMS-MAA metings.

(2) I kept polishing up things: examples; proofs, etc.

(3) I was also preoccupied with writing papers on other topics.

A third class of examples attracted my attention in my early research: *primitive* skew lattices, which consisted of exactly two $\mathcal{D}$-classes, an upper class and a lower class: $A > B$. As it turned out, a complete characterization of these primitive algebras is easily given.

Given $A > B$, A is partitioned by B-cosets in A and B is partitioned by A-cosets in B. Here, for each a in A, its *B-coset* is $B \vee a \vee B = \{b \vee a \vee b \mid b \in B\} \subseteq A$; likewise, for each b in B, it's *A-coset* is $A \wedge b \wedge A = \{a \wedge b \wedge a \mid a \in A\} \subseteq B$. Thus given a, a' in A, either $B \vee a \vee B = B \vee a' \vee B$ or both cosets are disjoint. Similar remarks hold for A-cosets in B. What is more, all cosets in A or B are mutually isomorphic. In particular, given any B-coset A_i in A and any A-coset B_j in B, an isomorphism $\varphi_{ij}\colon A_i \cong B_j$ is given by $\varphi_{ij}(a) = b$ for $a \in A$ and $b \in B$ iff $a > b$. This gives us a picture something like the one in Figure 2.

Thus, *all* cosets are mutually isomorphic with the *coset isomorphisms* determining $\wedge$ and $\vee$ *between* cosets in A and B. That is, for all a in A_i and all b in B_j:

$$a \wedge b = \varphi_{ij}(a) \wedge b \;\&\; b \wedge a = b \wedge \varphi_{ij}(a) \;\text{ and}$$
$$a \vee b = a \vee \varphi_{ij}^{-1}(b) \;\&\; b \vee a = \varphi_{ij}^{-1}(b) \vee a.$$

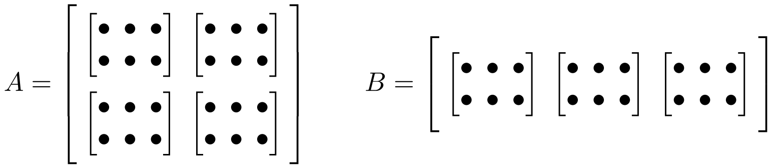


Figure 2: A coset decomposition showing isomorphic cosets.

Conversely, this characterization provided a general recipe for constructing primitive algebras.

In this situation, the *coset weight* $\gamma(A, B)$ is the common size of all cosets in $A \cup B$. The index $[A : B]$ of B in A is the number of B-cosets in A; dually the index $[B : A]$ of A in B is the number of A-cosets in B.

In the finite case, as with finite groups one thus has a *Lagrange-type* theorem:

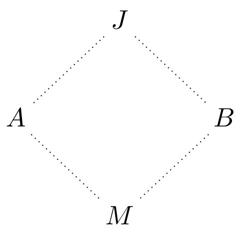
Theorem 0.1. *If A and B in the primitive skew lattice $A > B$ are both finite, then:*

$$|A| = [A : B]\gamma(A, B) \quad \textit{and} \quad |B| = [B : A]\gamma(A, B).$$

Corollary 0.2. *Given $A > B$, if $|A|$ & $|B|$ are finite and coprime, then $\gamma(A, B) = 1$ and*

$$\forall a \in A, \forall b \in B\colon (a \wedge b = b \wedge a = b \;\&\; a \vee b = b \vee a = a).$$

These algebras form the basis for an analysis of skew lattice structure – or one might say, of the *architecture* of skew lattices. For instance, consider a *skew diamond* of $\mathcal{D}$-classes in a skew lattice S. Here J and M are the join and meet $\mathcal{D}$-classes of $\mathcal{D}$-classes A and B.



If both $|A|, |B| < \infty$, then also both $|J|, |M| < \infty$. Indeed, both $|J|$ and $|M|$ divide $|A||B|$.

If S is cancellative, then $|A||B| = |J||M|$ (João Pita da Costa 2012 Dissertation [54]).

As consequences we have:

(1) The union of all singleton $\mathcal{D}$-classes forms a sublattice $\mathcal{Z}_S$ of S.

(2) $\mathcal{Z}_S$ is the center of S:

$$\begin{aligned}\mathcal{Z}_S &= \{x \in S \mid x \wedge y = y \wedge x \text{ for all } y \in S\}\\ &= \{x \in S \mid x \vee y = y \vee x \text{ for all } y \in S\}.\end{aligned}$$

(3) The union of all finite $\mathcal{D}$-classes is a subalgebra.

(4) Given prime p, the union of all $\mathcal{D}$-classes of p-power size is a subalgebra.

With the exception of Pita da Costa's result, the above formed part of the content the fourth paper:

- "The geometric structure of skew lattices" appeared in *Transactions of the American Mathematical Society* in 1993 [47].

First Contact! Some time after the publication of my paper on skew Boolean algebras, I received a letter from Robert J. Bignall of Monash University in Australia. In it I discovered that a paper entitled "Boolean skew algebras" had been published in 1980 by his dissertation advisor, William Cornish [15]. (Some may be aware that Bill Cornish was one of the first to publish work in response to the extension of Stone duality to bounded distributive lattice by Hillary Priestly at Oxford.) Bob Bignall's 1976 dissertation written in South Australia was entitled *Quasiprimal Varieties and Components of Universal Algebra* [5]. It began with a chapter entitled "Quasi-Boolean skew lattices". While not term equivalent to the skew Boolean algebras I had studied, both types of algebras were quite similar in spirit. Some will find it interesting that in his dissertation Bob studied sheaf theoretic representations of these algebras. His interest in noncommutative Boolean algebras also manifested itself in his 1991 paper, "A non-commutative multiple-valued logic", that appeared in the *Proceedings of the 21st International Symposium on Multiple-Valued Logic*, sponsored by the IEEE Computer Society [6]. Indeed he continued to author further papers in this area.

I also received a copy of a paper he had submitted to *Algebra Universalis*. It was about a class of algebras very much like my skew Boolean algebras except that his relative complement was different, and of course, there was a slightly different axiom system. Also, his algebras had close connections to what are called *discriminator algebras* which Stanley Burris had called "the most successful generalization of Boolean algebras to date" in his 1981 text on universal algebra [11].

Later on Bob visited me for a few days in Santa Barbara. One morning, after breakfast at a seaside restaurant, I shared some thoughts on how our two types of algebras could be merged. The means to do this was the concept of any two elements having a meet relative to the natural partial order – their *intersection* – as opposed to their skew meet (i.e. "meet" in a general noncommutative context). Bignall's difference essentially involved subtracting the intersection from one of the two given elements while mine involved subtracting their skew meet. This led us to the variety of skew Boolean algebras with intersections. As it turned out its subvariety of right-handed [or left-handed] algebras is term-equivalent to the variety of pointed discriminator algebras. Both types of algebras were the subject of our joint revision of Bob's earlier paper entitled "Skew Boolean algebras and discriminator varieties" that appeared in *Algebra Universalis* in 1995 [7]. The communicating editor was Stanley Burris.

It was during this time that Alfred Clifford passed away. A symposium in his honor was held at Tulane University where he had taught for many years. There I gave survey talk on recent developments in skew lattice theory. This talk was published as a survey article in the *Semigroup Forum* in 1996 [48]. This brought to six the number of articles I had published on skew lattices since 1989. I would not publish another until 2002.

In the meantime, much of my focus was on inverse monoids and especially the categorical foundations of symmetric inverse monoids and their duals. In particular, I coauthored a paper on dual symmetric inverse monoids with a colleague from Tasmania, Des FitzGerald, who happily was at this workshop. Our paper, "Dual symmetric inverse monoids and representation theory," appeared in 1998 in the *Journal of the Australian Mathematical Society* [31]. A main feature of inverse monoids is the fact that their elements have a natural partial order which in many cases has natural meets (or *intersections*, in our terminology). Natural meets received a good bit of attention in the papers I published during this period. My perspective on inverse monoids at this time was, no doubt, influenced by the paper coauthored with Bob Bignall. Connections between inverse monoids or inverse semigroups in general, and Boolean structures (often with intersections) has been a subject of study in recent years. For an extended exposition of these and related matters see Friedrich Wehrung's 2017 Springer monograph [60].

During this period, however, I started hearing from graduate students in Europe and Australia. One of the first was Gratiela Laslo, who was writing a dissertation on noncommutative lattices at Babes-Bolyai University in Cluj-Napoca, Romania. Her research, plus several insights from me, led to a seventh paper, co-authored with Gratiela and entitled "Green's equivalences on noncommutative lattices" that appeared in 2002 in the Szeged journal, *Acta Scientiarum Mathematicarum* [41]. Here, all involved algebras are generalizations of skew lattices with many results applying to skew lattices. The paper attempted to provide a coherent scheme consisting of four varieties into which nearly all of the previously studied types of noncommutative lattices could fit.

As it turns out, I wasn't Gratiela's only human connection to noncommutative lattices. She was also in contact with Professor Gheorghe Farcas of Petru Maior University in Targu Mures, Romania. Professor Farcas had published a number of papers on noncommutative lattices. But by the time I visited Gratiela in Targu Mures in 2005, he had been retired for some years and in ill health. Thus, regrettably, I never met him.

In the late 1990s I also started hearing from a protégé of Bob Bignall, Mathew Spinks. In an earlier paper, I had asked whether the dual pair of distributive identities that characterize distributive skew lattices are equivalent for skew lattices as they are for lattices. Spinks determined that they were not, publishing a set of four 9-element counter-examples in the *Semigroup Forum* in 2000 [58]. But there was more. Having had earlier access to his examples, I had noticed that they were non-symmetric. I asked Matthew if the two identities might be equivalent in the case of symmetric skew lattices. He initially found a computer-generated affirmative proof consisting of 757 steps. He was then able to reduce it to a 368-step proof that he published in a Monash University report: *Automated Deduction in Non-Commutative Lattice Theory* [57]. Several further reductions ensued. Finally a more standard "human" proof was obtained by Karin Cvetko-Vah and published in a short paper in the *Semigroup Forum* in 2006 [16]. Over the years Matthew and I have co-authored four papers:

- "Skew Boolean algebras derived from generalized Boolean algebras", in *Algebra Universalis* [49];
- "Cancellation in skew lattices" (with K. Cvetko-Vah and M. Kinyon), in *Order* [22];
- "Skew lattices and binary operations on functions" (with K. Cvetko-Vah), in *Journal of Applied Logic* [26];
- "Varieties of skew Boolean algebras with intersections", in *Journal of the Australian Mathematical Society* [50].

The last paper characterizes the lattice of subvarieties of this class of algebras. Spinks has also published very good papers with other individuals, including, of course, Bob Bignall.

Not all of these are about skew lattices. One of things that I appreciate about Matthew is his impressive knowledge of past and ongoing developments in universal algebra and logic. His ability to inject scholarly insights of relevance to a paper, or results of others that are critical to obtaining a proof or even a smoother proof, can make a decent paper good, and a good paper really good. It is interesting that both Matthew and Gratiela had initial connections to individuals who had engaged in serious research on noncommutative lattices.

This third graduate student was Karin Cvetko-Vah. I became aware of her in the early years of the new millennium, when she wrote and published several papers on multiplicative bands and skew lattices in rings. I recall reading her papers and discovering ideas and results that I had not considered. We first met at a Linear Algebra conference at Lake Bled in 2005. Since then Karin has written further papers about skew lattices in rings, three co-authored with me:

- "Associativity of the ∇-operation on bands in rings", in *Semigroup Forum* [23];
- "On maximal idempotent-closed subrings of $M_n(\mathbb{F})$", in *International Journal of Algebra and Computation* [24];
- "Rings whose idempotents form a multiplicative set", in *Communications in Algebra* [25].

The last two were on rings whose idempotents are closed under multiplication, and thus form, with additional operations, a skew Boolean algebra.

One of the things that has helped Karin and I work well together – besides the fact that she is very bright and hard-working – is her background in operator theory and in particular, matrix theory. Her dissertation advisor, Matjaž Omladič, was connected to a research group that included Peter Fillmore, Gordon MacDonald and Heydar Radjavi who among other things, studied multiplicative bands of idempotents in matrix rings. One result: *Every multiplicative band of idempotents in a matrix ring is simultaneously triangularizable.* Consequently *every skew lattice of idempotents in a matrix ring is simultaneously triangularizable.* Nice to know when you're looking for examples! In any case, with this background it's not that surprising that Karin and I might meet up.

Karin has authored and co-authored a number of important papers on the general structure of skew lattices. Besides her connection to Spinks' distributivity result, there is, e.g., her 2011 paper "On strongly symmetric skew lattices" that appeared in *Algebra Universalis* [17]. Another important contribution was also her involvement in research on duality theory extending the work of M. H. Stone and Hillary Priestly to skew Boolean algebras and strongly distributive skew lattices. Early in 2010, I mentioned to Karin that extending Stone duality for (generalized) Boolean algebras to a duality theory for skew Boolean algebras would be a worthy project. She brought this to the attention of two colleagues in Ljubljana, Andrej Bauer and Ganna Kudryavtseva. This led to a series of publications on duality that include:

- G. Kudryavtseva, "A refinement of Stone duality to skew Boolean algebras", in *Algebra Universalis* [36];
- A. Bauer and K. Cvetko-Vah, "Stone duality for skew Boolean algebras with intersections", in *Houston Journal of Mathematics* [2];
- A. Bauer, K. Cvetko-Vah, M. Gerkhe, G. Kudryavtseva and S. J. van Gool, "A noncommutative Priestly duality", in *Topology and Applications* [3].

Karin had met Mai Gehrke at a math conference in Switzerland and Sam van Gool was Mai's student. Further studies in duality are listed below. Karin has also explored connections to Church algebras (with Antonino Salibra) [29], skew Heyting algebras [18] and noncommutative toposes (with Jens Hemelaer and Lieven Le Bruyn) [21]. Again, see the references near the end.

Clearly a major contribution has been Karin's ability to engage the interest of others in some aspect of skew lattices. Indeed many are here because of an encounter with her. Besides those already mentioned, there is her wonderful student, João Pita da Costa, who we will mention from time to time.

As for engaging the interest of others in skew lattices, Matthew has also not been idle. In the summer of 2007 he attended a conference on automated deduction at the University of New Mexico. There he met Michael Kinyon, a broadly published mathematician working in various areas of algebra and even beyond. The two began discussing skew lattices. By 2008 Michael and I started having e-conversations, initially about cancellative skew lattices. Before long Matthew and Karin joined in. Long story short, this led to a sequence of three papers that extended significantly earlier research on skew lattice architecture and other aspects of skew lattice theory. They were all co-authored by Michael and me, at least.

- The previously mentioned, "Cancellation in skew lattices" (with K. Cvetko-Vah and M. Spinks) [22];
- "Categorical skew lattices", in *Order* [32];
- "Distributivity in skew lattices" (with J. Pita da Costa), in *Semigroup Forum* [33].

The first paper was a thorough study of cancellative skew lattices. To begin, they also form a subvariety. A characterization by Michael of these algebras via a finite list of forbidden algebras was also given, along with other nice results. One was Karin's "Parallelogram Laws" for cancellative skew lattices taken from her dissertation: given $\mathcal{D}$-classes A and B and their join and meet $\mathcal{D}$-classes J and M, one has $[J : B] = [A : M]$ and $[B : J] = [M : A]$. Likewise, $[J : A] = [B : M]$ and $[A : J] = [M : B]$.

A skew lattice is *categorical* when the nonempty composition of successive coset isomorphisms is also a coset isomorphism. Distributive skew lattices are categorical and in particular, skew lattices of idempotents in rings are categorical. Categorical skew lattices were studied in the second paper. Special attention was given to *strictly categorical* skew lattices where the composition of successive coset bijections arising in any chain of $\mathcal{D}$-classes $A > B > C$ is always nonempty. They include normal skew lattices and their conormal duals, as well as all primitive skew lattices. Categorical skew lattices form a proper subvariety of skew lattices with the strictly categorical ones forming a properly smaller subvariety. Here are some as yet unanswered queries:

- Do the normal and conormal subvarieties jointly generate the strictly categorical variety?
- What subvariety does the class of primitive skew lattices generate?

Here is a nice result: *a strictly categorical skew lattice S is distributive iff its maximal lattice image $S/\mathcal{D}$ is distributive*. A nice counting theorem quoted in this paper came from João's 2012 *Algebra Universalis* publication "Coset laws for categorical skew lattices" [53]: given a strictly categorical chain $A > B > C$ of $\mathcal{D}$-classes, if A and C are finite, then B is also finite; moreover, $[C : A] = [C : B] \times [B : A]$ and dually $[A : C] = [A : B] \times [B : C]$.

Our third paper on distributive skew lattices was coauthored with João and appeared in the *Semigroup Forum* in 2015. If S is distributive then

(1) $S/\mathcal{D}$ is distributive and

(2) each $\mathcal{D}$-class chain $A > B > C$ is distributive.

Condition (2), called *linear distributivity*, turns out to be a mild generalization of being strictly categorical. Now, what about the converse? Do (1) and (2) imply S is distributive? The answer is NO in general: Spinks' examples suffice. But it is YES, if S is also symmetric. (Here is another really crisp result about distributivity occurring in the presence of symmetry.) This and other aspects of distributivity are studied.

While on the topic of skew lattice architecture, further research in this area has been carried out by Karin and/or João. The relevant published papers, all appearing since 2010, are often recognized by such phrases as "coset structure" or "coset laws" appearing in the title. Again, nice counting theorems have arisen, as we have seen.

While working on my paper with Matthew on the lattice of varieties of skew Boolean algebras with intersections, the question arose as to whether free skew Boolean algebras in general have intersections. The answer is trivially yes in the finite case, but what about the infinite case? Oddly enough, free skew Boolean algebras had never been formally studied, probably due to the fact that so much else was going on. So I emailed several individuals, asking what do free SBAs look like and do they have intersections. Someone got right on the case, Ganna (Anya) Kudryavtseva, who had been very involved in the study of duality. This led to two papers:

- "Free skew Boolean algebras", in *International Journal of Algebra and Computation* [40];
- "Free skew Boolean intersection algebras and set partitions", in *Order* [38].

The first was co-authored by Anya and me, but the second was her work. And yes, free skew Boolean algebras do have intersections. To give a glimpse of what occurs in the finite case, the free left-handed skew Boolean algebra on n generators is, to within isomorphism:

$${}_L\mathbf{SBA}_n \cong \mathbf{1}^{\binom{n}{0}} \times \mathbf{2}^{\binom{n}{1}} \times \mathbf{3}_{\mathbf{L}}^{\binom{n}{2}} \times \cdots \times (\mathbf{n+1})_{\mathbf{L}}^{\binom{n}{n}}.$$

Here $(\mathbf{k+1})_{\mathbf{L}}$ is the primitive left-handed skew Boolean algebra on $\{0, 1, \ldots, k\}$ with 0 being the bottom element and $\{1, \ldots, k\}$ forming the upper $\mathcal{D}$-class. Similar decompositions in the finite case for algebras with intersection are given in Anya's paper. But instead of binomial coefficients $\binom{n}{k-1}$, the respective powers are given by Stirling numbers of the 2nd kind, $\left\{ {n+1 \atop k} \right\}$. (See [38, Theorem 28].)

Research on skew lattices and related subjects continues. We mention next a number of papers that have appeared (but not all!), loosely arranging them by topic. It is intended to give a sense of the current state of play. References are given at the end of the paper.

Further work on duality

- G. Kudryavtseva, "A dualizing object approach to noncommutative Stone duality", *Journal of the Australian Mathematical Society* [37];
- G. Kudryavtseva and M. V. Lawson, "Boolean sets, skew Boolean algebras and a non-commutative Stone duality", *Algebra Universalis* [39].

Partial function algebras

- J. Berendsen, D. N. Jansen, J. Schmaltz and F. W. Vaandrager, "The axiomatization of override and update", *Journal of Applied Logic* [4]. (This is related to the above mentioned paper by Cvetko-Vah, Leech and Spinks appearing in the same journal.)

Connections with logic, discriminator varieties and other systems

- R. J. Bignall and M. Spinks, "Propositional skew Boolean logic", in: *Proceedings 26th IEEE International Symposium on Multiple-Valued Logic*, IEEE Computer Society Press [8];
- R. J. Bignall and M. Spinks, "Implicative BCS-algebra subreducts of skew Boolean algebras", *Scientiae Mathematicae Japonicae* [9];
- J. Cirulis, "Nearlattices with an overriding operation", *Order* [14];
- K. Cvetko-Vah and A. Salibra, "The connection of skew Boolean algebras and discriminator varieties to Church algebras", *Algebra Universalis* [29];
- D. Saveliev, "Ultrafilter extensions of linearly ordered sets", *Order* [56];
- M. Spinks and R. Veroff, "Axiomatizing the skew Boolean propositional calculus", *Journal of Automated Reasoning* [59].

Cosets and skew lattice architecture

- K. Cvetko-Vah and J. Pita da Costa, "On coset laws for skew lattices in rings", *Novi Sad Journal of Mathematics* [27];
- K. Cvetko-Vah and J. Pita da Costa, "On coset laws for skew lattices", *Semigroup Forum* [20];
- J. Pita da Costa, "On the coset structure of a skew lattice", *Demonstratio Mathematica* [52];
- J. Pita da Costa, "Coset laws for categorical skew lattices", *Algebra Universalis* [53];
- J. Pita da Costa, "On the coset category of a skew lattice", *Demonstratio Mathematica* [55].

And beyond

- B. A. Alaba, M. Alamneh and Y. M. Gubena, "Skew semi-Heyting algebras", *International Journal of Computing Science and Applied Mathematics* [1];
- D. Carfi and K. Cvetko-Vah, "Skew lattices on the financial events plane", *Applied Sciences* [12];
- K. Cvetko-Vah, "On skew Heyting algebras", *Ars Mathematica Contemporanea* [18];
- K. Cvetko-Vah, "Noncommutative frames", *Journal of Algebra and Its Applications* [19];
- K. Cvetko-Vah, J. Hemelaer and L. Le Bruyn, "What is a noncommutative topos?", *Journal of Algebra and Its Applications* [21];
- K. Cvetko-Vah, M. Sadrzadeh, D. Kartsaklis, and B. Blundell, "Non-commutative logic for compositional distributional semantics", *Proceedings of the 24th Workshop on Logic, Languages, Information and Computation* [28];

- R. Koohnavard and A. Borumand Saeid, "(Skew) filters in residuated skew lattices", *Scientific Annals of Computer Science* [34];
- R. Koohnavard and A. Borumand Saeid, "(Skew) filters in residuated skew lattices II", *Honam Mathematical Journal* [35];
- L. Le Bruyn, "Covers of the arithmetic site" [43];
- Y. Zhi, X. Zhou and Q. Li, "Rough sets induced by ideals in skew lattices", *Journal of Intelligent and Fuzzy Systems* [61];
- Y. Zhi, X. Zhou and Q. Li, "Residuated skew lattices", *Information Science* [62].

(Again, these papers, and all mentioned in this article, are not intended to collectively give a comprehensive list of all publications related to skew lattices.)

Returning now to regular bands, as already indicated, regular bands and skew lattices are closely connected. I like to think that skew lattices are what regular bands can be when they grow up – just as semilattices can "grow" into lattices or even Boolean algebras. ☺ (This, of course, requires a nourishing environment, such as the multiplicative semigroup of some ring.) In any case, interest in regular bands is not limited to those studying semigroups or skew lattices. In their introductory remarks to *Cell Complexes, Poset Topology and the Representation Theory of Algebras Arising in Algebraic Combinatorics and Discrete Geometry* (to appear in the *Memoirs of the American Mathematical Society*) [51], Stuart Margolis, Franco Saliola and Benjamin Steinberg describe the relevance of left regular bands to various areas of mathematics, and mention many of the individuals involved along with selected relevant publications. For instance here are a few:

- K. S. Brown, "Semigroups, rings and Markov chains", *Journal of Theoretical Probability* [10];
- F. Chung and R. Graham, "Edge flipping in graphs", *Advances in Applied Mathematics* [13];
- P. Diaconis, "From shuffling cards to walking around the building: an introduction to modern Markov chain theory", *Proceedings of the International Congress of Mathematicians* [30];
- F. W. Lawvere, "Qualitative distinctions between some toposes of generalized graphs", *Categories in Computer Science and Logic, Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference* [42]. (Lawvere used the term "graphic monoid" for left regular band.)

The authors then further develop many of these connections in their monograph. In addition, the semigroup algebra $\mathcal{K}(B)$ of a finite left regular band B where $\mathcal{K}$ is a commutative ring with identity is studied along with homological aspects of its left module category. One of the things they discovered is that if the band B is finite and left regular, then $\mathcal{K}(B)$ has a *right identity*, that is, an element β such that $x\beta = x$ for all $x \in \mathcal{K}(B)$ [51, Theorem 4.2(2)]. Moreover, if this right identity is unique, then it is *the* multiplicative identity for $\mathcal{K}(B)$. Let's see why, using skew lattice theory.

First, suppose the elements of B are $b_1, b_2, \ldots, b_n$. Upon identifying each b_i with $1b_i$ in $\mathcal{K}(B)$, set $\beta = b_1 \vee b_2 \vee \cdots \vee b_n$, where $x \vee y = x + y - xy$. Then β lies in the top $\mathcal{D}$-class of the left-handed skew lattice S generated from B in $\mathcal{K}(B)$. Thus for all $b_j \in B$, $b_j \vee \beta = \beta$ since $(S, \vee)$ is right regular. Thus by duality, $b_j \wedge \beta = b_j$, that is, $b_j\beta = b_j$ in $\mathcal{K}(B)$. But if this holds for the generators (over $\mathcal{K}$) of $\mathcal{K}(B)$, then $x\beta = x$ for all $x \in \mathcal{K}(B)$. Thus β is a right identity for $\mathcal{K}(B)$. It is important to note that $b_n < \beta$ in the natural ordering of the idempotents in $\mathcal{K}(B)$. Thus β behaves like a 2-sided identity for at least b_n. Thus also $\mathcal{K}(B)$ has an identity, namely β, if all outcomes obtained by permuting the factors of $b_1 \vee b_2 \vee \cdots \vee b_n$ agree, since β then behaves like a 2-sided identity for all the b_i which collectively generate $\mathcal{K}(B)$. Conversely and trivially, if $\mathcal{K}(B)$ has a multiplicative identity, then it can only be β, no matter in what order it is assembled. Several comments:

(1) The above β expands to the noncommutative inclusion-exclusion expression:

$$\sum b_j - \sum_{i<j} b_i b_j + \sum_{i<j<k} b_i b_j b_k - \cdots + (-1)^{n+1} b_1 b_2 b_3 \cdots b_n.$$

(2) If $\{g_1, g_2, \ldots, g_m\}$ is a set of generators for B, then $\gamma = g_1 \vee g_2 \vee \cdots \vee g_m$ must be a right identity for $\mathcal{K}(B)$. γ will be a 2-sided identity if all outcomes obtained by permuting the factors of $g_1 \vee g_2 \vee \cdots \vee g_m$ agree.

(3) A further refinement: if we choose the set $\{m_1, m_2, \ldots, m_k\}$ of all elements in B that are maximal relative to the natural partial ordering of B ($e \geq f$ iff $ef = f = fe$) and repeat the process to get $\mu = m_1 \vee m_2 \vee \cdots \vee m_k$, then μ is also a right-identity that is a 2-sided identity if all outcomes obtained by permuting the factors of $m_1 \vee m_2 \vee \cdots \vee m_k$ agree. (This set of m_is is a subset of any set of generators of B.)

(4) Returning to the main argument, one need only assume that B is a left regular band for which $B/\mathcal{D}$ is finite with say n $\mathcal{D}$-classes. In this case $b_1, b_2, \ldots, b_n$ is a cross-section of elements, one chosen from each $\mathcal{D}$-class. The $\mathcal{D}$-class of $\beta = b_1 \vee b_2 \vee \cdots \vee b_n$ must be the maximal $\mathcal{D}$-class in the generated skew lattice S, due to the Clifford-McLean Theorem. Since S is left-handed, again β is a right identity for all elements in B and thus all elements in $\mathcal{K}(B)$.

In their monograph the authors characterize those left regular bands B for which an identity exists in *all* cases of $\mathcal{K}(B)$, i.e., for any commutative ring $\mathcal{K}$ with identity. Clearly, if identities always exist, this is true for $\mathbb{Z}(B)$ where $\mathbb{Z}$ is the ring of integers. But, as authors note, the converse is easily see to hold: if $\mathbb{Z}(B)$ has an identity, so must $\mathcal{K}(B)$ for any commutative ring $\mathcal{K}$ with identity. This is in their Theorem 4.15, where the authors also give a graph theoretic characterization of those B for which all $\mathcal{K}(B)$ have an identity. Applying it requires some insight into the behavior of B, as indeed do the methods of (2) and (3).

Assuming $\mathcal{K}$ is nontrivial, the map $b \mapsto 1b$ gives an easy isomorphic embedding of B into the multiplicative semigroup of $\mathcal{K}(B)$, at which location a skew lattice can be generated from the copy of B in $\mathcal{K}(B)$, if B is left or right regular, but not necessarily for all regular bands. But this simple method can be modified in the general case as follows. Given *any* regular band B with its respective maximal left and right regular images, B_L and B_R, the Kimura Theorem for regular bands initiates a chain of isomorphic embeddings from B into the multiplicative semigroup of a ring that is a product of semigroup rings:

$$B \to B_L \times B_R \to \mathcal{K}(B_L) \times \mathcal{K}(B_R).$$

In this ring, the image of B will generate a skew lattice S under the standard operations $x \wedge y = xy$ and $x \vee y = x + y - xy$. Thus, *every regular band B can be embedded in the* *reduct* $(S, \wedge)$ *of a skew lattice* $(S, \wedge, \vee)$. In this case B is embedded in a well-behaved skew lattice. When $B/\mathcal{D}$ is finite, while the relevant ring need not have a right or left identity (unless B is left or right regular), it will have a *middle identity* m such that $xmy = xy$ for all x, y in $\mathcal{K}(B_L) \times \mathcal{K}(B_R)$. In particular $xmx = x$ for all x in the generated skew lattice S. Such an m is given by *any* of the idempotents in the maximal $\mathcal{D}$-class of S (in the usual ordering of $\mathcal{D}$-classes). Thus the existence of a right, left or middle identity in the ring depends on the existence and behavior of a maximal $\mathcal{D}$-class in S. An identity occurs precisely when this class reduces to a single point.

The reader will have noticed various bits of mathematical genealogy relative to the history of noncommutative lattices. Let me say a few words about my genealogy, although in doing mine we'll "stray" into a larger arena. What follows are two genealogy sequences with the dates being when the individual received their PhD (see Figure 3). Both begin in

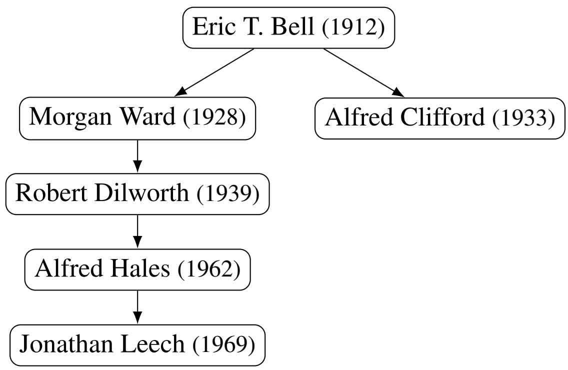


Figure 3: An advisor-student tree.

Pasadena, California with Eric Temple Bell directing dissertations at Caltech (California Institute of Technology) in the early 20th century. Indeed all you see occurs there until my advisor, Alfred Hales, received his PhD at Caltech and accepted a position at UCLA. While Bell's main interests were in number theory and related areas in algebra and analysis, with Dilworth we have arrived at a major figure in the developing theory of lattices. And while Hales may be more known for his work in combinatorics, especially Ramsey Theory (thus a co-winner of the Polya Prize), he made significant contributions to lattice theory. One surprising result, proved independently by Haim Gaifman, states that there exist countably generated complete Boolean algebras of arbitrarily high cardinality. The outside reader of my dissertation was another student of Bell, Alfred H. Clifford. By this time, he had moved to Tulane University, where he had already directed the dissertation of Naoki Kimura. But initially, after receiving his PhD he joined the Institute for Advanced Study, where he became an assistant to Hermann Weyl. Clifford was a master expositor, and in my early papers I benefitted greatly from his suggestions. (Two side-notes, courtesy of Professor Hales: Al and Alice Clifford were avid Bridge players as were the parents of Al Hales. Thus when both couples lived in Pasadena they knew each other. Also, both Alfreds attended Polytechnic School, a preparatory school in Pasadena, and though 31 years apart, in the middle grades both studied math under the nationally acclaimed teacher, Mary Ardis Schnebly.) Although they played different roles in my early career, to both Alfreds I owe a real debt of gratitude. Given another venue I would say more. But for now, to both gentlemen let me just say: Thank you! (And, of course, thank *you*, Mary Ardis Schnebly.)

As for E. T. Bell, he is known for a number of reasons. These include his study of Bell numbers that are named after him, although such a study was preceded in the notebooks of Ramanujan. (The n^{th} Bell number B_n is the number of distinct partitions of an n-element set.) Interestingly, Bell numbers appear in Anya Kudryavtseva's paper [38] where they are used to count the number of atomic $\mathcal{D}$-classes as well as the total number of atoms in a free left [right]-handed skew Boolean intersection algebra. Given n generators, these counts are respectively $B_{n+1} - 1$ and $B_{n+2} - 2B_{n+1}$. (Again, see [38, Theorem 28].)

In conclusion, in describing my journey into noncommutative lattice theory and in particular, skew lattices, I have focused not only on primary concepts and results, but also on the individuals involved in developing the current state of the subject, many of whom attended this workshop. Thankfully, I have not made this journey alone. To all of those who have been involved at its various stages, whether directly with me or not, I am grateful for your wonderful contributions. I must also thank Professors Tomaž Pisanski, Karin Cvetko-Vah and all others involved in the planning and running of the NCS2018 Workshop in Portorož and Piran – such a beautiful venue! In particular, thank you for making it possible for nearly all of my past coauthors to attend too. Your successful efforts are very much appreciated. And especially to Karin, thank you for your help in preparing the slides as well as the layout of this article. Again, it is much appreciated.

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P2.02
https://doi.org/10.26493/2590-9770.1259.9d5
(Also available at http://adam-journal.eu)

A graph-theoretic method to define any Boolean operation on partitions

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Received 4 August 2018, accepted 23 August 2018, published online 10 June 2019

Abstract

The lattice operations of join and meet were defined for set partitions in the nineteenth century, but no new logical operations on partitions were defined and studied during the twentieth century. Yet there is a simple and natural graph-theoretic method presented here to define any n-ary Boolean operation on partitions. An equivalent closure-theoretic method is also defined. In closing, the question is addressed of why it took so long for all Boolean operations to be defined for partitions.

Keywords: Set partitions, Boolean operations, graph-theoretic methods, closure-theoretic methods.
Math. Subj. Class.: 05A18, 03G10

1 Introduction

The lattice operations of join and meet were defined on set partitions during the late nineteenth century, and the lattice of partitions on a set was used as an example of a non-distributive lattice. But during the entire twentieth century, no new logical operations were defined on partitions.

> Equivalence relations are so ubiquitous in everyday life that we often forget about their proactive existence. Much is still unknown about equivalence relations. Were this situation remedied, the theory of equivalence relations could initiate a chain reaction generating new insights and discoveries in many fields dependent upon it.
>
> This paper springs from a simple acknowledgement: the only operations on the family of equivalence relations fully studied, understood and deployed are the binary join $\vee$ and meet $\wedge$ operations. [3, p. 445]

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Papers on the "logic" of equivalence relations [7] or partitions only involved the join and meet, and not the crucial logical operation of implication.

Yet, there is a general graph-theoretic method[1] by which any n-ary Boolean (or truth-functional) operation $f\colon \{T,F\}^n \to \{T,F\}$ can be used to define the corresponding n-ary operation $f\colon \prod(U)^n \to \prod(U)$ where $\prod(U)$ is the set of partitions on a set U.

A *partition* $\pi = \{B, B', \ldots\}$ on a set $U = \{u, u', \ldots\}$ is a set of disjoint non-empty subsets $B, B', \ldots$ of U, called *blocks*, whose union is U. The corresponding equivalence relation, denoted $\operatorname{indit}(\pi)$, is the set of ordered pairs of elements of U that are in the same block of π, and are called the *indistinctions* or *indits* of π, i.e.,

$$\operatorname{indit}(\pi) = \{(u, u') \in U \times U : \exists B \in \pi, u, u' \in B\}.$$

The complement $\operatorname{dit}(\pi) = U \times U - \operatorname{indit}(\pi)$ is the set of *distinctions* or *dits* of π, i.e., ordered pairs of elements in different blocks. As binary relations, the sets of distinctions or *ditsets* $\operatorname{dit}(\pi)$ of some partition π on U are called *partition* (or *apartness*) *relations.* Given partitions $\pi = \{B, B', \ldots\}$ and $\sigma = \{C, C', \ldots\}$ on U, the *refinement* relation is the partial order defined by:

$$\sigma \preceq \pi \quad \text{if} \quad \forall B \in \pi, \exists C \in \sigma, B \subseteq C.$$

At the top of the refinement partial order is the *discrete partition* $\mathbf{1} = \{\{u\} : u \in U\}$ of all singletons and at the bottom is the *indiscrete partition* $\mathbf{0} = \{U\}$ with only one block consisting of U. In terms of binary relations, the refinement partial order is just the inclusion partial order on ditsets, i.e., $\sigma \preceq \pi$ iff $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$. It should be noted that most of the previous literature on partitions (e.g., [1]) uses the opposite partial order of 'unrefinement' corresponding to the inclusion relation on equivalence relations—which reverses the definitions of the join and meet of partitions.

2 The join operation on partitions

The *join* $\pi \vee \sigma$ of partitions π and σ (least upper bound using the refinement partial order) is the partition whose blocks are the non-empty intersections $B \cap C$ of the blocks of π and σ (under the unrefinement ordering, it is the meet). In terms of ditsets, $\operatorname{dit}(\pi \vee \sigma) = \operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma)$. The general method for defining Boolean operations on partitions will be first illustrated with the join operation whose corresponding Boolean operation is disjunction with the truth table in Table 1.

Table 1: Truth table for disjunction.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Let $K(U)$ be the complete undirected graph on U. The links $u - u'$ corresponding to dits, i.e., $(u, u') \in \operatorname{dit}(\pi)$, of a partition are labelled with the 'truth value' T_π and

[1] The method is, strictly speaking, an algorithm only when U is finite.

corresponding to indits $(u,u') \in \operatorname{indit}(\pi)$ are labelled with the 'truth value' F_π. Given the two partitions π and σ, each link in the complete graph $K(U)$ is labelled with a pair of truth values. The *graph* $G(\pi \vee \sigma)$ *of the join* is obtained by putting a link $u - u'$ where the truth function applied to the pair of truth values on the link in $K(U)$ gives an F. Thus in the case at hand, the only links in $G(\pi \vee \sigma)$ are for the $u - u'$ labelled with F_π and F_σ in $K(U)$. Then the partition $\pi \vee \sigma$ is obtained as the connected components of its graph $G(\pi \vee \sigma)$. Thus u and u' are in the same block (connected component of $G(\pi \vee \sigma)$) if and only if the link $u - u'$ was labelled F_π and F_σ, i.e., u and u' were in the same block of π and in the same block of σ. Thus the graph-theoretic definition of the join reproduces the set-of-blocks definition of the join defined as having its blocks the non-empty intersections of the blocks of π and σ.

3 The meet operation on partitions

On the combined set of blocks $\pi \cup \sigma$ of π and σ, define the *overlap relation* $B \between C$ on two blocks if they have a non-empty intersection or overlap (see [8]). The reflexive-symmetric-transitive closure of this relation is an equivalence relation, and the union of the blocks in each equivalence class gives the blocks of the *meet* $\pi \wedge \sigma$. The corresponding truth-functional operation is conjunction with the truth table in Table 2.

Table 2: Truth table for conjunction.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

The same method is applied except that the links of the graph $G(\pi \wedge \sigma)$ are the ones for which the conjunction truth table gives an F when applied to the truth values on each link $u - u'$. Thus $G(\pi \wedge \sigma)$ contains a link $u - u'$ if $(u,u') \in \operatorname{indit}(\pi)$, $(u,u') \in \operatorname{indit}(\sigma)$, or both. Then the blocks of the partition $\pi \wedge \sigma$ are the connected components of the graph $G(\pi \wedge \sigma)$.

The proof that the graph-theoretic definition of the meet gives the usual set-of-blocks definition of the meet boils down to showing that: $B \in \pi$ and $C \in \sigma$ are contained in the same block of the usual meet $\pi \wedge \sigma$ (i.e., there is a chain of overlaps $B \between C' \between \cdots \between B' \between C$ connecting B and C) if and only for any $u \in B$ and $u' \in C$, u and u' are in the same connected component of $G(\pi \wedge \sigma)$. If any two blocks $B' \between C'$ overlap in the overlap chain, then there is an element $u'' \in B' \cap C'$ such any $u \in B'$ had a link $u - u''$ in $G(\pi \wedge \sigma)$ and similarly any $u' \in C'$ has a link $u'' - u'$ in $G(\pi \wedge \sigma)$. Hence the existence of an overlap chain connecting B and C implies that any $u \in B$ and $u' \in C$ are in the same connected component of $G(\pi \wedge \sigma)$. Conversely, if $u \in B$ and $u' \in C$ are in the same connected component of $G(\pi \wedge \sigma)$, then there is some chain of links $u = u_0 - u_1 - \cdots - u_{n-1} - u_n = u'$ where each link $u_i - u_{i+1}$ for $i = 0, \ldots, n-1$ has either $(u_i, u_{i+1}) \in \operatorname{indit}(\pi)$, $(u_i, u_{i+1}) \in \operatorname{indit}(\sigma)$, or both. Every link $u_i - u_{i+1}$ that is in one indit set but not the other, say, $(u_i, u_{i+1}) \in \operatorname{indit}(\pi)$ and $(u_i, u_{i+1}) \notin \operatorname{indit}(\sigma)$, establishes an overlap between the block of π containing u_i, u_{i+1} and the block of σ containing u_i as well as the different block of σ containing u_{i+1}. Thus the chain of links connecting $u \in B$ and $u' \in C$ establishes a chain of overlapping blocks connecting B and C.

4 The implication operation on partitions

The real beginning of the *logic* of partitions, as opposed to the lattice theory of partitions, was the discovery of the set-of-blocks definition of the implication operation $\sigma \Rightarrow \pi$ for partitions ([5, 6]). The intuitive idea is that $\sigma \Rightarrow \pi$ functions like an indicator or characteristic function to indicate which blocks B of π are contained in a block of σ. View the discretized version of $B \in \pi$, i.e., B replaced by the set of singletons of the elements of B, as the local version $\mathbf{1}_B$ of the discrete partition $\mathbf{1}$, and view the block B remaining whole as the local version $\mathbf{0}_B$ of the indiscrete partition $\mathbf{0}$. Then the partition implication as the inclusion indicator function is: the blocks of $\sigma \Rightarrow \pi$ are for any $B \in \pi$:

$$\begin{cases} \mathbf{1}_B & \text{if } \exists C \in \sigma, B \subseteq C \\ \mathbf{0}_B = B & \text{otherwise.} \end{cases}$$

In the case of the Boolean logic of subsets, for any subsets $S, T \subseteq U$, the conditional $S \supset T = S^c \cup T$ has the property: $S \supset T = U$ iff $S \subseteq T$, i.e., the conditional $S \supset T$ equals the top of the lattice of subsets of U iff the inclusion relation $S \subseteq T$ holds. Similarly, it is immediate that the corresponding relation holds in the partition case:

$$\sigma \Rightarrow \pi = \mathbf{1} \quad \text{iff} \quad \sigma \preceq \pi.$$

This set-of-blocks definition of the partition implication operation accounts for the important new non-lattice-theoretic properties revealed in the *algebra* of partitions $\prod(U)$ on U (defined with the join, meet, and implication as partition operations).

A logical formula in the language of join, meet, and implication is a *subset tautology* if for any non-empty universe U and any subsets of U substituted for the variables, the whole formula evaluates by the set-theoretic operations of join, meet, and implication (conditional) to the top U. Similarly, a formula in the same language is a *partition tautology* if for any universe U with $|U| > 1$ and for any partitions on U substituted for the variables, the whole formula evaluates by the partition operations of join, meet, and implication to the top $\mathbf{1}$ (the discrete partition). All partition tautologies are subset tautologies but not vice-versa. *Modus ponens* $(\sigma \wedge (\sigma \Rightarrow \pi)) \Rightarrow \pi$ is both a subset and partition tautology but Peirce's law, $((\sigma \Rightarrow \pi) \Rightarrow \sigma) \Rightarrow \sigma$, accumulation, $\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi))$, and distributivity, $((\pi \vee \sigma) \wedge (\pi \vee \tau)) \Rightarrow (\pi \vee (\sigma \wedge \tau))$, are examples of subset tautologies that are not partition tautologies. The importance of the implication for partition logic is emphasized by the fact that the only partition tautologies using only the lattice operations, e.g., $\pi \vee \mathbf{1}$, correspond to general lattice-theoretic identities, i.e., $\pi \vee \mathbf{1} = \mathbf{1}$ (see [9]).

The graph-theoretic method automatically gives a partition operation corresponding to the Boolean conditional or implication with the truth table in Table 3 and it is not trivial that the two definitions are the same. It may be helpful to restate the truth table in terms of the partitions; see Table 4.

For the graph-theoretic definition of $\sigma \Rightarrow \pi$, we again label the links $u - u'$ in the complete graph $K(U)$ with T_π if $(u, u') \in \operatorname{dit}(\pi)$ and F_π otherwise, and similarly for σ. Then we construct the graph $G(\sigma \Rightarrow \pi)$ by putting in a link $u - u'$ only in the case the link is labeled T_σ and F_π, i.e., $F_{\sigma \Rightarrow \pi}$. Then the partition $\sigma \Rightarrow \pi$ is the partition of connected components in the graph $G(\sigma \Rightarrow \pi)$.

Table 3: Truth table for conditional.

P	Q	$P \supset Q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 4: Implication truth table for partition 'truth values'.

σ	π	$\sigma \Rightarrow \pi$
T_σ	T_π	$T_{\sigma\Rightarrow\pi}$
T_σ	F_π	$F_{\sigma\Rightarrow\pi}$
F_σ	T_π	$T_{\sigma\Rightarrow\pi}$
F_σ	F_π	$T_{\sigma\Rightarrow\pi}$

To prove the graph-theoretic and set-of-blocks definitions equivalent, we might first note that if $(u, u') \in \operatorname{dit}(\pi)$, then T_π is assigned to that link in $K(U)$ so there is no link $u - u'$ in $G(\sigma \Rightarrow \pi)$. And if $(u, u') \in \operatorname{indit}(\pi)$ but also $(u, u') \in \operatorname{indit}(\sigma)$, then $T_{\sigma\Rightarrow\pi}$ is assigned to the link in $K(U)$ so again there is no link $u - u'$ in $G(\sigma \Rightarrow \pi)$. There is a link $u - u'$ in $G(\sigma \Rightarrow \pi)$ in and only in the following situation where $(u, u') \in \operatorname{indit}(\pi)$ and $(u, u') \in \operatorname{dit}(\sigma)$—which is exactly the situation when B is not contained in any block C of σ:

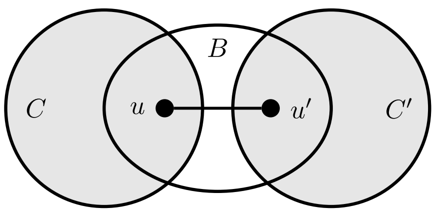

Figure 1: Links $u - u'$ in $G(\sigma \Rightarrow \pi)$.

Then for any other element $u'' \in B$ so that (u, u'') and $(u', u'') \in \operatorname{indit}(\pi)$, we must have either $(u, u'') \in \operatorname{dit}(\sigma)$ or $(u', u'') \in \operatorname{dit}(\sigma)$ so u'' is linked in $G(\sigma \Rightarrow \pi)$ to either u or to u'. Thus all the elements of B are in the same connected component of the graph $G(\sigma \Rightarrow \pi)$ whenever B is not contained in any block of σ. If, on the other hand, B is contained in some block C of σ, then any $u \in B$ cannot be linked to any other u'. In order to that F_π assigned to the link $u - u'$, the two elements have to both belong to B and thus since $B \subseteq C$, they both belong to C so F_σ and thus $T_{\sigma\Rightarrow\pi}$ is also assigned to that link. Thus when B is contained in a block $C \in \sigma$, then any point $u \in B$ is a disconnected component to itself in $G(\sigma \Rightarrow \pi)$ so B is discretized in the graph-theoretic construction of $\sigma \Rightarrow \pi$. Thus the graph-theoretic and set-of-blocks definitions of the partition implication are equivalent.

Example 4.1. Let $U = \{a, b, c, d\}$ so that $K(U) = K_4$ is the complete graph on four

$$\sigma = \{\{a\}, \{b, c, d\}\}$$
$$\pi = \{\{a, b\}, \{c, d\}\}$$
$$\sigma \Rightarrow \pi = \{\{a, b\}, \{c\}, \{d\}\}$$

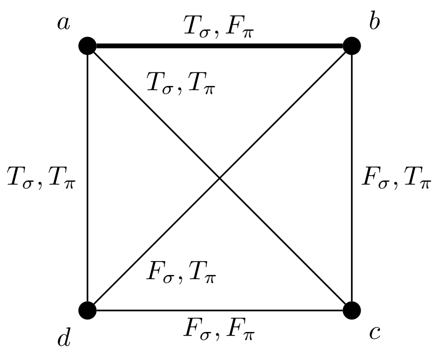

Figure 2: Example of graph for partition implication.

points. Let $\sigma = \{\{a\}, \{b, c, d\}\}$ and $\pi = \{\{a, b\}, \{c, d\}\}$ so we see immediately from the set-of-blocks definition, that the π-block of $\{c, d\}$ will be discretized while the π-block of $\{a, b\}$ will remain whole so the partition implication is $\sigma \Rightarrow \pi = \{\{a, b\}, \{c\}, \{d\}\}$. After labelling the links in $K(U)$, we see that only the $a - b$ link has the $F_{\sigma \Rightarrow \pi}$ 'truth value' so the graph $G(\sigma \Rightarrow \pi)$ has only that $a - b$ link (thickened in Figure 2). Then the connected components of $G(\sigma \Rightarrow \pi)$ give the same partition implication $\sigma \Rightarrow \pi = \{\{a, b\}, \{c\}, \{d\}\}$.

The partition implication is quite rich in defining new structures in the algebra of partitions (i.e., the lattice of partitions extended with other partition operations such as the implication). For instance, for a fixed partition π on U, all the partitions of the form $\sigma \Rightarrow \pi$ (for any partitions σ on U) form a Boolean algebra under the partition operations of implication, join, and meet, e.g., $(\sigma \Rightarrow \pi) \Rightarrow \pi$ is the negation of $\sigma \Rightarrow \pi$, called the *Boolean core* of the upper segment $[\pi, \mathbf{1}]$ in the partition algebra $\prod(U)$.

A *relation* is a subset of a product, and, dually, a *corelation* is a partition on a coproduct. Any partition π on U can be canonically represented as a relation: $\operatorname{dit}(\pi) \subseteq U \times U$. Dually any subset $S \subseteq U$ can be canonically represented as a corelation, namely the partition $\pi(S)$ on the coproduct (disjoint union) $U \uplus U$ where the only nonsingleton blocks in $\pi(S)$ are the pairs $\{u, u^*\}$ of u and its copy u^* for $u \notin S$. Using this corelation construction, any powerset Boolean algebra $\wp(U)$ can be canonically represented as the Boolean core of the upper segment $[\pi, \mathbf{1}]$ in the partition algebra $\prod(U \uplus U)$ where $\pi = \pi(\emptyset)$ is the partition on the disjoint union $U \uplus U$ whose blocks are all the pairs $\{u, u^*\}$ for each element $u \in U$ and its copy u^*. Each partition of the form $\sigma \Rightarrow \pi$ on $U \uplus U$ is $\pi(S)$ for some $S \subseteq U$ since $\sigma \Rightarrow \pi$ is essentially the characteristic function of some subset S of U with $\mathbf{1} \Rightarrow \pi = \pi(\emptyset)$ playing the role of the empty set $\emptyset$ and $\pi \Rightarrow \pi = \mathbf{1}_{U \uplus U}$ playing the role of U.

5 The general graph-theoretic method

Let $f\colon \{T, F\}^n \to \{T, F\}$ be an n-ary Boolean function and let $\pi_1, \ldots, \pi_n$ be n partitions on U. In order to define the corresponding n-ary partition operation $f(\pi_1, \ldots, \pi_n)$, we again consider the complete graph $K(U)$ and then use each partition π_i to label each link $u - u'$ with T_{π_i} if $(u, u') \in \operatorname{dit}(\pi_i)$ and F_{π_i} if $(u, u') \in \operatorname{indit}(\pi_i)$. Then on each link we may apply f to the n 'truth values' on the link and retain the link in $G(f(\pi_1, \ldots, \pi_n))$

if the result was $F_{f(\pi_1,\ldots,\pi_n)}$. The partition $f(\pi_1,\ldots,\pi_n)$ is obtained as the connected components of the graph $G(f(\pi_1,\ldots,\pi_n))$.

6 An equivalent closure-theoretic method

Given any subset $S \subseteq U \times U$, the *reflexive-symmetric-transitive (RST) closure* $\overline{S}$ is the intersection of all equivalence relations on U containing S. The 'topological' terminology of calling a subset *closed* if $S = \overline{S}$ is used even though the RST closure operator is *not* a topological closure operator since the union of two closed sets is not necessarily closed. The closed sets in $U \times U$ are the equivalence relations (or indit sets of partitions), and their complements, the *open* sets, are the partition relations (or ditsets of partitions). As usual, the interior operator $\operatorname{int}(S) = \left(\overline{S^c}\right)^c$ is the complement of the closure of the complement, and the open sets are the ones equalling their interiors.

The closure-theoretic method of defining Boolean operations on partitions will be illustrated using the symmetric difference or inequivalence operation $\pi \oplus \sigma$. Every n-ary Boolean operation can be defined by a truth table such as the one for symmetric difference in Table 5.

Table 5: Truth table for symmetric difference.

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

The disjunctive normal form (DNF) for the formula $P \oplus Q$ is given by the rows where the formula evaluates as T, i.e., $P \oplus Q = (P \wedge \neg Q) \vee (\neg P \wedge Q)$, while the DNF for the negation of the formula is given by the other rows where the formula evaluates as F, i.e., $\neg(P \oplus Q) = (P \wedge Q) \vee (\neg P \wedge \neg Q)$. Given two partitions π and σ on U, the closure-theoretic method of obtaining the partition $\pi \oplus \sigma$ is to start with the DNF for the negated Boolean formula and replace each unnegated variable by the corresponding ditset and each negated variable by the corresponding indit set—as well as replacing the disjunctions and conjunctions by the corresponding subset operations of union and intersection. Applied to $\neg(P \oplus Q) = (P \wedge Q) \vee (\neg P \wedge \neg Q)$, this procedure would yield

$$(\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)) \cup (\operatorname{indit}(\pi) \cap \operatorname{indit}(\sigma)) \subseteq U \times U.$$

Then the indit set of $\pi \oplus \sigma$ is obtained as the RST closure:

$$\operatorname{indit}(\pi \oplus \sigma) = \overline{(\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)) \cup (\operatorname{indit}(\pi) \cap \operatorname{indit}(\sigma))}$$

and the partition $\pi \oplus \sigma$ is the set of equivalence classes of this equivalence relation.

The graph-theoretic method of obtaining the partition $\pi \oplus \sigma$ would label each link $u - u'$ in $K(U)$ by the two 'truth values' given by π and σ, and then retain in the graph $G(\pi \oplus \sigma)$ the links where the truth values evaluated to $F_{\pi \oplus \sigma}$, namely the ones labelled with T_π, T_σ and F_π, F_σ. Then the partition $\pi \oplus \sigma$ is obtained as the connected components of the graph $G(\pi \oplus \sigma)$.

To see the equivalence between the two methods, note first that the links retained in $G(\pi \oplus \sigma)$ are precisely the pairs (u, u') in

$$(\mathrm{dit}(\pi) \cap \mathrm{dit}(\sigma)) \cup (\mathrm{indit}(\pi) \cap \mathrm{indit}(\sigma)).$$

The equivalence proof is completed by showing that taking connected components in the graph $G(\pi \oplus \sigma)$ is equivalent to taking the RST closure of

$$(\mathrm{dit}(\pi) \cap \mathrm{dit}(\sigma)) \cup (\mathrm{indit}(\pi) \cap \mathrm{indit}(\sigma)).$$

The elements u and u' are in the same connected component of $G(\pi \oplus \sigma)$ iff there is a chain of links $u = u_0 - u_1 - \ldots - u_{n-1} - u_n = u'$ in the graph $G(\pi \oplus \sigma)$ so each link has to be originally labelled T_π, T_σ or F_π, F_σ in the graph on $K(U)$. But the condition for (u, u') to be included in the RST closure

$$\overline{(\mathrm{dit}(\pi) \cap \mathrm{dit}(\sigma)) \cup (\mathrm{indit}(\pi) \cap \mathrm{indit}(\sigma))}$$

is that there is a chain of pairs $(u, u_1), (u_1, u_2), \ldots, (u_{n-1}, u')$ such that each pair is either in $\mathrm{dit}(\pi) \cap \mathrm{dit}(\sigma)$ or in $\mathrm{indit}(\pi) \cap \mathrm{indit}(\sigma)$. Hence the two methods give the same result.

The example suffices to illustrate the general closure-theoretic method and its equivalence to the graph-theoretic method of defining Boolean operations on partitions.

7 Relationships between Boolean operations on partitions

For two subset variables, there are $2^4 = 16$ binary Boolean operations on subsets—corresponding to the sixteen ways to fill in the truth table for a binary Boolean operation. Any compound Boolean function of two variables will be truth-table equivalent to one of the sixteen binary Boolean operations. For instance, the Pierce's Law formula $((Q \Rightarrow P) \Rightarrow Q) \Rightarrow Q$ defines a compound binary operation that is equivalent to the constant function T since it is a subset tautology. Certain subsets of the sixteen binary operations suffice to define all the binary operations, e.g., $\neg$ and $\vee$.

Matters are rather different for the Boolean operations on partitions. Using the graph-theoretic or the closure-theoretic method, partition versions of sixteen binary Boolean operations are easily defined. And certain combinations of the sixteen operations suffice to define all sixteen, e.g., $\vee$, $\wedge$, $\Rightarrow$, and $\oplus$ [5, 309–310 and f.n. 18]. But when the sixteen operations are compounded, still keeping to two variables, then the resulting binary partition operations does not necessarily reduce to one of the sixteen—due to the complicated compounding of the closure operations. For instance, the Pierce's Law formula $((\sigma \Rightarrow \pi) \Rightarrow \sigma) \Rightarrow \sigma$ for partitions is not equivalent to the constant function $\mathbf{1}$ since it is not a partition tautology. The topic of the total number of binary operations on partitions obtained by compounding the sixteen basic binary Boolean operations is one of many topics in partition logic that awaits future research.

8 Concluding Remarks

In conclusion, perhaps some remarks are in order as to why it took so long to extend the Boolean operations to partitions. The Boolean operations are normally associated with subsets of a set or, more specifically, with propositions. Boole originally defined his logic as the logic of subsets [2] of a universe set. It is then a theorem that the same set of subset tautologies is obtained as the truth-table tautologies. Perhaps because "logic" has been historically associated with propositions, the texts in mathematical logic throughout the twentieth century (to the author's knowledge) ignored the Boolean logic of subsets and started with the special case of the logic of propositions and then took the truth-table characterization as the *definition* of a tautology.

By the middle of the twentieth century, category theory was defined [4] and the category-theoretic duality was established between subobjects and quotient objects, e.g., between subsets of U and quotient sets (or equivalently equivalence relations or partitions) of U. The conceptual cost of restricting subset logic to the special case of propositional logic is that subsets have the category-theoretic dual concept of partitions while propositions have no such dual concept. Hence the focus on "propositional logic" did not lead to the search for the dual logic of partitions ([5, 6]) or to the simple and natural application of Boolean operations to partitions as well as subsets—which has been our topic here.

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P2.03
https://doi.org/10.26493/2590-9770.1342.109
(Also available at http://adam-journal.eu)

Groupoids on a skew lattice of objects

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To Jonathan Leech—Na zdravje!

Received 16 October 2018, accepted 11 December 2019, published online 27 December 2019

Abstract

Motivated by some alternatives to the classical logical model of boolean algebra, this paper deals with algebraic structures which extend skew lattices by locally invertible elements. Following the meme of the Ehresmann-Schein-Nambooripad theorem, we consider a groupoid (small category of isomorphisms) in which the set of objects carries the structure of a skew lattice. The objects act on the morphisms by left and right restriction and extension mappings of the morphisms, imitating those of an inductive groupoid. Conditions are placed on the actions, from which pseudoproducts may be defined. This gives an algebra of signature $(2, 2, 1)$, in which each binary operation has the structure of an orthodox semigroup. In the reverse direction, a groupoid of the kind described may be reconstructed from the algebra.

Keywords: Inductive groupoids, skew lattices, orthodox semigroups.

Math. Subj. Class.: 20L05, 20M19, 06B75

1 Non-commutative and non-idempotent lattice analogues

As non-classical logics have been developed for various knowledge domains, so various algebras have been proposed as extensions or alternatives to the classical model of boolean algebra. A significant one for this paper is the theory of skew lattices; for a contemporary account, see Leech's surveys [10, 11]. We provide the details of our notation in Section 3. Another proposal is that of MV-algebras, and their coördinatisation via inverse semigroups as described by Lawson and Scott [9]. Thus one theme is to allow sequential operations and hence non-commutative logical connectives, and another introduces non-idempotent

*The author thanks Vicky Gould, Michael Kinyon and Anja Kudryavtseva for their interest and input in discussions, both in person and by email, which helped convert a half-baked workshop idea into an interesting project.
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connectives. This note will consider a combination of these themes, by seeking reasonable structures which extend skew lattices by locally invertible elements.

The principal tool in this construction is based on the ideas behind the Ehresmann-Schein-Nambooripad (ESN) theorem, of which a full account is given in Chapter 4 of Lawson's book [8]: we consider a small category of isomorphisms in which the set of objects carries the structure of a skew lattice. We postulate that the objects act (partially) on the morphisms by left and right restriction and extension mappings of the morphisms, imitating those of an inductive groupoid. Certain reasonable conditions are postulated, and from these a suitable pseudoproduct is defined, much as in the inverse semigroup case, for each skew lattice operation (a non-commutative "meet" and "join"). This results in a total algebra involving two orthodox semigroups with a common set of idempotents isomorphic to the given skew lattice.

Because of the complexity involved in having two operations, we begin by considering a groupoid over a set of objects with a single band operation. A much more general situation has been studied, under the name of *weakly B-orthodox* semigroups, by Gould and Wang in [2], but because the present author has been unable to find this special case treated in the literature, a detailed account will be given here. Later sections deal with the pair of linked band operations, construct the total algebra described above, and show how the original groupoid may be recovered from the algebra. Aspects of the constructions which need further elaboration are noted in the final section.

2 Groupoids on a band of objects

Let us recall from [8] that an inverse semigroup is equivalent to an *inductive groupoid*, i.e.,

- a (small) category of isomorphisms with
- a meet operation on objects and
- a notion of restriction of a morphism to any of its subdomains.

We attempt something similar here, but changing the conditions on the set of objects. Let $\mathscr{G}$ be a groupoid with composition $\circ$ and B its set of objects, endowed with an associative and idempotent operation $\wedge$. Then $(B, \wedge)$ is known as a *lower* band, and possesses a pair of natural preorders: we write

- $a \leq_L b$ if and only if $a = a \wedge b$, and $a \leq_R b$ if and only if $a = b \wedge a$.

As usual, we may identify each object b with its identity $\mathbf{i}_b$, and write $\mathbf{d}g$ and $\mathbf{r}g$ for the domain and range maps in $\mathscr{G}$, thus: $\mathbf{d}g = g \circ g^{-1}$, $\mathbf{r}g = g^{-1} \circ g$. Suppose too that for each $a \in B$ there are *left* and *right restriction* (partial) operations ${}_a|$, $|_a \colon \mathscr{G} \to \mathscr{G}$ such that:

- ${}_a|g$ is defined whenever $a \leq_L \mathbf{d}g$, with ${}_a|g \colon a \to \mathbf{r}({}_a|g) \leq_L \mathbf{r}g$;

and (lateral-) dually,

- $g|_a$ is defined whenever $a \leq_R \mathbf{r}g$, with $g|_a \colon \mathbf{d}(g|_a) \to a,\ \mathbf{d}(g|_a) \leq_L \mathbf{d}g$.

Figure 1 shows the left and right restrictions. (There are analogous (in fact, vertically dual) requirements for extension operators, which will be dealt with more explicitly in Section 3.) Certain sensible axioms must be satisfied:

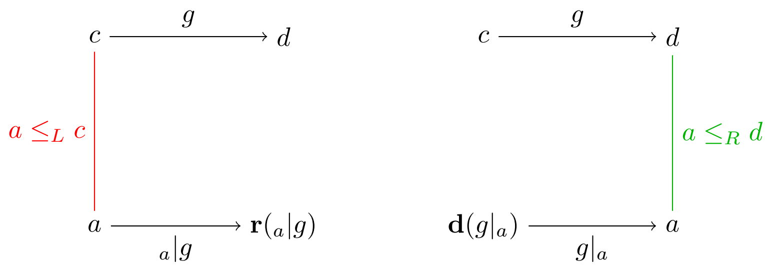

Figure 1: Left and right restriction operators.

(i) (identities) $\quad {}_{\mathbf{d}(g)}|g = g$;

(ii) (preorders) $\quad$ if $a \leq_L b$, then ${}_a|\mathbf{i}_b = \mathbf{i}_a$;

(iii) (transitivity) $\quad$ if $a \leq_L b \leq_L \mathbf{d}g$, then ${}_a|g = {}_{a \wedge b}|g = {}_a|({}_b|g)$;

(iv) (composition) $\quad$ if $f \circ g$ is defined (so that $\mathbf{r}f = \mathbf{d}g$), then

$$ {}_a|(f \circ g) = ({}_a|f) \circ ({}_{\mathbf{r}({}_a|f)}|g), $$

the right-hand composite being defined because $\mathbf{r}({}_a|f) \leq_L \mathbf{r}f = \mathbf{d}g$; see Figure 2.

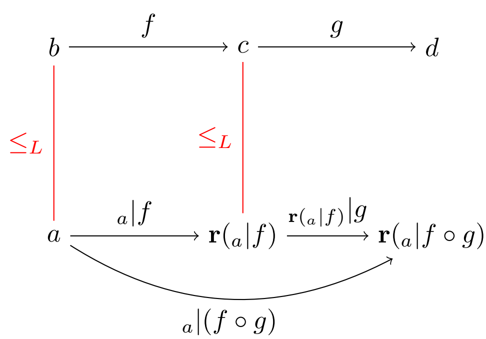

Figure 2: Restriction of a composite morphism.

2.1 Actions and conjugacy

Let us write, without prejudice, a^g as an alternative for $\mathbf{r}({}_a|g)$, and ${}^g a$ for $\mathbf{d}(g|_a)$. This lightens the notation, and emphasises the similarity to actions and conjugates. *Caution:* However, it should not be taken to mean that anything like $a^f = (f^{-1}|_a) \circ ({}_a|f)$, or $a^f = {}^{f^{-1}}a$, or ${}_a|f = f|_{a^f}$ necessarily hold: in general, $f|_{a^f} = f|_{\mathbf{r}f \wedge a^f}$.

What we do have, following from ${}_a|(f \circ g) = ({}_a|f) \circ ({}_{\mathbf{r}({}_a|f)}|g)$, is that

$$ {}_a|\mathbf{i}_b = \mathbf{i}_a = ({}_a|f) \circ ({}_{a^f}|f^{-1}), $$

so

$$({}_a|f)^{-1} = {}_{a^f}|f^{-1}, \qquad a^{\mathbf{i}_b} = a, \qquad \text{and} \qquad a^f = ({}_{a^f}|f^{-1}) \circ ({}_a|f);$$

moreover, $a^{f\circ g} = (a^f)^g$.

We want to link right and left "actions" by $({}_a|f)|_b = {}_a|(f|_b)$, but there is a little problem here, since one of the two sides of that equation may fail to be defined while the other is defined. We therefore seek to extend the conditionally-defined restrictions to total maps by the following device, based on the pseudoproduct construction familiar from the ESN theorem:

For $g \in \mathscr{G}$ and any $a \in B$, define ${}_a|g := {}_{a\wedge \mathbf{d}g}|g$, the right hand side being meaningful since $a \wedge \mathbf{d}g \leq_L \mathbf{d}g$. (Note that if $a \leq \mathbf{d}g$ already, the notations agree.) The next figure shows the situation (where, also by extension, we write a^g for the already-defined $(a \wedge \mathbf{d}g)^g$).

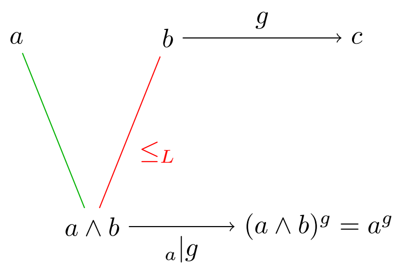

Figure 3: Generalised restriction of g to object a and action of g on a.

Then if $g = \mathbf{i}_b$, we have $a \wedge \mathbf{i}_b = {}_{a\wedge b}|\mathbf{i}_b = \mathbf{i}_{a\wedge b} = \mathbf{i}_a \wedge \mathbf{i}_b = \mathbf{i}_a \wedge b = \mathbf{i}_a|_{a\wedge b}$, and we may write $a \wedge g$ for ${}_a|g$ without conflict. A little re-writing of definitions shows that

$$(a \wedge b) \wedge g = {}_{a\wedge b}|g = {}_a|({}_b|g) = a \wedge (b \wedge g) \tag{2.1}$$

and

$$a \wedge \mathbf{d}g = \mathbf{d}(a \wedge g) = (a \wedge g) \circ (a \wedge g)^{-1}. \tag{2.2}$$

We complete our list of postulates with the previously-mentioned $({}_a|f)|_b = {}_a|(f|_b)$, which we now write as

- $(a \wedge f) \wedge b = a \wedge (f \wedge b)$, for all $a, b \in B$ and $f \in \mathscr{G}$.

(More fully, this is $({}_{a\wedge \mathbf{d}f}|f)|_{a^f\wedge b} = {}_{a\wedge^f b}|(f|_{b\wedge \mathbf{r}f})$.)

Next, we may extend the composition further, to a *pseudoproduct* $\otimes$: when $f : z \to a$ and $g : b \to c$, we define

$$f \otimes g := (f|_{a\wedge b}) \circ ({}_{a\wedge b}|g) = (f \wedge (a \wedge b)) \circ ((a \wedge b) \wedge g).$$

The pseudoproduct is defined for all pairs f, g.

Then $a \wedge f$ is actually just $\mathbf{i}_a \otimes f$. This is indeed an extension of meaning: when $f \circ g$ is defined, $f \otimes g = f \circ g$, and when $f = \mathbf{i}_a$ and $g = \mathbf{i}_b$,

$$f \otimes g = \mathbf{i}_a \otimes \mathbf{i}_b = \mathbf{i}_{a\wedge b} = \mathbf{i}_a \wedge \mathbf{i}_b;$$

so we may as well use just the one symbol $\wedge$ for $\otimes$, as it extends $\circ$ and the restrictions, as well as the original $\wedge$ on B. Let us check remaining non-trivial cases for associativity.

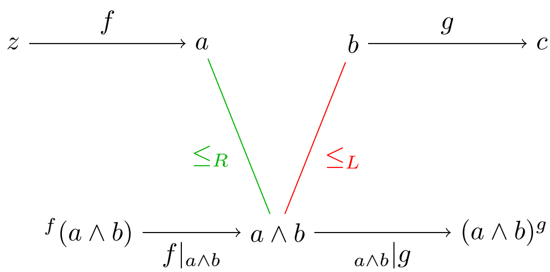

Figure 4: Diagram illustrating the pseudoproduct.

Lemma 2.1. *For all $f, g \in \mathscr{G}$ and $e \in B$, with $f\colon \mathbf{d}f \to a$ and $g\colon b \to \mathbf{r}g$,*

(i) $(f \wedge e) \wedge g = f \wedge (e \wedge g)$,

(ii) $e^{f\wedge g} = (e^f)^g$, *and*

(iii) ${}_e|(f \wedge g) = ({}_e|f) \wedge g$.

Proof. (i): By definition,

$$(f \wedge e) \wedge g = (f|_e) \wedge g = (f|_{a\wedge e})|_b \circ {}_{a\wedge e}|g = (f|_{a\wedge e\wedge b}) \circ ({}_{a\wedge e\wedge b}|g),$$

while $f \wedge (e \wedge g) = f|_{e\wedge b} \wedge {}_{e\wedge b}|g = f \wedge (e \wedge g)$.

(ii): We already have $e^{f\wedge g} = e^{(f|_b)\circ({}_a|g)} = (e^{f|_b})^{({}_a|g)}$.

Observe that $e \wedge \mathbf{d}(f|_b) \leq_R e \wedge \mathbf{d}f$, since $\leq_R$ is left compatible (Figure 5 may assist the reader). So $e \wedge f = e \wedge f|_b$ and $e^{f|_b} = e^f$. Likewise $e^f \wedge b \leq_L a \wedge b$ and $(e^f)^g = (e^f)^{({}_a|g)} = e^{f\wedge g}$.

(iii): Using $e^{f|_b} = e^f$ from (ii), we have

$$\begin{aligned} {}_e|(f \wedge g) &= {}_e|(f|_b \circ {}_a|g) = ({}_e|(f|_b)) \circ ({}_{e^{f|_b}}|({}_a|g)) \\ &= (({}_e|f)|_b) \circ ({}_{e^{f|_b}\wedge a}|g) = (({}_e|f)|_b) \circ ({}_{e^f}|g) \\ &= ({}_e|f) \wedge g. \end{aligned}$$

□

It remains to prove associativity in full generality:

Lemma 2.2. *For all $f, g, h \in \mathscr{G}$, $f \wedge (g \wedge h) = (f \wedge g) \wedge h$.*

Proof. First we establish that when $f \circ g$ is defined, $(f \circ g) \wedge h = f \wedge (g \wedge h)$. Let $r = \mathbf{r}g$ and $d = \mathbf{d}h$; we have

$$\begin{aligned} (f \circ g) \wedge h &= (f \circ g)|_d \circ {}_r|h = (f|_{{}^g d} \circ g|_d) \circ {}_r|h \\ &= f|_{{}^g d} \circ (g|_d \circ {}_r|h) = f|_{{}^g d} \circ (g \wedge h); \end{aligned}$$

and since ${}^g d = \mathbf{d}(g|_d) = \mathbf{d}(g \wedge h)$, the latter is indeed $f \wedge (g \wedge h)$. Now observe that, in the general case,

$$(f \wedge g) \wedge h = (f|_{\mathbf{d}g} \circ {}_{\mathbf{r}f}|g) \wedge h = f|_{\mathbf{d}g} \wedge ({}_{\mathbf{r}f}|g \wedge h)$$

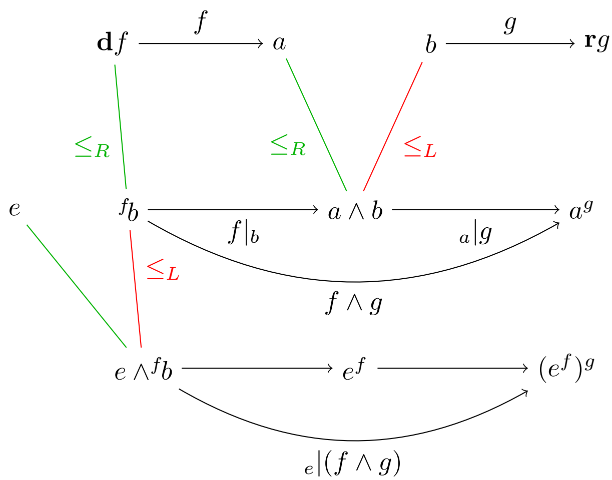

Figure 5: Diagram illustrating $e^{f \wedge g} = (e^f)^g$.

by the foregoing; and then, by Lemma 2.1(iii), we have

$$f|_{\mathbf{d}g} \wedge ({}_{\mathbf{r}f}|g \wedge h) = f|_{\mathbf{d}g} \wedge {}_{\mathbf{r}f}|(g \wedge h) = f \wedge (g \wedge h),$$

completing the proof. □

Theorem 2.3. *$S = (\mathscr{G}, \wedge)$ is an orthodox semigroup.*

Proof. Lemma 2.2 shows that S is a semigroup. S is regular, since $g \wedge g^{-1} \wedge g = g$ for any $g \in \mathscr{G}$. If $f \wedge f = f$, then $f = (f|_{b\wedge a}) \circ ({}_{b\wedge a}|f)$—see Figure 6—and in particular,

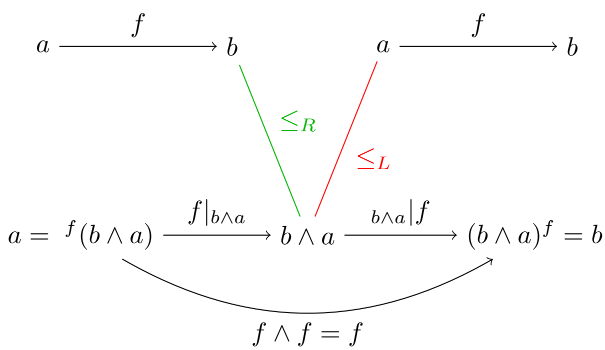

Figure 6: Diagram for an idempotent.

$a = {}^f(b \wedge a)$ and $b = (b \wedge a)^f$. Thus $({}_{b\wedge a}|f) \circ f^{-1}$ is defined and equal (by the composition axiom) to $({}_{b\wedge a}|f) \circ ({}_b|f^{-1}) = {}_{b\wedge a}|(f \circ f^{-1}) = {}_{b\wedge a}|\mathbf{i}_a = \mathbf{i}_{b\wedge a}$. So we have $\mathbf{i}_a = f \circ f^{-1} = (f|_{b\wedge a}) \circ \mathbf{i}_{b\wedge a} = f|_{b\wedge a}$, giving $f = \mathbf{i}_a$. Thus $E(S) = B$ and S is orthodox. □

It also follows that every idempotent is of the form $f \wedge f^{-1}$. With $s \in S$, put $s^+ = s \wedge s^{-1}$ and $s^- = s^{-1} \wedge s$. Clearly $(s^{-1})^+ = s^-$ and $(s^{-1})^- = s^+$, while $s^+ \mathscr{R}\, s\, \mathscr{L}\, s^-$

($\mathscr{R}$ and $\mathscr{L}$ being the usual Green's relations in S) and s^{-1} is the unique inverse of s such that $s^+ \mathrel{\mathscr{L}} s^{-1} \mathrel{\mathscr{R}} s^-$.

Theorem 2.4. *For all $s, t \in S$, there hold:*

(i) $s^+ \wedge s^+ = s^+ = (s^+)^+ = (s^+)^-$ *and* $s^- \wedge s^- = s^- = (s^-)^- = (s^-)^+$*;*

(ii) $s^+ \wedge s = s = s \wedge s^-$*;*

(iii) $s \wedge s = s$ *implies* $s = s^+ = s^-$*;*

(iv) $(s \wedge t)^+ = (s \wedge t^+)^+$ *and* $(s \wedge t)^- = (s^- \wedge t)^-$*;*

(v) $(s^+ \wedge t)^+ = s^+ \wedge t^+$ *and* $(s \wedge t^-)^- = s^- \wedge t^-$.

Proof. Parts (i) – (iii) follow by easy computation from the definitions and Theorem 2.3. The definition of the extended $\wedge$ in the new notation (see Figure 7) reads $s \wedge t = (s \wedge t^+) \circ (s^- \wedge t)$, and (iv) follows immediately. Part (v) is a consequence of (iv) with, respectively, s^+ for s and t^- for t. □

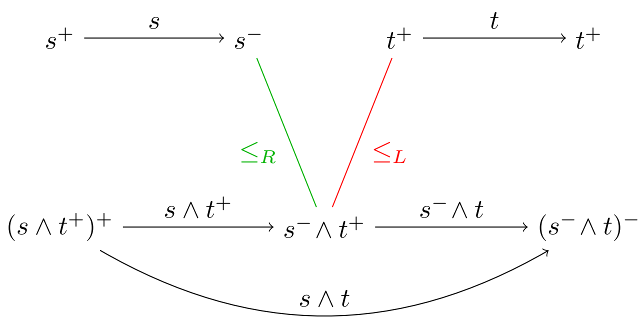

Figure 7: The pseudoproduct in $+/-$ notation.

Remarks 2.5. Theorem 2.4 sets out the object part of a functor imitating that of the ESN theorem. There may be another occasion to describe the morphism part, which should also involve examining the properties in the Theorem, since they include some of those forming the definitions of restriction and Ehresmann semigroups. In fact, $(s \wedge t)^+ \wedge s = s \wedge t^+$ and $t \wedge (s \wedge t)^- = s^- \wedge t$ hold in a restriction semigroup as defined by Kudryavtseva [6], but fail here unless B is a semilattice (in which case S is inverse). The restriction and Ehresmann classes are surveyed in [1], and one may see the directions in which the ideas have been taken more recently in [5] and [6]. This strand of research emphasises commuting idempotents, which distinguishes them from the present paper, where an element may have multiple left and right identities. This may appear a little strange, but is the price to be paid for dealing with *all* idempotents, not just a special subset. More general contexts have already been considered, as in [2, 12, 13], but the approach in hand is a natural and minimal extension of the inductive groupoid case, and returns to the spirit of groupoids as dealt with in another landmark paper—Lawson's [7].

Above all, our ultimate intent is to have B as a skew lattice. We deal with this in the next section, using the results above: beginning with a skew lattice $B = (B, \wedge, \vee)$, we dualise the whole process of Section 2 to extend the join operation $\vee$ to $\mathscr{G}$, resulting in an algebra $S = (\mathscr{G}, \vee, \wedge)$.

3 Skew lattices of objects

Let $\mathscr{G}$ be a groupoid with composition $\circ$ and B its set of objects, endowed with associative operations $\vee$ and $\wedge$ satisfying the absorptive axioms

$$a \vee (a \wedge b) = a = a \wedge (a \vee b), \qquad (a \wedge b) \vee b = b = (a \vee b) \wedge b$$

for a skew lattice [10, 11]. Then both $(B, \vee)$ and $(B, \wedge)$ are bands. Moreover each has a pair of natural preorders: in the *lower* band $(B, \wedge)$ we write (continuing on from the preceding Section 2)

- $a \leq_L b$ if and only if $a = a \wedge b$, and $a \leq_R b$ if and only if $a = b \wedge a$,

and additionally in the *upper* band $(B, \vee)$ we write

- $a \geq_L b$ if and only if $a = a \vee b$, and $a \geq_R b$ if and only if $a = b \vee a$.

We do not at this stage admit the usual convention that $\leq$ and $\geq$ are converse relations! The skew lattice absorptive axioms imply that $a = a \vee b \iff a \wedge b = b$ and $a \vee b = b \iff a = a \wedge b$, so that $a \leq_L b$ if and only if $a = a \wedge b$ if and only if $b \geq_R a$; which is to say that $\leq_L$ and $\geq_R$ are converse relations, as also $\leq_R$ and $\geq_L$. We write the relations in the form most suitable to the occasion. As a vertical dual to the set-up in Section 2, we postulate *left* and *right extension* operations denoted ${}^a|$, $|^a \colon \mathscr{G} \to \mathscr{G}$ such that

- ${}^a|g$ is defined whenever $a \geq_L \mathbf{d}g$, and ${}^a|g \colon a \to \mathbf{r}({}^a|g) \geq_L \mathbf{r}g$;

and again (lateral-) dually,

- $g|^a$ is defined whenever $a \geq_R \mathbf{r}g$, with $g|^a \colon \mathbf{d}(g|^a) \to a, \ \mathbf{d}(g|^a) \geq_L \mathbf{d}g$.

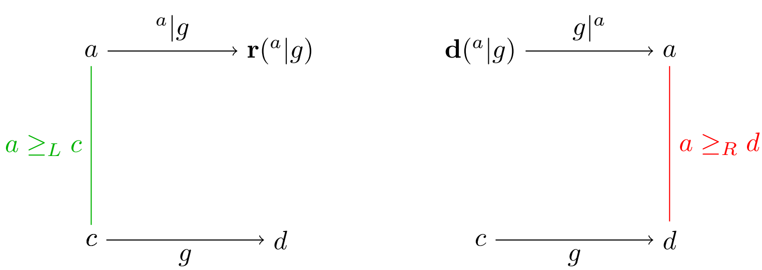

Figure 8: Left and right extension operators.

The relevant diagrams appear in Figure 8. Again we are able to write $a \vee g$ for ${}^a|g$ and by extension for ${}^{a \vee \mathbf{d}g}|g$, and a_g for $\mathbf{r}({}^a|g)$; similarly, ${}_g a = \mathbf{d}({}^a|g)$.

The postulates vertically dual to those of the preceding Section 2 are to hold also, and we list them here, using the abbreviated notation developed in Section 2 and without further explanation; moreover we only give one-sided forms, assuming the lateral duals hold by implication. Thus each postulate stands for a quartet (although some are self-dual or may have vertical and lateral duals equivalent).

(i) (identities) $\mathbf{d}g \vee g = g$;

(ii) (preorders) if $a \geq_R b$, then $a \vee \mathbf{i}_b = \mathbf{i}_{a \vee b}$;

(iii) (transitivity) if $a \geq_R b \geq_R \mathbf{d}g$, then $a \vee g = (a \vee b) \vee g = a \vee (b \vee g)$;

(iv) (composition) if $f \circ g$ is defined (so that $\mathbf{r}f = \mathbf{d}g$), then

$$a \vee (f \circ g) = (a \vee f) \circ (a_f \vee g);$$

(v) (dual of Theorem 2.3) $(a \vee f) \vee b = a \vee (f \vee b)$, for all $a, b \in B$ and $f \in \mathscr{G}$.

The vertical dual of the development in Section 2 extends the join operation $\vee$ to all of $\mathscr{G}$ and of course the dual results hold. In particular, we note that

$$a \vee f \vee f^{-1} = a \vee f \vee (a \vee f)^{-1}.$$

Moreover, extra postulates are required to establish compatibility conditions between the restriction and extension operators which reflect the skew lattice character of B.

Observe that when $f \circ g$ is defined,

$$f \vee g = f \wedge g = f \circ g;$$

in particular, $f \vee f^{-1} = f \wedge f^{-1} = \mathbf{d}f$, etc. From this point on, we write (to conform to precedent) f^* in place of f^{-1}, and may as well write ff^* for $f \circ f^* = f \wedge f^* = f \vee f^*$, etc. The identification of $\mathbf{i}_a$ with a also identifies a^* with $\mathbf{i}_a^{-1} = \mathbf{i}_a$ and so $(a \wedge f)^*$ with $a^f \wedge f^*$, and similarly $(a \vee f)^* = a^f \vee f^*$.

The restriction and extension operators should also be linked through the skew lattice orders. Consider any object $a \in B$ and morphism f; write $\mathbf{d}f = d = ff^*$ and $\mathbf{r}f = r = f^*f$, and set $b = r \vee a \geq_R r$. Then ${}_a|f\colon a \to a^f$ exists, and $a^f \leq_L r$, which is to say $r \geq_R a^f$, and so there is $({}_a|f)|^r\colon d' \to r$. When $f = \mathbf{i}_d$, we see that $d' = d$ so it is reasonable that this hold in general. See Figure 9.

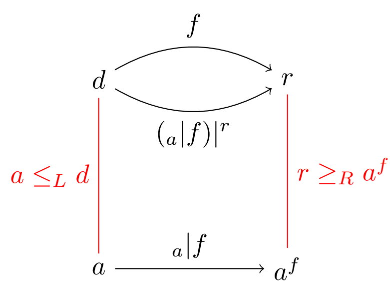

Figure 9: Restriction and extension operators linked.

Indeed we shall require, as a linking condition, that $({}_a|f)|^r = f$ and so we add to the previous list the axiom

(vi) $f = (a \wedge f) \vee f^* f$, or equivalently $f f^* = (a \wedge f) \vee f^*$.

We also assume the lateral and order duals, which are interpreted similarly. Note that when $f \in B$, $f = f^* = f f^*$, and this equation reduces to the absorptive identity

$$f = (a \wedge f) \vee f$$

of skew lattices.

We shall (tentatively) refer to a groupoid satisfying these conditions as a *skew inductive groupoid*. Theorem 2.3 applies and assures the existence of an algebra $(S, \vee, \wedge, *)$ arising from a skew inductive groupoid. We now seek to characterise such an algebra axiomatically.

4 Algebraic characterisation

Let $(S, \vee, \wedge, *)$ be an algebra of signature $(2, 2, 1)$, with $\vee, \wedge \colon S \times S \to S$ and ${}^* \colon S \to S$, that satisfy, for all $s, t \in S$:

(i) $(S, \vee)$ and $(S, \wedge)$ are associative (thus, semigroups);

(ii) $(s^*)^* = s$;

(iii) $s \vee s^* = s \wedge s^* = (s \wedge s^*)^*$;

(iv) $s \vee s^* \vee s = s = s \wedge s^* \wedge s$;

(v) $s \vee s = s$ or $s \wedge s = s$ implies $s = s^*$;

(vi) $s \vee s^* \vee (s \wedge t^* \wedge t) = s = s \wedge s^* \wedge (s \vee t^* \vee t)$ and lateral duals;

(vii) $s \vee s^* \vee t \vee t^* = s \vee s^* \vee t \vee (s \vee s^* \vee t)^*$ and duals;

(viii) $s^* \vee s = t \vee t^*$ implies

$$s \vee t \vee (s \vee t)^* = s \vee s^* \quad \text{and} \quad (s \vee t)^* \vee s \vee t = t^* \vee t \quad \text{and}$$
$$s \wedge t \wedge (s \wedge t)^* = s \wedge s^* \quad \text{and} \quad (s \wedge t)^* \wedge s \wedge t = t^* \wedge t.$$

The properties in Section 3, particularly Lemmas 2.1, 2.2 and Theorem 2.3, show that we were able to construct such an object from a skew inductive groupoid. Conversely, we have

Theorem 4.1. *Let $(S, \vee, \wedge, *)$ satisfy axioms (i)–(viii), and form a small category $\mathscr{C}$ as follows.*

- $\mathrm{Ob}(\mathscr{C}) = \{s \vee s^* : s \in S\}$,
- $\mathrm{Mor}(\mathscr{C}) = \{\widehat{s} = (s \vee s^*, s, s^* \vee s) : s \in S\}$,
- *when* $s^* \vee s = \mathbf{r}(\widehat{s}) = \mathbf{d}(\widehat{t}) = t \vee t^*$, $\widehat{s} \circ \widehat{t}$ *is defined and*

$$\widehat{s} \circ \widehat{t} = (s \vee s^*, st, t^* \vee t).$$

Then $\mathscr{C}$ is a skew inductive groupoid whose pseudoproduct gives an orthodox semigroup isomorphic with S.

Proof. From (vi) we have that $\mathrm{Ob}(\mathscr{C})$ is a skew lattice. Clearly composition when defined for triples is associative, and each $(s \vee s^*, s \vee s^*, s^* \vee s)$ is the identity at object $s \vee s^*$. Morphism $\widehat{s} = (s \vee s^*, s, s^* \vee s)$ has inverse $\widehat{s}^{-1} = (s^* \vee s, s^*, s \vee s^*)$. The restriction and extension operators must be defined: for a morphism $\widehat{s} = (s \vee s^*, s, s^* \vee s)$ and an object a such that $a \geq_L s \vee s^*$ (i.e. $a = a \vee s \vee s^*$), we set

$$^{a}|\widehat{s} = (a, a \vee s, (a \vee s)^* \vee a \vee s).$$

The r.h.s. is indeed in $\mathrm{Mor}(\mathscr{C})$: by equation (2.2), $(a \vee s) \vee (a \vee s)^* = (a \vee s) \vee s^* = a$ by hypothesis. Moreover,

$$\mathbf{r}(^{a}|\widehat{s}) \vee s \vee s^* = ((a \vee s)^* \vee a \vee s) \vee s \vee s^* = \mathbf{r}(^{a}|\widehat{s}),$$

so $\mathbf{r}(^{a}|\widehat{s}) \geq_L \mathbf{r}(s)$, as required for an extension operator.

Next, the postulates of Section 3 have to be verified. It is useful to observe that the right [left] component of a left- [right-]extended morphism depends solely on the middle component, and so may safely be left unspecified (written $\sim$) in certain calculations.

(i) ("identity") follows from regularity (axiom (iv)).

(ii) ("preorder") Assume $a = b \vee a$. By definition,

$$^{a}|\mathbf{i}_b = {}^{(a,a,a)}|(b, b, b) = (a \vee b, a \vee b, (a \vee b)^* \vee a \vee b) = \mathbf{i}_{a \vee b}.$$

(iii) ("transitivity") First, $^{b}|\widehat{s} = {}^{(b,b,b)}|(s \vee s^*, s, s^* \vee s) = (b \vee s \vee s^*, b \vee s, \sim)$, so

$$^{a}|(^{b}|\widehat{s}) = {}^{a}|(b \vee s \vee s^*, b \vee s, \sim) = (a \vee b \vee s \vee s^*, a \vee b \vee s, \sim) = {}^{(a \vee b)}|\widehat{s},$$

by associativity of S.

(iv) ("composition")

$$\begin{aligned}(^{a}|\widehat{s})|^{b} &= (a \vee s \vee s^*, a \vee s, (a \vee s)^* \vee a \vee s)|^{b} \\ &= ((a \vee s \vee b)(a \vee s \vee b)^*, a \vee s \vee b, (a \vee s)^* \vee a \vee s \vee b),\end{aligned}$$

while

$$\begin{aligned}^{a}|(\widehat{s}|^{b}) &= {}^{a}|((s \vee b)(s \vee b)^*, s \vee b, (s \vee s)^* \vee b) \\ &= (a \vee (s \vee b)(s \vee b)^*, a \vee s \vee b, (a \vee s \vee b)^* \vee a \vee s \vee b),\end{aligned}$$

and by (the lateral dual of) axiom (vii), these are equal.

(v) ("dual of Theorem 2.3") This follows from associativity in S.

In this manner we have constructed a groupoid $\widehat{S}$ over a skew lattice of objects. Now suppose that S arises from the original groupoid $\mathscr{G}$. The mapping $\mathscr{G} \to \widehat{S}$ given by $g \mapsto (\mathbf{d}g, g, \mathbf{r}g)$ is routinely an isomorphism, simply representing different ways of describing $\mathscr{G}$; the fact that it factors through S completes the proof. □

5 Models

Do such objects even exist? One special case occurs with $\mathscr{G}$ a true inductive groupoid and B a lattice. Such a combination gives rise to two inverse semigroups (monoids in fact), and an easy way to realise such an object is by taking the direct product of a group with a lattice. This could be the inspiration for a less trivial example, as follows.

Let a group G act by automorphisms on a band B. Then we may consider the semidirect product $S = G \ltimes B$ with base set $G \times B$ and multiplication, for $u, v \in G$ and $a, b \in B$,

$$(u, a)(v, b) = (uv, a^v \cdot b).$$

This situation was studied some time ago by Miklós Hartmann and Mária Szendrei [3, 4] and maybe others I have not yet found; and it seems to have been generalised in [2]. All we need to note here is that

- idempotents are exactly the elements $(1, a)$, and $E(S) \cong B$,
- S is regular with an involution $(u, a)^* = (u^{-1}, a^{u^{-1}})$, such that
- $(u, a)(u, a)^* = (u, a)(u^{-1}, a^{u^{-1}}) = (1, a^{u^{-1}}), \quad (u, a)^*(u, a) = (1, a)$
- $(u, a)(u, a)^*(u, a) = (u, a)$
- so S is orthodox but not inverse
- and $*$ is **not** an anti-automorphism.

On this last point, let us observe that $[(u, a)(v, b)]^* = (v^{-1}u^{-1}, a^{u^{-1}} \wedge b^{v^{-1}u^{-1}})$, so

$$[(u, a)(v, b)]^*[(u, a)(v, b)] = (1, (a \wedge b^{v^{-1}})^{u^{-1}}),$$

which reduces to $(1, a^{u^{-1}})$ precisely when $a = b^{v^{-1}}$, i.e., when

$$(u, a)^*(u, a) = (v, b)(v, b)^*.$$

In structural terms, these are both equivalent to $(u, a) \mathscr{R} (u, a)(v, b) \mathscr{L} (v, b)$. (This may also be relevant to criteria for composibility in the double-orthodox semigroup set-up.)

We may conventionally write a "normal form" ua for (u, a). Then $(u, a) = (u, \top)(1, a)$ when B has a top element $\top$, and so $S = GB$ and we have the factorisable case. Otherwise, $S \cup G$ is factorisable and S almost factorisable. See also Rida-e Zenab's recent article [14], and its references, for Zappa-Szép products of which this is also an example.

The map $\phi\colon S \to G,\ ua \mapsto u$ partitions S into blocks $S_u = u\phi^{-1}$, and S_u is isomorphic with B when given the sandwich multiplication (for $ua, ub \in S_u$), $ua \star ub = ua(u^{-1})ub = uab$; so S is a "group of (isomorphic) sandwich bands". Conversely, given such a $\{B_u : u \in G\}$ with connecting isomorphisms

$$\{\lambda_{u,v}, \rho_{u,v} : B_u \to B_v\}$$

satisfying the right axioms, one may reconstruct $S = \bigcup B_u$ with multiplication (for $s \in B_u, t \in B_v$) given by

$$s \cdot t = s\rho_{u,uv} \star t\lambda_{v,uv}$$

with $\star$ the multiplication in B_{uv}. (There is nothing special about this, it's just another description of a semidirect product.)

Then we can see what happens when we do it twice over, replacing $\cdot$ by $\wedge$ and $\vee$. (We will end up with an algebra of signature $(2, 2, 1)$.) Note that $(u, a) = u \wedge a = u \vee a$ in the normal form, and

$$ua \wedge (ua)^* = ua \wedge u^{-1}a^{u^{-1}} = 1 \wedge a^{u^{-1}},$$
$$(ua)^* \wedge ua = u^{-1}a^{u^{-1}} \wedge ua = 1 \wedge a;$$

and exactly the same with the $\vee$ operation. Thus $s \vee s^* = s \wedge s^* = ss^*$, etc. (using juxtaposition where either main operation may be applied). So the absorptive identity $a \vee (a \wedge b) = a$ is equivalent to

$$s^*s \vee (s^*s \wedge t^*t) = s^*s \qquad \text{and so to} \qquad s \vee (s^*s \wedge t^*t) = s;$$

and likewise for the lateral and order duals.

The theory of inverse semigroups suggests that we investigate an idempotent-separating $*$-congruence $\sim$ of such a $G \ltimes B$. If $ua \sim vb$ then $a \sim b = a$; so we are led to consider the subgroups $K_a := \{u \in G : ua = a\}$. Now

$$K_a \subseteq K_{a\vee b} \subseteq K_{(a\vee b)\wedge b} = K_b$$

for all $a, b \in B$; thus $K_a = K \trianglelefteq G$, say; and we may as well have started with G/K.

The groupoid version of $G \ltimes B$ may be presented as follows. Given a skew lattice B and a group G acting by automorphisms on B, make a category with objects from B and morphisms (b, g, b^g). The composition $(b, g, b^g) \circ (c, h, c^h)$ is defined exactly when $b^g = c$, and is given by $(b, g, b^g) \circ (c, h, c^h) = (b, gh, b^{gh})$. If one works it through, one has the pseudoproduct

$$(b, g, b^g) \otimes_\wedge (c, h, c^h) = (b \wedge c^{g-1}, gh, b^{gh} \wedge c^h),$$

which we may abbreviate $(g, b^g) \cdot (h, c^h) = (gh, b^{gh} \wedge c^h)$, the semidirect product.

6 Further comments

The restriction idea may provide another useful way of thinking about skew lattices. It remains to describe categories of orthodox semigroups with involutory inversion $*$ and of skew inductive groupoids, and functors establishing an equivalence between them. Refinement of the axioms may also be possible, and the relationships with the approach of actions (of objects on morphisms and morphisms on objects) should be explored. The connexions with restriction and Ehresmann semigroups need to be teased out. More "natural" or concrete examples would be desirable—for example, can they be found in rings or override algebras?

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THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P2.04
https://doi.org/10.26493/2590-9770.1309.6c0
(Also available at http://adam-journal.eu)

Ω-lattices from skew lattices

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Received 21 July 2019, accepted 27 September 2019, published online 28 December 2019

Abstract

Some notes on construction of Ω-lattices from special types of skew lattices, using lattices of weak congruences of skew lattices, are presented.

Keywords: *Ω-lattice, Ω-algebra, Ω-skew lattice, lattice-valued, weak congruence.*

Math. Subj. Class.: *08A30, 06B75*

1 Introduction

Weak congruences on algebras, as compatible, symmetric, transitive and weakly reflexive relations [4, 5, 14], have a distinguished role in structural investigations of algebras. Namely, a collection of all weak congruences of an algebra forms an algebraic lattice under inclusion and this lattice contains the lattice of all congruence relations as a filter generated by the identity (diagonal) relation, the lattice of all subalgebras (under isomorphism) as an ideal generated by the identity relation and also lattices of all congruences on all subalgebras.

Weak congruences are connected to Ω-valued algebraic structures (Ω is a complete lattice), Ω-valued here means that elements from the algebra carrier set are valued by elements of the lattice (i.e. the carrier set of the algebra is mapped to Ω). The denotation Ω is used since our research is connected also to Ω-sets introduced 1979. in a paper by Fourman and

*Research supported by Ministry of Education, Science and Technological Development, Republic of Serbia, Grant No. 174018.

†Corresponding author. Research supported by Ministry of Education, Science and Technological Development, Republic of Serbia, Grant No. 174013.

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Scott [6]. They used Ω-sets to model intuitionistic logic. The main connection between weak congruences is here the notion of Ω-valued equality, which is a symmetric and transitive function from A^2 to Ω, and this is a kind of weak equivalence (if considering algebraic structure it is a lattice valued weak congruence). In Fourman and Scott's research, Ω was a complete Heyting algebra, and here Ω is just a complete lattice.

Ω-sets are connected to non-classical predicate logics, and they are used in theoretical foundations of fuzzy set theory [7, 8]. Namely, a similar concept of algebras with an L-valued equality is introduced by Bělohlávek and Vychodil [3], where a complete residuated lattice L is used as a truth-values structure. By adding operations to this structure, so-called L-algebras are obtained. Notions and basics results, analogous to those in universal algebra are obtained in this framework, including a Birkhoff-like variety theorem.

Just to highlight that an L-valued equality is reflexive [3], which is not the case in our investigations, since we use a weak reflexivity. The weak reflexivity instead of the reflexivity enables a connection with weak congruences, which are here obtained as cut sets of an Ω-valued equality. Cut sets, as the inverse images (via valuating functions) of the filters in Ω are a useful tool in this setting. By cut-sets, main algebraic and set-theoretic notions and their properties are generalized from the classical notions to the lattice-valued framework.

In lattice-valued structures with a fuzzy equality [3], also identities were introduced (together with other universal-algebraic notions), in the sense of having a graded satisfiability. Our approach is that an identity holds if the corresponding lattice-theoretic formula is fulfilled (i.e. either identity holds or it does not hold). This notion was firstly introduced in [16], and then developed in [1]. Basic properties of Ω-algebras and representation theorems for Ω-algebras in general are proved in [12].

Since an identity can hold in a lattice-valued algebra, while the underlying classical algebra need not satisfy the same identity, this concept is used in the present paper to introduce Ω-lattices from skew lattices, constructing Ω-lattices using weak congruence lattices of skew lattices.

2 Preliminaries

We use basic notions and notations from universal algebra [2]. An *algebra* is denoted by $\mathcal{A} = (A, F)$, where A is a nonempty underlying set and F is a set of (fundamental) operations on A. Notions of subalgebras, subuniverses and congruences are also well known. In addition to congruences, we use *weak congruences* on $\mathcal{A}$ as symmetric, transitive and compatible relations on the algebra $\mathcal{A}$. Compatibility here implies also weak reflexivity: the property that all nullary operations - constants are in the relation to itself. A weak congruence on $\mathcal{A}$ is obviously a congruence on the subalgebra determined by its domain. The collection $\mathrm{Con}_w(\mathcal{A})$ of all weak congruences on an algebra $\mathcal{A}$ is an algebraic lattice under inclusion [4, 5, 14].

We also deal with *terms*, *term-operations*, and *identities* in the given language as formulas $t_1 \approx t_2$, where t_1, t_2 are terms in the same language.

Recall that a *closure system* on a nonempty set X is a collection of subsets of X closed under set intersections (including $\bigcap \emptyset$ which is X). It is a complete lattice under inclusion.

We also use notions and properties of Ω-valued sets and relations. All notions necessary for comprehending this text will be introduced in the sequel, other details can be found in papers dealing with lattice-valued structures (see e.g. [1, 8, 12, 15]).

Let $(\Omega, \wedge, \vee, \leqslant)$ be a complete lattice with the top and the bottom elements denoted by 1 and 0, respectively.

If A is a nonempty set, then an *Ω-valued function* (Ω-valued set) μ on A is a map $\mu\colon A \to \Omega$. For $x \in A$, $\mu(x)$ is a *degree of membership* of x to μ.

For $p \in L$, a *cut set* or a *p-cut* of an Ω-valued function $\mu\colon A \to \Omega$ is a subset μ_p of A which is the inverse image of the principal filter $\uparrow p$ in Ω:

$$\mu_p = \mu^{-1}(\uparrow p) = \{x \in X \mid \mu(x) \geqslant p\}.$$

An *Ω-valued* (binary) *relation R on A* is an Ω-valued function on A^2, i.e., it is a mapping $R\colon A^2 \to \Omega$. As above, for $p \in \Omega$, a *cut* R_p of R is the binary relation on A, which is the inverse image of $\uparrow p$: $R_p = R^{-1}(\uparrow p)$.

R is *symmetric* if $R(x,y) = R(y,x)$ for all $x, y \in A$, and *transitive* if $R(x,z) \wedge R(z,y) \leqslant R(x,y)$ for all $x, y, z \in A$.

A symmetric and transitive Ω-valued relation on A fulfills the *strictness* property:

$$R(x,y) \leqslant R(x,x) \wedge R(y,y). \tag{2.1}$$

The strictness is a kind of a *weak reflexivity* of R. Therefore, a symmetric and transitive Ω-valued relation on A is a *weak Ω-valued equivalence on A*.

Lemma 2.1 ([15])**.** *An Ω-valued binary relation R on A is a weak Ω-valued equivalence on A if and only if all cuts of R are weak equivalences (symmetric and transitive) relations on A.*

A weak Ω-valued equivalence R on A is a weak *Ω-valued equality*, if it satisfies the *separation property*:

$$R(x,y) = R(x,x) \quad \text{implies} \quad x = y. \tag{2.2}$$

Remark 2.2. The *separation* property is introduced in [6] by a weaker condition:

$$R(x,y) = R(x,x) = R(y,y) \quad \text{implies} \quad x = y.$$

If $\mathcal{A} = (A, F)$ is an algebra and $\mu\colon A \to \Omega$ an Ω-valued function on A, then μ is *compatible* with the operations in F, if for every n-ary operation $f \in F$, for all $a_1, \dots, a_n \in A$, and for every constant (nullary operation) $c \in F$

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f(a_1, \dots, a_n)), \quad \text{and} \quad \mu(c) = 1. \tag{2.3}$$

An Ω-valued function on A, compatible with the operations in F is also called a *fuzzy algebra*, or a *lattice valued algebra* [9, 13].

Further, an Ω-valued relation $R\colon A^2 \to \Omega$ on A is *compatible* with the operations in F if for every n-ary operation $f \in F$, for all $a_1, \dots, a_n, b_1, \dots, b_n \in A$, and for every constant $c \in F$

$$\bigwedge_{i=1}^{n} R(a_i, b_i) \leqslant R(f(a_1, \dots, a_n), f(b_1, \dots, b_n)), \quad \text{and} \quad R(c,c) = 1. \tag{2.4}$$

A weak Ω-valued equivalence on A which is compatible with the operations in F is called a weak Ω-valued congruence on $\mathcal{A}$.

Algebraic topics related to lattice-valued structures are here presented mostly from [1, 4, 5, 6, 8, 12, 14].

An Ω-*set* (originating from [6]) is a pair (A, E), where A is a nonempty set, and E is a symmetric and transitive Ω-valued relation on A, fulfilling the separation property (2.2).

We say that E is an Ω-*valued equality* in (A, E).

We also endow the Ω-set with an Ω-valued function $\mu\colon A \to \Omega$ defined by:

$$\mu(x) := E(x, x).$$

Throughout the text, we also say that (A, E) is a *lattice-valued set*, without particularly fixing the co-domain lattice.

Lemma 2.3 ([1]). *Every cut E_p, $p \in \Omega$, of the Ω-valued equality in an Ω-set (A, E) is an equivalence relation on the corresponding cut μ_p of μ.*

If $\mathcal{A} = (A, F)$ is an algebra, (A, E) is an Ω-set and E is compatible with the operations in F then a pair $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is an Ω-*algebra*. $\mathcal{A}$ is called the *underlying, basic algebra* of $\overline{\mathcal{A}}$.

Function μ defined above is then a lattice valued algebra in the sense of [9, 13] and for every $p \in \Omega$, the cut μ_p of μ is a subalgebra of $\mathcal{A}$ [1].

Proposition 2.4 ([1]). *Let $(\mathcal{A}, E)$ be an Ω-algebra. Then every cut of E is a weak congruence on $\mathcal{A}$, namely for $p \in \Omega$, E_p is a congruence on the subalgebra μ_p.*

Identities on Ω-algebras are introduced in [16] in a particular way:

Let $(\mathcal{A}, E)$ be an Ω-algebra and $u(x_1, \dots, x_n) \approx v(x_1, \dots, x_n)$, briefly $u \approx v$, be an identity in the type of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ *satisfies* the identity $u \approx v$ (i.e., this identity *holds* on $(\mathcal{A}, E)$) if

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E(u(a_1, \dots, a_n), v(a_1, \dots, a_n)), \tag{2.5}$$

for all $a_1, \dots, a_n \in A$ and the term-operations on $\mathcal{A}$ corresponding to the terms u and v respectively.

If the Ω-algebra $(\mathcal{A}, E)$ satisfies an identity, this identity need not hold on $\mathcal{A}$, but the converse obviously holds [1].

As an example of an Ω-algebra, we introduce here a notion of an Ω-lattice. This notion will be used in the sequel.

Let Ω be a complete lattice, let $\mathcal{A} = (A, \wedge, \vee)$ be an algebra with two binary operations (without any additional conditions) and let $E\colon A^2 \to \Omega$ be an Ω-valued equality on A (i.e., (A, E) is an Ω-set). Let E be compatible with operations $\wedge$ and $\vee$: for all $x, y, z, t \in A$,

$$E(x, y) \wedge E(z, t) \leqslant E(x \wedge z, y \wedge t) \quad \text{and}$$
$$E(x, y) \wedge E(z, t) \leqslant E(x \vee z, y \vee t).$$

An Ω-algebra $(\mathcal{A}, E)$ is an Ω-lattice, if it satisfies lattice identities:

$\ell 1$:	$x \wedge y \approx y \wedge x$	(*commutativity*)
$\ell 2$:	$x \vee y \approx y \vee x$	
$\ell 3$:	$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$	(*associativity*)
$\ell 4$:	$x \vee (y \vee z) \approx (x \vee y) \vee z$	
$\ell 5$:	$(x \wedge y) \vee x \approx x$	(*absorption*)
$\ell 6$:	$(x \vee y) \wedge x \approx x$	

In other words, for all $x, y, z \in A$, the following formulas are satisfied with the mapping $\mu\colon M \to \Omega$ defined by $\mu(x) = E(x, x)$:

$L1\colon$	$\mu(x) \wedge \mu(y) \leqslant E(x \wedge y, y \wedge x)$	*(commutative laws)*
$L2\colon$	$\mu(x) \wedge \mu(y) \leqslant E(x \vee y, y \vee x)$	
$L3\colon$	$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \wedge y) \wedge z, x \wedge (y \wedge z))$	*(associative laws)*
$L4\colon$	$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \vee y) \vee z, x \vee (y \vee z))$	
$L5\colon$	$\mu(x) \wedge \mu(y) \leqslant E((x \wedge y) \vee x, x)$	*(absorption laws)*
$L6\colon$	$\mu(x) \wedge \mu(y) \leqslant E((x \vee y) \wedge x, x)$	

Remark 2.5. The operations on the lattice Ω and on the basic structure for an Ω-lattice $(A, \wedge, \vee)$ are denoted with the same symbols $\vee$ and $\wedge$. However, the confusion can not arise since one structure is the domain and the other is the co-domain of the function μ and from the contest we can distinguish them.

The following propositions are true.

Theorem 2.6 ([1])**.** *Let $(\mathcal{A}, E)$ be an Ω-algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies all identities in $\mathcal{F}$ if and only if for every $p \in \Omega$ the quotient algebra μ_p/E_p satisfies the same identities.*

Corollary 2.7. *Let $(\mathcal{A}, E)$ be an Ω-algebra with two binary operations. Then, $(\mathcal{A}, E)$ is an Ω-lattice if and only if for all $p \in \Omega$ the quotient algebras μ_p/E_p are lattices.*

Corollary 2.8. *If a diagonal relation $\Delta_A = \{(a, a) \mid a \in A\}$ is a cut of E, then each identity fulfilled by an Ω-algebra $\overline{\mathcal{A}} = (\mathcal{A}, E)$ also holds on the underlying algebra $\mathcal{A}$.*

By Corollary 2.8, we are interested in Ω-algebras which do not contain a copy of the underlying algebra among quotient substructures. An Ω-algebra $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is said to be *proper* if Δ_A is not a cut of E.

Theorem 2.9 ([12])**.** *$\overline{\mathcal{A}} = (\mathcal{A}, E)$ is a proper Ω-algebra if and only if*

$$\text{there are } a, b \in A,\ a \neq b, \text{ such that} \quad E(a, b) \geqslant \bigwedge\{E(x, x) \mid x \in A\}. \tag{2.6}$$

2.1 Representation

This part is mostly from [12] and it will be used in the sequel in Ω-skew lattices.

Proposition 2.10 ([12])**.** *The collection of the cuts of E in an Ω-algebra $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is a closure system on A^2, a subposet of the weak congruence lattice $\mathrm{Con}_w(\mathcal{A})$ of $\mathcal{A}$.*

Theorem 2.11 ([12])**.** *Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\mathrm{Con}_w(\mathcal{A})$ such that for all $a, b \in A$, $a \neq b$,*

$$(a, b) \notin \bigcap\{R \in \mathcal{R} \mid (a, a) \in R\}. \tag{2.7}$$

Then there is a complete lattice Ω and an Ω-algebra $(\mathcal{A}, E)$ with the underlying algebra $\mathcal{A}$, such that $\mathcal{R}$ consists of cuts of E.

As proved in [12], the lattice Ω in Theorem 2.11 is the starting collection $\mathcal{R}$ of weak congruences ordered by the dual of inclusion, $\supseteq$. The required Ω-algebra is $(\mathcal{A}, E)$, where the Ω-valued equality $E\colon A^2 \to \Omega$ is defined by:

$$E(a, b) := \bigcap\{R \in \mathcal{R} \mid (a, b) \in R\} \quad \text{for all} \quad a, b \in A. \tag{2.8}$$

The lattice Ω and the corresponding Ω-algebra defined by means of (2.8), are said to be obtained by the *canonical construction* over the algebra $\mathcal{A}$; in this context, we say that an Ω-algebra $(\mathcal{A}, E)$, obtained by the canonical construction is a *canonical Ω-algebra*.

For a symmetric and transitive relation $R \subseteq A^2$, $\operatorname{dom} R := \{x \in A \mid (x, x) \in R\}$. By the construction in the proof of Theorem 2.11 [12], we get:

Corollary 2.12. *Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\mathrm{Con}_w(\mathcal{A})$ fulfilling condition (2.7). Let also $\mathcal{F}$ be a set of identities in the language of $\mathcal{A}$ and suppose that for every $R \in \mathcal{R}$, the algebra $\operatorname{dom} R/R$ fulfills these identities. Then there is a complete lattice Ω and an Ω-algebra $(\mathcal{A}, E)$, such that $\mathcal{R}$ consists of cuts of E and $(\mathcal{A}, E)$ satisfies $\mathcal{F}$.*

Let $(\mathcal{A}, E1)$ and $(\mathcal{A}, E2)$ be an Ω_1-valued algebra and an Ω_2-valued algebra respectively. We say that the structures $(\mathcal{A}, E1)$ and $(\mathcal{A}, E2)$ are *cut-equivalent* if their collections of quotient algebras over cuts of $E1$ and $E2$ coincide, i.e., if for every $p \in \Omega_1$ there is $q \in \Omega_2$ such that $\mu 1_p/E1_p = \mu 2_q/E2_q$ and vice versa.

Theorem 2.13 ([12])**.** *Let $\overline{\mathcal{A}} = (\mathcal{A}, E)$ be an Ω-algebra where Ω is an arbitrary complete lattice. Then there is a lattice and a lattice-valued algebra cut-equivalent with $\overline{\mathcal{A}}$, obtained by the canonical construction over $\mathcal{A}$.*

Shortly, the Ω-algebra $(\mathcal{A}, E)$ with the collection $\{E_p \mid p \in \Omega\}$ of cuts of E is fixed. Then a new lattice $\Omega 1$ is given by $\Omega 1 = (\{E_p \mid p \in \Omega\}, \supseteq)$ and $E1\colon A^2 \to \Omega 1$ is defined as $E1(x, y) := \bigcap\{R \in \Omega 1 \mid (x, y) \in R\}$. Then $(\mathcal{A}, E)$ and $(\mathcal{A}, E1)$ are cut-equivalent.

2.2 Skew lattices

For basic notions related to skew lattices given in the sequel, see [10, 11].

An idempotent semigroup (associative groupoid) $(S, \cdot)$ is a *band*, i.e., it fulfills identities $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ and $x \cdot x = x$. A commutative band is a *semilattice*. A *rectangular band* is a band that satisfies $x \cdot y \cdot x = x$.

A *skew lattice* is an algebra $(S, \vee, \wedge)$ with two binary *associative* operations, satisfying the *absorption laws*:

$$x \wedge (x \vee y) = x = (y \vee x) \wedge x,$$
$$x \vee (x \wedge y) = x = (y \wedge x) \vee x.$$

Duality principle clearly holds, as for lattices. Due to absorption laws, both operations are *idempotent*: $x \vee x = x$ and $x \wedge x = x$. Hence, the reduct algebras $(S, \vee)$ and $(S, \wedge)$ of a skew lattice are bands. In addition, in a skew lattice

$$x \wedge y = x \quad \text{if and only if} \quad x \vee y = y \qquad \text{and}$$
$$x \wedge y = y \quad \text{if and only if} \quad x \vee y = x.$$

Therefore, the natural partial order in a skew lattice S is defined as follows:

$$x \leqslant y \quad \text{if and only if} \quad x \wedge y = x = y \wedge x \qquad \text{or dually}$$
$$x \leqslant y \quad \text{if and only if} \quad x \vee y = y = y \vee x.$$

Clearly, lattices are skew lattices. A *rectangular skew lattice* is a skew lattice in which both band-reducts $(S, \wedge)$ and $(S, \vee)$ are rectangular bands and $x \wedge y = y \vee x$ for all $x, y \in S$.

Proposition 2.14 ([11])**.** *Let $(S, \vee, \wedge)$ be a skew lattice and θ a relation on S given by*

$$x \; \theta \; y \quad \text{if} \quad x \wedge y \wedge x = x \text{ and } y \wedge x \wedge y = y. \tag{2.9}$$

Then θ is a congruence relation on $(S, \vee, \wedge)$ and S/θ is the maximal lattice image of S. In addition, all congruence classes of θ are maximal rectangular skew lattices - subalgebras of S.

If $(S, \vee, \wedge)$ is a rectangular skew lattice, then clearly θ is A^2 and the factor lattice has a single class.

3 Ω-lattices from skew lattices

In this part we investigate Ω-lattices with the underlying algebras being skew lattices.

Following Theorem 2.13, we assume that all the Ω-algebras $(\mathcal{A}, E)$ are canonical, i.e., that the lattice Ω is a closure system in the weak congruence lattice $\mathrm{Con}_w(\mathcal{A})$ ordered dually to inclusion. Therefore, for all $p \in \Omega$, we have that $E_p = p$, and by the definition (2.8),

$$E(x, y) = \bigcap\{R \in \Omega \mid (x, y) \in R\} \qquad \text{for all } x, y \in A.$$

We start with a definition related to canonical Ω-algebras in general. Let $\mathcal{A}$ be an algebra and $\mathcal{J}$ a set of identities in the language of $\mathcal{A}$. We say that $\mathcal{A}$ is Ω-*simple with respect to* $\mathcal{J}$ if the only weak congruences in $\mathrm{Con}_w(\mathcal{A})$ such that all the identities in $\mathcal{J}$ hold in the quotient structures over these congruences are squares. Consequently, we say that $\mathcal{A}$ is Ω-*regular with respect to* $\mathcal{J}$ if it is not Ω-simple with respect to $\mathcal{J}$. A straightforward consequence of this definition is:

Proposition 3.1. *If $\mathcal{A}$ is an Ω-simple algebra with respect to the set $\mathcal{J}$ of identities, then there is an Ω-algebra $(\mathcal{A}, E)$ fulfilling all the identities in $\mathcal{J}$.*

By Proposition 2.14, a skew lattice possesses a congruence with respect to which the corresponding factor algebra is the greatest factor lattice and all the cosets are noncommutative. In case of rectangular skew lattices, this congruence is the square of the skew lattice and the corresponding factor lattice has one element. We use these facts in the sequel.

Proposition 3.2. *A rectangular skew lattice is an Ω-simple algebra with respect to commutative identities for both operations.*

Corollary 3.3. *Let $\mathcal{A} = (A, \wedge, \vee)$ be a rectangular skew lattice, let Ω be the closure system on the weak congruence lattice $\mathrm{Con}_w(\mathcal{A})$. Then, for the Ω-equality relation $E\colon A^2 \to \Omega$, $(\mathcal{A}, E)$ is an Ω-lattice if and only if Ω consists of squares of algebras.*

Proof. Since $\mathcal{A}$ is a rectangular skew lattice, also all the subalgebras are rectangular skew lattices. Hence, by the fact explained above, all the weak-congruences in the closure system that form Ω are squares of algebras. The proof now follows by Proposition 3.2. □

Corollary 3.4. *For every complete and atomic Boolean algebra Ω, there is a rectangular skew lattice $\mathcal{A} = (A, \wedge, \vee)$, such that $(\mathcal{A}, E)$ is a canonical Ω-lattice.*

Proof. Let Ω be a complete and atomic Boolean algebra isomorphic to 2^A, where A is a set and let a rectangular skew lattice $\mathcal{A} = (A, \wedge, \vee)$ be defined by $x \wedge y = x$ and $x \vee y = y$. Then, all the weak equivalence relations are weak congruences and all the subsets are subuniverse lattices. Let $\mathrm{Con}_w(\mathcal{A})$ be such a weak congruence lattice. Any closure system in this lattice in which all factor algebras (related to weak congruences) are commutative makes an Ω-lattice. The subset of all squares of subalgebras (which are also weak congruences) is closed under intersections. Indeed, $\emptyset$ is also a weak congruence, since there is no nullary operations in the similarity type of skew lattices. In case the family of subalgebras $\{B_i \mid i \in I\}$ has a nonempty intersection, then $\bigcap_{i\in I} B_i^2 = (\bigcap_{i\in I} B_i)^2$. By Proposition 3.2, if Ω consists of all squares, then it is an Ω-lattice. The lattice of squares of subuniverses is a sublattice of the weak congruence lattice and it is isomorphic to the subuniverse lattice which is isomorphic to the Boolean algebra Ω. $\square$

Let $\mathcal{A} = (A, \wedge, \vee)$ be a skew lattice. As mentioned above, there is a special congruence θ on $\mathcal{A}$, called *natural equivalence*, such that the following holds: the quotient algebra $\mathcal{A}/\theta$ is the maximal lattice image of $\mathcal{A}$, and all congruence classes of θ are maximal rectangular subalgebras of $\mathcal{A}$. This congruence is defined by $x \;\theta\; y$ iff both $x \vee y \vee x = x$ and $y \vee x \vee y = y$, or, dually, $x \;\theta\; y$ iff $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$ (see [11]).

Now, if $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, let us denote the restriction of θ on B by $\theta_{\mathcal{B}}$, that is $\theta_{\mathcal{B}} = \theta \cap B^2$. Then $\theta_{\mathcal{B}}$ is a congruence on $\mathcal{B}$, and moreover, it is exactly the natural equivalence on $\mathcal{B}$ (since $\forall x, y \in B\colon (x, y) \in \theta_{\mathcal{B}}$ iff $(x, y) \in \theta$ iff $x \vee y \vee x = x, y \vee x \vee y = y$). This means that $\mathcal{B}/\theta_{\mathcal{B}}$ is the maximal lattice image of $\mathcal{B}$, and all congruence classes of $\theta_{\mathcal{B}}$ are maximal rectangular subalgebras of $\mathcal{B}$.

It is easy to see that the set $\mathcal{R} = \{\theta_{\mathcal{B}} \mid \mathcal{B} \in \mathrm{Sub}(\mathcal{A})\} \cup \{A^2\}$ is a closure system in $\mathrm{Con}_w(\mathcal{A})$, therefore a complete lattice with respect to $\subseteq$ and its dual, $\supseteq$.

Indeed, let $\{\theta_{\mathcal{B}_i} \mid i \in I\}$ be a nonempty family in $\mathrm{Sub}\,\mathcal{A}$. Then,

$$\bigcap_{i\in I} \theta_{\mathcal{B}_i} = \bigcap_{i\in I} (\theta \cap {\mathcal{B}_i}^2) = \theta \cap \bigcap_{i\in I} {\mathcal{B}_i}^2 = \theta_{\mathcal{B}'},$$

where

$$\mathcal{B}' = \bigcap_{i\in I} \mathcal{B}_i \in \mathrm{Sub}(\mathcal{A}).$$

Intersection of empty family is by the definition A^2, and it belongs to the $\mathcal{R}$ by construction. Therefore $\mathcal{R}$ is a closure system in the weak congruence lattice.

Theorem 3.5. *Let $\mathcal{A} = (A, \wedge, \vee)$ be a skew lattice and $\Omega = (\mathcal{R}, \supseteq)$ be the closure system $\mathcal{R}$ defined above, ordered by the dual of inclusion, $\supseteq$. Then, the Ω-skew lattice $\overline{\mathcal{A}}$ obtained by the canonical construction over $\mathcal{A}$, is an Ω-lattice.*

Proof. Let us notice that for each $a \in A$, $(\{a\}, \wedge, \vee)$ is a subalgebra, due to the fact that operations are idempotent. So,

$$\bigcap \{R \in \Omega \mid (a, a) \in R\} = \{(a, a)\},$$

which means that the closure system satisfies condition (2.7) from Theorem 2.11. If we define $E\colon A^2 \longrightarrow \Omega$ by $E(a, b) = \bigcap \{R \in \Omega \mid (a, b) \in R\}$ for all $a, b \in A$, then the cuts of E are exactly the elements of Ω, that is, for each $R \in \Omega$, $E_R = R$. By construction, $\mathcal{R} = \{\theta_{\mathcal{B}} \mid \mathcal{B} \in \mathrm{Sub}(\mathcal{A})\} \cup \{A^2\}$, so $E_{\theta_{\mathcal{B}}} = \theta_{\mathcal{B}}$, for each $\mathcal{B} \in \mathrm{Sub}(\mathcal{A})$. Also, by

Theorem 2.11, Ω-valued relation E is an Ω-valued equality compatible with fundamental operations of $\mathcal{A}$, which means that $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is an Ω-skew lattice.

Furthermore, $E_{\theta_\mathcal{B}}$ is a congruence on subalgebra $\mu_{\theta_\mathcal{B}}$ of $\mathcal{A}$, by Proposition 2.4, and also a congruence on subalgebra $\mathcal{B}$, by the construction (for all $\theta_\mathcal{B} \in \Omega$, that is for all $\mathcal{B} \in \mathrm{Sub}\,\mathcal{A}$). This means $\mathcal{B} = \mu_{\theta_\mathcal{B}} = \{x \in A \mid (x, x) \in E_{\theta_\mathcal{B}}\}$, for all $\mathcal{B} \in \mathrm{Sub}\,\mathcal{A}$.

So, quotient algebras $\mu_{\theta_\mathcal{B}}/E_{\theta_\mathcal{B}}$ are, in fact $\mathcal{B}/\theta_\mathcal{B}$. As mentioned earlier, for each $\mathcal{B}$, $\theta_\mathcal{B}$ is a natural equivalence on $\mathcal{B}$, so the quotient $\mathcal{B}/\theta_\mathcal{B}$ is the maximal lattice image of $\mathcal{B}$, therefore a lattice. By the Theorem 2.6, the Ω-skew lattice $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is an Ω-lattice. $\square$

Remark 3.6. If $\mathcal{A}$ is a rectangular skew lattice, then the congruence θ is full, and so are all $\theta_\mathcal{B}$. The quotients are therefore trivial, and the obtained Ω-algebra is an Ω-lattice (as already proven). Let us notice that the Ω-lattice in this case satisfies every lattice-identity, since all the quotients do.

Remark 3.7. A skew lattice is called *primitive* if its congruence θ has exactly two blocks. If $\mathcal{A}$ is a primitive skew lattice, then its subalgebras are either rectangular or primitive. The reason for this is the following: the blocks themselves are rectangular subalgebras, and so are all subalgebras contained in either of the blocks. If a subalgebra is not contained in a block, the corresponding restriction of θ also has two blocks.

This means that the quotient algebras from the construction explained above are either trivial or two-element lattices $(\{0, 1\}, \wedge, \vee)$. The Ω-lattice, therefore, satisfies every identity satisfied by the two element lattice.

Example 3.8. We start from a three element skew lattice $\mathcal{A} = (A, \wedge, \vee)$, where $A = \{a, b, 0\}$, and the operations are defined by: $0 \wedge x = 0 = x \wedge 0$, $0 \vee x = x = x \vee 0$, $a \wedge b = b$, $b \wedge a = a$, $a \vee b = a$, $b \vee a = b$. Now, we define an Ω-lattice, where Ω is a lattice anti-isomorphic to the closure system on the weak congruence lattice of skew lattice $\mathcal{A}$. The lattice Ω is given in Figure 1. The canonical Ω-algebra $(\mathcal{A}, E)$ is defined as follows:

$$E(x, y) := \bigcap\{R \in \Omega \mid (x, y) \in R\} \qquad \text{for all } x, y \in A.$$

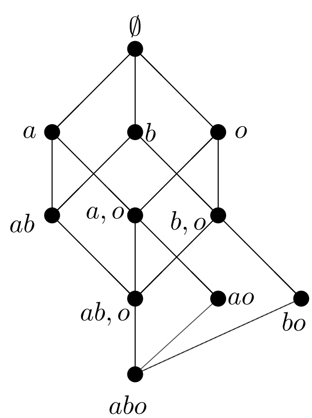

Figure 1: The lattice Ω.

This mapping is well defined since Ω is closed under intersection, and since Ω is canonical, for every $p \in \Omega$, $E_p = p$.

Table 1: The equality E and the related function μ.

(a)

E	0	a	b
0	0	$a0$	$b0$
a	$a0$	a	ab
b	$b0$	ab	b

(b)

$$\mu = \begin{pmatrix} 0 & a & b \\ 0 & a & b \end{pmatrix}$$

The equality E is given in Table 1, together with its diagonal, compatible function μ.

The cuts of μ are all subsets except $\{a, b\}$. The cuts of E are all diagonal relations on all cuts of μ , all squares of all subalgebras and $E_{ab,0}$ has two classes. Hence, the factor algebras μ_p/E_p are either one-element lattices or two element lattices. Hence, $(\mathcal{A}, E)$ is an Ω-lattice.

Example 3.9. Now we start from a rectangular skew lattice with four elements: $A = L\times R$, where $L = \{a, b\}$, $R = \{c, d\}$. The operation tables are given in Table 2(a) and 2(b). Here and in Figure 2 and Table 3 below, ordered couples are denoted ac, ad, bc, bd just for simplicity.

Table 2: The operation tables of the rectangular skew lattice.

(a)

$\vee$	ac	ad	bc	bd
ac	ac	ac	bc	bc
ad	ad	ad	bd	bd
bc	ac	ac	bc	bc
bd	ad	ad	bd	bd

(b)

$\wedge$	ac	ad	bc	bd
ac	ac	ad	ac	ad
ad	ac	ad	ac	ad
bc	bc	bd	bc	bd
bd	bc	bd	bc	bd

Let Ω be the lattice given by the diagram in Figure 2. Ω is a lattice anti-isomorphic to the closure system on the weak congruence lattice of the skew lattice $\mathcal{A}$. Some of the subalgebras of $\mathcal{A}$ (appearing in $\mathrm{Con}_w(\mathcal{A})$) are denoted by: $H = \{bd, ad\}$, $D = \{ac, bc\}$, $C = \{bc, bd\}$, $B = \{ac, ad\}$.

Let the Ω-equality and the related function μ be given in Table 3.

Table 3: The Ω-equality and the related function μ.

(a)

E	ac	ad	bc	bd
ac	ac	B^2	D^2	A^2
ad	B^2	ad	A^2	H^2
bc	D^2	A^2	bc	C^2
bd	A^2	H^2	C^2	bd

(b)

$$\mu = \begin{pmatrix} ac & ad & bc & bd \\ ac & ad & bc & bd \end{pmatrix}$$

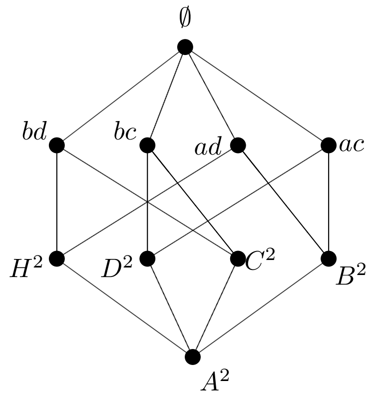

Figure 2: The lattice Ω.

Now we see that all the structures μ_z/E_z, $z \in \Omega$ are one-element, hence lattices. Hence, $(\mathcal{R} \times \mathcal{L}, E)$ is an Ω-lattice.

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P2.05
https://doi.org/10.26493/2590-9770.1305.f45
(Also available at http://adam-journal.eu)

On the coset structure of distributive skew lattices

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Received 14 July 2019, accepted 19 September 2019, published online 29 December 2019

Abstract

In the latest developments in the theory of skew lattices, the study of distributivity has been one of the main topics. The largest classes of examples of skew lattices thus far encountered are distributive. In this paper, we will discuss several aspects of distributivity in the absence of commutativity, and review recent related results in the context of the coset structure of skew lattices. We show that the coset perspective is essential to fully understand the nature of skew lattices and distributivity in particular. We will also discuss the combinatorial implications of these results and their impact in the study of skew lattices.

Keywords: Skew lattices, distributive structures, noncommutative structures, ordered structures, bands of semigroups.

Math. Subj. Class.: 06A75, 06B20, 06B75

1 Introduction

The study of skew lattices provides two perspectives that complement each other: in one perspective skew lattices are seen to be noncommutative variants of lattices; in the other they are viewed as double bands, two band structures occupying the same set, but are in some way dual to each other. Due to this, many of its basic concepts originate in either lattice theory or semigroup theory. Thus, e.g., skew lattices have a natural partial order similar to that occurring in lattice theory, but also the Green's equivalences that are fundamental in the study of bands. In recent developments in skew lattice theory, distributivity has been

*The author thanks the support of the Croatian Science Foundation's funding of the project EVOSOFT, with the reference UIP-2014-09-7945.

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one of the main topics [16]. When commutativity is no longer assumed, the concept of distributivity, inherited from lattice theory, has broken into various types of distributivity. Following Leech's early papers, and then in more recent publications, a group of mathematicians (K. Cvetko-Vah, M. Kinyon, J. Leech, M. Spinks and J. Pita Costa) have studied this topic. Much of this more recent study involves coset analysis which underlies the study of skew lattice architecture that was initiated by Leech in [15], and extended more recently by the researchers above, especially in [2, 6] and [10]. It is our aim to give a clear and concise overview of this work, while adding a few new results on the topic.

2 Preliminaries

A *skew lattice* is a set S with binary operations $\wedge$ and $\vee$ that are both idempotent and associative, and satisfy the absorption laws $x \wedge (x \vee y) = x = (y \vee x) \wedge x$ and their $\vee$-$\wedge$ duals. Given that $\wedge$ and $\vee$ are idempotent and associative, these laws are equivalent to the absorption dualities: $x \wedge y = x$ iff $x \vee y = y$ and its $\vee$-$\wedge$ dual. A *band* is a semigroup consisting of just idempotents, and a *semilattice* is a commutative band. When $\mathbf{S}$ is a commutative semigroup, the set $E(S)$ of all idempotents in $\mathbf{S}$ is a semilattice under the semigroup multiplication. When $\mathbf{S}$ is not commutative, $E(S)$ need not be closed under multiplication [8]. Recall that a band is *regular* if it satisfies the identity $xyxzx = xyzx$. Skew lattices can be seen as double regular bands since both band reducts $(S, \wedge)$ and $(S, \vee)$ are regular. Green's relations are basic equivalence relations on semigroups, introduced in [7]. For bands they are defined by: $x \,\mathcal{D}\, y$ iff $xyx = x$ and $yxy = y$; $x \,\mathcal{L}\, y$ iff $xy = x$ and $yx = y$; and $x \,\mathcal{R}\, y$ iff $xy = y$ and $yx = x$. Due to the absorption dualities, the *Green's relations* were defined for skew lattices in [12] by $\mathcal{R} = \mathcal{R}_\wedge = \mathcal{L}_\vee$, $\mathcal{L} = \mathcal{L}_\wedge = \mathcal{R}_\vee$ and $\mathcal{D} = \mathcal{D}_\wedge = \mathcal{D}_\vee$. In the literature, $\mathcal{D}$ is often called the *natural equivalence*. *Right-handed* skew lattices are the skew lattices for which $\mathcal{D} = \mathcal{R}$ while *left-handed* skew lattices satisfy $\mathcal{D} = \mathcal{L}$ [16]. Influenced by the natural quasiorders defined on bands [8], we define for skew lattices the following distinct concepts:

(i) the *natural partial order* defined by $x \geq y$ if $x \wedge y = y = y \wedge x$ or, dually, $x \vee y = x = y \vee x$;

(ii) the *natural preorder* defined by $x \succeq y$ if $y \wedge x \wedge y = y$ or, dually, $x \vee y \vee x = x$.

Observe that $x \,\mathcal{D}\, y$ iff $x \succeq y$ and $y \succeq x$.

A band $\mathbf{S}$ is *rectangular* if $xyx = x$ holds. A skew lattice is *rectangular* if both band reducts $(S, \wedge)$ and $(S, \vee)$ are rectangular. This is equivalent to $x \wedge y = y \vee x$ holding. For every skew lattice $\mathbf{S}$, $\mathcal{D}$ is a congruence, $\mathbf{S}/\mathcal{D}$ is the maximal lattice image of $\mathbf{S}$ and all congruence classes of $\mathcal{D}$ are maximal rectangular skew lattices in $\mathbf{S}$. Recall that a *chain* (or *totally ordered set*) is an ordered set where every pair of elements are (order) related, and an *antichain* is an ordered set where no two elements are (order) related. We call $\mathbf{S}$ a *skew chain* whenever $S/\mathcal{D}$ is a chain. All $\mathcal{D}$-classes are antichains.

A *primitive* skew lattice is a skew lattice $\mathbf{S}$ composed of two comparable $\mathcal{D}$-classes A and B that are comparable in that $A \geq B$ in $S/\mathcal{D}$, $a \succeq b$ for all $a \in A$ and $b \in B$. A *skew diamond* is a skew lattice composed by two incomparable $\mathcal{D}$-classes, A and B, a join $\mathcal{D}$-class $J = A \vee B$ and a meet $\mathcal{D}$-class $M = A \wedge B$. In particular, we express a primitive skew lattice as $\{\, A > B \,\}$ and the skew diamond $\mathbf{X}$ as $\{\, J > A, B > M \,\}$. Note that if A and B are incomparable $\mathcal{D}$-classes in a skew lattice $\mathbf{S}$, with $A \wedge B = M$ and $A \vee B = J$, then the $\mathcal{D}$-relation on the skew diamond $\mathbf{X} = \{\, J > A, B > M \,\}$ is the restriction of the

$\mathcal{D}$-relation on $\mathbf{S}$, and the $\mathcal{D}$-classes in the sub skew lattice $\mathbf{X}$ are exactly A, B, M and J. In this case $\{ J > A \}$, $\{ J > B \}$, $\{ A > M \}$ and $\{ B > M \}$ are primitive skew lattices.

Proposition 2.1 ([12]). *Let A and B be comparable $\mathcal{D}$-classes in a skew lattice $\mathbf{S}$ such that $A \geq B$. Then, for each $a \in A$, there exists $b \in B$ such that $a \geq b$, and dually, for each $b \in B$, there exists $a \in A$ such that $a \geq b$.*

Proposition 2.2 ([12]). *Let $\{ J > A, B > M \}$ be a skew diamond. Then, for every $a \in A$ there exists $b \in B$ such that $a \vee b = b \vee a$ in J and $a \wedge b = b \wedge a$ in M. Moreover,*

$$J = \{a \vee b \mid a \in A, b \in B \text{ and } a \vee b = b \vee a\} \quad \text{and}$$
$$M = \{a \wedge b \mid a \in A, b \in B \text{ and } a \wedge b = b \wedge a\}.$$

The *variety of distributive skew lattices* was first introduced in [12] by the defining identities:

(d1) $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$,

(d2) $x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$.

Skew lattices satisfying (d1) are called $\wedge$-*distributive* while skew lattices satisfying (d2) are called $\vee$-*distributive*. Clearly, the lattice $\mathbf{S}/\mathcal{D}$ is distributive in either case, because the distributivity of $\wedge$ will imply the distributivity of $\vee$ in the presence of commutativity. Since $x \vee y = y \wedge x$ in any $\mathcal{D}$-class, it is easily seen that in any skew lattice both (d1) and (d2) must hold when y and z are $\mathcal{D}$-related. We thus also have the following pair of "balanced" identities that are equivalent to (d1) and (d2) respectively.

Proposition 2.3 ([19]). *Let $\mathbf{S}$ be a skew lattice. Then $\mathbf{S}$ is distributive if and only if for all $a, b, c \in S$,*

(i) $a \wedge (c \vee b \vee c) \wedge a = (a \wedge c \wedge a) \vee (a \wedge b \wedge a) \vee (a \wedge c \wedge a)$ *and, dually,*

(ii) $a \vee (c \wedge b \wedge c) \vee a = (a \vee c \vee a) \wedge (a \vee b \vee a) \wedge (a \vee c \vee a)$.

In order to explore further the different concepts of distributivity occurring in the literature, consider the following axioms:

(d3) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(d4) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,

(d5) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,

(d6) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Skew lattices satisfying (d3) and (d4) are called *strongly distributive* while skew lattices satisfying (d5) and (d6) are called *co-strongly distributive*. Examples exist of skew lattices satisfying any combination of these four distributive identities (cf. [11]). A skew lattice satisfies all four distributive laws (d3) to (d6) if and only if it is the direct product of a rectangular skew lattice with a distributive lattice (cf. [11]). This is also equivalent to a skew lattice satisfying the following pair of identities:

(d7) $x \wedge (y \vee z) \wedge w = (x \wedge y \wedge w) \vee (x \wedge z \wedge w)$, (d8) $x \vee (y \wedge z) \vee w = (x \vee y \vee w) \wedge (x \vee z \vee w)$.

Skew lattices satisfying (d7) are called $\wedge$-*bidistributive* skew lattices while skew lattices satisfying (d8) are called $\vee$-*bidistributive* skew lattices. Clearly a number of implications are immediate:

$$((d3) \text{ and } (d4)) \Rightarrow (d7) \Rightarrow (d1), \text{ while } ((d5) \text{ and } (d6)) \Rightarrow (d8) \Rightarrow (d2).$$

It turns out that (d7) implies both (d1) and (d2) as does (d8), and thus also (d3) and (d4) together, as well as (d5) and (d6) together [14]. For lattices, all of these identities reduce to either (d3) or (d5), which in turn are equivalent. A skew lattice **S** is *quasi-distributive* if its maximal lattice image $\mathbf{S}/\mathcal{D}$ is distributive. Clearly this is the case for the types of distributive skew lattices above. In general, a skew lattice is quasi-distributive precisely when no copy of either 5-element non-distributive lattice $\mathbf{M}_5$ or $\mathbf{N}_5$ is a subalgebra [4]. Finally, a skew lattice **S** is *cancellative* if for all $x, y, z \in S$,

$$z \vee x = z \vee y \text{ and } z \wedge x = z \wedge z \text{ imply } x = y, \text{ and}$$
$$x \vee z = y \vee z \text{ and } x \wedge z = y \wedge z \text{ imply } x = y.$$

For lattices, being cancellative is equivalent to being distributive. In general, cancellative skew lattices are quasi-distributive, but need not be distributive. *Skew lattices of idempotents in rings are always cancellative.*

To understand further the connections between these variations on a distributive theme, we need several further concepts. A skew lattice is *symmetric* if commutativity is unambiguous in that for all $x, y \in S$,

$$x \wedge y = y \wedge x \text{ iff } x \vee y = y \vee x.$$

Our interest in symmetric skew lattices is due in part to the fact that *skew lattices of idempotents in rings are always distributive and symmetric*.

A skew lattice **S** is *normal* if it satisfies

$$x \wedge y \wedge z \wedge w = x \wedge z \wedge y \wedge w;$$

dually, if it satisfies

$$x \vee y \vee z \vee w = x \vee z \vee y \vee w$$

it is *conormal*. While (d1) and (d2) are not equivalent for skew lattices in general, for symmetric skew lattices we have:

Theorem 2.4 ([20, 3])**.** *Given a symmetric skew lattice, (d1) holds if and only if (d2) holds.*

Theorem 2.5 ([16])**.** *A skew lattice is strongly distributive in that it satisfies (d3) and (d4) if and only if it is distributive, symmetric and normal. It is co-strongly distributive in that it satisfies (d5) and (d6) if and only if it is distributive, symmetric and conormal. It is* $\wedge$-distributive *in that it satisfies (d7) if and only if it is distributive and normal. It is* $\vee$-distributive *in that it satisfies (d8) if and only if it is distributive and conormal. Finally, it is both* $\wedge$-distributive *and* $\vee$-distributive *if and only if it is the product of a distributive lattice and a rectangular skew lattice.*

A *skew Boolean algebra* is an algebra $\mathbf{S} = (S; \vee, \wedge, \setminus, 0)$ such that $(S; \vee, \wedge, 0)$ is a distributive, normal and symmetric skew lattice with a constant 0, called *zero*, satisfying $x \wedge 0 = 0 = 0 \wedge x$, and a binary operation $\setminus$ on $\mathbf{S}$, called *relative complement*, satisfying $(x \wedge y \wedge x) \vee (x \setminus y) = x$ and $(x \wedge y \wedge x) \wedge (x \setminus y) = 0$. Skew lattice reducts of skew Boolean algebras are strongly distributive. In general, a skew lattice can be embedded in the skew lattice reduct of a skew Boolean algebra if and only if it is strongly distributive [13]. Given a ring R whose set $E(R)$ of idempotents are closed under multiplication, then $E(R)$ forms a skew Boolean algebra under the operations:

$$\begin{aligned} e \wedge f &= ef, \\ e \vee f &= e + f + fe - efe - fef, \\ e \setminus f &= e - efe. \end{aligned}$$

It should be mentioned that strongly distributive skew lattices, and in particular skew Boolean algebras, are always cancellative.

Consider a skew lattice $\mathbf{S}$ consisting of exactly two $\mathcal{D}$-classes $A > B$. Given $b \in B$, the subset $A \wedge b \wedge A = \{a \wedge b \wedge a \mid a \in A\}$ of B is said to be a *coset* of A in B (or an *A-coset in B*). Similarly, a coset of B in A (or a B-coset in A) is any subset $B \vee a \vee B = \{b \vee a \vee b \mid b \in B\}$ of A, for a fixed $a \in A$. On the other hand, given $a \in A$, the *image set* of a in B is the set $a \wedge B \wedge a = \{a \wedge b \wedge a \mid b \in B\} = \{b \in B \mid b < a\}$. Dually, given $b \in B$ the set $b \vee A \vee b = \{a \in A \mid b < a\}$ is the image set of b in A (cf. [15, 17, 19]).

Theorem 2.6 ([15])**.** *Let* $\mathbf{S}$ *be a skew lattice with comparable* $\mathcal{D}$*-classes* $A > B$*. Then,* B *is partitioned by the cosets of* A *in* B *and the image set in* B *of any element* $a \in A$ *in* B *is a transversal of the cosets of* A *in* B*; dual remarks hold for any* $b \in B$ *and the cosets of* B *in* A *that determine a partition of* A*. Moreover, any coset* $B \vee a \vee B$ *of* B *in* A *is isomorphic to any coset* $A \wedge b \wedge A$ *of* A *in* B *under a natural bijection* φ *defined implicitly for any* $a \in A$ *and* $b \in B$ *by:* $x \in B \vee a \vee B$ *corresponds to* $y \in A \wedge b \wedge A$ *if and only if* $x \geq y$*. Thus, all cosets in* A *or* B *are mutually isomorphic.*

Furthermore, the operations $\wedge$ and $\vee$ on $A \cup B$ are determined jointly by the coset bijections and the rectangular structure of each $\mathcal{D}$-class. Even if A and B are unrelated so that one has a proper skew diamond, $\{ A \vee B > A, B > A \wedge B \}$, coset bijections play an important role in calculating both $a \vee b$ in $A \vee B$ and $a \wedge b$ in $A \wedge B$. (See [15, Lemma 1.3 and Theorem 3.3].) *Thus, even if the natural partial order does not completely determine the operations* $\wedge$ *and* $\vee$ *as it does for lattices, it still has a very significant role.*

All cosets and all image sets are rectangular sub-skew lattices (cf. [17]). E.g., $(A \wedge b \wedge A) \wedge (A \wedge b' \wedge A)$ quickly reduces to $A \wedge b \wedge b' \wedge A$ which in turn must also be $(A \wedge b' \wedge A) \vee (A \wedge b \wedge A)$. Moreover, coset equality is given by the following result:

Proposition 2.7 ([6])**.** *Let* $\mathbf{S}$ *be a skew lattice with comparable* $\mathcal{D}$*-classes* $A > B$ *and let* $y, y' \in B$*. Then the following are equivalent:*

(i) $A \wedge y \wedge A = A \wedge y' \wedge A$*;*

(ii) for all $x \in A$*,* $x \wedge y \wedge x = x \wedge y' \wedge x$*;*

(iii) there exists $x \in A$ *such that* $x \wedge y \wedge x = x \wedge y' \wedge x$*.*

Dual results hold for B-cosets in A, having similar statements.

The following propositions characterize important properties of skew lattices through coset structure flavored aspects of these algebras. They motivate the results in the context of coset laws characterization discussed later in this paper.

Proposition 2.8 ([15]). *A skew lattice $\mathbf{S}$ is symmetric if and only if given any skew diamond $\{ J > A, B > M \}$ in $\mathbf{S}$ and any $m, m' \in M$, $j, j' \in J$ the following equivalences hold:*

(a) $J \wedge m \wedge J = J \wedge m' \wedge J$ *iff* $A \wedge m \wedge A = A \wedge m' \wedge A$ *and* $B \wedge m \wedge B = B \wedge m' \wedge B$;

(b) $M \vee j \vee M = M \vee j' \vee M$ *iff* $A \vee j \vee A = A \vee j' \vee A$ *and* $B \vee j \vee B = B \vee j' \vee B$.

Proposition 2.9 ([18]). *A skew lattice $\mathbf{S}$ is normal iff for each comparable pair of $\mathcal{D}$-classes $A > B$ in $\mathbf{S}$ and all $x, x' \in B$, $A \wedge x \wedge A = A \wedge x' \wedge A$. Dually, $\mathbf{S}$ is conormal iff for all comparable pairs of $\mathcal{D}$-classes $A > B$ in $\mathbf{S}$ and all $x, x' \in A$, $B \vee x \vee B = B \vee x' \vee B$.*

Proposition 2.9 above essentially states that $\mathbf{S}$ is a normal skew lattice if and only if, for each comparable pair of $\mathcal{D}$-classes $A > B$ in $\mathbf{S}$ and for each $x \in A$, a unique $y \in B$ exists such that $y \leq x$. A dual result holds for conormal skew lattices.

3 Coset bijections and categorical skew lattices

A skew lattice is *categorical* if nonempty composites of coset bijections are coset bijections. That is, given $\mathcal{D}$-classes $A > B > C$ and coset bijections $\phi\colon B \vee a \vee B \to A \wedge b \wedge A$ and $\psi\colon C \vee b' \vee S \to B \wedge c \wedge B$ with

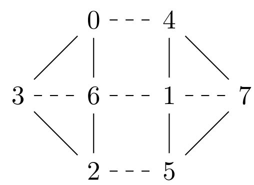
$$(A \wedge b \wedge A) \cap (C \vee b' \vee S) \neq \emptyset,$$

then the resulting nonempty composite $\psi\phi$ is a bijection between cosets in A and C. Rectangular and primitive skew lattices are trivially categorical. Skew lattices in rings and skew Boolean algebras provide nontrivial examples (cf. [15] and [1]). Clearly a skew lattice is categorical iff every skew chain of $\mathcal{D}$-classes $A > B > C$ in $\mathbf{S}$ is categorical. Indeed, this property is primarily about the skew chains of $\mathcal{D}$-classes in a skew lattice and about their coset bijections being well-behaved under composition. Upon adding empty partial bijections one indeed obtains a category of partial bijections whose objects are $\mathcal{D}$-classes and whose morphisms are given by coset bijections and the empty partial bijections.

Example 3.1. A minimal example of a non-categorical skew lattice is given by the 8-element left-handed skew chain given in Figure 1. In this example, considering the skew chain $\{ 0, 4 \} > \{ 3, 6, 1, 7 \} > \{ 2, 5 \}$, the coset bijections are the following:

$$\varphi_1\colon \{0,4\} \to \{3,1\}, \qquad \varphi_2\colon \{0,4\} \to \{6,7\}$$
$$\psi_1\colon \{3,7\} \to \{2,5\}, \qquad \psi_2\colon \{6,1\} \to \{2,5\}$$
$$\chi\colon \{0,4\} \to \{2,5\}$$

Observe that 0 has no image under $\psi_2 \circ \varphi_1$ and that $\chi(0) \in \{2, 5\}$ so that $\psi_2 \circ \varphi_1 \neq \chi$. The reader can find a detailed study of such examples in [9] where this skew lattice is called $\mathbf{X}_2$ and its right-handed version is called $\mathbf{Y}_2$.



$\wedge$	0	1	2	3	4	5	6	7
0	0	3	2	3	0	2	6	6
1	1	1	5	1	1	5	1	1
2	2	2	2	2	2	2	2	2
3	3	3	2	3	3	2	3	3
4	4	1	5	1	4	5	7	7
5	5	5	5	5	5	5	5	5
6	6	6	2	6	6	2	6	6
7	7	7	5	7	7	5	7	7

$\vee$	0	1	2	3	4	5	6	7
0	0	4	0	0	4	4	0	4
1	0	1	6	3	4	1	6	7
2	0	1	2	3	4	5	6	7
3	0	1	3	3	4	7	6	7
4	0	4	0	0	4	4	0	4
5	0	1	2	3	4	5	6	7
6	0	1	6	3	4	1	6	7
7	0	1	3	3	4	7	6	7

Figure 1: The admissible Hasse diagram of a left-handed non-categorical skew lattice.

A categorical skew lattice is *strictly categorical* if the compositions of coset bijections between comparable $\mathcal{D}$-classes $A > B > C$ are never empty. Rectangular and primitive skew lattices, as well as skew Boolean algebras are strictly categorical skew lattices (cf. [1]). Normal and conormal skew lattices are strictly categorical (cf. [9]). As strictly categorical skew lattices form a variety [9], sub skew lattices of strictly categorical skew lattices are also strictly categorical. Clearly a skew lattice is strictly categorical if and only if every skew chain of $\mathcal{D}$-classes $A > B > C$ in $\mathbf{S}$ is strictly categorical. In the strictly categorical case, the above category of coset bijections can be defined without the need of empty partial bijections.

Example 3.2. A minimal example of a categorical skew lattice that is not strictly categorical is given by the right-handed manifestation of the skew chain with three $\mathcal{D}$-classes in Figure 2. In fact, the composition of the coset bijections $\psi\colon \{1\} \to \{a\}$ and $\varphi'\colon \{b\} \to \{0\}$ is empty. Observe that $\chi\colon \{1\} \to \{0\}$ can be decomposed either by the composition of ψ and $\varphi\colon \{a\} \to \{0\}$, or by the composition of $\psi'\colon \{1\} \to \{b\}$ and φ' (cf. [9]).

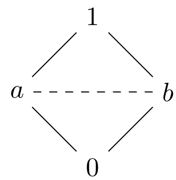
Categorical skew lattices and strictly categorical skew lattices are subvarieties of the variety of skew lattices (cf. [16]) as are the many classes of skew lattices considered in the previous section. What follows is a practical characterization of strictly categorical skew chains, followed by an immediate application.

Proposition 3.3 ([9])**.** *A skew chain $A > B > C$ is strictly categorical if and only if given $a \in A$, $b, b' \in B$ and $c \in C$ such that $a > b > c$ and $a > b' > c$, it follows that $b = b'$.*

Theorem 3.4. *A strictly categorical skew lattice $\mathbf{S}$ is distributive if and only if it is quasi-distributive, i.e., $\mathbf{S}/\mathcal{D}$ is distributive. In particular, a (co-)normal skew lattice is distributive if and only if it is quasi-distributive.*

Proof. We need only verify $\Leftarrow$. Suppose $\mathbf{S}$ is quasi-distributive. Then for all $x, y, z \in S$,

$$x \wedge (y \wedge z) \wedge x \;\mathcal{D}\; (x \wedge y \wedge x) \vee (x \wedge z \wedge x).$$



$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	b	a
b	0	a	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	a	1
b	b	b	b	1
1	1	1	1	1

Figure 2: The Cayley tables and the admissible Hasse diagram of a non-strictly categorical right-handed skew lattice.

Clearly $x > x \wedge (y \vee z) \wedge x$ and $x > (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$. On the other hand both of these polynomial terms are easily seen to be $> x \wedge z \wedge y \wedge x$. Thus by the above midpoint criterion, $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$. We illustrate the details for showing both terms being bigger than $x \wedge z \wedge y \wedge x$ by the following calculations:

$$
\begin{aligned}
x \wedge (y \vee z) \wedge x \wedge x \wedge z \wedge y \wedge x &= x \wedge (y \vee z) \wedge x \wedge z \wedge y \wedge x \\
&= x \wedge (y \vee z) \wedge z \wedge y \wedge x = x \wedge z \wedge y \wedge x,
\end{aligned}
$$

$$
\begin{aligned}
&[(x \wedge y \wedge x) \vee (x \wedge z \wedge x)] \wedge x \wedge z \wedge y \wedge x \\
&\qquad = [(x \wedge y \wedge x) \vee (x \wedge z \wedge x)] \wedge x \wedge z \wedge x \wedge y \wedge x \\
&\qquad = x \wedge z \wedge x \wedge y \wedge x = x \wedge z \wedge y \wedge x.
\end{aligned}
$$

The dual identities are similarly verified. □

Theorem 3.5 ([2]). *A skew chain* $\mathbf{S}$ *consisting of* $\mathcal{D}$*-classes* $A > B > C$ *is categorical if and only if for all elements* $a \in A$*,* $b \in B$ *and* $c \in C$ *satisfying* $a > b > c$*, one (and hence both) of the following equivalent statements hold:*

(i) $(A \wedge b \wedge A) \cap (C \vee b \vee C) = (C \vee a \vee C) \wedge b \wedge (C \vee a \vee C)$*;*

(ii) $(A \wedge b \wedge A) \cap (C \vee b \vee C) = (A \wedge c \wedge A) \vee b \vee (A \wedge c \wedge A)$*.*

Moreover, $\mathbf{S}$ *is strictly categorical iff in addition to (i) and (ii),*

$$(A \wedge b \wedge A) \cap (C \vee b' \vee C) \neq \emptyset,$$

for all $b, b' \in B$*.*

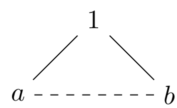
In conformity with [10], a nonempty intersection $(A \wedge b \wedge A) \cap (C \vee b' \vee C)$ of A-cosets and C-cosets in middle class B is called an *A-C coset in B*. Any such intersection equals $(A \wedge y \wedge A) \cap (C \vee y \vee C)$ for all y in this intersection.

Let $A \geq B$ be comparable $\mathcal{D}$ classes in a normal skew lattice $\mathbf{S}$ and let $a \in A$. Due to the normality of $\mathbf{S}$, Proposition 2.9 implies that, for all $a \in A$ there exists a unique $b \in B$

such that $a \geq b$. Since $a \wedge b \wedge a \leq a$ for all $b \in B$, $b \leq a$ in B if and only if $b = a \wedge b \wedge a$. Hence, given $b, b' \in B$ such that $b, b' \leq a$ in B, we have

$$\begin{aligned} b &= a \wedge b \wedge a = a \wedge b \wedge b' \wedge b \wedge a = a \wedge b \wedge b' \wedge b' \wedge b \wedge a \\ &= a \wedge b' \wedge b \wedge b \wedge b' \wedge a = a \wedge b' \wedge a = b'. \end{aligned}$$

Thus, normal skew lattices are strictly categorical. Indeed, the lower $\mathcal{D}$-class in any maximal primitive sub skew lattice of a normal skew lattice has exactly one coset. Thus, adjacent coset bijections are closed under composition and in particular have nonempty compositions. Dual remarks hold for conormal skew lattices. Strictly categorical skew lattices, however, need not be either normal or conormal. What follows is an example of a conormal, but not normal, skew lattice.



$\wedge$	a	b	1
a	a	b	a
b	a	b	b
1	a	b	1

$\vee$	a	b	1
a	a	a	1
b	b	b	1
1	1	1	1

Figure 3: The Cayley tables and the admissible Hasse diagram of a non-normal but strictly categorical right-handed skew lattice.

Example 3.6. Strictly categorical skew lattices need not be normal: the admissible Hasse diagram in Figure 3 represents a right-handed skew chain defined by the respective Cayley tables. It is strictly categorical but normality fails as the upper $\mathcal{D}$-class determines more then one coset in the lower $\mathcal{D}$-class.

Proposition 3.7 ([18])**.** *Let* $\mathbf{S}$ *be a skew lattice. The following statements are equivalent:*

(i) $\mathbf{S}$ *is categorical;*

(ii) for all distinct $\mathcal{D}$*-classes* $A > B > C$ *with elements* $b \in B$ *and* $c \in C$ *that satisfy* $b > c$*, the coset bijection* $\phi_{b,c}\colon C \vee b \vee C \to B \wedge c \wedge B$ *restricted to the corresponding* AC*-coset* $(A \wedge b \wedge A) \cap (C \vee b \vee C)$ *in* B *is a bijection of the later onto the* A*-coset* $A \wedge c \wedge A$ *in* C*;*

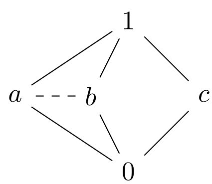
(ii′) for all distinct $\mathcal{D}$*-classes* $A > B > C$ *with elements* $a \in A$ *and* $b \in B$ *that satisfy* $a > b$*, the coset bijection* $\phi_{b,a}\colon A \wedge b \wedge A \to B \vee a \vee B$ *restricted to the corresponding* AC*-coset* $(A \wedge b \wedge A) \cap (C \vee b \vee C)$ *in* B *is a bijection of the latter onto the* C*-coset* $C \vee a \vee C$ *in* A*.*

The following results, based on research of Kinyon and Leech on distributive skew lattices (cf. [9] and [10]), reveal the relationship between distributivity and (strict) categoricity, and allow us to extend the ideas in Proposition 3.7 to coset laws in the categorical case and in particular in the distributive case.

Given a skew chain $A > B > C$, two elements $b, b' \in B$ are *AC-connected* if a finite sequence $b = b_0, b_1, \ldots, b_n = b'$ exists in B such that $A \wedge b_i \wedge A = A \wedge b_{i+1} \wedge A$ or $C \vee b_i \vee C = C \vee b_{i+1} \vee C$ for all $0 \leq i \leq n-1$. Clearly all elements in an A-coset [C-coset] in B are AC-connected.

Corollary 3.8. *Given a categorical skew chain $A > B > C$, all AC-cosets in B have the same size as all A-cosets in C and all C-cosets in A. Conversely, given a skew chain $A > B > C$ having this property, if the common cardinal size is also finite, then the skew chain is categorical.*

Given a skew chain $A > B > C$, an *AC-component* in B is a maximal AC-connected subset of B. B is AC-connected if all b, b' in it are AC-connected.



$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	b	0	a
b	0	a	b	0	b
c	0	0	0	c	c
1	0	a	b	c	1

$\vee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	a	1	1
b	b	b	b	1	1
c	c	1	1	c	1
1	1	1	1	1	1

Figure 4: The Cayley tables and admissible Hasse diagram of the right-handed skew lattice $\mathbf{NC}_5^R$.

Example 3.9. For instance, in Figure 1 the set $\{3, 6, 1, 7\}$ is an AC-component of the represented skew lattice for which $A = \{0, 4\}$ and $C = \{2, 5\}$. On the other hand, considering the skew lattice in Figure 4 with $A = \{1\}$ and $C = \{0\}$, the AC-components in this case are $\{a\}$, $\{b\}$ and $\{c\}$.

Proposition 3.10. *Let $A > B > C$ be a categorical skew chain. Then, B is a disjoint union of its AC-components. Every AC-component B' of B is the disjoint union of all A-cosets in B that are contained in B' and the disjoint union of all C-cosets in B that are contained in B', as well as the disjoint union of all the AC-cosets in B'. For each AC-component B', the union $A \cup B' \cup C$ forms a skew chain $A > B' > C$. In particular, $A > B > C$ is categorical if and only if $A > B' > C$ is categorical for each AC-component B'.*

The converse of this result doesn't hold as it was shown in [10] where the authors present a categorical skew chain $A > B > C$ with one unique AC-coset in B that is not distributive (and, therefore, not strictly categorical).

4 Strictly categorical skew lattices and orthogonality

Given skew lattice $\mathbf{S}$ with three equivalence classes A, B and C such that B is comparable to both A and C, we say that $x \in A$ is *covered* by a coset of C in B if the image set of x

in B is a subset of this coset of C. The dual definition is similar. A and C are *orthogonal* in B if and only if each $x \in A$ is covered by a coset of C in B and, dually, each $y \in C$ is covered by a coset of A in B.

Proposition 4.1 ([15]). *Let* **S** *be a skew lattice with three equivalence classes A, B and C such that B is comparable to both A and C. If A and C are orthogonal in B, then each coset of A in B has nonempty intersection with each coset of C in B. Moreover, all such coset intersections have common cardinality.*

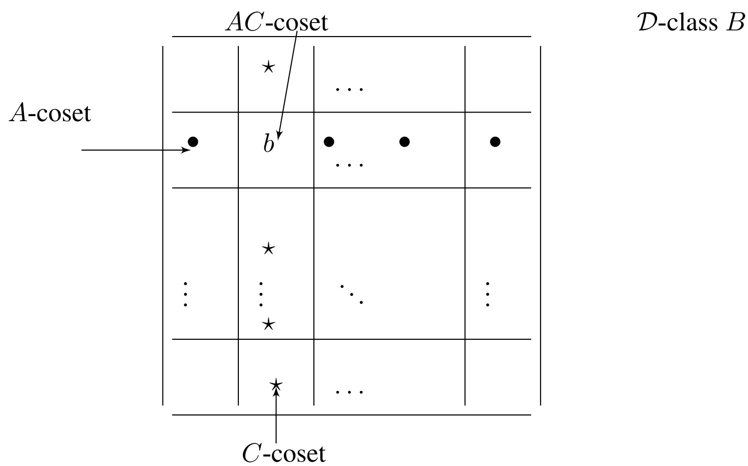


Figure 5: The partition diagram of the $\mathcal{D}$-class B by A-cosets and C-cosets, when A and C orthogonal in B.

Due to Proposition 4.1, we can visualize the above $\mathcal{D}$-class B in the orthogonal situation by the double partition diagram of Figure 5. We call a diagram representing intersection of partitions, like the one in Figure 5, a *partition diagram*. In this case, since $\mathcal{D}$-classes and cosets are rectangular skew lattices due to the rectangularity of cosets, we represent them by rectangles in this diagram. B is a doubly partitioned rectangle arising from a horizontal partition by one class of congruent rectangles, the A-cosets, and a vertical partition by a second class of congruent rectangles, the C-cosets. Their nonempty intersections are again called *AC-cosets* in B in conformity with [10]. It follows that if X and Y are respective A and C-cosets in B, then

$$|B| = \frac{|X| \cdot |Y|}{|X \cap Y|}.$$

This makes sense in the infinite case if $|Y|/|X \cap Y|$ is understood to be the size of the class of all cosets of the form $X \cap Y$ in Y. Since $|X|$ and $|Y|$ are respective divisors of $|A|$ and $|C|$, it follows that if both $|A|$ and $|C|$ are finite, then $|B|$ divides $|A| \cdot |C|$ and that $|B| \leq |A| \cdot |C|$ in general. The classical case of orthogonal behavior appeared in 1993 in [15].

Proposition 4.2 ([15]). *If* **S** *is a skew lattice with two non-comparable $\mathcal{D}$-classes A and B, then A and B are orthogonal in both their meet class M and in their join class J. The resulting AB-coset partitions of both J and M are refined by the coset partitions that J*

and M induce on each other. $\mathbf{S}$ *is symmetric precisely when both the AB-coset partition of M is the J-coset partition of M and, dually, the AB-coset partition of J is the M-coset partition of J. In this context, individual joins $a \vee b$ and meets $a \wedge b$ of elements $a \in A$ and $b \in B$ are determined by the orthogonality relationship.*

Lemma 4.3. *Let $A' > B'$ be a primitive subalgebra of a primitive skew lattice $A > B$ with $A' \subseteq A$ and $B' \subseteq B$. Then the B'-cosets in A' are precisely the nonempty intersections $A' \cap C$ of A' with some B-coset C in A. Likewise, the A'-cosets in B' are the nonempty intersections $B' \cap D$ of B' with some A-coset D in B.*

Proof. Given $a' \in A'$, $B' \vee a' \vee B' \subseteq A' \cap (B \vee a' \vee B)$, the intersection being nonempty. Conversely, let $A' \cap (B \vee a \vee B)$ be nonempty with $a \in A$. Then $B \vee a \vee B = B \vee a' \vee B$ for any a' in this intersection and hence in A'. Suppose $y = b \vee a' \vee b$ in A' for some $b \in B$. Then $y > b'$ for some b' in B' and $b \vee a' \vee b$ is in the same B-coset as $b' \vee a' \vee b'$. Since both are strictly bigger than b', they must be equal. That is, $A' \cap (B \vee a' \vee B) \subseteq B' \vee a' \vee B'$ and the first assertion follows. The second is similar. □

Corollary 4.4. *Let skew chain $A' > B' > C'$ be a subalgebra of a skew chain $A > B > C$. If the A-cosets and the C-cosets in B are orthogonal, then likewise the A'-cosets and the C'-cosets in B' are orthogonal.*

Proof. Given $c' \in C'$, by assumption all images of c' in B lie in some A-coset $A \wedge b \wedge A$ in B. Thus all images of $c' \in B'$ must lie in $A \wedge b \wedge A \cap B'$ which is an A'-coset in B', due to Lemma 4.3. Likewise, given $a' \in A'$, its images in B' must lie in some common C'-coset in B'. □

Clearly we now have:

Theorem 4.5. *If $A > B > C$ is a skew chain for which the A-cosets and the C-cosets in B are orthogonal, then $A > B > C$ is a strictly categorical skew chain, which is necessarily strictly categorical. In general, a skew lattice is strictly categorical if and only if all its skew chains have this orthogonal property.*

Proof. None of the $\mathbf{X}_n$ or the $\mathbf{Y}_n$ skew chains for $n > 1$ have the orthogonal property (see [9] for the construction of $\mathbf{X}_n$ and $\mathbf{Y}_n$). Hence by the Corollary 4.4 they cannot be subalgebras of this skew chain. It follows that $A > B > C$ is categorical. □

Proposition 4.6 ([10]). *Let $A > B > C$ be a strictly categorical skew chain, let $a \in A$, $c \in C$ such that $a > c$, and let $b \in B$ be the unique element such that $a > b > c$. Then, b lies jointly in the C-coset in B containing all images of a in B and in the A-coset in B containing all images of c in B.*

Theorem 4.7 ([9]). *Let $A > B > C$ be a strictly categorical skew chain.*

(i) For any $a \in A$, there exists $b \in B$ such that $a \wedge B \wedge a \subseteq C \vee b \vee C$;

(ii) Likewise, for any $c \in C$, there exists $b' \in B$ such that $c \vee B \vee c \subseteq A \wedge b' \wedge A$;

(iii) Given $a > c$, then the unique b in B such that $a > b > c$ lies jointly in the C-coset in B containing all images of a in B, and in the A-coset in B containing all images of c in B.

This is illustrated in Figure 5 where $b \in B$ lies in the intersection of the A-coset in B containing all images • of $c \in B$ and the C-coset in B containing all images $\star$ of $a \in B$.

Corollary 4.8 ([9]). *Given any categorical skew chain $A > B > C$ of $\mathcal{D}$-classes in* **S**, *the following statements are equivalent:*

(i) $A > B > C$ *is strictly categorical;*

(ii) Given $a \in A$, $c \in C$ and a coset bijection $\chi\colon C \vee a \vee C \to A \wedge c \wedge A$, unique coset bijections $\varphi\colon B \vee a \vee B \to A \wedge b \wedge A$ and $\psi\colon C \vee b \vee C \to B \wedge c \wedge B$ exist such that $\chi = \psi \circ \varphi$.

We now turn our attention to the relationship between orthogonality and strict categoricity, and the consequences of this relationship in the study of the coset structure of skew chains.

Theorem 4.9. *Given a skew chain $A > B > C$ in a skew lattice* **S**, *A and C are orthogonal in B if and only if $A > B > C$ is strictly categorical.*

Proof. Due to Proposition 4.1, the orthogonality of A and C in B implies that each coset of A in B has nonempty intersection with each coset of C in B. Proposition 3.5 then implies that the skew chain $A > B > C$ is strictly categorical.

Conversely, let $A > B > C$ be a strictly categorical skew chain and X be a order component of A in B. Then, it follows that the orthogonality of A and C in B is equivalent to the conditions (i) and (ii) of Theorem 4.7. Clearly, if A and C are orthogonal in B then both (i) and (ii) hold. Conversely, assume that (i) and (ii) hold and let K be a order component of A in B. Let $x \in A$. By the assumptions, $x \wedge B \wedge x$ lies in a unique C-coset in B, say $C \vee b \vee C$ for some $b \in B$. Let $y \in K$. Then, there exists a sequence of image sets $x_i \wedge B \wedge x_i$ with $x = x_1, x_2, \ldots, x_n$ such that $y \in x_n \wedge B \wedge x_n$,

$$(x_i \wedge B \wedge x_i) \cap (x_j \wedge B \wedge x_j) \neq \emptyset$$

for every $1 \leq i, j \leq n$ and $K = \bigcup_{1 \leq i \leq n}(x_i \wedge B \wedge x_i)$. Thus

$$\bigcap_{i \in I} \Big((x_i \wedge B \wedge x_i) \cap (C \vee b \vee C) \Big) \neq \emptyset$$

so that $K = \bigcup_{1 \leq i \leq n}(x_i \wedge B \wedge x_i) \subseteq (C \vee b \vee C)$, by Theorem 2.6. Hence, A and C are orthogonal in B. □

Corollary 4.10. *Let* **S** *be a skew Boolean algebra. Then, A and C are orthogonal in B for all skew chains $A > B > C$ in* **S**.

Proof. All skew Boolean algebras are normal skew lattices by definition. Thus, they are strictly categorical skew lattices due to normality. The conclusion now follows from Theorem 4.9. □

Remark 4.11. In [10], the orthogonal property for skew chains $A > B > C$ (A-cosets in B and C-cosets in B are orthogonal) is something that is added onto the categorical property. Thus, a skew lattice is strictly categorical iff it is categorical and its skew chains have the orthogonal property as in Theorem 4.5. But the orthogonal property for skew chains by itself implies that the skew lattice is categorical as in Theorem 4.9.

We conclude this section with a weakened version of orthogonal behavior. Given a categorical skew chain $A > B > C$, the A-cosets in B are *amenable* to the C-cosets in B if in any AC-component B' of B each A-coset meets each C-coset. Amenability is of *coset nature* as it is shown in the next result.

Proposition 4.12. *Let $A > B > C$ be a categorical skew chain. The A-cosets in B are amenable to the C-cosets in B if and only if, in any AC-component B' of B, the following equivalence holds for all $x, y \in B'$:*

$$A \wedge x \wedge A = A \wedge u \wedge A \text{ and } C \vee y \vee C = C \vee u \vee C \text{ for some } u \in B'$$

iff

$$A \wedge y \wedge A = A \wedge v \wedge A \text{ and } C \vee x \vee C = C \vee v \vee C \text{ for some } v \in B'.$$

Proof. Let B' be a AC component of B and $x, y \in B'$. Then, x and y are AC-connected. Assume that $A \wedge x \wedge A = A \wedge u \wedge A$ and $C \vee y \vee C = C \vee u \vee C$ for some $u \in B'$. Thus, the amenability of the A-cosets in B to the C-cosets in B implies the existence of $v \in (A \wedge y \wedge A) \cap (C \vee x \vee C)$ and, therefore,

$$A \wedge y \wedge A = A \wedge v \wedge A \text{ and } C \vee x \vee C = C \vee v \vee C.$$

Conversely, let $x, y \in B$ such that x, y are AC-connected. Suppose, without loss of generality that $u \in B$ exists such that $A \wedge x \wedge A = A \wedge u \wedge A$ and $C \vee y \vee C = C \vee u \vee C$. Thus, $A \wedge y \wedge A = A \wedge v \wedge A$ and $C \vee x \vee C = C \vee v \vee C$ for some $v \in B$. As x and y are arbitrary elements in B, each A-coset meets each C-coset in the AC-component of B where x and y belong. □

This is illustrated by the coset diagram in Figure 5 where the rows denote parts of A-cosets and the columns parts of C-cosets.

Proposition 4.13 ([10]). *Given a categorical skew chain $A > B > C$, the A-cosets and C-cosets in B are amenable if only only if for each AC-component B' of B, the skew chain $A > B' > C$ is strictly categorical. Furthermore, for all $a \in A$ and $c \in C$ such that $a > c$, the unique element $b \in B'$ such that $a > b > c$ lies jointly in the C-coset in B' containing all images of a in B' and in the A-coset in B' containing all images of c in B'.*

According to the Proposition 4.13 above, whenever A-cosets and C-cosets in B are amenable in a categorical skew chain $A > B > C$, A and C are "orthogonal" in each AC-component B' of B, in the sense of Remark 4.11. Amenability, unlike orthogonality, does not by itself insure that a skew chain is categorical (cf. [10]).

Proposition 4.14 ([18]). *A skew lattice is categorical if and only if all its skew chains are categorical. Furthermore, given a skew chain $A > B > C$ in a skew lattice $\mathbf{S}$, the following statements are equivalent:*

(i) $\{ A > B > C \}$ *is categorical,*

(ii) For all distinct $c, c' \in C$,

$$A \wedge c \wedge A = A \wedge c' \wedge A \quad \text{if and only if} \quad B \wedge c \wedge B = B \wedge c' \wedge B$$

and, for any b, b' in some C-coset in B such that $b > c$ and $b' > c'$,

$$A \wedge b \wedge A = A \wedge b' \wedge A$$

[where $b' = c' \vee b \vee c'$].

(iii) For all distinct $a, a' \in A$,

$$C \vee a \vee C = C \vee a' \vee C \quad \textit{if and only if} \quad B \vee a \vee B = B \vee a' \vee B$$

and, for any b, b' in some A-coset in B such that $a > b$ and $a' > b'$,

$$C \vee b \vee C = C \vee b' \vee C$$

[where $b' = a' \wedge b \wedge a'$].

5 Coset laws for distributive skew lattices

The following results deriving from Proposition 3.7 make use of identities involving cosets to characterize categorical and strictly categorical skew lattices by the description of their coset structure. These will lead us to the recent achievements by Leech and Kinyon in [10] that can be revisited in this coset structure context permitting us a similar characterization for distributive skew lattices.

Remark 5.1. Given $A > B > C$, let $a > c$ with $a \in A$ and $c \in C$ and let $b \in B$. We can use b to produce some $y \in B$ such that $a > y > c$ in two ways: either set $y = a \wedge (c \vee b \vee c) \wedge a$ or else set $y = c \vee (a \wedge b \wedge a) \vee c$. In general they need not give the same outcome.

Lemma 5.2 ([10])**.** *A skew chain $A > B > C$ is distributive if and only if for all $a \in A$, $b \in B$ and $c \in C$ such that $a > c$,*

$$a \wedge (c \vee b \vee c) \wedge a = c \vee (a \wedge b \wedge a) \vee c. \tag{5.1}$$

When this condition holds, the common outcome is the same for all b in a common AC-component of B and is the unique element y in that component such that $a > y > c$.

Theorem 5.3. *Given an skew chain $A > B > C$ the following are equivalent:*

(i) $A > B > C$ is distributive;

(ii) For all AC-components B' in B, $A > B' > C$ is strictly categorical.

(iii) $A > B > C$ is categorical and the A-cosets in B are amenable to the C-cosets in B.

In particular we have the following immediate consequence of Theorem 3.4:

Corollary 5.4. *All strictly categorical skew chains are distributive.*

Though, the converse statement to Corollary 5.4 does not hold: the skew lattice represented in Figure 2 is an example of a distributive skew lattice that is not strictly categorical having singular AC-components.

In the remainder of this section we turn our attention to the image set of an element in a coset and its role of a transversal of cosets permitting us to count those cosets. Transversals allows us to define the *index* of B in A, first presented in [4] and denoted by $[A : B]$, as the cardinality of the image set $b \vee A \vee b$, for any $b \in B$. Dually, we define the index of A in B, denoted by $[B : A]$, as the cardinality of the image set $a \wedge B \wedge a$, for any $a \in A$. The index $[A : B]$ equals the cardinality of the set of all B-cosets in A, and $[B : A]$ equals the cardinality of the set of all A-cosets in B. As all A-cosets in B and all B-cosets in A have a common size due to coset decomposition, we name this number the *order of the A-coset in B* (or the order of the B-coset in A), denoting it by $\omega[A, B]$ or, equivalently, by $\omega[B, A]$ (denoted by $c[A, B]$ in [12]).

Example 5.5. The situation in $A > B$ is illustrated in the following figure where the upper eggboxes represent the B-cosets in A and the lower eggboxes represent the A-cosets in B. In this case, $|A| = 18$ and $|B| = 12$ with $[A, B] = 3$, $[B, A] = 2$ and $\omega[A, B] = 6$.

Lemma 5.6 ([6]). *Given a skew lattice* $\mathbf{S}$ *with comparable* $\mathcal{D}$*-classes* $X > Y$ *consider the set of all* Y*-cosets in* X, $\{X_i \mid i \leq [X : Y]\}$ *and the set of* X*-cosets in* Y, $\{Y_j \mid j \leq [Y : X]\}$. *Then,* X *is finite if and only if* $[X : Y]$ *and* $\omega[X, Y]$ *are finite and, in that case,*

$$|X| = [X : Y] \cdot \omega[X, Y].$$

Similar remarks hold regarding the finitude of Y *and, likewise,* $|Y| = [Y : X] \cdot \omega[X, Y]$ *whenever* Y *is finite.*

The nature of the coset structure of a skew lattice permits such instances of combinatorial implications that are arise frequently in the literature. These combinatorial properties enabled us to derive coset laws to characterize varieties of symmetric skew lattices and cancellative skew lattices, in [15] and [4] respectively, or in the first author's research in [5, 6, 18], and his PhD thesis in [19]. Similar characterizations are also available for normal (and conormal) skew lattices, as a direct consequence of Proposition 2.9.

Proposition 5.7 ([18]). *Let* $\mathbf{S}$ *be a skew lattice. Then* $\mathbf{S}$ *is normal if and only if* $[B : A] = 1$ *and thus* $\omega[A, B] = |B|$ *for all comparable* $\mathcal{D}$*-classes* $A > B$ *in* $\mathbf{S}$. *Dually,* $\mathbf{S}$ *is conormal if and only if* $[A : B] = 1$ *and thus* $\omega[A, B] = |A|$ *for all comparable* $\mathcal{D}$*-classes* $A > B$ *in* $\mathbf{S}$.

We turn to examine the more general case of strictly categorical skew lattices. From Theorem 5.3 we are able to achieve counting results regarding distributive skew lattices.

Proposition 5.8 ([18]). *Given a skew chain* $A > B > C$ *in a skew lattice* $\mathbf{S}$ *with both* A *and* C *finite, then*

$$|B| \geq \frac{\omega[A, B] \cdot \omega[B, C]}{\omega[A, C]}.$$

Furthermore, $A > B > C$ *is strictly categorical if and only if*

$$|B| = \frac{\omega[A, B] \cdot \omega[B, C]}{\omega[A, C]} \tag{5.2}$$

so that in particular B *is also finite. With* B *finite,* $A > B > C$ *is strictly categorical if and only if*

$$|B| = \omega[A, C] \cdot [B : A] \cdot [B : C]. \tag{5.3}$$

Proposition 5.9. *Let* $A > B > C$ *be a skew chain where* A *and* C *are finite. Then,* $A > B > C$ *is distributive if and only if*

$$|B_i| = \frac{\omega[A, B] \cdot \omega[B, C]}{\omega[A, C]}$$

for each AC*-component* B_i *of* B.

Corollary 5.10. *Let* $A > B > C$ *be a skew chain where* A *and* C *are finite. If* B *has* $n < \infty$ AC*-components, then* $A > B > C$ *is distributive if and only if*

$$|B| = n\frac{\omega[A, B] \cdot \omega[B, C]}{\omega[A, C]}.$$

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P2.06
https://doi.org/10.26493/2590-9770.1333.152
(Also available at http://adam-journal.eu)

Regular antilattices

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Received 18 November 2019, accepted 21 December 2019, published online 30 December 2019

Abstract

Antilattices $(S; \vee, \wedge)$ for which the Green's equivalences $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$ are all congruences of the entire antilattice are studied and enumerated.

Keywords: Noncommutative lattice, antilattice, Green's equivalences, lattice of subvarieties, enumeration, partition, composition.

Math. Subj. Class.: 06B75, 05A15, 05A17, 03G10, 11P99

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†Michael Kinyon was supported by the Simons Foundation Collaboration Grant 359872.

‡Work of Tomaž Pisanski is supported in part by the ARRS grants P1-0294, J1-7051, N1-0032, and J1-9187.

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1 Introduction

In the study of noncommutative lattices, lattices still play an important role. They are the commutative cases of the algebras being considered and indeed play an important role in the general theory of that larger class of algebras. (As with rings, "noncommutative" is understood inclusively to mean *not necessarily commutative*). But also, typically, a second subclass of algebras exists that plays counterpoint to the subclass of lattices. It has become common to refer to their members as "antilattices." Typically they resist any kind of nontrivial commutative behavior. That is, an instance of $xy = yx$ for a relevant binary operation can occur only when $x = y$. Antilattices, however, are not without their special charm. Indeed, they have been studied in connection with magic squares and finite planes. (See [8].)

In this paper we study the class of regular antilattices for which the Green's equivalences are congruences. Precise definitions occur in Section 2 where basic concepts such as bands, quasilattices and the condition of regularity are described, along with some relevant preliminary results.

Regular antilattices themselves are the focus of Section 3. The main results are a very precise decomposition given in Theorem 3.3 and its several consequences. A closer look at the lattice of subvarieties (see Figure 1) occurs in the final fourth section.

The reader seeking further information on bands is referred to the presentations given in Clifford and Preston [4], in Grillet [5] and in Howie [6]. For further background on skew lattices and quasilattices, see [7] and [9]. The basic facts of universal algebras, and in particular varieties, may be found in the second chapter of [2].

2 Preliminary concepts and results

A *band* is a semigroup $(S; \cdot)$ for which all elements are idempotent, that is, $xx = x$ holds. A band is *rectangular* if it satisfies the identity $xyz = xz$, or equivalently, $xyx = x$. (As often occurs, if just a single binary operation is involved, its appearance is suppressed in equations.) A *semilattice* is a commutative band ($xy = yx$). Clearly, rectangular semilattices form the class of trivial 1-point bands. Indeed both classes are structural opposites that play important roles in the general structure of bands. To see how and to set the stage for further preliminaries requires the use of Green's relations, defined first for bands.

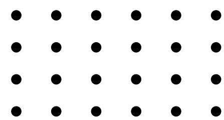
$$\begin{array}{llll} \mathcal{D}: & x\,\mathcal{D}\,y & \text{iff} & \text{both } xyx = x \text{ and } yxy = y; \\ \mathcal{L}: & x\,\mathcal{L}\,y & \text{iff} & \text{both } xy = x \text{ and } yx = y; \\ \mathcal{R}: & x\,\mathcal{R}\,y & \text{iff} & \text{both } xy = y \text{ and } yx = x. \end{array}$$

For bands, $\mathcal{L}$ and $\mathcal{R}$ commute under the usual composition of relations, with the common outcome being $\mathcal{D}$, i.e., $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{R} \vee \mathcal{L} = \mathcal{D}$. Here $\mathcal{R} \vee \mathcal{L}$ denotes the join of the two relations. Moreover, we have the following fundamental result of Clifford and McLean [3, 10]:

Theorem 2.1. *Given a band $(S; \cdot)$, the relation $\mathcal{D}$ is a congruence for which $S/\mathcal{D}$ is the maximal semilattice image and each $\mathcal{D}$-class of S is a maximal rectangular subalgebra of S. In brief, every band is a semilattice of rectangular bands.*

So what do rectangular bands look like? First there are two basic cases. A *left-zero band* is a band $(L; \cdot)$ with the trivial composition: $xy = x$. A *right-zero band* is a band $(R; \cdot)$ with the trivial composition: $xy = y$. In other words, we either have just a single

$\mathcal{L}$-class or just a single $\mathcal{R}$-class. Finally, there are products of both types, $L \times R$, and up to isomorphism, that is it. Thus a rectangular band may be pictured as a rectangular grid consisting of rows that are $\mathcal{R}$-classes and columns that are $\mathcal{L}$-classes.



The product xy of elements x and y is the unique element in the row of x and the column of y. Given a rectangular band $(S; \cdot)$ and x in S, if L denotes the $\mathcal{L}$-class of x and R denotes the $\mathcal{R}$-class of S, and $\varphi\colon L \times R \to S$ is defined by $\varphi(u, v) = uv \in S$, then φ is an isomorphism of rectangular bands. Rectangular bands are precisely the bands that are *anti-commutative* in that $xy = yx$ iff $x = y$.

While bands have a very simple local structure – their rectangular $\mathcal{D}$-classes – it is not immediately clear how elements from different $\mathcal{D}$-classes combine under the binary operation.

A band is *regular* if the relations $\mathcal{L}$ and $\mathcal{R}$ are both congruences. Semilattices and rectangular bands are both regular. In the semilattice case $\mathcal{L}$ and $\mathcal{R}$ reduce to the identity relation, so that regularity is trivial. One might expect all bands to be regular, but that is not so. In the rectangular case there is more: $\mathcal{L}$ and $\mathcal{R}$ commute under composition, not only with each other, but with every congruence θ:

$$\mathcal{L} \circ \theta = \theta \circ \mathcal{L} = \theta \vee \mathcal{L} \quad \text{and} \quad \mathcal{R} \circ \theta = \theta \circ \mathcal{R} = \theta \vee \mathcal{R}.$$

A *double band* is an algebra $(S; \vee, \wedge)$ for which both reducts $(S; \vee)$ and $(S; \wedge)$ are bands. A lattice is thus a double band where both $(S; \vee)$ and $(S; \wedge)$ are semilattices that jointly satisfy the standard *absorption* identities for a lattice: $x \wedge (x \vee y) = x = x \vee (x \wedge y)$. A very general class of noncommutative lattices is as follows. A *quasilattice* is a double band that satisfies the following (modified) absorption identities:

$$x \wedge (y \vee x \vee y) \wedge x = x = x \vee (y \wedge x \wedge y) \vee x.$$

Note that if commutativity is assumed, both identities reduce to the absorption identities for a lattice.

A *skew lattice* is a noncommutative lattice that satisfies the dual absorption identities:

$$x \wedge (x \vee y) = x = (x \vee y) \wedge x,$$
$$x \vee (x \wedge y) = x = (x \wedge y) \vee x.$$

A skew lattice is a quasilattice, but not conversely. In a quasilattice, both operations share common $\mathcal{D}$-classes that also form subalgebras, although on these classes both operations need not agree! Clearly, for a quasilattice $(S; \vee, \wedge)$, $\mathcal{D}$ is a congruence. Indeed, $S/\mathcal{D}$ is the maximal lattice image of S. This leads us to:

Definition 2.2. An *antilattice* is a double band $(S; \vee, \wedge)$ for which both reducts, $(S; \vee)$ and $(S; \wedge)$, are rectangular bands, i.e., satisfy the identity $xyz = xz$ or equivalently $xyx = x$.

An antilattice is trivially a quasilattice. Conversely, each $\mathcal{D}$-class of a quasilattice is a subalgebra that is an antilattice.

If the antilattice is a skew lattice, it is also called a *rectangular skew lattice*. As an antilattice, it is characterized by $x \wedge y = y \vee x$. $\mathcal{D}$-classes of skew lattices are always rectangular skew lattices.

Similar to bands, a version of the Clifford-McLean Theorem holds:

Theorem 2.3. *Given a quasilattice $(S; \vee, \wedge)$, the relation $\mathcal{D}$ is the same for both, $(S; \vee)$ and $(S; \wedge)$. For this common congruence, the quotient algebra $S/\mathcal{D}$ is the maximal lattice image and each $\mathcal{D}$-class of S is a maximal sub-antilattice of S. In brief, every quasilattice is a lattice of antilattices. (Compare Corollary 3 of [7]; see also [9].)*

Antilattices have been studied, not only due to their connection to quasilattices, but also in connection with magic squares and finite planes. (See [8].)

Like quasilattices and semigroups, by definition antilattices do not have prescribed constants, thus making the empty set a viable subalgebra. In so doing, this allows for the existence of a complete lattice of subalgebras for any given antilattice.

3 Regular antilattices

Given an antilattice $(S; \vee, \wedge)$, both reducts $(S; \vee)$ and $(S; \wedge)$ are regular in that $\mathcal{L}_{(\vee)}$ and $\mathcal{R}_{(\vee)}$ are congruences on $(S; \vee)$, and likewise $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$ are congruences on $(S; \wedge)$. The antilattice is *regular* if all four equivalences are congruences for the whole algebra. In general, a quasilattice $(S; \vee, \wedge)$ is *regular* if $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$ are congruences of $(S; \vee, \wedge)$. Skew lattices are regular, but in general, quasilattices need not be regular.

Theorem 3.1. *Regular antilattices form a subvariety of the variety of antilattices.*

Proof. We show that antilattices for which $\mathcal{L}_{(\vee)}$ is a congruence form a subvariety. To begin, in an antilattice S, $x \,\mathcal{L}_{(\vee)}\, u \vee x$ holds for all $u, x \in S$, and conversely, if $x \,\mathcal{L}_{(\vee)}\, x'$, then trivially, $x' = x' \vee x$. Since $\mathcal{L}_{(\vee)}$ is already a congruence on the reduct $(S; \vee)$, for $\mathcal{L}_{(\vee)}$ to be a congruence on $(S; \wedge)$, precisely the following identities need to hold:

$$(y \wedge x) \vee [y \wedge (u \vee x)] = y \wedge x \quad \& \quad [y \wedge (u \vee x)] \vee (y \wedge x) = y \wedge (u \vee x)$$

and

$$(x \wedge y) \vee [(u \vee x) \wedge y] = x \wedge y \quad \& \quad [(u \vee x) \wedge y] \vee (y \wedge x) = (u \vee x) \wedge y.$$

Thus this class of antilattices indeed forms a subvariety. Similar remarks verify the same claim for $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$. The theorem now follows. □

Since $\mathcal{L}$ and $\mathcal{R}$ commute under composition with all congruences on a rectangular band, $\mathcal{L}_{(\vee)} \circ \mathcal{L}_{(\wedge)} = \mathcal{L}_{(\wedge)} \circ \mathcal{L}_{(\vee)}$; $\mathcal{L}_{(\vee)} \circ \mathcal{R}_{(\wedge)} = \mathcal{R}_{(\wedge)} \circ \mathcal{L}_{(\vee)}$; $\mathcal{R}_{(\vee)} \circ \mathcal{L}_{(\wedge)} = \mathcal{L}_{(\wedge)} \circ \mathcal{R}_{(\vee)}$; and $\mathcal{R}_{(\vee)} \circ \mathcal{R}_{(\wedge)} = \mathcal{R}_{(\wedge)} \circ \mathcal{R}_{(\vee)}$ hold for regular antilattices. All four outcomes are thus congruences on the antilattice, and indeed form the join congruences of the respective pairs of congruences.

An antilattice $(S; \vee, \wedge)$ is *flat* if the reduct $(S; \vee)$ is either a left-0 semigroup ($x \vee y = x$) or a right-0 semigroup ($x \vee y = y$), and likewise the reduct $(S; \wedge)$ is either a left-0 semigroup ($x \wedge y = x$) or a right-0 semigroup ($x \wedge y = y$). That is, for each operation, either $\mathcal{D} = \mathcal{L}$ or $\mathcal{D} = \mathcal{R}$. Clearly there are 4 distinct classes of flat antilattices:

- the class $\mathbf{A}_{\mathcal{LL}}$ of all antilattices where $x \vee y = x = x \wedge y$.

- the class $\mathbf{A}_{\mathcal{LR}}$ of all antilattices where $x \vee y = x$ but $x \wedge y = y$.
- the class $\mathbf{A}_{\mathcal{RL}}$ of all antilattices where $x \vee y = y$ but $x \wedge y = x$.
- the class $\mathbf{A}_{\mathcal{RR}}$ of all antilattices where $x \vee y = y = x \wedge y$.

Clearly *each class is a subvariety of regular antilattices*. What is more:

Lemma 3.2. *Flat antilattices S and T of the same class are isomorphic if and only if they have the same cardinality. When the latter is the case, an isomorphism is given by any bijection between S and T.*

Theorem 3.3 (Decomposition Theorem)**.** *Every nonempty regular antilattice $(S; \vee, \wedge)$ factors into the direct product $S_{\mathcal{LL}} \times S_{\mathcal{LR}} \times S_{\mathcal{RL}} \times S_{\mathcal{RR}}$ of its four maximal flat images, one from each class above, with the respective factors being unique up to isomorphism.*

Proof. The factorization is obtained by first factoring with respect to say $\vee$:

$$S \cong S/\mathcal{R}_{(\vee)} \times S/\mathcal{L}_{(\vee)}$$

to get two factors for which the $\vee$-operation is flat. Then similarly factor both factors further with respect to the relevant $\mathcal{R}_{(\wedge)}$ and $\mathcal{L}_{(\wedge)}$ congruences to get four flat factors:

$$S \cong S_{\mathcal{LL}} \times S_{\mathcal{LR}} \times S_{\mathcal{RL}} \times S_{\mathcal{RR}}$$

where: $S_{\mathcal{LL}} = S/(\mathcal{R}_{(\vee)} \vee \mathcal{R}_{(\wedge)})$; $S_{\mathcal{LR}} = S/(\mathcal{R}_{(\vee)} \vee \mathcal{L}_{(\wedge)})$; $S_{\mathcal{RL}} = S/(\mathcal{L}_{(\vee)} \vee \mathcal{R}_{(\wedge)})$ and $S_{\mathcal{RR}} = S/(\mathcal{L}_{(\vee)} \vee \mathcal{L}_{(\wedge)})$. □

Further factorization can take place. But first, given a positive integer n, let $\mathbf{n}_{\mathcal{LL}}$, $\mathbf{n}_{\mathcal{LR}}$, $\mathbf{n}_{\mathcal{RL}}$ and $\mathbf{n}_{\mathcal{RR}}$ denote the relevant flat antilattices on the set $\{1, 2, 3, \ldots, n\}$. This leads us to the following finite version of the Decomposition Theorem:

Theorem 3.4. *Let $(S; \vee, \wedge)$ be a nonempty finite regular antilattice with the above factorization $S_{\mathcal{LL}} \times S_{\mathcal{LR}} \times S_{\mathcal{RL}} \times S_{\mathcal{RR}}$. If $n_{\mathcal{LL}} = |S_{\mathcal{LL}}|$, $n_{\mathcal{LR}} = |S_{\mathcal{LR}}|$, etc., then*

$$S \cong \mathbf{n}_{\mathcal{LL}} \times \mathbf{n}_{\mathcal{LR}} \times \mathbf{n}_{\mathcal{RL}} \times \mathbf{n}_{\mathcal{RR}}.$$

Clearly these four parameters characterize $(S; \vee, \wedge)$. It is also clear that factorization can continue on each of the four factors. For instance say $|S_{\mathcal{LL}}| = 180 = 4 \times 5 \times 9$. Then we have $S_{\mathcal{LL}} \cong (\mathbf{2}_{\mathcal{LL}})^2 \times (\mathbf{3}_{\mathcal{LL}})^2 \times \mathbf{5}_{\mathcal{LL}}$. Up to isomorphism, the only 1-point algebra is the trivial algebra $\mathbf{1} = \{0\}$.

Corollary 3.5. *A regular antilattice $(S; \vee, \wedge)$ is directly irreducible iff $|S|$ is either 1 or a prime. Every finite regular antilattice of order > 1 thus factors into a direct product of finitely many flat antilattices of prime order.*

Corollary 3.6. *A regular antilattice $(S; \vee, \wedge)$ is subdirectly irreducible iff either $|S| = 1$ or $|S| = 2$. Every (finite) regular antilattice of order > 1 is thus isomorphic to a subdirect product of (finitely) many flat antilattices of order 2.*

A sub-(pseudo)variety of regular antilattices is *positive*, if it is not the sub-variety $\{\emptyset\}$.

Corollary 3.7. *The lattice of all positive subvarieties of regular antilattices is a Boolean algebra with 16 elements and 4 atoms: $\mathbf{A}_{\mathcal{LL}}$, $\mathbf{A}_{\mathcal{LR}}$, $\mathbf{A}_{\mathcal{RL}}$ and $\mathbf{A}_{\mathcal{RR}}$ (see Figure 1).*

Corollary 3.8. *The lattice of all positive sub-pseudovarieties of finite regular antilattices is a Boolean algebra with* 16 *elements and the four atoms as above, but with their respective classes now restricted to finite algebras:* ${}_f\mathbf{A}_{\mathcal{LL}}$, ${}_f\mathbf{A}_{\mathcal{LR}}$, ${}_f\mathbf{A}_{\mathcal{RL}}$ *and* ${}_f\mathbf{A}_{\mathcal{RR}}$.

We will take a closer look at the positive subvarieties involved in the fourth section.

What can one say about the congruence lattice of a regular antilattice? To begin observe that the four classes of flat antilattces are mutually term equivalent with each other and with the class of all left-zero semigroups and also the class of all right-zero semigroups. In all these special cases the congruence lattice is precisely the full lattice $\Pi(S)$ of all equivalences of the underlying set S. Following the situation for rectangular bands in general, we have:

Theorem 3.9. *Let a nonempty regular antilattice* $(S; \vee, \wedge)$ *be factored into the direct product of its four maximal flat images:* $S_{\mathcal{LL}} \times S_{\mathcal{LR}} \times S_{\mathcal{RL}} \times S_{\mathcal{RR}}$. *Then the congruence lattice of* S *is given by* $\Pi(S_{\mathcal{LL}}) \times \Pi(S_{\mathcal{LR}}) \times \Pi(S_{\mathcal{RL}}) \times \Pi(S_{\mathcal{RR}})$. *That is, if the elements if* S *are expressed as* 4*-tuples* (x, y, z, w) *given by the factorization, then each congruence* θ *on* S *can be represented as a* 4*-tuple* $(\theta_{\mathcal{LL}}, \theta_{\mathcal{LR}}, \theta_{\mathcal{RL}}, \theta_{\mathcal{RR}})$ *of congruences on each factor in that:*

$$(x, y, z, w)\,\theta\,(x', y', z', w') \quad \textit{iff} \quad x\,\theta_{\mathcal{LL}}\,x', y\,\theta_{\mathcal{LR}}\,y', z\,\theta_{\mathcal{RL}}\,z' \,\&\, w\,\theta_{\mathcal{RR}}\,w'.$$

Conversely, in this manner every such 4*-tuple of congruences defines a congruence on the full antilattice* S.

In similar fashion:

Theorem 3.10. *Given a nonempty regular antilattice* S *with factorization* $S_{\mathcal{LL}} \times S_{\mathcal{LR}} \times S_{\mathcal{RL}} \times S_{\mathcal{RR}}$, *if* $a = |S_{\mathcal{LL}}|$, $b = |S_{\mathcal{LR}}|$, $c = |S_{\mathcal{RL}}|$ *and* $d = |S_{\mathcal{RR}}|$, *then the number of subalgebras of* S *is:*

$$1 + (2^a - 1)(2^b - 1)(2^c - 1)(2^d - 1).$$

One can ask: *given a positive integer* $n \geq 1$, *up to isomorphism, how many nonisomorphic regular antilattices are there of size* n? By Theorem 3.4 it is the number $\rho(n)$ of 4-fold positive factorizations $abcd$ of n, where the order of the factors a, b, c, d is important. Here a is the size of the $\mathcal{LL}$-factor, b is the size of the $\mathcal{LR}$-factor, etc.

To begin, thanks to Corollary 3.5, given the prime power factorization $n = 2^{e_2}3^{e_3}5^{e_5} \cdots p_k^{e_{p_k}}$:

$$\rho(n) = \rho(2^{e_2})\rho(3^{e_3})\rho(5^{e_5}) \cdots \rho(p_k^{e_{p_k}}).$$

Thus things can be reduced to calculating $\rho(p^e)$ for any prime power p^e .

From a combinatorial perspective, this is equivalent to asking in how many distinct ways can e identical balls be distributed into 4 labeled boxes. This question has a simple answer:

$$\rho(p^e) = \binom{e+3}{3}.$$

By putting all these together we obtain the following closed formula for $\rho(n)$:

Theorem 3.11. *Let* n *have the following prime power factorization* $n = 2^{e_2}3^{e_3}5^{e_5} \cdots p_k^{e_{p_k}}$: *Then*

$$\rho(n) = \binom{e_2+3}{3}\binom{e_3+3}{3}\binom{e_5+3}{3} \cdots \binom{e_{p_k}+3}{3}.$$

See also Table 1.

One can ask a more general question: In how many distinct ways can e identical balls be distributed into k labeled boxes? This question has analogous answer, namely $\binom{e+k-1}{k-1}$; see, for instance [1]. We will use some of these in the next section. Note that such an ordered partition of an integer n into k possibly empty parts is sometimes called a *composition*; see [1].

4 Semi-flat antilattices and other subvarieties

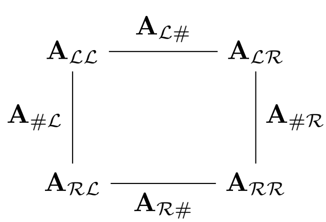
An antilattice $(S; \vee, \wedge)$ is *semi-flat* if either $(S; \vee)$ or $(S; \wedge)$ is flat. Flat antilattices are trivially semi-flat. The class of all semi-flat antilattices consists of four distinct subclasses that are not necessarily disjoint:

- $\mathbf{A}_{\mathcal{L}\#}$, the class of all antilattices $(S; \vee, \wedge)$ s.t. $(S; \vee)$ is a left-0 band;
- $\mathbf{A}_{\mathcal{R}\#}$, the class of all antilattices $(S; \vee, \wedge)$ s.t. $(S; \vee)$ is a right-0 band;
- $\mathbf{A}_{\#\mathcal{L}}$, the class of all antilattices $(S; \vee, \wedge)$ s.t. $(S; \wedge)$ is a left-0 band;
- $\mathbf{A}_{\#\mathcal{R}}$, the class of all antilattices $(S; \vee, \wedge)$ s.t. $(S; \wedge)$ is a right-0 band.

Theorem 4.1. *These four classes are subvarieties of the variety of regular antilattices.*

Proof. First observe that each is at least a subvariety in the variety of all antilattices. We show this for $\mathbf{A}_{\mathcal{L}\#}$, the other cases being similar. The identity characterizing $\mathbf{A}_{\mathcal{L}\#}$ in the variety of antilattices is clearly $x \vee y = x$. Thus $\mathbf{A}_{\mathcal{L}\#}$ is indeed a subvariety of antilattices. To see that all semi-flat antilattices are regular, again we need only consider, say, $\mathbf{A}_{\mathcal{L}\#}$. So let $(S; \vee, \wedge)$ be an antilattice for which $(S; \vee)$ is a left zero-band. Thus $\mathcal{L}_{(\vee)}$ is the universal equivalence ∇ on S, and thus trivially a congruence on $(S; \wedge)$ while $\mathcal{R}_{(\vee)}$ is the identity equivalence and thus again trivially a congruence on $(S; \wedge)$. Next consider $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$. Being congruences on $(S; \wedge)$, they are at least equivalences on S. But all equivalences on S are congruences on the left zero-band $(S; \vee)$, and thus $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$ are congruences on $(S; \vee, \wedge)$. □

Consider next the following diagram.



The four flat varieties occupy the middle rectangle. If two distinct flat varieties are adjacent on this rectangle, their join variety is the semi-flat variety labeling the line between them. But if they are diagonal opposites, we have the following:

- $\mathbf{A}_{\mathcal{L}\mathcal{L}} \vee \mathbf{A}_{\mathcal{R}\mathcal{R}}$ = the subvariety of antilattices for which $x \vee y = x \wedge y$.
- $\mathbf{A}_{\mathcal{L}\mathcal{R}} \vee \mathbf{A}_{\mathcal{R}\mathcal{L}}$ = the subvariety of antilattices for which $x \vee y = y \wedge x$.

These are the antilattice subvarieties that are, respectively, skew* lattices or skew lattices.

Next are the four double joins. Consider $\mathbf{A}_{\mathcal{LL}} \vee \mathbf{A}_{\mathcal{LR}} \vee \mathbf{A}_{\mathcal{RL}}$. It consists of regular antilattices for which the $\mathbf{A}_{\mathcal{RR}}$-factor is trivial. Since $\vee$ and $\wedge$ are idempotent, this reduces to no nontrivial $\mathbf{A}_{\mathcal{RR}}$-subalgebra occurring in the given antilattice. More briefly, no copy of $\mathbf{2}_{\mathcal{RR}}$ occurs as a subalgebra. This is guaranteed by the identity $x \vee (x \wedge y) = x$ (that is equivalent to the implication: $u\,\mathcal{R}_{(\wedge)}\,v \Rightarrow u\,\mathcal{L}_{(\vee)}\,v$) along with its $\vee - \wedge$ dual. This subvariety is, of course, the Boolean complement $\mathbf{A}^C_{\mathcal{RR}}$ of $\mathbf{A}_{\mathcal{RR}}$. The three other double joins are treated similarly to obtain:

$$\mathbf{A}_{\mathcal{LL}} \vee \mathbf{A}_{\mathcal{LR}} \vee \mathbf{A}_{\mathcal{RR}} = \mathbf{A}^C_{\mathcal{RL}},$$
$$\mathbf{A}_{\mathcal{LL}} \vee \mathbf{A}_{\mathcal{RL}} \vee \mathbf{A}_{\mathcal{RR}} = \mathbf{A}^C_{\mathcal{LR}},$$
$$\mathbf{A}_{\mathcal{LR}} \vee \mathbf{A}_{\mathcal{RL}} \vee \mathbf{A}_{\mathcal{RR}} = \mathbf{A}^C_{\mathcal{LL}}.$$

Finally, above these four lies the full variety of all regular antilattices and just below the four flat cases lies the variety of trivial 1-point algebras. The resulting lattice of all subvarieties of regular antilattices is, of course, isomorphic to the lattice of all subsets of any 4-element set, which brings us back to Corollary 3.7.

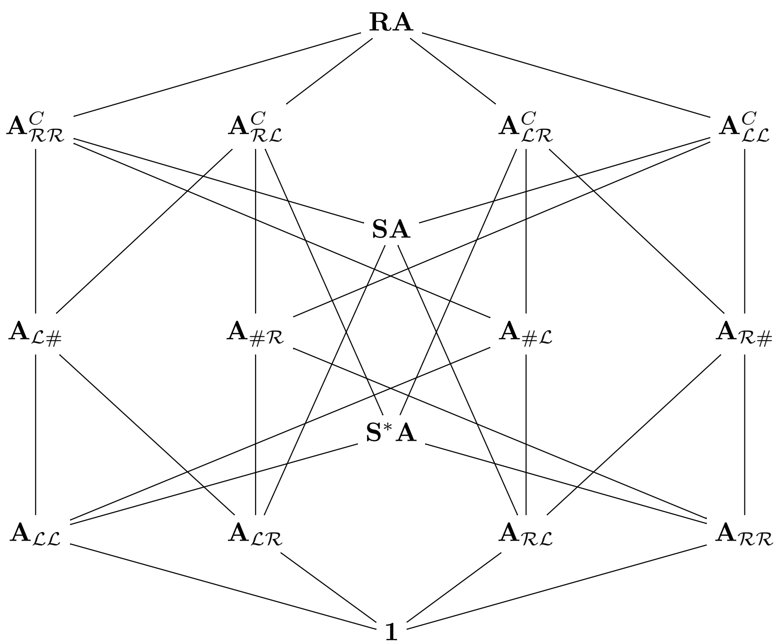


Figure 1: The Hasse diagram of the Boolean lattice of all positive subvarieties of antilattices.

The Hasse diagram of this Boolean lattice is explained in Table 1.

Equipped with all necessary tools we may now perform enumeration of the pseudo-variety of finite regular antilattices and their sub-pseudo-varieties; see Table 2. These sequences can be found in OEIS [11].

Table 1: The 16 positive subvarieties of regular antilattices.

Symbol	Subvariety	$\rho(p^e)$
$\mathbf{RA}$	regular antilattices	$\binom{e+3}{3}$
$\mathbf{A}^C_{\mathcal{RR}}$	complement of $\mathbf{A}_{\mathcal{RR}}$	$\binom{e+2}{2}$
$\mathbf{A}^C_{\mathcal{RL}}$	complement of $\mathbf{A}_{\mathcal{RL}}$	$\binom{e+2}{2}$
$\mathbf{A}^C_{\mathcal{LR}}$	complement of $\mathbf{A}_{\mathcal{LR}}$	$\binom{e+2}{2}$
$\mathbf{A}^C_{\mathcal{LL}}$	complement of $\mathbf{A}_{\mathcal{LL}}$	$\binom{e+2}{2}$
$\mathbf{SA}$	skew antilattices	$\binom{e+1}{1} = e+1$
$\mathbf{A}_{\mathcal{L}\#}$	$\mathcal{L}^*$ semi-flat	$\binom{e+1}{1} = e+1$
$\mathbf{A}_{\#\mathcal{R}}$	$^*\mathcal{R}$ semi-flat	$\binom{e+1}{1} = e+1$
$\mathbf{A}_{\#\mathcal{L}}$	$^*\mathcal{L}$ semi-flat	$\binom{e+1}{1} = e+1$
$\mathbf{A}_{\mathcal{R}\#}$	$\mathcal{R}^*$ semi-flat	$\binom{e+1}{1} = e+1$
$\mathbf{S}^*\mathbf{A}$	skew* antilattices	$\binom{e+1}{1} = e+1$
$\mathbf{A}_{\mathcal{LL}}$	$\mathcal{LL}$-flat	$\binom{e+0}{0} = 1$
$\mathbf{A}_{\mathcal{LR}}$	$\mathcal{LR}$-flat	$\binom{e+0}{0} = 1$
$\mathbf{A}_{\mathcal{RL}}$	$\mathcal{RL}$-flat	$\binom{e+0}{0} = 1$
$\mathbf{A}_{\mathcal{RR}}$	$\mathcal{RR}$-flat	$\binom{e+0}{0} = 1$
$\mathbf{1}$	trivial antilattice	1 if $e = 0$

Table 2: Enumeration of small regular antilattices and their subvarieties.

n	$\mathbf{RA}$	$\mathbf{A}^C_{\mathcal{RR}}, \mathbf{A}^C_{\mathcal{RL}}, \mathbf{A}^C_{\mathcal{LR}}, \mathbf{A}^C_{\mathcal{LL}}$	$\mathbf{SA}, \mathbf{A}_{\mathcal{L}\#}, \mathbf{A}_{\#\mathcal{R}}, \mathbf{A}_{\#\mathcal{L}}, \mathbf{A}_{\mathcal{R}\#}, \mathbf{S}^*\mathbf{A}$	$\mathbf{A}_{\mathcal{LL}}, \mathbf{A}_{\mathcal{LR}}, \mathbf{A}_{\mathcal{RL}}, \mathbf{A}_{\mathcal{RR}}$	$\mathbf{1}$
OEIS	A007426	A007425	A000005	A000012	
1	1	1	1	1	1
2	4	3	2	1	0
3	4	3	2	1	0
4	10	6	3	1	0
5	4	3	2	1	0
6	16	9	4	1	0
7	4	3	2	1	0
8	20	10	4	1	0
9	10	6	3	1	0
10	16	9	4	1	0
11	4	3	2	1	0
12	40	18	6	1	0
13	4	3	2	1	0
14	16	9	4	1	0
15	16	9	4	1	0
16	35	15	5	1	0

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P2.07
https://doi.org/10.26493/2590-9770.1323.ceb
(Also available at http://adam-journal.eu)

On noncommutative generalisations of Boolean algebras*

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Received 22 August 2019, accepted 7 December 2019, published online 30 December 2019

Abstract

Skew Boolean algebras (SBA) and Boolean-like algebras (nBA) are one-pointed and n-pointed noncommutative generalisation of Boolean algebras, respectively. We show that any nBA is a cluster of n isomorphic right-handed SBAs, axiomatised here as the variety of skew star algebras. The variety of skew star algebras is shown to be term equivalent to the variety of nBAs. We use SBAs in order to develop a general theory of multideals for nBAs. We also provide a representation theorem for right-handed SBAs in terms of nBAs of n-partitions.

Keywords: Skew Boolean algebras, Boolean-like algebras, Church algebras, multideals.

Math. Subj. Class.: 06E75, 03G05, 08B05, 08A30

1 Introduction

Boolean algebras are the main example of a well-behaved double-pointed variety – meaning a variety $\mathcal{V}$ whose type includes two distinct constants $0, 1$ in every nontrivial $\mathbf{A} \in \mathcal{V}$. Since there are other double-pointed varieties of algebras that have Boolean-like features, in [15, 23] the notion of Boolean-like algebra (of dimension 2) was introduced as a generalisation of Boolean algebras to a double-pointed but otherwise arbitrary similarity type.

*The results of this paper were presented at Non-commutative structures 2018: A workshop in honor of Jonathan Leech. The authors would like to thank the anonymous reviewer for the constructive comments, which helped us to improve the presentation of the paper.

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The idea behind this approach was that a Boolean-like algebra of dimension 2 is an algebra $\mathbf{A}$ such that every $a \in A$ is 2-central in the sense of Vaggione [27], meaning that $\theta(a, 0)$ and $\theta(a, 1)$ are complementary factor congruences of $\mathbf{A}$. Central elements can be given an equational characterisation through the ternary operator q satisfying the fundamental properties of the if-then-else connective of computer science. Algebraic analogues of the if-then-else construction have been studied extensively in the literature; the best known of these realisations is the ternary discriminator function $t\colon A^3 \to A$ of general algebra [28], defined for all $a, b, c \in A$ by $t(a, b, c) = c$ if $a = b$ and a otherwise. Varieties generated by a class of algebras with a common discriminator term are called discriminator varieties and are the most successful generalisation of Boolean algebras to date ([7, Section IV.9]).

It turns out that some important properties of Boolean algebras are shared by n-pointed algebras whose elements satisfy all the equational conditions of n-central elements through an operator q of arity $n + 1$ satisfying the fundamental properties of a generalised if-then-else connective. These algebras, and the varieties they form, were termed *Boolean-like algebras of dimension* n (nBA, for short) in [6]. Varieties of nBAs have many remarkable properties in common with the variety of Boolean algebras. In particular, any variety of nBAs with compatible operations is generated by the nBAs of finite cardinality n. In the pure case (i.e., when the type includes just the generalised if-then-else q and the n constants $\mathsf{e}_1, \dots, \mathsf{e}_n$), there is (up to isomorphism) a unique nBA $\mathbf{n}$ of cardinality n, so that any pure nBA is isomorphic to a subalgebra of $\mathbf{n}^I$, for a suitable set I. Another remarkable property of the 2-element Boolean algebra is the definability of all finite Boolean functions in terms of the connectives AND, OR, NOT. This property is inherited by the algebra $\mathbf{n}$: all finite functions on the universe of $\mathbf{n}$ are term-definable, so that the variety of pure nBAs is primal. More generally, a variety of an arbitrary type with one generator is primal if and only if it is a variety of nBAs.

Lattices and boolean algebras have been generalised in other directions: in the last decades weakenings of lattices where the meet and join operations may fail to be commutative have attracted the attention of various researchers. A non-commutative generalisation of lattices, probably the most interesting and successful, is the concept of *skew lattice* [16] along with the related notion of *skew Boolean algebra* (SBA) (the interested reader is referred to [3, 17, 18] or [26] for a comprehensive account). Here, a SBA is a symmetric skew lattice with zero in the sense of Leech [16], structurally enriched with an implicative BCS-difference [4] operation. Roughly speaking, a SBA is a non-commutative analogue of a generalised Boolean algebra. The significance of SBAs is revealed by a result of Leech [17], stating that any right-handed SBA can be embedded into some SBA of partial functions. This result has been revisited and further explored in [1] and [14], showing that any SBA is dual to a sheaf over a locally-compact Boolean space.

SBAs are also closely related to discriminator varieties (see [3, 9] for the one-pointed case and [23] for the double-pointed one). Seminal results of Bignall and Leech [3] show that every algebra in a one-pointed discriminator variety can be presented, up to term equivalence, as a skew Boolean intersection algebra (SBIA) with compatible operations. SBIAs are closely related to the SBAs of Leech [17]. Every SBIA has a SBA term reduct, but not conversely.

The present paper explores the connection between skew Boolean algebras and Boolean-like algebras of dimension n. We prove that any nBA $\mathbf{A}$ contains a symmetric $\cap$-skew cluster of right-handed SBIAs $S_1^{\cap}(\mathbf{A}), \dots, S_n^{\cap}(\mathbf{A})$, called its $\cap$-*skew reducts*. Interestingly, every permutation σ of the symmetric group S_n determines a bunch of isomorphisms

$$\mathrm{S}_1^{\cap}(\mathbf{A}) \cong \mathrm{S}_{\sigma 1}^{\cap}(\mathbf{A}), \dots, \mathrm{S}_n^{\cap}(\mathbf{A}) \cong \mathrm{S}_{\sigma n}^{\cap}(\mathbf{A})$$

which shows the inner symmetry of the nBAs. Every nBA has also a skew cluster $S_1(\mathbf{A}), \dots, S_n(\mathbf{A})$ of isomorphic right-handed SBAs, called its *skew reducts*, which are the skew Boolean algebra reducts of members of the $\cap$-skew cluster of $\mathbf{A}$. The skew reducts of a nBA are so deeply correlated that they allow us to recover the full structure of the nBA. We introduce a new variety of algebras, called *skew star* algebras, equationally axiomatising a bunch of skew Boolean algebras and their relationships, and we prove that it is term equivalent to the variety of nBAs. We also provide a representation theorem for right-handed skew Boolean algebras in terms of nBAs of n-partitions. This result follows on combining Leech's example [17] showing that every right-handed skew Boolean algebra can be embedded in an algebra of partial functions with codomain $\{1, 2\}$ with the result given in [6] that every nBA is isomorphic to a nBA of n-partitions.

The notion of ideal plays an important role in order theory and universal algebra. Ideals, filters and congruences are interdefinable in Boolean algebras. In the case of nBAs, the couple ideal-filter is replaced by *multideals*, which are tuples $(I_1, \dots, I_n)$ of disjoint skew Boolean ideals satisfying some compatibility conditions that extend in a conservative way those of the Boolean case. We show that there exists a bijective correspondence between multideals and congruences on nBAs, rephrasing the well known correspondence of the Boolean case. The proof of this result makes an essential use of the notion of a coordinate, originally defined in [6] and rephrased here in terms of the operations of the skew reducts. Any element a of a nBA $\mathbf{A}$ univocally determines a n-tuple of elements of the canonical inner Boolean algebra $\mathbf{B}$ of $\mathbf{A}$, its coordinates, codifying a as a "linear combination". In the Boolean case, there is a bijective correspondence between maximal ideals and homomorphisms onto $\mathbf{2}$. In the last section of the paper we show that every multideal can be extended to an ultramultideal, and that there exists a bijective correspondence between ultramultideals and homomorphisms onto $\mathbf{n}$. Moreover, ultramultideals are proved to be exactly the prime multideals.

2 Preliminaries

The notation and terminology in this paper are pretty standard. For concepts, notations and results not covered hereafter, the reader is referred to [7, 21] for universal algebra, to [17, 18, 26] for skew Boolean algebras and to [6, 15, 23] for nBAs.

2.1 Algebras

If τ is an algebraic type, an algebra $\mathbf{A}$ of type τ is called *a τ-algebra*, or simply an algebra when τ is clear from the context. An algebra is *trivial* if its carrier set is a singleton set.

Superscripts that mark the difference between operations and operation symbols will be dropped whenever the context is sufficient for a disambiguation.

$\mathrm{Con}(\mathbf{A})$ is the lattice of all congruences on $\mathbf{A}$, whose bottom and top elements are, respectively, $\Delta = \{(a, a) : a \in A\}$ and $\nabla = A \times A$. Given $a, b \in A$, we write $\theta(a, b)$ for the smallest congruence θ such that $(a, b) \in \theta$.

We say that an algebra $\mathbf{A}$ is:

(i) *subdirectly irreducible* if the lattice $\mathrm{Con}(\mathbf{A})$ has a unique atom;

(ii) *simple* if $\mathrm{Con}(\mathbf{A}) = \{\Delta, \nabla\}$;

(iii) *directly indecomposable* if $\mathbf{A}$ is not isomorphic to a direct product of two nontrivial algebras.

A class $\mathcal{V}$ of τ-algebras is a *variety* (equational class) if it is closed under subalgebras, direct products and homomorphic images. If K is a class of τ-algebras, the variety $\mathcal{V}(K)$ generated by K is the smallest variety including K. If $K = \{\mathbf{A}\}$ we write $\mathcal{V}(\mathbf{A})$ for $\mathcal{V}(\{\mathbf{A}\})$.

Following Blok and Pigozzi [5], two elements a, b of an algebra $\mathbf{A}$ are said to be *residually distinct* if they have distinct images in every non-trivial homomorphic image of $\mathbf{A}$.

We say that a variety $\mathcal{V}$ is n-*pointed* iff it has at least n nullary operators that are residually distinct in any nontrivial member of $\mathcal{V}$. Boolean algebras are the main example of a double-pointed variety.

A one-pointed variety $\mathcal{V}$ is 0-*regular* if the congruences of algebras in $\mathcal{V}$ are uniquely determined by their 0-classes. Fichtner [10] has shown that a one-pointed variety is 0-regular if and only if there exist binary terms $d_1(x, y), \dots, d_n(x, y)$ satisfying the following two conditions:

- $d_i(x, x) = 0$ for every $i = 1, \dots, n$;
- $d_1(x, y) = d_2(x, y) = \cdots = d_n(x, y) = 0 \;\Rightarrow\; x = y$.

2.1.1 Notations

If A is a set and $X \subseteq A$, then $\overline{X}$ denotes the set $A \setminus X$.

Let $\hat{n} = \{1, \dots, n\}$ and q be an operator of arity $n + 1$. If $d_1, \dots, d_k$ is a partition of $\hat{n}$ and $a, b_1 \dots, b_k \in A$, then

$$q(a, b_1/d_1, \dots, b_k/d_k) \tag{2.1}$$

denotes $q(a, c_1, \dots, c_n)$, where for all $1 \le i \le n$, $c_i = b_j$ iff $i \in d_j$. Notice that $q(a, b_1/d_1, \dots, b_k/d_k)$ is well-defined as $d_1, \dots, d_k$ partition $\hat{n}$. If d_j is a singleton $\{i\}$, then we write b/i for b/d_j. If $d_i = \hat{n} \setminus d_r$ is the complement of d_r, then we may write $b/\overline{d}_r$ for b/d_i. The notation (2.1) will be used extensively throughout the paper, mainly to define derived term operations in the context of nBAs.

2.2 Factor congruences and decomposition

Directly indecomposable algebras play an important role in the characterisation of the structure of a variety of algebras. For example, if the class of indecomposable algebras in a Church variety (see Section 3.1 and [23]) is universal, then any algebra in the variety is a weak Boolean product of directly indecomposable algebras. In this section we summarize the basic ingredients of factorisation: tuples of complementary factor congruences and decomposition operators (see [21]).

Definition 2.1. A sequence $(\phi_1, \dots, \phi_n)$ of congruences on a τ-algebra $\mathbf{A}$ is a n-tuple of complementary factor congruences exactly when:

(1) $\bigcap_{1 \le i \le n} \phi_i = \Delta$;

(2) $\forall (a_1, \dots, a_n) \in A^n$, there is $u \in A$ such that $a_i \; \phi_i \; u$, for all $1 \le i \le n$.

Such an element u such that $a_i\ \phi_i\ u$ for every i is unique by Definition 2.1(1).

If $(\phi_1, \dots, \phi_n)$ is a n-tuple of complementary factor congruences on $\mathbf{A}$, then the function $f\colon \mathbf{A} \to \prod_{i=1}^n \mathbf{A}/\phi_i$, defined by $f(a) = (a/\phi_1, \dots, a/\phi_n)$, is an isomorphism. Moreover, every factorisation of $\mathbf{A}$ in n factors univocally determines a n-tuple of complementary factor congruences.

A pair (ϕ_1, ϕ_2) of congruences is a pair of complementary factor congruences if and only if $\phi_1 \cap \phi_2 = \Delta$ and $\phi_1 \circ \phi_2 = \nabla$. The pair (Δ, ∇) corresponds to the product $\mathbf{A} \cong \mathbf{A} \times \mathbf{1}$, where $\mathbf{1}$ is a trivial algebra; obviously $\mathbf{1} \cong \mathbf{A}/\nabla$ and $\mathbf{A} \cong \mathbf{A}/\Delta$.

A *factor congruence* is any congruence which belongs to a pair of complementary factor congruences. The set of factor congruences of $\mathbf{A}$ is not, in general, a sublattice of $\mathrm{Con}(\mathbf{A})$.

Notice that, if $(\phi_1, \dots, \phi_n)$ is a n-tuple of complementary factor congruences, then ϕ_i is a factor congruence for each $1 \leq i \leq n$, because the pair $(\phi_i, \bigcap_{j \neq i} \phi_j)$ is a pair of complementary factor congruences.

It is possible to characterise n-tuples of complementary factor congruences in terms of certain algebra homomorphisms called *decomposition operators* (see [21, Definition 4.32] for additional details).

Definition 2.2. An *n-ary decomposition operator* on a τ-algebra $\mathbf{A}$ is a function $f\colon A^n \to A$ satisfying the following conditions:

(D1) $f(x, x, \dots, x) = x$;

(D2) $f(f(x_{11}, x_{12}, \dots, x_{1n}), \dots, f(x_{n1}, x_{n2}, \dots, x_{nn})) = f(x_{11}, \dots, x_{nn})$;

(D3) f is an algebra homomorphism from $\mathbf{A}^n$ to $\mathbf{A}$:

$$f(g(x_{11}, x_{12}, \dots, x_{1k}), \dots, g(x_{n1}, x_{n2}, \dots, x_{nk})) \\ = g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1k}, \dots, x_{nk})),$$

for every $g \in \tau$ of arity k.

There is a bijective correspondence between n-tuples of complementary factor congruences and n-ary decomposition operators, and thus, between n-ary decomposition operators and factorisations of an algebra in n factors.

Theorem 2.3. *Any n-ary decomposition operator $f\colon \mathbf{A}^n \to \mathbf{A}$ on an algebra $\mathbf{A}$ induces a n-tuple of complementary factor congruences $\phi_1, \dots, \phi_n$, where each $\phi_i \subseteq A \times A$ is defined by:*

$$a\ \phi_i\ b \quad \text{iff} \quad f(a, \dots, a, b, a, \dots, a) = a \qquad (b \text{ at position } i).$$

Conversely, any n-tuple $\phi_1, \dots, \phi_n$ of complementary factor congruences induces a decomposition operator f on $\mathbf{A}$: $f(a_1, \dots, a_n) = u$ iff $a_i\ \phi_i\ u$ for all i.

We say that two functions $f\colon A^m \to A$ and $g\colon A^n \to A$ *commute* (see [21, Definition 4.34]) if

$$f(g(x_{11}, \dots, x_{1n}), \dots, g(x_{m1}, \dots, x_{mn})) \\ = g(f(x_{11}, \dots, x_{m1}), \dots, f(x_{1n}, \dots, x_{mn})).$$

In this case, f is a homomorphism from $(A, g)^m$ into (A, g) and g is a homomorphism from $(A, f)^n$ into (A, f).

The following proposition is [21, Exercise 4.38(15)].

Proposition 2.4. *Let f and g be an m-ary and an n-ary decomposition operator of an algebra $\mathbf{A}$. Then $f(g(x_{11},\dots,x_{1n}),\dots,g(x_{m1},\dots,x_{mn}))$ is a decomposition operator of $\mathbf{A}$ if and only if f and g commute.*

The variables occurring in $f(g(x_{11},\dots,x_{1n}),\dots,g(x_{m1},\dots,x_{mn}))$ may not all be distinct, as explained in the following proposition.

Proposition 2.5. *If f is a n-ary decomposition operator and $d_1,\dots,d_k$ ($k \geq 2$) is a partition of $\hat{n} = \{1,\dots,n\}$, then the map h, defined by*

$$h(y_1,\dots,y_k) = f(z_1,\dots,z_n), \textit{ where for all } 1 \leq i \leq n, z_i = y_j \textit{ iff } i \in d_j,$$

is a k-ary decomposition operator.

2.3 Factor elements

The notion of decomposition operator and of factorisation can sometimes be internalised: some elements of the algebra, the so called factor elements, can embody all the information codified by a decompostion operator.

Let $\mathbf{A}$ be a τ-algebra, where we distinguish a $(n+1)$-ary term operation q.

Definition 2.6. We say that an element e of $\mathbf{A}$ is a *factor element with respect to q* if the n-ary operation $f_e\colon A^n \to A$, defined by

$$f_e(a_1,\dots,a_n) = q^{\mathbf{A}}(e,a_1,\dots,a_n), \text{ for all } a_i \in A,$$

is a n-ary decomposition operator (that is, f_e satisfies identities (D1) – (D3) of Definition 2.2).

An element e of $\mathbf{A}$ is a factor element if and only if the tuple of relations $(\phi_1,\dots,\phi_n)$, defined by $a\ \phi_i\ b$ iff $q(e,a,\dots,a,b,a,\dots,a) = a$ (b at position i), constitute a n-tuple of complementary factor congruences of $\mathbf{A}$.

By [9, Proposition 3.4] the set of factor elements is closed under the operation q: if $a, b_1,\dots,b_n \in A$ are factor elements, then $q(a,b_1,\dots,b_n)$ is also a factor element.

We notice that

- different factor elements may define the same tuple of complementary factor congruences;
- there may exist n-tuples of complementary factor congruences that do not correspond to any factor element.

In Section 3 we describe a class of algebras, called Church algebras of dimension n, where the $(n+1)$-ary operator q induces a bijective correspondence between a suitable subset of factor elements, the so-called n-central elements, and the set of all n-ary decomposition operators.

2.4 Skew Boolean algebras

We review here some basic definitions and results on *skew lattices* [16] and *skew Boolean algebras* [17].

Definition 2.7. A *skew lattice* is an algebra $\mathbf{A} = (A, \vee, \wedge)$ of type $(2, 2)$, where both $\vee$ and $\wedge$ are associative, idempotent binary operations, connected by the absorption laws: $x \vee (x \wedge y) = x = x \wedge (x \vee y)$; and $(y \wedge x) \vee x = x = (y \vee x) \wedge x$.

The absorption conditions are equivalent to the following pair of biconditionals: $a \vee b = b$ iff $a \wedge b = a$; and $a \vee b = a$ iff $a \wedge b = b$.

In any skew lattice we define the following relations:

1. $a \leq b$ iff $a \wedge b = a = b \wedge a$.
2. $a \preceq_{\mathcal{D}} b$ iff $a \wedge b \wedge a = a$.
3. $a \preceq_{\mathcal{L}} b$ iff $a \wedge b = a$.
4. $a \preceq_{\mathcal{R}} b$ iff $b \wedge a = a$.

The relation $\leq$ is a partial ordering, while the relations $\preceq_{\mathcal{D}}, \preceq_{\mathcal{L}}, \preceq_{\mathcal{R}}$ are preorders. The equivalences $\mathcal{D}$, $\mathcal{L}$ and $\mathcal{R}$, respectively induced by $\preceq_{\mathcal{D}}$, $\preceq_{\mathcal{L}}$ and $\preceq_{\mathcal{R}}$, are congruences. For more details see Schein [24].

A skew lattice is *right-handed* (*left-handed*) if $\mathcal{R} = \mathcal{D}$ ($\mathcal{L} = \mathcal{D}$). The following conditions are equivalent for a skew lattice $\mathbf{A}$:

(a) $\mathbf{A}$ is right-handed (left-handed);

(b) for all $a, b \in A$, $a \wedge b \wedge a = b \wedge a$ ($a \wedge b \wedge a = a \wedge b$).

Observe that

(i) The quotient $\mathbf{A}/\mathcal{D}$ is the maximal lattice image of $\mathbf{A}$. This is the skew-lattice theoretic analogue [16, Theorem 1.7] of the well-known Clifford-McLean theorem for bands.

(ii) The algebras $\mathbf{A}/\mathcal{L}$ and $\mathbf{A}/\mathcal{R}$ are the maximal right-handed and left-handed images of $\mathbf{A}$ respectively.

(iii) The skew lattice $\mathbf{A}$ is the fibered product of its maximal right-handed image $\mathbf{A}/\mathcal{L}$ with its maximal left-handed $\mathbf{A}/\mathcal{R}$ over its maximal lattice image. This result is the skew-lattice theoretic analogue [16, Theorem 1.15] of the Kimura factorisation theorem for idempotent semigroups.

In a skew lattice elements commuting under $\vee$ need not commute under $\wedge$ and vice-versa. A skew lattice, satisfying $x \wedge y = y \wedge x$ if and only if $x \vee y = y \vee x$ for all x and y, is called *symmetric*. Symmetric skew lattices form a variety characterised by the following identities (see [25, Theorem SSL-6]):

$$x \vee y \vee (x \wedge y) = (y \wedge x) \vee y \vee x; \qquad x \wedge y \wedge (x \vee y) = (y \vee x) \wedge y \wedge x.$$

The two most significant classes of examples, skew lattices of idempotents in rings (see, e.g., [16]) and skew Boolean algebras (see [17] and Definition 2.8 below), consist of symmetric skew lattices.

If we expand skew lattices by a subtraction operation and a constant 0, we get the following non-commutative variant of Boolean algebras (see [17]).

Definition 2.8. A *skew Boolean algebra* (SBA, for short) is an algebra $\mathbf{A} = (A, \vee, \wedge, \setminus, 0)$ of type $(2, 2, 2, 0)$ such that:

(S1) its reduct $(A, \vee, \wedge)$ is a skew lattice satisfying

- Normality: $x \wedge y \wedge z \wedge x = x \wedge z \wedge y \wedge x$;
- Symmetry: $x \wedge y = y \wedge x$ iff $x \vee y = y \vee x$;
- Distributivity: $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$ and $x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$;

(S2) 0 is left and right absorbing w.r.t. skew lattice meet;

(S3) the operation $\setminus$ satisfies the identities

$$(x \wedge y \wedge x) \vee (x \setminus y) = x = (x \setminus y) \vee (x \wedge y \wedge x);$$
$$x \wedge y \wedge x \wedge (x \setminus y) = 0 = (x \setminus y) \wedge x \wedge y \wedge x.$$

Every SBA is strongly distributive, i.e., it satisfies the identities $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(y \vee z) \wedge x = (y \wedge x) \vee (z \wedge x)$.

It can be seen that, for every $a \in A$, the natural partial order of the subalgebra $a \wedge A \wedge a = \{a \wedge b \wedge a : b \in A\} = \{b : b \leq a\}$ of $\mathbf{A}$ is a Boolean lattice. Indeed, the algebra $(a \wedge A \wedge a, \vee, \wedge, 0, a, \neg)$, where $\neg b = a \setminus b$ for every $b \leq a$, is a Boolean algebra with minimum 0 and maximum a.

Notice that

- The normal axiom implies the commutativity of $\wedge$ and $\vee$ in the interval $a \wedge A \wedge a$.
- Axiom (S2) expresses that 0 is the minimum of the natural partial order on A.
- Axiom (S3) implies that, for every $b \in a \wedge A \wedge a$, the element $a \setminus b$ is the complement of b in the Boolean lattice $a \wedge A \wedge a$. We point out here that $a \setminus b$ is in fact a kind of relative complement that acts 'locally' on subalgebras of the form $a \wedge A \wedge a$.

An element m of a SBA $\mathbf{A}$ is *maximal* if $a \preceq_{\mathcal{D}} m$ for every $a \in A$ (i.e., $a \wedge m \wedge a = a$, for every $a \in A$). When they exist, maximal elements form an equivalence class (modulo $\mathcal{D}$) called the *maximal class*. If $\mathbf{A}$ is a SBA, then $\mathbf{A}/\mathcal{D}$, where $\mathcal{D}$ is the Clifford-McLean congruence on $\mathbf{A}$, is a Boolean algebra iff $\mathbf{A}$ has a maximal class. Skew Boolean algebras with a maximal class thus constitute a very specialised class of skew Boolean algebras. It is known that every skew Boolean algebra embeds into a skew Boolean algebra with a maximal class.

A nonempty subset I of a SBA $\mathbf{A}$ closed under $\vee$ is an *ideal* of $\mathbf{A}$ (see [19, Section 4]) if it satisfies one of the following equivalent conditions:

- $a \in A$, $b \in I$ and $a \preceq_{\mathcal{D}} b$ imply $a \in I$;
- $a \in A$ and $b \in I$ imply $a \wedge b, b \wedge a \in I$;
- $a \in A$ and $b \in I$ imply $a \wedge b \wedge a \in I$.

Given a congruence ϕ on a SBA, the equivalence class $0/\phi$ is an ideal. However, congruences on a SBA are not in general in 1-1 correspondence with ideals. In particular, the congruence lattices of SBAs may satisfy no special lattice identities and they need not be congruence n-permutable for any $n \geq 2$.

2.5 Skew Boolean algebras with intersections

Skew Boolean algebras such that every finite subset of their universe has an infimum w.r.t. the underlying natural partial ordering of the algebra stand out for their significance. We denote the infimum of a and b w.r.t. the natural partial order by $a \cap b$ and refer to the operation $\cap$ as *intersection* in order to distinguish it from the skew lattice meet $\wedge$. It turns out that SBAs augmented with the additional operation $\cap$ can be given an equational characterisation provided we include the operation $\cap$ into the signature.

Definition 2.9. A *skew Boolean* $\cap$*-algebra* (SBIA, for short) is an algebra $\mathbf{A} = (A; \vee, \wedge, \cap, \setminus, 0)$ of type $(2, 2, 2, 2, 0)$ such that:

(i) The reduct $(A; \vee, \wedge, \setminus, 0)$ is a SBA and the reduct $(A; \cap)$ is a meet semilattice;

(ii) $\mathbf{A}$ satisfies the identities $x \cap (x \wedge y \wedge x) = x \wedge y \wedge x$ and $x \wedge (x \cap y) = x \cap y = (x \cap y) \wedge x$.

The next theorem by Bignall and Leech [3], which we present in its simplest form, provides a powerful bridge between the theories of SBAs and pointed discriminator varieties.

Theorem 2.10. *The variety of type* $(3, 0)$ *generated by the class of all one-pointed discriminator algebras* $(A; t, 0)$*, where* t *is the discriminator function on* A *and* 0 *is a constant, is term equivalent to the variety of right handed SBIAs.*

2.6 A term equivalence result for skew Boolean algebras

In [9] Cvetko-Vah and the second author have introduced the variety of semicentral right Church algebras (SRCAs) and have shown that the variety of right-handed SBAs is term equivalent to the variety of SRCAs. It is worth noticing that, in SRCAs, a single ternary operator q replaces all the binary operators of SBAs.

An algebra $\mathbf{A} = (A, q, 0)$ of type $(3, 0)$ is called a *right Church algebra* (RCA, for short) if it satisfies the identity $q(0, x, y) = y$.

Definition 2.11. Let $\mathbf{A} = (A, q, 0)$ be a RCA. An element $a \in A$ is called *semicentral* if it is a factor element (w.r.t. q) satisfying $q(a, a, 0) = a$.

Lemma 2.12 ([9, Proposition 3.9])**.** *Let* $\mathbf{A} = (A, q, 0)$ *be an RCA. Every semicentral element* $e \in A$ *determines a pair of complementary factor congruences:*

$$\phi_e = \{(a, b) : q(e, a, b) = a\} \quad \textit{and} \quad \bar{\phi}_e = \{(a, b) : q(e, a, b) = b\}$$

such that $\phi_e = \theta(e, 0)$*, the least congruence of* $\mathbf{A}$ *equating* e *and* 0*.*

Definition 2.13. An algebra $\mathbf{A} = (A, q, 0)$ of type $(3, 0)$ is called a *semicentral* RCA (SRCA, for short) if every element of A is semicentral.

To help the reader in understanding the term equivalence of SRCAs and right-handed SBAs, it is perhaps useful to provide an explicit axiomatisation of SRCAs. Such an axiomatisation is not long:

1. $q(0, x, y) = y$;
2. $q(w, w, 0) = w$;

3. $q(w, y, y) = y$;
4. $q(w, q(w, x, y), z) = q(w, x, z)$;
5. $q(w, x, q(w, y, z)) = q(w, x, z)$;
6. $q(w, q(y_1, y_2, y_3), q(z_1, z_2, z_3)) = q(q(w, y_1, z_1), q(w, y_2, z_2), q(w, y_3, z_3))$.

The last five identities equationally formalise that the element w is semicentral.

Theorem 2.14 ([9]). *The variety of right-handed SBAs is term equivalent to the variety of SRCAs.*

The proof is based on the following correspondence between the algebraic similarity types of SBAs and of SRCAs:

$$\begin{aligned} q(x, y, z) &\rightsquigarrow (x \wedge y) \vee (z \setminus x) \\ x \vee y &\rightsquigarrow q(x, x, y) \\ x \wedge y &\rightsquigarrow q(x, y, 0) \\ x \setminus y &\rightsquigarrow q(y, 0, x). \end{aligned}$$

The natural partial order and preorder of a SRCA are the partial order $\leq$ and the preorder $\preceq_{\mathcal{D}} = \preceq_{\mathcal{R}}$ of its corresponding SBA.

Example 2.15 (see [8, 9]). Let $\mathcal{F}(X, Y)$ be the set of all partial functions from X into Y. The algebra $\mathbf{F} = (\mathcal{F}(X, Y), q, 0)$ is a SRCA, where

- $0 = \emptyset$ is the empty function;
- For all functions $f\colon F \to Y$, $g\colon G \to Y$ and $h\colon H \to Y$ ($F, G, H \subseteq X$),

$$q(f, g, h) = g|_{G\cap F} \cup h|_{H\cap \overline{F}}.$$

By Theorem 2.14 $\mathbf{F}$ is term equivalent to the right-handed SBA with universe $\mathcal{F}(X, Y)$, whose operations are defined as follows:

$$f \wedge g = g|_{G\cap F}; \qquad f \vee g = f \cup g|_{G\cap \overline{F}}; \qquad g \setminus f = g|_{G\cap \overline{F}}.$$

3 Boolean-like algebras of finite dimension

Some important properties of Boolean algebras are shared by n-pointed algebras whose elements satisfy all the equational conditions of n-central elements through an operator q of arity $n + 1$ satisfying the fundamental properties of a generalised if-then-else connective. These algebras, and the varieties they form, were termed Boolean-like algebras of dimension n in [6].

3.1 Church algebras of finite dimension

In this section we recall from [6] the notion of a Church algebra of dimension n. These algebras have n nullary operations $\mathsf{e}_1, \dots, \mathsf{e}_n$ ($n \geq 2$) and an operation q of arity $n + 1$ (a sort of "generalised if-then-else") satisfying the identities $q(\mathsf{e}_i, x_1, \dots, x_n) = x_i$. The operator q induces, through the so-called n-central elements, a decomposition of the algebra into n factors.

Definition 3.1. Algebras of type τ, equipped with at least n nullary operations $\mathsf{e}_1, \dots, \mathsf{e}_n$ ($n \geq 2$) and a term operation q of arity $n+1$ satisfying $q(\mathsf{e}_i, x_1, \dots, x_n) = x_i$, are called *Church algebras of dimension* n (nCA, for short); nCAs admitting only the $(n+1)$-ary q operator and the n constants $\mathsf{e}_1, \dots, \mathsf{e}_n$ are called *pure* nCAs.

If $\mathbf{A}$ is an nCA, then $\mathbf{A}_0 = (A, q, \mathsf{e}_1, \dots, \mathsf{e}_n)$ is the *pure reduct* of $\mathbf{A}$.

Church algebras of dimension 2 were introduced as Church algebras in [20] and studied in [23]. Examples of Church algebras of dimension 2 are Boolean algebras (with $q(x, y, z) = (x \wedge y) \vee (\neg x \wedge z))$ or rings with unit (with $q(x, y, z) = xy + z - xz$). Next, we present some examples of Church algebra having dimension greater than 2.

Example 3.2 (*Semimodules*). Let R be a semiring and V be an R-semimodule generated by a finite set $E = \{\mathsf{e}_1, \dots, \mathsf{e}_n\}$. Then we define an operation q of arity $n+1$ as follows (for all $\mathbf{v} = \sum_{j=1}^{n} v_j \mathsf{e}_j$ and $\mathbf{w}^i = \sum_{j=1}^{n} w_j^i \mathsf{e}_j$):

$$q(\mathbf{v}, \mathbf{w}^1, \dots, \mathbf{w}^n) = \sum_{i=1}^{n} v_i \mathbf{w}^i.$$

Under this definition, V becomes a nCA. As a concrete example, if B is a Boolean algebra, B^n is a semimodule (over the Boolean ring B) with the following operations: $(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 \vee b_1, \dots, a_n \vee b_n)$ and $b(a_1, \dots, a_n) = (b \wedge a_1, \dots, b \wedge a_n)$. B^n is also called a Boolean vector space (see [11, 12]).

Example 3.3 (n-*Sets*). Let I be a set. A n-*subset* of I is a sequence $(Y_1, \dots, Y_n)$ of subsets Y_i of I. We denote by $\mathrm{Set}_n(I)$ the family of all n-subsets of I. $\mathrm{Set}_n(I)$ becomes a pure nCA if we define an $(n+1)$-ary operator q and n constants $\mathsf{e}_1, \dots, \mathsf{e}_n$ as follows, for all n-subsets $\mathbf{y}^i = (Y_1^i, \dots, Y_n^i)$:

$$q(\mathbf{y}^0, \mathbf{y}^1, \dots, \mathbf{y}^n) = (\bigcup_{i=1}^{n} Y_i^0 \cap Y_1^i, \dots, \bigcup_{i=1}^{n} Y_i^0 \cap Y_n^i);$$
$$\mathsf{e}_1 = (I, \emptyset, \dots, \emptyset), \dots, \mathsf{e}_n = (\emptyset, \dots, \emptyset, I).$$

In [27], Vaggione introduced the notion of *central element* to study algebras whose complementary factor congruences can be replaced by certain elements of their universes. Central elements coincide with central idempotents in rings with unit and with members of the centre in ortholattices.

Theorem 3.4 ([6]). *If $\mathbf{A}$ is a nCA of type τ and $c \in A$, then the following conditions are equivalent:*

1. *c is a factor element (w.r.t. q) satisfying the identity $q(c, \mathsf{e}_1, \dots, \mathsf{e}_n) = c$;*

2. *the sequence of congruences $(\theta(c, \mathsf{e}_1), \dots, \theta(c, \mathsf{e}_n))$ is a n-tuple of complementary factor congruences of $\mathbf{A}$;*

3. *for all $a_1, \dots, a_n \in A$, $q(c, a_1, \dots, a_n)$ is the unique element such that*

$$a_i \; \theta(c, \mathsf{e}_i) \; q(c, a_1, \dots, a_n),$$

for all $1 \leq i \leq n$;

4. *The function* f_c*, defined by* $f_c(a_1, \dots, a_n) = q(c, a_1, \dots, a_n)$ *for all* $a_1, \dots, a_n \in A$*, is a* n*-ary decomposition operator on* $\mathbf{A}$ *such that* $f_c(\mathsf{e}_1, \dots, \mathsf{e}_n) = c$*.*

Definition 3.5. If $\mathbf{A}$ is a nCA, then $c \in A$ is called *n-central* if it satisfies one of the equivalent conditions of Theorem 3.4. A n-central element c is *nontrivial* if $c \notin \{\mathsf{e}_1, \dots, \mathsf{e}_n\}$.

Every n-central element $c \in A$ induces a decomposition of $\mathbf{A}$ as a direct product of the algebras $\mathbf{A}/\theta(c, \mathsf{e}_i)$, for $i \leq n$.

The set of all n-central elements of a nCA $\mathbf{A}$ is a subalgebra of the pure reduct of $\mathbf{A}$. We denote by $\mathbf{Ce}_n(\mathbf{A})$ the algebra $(\mathrm{Ce}_n(\mathbf{A}), q, \mathsf{e}_1, \dots, \mathsf{e}_n)$ of all n-central elements of an nCA $\mathbf{A}$.

Factorisations of arbitrary algebras in n factors may be studied in terms of n-central elements of suitable nCAs of functions, as explained in the following example.

Example 3.6. Let $\mathbf{A}$ be an arbitrary algebra of type τ and F be a set of functions from A^n into A, which includes the projections $\mathsf{e}_i^{\mathbf{F}}$ and all constant functions f_b ($b \in A$):

(1) $\mathsf{e}_i^{\mathbf{F}}(a_1, \dots, a_n) = a_i$, for every $a_1, \dots, a_n \in A$;

(2) $f_b(a_1, \dots, a_n) = b$, for every $a_1, \dots, a_n \in A$;

and it is closed under the following operations (for all $f, h_i, g_j \in F$ and all $a_1, \dots, a_n \in A$):

(3) $q^{\mathbf{F}}(f, g_1 \dots, g_n)(a_1, \dots, a_n) = f(g_1(a_1, \dots, a_n) \dots, g_n(a_1, \dots, a_n))$.

(4) $\sigma^{\mathbf{F}}(h_1, \dots, h_k)(a_1, \dots, a_n) = \sigma^{\mathbf{A}}(h_1(a_1, \dots, a_n), \dots, h_k(a_1, \dots, a_n))$, for every $\sigma \in \tau$ of arity k.

The algebra $\mathbf{F} = (F, \sigma^{\mathbf{F}}, q^{\mathbf{F}}, \mathsf{e}_1^{\mathbf{F}}, \dots, \mathsf{e}_n^{\mathbf{F}})_{\sigma \in \tau}$ is a nCA. It is possible to prove that a function $f \in F$ is a n-central element of $\mathbf{F}$ if and only if f is a n-ary decomposition operator on the algebra $\mathbf{A}$ commuting (see Section 2.2) with every function $g \in F$. The reader may consult [22] for the case $n = 2$.

3.2 Boolean-like algebras

Boolean algebras are Church algebras of dimension 2 all of whose elements are 2-central. It turns out that, among the n-dimensional Church algebras, those algebras all of whose elements are n-central inherit many of the remarkable properties that distinguish Boolean algebras. We now recall from [6] the notion of Boolean-like algebras of dimension n, the main subject of study of this paper.

In [6] nBAs are studied in the general case of an arbitrary similarity type. Here, we restrict ourselves to consider the *pure* case, where q is the unique operator of the algebra.

Definition 3.7. A pure nCA $\mathbf{A} = (A, q, \mathsf{e}_1, \dots, \mathsf{e}_n)$ is called a *Boolean-like algebra of dimension* n (nBA, for short) if every element of A is n-central.

The class of all nBAs is a variety axiomatised by the following identities:

(B0) $q(\mathsf{e}_i, x_1, \dots, x_n) = x_i$ ($i = 1, \dots, n$).

(B1) $q(y, x, \dots, x) = x$.

(B2) $q(y, q(y, x_{11}, x_{12}, \dots, x_{1n}), \dots, q(y, x_{n1}, x_{n2}, \dots, x_{nn})) = q(y, x_{11}, \dots, x_{nn})$.

(B3) $q(y, q(x_{10}, \dots, x_{1n}), \dots, q(x_{n0}, \dots, x_{nn}))$
$$= q(q(y, x_{10}, \dots, x_{n0}), \dots, q(y, x_{1n}, \dots, x_{nn})).$$

(B4) $q(y, \mathsf{e}_1, \dots, \mathsf{e}_n) = y$.

In the following lemma we show that every nontrivial nBA has at least n elements.

Lemma 3.8. *The constants e_i ($1 \le i \le n$) are pairwise residually distinct in every nontrivial nBA.*

Proof. Let $\mathbf{A}$ be a nontrivial nBA such that $\mathsf{e}_k = \mathsf{e}_j$ for some $k \neq j$. If $a_1, \dots, a_n \in A$ with $a_k \neq a_j$, then $a_k = q(\mathsf{e}_k, a_1, \dots, a_n) = q(\mathsf{e}_j, a_1, \dots, a_n) = a_j$, providing a contradiction. □

Boolean-like algebras of dimension 2 were introduced in [23] with the name "Boolean-like algebras". *Inter alia*, it was shown in that paper that the variety of Boolean-like algebras of dimension 2 is term-equivalent to the variety of Boolean algebras.

Example 3.9. The algebra $\mathbf{Ce}_n(\mathbf{A})$ of all n-central elements of a nCA $\mathbf{A}$ of type τ is a canonical example of nBA (see the remark after Definition 3.5).

Example 3.10. The algebra $\mathbf{n} = (\{\mathsf{e}_1, \dots, \mathsf{e}_n\}, q^{\mathbf{n}}, \mathsf{e}_1^{\mathbf{n}}, \dots, \mathsf{e}_n^{\mathbf{n}})$, where $q^{\mathbf{n}}(\mathsf{e}_i, x_1, \dots, x_n) = x_i$ for every $i \le n$, is a nBA.

Example 3.11 (*n-Partitions*)**.** Let I be a set. An *n-partition* of I is a n-subset $(Y^1, \dots, Y^n)$ of I such that $\bigcup_{i=1}^n Y^i = I$ and $Y^i \cap Y^j = \emptyset$ for all $i \neq j$. The set of n-partitions of I is closed under the q-operator defined in Example 3.3 and constitutes the algebra of all n-central elements of the pure nCA $\mathbf{Set}_n(I)$ of all n-subsets of I. Notice that the algebra of n-partitions of I, denoted by $\mathbf{Par}_n(I)$, can be proved isomorphic to the nBA $\mathbf{n}^I$ (the Cartesian product of I copies of the algebra $\mathbf{n}$).

The variety BA of Boolean algebras is semisimple as every $\mathbf{A} \in$ BA is subdirectly embeddable into a power of the 2-element Boolean algebra, which is the only subdirectly irreducible (in fact, simple) member of BA. This property finds an analogue in the structure theory of nBAs.

Theorem 3.12 ([6])**.** *The algebra $\mathbf{n}$ is the unique subdirectly irreducible (in fact, simple) nBA and it generates the variety of nBAs.*

The next corollary shows that, for any $n \ge 2$, the nBA $\mathbf{n}$ plays a role analogous to the Boolean algebra $\mathbf{2}$ of truth values.

Corollary 3.13. *Every nBA $\mathbf{A}$ is isomorphic to a subdirect power of $\mathbf{n}^I$, for some set I.*

A subalgebra of the nBA $\mathbf{Par}_n(I)$ of the n-partitions on a set I, defined in Example 3.11, is called a *field of n-partitions on I*. The Stone representation theorem for nBAs follows.

Corollary 3.14. *Any nBA is isomorphic to a field of n-partitions on a suitable set I.*

One of the most remarkable properties of the 2-element Boolean algebra, called *primality* in universal algebra [7, Section 7 in Chapter IV], is the definability of all finite Boolean functions in terms of the connectives AND, OR, NOT. This property is inherited by nBAs. An algebra of cardinality n is primal if and only if it admits the nBA $\mathbf{n}$ as a subreduct.

Definition 3.15. Let $\mathbf{A}$ be a nontrivial algebra. $\mathbf{A}$ is *primal* if it is of finite cardinality and, for every function $f\colon A^k \to A$ ($k \geq 0$), there is a k-ary term t such that for all $a_1, \dots, a_k \in A$, $f(a_1, \dots, a_k) = t^{\mathbf{A}}(a_1, \dots, a_k)$.

A variety $\mathcal{V}$ is primal if $\mathcal{V} = \mathcal{V}(\mathbf{A})$ for a primal algebra $\mathbf{A}$.

Theorem 3.16 ([6])**.**

(i) The variety $n\mathrm{BA} = \mathcal{V}(\mathbf{n})$ *is primal;*

(ii) Let $\mathbf{A}$ *be a finite algebra of cardinality* n*. Then* $\mathbf{A}$ *is primal if and only if it admits the algebra* $\mathbf{n}$ *as a subreduct.*

We would like to point out here that when an algebra $\mathbf{A}$ is primal, the choice of fundamental operations is a matter of taste and convenience (since any set of functionally complete operations would serve), and hence is typically driven by applications.

4 Skew Boolean algebras and nBAs

In this section we prove that any nBA $\mathbf{A}$ contains a symmetric $\cap$-skew cluster of right-handed SBIAs $S_1^{\cap}(\mathbf{A}), \dots, S_n^{\cap}(\mathbf{A})$. The algebra $S_i^{\cap}(\mathbf{A})$, called the $\cap$-skew i-reduct of $\mathbf{A}$, has e_i as a bottom element, and the other constants $\mathsf{e}_1, \dots, \mathsf{e}_{i-1}, \mathsf{e}_{i+1}, \dots, \mathsf{e}_n$ as maximal elements. Rather interestingly, every permutation σ of the symmetric group S_n determines a bunch of isomorphisms

$$\mathrm{S}_1^{\cap}(\mathbf{A}) \cong \mathrm{S}_{\sigma 1}^{\cap}(\mathbf{A}), \dots, \mathrm{S}_n^{\cap}(\mathbf{A}) \cong \mathrm{S}_{\sigma n}^{\cap}(\mathbf{A})$$

which shows the inner symmetry of the nBAs. Every nBA has also a skew cluster $S_1(\mathbf{A})$, $\dots, S_n(\mathbf{A})$ of isomorphic right-handed SBAs, which are the skew Boolean algebra reducts of members of the $\cap$-skew cluster of $\mathbf{A}$. We conclude the section with a general representation theorem for right-handed SBAs in terms of nBAs of n-partitions.

4.1 The skew reducts of a nBA

In [9] it is shown that the variety of SBAs is term equivalent to the variety of SRCAs (see Section 2.6), whose type contains only a ternary operator and a nullary operator. Here we use the $(n+1)$-ary operator q of a nBA $\mathbf{A}$ to define ternary operators $t_1, \dots, t_n$ such that the reducts (A, t_i, e_i) are isomorphic SRCAs. Their term equivalent SBAs are all isomorphic reducts of $\mathbf{A}$, too. We also show that these isomorphic SBAs are in their turn reducts of isomorphic SBIAs.

For every $i \in \hat{n}$, we denote by $\bar{i}$ the set $\hat{n} \setminus \{i\}$.

In the following definition we use the $(n+1)$-ary operator q of nBAs to introduce some term operations needed to define the above-described reducts of nBAs.

Definition 4.1. Let $\mathbf{A} = (A, q, \mathsf{e}_1, \dots, \mathsf{e}_n)$ be a nBA. Given $1 \leq i \leq n$, we define the following term operations:

- $t_i(x,y,z) = q(x, y/\bar{i}, z/i)$;
- $x \wedge_i y = t_i(x, y, \mathsf{e}_i)$;
- $x \vee_i y = t_i(x, x, y)$;
- $x \setminus_i y = t_i(y, \mathsf{e}_i, x)$;
- $d_i(x,y) = q(x, t_1(y, x \vee_i y, \mathsf{e}_i), t_2(y, x \vee_i y, \mathsf{e}_i), \dots, t_n(y, x \vee_i y, \mathsf{e}_i))$;
- $x \cap_i y = q(x, t_1(y, \mathsf{e}_i, x), t_2(y, \mathsf{e}_i, x), \dots, t_n(y, \mathsf{e}_i, x))$.

We now define three reducts of a nBA $\mathbf{A}$, for each $1 \le i \le n$.

Definition 4.2. Let $\mathbf{A}$ be a nBA. We define the following three reducts of $\mathbf{A}$:

(i) The right Church i-reduct $R_i(\mathbf{A}) = (A, t_i, \mathsf{e}_i)$.

(ii) The skew i-reduct $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathsf{e}_i)$.

(iii) The $\cap$-skew i-reduct $S_i^{\cap}(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathsf{e}_i, \cap_i)$.

In the remaining part of this subsection we will prove that $R_i(\mathbf{A})$ is a SRCA, $S_i^{\cap}(\mathbf{A})$ is a e_i-regular (w.r.t. d_i) right-handed SBIA, and $S_i(\mathbf{A})$ is a right-handed SBA.

In the following lemmas we prove some properties of the term operations introduced in Definition 4.1.

Lemma 4.3. *The term operations of Definition 4.1 satisfy the following conditions when they are interpreted in the generator* $\mathbf{n}$ *of the variety* n*BA:*

$$t_i(a,b,c) = \begin{cases} c & \text{if } a = \mathsf{e}_i \\ b & \text{if } a \ne \mathsf{e}_i \end{cases}; \quad a \wedge_i b = \begin{cases} \mathsf{e}_i & \text{if } a = \mathsf{e}_i \\ b & \text{if } a \ne \mathsf{e}_i \end{cases}; \quad a \vee_i b = \begin{cases} b & \text{if } a = \mathsf{e}_i \\ a & \text{if } a \ne \mathsf{e}_i \end{cases};$$

$$a \setminus_i b = \begin{cases} a & \text{if } b = \mathsf{e}_i \\ \mathsf{e}_i & \text{if } b \ne \mathsf{e}_i \end{cases}; \quad d_i(a,b) = \begin{cases} \mathsf{e}_i & \text{if } a = b \\ a \vee_i b & \text{if } a \ne b \end{cases}; \quad a \cap_i b = \begin{cases} a & \text{if } a = b \\ \mathsf{e}_i & \text{if } a \ne b \end{cases}.$$

Proof. The proof is trivial for $t_i, \wedge_i, \vee_i, \setminus_i$. We now prove the relation for $d_i(a,b)$. We distinguish three cases.

- $(a = b)$:

$$\begin{aligned} d_i(a,a) &= q(a, t_1(a, a \vee_i a, \mathsf{e}_i), \dots, t_n(a, a \vee_i a, \mathsf{e}_i)) \\ &=_{\text{(B2)}} q(a, \mathsf{e}_i, \dots, \mathsf{e}_i) =_{\text{(B1)}} \mathsf{e}_i. \end{aligned}$$

- $(a = \mathsf{e}_k$ and $a \ne b)$:

$$\begin{aligned} d_i(\mathsf{e}_k, b) &= q(\mathsf{e}_k, t_1(b, \mathsf{e}_k \vee_i b, \mathsf{e}_i), \dots, t_n(b, \mathsf{e}_k \vee_i b, \mathsf{e}_i)) \\ &= t_k(b, \mathsf{e}_k \vee_i b, \mathsf{e}_i) =_{(b \ne \mathsf{e}_k)} \mathsf{e}_k \vee_i b = a \vee_i b. \end{aligned}$$

By definition of $\cap_i$ it is trivial to prove $a \cap_i a = a$. If $a = \mathsf{e}_k \ne b$, then we have:

$$\begin{aligned} a \cap_i b &= q(a, t_1(b, \mathsf{e}_i, a), t_2(b, \mathsf{e}_i, a), \dots, t_n(b, \mathsf{e}_i, a)) \\ &=_{(a = \mathsf{e}_k)} t_k(b, \mathsf{e}_i, a) =_{(b \ne \mathsf{e}_k)} \mathsf{e}_i. \end{aligned}$$

□

Lemma 4.4. *The following identities hold in every nBA:*

(1) $t_i(\mathsf{e}_i, x, y) = y$ *and* $t_i(\mathsf{e}_j, x, y) = x$*, for every* $j \neq i$*;*

(2) $$\begin{aligned} q(x, y_1, \dots, y_n) &= t_1(x, t_2(x, t_3(x, \dots t_n(x, z, y_n) \dots, y_3), y_2), y_1) \\ &= t_1(x, t_2(x, t_3(x, \dots t_{n-1}(x, y_n, y_{n-1}) \dots, y_3), y_2), y_1). \end{aligned}$$

(3) $\wedge_i$ *and* $\vee_i$ *are idempotents;*

(4) $d_i(x, x) = \mathsf{e}_i$ *and* $x \cap_i x = x$*;*

(5) $t_i(d_i(x, y), x, y) = x$ *and* $t_i(x \cap_i y, y, x) = x$*;*

(6) $(A, \cap_i, \mathsf{e}_i)$ *is a meet semilattice with bottom* e_i*;*

(7) $x \cap_i (x \wedge_i y \wedge_i x) = x \wedge_i y \wedge_i x$ *and* $x \wedge_i (x \cap_i y) = x \cap_i y = (x \cap_i y) \wedge_i x$*;*

(8) $d_i(x, y) = (x \vee_i y) \setminus_i (x \cap_i y)$*;*

(9) $x \cap_i y = (x \wedge_i y) \setminus_i d_i(x, y)$*.*

Proof. The identities are checked in the generator $\mathbf{n}$ of the variety $n\mathrm{BA}$.

(2): First we have:

$$\begin{aligned} &t_1(\mathsf{e}_k, t_2(\mathsf{e}_k, t_3(\mathsf{e}_k, \dots t_n(\mathsf{e}_k, c, b_n) \dots, b_3), b_2), b_1) \\ &\qquad = t_2(\mathsf{e}_k, t_3(\mathsf{e}_k, \dots t_n(\mathsf{e}_k, c, b_n) \dots, b_3), b_2) \\ &\qquad = \dots = t_k(\mathsf{e}_k, \dots t_n(\mathsf{e}_k, c, b_n) \dots, b_k) = b_k = q(\mathsf{e}_k, b_1, \dots, b_n). \end{aligned}$$

If $k \neq n$, a similar computation gives:

$$t_1(\mathsf{e}_k, t_2(\mathsf{e}_k, t_3(\mathsf{e}_k, \dots t_{n-1}(\mathsf{e}_k, b_n, b_{n-1}) \dots, b_3), b_2), b_1) = b_k.$$

If $k = n$, then we have:

$$\begin{aligned} &t_1(\mathsf{e}_n, t_2(\mathsf{e}_n, t_3(\mathsf{e}_n, \dots t_{n-1}(\mathsf{e}_n, b_n, b_{n-1}) \dots, b_3), b_2), b_1) \\ &\qquad = \dots = t_{n-1}(\mathsf{e}_n, b_n, b_{n-1}) = b_n. \end{aligned}$$

(3) – (4): Trivial by Lemma 4.3.

(5): If $a = b$, then the conclusion $t_i(d_i(a, a), a, a) = a$ and $t_i(a \cap_i a, a, a) = a$ is trivial by (4). Let now $a \neq b$.

- $t_i(a \cap_i b, b, a) =_{\text{(Lemma 4.3)}} t_i(\mathsf{e}_i, b, a) = a$.
- If $a = \mathsf{e}_i$ then $t_i(d_i(\mathsf{e}_i, b), \mathsf{e}_i, b) =_{\text{(Lemma 4.3)}} t_i(b, \mathsf{e}_i, b) =_{(b \neq \mathsf{e}_i)} \mathsf{e}_i = a$.
- If $a \neq \mathsf{e}_i$, then $t_i(d_i(a, b), a, b) =_{\text{(Lemma 4.3)}} t_i(a, a, b) =_{(a \neq \mathsf{e}_i)} a$.

(8):

$$\begin{aligned} (a \vee_i b) \setminus_i (a \cap_i b) &= t_i(a \cap_i b, \mathsf{e}_i, a \vee_i b) \\ &= \begin{cases} t_i(a, \mathsf{e}_i, a) = \mathsf{e}_i & \text{if } a = b \\ t_i(\mathsf{e}_i, \mathsf{e}_i, a \vee_i b) = a \vee_i b & \text{if } a \neq b \end{cases} \\ &= d_i(a, b). \end{aligned}$$

(9): First we have: $(a \wedge_i a) \setminus_i d_i(a,a) = t_i(\mathsf{e}_i, \mathsf{e}_i, a) = a = a \cap_i a$. If $a \neq b$, then

$$(a \wedge_i b) \setminus_i d_i(a,b) = t_i(d_i(a,b), \mathsf{e}_i, a \wedge_i b) = t_i(a \vee_i b, \mathsf{e}_i, a \wedge_i b) = \mathsf{e}_i = a \cap_i b,$$

because by Lemma 4.3, $a \vee_i b \neq \mathsf{e}_i$ if $a \neq b$.

(6) and (7) can be similarly checked in the generator $\mathbf{n}$ of the variety nBA by using Lemma 4.3. □

Lemma 4.5. *Let $\mathbf{A}$ be a nBA, and $a, b \in A$. Then we have:*

(i) $t_i(a, -, -)$ *is a* 2*-ary decomposition operator on* $\mathbf{A}$*.*

(ii) $d_i(a,b) = \mathsf{e}_i \;\Rightarrow\; a = b$*.*

Proof. (i): The binary operator $t_i(a,-,-)$ is a decomposition operator, because it is obtained by the n-ary decomposition operator $q(a,-,\dots,-)$ equating some of its coordinates (see [21] and Proposition 2.5).

(ii): Let $d_i(a,b) = \mathsf{e}_i$. Then $a =_{(\text{Lemma } 4.4(5))} t_i(d_i(a,b), a, b) = t_i(\mathsf{e}_i, a, b) = b$. □

We now characterise the reducts $R_i(\mathbf{A})$, $S_i(\mathbf{A})$ and $S_i^{\cap}(\mathbf{A})$ of a nBA $\mathbf{A}$ (see Definition 4.2).

Proposition 4.6. *Let $\mathbf{A}$ be a nBA. Then the following conditions hold:*

(i) The right Church i-reduct $R_i(\mathbf{A}) = (A, t_i, \mathsf{e}_i)$ of $\mathbf{A}$ is a SRCA;

(ii) The skew i-reduct $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathsf{e}_i)$ of $\mathbf{A}$ is a right-handed SBA.

Proof. By Lemma 4.4(3) and Lemma 4.5(i) every element of A is a factor element (w.r.t. t_i) that is $\wedge_i$-idempotent. Then every element of A is semicentral, so that $R_i(\mathbf{A})$ is a SRCA. By Theorem 2.14 the skew i-reduct $S_i(\mathbf{A})$ is a right-handed SBA. □

Hereafter, we denote by $\preceq^i_{\mathcal{D}}, \preceq^i_{\mathcal{L}}, \preceq^i_{\mathcal{R}}$ and $\leq^i$ the natural preorders and order of the SBA $S_i(\mathbf{A})$ (see Section 2.4). Since $S_i(\mathbf{A})$ is right-handed $\preceq^i_{\mathcal{D}}$ and $\preceq^i_{\mathcal{R}}$ coincide.

Proposition 4.7. *The elements $\mathsf{e}_1, \dots, \mathsf{e}_{i-1}, \mathsf{e}_{i+1}, \dots, \mathsf{e}_n$ are maximal elements of $S_i(\mathbf{A})$.*

Proof. We show the maximality of the elements $\mathsf{e}_1, \dots, \mathsf{e}_{i-1}, \mathsf{e}_{i+1}, \dots, \mathsf{e}_n$ with respect to the natural preorder $\preceq^i_{\mathcal{D}}$ of the SBA $S_i(\mathbf{A})$, defined by $a \preceq^i_{\mathcal{D}} b$ iff $a \wedge_i b \wedge_i a = a$. If $k \neq i$ and $a \in A$, then $a \wedge_i \mathsf{e}_k \wedge_i a = a \wedge_i a = a$, because $\mathsf{e}_k \wedge_i a = t_i(\mathsf{e}_k, a, \mathsf{e}_i) = a$ by Lemma 4.4(1). □

By Proposition 4.7 the skew i-reduct $S_i(\mathbf{A})$ has a maximal class M with $\mathsf{e}_1, \dots, \mathsf{e}_{i-1}, \mathsf{e}_{i+1}, \dots, \mathsf{e}_n \in M$. Then the algebra $S_i(\mathbf{A})/\mathcal{D}_i$ is a Boolean algebra, where $\mathcal{D}_i$ is the equivalence induced by $\preceq^i_{\mathcal{D}}$.

Proposition 4.8. *The $\cap$-skew i-reduct $S_i^{\cap}(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathsf{e}_i, \cap_i)$ of $\mathbf{A}$ is a e_i-regular right-handed SBIA.*

Proof. By Proposition 4.6 the skew i-reduct $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathsf{e}_i)$ is a right-handed SBA. By Lemma 4.4(6),(7) and by Definition 2.9 the $\cap$-skew i-reduct $S_i^{\cap}(\mathbf{A})$ is a right-handed SBIA. By Lemma 4.4(8) we have that $d_i(x,y) = (x \vee_i y) \setminus_i (x \cap_i y)$ is a term operation in the type of SBIAs. Then the e_i-regularity w.r.t. d_i follows from Lemma 4.4(4) and Lemma 4.5(ii) (see Section 2.1 for the definition of regularity). □

Remark 4.9. Skew Boolean algebras, whose underlying natural partial ordering is a meet semilattice, cannot be equationally axiomatised in the type of SBAs. Therefore, skew Boolean $\cap$-algebras of Definition 2.9 are equationally axiomatised in the type of SBAs enriched with a binary operator $\cap$ of intersection. Rather interestingly, if $\mathbf{A}$ is a nBA the term operation $\cap_i$ is definable in terms of $\wedge_1, \vee_1, \backslash_1, \mathsf{e}_1, \dots, \wedge_n, \vee_n, \backslash_n, \mathsf{e}_n$. This follows from Definition 4.1, Lemma 4.4(2), and Theorem 2.14. Then the following question is natural. Let $\mathbf{A} = (A, \wedge, \vee, \backslash, 0)$ be a SBA, whose underlying natural partial ordering is a meet semilattice $(A, \cap, 0)$ with bottom. Does there exist a bunch of SBAs $\mathbf{A}_1 = (A, \wedge_1, \vee_1, \backslash_1, 0_1), \dots, \mathbf{A}_k = (A, \wedge_k, \vee_k, \backslash_k, 0_k)$ such that the meet operation $\cap$ is definable in terms of the skew Boolean operations of $\mathbf{A}_1, \dots, \mathbf{A}_k$? A further analysis of this question will be given in Section 5.

4.2 A bunch of isomorphisms

It turns out that all the $\cap$-skew reducts of a nBA $\mathbf{A}$ are isomorphic. In order to prove this, we study the action of the symmetric group S_n on $\mathbf{A}$. The first part of this section is rather technical.

Let $\mathbf{A}$ be a nBA. For every permutation σ of the symmetric group S_n and $a, b_1, \dots, b_n \in A$, we define a sequence u_s $(1 \le s \le n+1)$ parametrised by another permutation τ:

$$u_{n+1} = b_{\tau n}; \qquad u_s = t_{\tau s}(a, u_{s+1}, b_{\sigma\tau s}) \quad (1 \le s \le n).$$

In the following lemma we prove that u_1 is independent of the permutation τ.

Notice that $u_n = q(a, b_{\tau n}/\overline{\tau n}, b_{\sigma\tau n}/\tau n)$ and $u_s = q(a, u_{s+1}/\overline{\tau s}, b_{\sigma\tau s}/\tau s)$.

Lemma 4.10. *For every $1 \le s \le n$ we have:*

$$u_s = q(a, b_{\tau n}/\{\tau 1, \tau 2 \dots, \tau(s-1)\}, b_{\sigma\tau s}/\tau s, b_{\sigma\tau(s+1)}/\tau(s+1), \dots, b_{\sigma\tau n}/\tau n).$$

Then $u_1 = q(a, b_{\sigma\tau 1}/\tau 1, b_{\sigma\tau 2}/\tau 2, \dots, b_{\sigma\tau n}/\tau n) = q(a, b_{\sigma 1}, b_{\sigma 2}, \dots, b_{\sigma n})$.

Proof. Assume that

$$u_{s+1} = q(a, b_{\tau n}/\{\tau 1, \tau 2 \dots, \tau s\}, b_{\sigma\tau(s+1)}/\tau(s+1), \dots, b_{\sigma\tau n}/\tau n).$$

Then we have:

$$\begin{aligned} u_s &= t_{\tau s}(a, u_{s+1}, b_{\sigma\tau s}) \\ &= q(a, u_{s+1}/\overline{\tau s}, b_{\sigma\tau s}/\tau s) \\ &=_{(B2)} q(a, b_{\tau n}/\{\tau 1, \tau 2 \dots, \tau(s-1)\}, b_{\sigma\tau s}/\tau s, b_{\sigma\tau(s+1)}/\tau(s+1), \dots, b_{\sigma\tau n}/\tau n). \square \end{aligned}$$

We define

$$a^\sigma = q(a, \mathsf{e}_{\sigma 1}, \mathsf{e}_{\sigma 2}, \dots, \mathsf{e}_{\sigma n}).$$

The transposition (ij) exchanges i and j: $(ij)(i) = j$ and $(ij)(j) = i$.

Lemma 4.11. *The following conditions hold in every nBA, for all permutations σ, τ and indices $i \ne j$:*

(1) The 2-ary decomposition operators $t_i(x, -, -)$ and $t_j(x, -, -)$ commute:

$$\begin{aligned} t_i(x, t_j(x, y, z), t_j(x, u, w)) &= t_j(x, t_i(x, y, u), t_i(x, z, w)) \\ &= q(x, y/\overline{\{i, j\}}, z/j, u/i); \end{aligned}$$

(2) $t_i(x, t_j(x,y,z), u) = t_j(x, t_i(x,y,u), z) = q(x, y/\overline{\{i,j\}}, z/j, u/i)$;

(3) $q(x, y_{\sigma 1}, \dots, y_{\sigma n})$
$= t_{\tau 1}(x, t_{\tau 2}(x, t_{\tau 3}(x, \dots t_{\tau n}(x, y_{\tau n}, y_{\sigma\tau n}) \dots, y_{\sigma\tau 3})), y_{\sigma\tau 2}), y_{\sigma\tau 1})$;

(4) $x^\sigma = t_{\tau 1}(x, t_{\tau 2}(x, t_{\tau 3}(x, \dots t_{\tau n}(x, \mathsf{e}_{\tau n}, \mathsf{e}_{\sigma\tau n}) \dots, \mathsf{e}_{\sigma\tau 3})), \mathsf{e}_{\sigma\tau 2}), \mathsf{e}_{\sigma\tau 1})$;

(5) $q(x^\sigma, y_1, \dots, y_n) = q(x, y_{\sigma 1}, \dots, y_{\sigma n})$;

(6) $x^{(ij)} = t_i(x, t_j(x, x, \mathsf{e}_i), \mathsf{e}_j) = t_j(x, t_i(x, x, \mathsf{e}_j), \mathsf{e}_i)$;

(7) $q(x, y_1, \dots, y_n)^\sigma = q(x, (y_1)^\sigma, \dots, (y_n)^\sigma)$;

(8) $x^{\tau\circ\sigma} = (x^\sigma)^\tau$.

Proof. Let $\mathbf{A}$ be a nBA and $a, b, c, d, e, b_1, \dots, b_n$ be elements of A.

(1):

$$\begin{aligned} t_i(a, t_j(a,b,c), t_j(a,d,e)) &= q(a, t_j(a,b,c)/\bar{i}, t_j(a,d,e)/i) \\ &= q(a, q(a, b/\bar{j}, c/j)/\bar{i}, q(a, d/\bar{j}, e/j)/i) \\ &=_{(B2)} q(a, b/\overline{\{i,j\}}, c/j, d/i). \end{aligned}$$

By symmetry we also get $t_j(a, t_i(a,b,d), t_i(a,c,e)) = q(a, b/\overline{\{i,j\}}, c/j, d/i)$.

(2): $t_i(a, t_j(a,b,c), d) = t_i(a, t_j(a,b,c), t_j(a,d,d)) =_{(1)} q(a, b/\overline{\{i,j\}}, c/j, d/i)$, and similarly $t_j(a, t_i(a,b,d), c) = t_j(a, t_i(a,b,d), t_i(a,c,c)) = q(a, b/\overline{\{i,j\}}, c/j, d/i)$.

(3): By Lemma 4.10, $u_1 = q(a, b_{\sigma 1}, \dots, b_{\sigma n})$. Then the conclusion follows from the unfolding of the definition of u_1:

$$u_1 = t_{\tau 1}(a, u_2, b_{\sigma\tau 1}) = t_{\tau 1}(a, t_{\tau 2}(a, u_3, b_{\sigma\tau 2}), b_{\sigma\tau 1}) = \cdots$$

(4): Follows from (3) by putting $y_k = \mathsf{e}_k$.

(5): $q(a^\sigma, b_1, \dots, b_n) = q(q(a, \mathsf{e}_{\sigma 1}, \dots, \mathsf{e}_{\sigma n}), b_1, \dots, b_n) =_{(B3)} q(a, b_{\sigma 1}, \dots, b_{\sigma n})$.

(6):

$$\begin{aligned} t_i(a, t_j(a,a,\mathsf{e}_i), \mathsf{e}_j) &= t_j(a, t_i(a,a,\mathsf{e}_j), \mathsf{e}_i) && \text{by (2)} \\ &= q(a, a/\overline{\{i,j\}}, \mathsf{e}_i/j, \mathsf{e}_j/i) && \text{by (2)} \\ &= q(a, q(a, \mathsf{e}_1, \dots, \mathsf{e}_n)/\overline{\{i,j\}}, \mathsf{e}_i/j, \mathsf{e}_j/i) && \text{by (B4)} \\ &= a^{(ij)} && \text{by (B2)} \end{aligned}$$

(7):

$$\begin{aligned} q(a, b_1, \dots, b_n)^\sigma &= q(q(a, b_1, \dots, b_n), \mathsf{e}_{\sigma 1}, \mathsf{e}_{\sigma 2}, \dots, \mathsf{e}_{\sigma n}) \\ &=_{(B3)} q(a, q(b_1, \mathsf{e}_{\sigma 1}, \mathsf{e}_{\sigma 2}, \dots, \mathsf{e}_{\sigma n}), \dots, q(b_n, \mathsf{e}_{\sigma 1}, \mathsf{e}_{\sigma 2}, \dots, \mathsf{e}_{\sigma n})) \\ &= q(a, (b_1)^\sigma, \dots, (b_n)^\sigma). \end{aligned}$$

(8):

$$\begin{aligned} (a^\sigma)^\tau &= q(a, \mathsf{e}_{\sigma 1}, \mathsf{e}_{\sigma 2}, \dots, \mathsf{e}_{\sigma n})^\tau \\ &=_{(7)} q(a, (\mathsf{e}_{\sigma 1})^\tau, \dots, (\mathsf{e}_{\sigma n})^\tau) \\ &= q(a, \mathsf{e}_{\tau(\sigma 1)}, \dots, \mathsf{e}_{\tau(\sigma n)}) \\ &= a^{\tau\circ\sigma}. \end{aligned}$$

□

Theorem 4.12. *For every transposition* $(rk) \in S_n$*, the map* $x \mapsto x^{(rk)}$ *defines an isomorphism from* $\mathrm{S}^{\cap}_r(\mathbf{A})$ *onto* $\mathrm{S}^{\cap}_k(\mathbf{A})$*.*

Proof. Let $\sigma = (rk)$ in this proof. The map $x \mapsto x^\sigma$ is bijective, because

$$(a^\sigma)^\sigma =_{\text{(Lemma 4.11(8))}} a^{\sigma\circ\sigma} = a^{\mathrm{Id}} = q(a, \mathsf{e}_1, \dots, \mathsf{e}_n) =_{\text{(B4)}} a,$$

for every $a \in A$. We now prove that $x \mapsto x^\sigma$ is a homomorphism of SBIAs. We recall from Definition 4.1 that the operations $\wedge_r, \vee_r, \setminus_r$ are defined in terms of t_r. Then to get the conclusion it is sufficient to prove the following equalities, for all $a, b, c \in A$: $t_r(a, b, c)^\sigma = t_k(a^\sigma, b^\sigma, c^\sigma)$, $(a \cap_r b)^\sigma = a^\sigma \cap_k b^\sigma$ and $(\mathsf{e}_r)^\sigma = \mathsf{e}_k$:

- $$\begin{aligned} t_r(a, b, c)^\sigma &=_{\text{(Lemma 4.11(7))}} t_r(a, b^\sigma, c^\sigma) = t_r((a^\sigma)^\sigma, b^\sigma, c^\sigma) = q((a^\sigma)^\sigma, b^\sigma/\overline{r}, c^\sigma/r) \\ &=_{\text{(Lemma 4.11(5))}} q(a^\sigma, b^\sigma/\overline{k}, c^\sigma/k) = t_k(a^\sigma, b^\sigma, c^\sigma). \end{aligned}$$
- $(\mathsf{e}_r)^\sigma = \mathsf{e}_{\sigma r} = \mathsf{e}_k$.
- $$\begin{aligned} (a \cap_r b)^\sigma &= q(a, t_1(b, \mathsf{e}_r, a), \dots, t_n(b, \mathsf{e}_r, a))^\sigma \\ &= q(a, t_1(b, \mathsf{e}_r, a)^\sigma, \dots, t_n(b, \mathsf{e}_r, a)^\sigma) \\ &= q(a, t_1(b, (\mathsf{e}_r)^\sigma, a^\sigma), \dots, t_n(b, (\mathsf{e}_r)^\sigma, a^\sigma)) \\ &= q(a, t_1(b, \mathsf{e}_k, a^\sigma), \dots, t_n(b, \mathsf{e}_k, a^\sigma)) \\ &= q((a^\sigma)^\sigma, t_1((b^\sigma)^\sigma, \mathsf{e}_k, a^\sigma), \dots, t_n((b^\sigma)^\sigma, \mathsf{e}_k, a^\sigma)) \\ &= q((a^\sigma)^\sigma, \dots, t_r((b^\sigma)^\sigma, \mathsf{e}_k, a^\sigma), \dots, t_k((b^\sigma)^\sigma, \mathsf{e}_k, a^\sigma), \dots) \\ &= q(a^\sigma, \dots, t_k((b^\sigma)^\sigma, \mathsf{e}_k, a^\sigma), \dots, t_r((b^\sigma)^\sigma, \mathsf{e}_k, a^\sigma), \dots) \\ &= q(a^\sigma, \dots, t_r(b^\sigma, \mathsf{e}_k, a^\sigma), \dots, t_k(b^\sigma, \mathsf{e}_k, a^\sigma), \dots) \\ &= a^\sigma \cap_k b^\sigma. \end{aligned}$$ □

4.3 A general representation theorem for right-handed SBAs

In this section we show that, for every $n \geq 3$, there is a representation of an arbitrary right-handed SBA within the skew i-reduct of a suitable nBA of n-partitions (described in Example 3.11). The theorem also provides a new proof that every SBA can be embedded into a SBA with a maximal class (see Proposition 4.7).

Theorem 4.13. *Let* $n \geq 3$*. Then every right-handed SBA can be embedded into the skew* i*-reduct* $S_i(\mathbf{A})$ *of a suitable* n*BA* $\mathbf{A}$ *of* n*-partitions.*

Proof. (a) By [17, Corollary 1.14] every right-handed SBA can be embedded into an algebra of partial functions with codomain the set $\{1, 2\}$ (see Example 2.15), where $0 = \emptyset$ is the empty function, $f \wedge g = g|_{G\cap F}$, $f \vee g = f \cup g|_{G\cap\overline{F}}$ and $g \setminus f = g|_{G\cap\overline{F}}$ (with F, G and H the domains of the functions f, g, h, respectively).

(b) By Corollary 3.14 every nBA is isomorphic to a nBA of n-partitions of a suitable set I (see Examples 3.3 and 3.11). If $P = (P_1, \dots, P_n)$ and $Q = (Q_1, \dots, Q_n)$ are n-partitions of I, then

$$\begin{aligned} P \wedge_i Q &= t_i(P, Q, \mathsf{e}_i) = q(P, Q/\overline{i}, \mathsf{e}_i/i) \\ &= (\overline{P_i} \cap Q_1, \dots, P_i \cup (\overline{P_i} \cap Q_i), \dots, \overline{P_i} \cap Q_n). \end{aligned} \tag{4.1}$$

The other operations can be similarly defined.

(c) We define an injective function * between the set of partial functions from a set I into $\{1,2\}$ and the set of n-partitions of I. If $f\colon I \rightharpoonup \{1,2\}$ is a partial function, then $f^* = (P_1,\dots,P_n)$ is the following n-partition of I: $P_1 = f^{-1}(1)$, $P_2 = f^{-1}(2)$, $P_i = I\setminus \mathrm{dom}(f)$ and $P_k = \emptyset$ for any $k \neq 1,2,i$.

(d) The map * preserves the meet. Let $f\colon F \to \{1,2\}$ and $g\colon G \to \{1,2\}$ ($F, G \subseteq I$) be functions. Then we derive $(f\wedge g)^* = f^* \wedge_i g^*$ as follows:

$$\begin{aligned}
f^* &= (f^{-1}(1), f^{-1}(2), \emptyset,\dots,\emptyset,\overline{F},\emptyset,\dots,\emptyset)\\
g^* &= (g^{-1}(1), g^{-1}(2), \emptyset,\dots,\emptyset,\overline{G},\emptyset,\dots,\emptyset)\\
(f\wedge g)^* &= (g|_{G\cap F})^*\\
&= (F\cap g^{-1}(1), F\cap g^{-1}(2),\emptyset,\dots,\emptyset,\overline{G}\cup\overline{F},\emptyset,\dots,\emptyset)\\
&= (F\cap g^{-1}(1), F\cap g^{-1}(2),\emptyset\dots,\overline{F}\cup(F\cap\overline{G}),\emptyset,\dots,\emptyset)\\
&= f^*\wedge_i g^*.
\end{aligned}$$

Similarly for the other operations. □

5 Skew star algebras

The skew reducts of a nBA are so deeply related that they allow us to recover the full structure of the nBA. It is worthwhile introducing a new variety of algebras, called *skew star* algebras, equationally axiomatising n isomorphic SBAs and their relationships. In the main result of this section we prove that the variety of skew star algebras is term equivalent to the variety of nBAs.

By Lemma 4.4(2) we have that the identity

$$q(x,y_1,\dots,y_n) = t_1(x,t_2(x,t_3(x,\dots t_{n-1}(x,y_n,y_{n-1})\dots,y_3),y_2),y_1)$$

holds in every nBA. It follows that

$$t_n(x,y,z) = t_1(x,t_2(x,t_3(x,\dots t_{n-1}(x,z,y)\dots,y),y),y),$$

so that t_n is term definable by the remaining t_i ($1\le i\le n-1$). This is one of the reasons for introducing $n-1$ (and not n) ternary operators in the following definition. Another reason is technical simplification.

Definition 5.1. An algebra $\mathbf{B} = (B,t_1,\dots,t_{n-1},0_1,\dots,0_n)$, where t_i is ternary and 0_j is a nullary operator, is called a *skew star algebra* if the following conditions hold, for every $1\le i,k\le n-1$ and $1\le j\le n$:

(N0) $(B,t_i,0_i)$ is a SRCA.

(N1) $t_i(0_j,y,z) = y$ ($i\neq j$).

(N2) $t_1(x,t_2(x,t_3(x,(\dots t_{n-1}(x,0_n,0_{n-1})\dots),0_3),0_2),0_1) = x$.

(N3) $t_i(x,t_k(x,y,z),t_k(x,u,w)) = t_k(x,t_i(x,y,u),t_i(x,z,w))$ ($i\neq k$).

(N4) $t_i(x,t_k(x,y,z),u) = t_k(x,t_i(x,y,u),z)$ ($i\neq k$).

(N5) $t_i(x,-,-)$ is a homomorphism of the algebra $(B,t_k,0_k) \times (B,t_k,0_k)$ into $(B,t_k,0_k)$:

$$t_i(x,t_k(y_1,y_2,y_3),t_k(z_1,z_2,z_3)) = t_k(t_i(x,y_1,z_1),t_i(x,y_2,z_2),t_i(x,y_3,z_3)).$$

Skew star algebras constitute a variety of algebras.

The identities characterising skew star algebras deserve some explanation. Let $\mathbf{B}_i = (B,t_i,0_i)$ $(i = 1,\dots,n-1)$ be a family of SRCAs having the same universe B and such that $0_1,\dots,0_{n-1}$ are distinct elements of B. Let 0_n be another element of B distinct from $0_1,\dots,0_{n-1}$. Let $\mathbf{B} = (B,t_1,\dots,t_{n-1},0_1,\dots,0_n)$ be the algebra collecting the basic operations of the algebras $\mathbf{B}_i$ and the constant 0_n. Roughly speaking, the structure of an nBA on $\mathbf{B}$ with respect to the term operation q_t, defined by

$$q_t(x,y_1,\dots,y_n) := t_1(x,t_2(x,t_3(x,\dots t_{n-1}(x,y_n,y_{n-1})\dots,y_3),y_2),y_1), \tag{5.1}$$

can be recovered from the cluster of SRCAs $\mathbf{B}_i$ if (N1) – (N5) hold:

(i) (N1) implies that $\mathbf{B}$ is a nCA with respect to the operation q_t.

(ii) Since $\mathbf{B}_i$ is a SRCA, then, for every $b \in B$, the function $t_i(b,-,-)$ satisfies conditions (D1) and (D2) of Definition 2.2. Then, axiom (N5) implies that, for every $b \in B$, the binary functions $t_1(b,-,-),\dots,t_{n-1}(b,-,-)$ are 2-ary decomposition operators of the nCA $\mathbf{B}$.

(iii) (N3) means that the decomposition operators $t_1(b,-,-),\dots,t_{n-1}(b,-,-)$ are pairwise commuting. Hence, by Proposition 2.4 and by Proposition 2.5 the n-ary operator $q_t(b,-,\dots,-)$ (see (5.1) above) is a n-ary decomposition operator of the nCA $\mathbf{B}$.

(iv) (N2) implies that the factor element b satisfies the identity $q_t(b,0_1,\dots,0_n) = b$. Then b is a n-central element of the nCA $\mathbf{B}$, for every $b \in B$. We conclude that axioms (N1), (N2), (N3) and (N5) collectively imply that $\mathbf{B}$ is a nBA (w.r.t. q_t).

(v) Axiom (N4) is used to recover the ternary operations t_i $(1 \le i \le n-1)$ from q_t, i.e., $t_i(a,b,c) = q_t(a,b/\bar{i},c/i)$.

We now are going to prove that the variety of skew star algebras and of nBAs are term equivalent. Consider the following correspondence between the algebraic similarity types of nBAs and of skew star algebras.

- Beginning on the nBA side: $t_i(x,y,z) := q(x,y/\bar{i},z/i)$ $(1 \le i \le n-1)$ and $0_j := \mathsf{e}_j$ $(1 \le j \le n)$.
- Beginning on the skew star algebra side:

$$q_t(x,y_1,\dots,y_n) := t_1(x,t_2(x,t_3(x,\dots t_{n-1}(x,y_n,y_{n-1})\dots,y_3),y_2),y_1);$$
$$\mathsf{e}_j := 0_j.$$

If $\mathbf{B}$ is a skew star algebra, then $\mathbf{B}^\bullet = (B;q_t,\mathsf{e}_1,\dots,\mathsf{e}_n)$ denotes the corresponding algebra in the similarity type of nBAs.
Similarly, if $\mathbf{A}$ is a nBA, then $\mathbf{A}^* = (A;t_1,\dots,t_{n-1},0_1,\dots,0_n)$ denotes the corresponding algebra in the similarity type of skew star algebras.

It is not difficult to prove the following theorem.

Theorem 5.2. *The above correspondences define a term equivalence between the varieties of nBAs and of skew star algebras. More precisely,*

(i) If $\mathbf{A}$ is a nBA, then $\mathbf{A}^*$ is a skew star algebra;

(ii) If $\mathbf{B}$ is a skew star algebra, then $\mathbf{B}^\bullet$ is a nBA;

(iii) $(\mathbf{A}^*)^\bullet = \mathbf{A}$;

(iv) $(\mathbf{B}^\bullet)^* = \mathbf{B}$.

Proof. (i): (N0) derives from Proposition 4.6, while (N1) comes from Lemma 4.4(1). (N2) follows from

$$t_1(x, t_2(x, t_3(x, \dots t_n(x, 0_{n-1}, 0_n) \dots, 0_3), 0_2), 0_1) =_{(B2)} q(x, 0_1, \dots, 0_n) =_{(B4)} x.$$

(N3) is a consequence of Lemma 4.11(1). For (N4) we apply Lemma 4.11(2). (N5) follows from (B3).

(ii): (B0) derives from (N0) and (N1). By (N0) and (N5), $t_i(x, -, -)$ $(1 \leq i \leq n)$ is a decomposition operator on $\mathbf{B}$. Then, for every $b \in B$, $q_t(b, -, \dots, -)$ is a n-ary decomposition operator on $\mathbf{B}^\bullet$ (i.e., (B1) – (B3) hold), because commuting decomposition operators are closed under composition (see [21], Proposition 2.4 and Proposition 2.5). (B4) is a consequence of (N2).

(iii): Let $\mathbf{A}$ be a nBA. Since $t_i(x, y, z) = q(x, y/\bar{i}, z/i)$, then by (B2) we have:

$$\begin{aligned} q_t(x, y_1, \dots, y_n) &= t_1(x, t_2(x, \dots), y_1) \\ &= q(x, y_1, t_2(x, \dots), \dots, t_2(x, \dots)) \\ &= q(x, y_1, y_2, t_3(x, \dots), \dots, t_3(x, \dots)) \\ &\vdots \\ &= q(x, y_1, \dots, y_n). \end{aligned}$$

(iv): Let $\mathbf{B} = (B, t_1, \dots, t_{n-1}, 0_1, \dots, 0_n)$ be a skew star algebra. The conclusion $(\mathbf{B}^\bullet)^* = \mathbf{B}$ follows because by (N4) we obtain that $t_i(x, y, z) = q_t(x, y/\bar{i}, z/i)$ for every $1 \leq i \leq n-1$. □

6 Multideals

The notion of ideal plays an important role in order theory and universal algebra. Ideals, filters and congruences are interdefinable in Boolean algebras. For every Boolean ideal I, we have that $a \in I$ if and only if $\neg a \in \neg I$ if and only if $a\ \theta_I\ 0$ if and only if $\neg a\ \theta_I\ 1$. In the case of nBAs, the couple $(I, \neg I)$ is replaced by a n-tuple $(I_1, \dots, I_n)$ satisfying some compatibility conditions that extend in a conservative way those of the Boolean case.

Definition 6.1. Let $\mathbf{A}$ be a nBA. A *multideal* is a n-partition $(I_1, \dots, I_n)$ of a subset I of A such that

(m1) $\mathsf{e}_k \in I_k$;

(m2) $a \in I_r, b \in I_k$ and $c_1, \dots, c_n \in A$ imply $q(a, c_1, \dots, c_{r-1}, b, c_{r+1}, \dots, c_n) \in I_k$;

(m3) $a \in A$ and $c_1, \dots, c_n \in I_k$ imply $q(a, c_1, \dots, c_n) \in I_k$.

The set I is called the *carrier* of the multideal. An *ultramultideal* of $\mathbf{A}$ is a multideal whose carrier is A.

The following lemma, whose proof is straightforward, shows the appropriateness of the notion of multideal. In Section 7 we show that there exists a bijective correspondence between multideals and congruences.

Lemma 6.2. *If θ is a proper congruence on a nBA $\mathbf{A}$, then $I(\theta) = (\mathsf{e}_1/\theta, \dots, \mathsf{e}_n/\theta)$ is a multideal of $\mathbf{A}$.*

Multideals extend to the n-ary case the fundamental notions of Boolean ideal and filter, as shown in the following proposition.

Recall from [23] that a 2BA $\mathbf{A} = (A, q, \mathsf{e}_1, \mathsf{e}_2)$ is term equivalent to the Boolean algebra $\mathbf{A}^* = (A, \wedge, \vee, \neg, 0, 1)$, where $0 = \mathsf{e}_2$, $1 = \mathsf{e}_1$, $x \wedge y = q(x, y, 0)$, $x \vee y = q(x, 1, y)$, $\neg x = q(x, 0, 1)$. We remind the reader here how q is recovered from the Boolean algebra operations: $q(x, y, z) = (x \wedge y) \vee (\neg x \wedge z)$.

Proposition 6.3. *Let $\mathbf{A}$ be a 2BA, and $I_1, I_2 \subseteq A$. Then (I_1, I_2) is a multideal of $\mathbf{A}$ if and only if I_2 is an ideal of $\mathbf{A}^*$, and $I_1 = \neg I_2$ is the filter associated to I_2 in $\mathbf{A}^*$.*

Proof. If (I_1, I_2) is a multideal, then $0 = \mathsf{e}_2 \in I_2$ and $1 = \mathsf{e}_1 \in I_1$. Moreover, if $a, b \in I_2$ then $a \vee b = q(a, 1, b) \in I_2$ by (m2), and if $a \in I_2$ and $b \in A$, then $b \wedge a = q(b, a, 0) \in I_2$ by (m3). It follows that I_2 is a Boolean ideal. By (m2) and the definition of $\neg$ we have $I_1 \supseteq \{\neg a \mid a \in I_2\}$ and $I_2 \supseteq \{\neg a \mid a \in I_1\}$. Then from $\neg\neg a = a$ it follows that $I_1 = \{\neg a \mid a \in I_2\}$.

Conversely, if I_2 is a Boolean ideal of $\mathbf{A}^*$ and $I_1 = \neg I_2$, then the condition (m1) is clearly satisfied. Concerning (m2), it is worth noticing that $q(a, c, b) = (a \wedge c) \vee (\neg a \wedge b)$. Then if $a \in I_2, b \in I_1, c \in A$ (for instance, the other 3 cases being similar to this one), we have that $\neg a \in I_1$, so that $\neg a \wedge b \in I_1$ and we conclude that $(a \wedge c) \vee (\neg a \wedge b) = q(a, c, b) \in I_1$. Concerning (m3), if $a, b \in I_2$ and $c \in A$ then $c \wedge a, \neg c \wedge b \in I_2$, hence $(c \wedge a) \vee (\neg c \wedge b) = q(c, a, b) \in I_2$. If $a, b \in I_1$ and $c \in A$, then

$$\begin{aligned}(c \wedge a) \vee (\neg c \wedge b) &\geq (c \wedge (a \wedge b)) \vee (\neg c \wedge (a \wedge b)) \\ &= (c \vee \neg c) \wedge (a \wedge b) = a \wedge b \in I_1,\end{aligned}$$

so that $(c \wedge a) \vee (\neg c \wedge b) = q(c, a, b) \in I_1$. □

In the n-ary case, multideals of $\mathbf{A}$ may be characterised as n-tuples of ideals in the skew i-reducts $S_i(\mathbf{A})$ of $\mathbf{A}$, satisfying the conditions expressed in the following proposition.

Proposition 6.4. *Let $\mathbf{A}$ be a nBA and $(I_1, \dots, I_n)$ be a n-partition of a subset I of A. Then $(I_1, \dots, I_n)$ is a multideal if and only if the following conditions are satisfied:*

(I1) $\mathsf{e}_r \in I_r$;

(I2) $a \in I_r$, $b \in I_k$ *and* $c \in A$ *imply* $t_r(a, c, b) \in I_k$;

(I3) $a, b \in I_r$ *and* $c \in A$ *imply* $t_k(c, a, b) \in I_r$, *for all* k.

Proof. Showing that a multideal satisfies (I1), (I2) and (I3) is straightforward. A n-partition satisfying (I1), (I2) and (I3), trivially verifies (m1). Concerning (m2), let us suppose that

$a \in I_r$, $b \in I_k$ and $c_1, \dots, c_n \in A$. In order to show that $q(a, c_1, \dots, c_{r-1}, b, c_{r+1}, \dots, c_n) \in I_k$, we apply Lemma 4.11(4):

$$q(x, c_{\sigma 1}, \dots, c_{\sigma n}) = t_{\tau 1}(x, t_{\tau 2}(x, t_{\tau 3}(x, \dots t_{\tau n}(x, c_{\tau n}, c_{\sigma \tau n}) \dots, c_{\sigma \tau 3})), c_{\sigma \tau 2}), c_{\sigma \tau 1})$$

in the case $\sigma = \mathrm{Id}, \tau = (1r)$, and we get

$$q(a, c_1, \dots, c_{r-1}, b, c_{r+1}, \dots, c_n) = t_r(a, t_2(a, \dots, c_2), b) \in I_k, \qquad \text{by (I2).}$$

Concerning (m3), let $a_1, \dots, a_n \in I_k$ and $b \in A$. By Lemma 4.4(2) we have

$$q(b, a_1, \dots, a_n) = t_1(b, t_2(b, t_3(b, \dots t_{n-1}(b, a_n, a_{n-1}) \dots, a_3), a_2), a_1).$$

By applying (I3) n times, we conclude that $q(b, a_1, \dots, a_n) \in I_k$, since $t_{n-1}(b, a_n, a_{n-1}) \in I_k$, hence $t_{n-2}(b, t_{n-1}(b, a_n, a_{n-1}), a_{n-2}) \in I_k$, and so on. □

By using the characterisation of Proposition 6.4 it is easy to see that the components of a multideal are ideals of the SBA corresponding to their index.

Recall from Section 2.4 the notion of an ideal of a SBA.

Corollary 6.5. *If $(I_1, \dots, I_n)$ is a multideal of a nBA $\mathbf{A}$ and $1 \leq i \leq n$, then I_i is an ideal of the skew i-reduct $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \backslash_i, \mathsf{e}_i)$.*

Proof. Since $S_i(\mathbf{A})$ is right-handed, a non empty set $K \subseteq A$ is an ideal of $S_i(\mathbf{A})$ if and only if, for all $a, b \in K$ and $c \in A$, $a \vee_i b \in K$ and $a \wedge_i c \in K$ (see Section 2.4). Given $a, b \in I_i$ and $c \in A$, we have $a \vee_i b = t_i(a, a, b) \in I_i$ and $a \wedge_i c = t_i(a, c, \mathsf{e}_i) \in I_i$, by using in both cases condition (I2) of Proposition 6.4 (notice that $\mathsf{e}_i \in I_i$, by (I1)). □

Lemma 6.6. *The carrier I of a multideal $(I_1, \dots, I_n)$ of a nBA $\mathbf{A}$ is a subalgebra of $\mathbf{A}$.*

Proof. The constants $\mathsf{e}_1, \dots, \mathsf{e}_n$ belong to I by (m1). If $a \in I_r$ and $b \in I_k$, then $q(a, c_1, \dots, c_{r-1}, b, c_{r+1}, \dots, c_n) \in I_k$, for all $c_1, \dots, c_n \in A$, by (m2). Hence I is a subalgebra of $\mathbf{A}$. □

Any component I_i of a multideal $(I_1, \dots, I_n)$ determines the multideal completely, as shown in the following lemma.

Lemma 6.7. *If $(I_1, \dots, I_n)$ is a multideal of a nBA $\mathbf{A}$, then $I_k = I_r^{(rk)}$ for all r, k.*

Proof. Let $a \in I_r$. Then $a^{(rk)} = t_r(a, t_k(a, a, \mathsf{e}_r), \mathsf{e}_k) \in I_k$ by Lemma 4.11(5) and Proposition 6.4(I2). Then we have

$$I_r^{(rk)} \subseteq I_k; \qquad I_k^{(rk)} \subseteq I_r.$$

The conclusion follows because $(a^{(rk)})^{(rk)} = a$, by Lemma 4.11(6) and (B4). □

Multideals are closed under arbitrary nonempty componentwise intersection. The minimum multideal is the sequence $(\{\mathsf{e}_k\})_{k \in \hat{n}}$. Given a nBA $\mathbf{A}$, and $A_1, \dots, A_n \subseteq A$, let us consider the set $\mathcal{A}$ of multideals containing $(A_1, \dots, A_n)$. The *ideal closure* of $(A_1, \dots, A_n)$ is the componentwise intersection of the elements of $\mathcal{A}$, if $\mathcal{A} \neq \emptyset$. Otherwise, the ideal closure of $(A_1, \dots, A_n)$ is the constant n-tuple $I^\top = (A, \dots, A)$, that we consider as a degenerate multideal, by a small abuse of terminology.

As a matter of fact, $I^\top$ is the only degenerate multideal.

Lemma 6.8. *Let $\mathbf{A}$ be a nBA and $I = (I_1, \dots, I_n)$ be a tuple of subsets of A satisfying the closure properties of Definition 6.1. The following are equivalent:*

(i) there exist $a \in A$ and $r \neq k$ such that $a \in I_r \cap I_k$.

(ii) there exist $r \neq k$ such that $\mathsf{e}_k \in I_r$.

(iii) $I = I^{\top}$.

Proof. (i) $\Rightarrow$ (ii): Since $a \in I_k$, by Lemma 6.7 we have that $a^{(rk)} \in I_r$. By Definition 6.1(m2), we conclude that $q(a, \mathsf{e}_k/\overline{r}, a^{(rk)}/r) =_{(B2)} q(a, \mathsf{e}_k, \dots, \mathsf{e}_k) = \mathsf{e}_k \in I_r$.

(ii) $\Rightarrow$ (iii): Given $b \in A$, we have $b = q(\mathsf{e}_k, b/\overline{r}, \mathsf{e}_r/r) \in I_r$ by Definition 6.1(m2). Hence $I_r = A$ and the result follows from Lemma 6.7 since $A^{(rk)} = A$ for all $1 \leq k \leq n$.

(iii) $\Rightarrow$ (i): Trivial. □

7 The relationship between multideals and congruences

For any congruence θ on a nBA, the equivalence classes e_i/θ form a multideal (see Lemma 6.2), exactly as in the Boolean case where $0/\theta$ is an ideal and $1/\theta$ the corresponding filter. Conversely, in the Boolean case, any ideal I (resp. filter F) defines the congruence $x\ \theta_I\ y \Leftrightarrow x \oplus y \in I$ (resp. $x\ \theta_F\ y \Leftrightarrow x \leftrightarrow y \in F$), where $x \oplus y = (x \wedge \neg y) \vee (\neg x \wedge y)$ and $x \leftrightarrow y = (\neg x \vee y) \wedge (x \vee \neg y)$. Rephrasing this latter correspondence in the n-ary case is a bit more complicated.

7.1 The Boolean algebra of coordinates

Let $\mathbf{A}$ be a nBA, $a \in A$ and $i \in \hat{n}$. We consider the factor congruence $\theta^i_a = \theta(a, \mathsf{e}_i) = \{(x, y) : t_i(a, x, y) = x\}$ generated by a. By Lemma 6.2 the tuple $(\mathsf{e}_1/\theta^i_a, \dots, \mathsf{e}_n/\theta^i_a)$ is a multideal of $\mathbf{A}$.

We recall that $\preccurlyeq^i_{\mathcal{R}}$ and $\leq^i$ denote the preorder and the partial order of the SBA $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathsf{e}_i)$, respectively (see Section 2.4).

Lemma 7.1. $\mathsf{e}_i/\theta^i_a = \{b \in A : b \preccurlyeq^i_{\mathcal{R}} a\}$.

Proof. By definition of θ^i_a, we have $b\ \theta^i_a\ c$ iff $t_i(a, b, c) = b$. Then $b \in \mathsf{e}_i/\theta^i_a$ iff $b = t_i(a, b, \mathsf{e}_i) = a \wedge_i b$ iff $b \preccurlyeq^i_{\mathcal{R}} a$ (by definition of $\preccurlyeq^i_{\mathcal{R}}$). □

The following proposition is a consequence of [9, Proposition 4.15].

Proposition 7.2.

(i) The set e_i/θ^i_a is a subalgebra of the right Church i-reduct (A, t_i, e_i).

(ii) The algebra $(\mathsf{e}_i/\theta^i_a, t_i, \mathsf{e}_i, a)$ is a 2CA.

(iii) The set $\downarrow_i a = \{b : b \leq^i a\}$ is the Boolean algebra of 2-central elements of $(\mathsf{e}_i/\theta^i_a, t_i, \mathsf{e}_i, a)$.

Proof. (i): Let $b, c, d \in \mathsf{e}_i/\theta^i_a$. Then $b\ \theta^i_a\ \mathsf{e}_i$, $c\ \theta^i_a\ \mathsf{e}_i$ and $d\ \theta^i_a\ \mathsf{e}_i$. By applying the properties of the congruences, we derive $t_i(b, c, d)\ \theta^i_a\ t_i(\mathsf{e}_i, \mathsf{e}_i, \mathsf{e}_i) = \mathsf{e}_i$.

(ii): By $t_i(a, b, c) = b$ and $t_i(\mathsf{e}_i, b, c) = c$, for every $b, c \in \mathsf{e}_i/\theta^i_a$.

(iii): By Lemma 4.5(i) b is a factor element for every $b \in \mathsf{e}_i/\theta^i_a$. Then b is 2-central iff $t_i(b, a, \mathsf{e}_i) = b$ iff $b \wedge_i a = b$ iff $b \leq^i a$, because $a \wedge_i b = t_i(a, b, \mathsf{e}_i) = b$ for all $b \in \mathsf{e}_i/\theta^i_a$. □

Notice that a is maximal (w.r.t. $\leq^i$) because, if $a \leq^i b \in \mathsf{e}_i/\theta^i_a$, then $a = a \wedge_i b = t_i(a, b, \mathsf{e}_i) = b$.

We now specialise the above construction to the case $a = \mathsf{e}_j$ for a given $j \neq i$.

Definition 7.3. Let $\mathbf{A}$ be a nBA and $i \neq j$. The *Boolean centre of* $\mathbf{A}$, denoted by $\mathbf{B}_{ij}$, is the Boolean algebra of 2-central elements of the 2CA $(A, t_i, \mathsf{e}_i, \mathsf{e}_j)$.

By Proposition 7.2(iii) the carrier set of $\mathbf{B}_{ij}$ is the set $\downarrow_i \mathsf{e}_j = \{b \in A : b \leq^i \mathsf{e}_j\}$ and we call *Boolean* any element of $\mathbf{B}_{ij}$.

Remark 7.4. The Boolean algebra $\mathbf{B}_{ij}$ was defined in [6] in a different but equivalent way (see [6, Section 6.1, Lemma 7(iii)]).

Lemma 7.5. *Let $\mathbf{A}$ be a nBA, $\mathbf{B}_{ij}$ be the Boolean centre of $\mathbf{A}$, $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathsf{e}_i)$ be the skew i-reduct of $\mathbf{A}$, and $i \neq j$. Then, for all $b, c \in B_{ij}$, we have $b \vee_i c = t_i(b, \mathsf{e}_j, c)$.*

Proof. $b \vee_i c = t_i(b, b, c) =_{(b \leq^i \mathsf{e}_j)} t_i(b, t_i(b, \mathsf{e}_j, \mathsf{e}_i), c) =_{\text{(B2)}} t_i(b, \mathsf{e}_j, c)$. □

By Lemma 7.5 and by [23] the Boolean operations on $\mathbf{B}_{ij}$ are $\wedge_i, \vee_i, \neg_{ij}$, where $\wedge_i, \vee_i$ are the corresponding operation of $S_i(\mathbf{A})$ restricted to $\mathbf{B}_{ij}$, and $\neg_{ij}(b) = t_i(b, \mathsf{e}_i, \mathsf{e}_j)$ for every $b \in B_{ij}$.

In [6] a representation theorem is proved, showing that any given nBA $\mathbf{A}$ can be embedded into the nBA of the n-central elements of the Boolean vector space $B_{ij} \times \ldots \times B_{ij} = (B_{ij})^n$ (see Example 3.2). The proof of this result makes an essential use of the notion of *coordinates* of elements of $\mathbf{A}$, that are n-tuples of elements of $(B_{ij})^n$, codifying the elements of $\mathbf{A}$ as linear combinations (see Lemma 7.12(5)). In this paper, the notion of coordinate is again a central one, being used to define the congruence associated to a multideal. In order to highlight their relationship with the skew reducts of $\mathbf{A}$, here we define the coordinates in terms of the operations t_k.

Definition 7.6. The *coordinates* of $a \in A$ are the elements $a_\mathsf{k} = t_k(a, \mathsf{e}_i, \mathsf{e}_j)$, for $1 \leq k \leq n$.

Notice that $a_\mathsf{k} \in B_{ij}$ for every $1 \leq k \leq n$, because $a_\mathsf{k} \leq^i \mathsf{e}_j$:

$$a_\mathsf{k} \wedge_i \mathsf{e}_j = t_i(t_k(a, \mathsf{e}_i, \mathsf{e}_j), \mathsf{e}_j, \mathsf{e}_i) =_{\text{(B3)}} q(a, \mathsf{e}_i/\bar{k}, \mathsf{e}_j/k) = t_k(a, \mathsf{e}_i, \mathsf{e}_j) = a_\mathsf{k}.$$

Lemma 7.7. *Let $a, b, b_1, \ldots, b_n \in A$. We have:*

(i) $a_\mathsf{k} \wedge_i a_\mathsf{r} = \mathsf{e}_i$ *for all* $k \neq r$.

(ii) $a_1 \vee_i a_2 \vee_i \cdots \vee_i a_\mathsf{n} = \mathsf{e}_j$.

(iii) $q(a, b_1, \ldots, b_n)_\mathsf{k} = q(a, (b_1)_\mathsf{k}, \ldots, (b_n)_\mathsf{k}) = (a_1 \wedge_i (b_1)_\mathsf{k}) \vee_i \cdots \vee_i (a_\mathsf{n} \wedge_i (b_n)_\mathsf{k})$.

(iv) $(a \wedge_i b)_\mathsf{k} = a \wedge_i b_\mathsf{k}$, *for every* $k \neq i$.

(v) $a_\mathsf{k} \wedge_i a = a_\mathsf{k} \wedge_i \mathsf{e}_k$, *for every* $k \neq i$.

(vi) $a_\mathsf{i} \wedge_i a = \mathsf{e}_i$.

(vii) If $a \in B_{ij}$, then

$$a_{\mathsf{k}} = \begin{cases} \neg_{ij}(a) & \text{if } k = i \\ a & \text{if } k = j \\ \mathsf{e}_i & \text{otherwise} \end{cases}$$

Proof. (i) – (vi): It is sufficient to check in the generator $\mathbf{n}$ of the variety nBA, where $B_{ij} = \{\mathsf{e}_i, \mathsf{e}_j\}$, $(\mathsf{e}_r)_{\mathsf{r}} = \mathsf{e}_j$ and $(\mathsf{e}_r)_{\mathsf{k}} = \mathsf{e}_i$ if $r \neq k$.

(vii): ($k = i$): By definition of $\neg_{ij}$:

$$a_{\mathsf{i}} = t_i(a, \mathsf{e}_i, \mathsf{e}_j) = \neg_{ij}(a).$$

($k \neq i, j$):

$$a_{\mathsf{k}} = t_k(a, \mathsf{e}_i, \mathsf{e}_j) = t_k(a \wedge_i \mathsf{e}_j, \mathsf{e}_i, \mathsf{e}_j) = t_k(t_i(a, \mathsf{e}_j, \mathsf{e}_i), \mathsf{e}_i, \mathsf{e}_j) = t_i(a, \mathsf{e}_i, \mathsf{e}_i) = \mathsf{e}_i.$$

($k = j$):

$$a_{\mathsf{j}} = t_j(a, \mathsf{e}_i, \mathsf{e}_j) = t_j(a \wedge_i \mathsf{e}_j, \mathsf{e}_i, \mathsf{e}_j) = t_j(t_i(a, \mathsf{e}_j, \mathsf{e}_i), \mathsf{e}_i, \mathsf{e}_j) = t_i(a, \mathsf{e}_j, \mathsf{e}_i) = a. \quad \square$$

Proposition 7.8. *The following conditions are equivalent for an element $a \in A$:*

(a) a is Boolean;

(b) $a \wedge_i \mathsf{e}_j = a$;

(c) $a = b_{\mathsf{k}}$, for some $b \in A$ and index $1 \leq k \leq n$;

(d) $a = a_{\mathsf{j}}$;

(e) $a_{\mathsf{k}} = \mathsf{e}_i$, for every $k \neq i, j$;

(f) $a = (a_{\mathsf{i}})_{\mathsf{i}}$.

Proof. (a) ⇔ (b): We have that $a \leq^i \mathsf{e}_j$ iff $a \wedge_i \mathsf{e}_j = a$ and $\mathsf{e}_j \wedge_i a = a$. The conclusion is obtained because the latter equality is trivially true.

(c) ⇒ (b):

$$b_{\mathsf{k}} \wedge_i \mathsf{e}_j = t_i(t_k(b, \mathsf{e}_i, \mathsf{e}_j), \mathsf{e}_j, \mathsf{e}_i) =_{\text{(B3)}} q(b, \mathsf{e}_i/\bar{k}, \mathsf{e}_j/k) = t_k(b, \mathsf{e}_i, \mathsf{e}_j) = b_{\mathsf{k}}.$$

(b) ⇒ (d): If $a \wedge_i \mathsf{e}_j = a$, then we have:

$$\begin{aligned} a_{\mathsf{j}} &= t_j(a, \mathsf{e}_i, \mathsf{e}_j) = t_j(a \wedge_i \mathsf{e}_j, \mathsf{e}_i, \mathsf{e}_j) = t_j(t_i(a, \mathsf{e}_j, \mathsf{e}_i), \mathsf{e}_i, \mathsf{e}_j) \\ &=_{\text{(B3)}} t_i(a, \mathsf{e}_j, \mathsf{e}_i) = a \wedge_i \mathsf{e}_j = a. \end{aligned}$$

(d) ⇒ (c): Trivial.

(a) ⇒ (e): By Lemma 7.7(vii).

(e) ⇒ (d): By Lemma 7.7(ii) the join of all coordinates of $a \in A$ in $\mathbf{B}_{ij}$ is the top element e_j. By hypothesis (e) we derive $a_{\mathsf{i}} \vee_i a_{\mathsf{j}} = \mathsf{e}_j$. Then, by applying the strong distributive property of $\wedge_i$ w.r.t. $\vee_i$ in the SBA $S_i(\mathbf{A})$, we obtain:

$$\begin{aligned} a &= \mathsf{e}_j \wedge_i a =_{\text{(Lemma 7.7(ii))}} (a_{\mathsf{i}} \vee_i a_{\mathsf{j}}) \wedge_i a = (a_{\mathsf{i}} \wedge_i a) \vee_i (a_{\mathsf{j}} \wedge_i a) \\ &=_{\text{(Lemma 7.7(vi))}} \mathsf{e}_i \vee_i (a_{\mathsf{j}} \wedge_i a) = a_{\mathsf{j}} \wedge_i a =_{\text{(Lemma 7.7(v))}} a_{\mathsf{j}} \wedge_i \mathsf{e}_j =_{(a_{\mathsf{j}} \leq^i \mathsf{e}_j)} a_{\mathsf{j}}. \end{aligned}$$

(f) ⇔ (b): $(a_{\mathsf{i}})_{\mathsf{i}} = t_i(t_i(a, \mathsf{e}_i, \mathsf{e}_j), \mathsf{e}_i, \mathsf{e}_j) =_{\text{(B3)}} t_i(a, \mathsf{e}_j, \mathsf{e}_i) = a \wedge_i \mathsf{e}_j$. Then $(a_{\mathsf{i}})_{\mathsf{i}} = a$ iff $a \wedge_i \mathsf{e}_j = a$. $\square$

By Lemma 7.7(iv) and Proposition 7.8(d) $a \wedge_i b$ is a Boolean element, for every $a \in A$ and $b \in B_{ij}$.

7.2 The congruence defined by a multideal

Let $\mathbf{A}$ be a nBA and $\mathbf{B}_{ij}$ be the Boolean centre of $\mathbf{A}$.

Lemma 7.9. *Let I be a multideal on $\mathbf{A}$. Then $I_* = B_{ij} \cap I_i$ is a Boolean ideal and $I^* = B_{ij} \cap I_j$ is the Boolean filter complement of I_*.*

Proof. Recall that, in $\mathbf{B}_{ij}$, e_i is the bottom element, e_j is the top element and $b \in B_{ij}$ iff $b \wedge_i \mathsf{e}_j = b$. We prove that I_* is a Boolean ideal. First $\mathsf{e}_i \in I_*$. If $b, c \in I_*$ and $d \in B_{ij}$, then we prove that $b \vee_i c$ and $b \wedge_i d$ belong to I_*. By Proposition 6.4(I2) $b \vee_i c$ and $b \wedge_i d$ belong to I_i. Moreover, since $b, c, d \in B_{ij}$ and $\wedge_i, \vee_i$ are respectively the meet and the join of $\mathbf{B}_{ij}$, then $b \vee_i c$ and $b \wedge_i d$ also belong to $\mathbf{B}_{ij}$. We now show that I^* is the Boolean filter complement of I_*.

($b \in I_* \Rightarrow \neg_{ij} b \in I^*$): As $b \in I_i \cap B_{ij}$, then by Proposition 6.4(I2)

$$\neg_{ij} b = t_i(b, \mathsf{e}_i, \mathsf{e}_j) \in I_j \cap B_{ij}.$$

($\neg_{ij} b \in I^* \Rightarrow b \in I_*$): As $t_i(b, \mathsf{e}_i, \mathsf{e}_j) \in I_j$, then

$$b = \neg_{ij}\neg_{ij} b = t_i(t_i(b, \mathsf{e}_i, \mathsf{e}_j), \mathsf{e}_i, \mathsf{e}_j) \in I_i. \qquad \square$$

The following lemma characterises multideals in terms of coordinates.

Lemma 7.10. *Let $(I_1, \dots, I_n)$ be a multideal on a nBA $\mathbf{A}$ and let $b \in A$. Then we have:*

(a) $b \in I_r$ if and only if the coordinate b_r of b belongs to I_j.

(b) If $b \in I_i$, then the coordinate b_k of b belongs to I_i, for every $k \neq i$.

Proof. (a): We start with $r = i$.

($\Rightarrow$): Let $b \in I_i$. By Proposition 6.4(I2) we have $\mathsf{b}_\mathsf{i} = t_i(b, \mathsf{e}_i, \mathsf{e}_j) \in I_j$, because $b \in I_i$ and $\mathsf{e}_j \in I_j$.

($\Leftarrow$): By hypothesis $\mathsf{b}_\mathsf{i} \in I_j$. Then by Lemma 6.7 $\mathsf{b}_\mathsf{i}^{(ij)} \in I_i$. Now

$$\begin{aligned} \mathsf{b}_\mathsf{i}^{(ij)} &= q(\mathsf{b}_\mathsf{i}, \mathsf{e}_{(ij)1}, \dots, \mathsf{e}_{(ij)n}) = q(t_i(b, \mathsf{e}_i, \mathsf{e}_j), \mathsf{e}_{(ij)1}, \dots, \mathsf{e}_{(ij)n}) \\ &=_{\text{(B3)}} t_i(b, \mathsf{e}_j, \mathsf{e}_i) = b \wedge_i \mathsf{e}_j \in I_i. \end{aligned}$$

We conclude $b \in I_i$ by applying Proposition 6.4(I2) to $t_i(b \wedge_i \mathsf{e}_j, b, \mathsf{e}_i)$, because $b \wedge_i \mathsf{e}_j \in I_i$ and $b =_{\text{(Lemma 4.4(3))}} b \wedge_i b =_{(b = \mathsf{e}_j \wedge_i b)} b \wedge_i \mathsf{e}_j \wedge_i b = t_i(b \wedge_i \mathsf{e}_j, b, \mathsf{e}_i)$.

We analyse $r \neq i$. Let σ be equal to the transposition (ir). By definition of b^σ we derive $(b^\sigma)_\mathsf{i} = t_i(q(b, \mathsf{e}_{\sigma 1}, \dots, \mathsf{e}_{\sigma n}), \mathsf{e}_i, \mathsf{e}_j) =_{\text{(B3)}} q(b, \mathsf{e}_j/r, \mathsf{e}_i/\bar{r}) = t_r(b, \mathsf{e}_i, \mathsf{e}_j) = \mathsf{b}_\mathsf{r}$. Then, $\mathsf{b}_\mathsf{r} \in I_j \Leftrightarrow \mathsf{b}_\mathsf{r} = (b^\sigma)_\mathsf{i} \in I_j \Leftrightarrow b^\sigma \in I_i \Leftrightarrow_{\text{(Lemma 6.7)}} b = (b^\sigma)^\sigma \in I_r$.

(b): By Proposition 6.4(I2), $k \neq i$ and $b \in I_i$ we get $\mathsf{b}_\mathsf{k} = t_k(b, \mathsf{e}_i, \mathsf{e}_j) \in I_i$. $\square$

We consider the homomorphism $f_I \colon \mathbf{B}_{ij} \to \mathbf{B}_{ij}/I_*$ and we define on A the following equivalence relation:

$$b\ \theta_I\ c \Leftrightarrow \forall k. f_I(\mathsf{b}_\mathsf{k}) = f_I(\mathsf{c}_\mathsf{k}),$$

where $\mathsf{b}_\mathsf{k}, \mathsf{c}_\mathsf{k}$ are the k-coordinates of b and c, respectively (see Definition 7.6).

Proposition 7.11. *θ_I is a congruence on $\mathbf{A}$.*

Proof. Let $a, b, c_1, d_1, \dots, c_n, d_n$ be elements of A such that $a\ \theta_I\ b$ and $c_k\ \theta_I\ d_k$, for every k. Then $q(a, c_1, \dots, c_n)\ \theta_I\ q(b, d_1, \dots, d_n)$ iff

$$\forall k. f_I(q(a, c_1, \dots, c_n)_\mathsf{k}) = f_I(q(b, d_1, \dots, d_n)_\mathsf{k}).$$

The conclusion follows, because f_I is a Boolean homomorphism and

$$q(a, c_1, \dots, c_n)_\mathsf{k} =_{\text{(Lemma 7.7(iii))}} q(a, (c_1)_\mathsf{k}, \dots, (c_n)_\mathsf{k}) =_{\text{(Lemma 7.7(iii))}} (a_1 \wedge_i (c_1)_\mathsf{k}) \vee_i \cdots \vee_i (a_\mathsf{n} \wedge_i (c_n)_\mathsf{k}).$$ □

We define a new term operation to be used in Theorem 7.13:

$$x +_i y = q(x, t_i(y, \mathsf{e}_i, \mathsf{e}_1), \dots, t_i(y, \mathsf{e}_i, \mathsf{e}_{i-1}), y, t_i(y, \mathsf{e}_i, \mathsf{e}_{i+1}), \dots, t_i(y, \mathsf{e}_i, \mathsf{e}_n)),$$

where y is at position i.

Lemma 7.12. *Let $a, b \in A$ and $a_1, a_2, \dots, a_\mathsf{n}$ be the coordinates of a. Then*

(1) $a +_i \mathsf{e}_i = \mathsf{e}_i +_i a = a$;

(2) $a +_i b = b +_i a$;

(3) $a +_i \mathsf{e}_k = \mathsf{e}_k +_i a = a_\mathsf{i} \wedge_i \mathsf{e}_k$ $(k \neq i)$.

(4) $a +_i a = \mathsf{e}_i$;

(5) The value of the expression $E \equiv (a_1 \wedge_i \mathsf{e}_1) +_i ((a_2 \wedge_i \mathsf{e}_2) +_i (\cdots +_i (a_\mathsf{n} \wedge_i \mathsf{e}_n)) \cdots)$ *is independent of the order of its parentheses. Without loss of generality, we write* $(a_1 \wedge_i \mathsf{e}_1) +_i (a_2 \wedge_i \mathsf{e}_2) +_i \cdots +_i (a_\mathsf{n} \wedge_i \mathsf{e}_n)$ *for the expression E. Then we have:*

$$(a_1 \wedge_i \mathsf{e}_1) +_i (a_2 \wedge_i \mathsf{e}_2) +_i \cdots +_i (a_\mathsf{n} \wedge_i \mathsf{e}_n) = a.$$

(6) If a and b have the same coordinates, then $a = b$.

Proof. (1):

$$a +_i \mathsf{e}_i = q(a, t_i(\mathsf{e}_i, \mathsf{e}_i, \mathsf{e}_1), \dots, t_i(\mathsf{e}_i, \mathsf{e}_i, \mathsf{e}_{i-1}), \mathsf{e}_i, t_i(\mathsf{e}_i, \mathsf{e}_i, \mathsf{e}_{i+1}), \dots, t_i(\mathsf{e}_i, \mathsf{e}_i, \mathsf{e}_n)) = q(a, \mathsf{e}_1, \dots, \mathsf{e}_n) =_{\text{(B4)}} a.$$

(2):

$$\begin{aligned} a +_i b &= q(a, t_i(b, \mathsf{e}_i, \mathsf{e}_1), \dots, t_i(b, \mathsf{e}_i, \mathsf{e}_{i-1}), b, t_i(b, \mathsf{e}_i, \mathsf{e}_{i+1}), \dots, t_i(b, \mathsf{e}_i, \mathsf{e}_n)) \\ &=_{\text{(B4)}} q(t_i(b, a, a), t_i(b, \mathsf{e}_i, \mathsf{e}_1), \dots, t_i(b, \mathsf{e}_i, \mathsf{e}_{i-1}), \\ &\qquad q(b, \mathsf{e}_1, \dots, \mathsf{e}_n), t_i(b, \mathsf{e}_i, \mathsf{e}_{i+1}), \dots, t_i(b, \mathsf{e}_i, \mathsf{e}_n)) \\ &=_{\text{(B3)}} q(b, t_i(a, \mathsf{e}_i, \mathsf{e}_1), \dots, t_i(a, \mathsf{e}_i, \mathsf{e}_{i-1}), \\ &\qquad q(a, \mathsf{e}_1, \dots, \mathsf{e}_n), t_i(a, \mathsf{e}_i, \mathsf{e}_{i+1}), \dots, t_i(a, \mathsf{e}_i, \mathsf{e}_n)) \\ &= b +_i a. \end{aligned}$$

(3):

$$\begin{aligned} a +_i \mathsf{e}_k &= q(a, t_i(\mathsf{e}_k, \mathsf{e}_i, \mathsf{e}_1), \dots, t_i(\mathsf{e}_k, \mathsf{e}_i, \mathsf{e}_{i-1}), \mathsf{e}_k, t_i(\mathsf{e}_k, \mathsf{e}_i, \mathsf{e}_{i+1}), \dots, t_i(\mathsf{e}_k, \mathsf{e}_i, \mathsf{e}_n)) \\ &=_{(k \neq i)} q(a, \mathsf{e}_i, \dots, \mathsf{e}_i, \mathsf{e}_k, \mathsf{e}_i, \dots, \mathsf{e}_i) = t_i(a, \mathsf{e}_i, \mathsf{e}_k) = \mathsf{e}_k +_i a. \end{aligned}$$

Moreover, $a_{\mathsf{i}} \wedge_i \mathsf{e}_k = t_i(t_i(a, \mathsf{e}_i, \mathsf{e}_j), \mathsf{e}_k, \mathsf{e}_i) =_{(B3)} t_i(a, \mathsf{e}_i, \mathsf{e}_k)$.

(4):

$$\begin{aligned} a +_i a &= q(a, t_i(a, \mathsf{e}_i, \mathsf{e}_1), \dots, t_i(b, \mathsf{e}_i, \mathsf{e}_{i-1}), a, t_i(a, \mathsf{e}_i, \mathsf{e}_{i+1}), \dots, t_i(a, \mathsf{e}_i, \mathsf{e}_n)) \\ &=_{(B2)} q(a, \mathsf{e}_i, \dots, \mathsf{e}_i, a, \mathsf{e}_i, \dots, \mathsf{e}_i) \\ &=_{(B4)} q(a, \mathsf{e}_i, \dots, \mathsf{e}_i, q(a, \mathsf{e}_1, \dots, \mathsf{e}_n), \mathsf{e}_i, \dots, \mathsf{e}_i) \\ &= q(a, \mathsf{e}_i, \dots, \mathsf{e}_i) = \mathsf{e}_i. \end{aligned}$$

(5): It is easy to check the identity in the generator $\mathbf{n}$ of the variety nBA.

(6): is a consequence of (5). □

Theorem 7.13. *Let ϕ be a congruence, $I(\phi) = (\mathsf{e}_1/\phi, \dots, \mathsf{e}_n/\phi)$ be the multideal of $\mathbf{A}$ determined by ϕ, $H = (H_1, \dots, H_n)$ be a multideal and θ_H be the congruence on $\mathbf{A}$ determined by H. Then*

$$\theta_{I(\phi)} = \phi \quad \textit{and} \quad I(\theta_H) = H.$$

Proof. We first prove $I(\theta_H)_k = H_k$, for all k. Recall that $\mathsf{e}_j = (\mathsf{e}_i)_{\mathsf{i}}$ is the i-coordinate of e_i and $\mathsf{e}_i = (\mathsf{e}_i)_{\mathsf{k}}$ is the k-coordinate of e_i for every $k \neq i$.

(1) First we provide the proof for $k = i$. Let $a \in I(\theta_H)_i$. If $a\ \theta_H\ \mathsf{e}_i$ then $f_H(a_{\mathsf{i}}) = f_H((\mathsf{e}_i)_{\mathsf{i}}) = f_H(\mathsf{e}_j)$, that implies $a_{\mathsf{i}} \in H^*$. By Lemma 7.10(a) we get the conclusion $a \in H_i$.

For the converse, let $a \in H_i$. By Lemma 7.10(a) we have $a_{\mathsf{i}} \in H^*$ and by Lemma 7.10(b) $a_{\mathsf{k}} \in H_*$ for all $k \neq i$. This implies $f_H(a_{\mathsf{i}}) = f_H(\mathsf{e}_j) = f_H((\mathsf{e}_i)_{\mathsf{i}})$ and $f_H(a_{\mathsf{k}}) = f_H(\mathsf{e}_i) = f_H((\mathsf{e}_i)_{\mathsf{k}})$ for all $k \neq i$, that implies $a\ \theta_H\ \mathsf{e}_i$. Since $I(\theta_H)_i = \mathsf{e}_i/\theta_H$, we conclude.

(2) Let now $k \neq i$. By Lemma 6.7 we have $H_k = H_i^{(ik)}$. Let $a \in H_k$. Then $a = b^{(ik)}$ for some $b \in H_i$. As, by (1), $b \theta_H \mathsf{e}_i$, then we have $a = b^{(ik)} \theta_H (\mathsf{e}_i)^{(ik)} = \mathsf{e}_k$. Since $I(\theta_H)_k = \mathsf{e}_k/\theta_H$, we conclude. Now, assuming $a \theta_H \mathsf{e}_k$, we have: $b = (a)^{(ik)} \theta_H (\mathsf{e}_k)^{(ik)} = \mathsf{e}_i$. Then $b \in H_i$ and $a = b^{(ik)} \in H_k$.

Let ϕ be a congruence.

(a) Let $a\ \phi\ b$. Then $\forall h.\ a_{\mathsf{h}}\ \phi\ b_{\mathsf{h}}$. Since ϕ restricted to $\mathbf{B}_{ij}$ is also a Boolean congruence, then we obtain $(a_{\mathsf{h}} \oplus_{ij} b_{\mathsf{h}})\ \phi\ \mathsf{e}_i$, where $\oplus_{ij}$ denotes the symmetric difference in the Boolean centre $\mathbf{B}_{ij}$. We now prove that $a\ \theta_{I(\phi)}\ b$. We have $a\ \theta_{I(\phi)}\ b$ iff $\forall h.\ f_{I(\phi)}(a_{\mathsf{h}}) = f_{I(\phi)}(b_{\mathsf{h}})$ iff $\forall h.\ a_{\mathsf{h}} \oplus_{ij} b_{\mathsf{h}} \in I(\phi)_* = B_{ij} \cap \mathsf{e}_i/\phi$ iff $\forall h.\ a_{\mathsf{h}} \oplus_{ij} b_{\mathsf{h}} \in \mathsf{e}_i/\phi$ iff $\forall h.\ (a_{\mathsf{h}} \oplus_{ij} b_{\mathsf{h}})\ \phi\ \mathsf{e}_i$. This last relation is proved above and we conclude $a\ \theta_{I(\phi)}\ b$.

(b) Let $a\ \theta_{I(\phi)}\ b$. Then $\forall h.\ a_{\mathsf{h}} \oplus_{ij} b_{\mathsf{h}} \in \mathsf{e}_i/\phi$ that implies $\forall h.\ a_{\mathsf{h}}\ \phi\ b_{\mathsf{h}}$, because ϕ restricted to $\mathbf{B}_{ij}$ is a Boolean congruence. Since by Lemma 7.12(5) there is a n-ary term u such that $a = u(a_1, \dots, a_{\mathsf{n}})$ and $b = u(b_1, \dots, b_{\mathsf{n}})$, then we conclude $a\ \phi\ b$ by using $\forall h.\ a_{\mathsf{h}}\ \phi\ b_{\mathsf{h}}$. □

7.3 Ultramultideals

In the Boolean case, there is a bijective correspondence between maximal ideals and homomorphisms onto $\mathbf{2}$. In this section we show that every multideal can be extended to an ultramultideal, and that there exists a bijective correspondence between ultramultideals and homomorphisms onto $\mathbf{n}$. We also show that prime multideals coincide with ultramultideals.

Let $(I_1, \dots, I_n)$ be a multideal of a nBA $\mathbf{A}$ and U be a Boolean ultrafilter of $\mathbf{B}_{ij}$ that extends $I^* = B_{ij} \cap I_j$, and so the maximal ideal $\bar{U} = B_{ij} \setminus U$ extends $I_* = B_{ij} \cap I_i$.

Lemma 7.14. *For all $a \in A$, there exists a unique k such that $a_{\mathsf{k}} \in U$.*

Proof. By Lemma 7.7(ii) the meet of two distinct coordinates is the bottom element e_i. Then at most one coordinate may belong to U. On the other hand, if all coordinates belong to $\bar{U}$, then the top element e_j belong to $\bar{U}$. □

Let $(G_k)_{k\in\hat{n}}$ be the sequence such that $G_k = \{a \in A : a_{\mathsf{k}} \in U\}$, which, by Lemma 7.14, is well defined.

Lemma 7.15. *$(G_k)_{k\in\hat{n}}$ is a ultramultideal which extends $(I_k)_{k\in\hat{n}}$.*

Proof. (m1): $\mathsf{e}_k \in G_k$ because $(\mathsf{e}_k)_{\mathsf{k}} = \mathsf{e}_j \in U$.
(m2): Let $a \in G_r$, $b \in G_k$, and $c_1, \dots, c_n \in A$. By Lemma 7.7(ii),

$$q(a, c_1, \dots, c_{r-1}, b, c_{r+1} \dots, c_n)_{\mathsf{k}} = [\bigvee_{s\neq r} (a_{\mathsf{s}} \wedge_i (c_s)_{\mathsf{k}})] \ \vee_i \ (a_{\mathsf{r}} \wedge_i b_{\mathsf{k}}).$$

Since $a_{\mathsf{r}}, b_{\mathsf{k}} \in U$, then $a_{\mathsf{r}} \wedge_i b_{\mathsf{k}} \in U$, and so $a_{\mathsf{r}} \wedge_i b_{\mathsf{k}} \sqsubseteq [\bigvee_{s\neq r}(a_{\mathsf{s}} \wedge_i (c_s)_{\mathsf{k}})] \vee_i (a_{\mathsf{r}} \wedge_i b_{\mathsf{k}}) \in U$, where $\sqsubseteq$ is the Boolean order of the Boolean algebra $\mathbf{B}_{ij}$. Hence,

$$q(a, c_1, \dots, c_{r-1}, b, c_{r+1} \dots, c_n) \in G_k.$$

(m3): It can be proved similarly.
We now prove that $(G_k)_{k\in\hat{n}}$ extends $(I_k)_{k\in\hat{n}}$. It is sufficient to show that, for every $a \in I_k$, we have that $a_{\mathsf{k}} \in U$. We get the conclusion by Lemma 7.10(a). □

Theorem 7.16.

(i) Every multideal can be extended to an ultramultideal.

(ii) There is a bijective correspondence between ulramultideals and homomorphisms onto $\mathbf{n}$.

Proof. (i) follows from Lemma 7.15. Regarding (ii), we remark that the algebra $\mathbf{n}$ is the unique simple nBA. □

We conclude this section by characterising prime multideals.

Definition 7.17. We say that a multideal $(I_1, \dots, I_n)$ is *prime* if, for every i, $a \wedge_i b \in I_i$ implies $a \in I_i$ or $b \in I_i$.

Proposition 7.18. *A multideal is prime iff it is an ultramultideal.*

Proof. (⇒): Let $(I_1, \dots, I_n)$ be a prime multideal. If $a \in B_{ij}$, then $a \wedge_i \neg_{ij}(a) = \mathsf{e}_i \in I_i$. Then either a or $\neg_{ij}(a) \in I_i$. This implies that $I_* = B_{ij} \cap I_i$ is a maximal Boolean ideal and the complement $I^* = B_{ij} \cap I_j$ is a Boolean ultrafilter.

Let now $b \in A$ such that $b \notin I = \bigcup_{k=1}^n I_k$. By Lemma 7.10(a) we have that $b \in I_r$ iff $b_{\mathsf{r}} \in I^*$. Then $b_{\mathsf{r}} \notin I^*$ for all r. Since I^* is a Boolean ultrafilter, then $b_{\mathsf{r}} \in I_*$ for all r. Hence $\mathsf{e}_j = \bigvee_{r=1}^n b_{\mathsf{r}} \in I_*$, contradicting the fact that the top element does not belong to a maximal ideal. In conclusion, $b \in I = \bigcup_{k=1}^n I_k$ for an arbitrary b, so that $I = A$.

(⇐): Let I be an ultramultideal. Let $a \wedge_i b \in I_i$ with $a \in I_r$ and $b \in I_k$ (with $r \neq i$ and $k \neq i$). Then by property (m2) of multideals we get $a \wedge_i b = t_i(a, b, \mathsf{e}_i) = q(a, b, \dots, b, \mathsf{e}_i, b, \dots, b) \in I_k$. Contradiction. □

Conclusion

Boolean-like algebras have been introduced in [6, 23] as a generalisation of Boolean algebras to any finite number of truth values. Boolean-like algebras provide a new characterisation of primal varieties exhibiting a perfect symmetry of the values of the generator of the variety. In this paper we have investigated the relationships between skew Boolean algebras and Boolean-like algebras. We have shown that any n-dimensional Boolean-like algebra is a skew cluster of n isomorphic right-handed skew Boolean algebras, and that the variety of skew star algebras is term equivalent to the variety of Boolean-like algebras. Moreover, we have got a representation theorem for right-handed skew Boolean algebras, and developed a general theory of multideals for Boolean-like algebras. Several further topics are worth mentioning:

1. How is the duality theory of SBAs and BAs related to a possible duality theory of nBAs (a Stone-like topology on ultramultideals).

2. Find a more satisfactory axiomatisation of skew star algebras.

3. Each SBA living inside a nBA has a bottom element 0 and several maximal elements. The construction could be made symmetric, by defining "skew-like" algebras having several minimal and several maximal elements.

4. For each nBA $\mathbf{A}$, the algebras $S_1(\mathbf{A}), \dots, S_n(\mathbf{A})$, constituting the skew cluster of $\mathbf{A}$, are isomorphic. This result is also of technical interest for the following open problem in the theory of skew Boolean algebras:

   Problem 7.19. Given a SBA $\mathbf{A}$ with a maximal class $M \supseteq \{m_1, m_2\}$, let $\mathbf{A}_{m_1}$ and $\mathbf{A}_{m_2}$ be the algebras obtained from $\mathbf{A}$ on distinguishing the elements m_1 and m_2 respectively. Are the algebras $\mathbf{A}_{m_1}$ and $\mathbf{A}_{m_2}$ isomorphic?

   This problem is part of the folklore and it does not appear in any published work to date. It is implicit in Leech [17], where both skew Boolean algebras (as they are now understood) and skew Boolean algebras possessing a maximal class are introduced.

   The difficulty in obtaining a solution to Problem 7.19 evidently lies in constructing the required isomorphism. For skew Boolean $\cap$-algebras, a related problem has been considered and resolved in the positive by Bignall [2]. The proof exploits sheaf (Boolean product) representations to obtain the desired isomorphism; as skew Boolean algebras admit only a weak Boolean product representation, the proof does not seem readily adaptable.

   Problem 7.19 is of purely technical interest in the theory of skew Boolean algebras. However, it assumes greater prominence in logics arising from (structurally enriched) skew Boolean algebras. Very roughly speaking, let S be an algebraisable logic arising from a quasivariety K of 1-regular (necessarily structurally enriched) skew Boolean algebras. Given n residually distinct constant terms of K, $1 < n < \omega$ (working with the finite case for simplicity), S admits $n-1$ negation connectives via implication into m, for each m a constant term distinct from 1. A positive solution to Problem 7.19 would imply that these $n-1$ negations are not essentially different, and hence that it is enough to fix a single such negation univocally when studying S; whereas a negative solution to Problem 7.19 would imply that these $n-1$ negations are all distinct, and hence that they must all be accounted for in any study of S.

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ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P2.08
https://doi.org/10.26493/2590-9770.1324.5b2
(Also available at http://adam-journal.eu)

Dual binary discriminator varieties*

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Received 22 August 2019, accepted 28 November 2019, published online 30 December 2019

Abstract

Left normal bands, strongly distributive skew lattices, implicative BCS-algebras, skew Boolean algebras, skew Boolean intersection algebras, and certain other non-commutative structures occur naturally as term reducts in the study of ternary discriminator algebras and the varieties that they generate, giving rise thereby to various classes of *pointed discriminator varieties*[1] that generalise the class of pointed ternary discriminator varieties. For each such class of varieties there is a corresponding *pointed discriminator function* that generalises the ternary discriminator. In this paper some of the classes of pointed discriminator varieties that are contained in the class of dual binary discriminator varieties are characterised. A key unifying property is that the principal ideals of an algebra in a dual binary discriminator variety are entirely determined by the dual binary discriminator term for that variety.

Keywords: Dual binary discriminator, binary discriminator, ternary discriminator, skew lattice.

Math. Subj. Class.: 06E75, 06B75, 06-06

*The results of this paper were presented at Non-commutative structures 2018: A workshop in honor of Jonathan Leech.

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[1]In the literature “discriminator”, “discriminator algebra”, and “discriminator variety” are normally used to refer to the ternary discriminator function, a ternary discriminator algebra, and a ternary discriminator variety respectively. In this paper these terms are used to refer more generally to various functions, algebras, and varieties that generalise the ternary discriminator, ternary discriminator algebras, and ternary discriminator varieties.

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1 Introduction

Recall that the *ternary discriminator* on a set A is the function $t\colon A^3 \to A$ defined for all elements $a, b, c \in A$ by $t(a, b, c) = a$ if $a \neq b$, and c otherwise. A *ternary discriminator algebra* is an algebra $\mathbf{A}$ for which there exists a term $t(x, y, z)$ in the language of $\mathbf{A}$ that realises the ternary discriminator on A. A *ternary discriminator variety* is a variety generated by a class K of ternary discriminator algebras, for which there exists a term $t(x, y, z)$ realising the ternary discriminator function on each $\mathbf{A} \in \mathsf{K}$. Ternary discriminator varieties generalise Boolean algebras and have been widely studied; see [15, Chapter IV§9].

A key property of the ternary discriminator term $t(x, y, z)$ for a ternary discriminator variety V is that it determines the congruences on each algebra $\mathbf{A} \in \mathsf{V}$, in the sense that the congruences of $\mathbf{A}$ are precisely those of the term reduct $\langle A; t^{\mathbf{A}}\rangle$. This motivates the following definition.

Definition 1.1. The *generic*[2] *ternary discriminator variety* TD is the variety of algebras of similarity type $\langle 3\rangle$ generated by the class of all algebras of the form $\mathbf{A} = \langle A; t\rangle$, where the ternary operation t is the ternary discriminator function on A.

Algebras in ternary discriminator varieties have a number of strong congruence properties. In particular, they are congruence-distributive, congruence-permutable, congruence-regular, and congruence-uniform. Moreover, every compact congruence is a principal factor congruence. Consequently, every algebra in a ternary discriminator variety can be represented as a Boolean product of ternary discriminator algebras; for details see [15, Chapter IV§9].

Examples of ternary discriminator varieties include varieties generated by a primal algebra (and thus the variety of Boolean algebras), monadic algebras, cylindric algebras of dimension n, and skew Boolean intersection algebras. The latter are used as a paradigmatic example of a (pointed) ternary discriminator variety in this paper. Briefly, a skew Boolean intersection algebra (SBIA) is a skew lattice with additional operations such that each principal subalgebra $a \wedge A \wedge a$ is a Boolean lattice, and for which finite meets exist with respect to the natural skew lattice partial order. For a more detailed definition and some key properties of SBIAs see [7].

While the generic ternary discriminator variety is a useful concept (see for example [14] and [29]), it can be somewhat unintuitive to work directly with the ternary discriminator term. In practice, almost all natural examples of ternary discriminator varieties have at least one constant term,[3] which facilitates the definition and use of more familiar binary terms. In particular, it is shown in [7] that every algebra $\mathbf{A}$ in a *pointed* ternary discriminator variety, that is, a ternary discriminator variety with a constant term, has a right handed (and thus also a left handed) SBIA term reduct that has the same congruences as $\mathbf{A}$. This follows from the observation that the variety of left (or right) handed SBIAs is, up to term equivalence, the *generic pointed ternary discriminator variety*, namely the variety TD_0 generated by the class of all algebras of type $\langle 3, 0\rangle$, having the form $\langle A; t, 0\rangle$, where the operation t is the ternary discriminator on A.

More generally, if an algebra $\mathbf{A}$ is a member of a (not necessarily pointed) ternary discriminator variety V with ternary discriminator term $t(x, y, z)$ and c is an arbitrary element

[2] Alternatively called *pure* by some authors; see for example [14].

[3] Two important exceptions [36, Corollary 4.31] are the varieties SA_3 and BN_4 arising respectively as the equivalent quasivariety semantics (in the sense of Blok and Pigozzi [10]) of the 3-valued relevant logic with mingle $\mathbf{RM}_3$ [5, §26.9, §29.12] and its 4-valued cousin, the logic $\mathbf{BN}_4$ of Brady [13].

of its base set A, then the polynomial reduct $\mathbf{A}_c = \langle A; \vee_c, \wedge_c, \backslash_c, \cap_c, c\rangle$ is a left handed skew Boolean intersection algebra[4] such that **Con A** = **Con** $\mathbf{A}_c$, with operations defined by[5]

$$a \vee_c b := t(b, c, a); \qquad a \wedge_c b := t(b, t(b, c, a), a);$$
$$a/_c b := t(a, b, c); \qquad a\backslash_c b := t(c, b, a), \text{ and}$$
$$a \cap_c b := t(c, t(c, b, a), a) = a/_c(a/_c b).$$

The various reducts of $\mathbf{A}_c$ are also familiar structures. For example, $\langle A; \wedge_c, c\rangle$ is a left normal band; $\langle A; \backslash_c, c\rangle$ is an implicative BCS-algebra; $\langle A; /_c, c\rangle$ is an implicative BCK-algebra; $\langle A; \vee_c, \wedge_c, c\rangle$ is a left handed strongly distributive skew lattice with zero c; and $\langle A; \vee_c, \wedge_c, \backslash_c, c\rangle$ is a left handed skew Boolean algebra.

Up to isomorphism the structure of each of these derived algebras is independent of the choice of c. This is shown by the following result from [6]. It can be proved using the Boolean product representations of $\mathbf{A}_c$ and $\mathbf{A}_d$.

Theorem 1.2. *Let* V *be a ternary discriminator variety, with* $\mathbf{A} \in \mathsf{V}$. *Then for all* $c, d \in A$, $\mathbf{A}_c \cong \mathbf{A}_d$.

Every congruence on an algebra must also be a congruence on each of its reducts. Since the congruence lattice of every algebra is always a complete sublattice of the lattice of equivalence relations on its base set, the congruence lattice of $\mathbf{A}_c$, and thus of $\mathbf{A}$, is a sublattice of the congruence lattice of each reduct of $\mathbf{A}_c$. Of course, such reducts do not in general have amenable congruence properties. In particular, their congruence lattices may satisfy no special lattice identities and they need not be congruence n-permutable for any $n \geq 2$. However, it follows from Theorem 2.19 in the next section that whenever $\wedge_c$ is included as one of its operations such a reduct has the same principal ideals as $\mathbf{A}_c$. Moreover, for each such reduct there exists a corresponding function that generalises the ternary discriminator, and each of these generalised discriminator functions gives rise to a class of pointed discriminator varieties that generalises the class of pointed ternary discriminator varieties. The class of dual binary discriminator varieties and its subclass of binary discriminator varieties, which have been studied by a number of authors, are examples.

Section 2 of this paper provides a new characterisation of the class of dual binary discriminator varieties (Theorem 2.19). In subsequent sections a number of its pointed discriminator variety subclasses are described and characterised. These are the classes of binary, skew, skew Boolean, multiplicative, pointed fixedpoint, and pointed ternary discriminator varieties. It is shown that the principal ideals of algebras in such varieties are entirely determined by their dual binary discriminator term. Various characterisations, including some that are purely ideal-theoretic in nature, are obtained for the classes of binary, skew Boolean, pointed fixedpoint, and pointed ternary discriminator varieties; see Theorems 3.2, 6.6, 8.1, and 9.1 respectively.

2 The class of dual binary discriminator varieties

Binary and dual binary discriminator varieties were introduced in [16]. The next three definitions are based on that paper, with some minor differences in the terminology and

[4] Note that $\mathbf{A}_c$ is term equivalent to the algebra $\langle A; \vee_c, \wedge_c, /_c, c\rangle$.

[5] We follow the normal convention of writing $t(a, b, c)$ rather than $t^{\mathbf{A}}(a, b, c)$ for the realisation in an algebra $\mathbf{A}$ of a term $t(x, y, z)$, provided that there is no ambiguity about which algebra is intended.

notation used.

Definition 2.1. Let A be a non-empty set and let $0 \in A$. The *dual binary* 0*-discriminator* on A is the binary function $\wedge$ defined for all $a, b \in A$ by $a \wedge b = a$, if $b \neq 0$, and 0 otherwise. 0 is called the *discriminating element*.

Definition 2.2. A *dual binary discriminator algebra* is an algebra $\mathbf{A}$ for which there exists a binary term $x \wedge y$ and a constant term $\mathbf{0}$ in the language of $\mathbf{A}$ that induce the dual binary 0-discriminator and its discriminating element 0 respectively on the base set A of $\mathbf{A}$.

Definition 2.3. A *dual binary discriminator variety* is a variety V with a binary term $x \wedge y$ and a constant term $\mathbf{0}$ in the language of V such that V is generated by a class K of dual binary discriminator algebras, with the terms $x \wedge y$ and $\mathbf{0}$ inducing the dual binary 0-discriminator and its discriminating element respectively on each $\mathbf{A} \in \mathsf{K}$.

The constant term in Definition 2.3 is referred to as the *discriminating constant* of the variety. A dual binary discriminator variety with discriminating constant $\mathbf{0}$ is called a *dual binary* $\mathbf{0}$*-discriminator variety*. Similarly, a dual binary discriminator algebra with discriminating element 0 is called a *dual binary* 0*-discriminator algebra*.

Natural examples of dual binary discriminator varieties are common and diverse, and include normal bands with zero, semilattices with zero, strongly distributive skew lattices with zero, pseudocomplemented semilattices, bounded distributive lattices, Stone algebras, skew Boolean algebras, skew Boolean intersection algebras, and many others.

Definition 2.4. The *generic dual binary discriminator variety*, denoted by DBD, is the variety of similarity type $\langle 2, 0\rangle$ generated by the class of all dual binary discriminator algebras of the form $\mathbf{A} = \langle A; \wedge, 0\rangle$, with $\wedge$ being the dual binary 0-discriminator on A.

Recall that an idempotent semigroup $\mathbf{A} = \langle A; \cdot\rangle$ (i.e. a *band*) is *normal* if it satisfies the identity $xyzx \approx xzyx$. $\mathbf{A}$ is *left normal* (resp. *right normal*) if it satisfies $xyz \approx xzy$ (resp. $xyz \approx yxz$). A *band with zero* is an algebra $\mathbf{A} = \langle A; \cdot, 0\rangle$ of similarity type $\langle 2, 0\rangle$ with a band operation $\cdot$ and a constant $\mathbf{0}$, satisfying the band identities plus the identities $x\mathbf{0} \approx \mathbf{0}x \approx \mathbf{0}$. By Schein [32] the only subdirectly irreducible normal bands with zero are (up to isomorphism) $\mathbf{S_0}$, the 2-element meet semilattice with zero; $\mathbf{L}$, the three-element left normal band with zero that has no non-trivial two-sided semigroup ideals; and $\mathbf{R}$, the three-element right normal band with zero that has no non-trivial two-sided semigroup ideals. It is easily seen that the term xyx induces the dual binary 0-discriminator $\wedge$ on each of these algebras.

Since the identity $xyx \approx xy$ holds for left normal bands, the semigroup operation realises the dual binary 0-discriminator on $\mathbf{L}$. We denote the variety of left normal bands with zero by $\mathbf{LNB_0}$. Since $\mathbf{S_0}$ and $\mathbf{L}$ are the only subdirectly irreducible members of $\mathbf{LNB_0}$ and $\mathbf{S_0}$ is a subagebra of $\mathbf{L}$, it follows that $\mathsf{LNB}_0 = \mathbf{HSP}(\{\mathbf{L}\})$, the variety generated by $\mathbf{L}$.

Proposition 2.5. $\mathsf{DBD} = \mathsf{LNB}_0$.

Proof. It is straightforward to check that every dual binary 0-discriminator algebra $\langle A; \wedge, 0\rangle$ in DBD is an idempotent semigroup with zero that satisfies the left normal band identity $x \wedge y \wedge z \approx x \wedge z \wedge y$. Let K denote the class of all dual binary discriminator algebras in DBD. Then $\mathsf{K} \subseteq \mathsf{LNB}_0$ and hence $\mathbf{HSP}(\mathsf{K}) = \mathsf{DBD} \subseteq \mathsf{LNB}_0$. But $\mathbf{L} \in \mathsf{K}$, so $\mathbf{HSP}(\{\mathbf{L}\}) = \mathsf{LNB}_0 \subseteq \mathsf{DBD}$. □

It is convenient to use the term **0**-*band* for a band with a zero element that is the realisation of a constant term **0**.

Corollary 2.6. *Every algebra* $\mathbf{A}$ *in a dual binary* $\mathbf{0}$*-discriminator variety has a left normal* $\mathbf{0}$*-band term reduct* $\langle A; \wedge, 0\rangle$*, where* $\wedge$ *is the operation induced by the dual binary discriminator term.*

Clearly, the dual binary discriminator term for a given dual binary **0**-discriminator variety is unique up to identity of terms. However, it is possible for a variety to be a dual binary discriminator variety with respect to more than one constant. For example, the variety L_0^1 of bounded distributive lattices $\langle L; \vee, \wedge, 0, 1\rangle$ is both a dual binary **0**-discriminator variety, with dual binary discriminator term $x \wedge y$, and a dual binary **1**-discriminator variety, with dual binary discriminator term $x \vee y$. L_0^1 is generated by the two-element bounded distributive lattice, which is both a dual binary 0-discriminator algebra and a dual binary 1-discriminator algebra.

Let $\mathbf{A}$ be an algebra in a dual binary **0**-discriminator variety with dual binary **0**-discriminator term $x \wedge y$. It is well-known from semigroup theory (and straightforward to prove) that the binary relation $\preceq$ defined for all $a, b \in A$ by $a \preceq b$ if $a \wedge b = a$ is a preorder on A, and that the binary relation $\leq$ defined for all a, b by $a \leq b$ if $b \wedge a = a$ is a partial order.[6] Observe that $\leq\ \subseteq\ \preceq$ and $0 \leq a$ for all $a \in A$, where 0 is the element induced by the discriminating constant **0**. The equivalence relation Ξ given by $a\ \Xi\ b$ if $a \preceq b$ and $b \preceq a$ is referred to as the *Clifford-Mclean* relation on A. It is a congruence on the left normal **0**-band reduct of $\mathbf{A}$, with $\langle A; \wedge, 0\rangle/\Xi$ being the maximal meet semilattice homomorphic image of $\langle A; \wedge, 0\rangle$.

Definition 2.7. An element $m \in A$ is called *maximal* if $a \preceq m$ for all $a \in A$.

For example, every non-zero element of a dual binary 0-discriminator algebra is maximal. Clearly, when the set M of maximal elements of an algebra $\mathbf{A}$ is non-empty it forms an equivalence class of Ξ.

Recall from [20] that a term $t(\vec{x}, \vec{y})$ is an *ideal term in* $\vec{y}$ for a class K of algebras with respect to a constant term **0** if $\mathsf{K} \models t(\vec{x}, \vec{\mathbf{0}}) \approx \mathbf{0}$, where $\vec{x}$ and $\vec{y}$ denote sequences of variables. A non-empty subset I of $\mathbf{A} \in \mathsf{K}$ is a **0**-*ideal* of $\mathbf{A}$ (or just an *ideal* when there is no ambiguity regarding which constant term is intended) if $0 = \mathbf{0}^{\mathbf{A}} \in I$ and for every $\vec{a} \in A$ and $\vec{b} \in I$, $t^{\mathbf{A}}(\vec{a}, \vec{b}) \in I$ whenever $t(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$ for K. The ideals of an algebra $\mathbf{A}$ form an algebraic lattice under set inclusion, so for every $X \subseteq A$ the smallest ideal containing X exists. It is denoted by $\langle X\rangle$ and is called the *ideal generated by* X. When $X = \{a\}$ this ideal is called *principal* and is denoted by $\langle a\rangle$. We denote the set of all ideals of $\mathbf{A}$ by Id $\mathbf{A}$, and the lattice of ideals of $\mathbf{A}$ by **Id** $\mathbf{A}$. Clearly, for every congruence ψ, $[0]\psi = \{a \mid a\ \psi\ 0\}$ is always an ideal. However, it is not always the case that an ideal is a congruence class. If every ideal of $\mathbf{A}$ is the 0-class of a congruence on $\mathbf{A}$, then $\mathbf{A}$ is said to be *normal* or to have normal ideals.

Definition 2.8. Given a language with a constant **0**, a term $t(x_1, \ldots, x_n)$ is called **0**-*reflexive* if it satisfies the identity $t(\mathbf{0}, \ldots, \mathbf{0}) \approx \mathbf{0}$. An algebra $\mathbf{A}$ with a constant term **0** in its language is said to be *reflexive* if $\{0\} = \{\mathbf{0}^{\mathbf{A}}\}$ is a one-element sub-universe, that is, if $f_\gamma(0, \ldots, 0) = 0$ for each operation f_γ of $\mathbf{A}$. A class K of algebras with a constant term **0** is *reflexive* if every member of K is reflexive.

[6] For a detailed discussion of the various order relations (called Green's preorders) on semigroups in general see [33, Section 0].

Clearly, a reflexive algebra must have (up to term equivalence) exactly one constant term in its language. Thus, a reflexive dual binary discriminator variety has, up to term equivalence, exactly one discriminating constant and one dual binary discriminator term. The generic dual binary discriminator variety is an example.

Lemma 2.9. *Let* V *be a dual binary* $\mathbf{0}$*-discriminator variety with dual binary discriminator term* $x \wedge y$ *and let* $t(x_1, \ldots, x_n)$*, where* $n > 0$*, be a term in the language of* V*. Then*

1. V *satisfies every identity of the form*

$$t(x_1, \ldots, x_n) \wedge y \approx t(x_1 \wedge y, \ldots, x_n \wedge y) \wedge y.$$

2. *When* $t(x_1, \ldots, x_n)$ *is* $\mathbf{0}$*-reflexive it satisfies the identity*

$$t(x_1, \ldots, x_n) \wedge y \approx t(x_1 \wedge y, \ldots, x_n \wedge y).$$

Proof. It is straightforward to verify that these identities hold on every member of V that is in the class of dual binary discriminator algebras that generates V. □

Definition 2.10. An algebra $\mathbf{A}$ in a variety with a constant term $\mathbf{0}$ is called $\mathbf{0}$-*ideal simple* if its only $\mathbf{0}$-ideals are $\{\mathbf{0}^{\mathbf{A}}\}$ and A.

For the remainder of this paper, in order to simplify the notation and unless stated otherwise, we regard a dual binary discriminator variety as having just one discriminating constant, which will normally be denoted by $\mathbf{0}$. An ideal term of such a dual binary discriminator variety V means an ideal term with respect to $\mathbf{0}$, while an ideal of an algebra $\mathbf{A} \in \mathsf{V}$ means a $\mathbf{0}$-ideal. In a similar fashion, an ideal simple algebra in V means one that is $\mathbf{0}$-ideal simple.

Lemma 2.11. *Every dual binary* 0*-discriminator algebra* $\mathbf{A}$ *is ideal simple.*

Proof. Clearly $x \wedge y$, the dual binary discriminator term for $\mathbf{A}$, is an ideal term in y. Suppose $I \in A$ is such that $I \neq \{0\}$. Let $b \in I$ be such that $b \neq 0$. Then for all $a \in A$, $a = a \wedge b \in I$. Thus $I = A$. Hence $\mathbf{A}$ is ideal simple. □

Thus a non-trivial dual binary 0-discriminator algebra has exactly two equivalence classes under the relation Ξ, namely $\{0\}$ and $A \setminus \{0\}$. We say that such an algebra is *flat*, since it is order isomorphic to a flat Scott domain.[7]

The universal algebraic notions of a semisimple algebra and a semisimple variety (see [15, Chapter IV§12]) have exact ideal-theoretic analogues.

Definition 2.12. An algebra is said to be *ideal semisimple* if it is isomorphic to a subdirect product of ideal simple algebras. A variety V is *ideal semisimple* if every member of V is ideal semisimple.

The proof of the following lemma is directly analogous to the proof of [15, Lemma IV§12.2] characterising semisimple varieties.

[7] In the literature, a skew lattice having exactly two Clifford-Maclean equivalence classes is said to be *primitive*. In general neither of these classes need be a singleton. However, if $\mathbf{A}$ is a primitive skew lattice with zero then the lower equivalence class is a singleton, and in that case $\mathbf{A}$ is order isomorphic to a flat Scott domain.

Lemma 2.13. *A variety* V *is ideal semisimple if and only if every subdirectly irreducible member of* V *is ideal simple.*

Examples of ideal semisimple dual binary discriminator varieties include normal bands with zero, strongly distributive skew lattices with zero, and skew Boolean algebras; see [26, 27].

In [3] Agliano and Ursini introduce the notion of equationally definable principal ideals for arbitrary varieties with a constant term. A variety V with a constant term $\mathbf{0}$ has *equationally definable principal ideals* (*EDPI*) if there exist pairs of binary terms $p_i, q_i, i = 1, \dots, n$ such that for every $\mathbf{A} \in \mathsf{V}$ and all $a, b \in A$, $a \in \langle b \rangle$ if and only if $p_i(a, b) = q_i(a, b)$ for $i = 1, \dots, n$, where $\langle b \rangle$ denotes the principal ideal generated by b. A related notion was subsequently considered by van Alten in [4] as follows:

Definition 2.14. A class K of algebras is said to have *EDPI** if there exists an ideal term $t(x, y)$ in y such that for each $\mathbf{A} \in \mathsf{K}$ and all $a, b \in A$, $a \in \langle b \rangle$ if and only if $a = t^{\mathbf{A}}(a, b)$.

The following key result follows from [4, Theorems 4.2 and 4.3].

Theorem 2.15. *Let* V *be a variety with constant term* $\mathbf{0}$ *generated by a class* K. *The following are equivalent.*

1. V *has EDPI.*
2. V *has EDPI**.
3. K *has EDPI**.

Proposition 2.16. *Every dual binary discriminator variety* V *has EDPI. A term in the language of* V *witnesses EDPI** *if and only if it is, up to term identity, the dual binary discriminator term.*

Proof. If K is a class of dual binary discriminator algebras, then by Lemma 2.11 its members are ideal simple and it is clear that the dual binary discriminator term witnesses *EDPI** for K, so by Theorem 2.15 V has *EDPI*. Suppose that $t(x, y)$ is an ideal term in y witnessing *EDPI** for V. Then for every dual binary discriminator algebra $\mathbf{A} \in \mathsf{V}$ and for all $a, b \in A$, $a \in \langle b \rangle$ if and only if $a = t^{\mathbf{A}}(a, b)$. Since $\mathbf{A}$ is ideal simple, this implies that $t^{\mathbf{A}}(a, b) = a$ when $b \neq 0$, with 0 being the realisation in A of the discriminating constant of V. Also, $t^{\mathbf{A}}(a, 0) = 0$, since $t(x, y)$ is an ideal term in y. Thus $t^{\mathbf{A}}(a, b)$ is the dual binary discriminator on A, and hence $t(x, y)$ is the dual binary discriminator term for V, since V is generated by a class of dual binary discriminator algebras. □

Corollary 2.17. *The principal ideals of every algebra in a dual binary* $\mathbf{0}$*-discriminator variety coincide with those of its left normal* $\mathbf{0}$*-band term reduct.*

In particular, every principal ideal $\langle b \rangle$ of an algebra $\mathbf{A}$ in a dual binary discriminator variety has the form $\langle b \rangle = \{a \in A \mid a \wedge b = a\} = \{a \in A \mid a \preceq b\}$, and so every ideal I is a down set with respect to the natural preorder, that is, if $b \in I$ and $a \preceq b$ then $a \in I$.

Definition 2.18. Let $\mathbf{A}$ be an algebra with a left normal $\mathbf{0}$-band term reduct $\mathbf{A_0} = \langle A; \wedge, 0 \rangle$. $\mathbf{A}$ has *ideal-compatible operations* if the principal $\mathbf{0}$-ideals of $\mathbf{A}$ coincide with those of $\mathbf{A_0}$. A class K of algebras with a left normal $\mathbf{0}$-band term is said to have *ideal-compatible operations* if every $\mathbf{A} \in \mathsf{K}$ has ideal-compatible operations with respect to its left normal $\mathbf{0}$-band term reduct.

Theorem 2.19. *Every dual binary discriminator variety is term equivalent to a variety of left normal bands with ideal-compatible operations. A variety* V *with a constant term* $\mathbf{0}$ *is a dual binary* $\mathbf{0}$*-discriminator variety if and only if it has EDPI and is generated by a class of* $\mathbf{0}$*-ideal simple algebras.*

Proof. The first statement is clear in view of Corollary 2.17. For the second statement, suppose that V is a dual binary $\mathbf{0}$-discriminator variety. Then by Proposition 2.16 and Theorem 2.15 V has *EDPI* and the dual binary $\mathbf{0}$-discriminator term for V witnesses *EDPI**. By Lemma 2.11, every dual binary 0-discriminator algebra is ideal simple, so V is generated by a class of ideal simple algebras. Conversely, if V has *EDPI* and is generated by a family K of ideal simple algebras, then by Theorem 2.15 the members of K have *EDPI**. If $t(x, y)$ is a term in the language of V that witnesses *EDPI** then it follows from the proof of Proposition 2.16 that $t(x, y)$ realises the dual binary 0-discriminator function on each member of K. Hence V is a dual binary $\mathbf{0}$-discriminator variety. □

2.1 Central elements

Let V be a dual binary $\mathbf{0}$-discriminator variety, with $\mathbf{A} \in \mathsf{V}$. Denote the element $\mathbf{0}^{\mathbf{A}}$ by 0. For each $c \in A$ let Ψ_c denote the binary relation on A given by $a\ \Psi_c\ b$ if $a \wedge c = b \wedge c$. It follows from Lemma 2.9 that Ψ_c is a congruence on $\mathbf{A}$; see also [16, Theorem 5.3]. Let Θ_c denote the smallest congruence on $\mathbf{A}$ that identifies the elements 0 and c. Since $a\ \Psi_c\ (a \wedge c)\ \Theta_c\ 0$ for all $a \in A$, it follows that $\Psi_c \vee \Theta_c = \iota_{\mathbf{A}}$, the largest congruence on $\mathbf{A}$.

Definition 2.20. An element c of an algebra $\mathbf{A}$ in a dual binary $\mathbf{0}$-discriminator variety is said to be *central* if Θ_c and Ψ_c are complementary factor congruences of $\mathbf{A}$.

In general, Ψ_c and Θ_c will be complementary factor congruences when $\Theta_c \circ \Psi_c = \Psi_c \circ \Theta_c = \iota_{\mathbf{A}}$, and $\Psi_c \wedge \Theta_c = \omega_{\mathbf{A}}$. Since $\Psi_c \vee \Theta_c = \iota_{\mathbf{A}}$ for every c, a sufficient condition for Ψ_c and Θ_c to be factor congruences is that $\Psi_c \wedge \Theta_c = \omega_{\mathbf{A}}$. When $c = 0$, $\Psi_c = \iota_{\mathbf{A}}$, while $\Theta_c = \omega_{\mathbf{A}}$, the smallest congruence on $\mathbf{A}$. On the other hand, if c is a maximal element, then $\Psi_c = \omega_{\mathbf{A}}$, while $\Theta_c = \iota_{\mathbf{A}}$, so 0, and maximal elements when they exist, are examples of central elements.

Since the concept of a central element considered in this paper does not require algebras to have elements that are residually distinct, it differs from the notion of a central element due to Vaggione [38]. In the case of algebras in dual binary discriminator varieties having a second constant term that is residually distinct from the discriminating constant, the central elements in the sense of Vaggione are the same as the central elements considered in this paper. In view of that, the following definition is useful.[8]

Definition 2.21. A dual binary $\mathbf{0}$-discriminator variety V is said to be *double pointed* if there exists a constant term $\mathbf{1}$ in the language of V that is *residually distinct* from $\mathbf{0}$; that is, $\Theta_1 = \iota_{\mathbf{A}}$ for all $\mathbf{A} \in \mathsf{V}$, where $1 = \mathbf{1}^{\mathbf{A}}$.

Examples of double pointed dual binary discriminator varieties include bounded distributive lattices, pseudocomplemented semilattices, Stone algebras, and Boolean algebras. Many examples that are double pointed ternary discriminator varieties arise in the study

[8]In conformance with our notation, Θ_1 abbreviates $\Theta(0, 1)$, the smallest congruence that identifies the elements 0 and 1.

of discriminator logics, since double pointedness ensures the existence of logical negation. For details, see [35].

Proposition 2.22. *A dual binary* $\mathbf{0}$*-discriminator variety* V *is double pointed if and only if there exists a constant term* $\mathbf{1}$ *in the language of* V *such that* $\mathsf{V} \models x \wedge \mathbf{1} \approx x$*. In that case the element* $\mathbf{1}^{\mathbf{A}} \in \mathbf{A}$ *is both maximal and central for every* $\mathbf{A} \in \mathsf{V}$*.*

Proof. Let V be a double pointed dual binary $\mathbf{0}$-discriminator variety and let $\mathbf{1}$ be a constant term that is residually distinct from $\mathbf{0}$. Let $\mathbf{A} \in \mathsf{V}$ be a non-trivial dual binary 0-discriminator algebra. Then $\mathbf{A}$ has at least two elements, so $\Theta_1 = \iota_{\mathbf{A}}$ implies that $1 \neq 0$, where $1 = \mathbf{1}^{\mathbf{A}}$ and $0 = \mathbf{0}^{\mathbf{A}}$. But then $a \wedge 1 = a$ for all $a \in \mathbf{A}$, since $\wedge$ is the dual binary 0-discriminator on A. Hence the identity $x \wedge \mathbf{1} \approx x$ is satisfied by a class of algebras that generates V.

Conversely, if V has a constant term $\mathbf{1}$ such that $\mathsf{V} \models x \wedge \mathbf{1} \approx x$, then the element $1 = \mathbf{1}^{\mathbf{A}}$ is maximal for every $\mathbf{A} \in \mathsf{V}$, so $\Theta_1 = \iota_{\mathbf{A}}$ for every $\mathbf{A} \in \mathsf{V}$. The second statement of the proposition follows because maximal elements are always central. □

When a variety V is a ternary discriminator variety every element of an algebra $\mathbf{A} \in \mathsf{V}$ is central. However, the converse does not hold. For example, every element of a skew Boolean algebra is central (see Proposition 6.2), but the variety of skew Boolean algebras is not a ternary discriminator variety. The following lemma identifies a necessary (but not sufficient) condition for every element of every algebra in V to be central.

Lemma 2.23. *Let* V *be a dual binary* $\mathbf{0}$*-discriminator variety. If the congruences* Θ_c *and* Ψ_c *permute for every* $\mathbf{A} \in \mathsf{V}$ *and all* $c \in A$ *then there exists a binary term* $s(x, y)$ *which satisfies the identities* $s(x, \mathbf{0}) \approx x$ *and* $s(x, x) \approx \mathbf{0}$*.*

Proof. Let $\mathbf{F}(x, y)$ denote the free V-algebra on free variables x and y. Assume that $\Theta_y \circ \Psi_y = \iota_{\mathbf{F}(x,y)}$. Then there exists an element $s = s(x, y)$ of $F(x, y)$ such that $x \;\Theta_y\; s(x, y) \;\Psi_y\; 0$. Since $y \equiv 0(\Theta_y)$ this implies that $x = s(x, 0)$. Also, $s(x, y) \;\Psi_y\; 0$ implies that $s(x, y) \wedge y = 0 \wedge y = 0$. Now $\mathbf{F}(x, y)$ is free in x and y, so $s(x, \mathbf{0}) \approx x$ and $s(x, y) \wedge y \approx \mathbf{0}$ are identities of V. Since $s(x, y)$ is a $\mathbf{0}$-reflexive term, $\mathsf{V} \models \mathbf{0} \approx s(x, y) \wedge y \approx s(x \wedge y, y \wedge y)$, by Lemma 2.9. Putting $x = y$ gives $\mathsf{V} \models s(x, x) \approx \mathbf{0}$. Thus, when V has the property that $\Theta_c \circ \Psi_c = \iota_{\mathbf{A}}$ for every $\mathbf{A} \in \mathsf{V}$ and $c \in A$, a binary term satisfying the stated identities must exist. □

A term $s(x, y)$ satisfying the identities of Lemma 2.23 is called $\mathbf{0}$-*subtractive*; see [37]. A variety V with a constant term $\mathbf{0}$ is called *subtractive at* $\mathbf{0}$, or $\mathbf{0}$-*subtractive*, if it has a $\mathbf{0}$-subtractive term. An algebra $\mathbf{A}$ with a constant term $\mathbf{0}$ is $\mathbf{0}$-*subtractive* if the variety $\mathbf{HSP}(\{\mathbf{A}\})$ is $\mathbf{0}$-subtractive. Subtractive algebras have normal $\mathbf{0}$-ideals and are *congruence-permutable at* 0, that is, $[0]\theta \circ \psi = [0]\psi \circ \theta$ for every pair of congruences θ and ψ, where $0 = \mathbf{0}^{\mathbf{A}}$. Conversely, a variety V with a constant $\mathbf{0}$ and the property that every $\mathbf{A} \in \mathsf{V}$ is congruence-permutable at $\mathbf{0}^{\mathbf{A}}$ has a $\mathbf{0}$-subtractive term; see [37, Proposition 1.2]. Such a variety is therefore also called $\mathbf{0}$-*permutable*, or *congruence-permutable at* $\mathbf{0}$. When a dual binary $\mathbf{0}$-discriminator variety is subtractive at $\mathbf{0}$ we simply say that it is subtractive.

3 Binary discriminator varieties

The definitions of the binary discriminator function, a binary discriminator algebra, a binary discriminator variety, and the generic binary discriminator variety mirror Definitions 2.1, 2.2, 2.3, and 2.4 in the previous section. Thus, given a set A with element $0 \in A$, the *binary* 0*-discriminator* on A is the function $\backslash\colon A^2 \to A$ defined for $a, b \in A$ by $a\backslash b = a$ if $b = 0$, and 0 otherwise. A *binary discriminator algebra* is an algebra $\mathbf{A}$ for which there exists a a binary term $x\backslash y$ and a constant term $\mathbf{0}$ that induce the binary 0-discriminator and its discriminating element $0 = \mathbf{0}^{\mathbf{A}}$ on the base set A of $\mathbf{A}$. A *binary discriminator variety* is a variety generated by a class K of binary discriminator algebras, with terms $x\backslash y$ and $\mathbf{0}$ inducing the binary 0-discriminator and its discriminating constant 0 and on each $\mathbf{A} \in \mathsf{K}$. The *generic binary discriminator variety* is the variety of similarity type $\langle 2, 0\rangle$ generated by the class of all algebras of the form $\mathbf{A} = \langle A; \backslash, 0\rangle$, with $\backslash$ being the binary 0-discriminator on A.

Lemma 3.1 (cf. [16, Theorem 2.1])**.** *A variety* V *is a binary* $\mathbf{0}$*-discriminator variety if and only if* V *is a* $\mathbf{0}$*-subtractive dual binary* $\mathbf{0}$*-discriminator variety.*

Proof. Suppose V is a binary $\mathbf{0}$-discriminator variety. It is immediate that the binary discriminator term $x\backslash y$ witnesses $\mathbf{0}$-subtractivity. Moreover, $a\backslash(a\backslash b)$ is the dual binary 0-discriminator $a \wedge b$ on every binary 0-discriminator algebra in V. On the other hand, if V is a dual binary $\mathbf{0}$-discriminator variety with a $\mathbf{0}$-subtractive term $s(x, y)$ then it is easily checked that $s(x, x \wedge y)$ realizes the binary 0-discriminator on every dual binary 0-discriminator algebra in V. □

It was shown in [8] that the generic binary discriminator variety is the variety iBCS of *implicative BCS-algebras* of type $\langle 2, 0\rangle$, axiomatised by the identities

$$\mathsf{iBCS}\colon\quad \begin{array}{cc} x\backslash x \approx \mathbf{0} & (x\backslash y)\backslash z \approx (x\backslash z)\backslash y \\ (x\backslash y)\backslash z \approx (x\backslash z)\backslash(y\backslash z) & x\backslash(y\backslash x) \approx x \end{array}$$

It was also shown there that iBCS is generated as a variety by the three-element binary discriminator algebra $\mathbf{B}_2 = \langle\{0, 1, 2\}; \backslash, 0\rangle$, that is, $\mathsf{iBCS} = \mathbf{HSP}(\{\mathbf{B}_2\})$. Implicative BCS-algebras are precisely the $\langle\backslash, 0\rangle$-subreducts of pseudocomplemented semilattices, where $a\backslash b = a \wedge b^*$ for each pseudocomplemented semilattice $\langle A; \wedge, ^*, 0\rangle$ and all $a, b \in A$. As such, they occur widely as subreducts of algebras such as Stone algebras, linearly ordered Heyting algebras, pseudocomplemented semilattices, skew Boolean algebras, strict basic logic algebras, product logic algebras, and algebras in residually finite varieties of basic logic algebras.

If $\mathbf{A} \in \mathsf{iBCS}$ then the Clifford-McLean equivalence relation Ξ is a congruence on $\mathbf{A}$ and $\mathbf{A}/\Xi \in \mathsf{iBCK}$, the variety of *implicative BCK-algebras*, which is the reflective subvariety of iBCS axiomatised relative to iBCS by the identity $x\backslash(x\backslash y) \approx y\backslash(y\backslash x)$.

Combining Theorem 2.19 and Lemma 3.1 yields the following.

Theorem 3.2. *Every binary discriminator variety is term equivalent to a variety of implicative BCS-algebras with ideal-compatible operations. A variety* V *with a constant term* $\mathbf{0}$ *is a binary* $\mathbf{0}$*-discriminator variety if and only if* V *is subtractive at* $\mathbf{0}$*, has EDPI and is generated by a class of* $\mathbf{0}$*-ideal simple algebras.*

Let $\mathbf{A}$ be an algebra in a binary $\mathbf{0}$-discriminator variety V. Recall that Θ_c denotes the smallest congruence on $\mathbf{A}$ equating the elements 0 and c, where 0 is the realisation of the discriminating constant $\mathbf{0}$ of V. Subtractivity ensures that every $\mathbf{0}$-ideal I of $\mathbf{A}$ is a congruence class, so it is meaningful to let Θ_I denote the smallest congruence θ of $\mathbf{A}$ such that $[0]\theta = I$.[9] It turns out that Θ_I has a simple characterisation that depends only on the binary discriminator term. This means that some key structural properties of algebras in binary discriminator varieties can be conveniently studied by restricting attention to their iBCS-algebra term reducts; see for example [9], where the following result is proved.

Theorem 3.3. *Let* V *be a binary* $\mathbf{0}$*-discriminator variety and let* $\mathbf{A} \in \mathsf{V}$*. For all* $a, b, c \in A$*,*

1. $a \equiv b \bmod \Theta_c$ *if and only if* $a \backslash c = b \backslash c$*.*
2. $[0]\Theta_c = \langle c \rangle = \{a \in A \mid a \backslash c = 0\} = \{a \in A \mid a \wedge c = a\}$*.*
3. *For every ideal* I*,* $a \equiv b \bmod \Theta_I$ *if and only if* $a \backslash c = b \backslash c$ *for some* $c \in I$*.*
4. *For all* $\Psi \in \mathrm{Con}\,\mathbf{A}$*,* $\Theta_I \vee \Psi = \Theta_I \circ \Psi \circ \Theta_I$*.*

Let $\mathbf{A} \in \mathsf{V}$, where V is a binary $\mathbf{0}$-discriminator variety. Since V is $\mathbf{0}$-subtractive, an element $c \in A$ will be central if and only if $\Psi_c \wedge \Theta_c = \omega_{\mathbf{A}}$. Let QB_2 denote the quasi-identity $x \wedge z \approx y \wedge z \ \&\ x \backslash z \approx y \backslash z \Rightarrow x \approx y$. The next result is immediate.

Theorem 3.4. *Let* V *be a binary* $\mathbf{0}$*-discriminator variety. The following are equivalent.*

1. *For all* $\mathbf{A} \in \mathsf{V}$*, every* $c \in A$ *is central.*
2. *For all* $\mathbf{A} \in \mathsf{V}$ *and* $a, b, c \in A$*,* $a \wedge c = b \wedge c$ *and* $a \backslash c = b \backslash c$ *implies* $a = b$*.*
3. $\mathsf{V} \models QB_2$*.*

We call a binary $\mathbf{0}$-discriminator variety satisfying the equivalent conditions of Theorem 3.4 a QB_2 *variety*. This terminology reflects the fact that the quasi-variety generated by $\mathbf{B}_2$, the three-element binary 0-discriminator algebra, is axiomatised by the iBCS identities together with the QB_2 quasi-identity. Most natural examples of binary discriminator varieties are QB_2 varieties.[10] Such varieties are of interest because their members have weak Boolean product representations. A number of examples of weak Boolean product representations of algebras in QB_2 varieties appear in the literature; see, for example, [18, 23], or [30].

4 Some other pointed discriminator functions

In the following definitions we follow the convention of using infix notation for functions in two variables.

Definition 4.1. Let A be a set and let $0 \in A$. Then the

- *skew* 0*-discriminator* on A is the function s defined for all $a, b, c \in A$ by

$$s(a,b,c) = \begin{cases} c & \text{if } c \neq 0, \\ a & \text{if } c = 0 \text{ and } b \neq 0, \\ 0 & \text{otherwise;} \end{cases}$$

- *multiplicative* 0*-discriminator* on A is the function q defined for all $a, b, c \in A$ by

$$q(a,b,c) = \begin{cases} a & \text{if } c \neq 0 \text{ and } a = b, \\ 0 & \text{otherwise;} \end{cases}$$

[9] Thus Θ_I is I^δ in the terminology of Agliano and Ursini [2].

[10] Two significant exceptions are the varieties of implicative BCS-algebras and pseudocomplemented semilattices.

- *pointed fixedpoint* 0*-discriminator* on A is the function f defined for all $a, b, c \in A$ by
$$f(a,b,c) = \begin{cases} c & \text{if } a = b, \\ 0 & \text{otherwise;} \end{cases}$$
- *skew Boolean* 0*-discriminator* on A is the function w defined for all $a, b, c \in A$ by
$$w(a,b,c) = \begin{cases} 0 & \text{if } c \neq 0, \\ b & \text{if } c = 0 \text{ and } b \neq 0, \\ a & \text{otherwise;} \end{cases}$$
- *meet* 0*-discriminator* on A is the function $\cap$ defined for all $a, b \in A$ by
$$a \cap b = \begin{cases} a & \text{if } a = b, \\ 0 & \text{otherwise;} \end{cases}$$
- *monoidal* 0*-discriminator* on A is the function $\vee$ defined for all $a, b \in A$ by
$$a \vee b = \begin{cases} b & \text{if } b \neq 0, \\ a & \text{otherwise.} \end{cases}$$
- *Implicative BCK difference* (briefly, *iBCK difference*) is the function $/$ defined for all $a, b \in A$ by
$$a/b = \begin{cases} a & \text{if } a \neq b, \\ 0 & \text{otherwise.} \end{cases}$$

Implicative BCK difference may alternatively be defined in terms of the binary and meet 0-discriminators: $a/b = a\backslash(a \cap b)$. For each of the 0-discriminator functions we have the associated notions of a discriminator algebra, discriminator variety, and generic discriminator variety, with definitions analogous to those for the corresponding dual binary and binary 0-discriminator constructs. Some examples of skew, skew Boolean, multiplicative, and pointed fixedpoint discriminator varieties are provided in the next four sections.

All of these pointed discriminator functions are to some extent interdefinable. For example, each of the cited 0-discriminator functions with two arguments can be written as a composition using just the ternary discriminator and the element 0, while each of the cited 0-discriminator functions with three arguments can be written as a composition of the cited 0-discriminator functions with two arguments. In particular, we have the following.

$$a \wedge b = t(b, t(b,0,a), a) \qquad q(a,b,c) = (a \cap b) \wedge c$$
$$a \vee b = t(b,0,a) \qquad s(a,b,c) = (a \wedge b) \vee c$$
$$a \cap b = t(a, t(a,b,0), 0) \qquad w(a,b,c) = (a \vee b)\backslash c$$
$$a \vee b = w(a,b,0) = s(a,a,b) \qquad a\backslash b = t(0,b,a)$$
$$a\backslash b = w(a,a,b) = f(b,0,a) \qquad a/b = t(a,b,0)$$
$$a \cap b = q(a,b,b) = f(a,b,a)$$
$$a \wedge b = f(0, f(b,0,a), a) = w(0,a,w(0,a,b)) = s(a,b,0) = q(a,a,b) = a\backslash(a\backslash b)$$
$$f(a,b,c) = (c\backslash(a\backslash(a \cap b)))\backslash(b\backslash(a \cap b)) = (c\backslash(a/b))\backslash(b/a)$$
$$t(a,b,c) = f(a,b,c) \vee (a\backslash(a \cap b)) = ((c\backslash(a/b))\backslash(b/a)) \vee (a/b)$$

In view of these equalities and Theorem 2.19 the following is immediate.

Proposition 4.2. *Every pointed ternary, pointed fixedpoint, skew Boolean, skew, multiplicative, and binary* **0***-discriminator variety is also a dual binary* **0***-discriminator variety. Hence, every such variety has both ideal-compatible operations and EDPI. Moreover, every pointed ternary discriminator variety is also a monoidal discriminator variety, a binary discriminator variety, a meet discriminator variety, a multiplicative discriminator variety, a skew discriminator variety, a skew Boolean discriminator variety, and a pointed fixedpoint discriminator variety.*

As a consequence of Proposition 4.2 and the displayed equalities, the various classes of pointed discriminator varieties can be ordered by class inclusion, as shown in Figure 1. For each class, the figure also shows which of the pointed discriminator terms with two arguments are definable in the varieties making up that class. Note that there are a number of subclasses of the class of dual binary discriminator varieties that are not included in the diagram; for example, the class of pointed dual discriminator varieties and the class of multiplicative skew discriminator varieties described in Section 7.

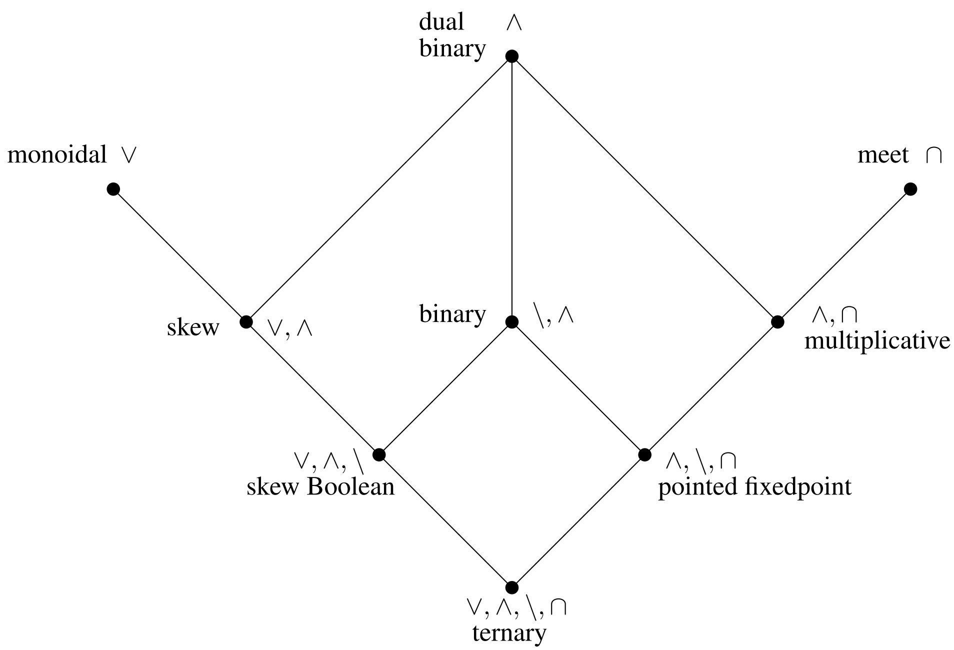

Figure 1: Some classes of pointed discriminator varieties.

4.1 Additivity

Definition 4.3. Let K be a class of algebras with a constant term $\mathbf{0}$ in its language. A binary term $x + y$ is called *additive with respect to* $\mathbf{0}$, or $\mathbf{0}$-*additive*, if $\mathsf{K} \models x + \mathbf{0} \approx x$ and $\mathsf{K} \models \mathbf{0} + x \approx x$. An algebra $\mathbf{A}$ is $\mathbf{0}$-additive if $\{\mathbf{A}\}$ has a $\mathbf{0}$-additive term.[11]

[11]The terminology *additive* is preferred over *monoidal* in this paper in view of the connection with direct summands, and to avoid confusion with the monoidal discriminator.

A variety that is reflexive and additive with respect to a constant term $\mathbf{0}$ has the property that a direct product $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ of two algebras decomposes into the direct sum of two subalgebras $\mathbf{A} = \mathbf{B}_1 \oplus \mathbf{C}_1$, where $\mathbf{B_1} \cong \mathbf{B}$ and $\mathbf{C_1} \cong \mathbf{C}$. (See [28, §2] for the definition of a direct sum and an outline of its properties.) In view of [28, Theorem 2] the converse is also true; a variety in which every product of two algebras decomposes into the direct sum of two subalgebras must have a constant term $\mathbf{0}$ in its language such that it is both reflexive and $\mathbf{0}$-additive.

We omit the prefix and say that a dual binary discriminator variety is additive when there is no ambiguity regarding which discriminating constant is intended. Skew, skew Boolean, and pointed ternary discriminator varieties are additive, so reflexive members of them that are isomorphic to finite direct products can be represented as a direct sum of subalgebras.

5 Skew discriminator varieties

A *skew lattice* $\langle A; \vee, \wedge \rangle$ is an algebra with two associative and idempotent binary operations $\vee$ and $\wedge$, satisfying the dual pair of absorption laws $x \wedge (x \vee y) \approx x \approx (y \vee x) \wedge x$ and $x \vee (x \wedge y) \approx x \approx (y \wedge x) \vee x$. For precise details, see [25]. By a *strongly distributive skew lattice* is meant a skew lattice that is symmetric, normal, and distributive; for details about such algebras see [27, §3].

Proposition 5.1. *The generic skew discriminator variety is term equivalent to the variety of left handed strongly distributive skew lattices with zero. Thus every skew discriminator variety is term equivalent to a variety of strongly distributive skew lattices with zero and with ideal-compatible operations.*

Proof. The skew 0-discriminator on a pointed set $A \supseteq \{0\}$ can be written as a composition of the dual binary and monoidal 0-discriminators: $s(a, b, c) = (a \wedge b) \vee c$. Conversely, we have $a \wedge b = s(a, b, 0)$ and $a \vee b = s(a, a, b)$. Thus any skew 0-discriminator algebra in the generic skew $\mathbf{0}$-discriminator variety is term equivalent to an algebra of the form $\mathbf{A} = \langle A; \vee, \wedge, 0 \rangle$. The remainder of the proof is analogous to the proof of Proposition 2.5. It is straightforward to verify that an algebra such as $\mathbf{A}$ is a primitive, and therefore flat, left handed strongly distributive skew lattice with zero and that every flat left handed strongly distributive skew lattice with zero has this form. On the other hand, it follows from [27, Theorem 3.2] that every subdirectly irreducible strongly distributive skew lattice with a zero is flat. Hence the generic skew discriminator variety is term equivalent to the variety of left handed strongly distributive skew lattices with zero. The second assertion of the proposition now follows from Theorem 2.19. □

More generally, if $\mathbf{A} = \langle A; \vee, \wedge, 0 \rangle$ is a flat strongly distributive skew lattice with a zero, then the skew 0-discriminator can be defined on A by

$$s(a, b, c) = c \vee (a \wedge b \wedge a) \vee c.$$

Examples of skew discriminator varieties that are not ternary discriminator or skew Boolean discriminator varieties thus include strongly distributive skew lattices with zero and hence also distributive lattices with zero, as well as certain varieties in which each member has a bounded distributive lattice term reduct, such as the variety of Q-distributive lattices introduced in [17].

While the term $x \vee y$ witnesses additivity for every skew discriminator variety, an additive dual binary discriminator variety need not be a skew discriminator variety. However, an additive binary discriminator variety is always a skew discriminator variety. In fact, rather more is true.

Proposition 5.2. *A variety* V *with a constant term* $\mathbf{0}$ *is a* $\mathbf{0}$*-additive binary* $\mathbf{0}$*-discriminator variety if and only if it is a skew Boolean* $\mathbf{0}$*-discriminator variety. Hence a subtractive skew discriminator variety is always a skew Boolean discriminator variety.*

Proof. Let V be a binary $\mathbf{0}$-discriminator variety with a $\mathbf{0}$-additive term $x+y$ and a binary $\mathbf{0}$-discriminator term $x \backslash y$. Let $\mathbf{A}$ be a binary 0-discriminator member of V. A straightforward case-splitting argument shows that the term $((x\backslash y)+y)\backslash z$ realises the skew Boolean 0-discriminator on A, with the sub-term $(x\backslash y)+y$ realising the monoidal 0-discriminator on A. Hence V is a skew Boolean $\mathbf{0}$-discriminator variety. Conversely, if V is a skew Boolean $\mathbf{0}$-discriminator variety with skew Boolean $\mathbf{0}$-discriminator term $w(x,y,z)$ then, by considering their realisations on a skew Boolean 0-discriminator algebra in V, it is easy to verify that $w(x,\mathbf{0},y)$ is a binary $\mathbf{0}$-discriminator term, while $w(x,y,\mathbf{0})$ is a $\mathbf{0}$-additive term. □

6 Skew Boolean discriminator varieties

A *skew Boolean algebra* may be regarded as an algebra $\mathbf{A} = \langle A; \vee, \wedge, \backslash, 0\rangle$, where the reducts $\langle A; \vee, \wedge\rangle$ and $\langle A; \backslash, 0\rangle$ are respectively a strongly distributive skew lattice and an implicative BCS-algebra, such that $\mathbf{A} \models x \wedge y \wedge x \approx x\backslash(x\backslash y)$. This identity ensures that the natural preorders on the two reducts coincide. For an alternative definition, and further details about the variety of skew Boolean algebras, see [26].

By [26, Theorem 1.13], there are, up to isomorphism, just three subdirectly irreducible skew Boolean algebras. Moreover, each of these algebras is flat. Given a flat skew Boolean algebra $\mathbf{A}$, it is straightforward to verify that the ternary function w defined for all $a,b,c \in A$ by $w(a,b,c) = (b \vee a \vee b)\backslash c$ is the skew Boolean 0-discriminator on A. It follows that skew Boolean algebras constitute a skew Boolean discriminator variety.

Proposition 6.1. *The generic skew Boolean discriminator variety is term equivalent to the class of left handed skew Boolean algebras. Thus, every skew Boolean discriminator variety is term equivalent to a variety of skew Boolean algebras with ideal-compatible operations.*

Proof. If $\mathbf{A}$ is a skew Boolean 0-discriminator algebra with a skew Boolean 0-discriminator w, the left handed skew Boolean algebra operations may be defined for all $a,b,c \in A$ by $a \wedge b = w(0,a,w(0,a,b))$, $a \vee b = w(a,b,0)$ and $a\backslash b = w(0,a,b)$. Conversely, if $\mathbf{A}$ is an ideal simple left handed skew Boolean algebra then the skew Boolean 0-discriminator on A is given for all $a,b,c \in A$ by $w(a,b,c) = (a \vee b)\backslash c$. The result now follows in a similar manner to Proposition 2.5, since every subdirectly irreducible skew Boolean algebra is flat and thus ideal simple. The second statement of the Proposition follows from Theorem 2.19. □

Proposition 6.2. *Every skew Boolean discriminator variety satisfies the* QB_2 *quasi-identity. Thus every element of an algebra in skew Boolean discriminator variety is central.*

Proof. By [26, Theorem 1.13], there are, up isomorphism, just three subdirectly irreducible skew Boolean algebras. Moreover, these algebras are ideal simple and have at most three elements, so their implicative BCS-algebra reducts are isomorphic to either $\mathbf{B}_2$, the three-element implicative BCS-algebra, or to the two-element implicative BCK-algebra, which is a subalgebra of $\mathbf{B}_2$. Therefore, every subdirectly irreducible skew Boolean algebra must satisfy QB_2. Birkhoff's Theorem (see [15, Theorem II§9.6]) ensures that every skew Boolean algebra is isomorphic to a subdirect product of subdirectly irreducible skew Boolean algebras. Thus every skew Boolean algebra must satisfy QB_2, since the satisfaction of quasi-identities is preserved under the taking of subdirect products. The result now follows from Theorem 3.4 and Proposition 6.1. □

Apart from skew Boolean algebras, examples of skew Boolean discriminator varieties include Stone algebras, double Stone algebras, Kleene-Stone algebras, strict basic logic algebras, and many others, including every pointed ternary discriminator variety. Skew Boolean discriminator varieties have a close connection with *Church algebras*, namely algebras that have a ternary term $q(x, y, z)$ and two constant terms $\mathbf{0}$ and $\mathbf{1}$ in their language satisfying the identities $q(\mathbf{1}, x, y) \approx x$ and $q(\mathbf{0}, x, y) \approx y$; see [18]. The next result is inspired by [31, Proposition 3.2].

Proposition 6.3. *Let* V *be a double-pointed skew Boolean* $\mathbf{0}$*-discriminator variety. Then* V *is a variety of Church algebras.*

Proof. Let $\mathbf{1}$ be a constant term that is residually distinct from $\mathbf{0}$. By Proposition 2.22, $\mathsf{V} \models x \wedge \mathbf{1} \approx x$, and for every $\mathbf{A} \in \mathsf{V}$ the element $1 = \mathbf{1}^{\mathbf{A}} \in A$ is maximal. Let x' abbreviate the term $\mathbf{1}\backslash x$ and let $q(x, y, z)$ denote the ternary term $(y \wedge x) \vee (z \wedge x')$. Then for all $a, b \in A$ we have

$$\begin{aligned} q^{\mathbf{A}}(0, a, b) &= (a \wedge 0) \vee (b \wedge 0') = 0 \vee (b \wedge (1\backslash 0)) \\ &= 0 \vee (b \wedge 1) = 0 \vee b = b, \end{aligned}$$

and

$$\begin{aligned} q^{\mathbf{A}}(1, a, b) &= (a \wedge 1) \vee (b \wedge 1') = a \vee (b \wedge (1\backslash 1)) \\ &= a \vee (b \wedge 0) = a \vee 0 = a. \end{aligned}$$

Hence V is a variety of Church algebras. □

In the particular case of semicentral right Church algebras there is an even closer correspondence. Briefly, V is a variety of *semicentral right Church algebras* if its language includes a constant term $\mathbf{0}$ and a ternary term $q(x, y, z)$ satisfying $q(0, x, y) \approx y$, such that for every $\mathbf{A} \in \mathsf{V}$, all elements of $\mathbf{A}$ are semicentral. For details see [18].

Proposition 6.4. *The class of skew Boolean discriminator varieties coincides with the class of varieties of semicentral right Church algebras.*[12]

Proof. Let V be a variety of semicentral right Church algebras, with right Church algebra term $q(x, y, z)$. By [18, Lemma 4.5], $\mathbf{A} \in \mathsf{V}$ is directly indecomposable if and only if for all $a, b, c \in A$, $q(a, b, c) = b$ if $a \neq 0$ and c otherwise. Let $w(a, b, c) = q(c, 0, q(b, b, a))$.

[12] The authors are grateful to the referee for pointing out this result.

Then for all $a, b, c \in A$, $w(a, b, c) = 0$ if $c \neq 0$, and $q(b, b, a)$ otherwise. But $q(b, b, a) = b$ if $b \neq 0$, and a otherwise. Hence $w(a, b, c)$ is the skew Boolean 0-discriminator on A. Since every subdirectly irreducible algebra is directly indecomposable it follows that the term $q(z, \mathbf{0}, q(y, y, x))$ realises the skew Boolean 0-discriminator on a class of algebras that generates V. Thus V is a skew Boolean discriminator variety.

Conversely, suppose that V is a skew Boolean discriminator variety with skew Boolean discriminator term $w(x, y, z)$ and discriminating constant term $\mathbf{0}$. Let $q(x, y, z)$ be the term $w(w(y, y, w(y, y, x)), w(z, z, x), \mathbf{0})$ and suppose that $\mathbf{A} \in \mathsf{V}$ is a skew Boolean 0-discriminator algebra. Then for all $a, b, c \in A$,

$$\begin{aligned} q(a, b, c) &= w(w(b, b, w(b, b, a)), w(c, c, a), 0) \\ &= w(w(b, b, 0), 0, 0) = w(b, 0, 0) = b \end{aligned}$$

when $a \neq 0$, while

$$\begin{aligned} q(a, b, c) &= w(w(b, b, w(b, b, 0)), w(c, c, 0), 0) \\ &= w(w(b, b, b), c, 0) = w(0, c, 0) = c \end{aligned}$$

when $a = 0$. Thus $\mathbf{A}$ is a directly indecomposable semicentral right Church algebra. Since V is generated by a class of such algebras, it must be a variety of semicentral right Church algebras. □

In [18] the *variety of pure semicentral right Church algebras* is defined to be the variety of type $\langle 3, 0\rangle$ comprising all semicentral right church algebras of the form $\langle A; q, 0\rangle$, with q being its right Church algebra operation. Combining Propositions 6.1 and 6.4 with [18, Theorem 4.6] yields the following.

Corollary 6.5. *The generic skew Boolean discriminator variety, the variety of pure semicentral right Church algebras, the variety of left handed skew Boolean algebras, and the variety of right handed skew Boolean algebras are all term equivalent.*

Every skew Boolean discriminator variety is additive, as witnessed by the term $x \vee y$. As a consequence, every principal ideal $\langle c\rangle$ of a reflexive algebra $\mathbf{A}$ in a skew Boolean discriminator variety is a direct summand. Its complementary direct summand is the ideal $\mathrm{ann}(c) = \{a \in A \mid a \wedge c = 0\}$. We remark that there is a converse to this result: a reflexive variety V with the property that the principal ideals of every member of V are direct summands must be a skew Boolean discriminator variety.

Theorem 6.6. *Let V be a variety with constant $\mathbf{0}$. The following are equivalent.*

1. *V is a skew Boolean $\mathbf{0}$-discriminator variety.*
2. *V is an additive binary $\mathbf{0}$-discriminator variety.*
3. *V is a subtractive skew $\mathbf{0}$-discriminator variety.*
4. *V is an additive and subtractive dual binary $\mathbf{0}$-discriminator variety.*
5. *V is additive and subtractive at $\mathbf{0}$, has EDPI and is generated by a class of $\mathbf{0}$-ideal simple algebras.*

Proof. Combine Theorem 3.2 and Proposition 5.2. □

Corollary 6.7. *A congruence-permutable dual binary* $\mathbf{0}$*-discriminator variety is additive and subtractive, and hence is a skew Boolean* $\mathbf{0}$*-discriminator variety.*

Proof. Let V be a congruence-permutable dual binary $\mathbf{0}$-discriminator variety. By a theorem of Mal'cev (see [15, Theorem II§12.2]), there exists a term $p(x, y, z)$ in the language of V such that $\mathsf{V} \models p(x, y, y) \approx p(y, y, x) \approx x$. Let $x + y$ be the term $p(x, \mathbf{0}, y)$ and let $s(x, y)$ be the term $p(x, y, 0)$. Then $x + \mathbf{0} \approx p(x, \mathbf{0}, \mathbf{0}) \approx x$, and $\mathbf{0} + x \approx p(\mathbf{0}, \mathbf{0}, x) \approx x$, so V is additive. Also $s(x, \mathbf{0}) \approx p(x, \mathbf{0}, \mathbf{0}) \approx x$ and $s(x, x) \approx p(x, x, \mathbf{0}) \approx \mathbf{0}$, so V is subtractive. Thus V is a skew Boolean $\mathbf{0}$-discriminator variety. □

7 Multiplicative discriminator varieties

Definition 7.1. An algebra $\mathbf{A}$ in a dual binary discriminator variety V has *intersections* if finite meets exist under the natural dual binary discriminator partial order on A. A dual binary discriminator variety V is said to have *intersections* if every member of V has intersections.

The variety of skew Boolean intersection algebras introduced in [7] is an example of a dual binary discriminator variety with intersections.

Lemma 7.2. *Let* V *be a multiplicative* $\mathbf{0}$*-discriminator variety. Then there exist terms* $x \cap y$ *and* $x \wedge y$ *that induce the meet* 0*-discriminator and the dual binary* 0*-discriminator respectively on the multiplicative* 0*-discriminator algebras in* V.

Proof. Let $q(x, y, z)$ be the multiplicative discriminator term for V. Put $x \cap y = q(x, y, x)$ and $x \wedge y = q(x, x, y)$. Let $\mathbf{A} \in \mathsf{V}$ be a multiplicative 0-discriminator algebra with discriminating element $0 = \mathbf{0}^{\mathbf{A}}$. Then for all $a, b \in A$, $q^{\mathbf{A}}(a, b, a) = a$ if $a = b$ and 0 otherwise; while $q^{\mathbf{A}}(a, a, b) = a$ if $b \neq 0$ and 0 otherwise. Hence these functions are respectively the meet and the dual binary 0-discriminators on A. □

Theorem 7.3. *Let* V *be a dual binary* $\mathbf{0}$*-discriminator variety with dual binary discriminator term* $x \wedge y$*. Then* V *is a multiplicative* $\mathbf{0}$*-discriminator variety if and only if there exists a binary term* $x \cap y$ *such that* V *satisfies the following identities:*

$$x \cap \mathbf{0} \approx \mathbf{0} \cap x \approx \mathbf{0} \qquad x \cap y \approx y \cap x \qquad x \cap (y \cap z) \approx (x \cap y) \cap z$$
$$x \cap x \approx x \qquad x \wedge (x \cap y) \approx x \cap y \qquad (x \wedge z) \cap (y \wedge z) \approx (x \cap y) \wedge z$$

Moreover, every dual binary $\mathbf{0}$*-discriminator variety with such a term has intersections and is a meet* $\mathbf{0}$*-discriminator variety.*

Proof. Let V be a multiplicative $\mathbf{0}$-discriminator variety and suppose that $\mathbf{A} \in \mathsf{V}$ is a multiplicative 0-discriminator algebra. By Lemma 7.2 there are terms $x \wedge y$ and $a \cap y$ that realise the dual binary and meet 0-discriminators on $\mathbf{A}$. Straightforward case-splitting arguments show that the displayed identities hold on $\mathbf{A}$ and hence they are identities of V, since it is a variety generated by a family of such algebras.

Conversely, if V is a dual binary $\mathbf{0}$-discriminator variety with dual binary $\mathbf{0}$-discriminator term $x \wedge y$, and a term $x \cap y$ such that the displayed identities are satisfied, then these identities imply that for every $\mathbf{A} \in \mathsf{V}$ the term reduct $\langle A; \cap, 0\rangle$ is a meet semilattice with zero. Moreover, the identities also imply that for all $a, b \in A$, $a \cap b \leq a$ and $a \cap b \leq b$ under the natural dual binary discriminator partial order. Suppose that $c \in A$ is such that

$c \leq a$ and $c \leq b$, so that $a \wedge c = c$ and $b \wedge c = c$. Then $c = (a \wedge c) \cap (b \wedge c) = (a \cap b) \wedge c$, which implies that $c \leq a \cap b$. Thus $a \cap b$ is the greatest lower bound of a and b with respect to the natural dual binary discriminator partial order and hence V has intersections.

To see that V is a meet $\mathbf{0}$-discriminator variety, let $\mathbf{A}$ be a dual binary $\mathbf{0}$-discriminator algebra in V. Let $a, b \in A$. Now $a \cap b$ is the meet of a and b under the natural dual binary discriminator partial order on A. But $\mathbf{A}$ is order isomorphic to a flat domain, which implies that $a \cap b = 0$ when $a \neq b$. Also $a \cap a = a$. Thus the term $x \cap y$ realises the meet $\mathbf{0}$-discriminator on a class of algebras that generates V. □

In view of this result we say that a dual binary discriminator variety V is *multiplicative* if it has a binary term $x \cap y$ such that for every $\mathbf{A} \in \mathsf{V}$ and all $a, b \in A$, $a \cap b$ is the meet of the elements a and b under the natural dual binary discriminator partial order on A.

Example 7.4. The *dual discriminator* on a set A is the ternary function $d\colon A^3 \to A$ given for all $a, b, c \in A$ by $d(a, b, c) = a$ if $a = b$, and c otherwise; see [19]. Let V be a pointed dual discriminator variety, with dual discriminator term $d(x, y, z)$ and a constant term $\mathbf{0}$. If $\mathbf{A} \in \mathsf{V}$ is a dual discriminator algebra then $d^{\mathbf{A}}(a, b, 0) = a$ if $a = b$, and 0 otherwise, while $d^{\mathbf{A}}(0, b, a) = a$ if $b \neq 0$, and 0 otherwise, so $d(x, y, 0)$ and $d(0, y, x)$ are respectively meet and dual binary discriminator terms for V, with the multiplicative discriminator term for V being $d(0, z, d(x, y, 0))$.

Jonathan Leech [24] has shown that a multiplicative skew discriminator variety is a pointed dual discriminator variety (and hence is congruence distributive). The converse does not hold, since the generic pointed dual discriminator variety is not additive.

8 Pointed fixedpoint discriminator varieties

Fixedpoint discriminator varieties arise in algebraic logic and were introduced by W. Blok and D. Pigozzi in [11]. Pointed fixedpoint discriminator varieties were introduced in [1], where they are called *dual fixedpoint discriminator varieties*, and independently in [34]. The generic pointed fixedpoint discriminator variety is (up to term equivalence) the variety iBCSK of *implicative BCSK-algebras*, introduced in [34]. An implicative BCSK-algebra is an algebra $\mathbf{A} = \langle A; /, \backslash, 0\rangle$ of type $\langle 2, 2, 0\rangle$, where $\langle A; /, 0\rangle$ is an implicative BCK-algebra, $\langle A; \backslash, 0\rangle$ is an implicative BCS-algebra, such that the natural partial orders on each of these term reducts coincide. An equational base for the variety iBCSK may be obtained by taking the iBCS and iBCK identities, together with the identities $(x \backslash y)/x \approx \mathbf{0}$ and $x \wedge (x/y) \approx x/y$. Humberstone [21, 22] has extensively investigated the deductive system canonically associated with the variety iBCSK from the perspective of the normal modal logic $\mathbf{S5}$.

Recall that an algebra with a constant term $\mathbf{0}$ is $\mathbf{0}$-*regular* if for every two congruences θ and ψ, $[0]\theta = [0]\psi$ implies $\theta = \psi$. A variety with a constant term $\mathbf{0}$ is $\mathbf{0}$-regular if every member of it is $\mathbf{0}$-regular. A variety V is said to be *ideal determined at* $\mathbf{0}$ if every ideal of an algebra $\mathbf{A} \in \mathsf{V}$ is the 0-class of a unique congruence relation; see [20, Definition 1.3]. Clearly, every algebra in such a variety has the property that its lattice of $\mathbf{0}$-ideals is isomorphic to its lattice of congruences. By [20, Corollary 1.9] a variety V with a constant term $\mathbf{0}$ is ideal determined at $\mathbf{0}$ if and only if it is both subtractive at $\mathbf{0}$ and $\mathbf{0}$-regular.

Implicative BCSK-algebras are $\mathbf{0}$-regular and, since the iBCS and iBCK operations are both subtractive at $\mathbf{0}$, the variety iBCSK is ideal determined. Moreover, iBCSK is semisimple, that is, every subdirectly irreducible member of iBCSK is simple. Full details appear in [34]. Since every pointed fixedpoint discriminator variety is term equivalent to a variety of iBCSK-algebras with ideal-compatible operations, it follows that such a variety must be ideal determined at its discriminating constant, and thus must also be semi-simple. In summary:

Theorem 8.1. *The following are equivalent for a variety with constant* $\mathbf{0}$.

1. V *is a pointed fixedpoint* $\mathbf{0}$*-discriminator variety.*
2. V *is a subtractive multiplicative* $\mathbf{0}$*-discriminator variety.*
3. V *is a multiplicative binary* $\mathbf{0}$*-discriminator variety.*
4. V *is a* $\mathbf{0}$*-regular binary* $\mathbf{0}$*-discriminator variety.*
5. V *is an ideal determined dual binary* $\mathbf{0}$*-discriminator variety.*
6. V *is ideal determined at* $\mathbf{0}$ *and is semi-simple with EDPI.*

The equivalence of 1 and 6 was shown independently in [1, Theorem 4.8]. We remark that for the double-pointed analogue of Theorem 8.1, further equivalences are possible: in particular, fundamental connections can be established with the pseudo-interior algebras of Blok and Pigozzi [12].

9 Pointed ternary discriminator varieties

Pointed ternary discriminator varieties can be characterised in many different ways. Note that a ternary discriminator variety is a dual binary $\mathbf{0}$-discriminator variety for each constant term $\mathbf{0}$ in its language.

Theorem 9.1. *For each constant term* $\mathbf{0}$ *in the language of a variety* V *the following are equivalent.*

1. V *is a pointed ternary discriminator variety.*
2. V *is term equivalent to a variety of skew Boolean intersection algebras with ideal-compatible operations.*
3. V *is a multiplicative skew Boolean* $\mathbf{0}$*-discriminator variety.*
4. V *is an ideal determined skew* $\mathbf{0}$*-discriminator variety.*
5. V *is a multiplicative and subtractive skew* $\mathbf{0}$*-discriminator variety*
6. V *is an additive and multiplicative binary* $\mathbf{0}$*-discriminator variety.*
7. V *is an additive, subtractive and multiplicative dual binary* $\mathbf{0}$*-discriminator variety.*
8. V *is an additive pointed fixedpoint* $\mathbf{0}$*-discriminator variety.*
9. V *is a congruence-permutable multiplicative* $\mathbf{0}$*-discriminator variety.*

Proof. Let V be a pointed ternary discriminator variety with constant term $\mathbf{0}$. By Theorem 1.2 and Proposition 4.2, V is a dual binary $\mathbf{0}$-discriminator variety. By [7, Theorem 4.7], the generic pointed ternary discriminator variety is term equivalent to the variety of left handed skew Boolean intersection algebras, so it follows from Theorem 2.19 that V must be term equivalent to a variety of skew Boolean intersection algebras with operations that are ideal-compatible with respect to its dual binary $\mathbf{0}$-discriminator term. Thus

1 implies 2. Now 2 implies 3 because every skew Boolean intersection algebra is a multiplicative skew Boolean algebra in view of [7, §4] and Theorem 7.3.

Since every ternary **0**-discriminator variety is also a skew, skew Boolean, binary, dual binary, and pointed fixedpoint **0**-discriminator variety, the equivalence of 3, 4, 5, 6, 7 and 8 follows directly from Theorems 3.2, 6.6, 7.3, and 9.1.

In view of Corollary 6.7, a congruence-permutable dual binary **0**-discriminator variety is skew Boolean **0**-discriminator variety, so 9 implies 3. Ternary discriminator varieties are congruence-permutable by [15, Theorem IV§9.4] and a ternary **0**-discriminator variety is a multiplicative **0**-discriminator variety by Proposition 4.2, so 1 implies 9.

To complete the proof it is sufficient to show that 3 implies 1, so assume that V is a multiplicative skew Boolean **0**-discriminator variety. Then by Proposition 6.1 V is term equivalent to a variety of skew Boolean algebras with ideal-compatible operations. By Theorem 7.3 these algebras have intersections, that are witnessed by a binary meet **0**-discriminator term $x \cap y$. Thus, by [7, Theorem 4.4], when the meet discriminator term is included in their type, they are members of the ternary discriminator variety of skew Boolean intersection algebras. Hence the ideal simple members of V are ternary discriminator algebras, and since it is generated by its ideal simple members, V must be a pointed ternary discriminator variety. □

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THE ART OF DISCRETE AND APPLIED MATHEMATICS

ISSN 2590-9770
The Art of Discrete and Applied Mathematics 2 (2019) #P2.09
https://doi.org/10.26493/2590-9770.1332.594
(Also available at http://adam-journal.eu)

Open problems from NCS 2018

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Received 14 July 2019, accepted 19 September 2019, published online 31 December 2019

Abstract

All participants in the workshop NCS 2018 were invited to submit a list of open problems, typically problems relevant to the subject matter of their talk. These problems follow, grouped according the particular individual presenting them. These individuals appear with their problem sets in alphabetical order, the sole exception being that the Honoree of the workshop appears first. Some of the presenters give a bit of background to accompany their problems. In other cases, such as myself, the presenter assumes that enough background was given in their talk as published herein. Thus the reader is invited to refer back to the relevant article. It is hoped that in the months to follow some of these problems will be solved, or at least, considerable light will be shed on them.

Keywords: Skew lattice, ring, primitive skew lattice, dual congruence distributivity, orthodox semigroup, sheaf representation, left regular band, quasilattice, paralattice, covering lattice, skew Boolean algebra, congruence lattice, antilattice, skew Heyting algebra, coset structure, comodernistic lattice.

Math. Subj. Class.: 06A75, 06B20, 06B75

1 Jonathan Leech

Problem 1.1. The following problem considers skew lattices in the context of rings.

(1) Given a finite left (right) regular band B, must the skew lattice generated from B in the semigroup ring $\mathbb{Z}(B)$ be finite also?

(2) If B is the free left (right) regular band on generators, $x_1, x_2, \ldots, x_n$, what must its generated skew lattice in $\mathbb{Z}(B)$ look like… in detail?

*The author thanks the support of the Croatian Science Foundation's funding of the project EVOSOFT, with the reference UIP-2014-09-7945.

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The $\mathbb{Z}$-coefficient case has implications for the case of an arbitrary commutative ring K with identity being used as the coefficient ring. Of course, if the generated skew lattice in (2) is finite for all n, then the answer to (1) is YES. In this case, for any finite left regular band in any ring, the generated skew lattice is finite.

Problem 1.2. Find necessary and sufficient conditions for a skew lattice S to be embedded into a ring. That is for S to have an isomorphic copy consisting of idempotents in a ring with the operations being multiplication and either the quadratic or cubic join. Preconditions include: being distributive (and thus categorical); being cancellative (and thus symmetric). Clearly skew Boolean algebras and their duals can be embedded into a ring. More generally, strongly distributive skew lattices [satisfying $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$] and dual strongly distributive lattices can be embedded into a ring. Are there other sufficient conditions?

Problem 1.3. In particular, find necessary and sufficient conditions for a primitive skew lattice to be embedded into a ring. A primitive skew lattice is both distributive and cancellative, Perhaps all primitive skew lattices can be so embedded. If not, then a third general requirement for a skew lattice to be embedded into a ring is that all of its primitive subalgebras can be so embedded. More generally, find necessary and sufficient conditions for skew chains $A > B > C$ or $A > B > C > D$ or longer, to be so embedded.

Problem 1.4. What can be said about the subvariety of skew lattices generated from all primitive skew lattices? It must be: distributive (and thus categorical), cancellative (and thus symmetric) and strictly categorical. What else? Again, what are necessary and sufficient conditions for a skew lattice to be in this subvariety?

Problem 1.5. What can one say about the subvariety of skew lattices generated jointly from the varieties of normal and conormal skew lattices?

2 Robert J. Bignall

Problem 2.1. Find a Mal'cev condition (see [21]) for dual congruence distributivity at a constant.

Problem 2.2. Characterise the class of dual binary discriminator varieties that are dually congruence distributive at their discriminating constant.

Background

Let K be a class of algebras of some given similarity type and let $\mathbf{0}$ be a constant term. Given an algebra $\mathbf{A} \in \mathsf{K}$ and $\Theta \in \operatorname{Con} \mathbf{A}$, denote the congruence class $\{a \in A \mid a \;\Theta\; 0\}$ by $[0]\Theta$, where $0 = \mathbf{0}^{\mathbf{A}}$. $\mathbf{A}$ is said to be *congruence distributive at* $\mathbf{0}$, or *congruence* $\mathbf{0}$*-distributive* if for all $\Theta, \Psi, \Phi \in \operatorname{Con} \mathbf{A}$

$$[0](\Theta \vee \Psi) \wedge \Phi = [0](\Theta \wedge \Phi) \vee (\Psi \wedge \Phi).$$

$\mathbf{A}$ is said to be *dually congruence distributive at* $\mathbf{0}$, or *dually congruence* $\mathbf{0}$*-distributive* if for all $\Theta, \Psi, \Phi \in \operatorname{Con} \mathbf{A}$

$$[0](\Theta \wedge \Psi) \vee \Phi = [0](\Theta \vee \Phi) \wedge (\Psi \vee \Phi).$$

The class K is congruence $\mathbf{0}$-distributive (resp. dually congruence $\mathbf{0}$-distributive) if every $\mathbf{A} \in \mathsf{K}$ is congruence $\mathbf{0}$-distributive (resp. dually congruence $\mathbf{0}$-distributive); see [2]. These properties are described as being *ideal-theoretic*, because $[0]\Theta$ is an ideal of $\mathbf{A}$ with respect to the constant $\mathbf{0}$.

Dual congruence $\mathbf{0}$-distributivity is a stronger condition than congruence $\mathbf{0}$-distributivity. For example, meet semilattices with $\mathbf{0}$, or more generally normal bands with $\mathbf{0}$, are congruence $\mathbf{0}$-distributive but not dually congruence $\mathbf{0}$-distributive. On the other hand, skew Boolean algebras are dually congruence $\mathbf{0}$-distributive, and hence are also congruence $\mathbf{0}$-distributive. Skew Boolean algebras, or more generally algebras in skew Boolean discriminator varieties, enjoy a number of other strong ideal-theoretic properties, despite the fact that their congruence lattices in general satisfy no lattice identities.

I. Chajda [2] has given a Mal'cev condition for the class of congruence $\mathbf{0}$-distributive varieties, modelled on Bjarni Jónsson's well-know Mal'cev condition for the class of congruence distributive varieties [11]. However, it would appear that no-one has as yet solved the harder problem of finding a Mal'cev condition for the class of dually congruence $\mathbf{0}$-distributive varieties (or dually congruence $\mathbf{0}$-distributive **SP** classes; that is classes of algebras that are closed under the taking of products, subalgebras and isomorphisms.) The congruence lattices of algebras in dual binary discriminator varieties do not in general have any special properties. However, the ideals of such algebras play a central role. Another significant ideal theoretic property is congruence permutability at $\mathbf{0}$. A class K of algebras with a constant $\mathbf{0}$ is congruence permutable at $\mathbf{0}$ if for any $\mathbf{A} \in \mathsf{K}$ and any $\Theta, \Phi \in \mathrm{Con}\,\mathbf{A}$,

$$[0]\Theta \circ \Phi = [0]\Phi \circ \Theta.$$

K is *arithmetic at* $\mathbf{0}$ if it is both congruence distributive and congruence permutable at $\mathbf{0}$, in which case it is also dually congruence distributive at $\mathbf{0}$.

The class of binary discriminator varieties has an entirely ideal-theoretic characterisation as that class of pointed varieties that are arithmetic at their constant term, have equationally definable principal ideals, and are generated by a sub-class of ideal simple members. A worthwhile but as yet incomplete research program would be to characterise each pointed discriminator variety sub-class of the class of dual binary discriminator varieties in terms of the ideal theoretic properties of its members.

3 Des FitzGerald

Problem 3.1. Let A be an algebra and S its monoid of endomorphisms. It is known that if S has commuting idempotents then A has the properties:

(RI) the intersection of two retracts of A is also a retract,

(UR) to each retract of A corresponds a unique idempotent with that range,

and their duals. Is the converse true? This was posed as Problem 7.4 of [10] where a proof of the forwards implication may be found.

Problem 3.2. My paper [5] necessarily included (in Section 2) a development of groupoids over a band of objects. It seemed likely that this topic had been considered already, but I have since found nothing directly comparable in the literature. Hence the problem: describe the class of orthodox semigroups which are obtained as groupoids over a band of objects in the manner shown there.

Background for 3.1

First note that every homomorphism (written as a right mapping) in a variety has a factorisation $f = hj$ with h surjective and j injective. If there are $h\colon A \to R$, $j\colon R \to A$ such that $jh = 1_R$, then h is called a *retraction* (and in fact is surjective) and j is a *section* (and is injective). R is a *retract* of A and the congruence $\rho_h := h \circ h^{-1}$ is a *coretract* of A. Of course $R \cong A/\rho_h$, so the distinction between retracts and coretracts is solely whether we consider R as a subalgebra or a quotient algebra. Put $f = hj$; then $jh = 1_R$ is equivalent to $f^2 = f$ (this is easy folklore).

Now (RI) and (UR) are as stated, but to unpick my little concealment, we could rephrase (UR) as: to each section $j\colon R \to A$ corresponds a unique right inverse $h\colon A \to R$. Then the duals are

(RI*) the join of two coretracts of A is a coretract of A;

(UR*) each retraction $h\colon A \to R$ has a unique section (left inverse).

I know no reference for my claim that CI: $S = \operatorname{End} A$ has commuting idempotents implies all of these; here's my proof.

Proof. Let two retracts of A be determined (as Af and Ag) by $f, g \in S$ with $f = f^2$, $g = g^2$, and $fg = gf$. Then $(fg)^2 = fg$ and Afg is a retract (and ρ_{fg} is a coretract). Clearly

$$Afg = Agf \subseteq Af, Ag$$

and so $Afg \subseteq Af \cap Ag$. But if $x \in Af \cap Ag$ then $x = xf = xg = xfg$, so $x \in Afg$. This proves $Af \cap Ag = Afg$, a retract, and establishes (RI). Similarly, any two coretracts of A are of the forms ρ_f and ρ_g, with f, g idempotents of $\operatorname{End} A$. Then $\rho_f \vee \rho_g \subseteq \rho_{fg}$ ($xf = x'f \Rightarrow xfg = x'fg$, etc.) But if $xfg = x'fg$, we have

$$x \;\rho_f\; xf \;\rho_g\; x'f \;\rho_f\; x' \qquad \text{and} \qquad x \;\rho_g\; xg \;\rho_f\; x'g \;\rho_g\; x',$$

and thus $(x, x') \in \rho_f \vee \rho_g$ whence $\rho_{fg} \subseteq \rho_f \vee \rho_g$. Thus $\rho_f \vee \rho_g = \rho_{fg}$, a coretract; so (RI*) holds.

Now for (UR). Suppose there are $f = f^2$, $g = g^2$ such that the retract $R = Af = Ag$. Then for any $x \in A$, there are $y, z \in A$ such that $xf = yg, xg = zf$. Then

$$\begin{aligned} xfg &= yg = xf \\ &= xgf = zf = xg, \end{aligned}$$

and so $f = g$ and we see (UR). To (UR*): Suppose there is $h\colon A \to R$ and sections $j, k\colon R \to A$ such that $jh = kh = 1_R$. Then hj, hk are idempotents in $\operatorname{End} A$ and we have in turn $hjhk = hkhj$, $hk = hj$, and $k = j$. ☐

4 Sam van Gool

Problem 4.1. What is the relationship, if any, between the duality for sheaf representations of distributive-lattice-ordered algebras of Gehrke and van Gool [6] and the duality for strongly distributive skew lattices of Bauer, Cvetko Vah, Gehrke, van Gool, and Kudryavtseva [1]? Sheaves play prominent but apparently distinct roles in these two dualities; any connection between the two could be an interesting direction for further research.

5 Michael Kinyon

5.1 Left regular bands and left-handed skew lattices as ordered sets

When I've given talks about skew lattices to nonexperts, one of the things that typically bothers members of the audience is that unlike lattices, skew lattices do not have a nice order theoretic definition.

There was one attempt to give such a characterization by Gerhardts [7], who studied a variety of noncommutative lattices he called *Fastverbanden*. They have an unintuitive definition, but it is not difficult to show that they are precisely what we now call left-handed binormal (normal and conormal) skew lattices. Gerhardts attempted to characterize *Fastverbanden* entirely in terms of their natural partial order and their natural preorder. Unfortunately, [7] has at least one mistake in it. However, the definitions can be modified to get his idea to work. For simplicity, I will just focus on bands here instead of skew lattices.

The basic structure $(X, \preceq, \leq)$ consists of a set X with a preorder $\preceq$ and a partial order $\leq$ which refines $\preceq$ (that is, $\leq\, \subseteq\, \preceq$). Let $\mathcal{L}$ denote the equivalence relation associated to $\preceq$. We assume the following:

$$\text{If } x \,\mathcal{L}\, y \text{ and } x, y \leq z, \text{ then } x = y. \qquad (*)$$

For $a, b \in X$, an element $c \in X$ is said to be a *lower bound* of the pair (a, b) if $c \leq a$ and $c \preceq b$. We say that c is an *infimum* of (a, b) if

(i) c is a lower bound of (a, b), and

(ii) if $x \preceq a, b$, then $x \preceq c$.

It is easy to prove that a pair (a, b) can have at most one infimum. (The definition of infimum is where I differ from Gerhardts.)

A *left normal band* $(X, \leq, \preceq)$ (in the order theoretic sense) is a structure as above such that every ordered pair (a, b) of elements has an infimum, which we denote by $a \wedge b$. We then can prove the expected theorem: left normal bands in the order theoretic sense are precisely the same as left normal bands in the algebraic sense.

It is easy to get the corresponding result for left-handed binormal skew lattices, and it is also not hard to generalize from the one-sided case to normal bands by using both the $\mathcal{L}$- and $\mathcal{R}$-preorders.

Left normality essentially comes from property $(*)$ above.

Problem 5.1. Generalize the above characterization to *left regular* bands. What property should replace $(*)$?

5.2 Quasilattices and paralattices

Recall that a double band $(X, \vee, \wedge)$ is a

- *quasilattice* if the natural preorders dualize each other,
- *paralattice* if the natural partial orders dualize each other.

Both classes of noncommutative lattices form varieties. Quasilattices are axiomatized by

$$x \wedge (y \vee x \vee y) \wedge x = x = x \vee (y \wedge x \wedge y) \vee x,$$

while paralattices are axiomatized by

$$(x \vee y \vee x) \wedge x \wedge (x \vee y \vee x) = x = (x \wedge y \wedge x) \vee x \vee (x \wedge y \wedge x).$$

(These are not the four identities axiomatizing paralattices in [12] but it is easy to see these two are equivalent to those.)

Let us say that a double band $(X, \vee, \wedge)$ is a ____*-lattice* if it satisfies the following implications:

$$\begin{aligned} x \leq_\wedge y \quad &\Rightarrow \quad y \preceq_\vee x \\ x \leq_\vee y \quad &\Rightarrow \quad y \preceq_\wedge x \end{aligned}$$

(Part of the problem here is to find a suitable name for this class of noncommutative lattices.) It is clear that ____-lattices include both quasilattices and paralattices as special cases. It is also straightforward to show that ____-lattices form a variety axiomatized by

$$x \wedge (x \vee y \vee x) \wedge x = x = x \vee (x \wedge y \wedge x) \vee x.$$

Problem 5.2 (Suggested by J. Leech)**.** Describe the join variety of the varieties of paralattices and quasilattices.

6 Jurij Kovič

Problem 6.1. Define the concept of (and develop a theory for) a covering of a (skew) lattice. Use the analogy with covering graphs (voltages).

Background

The concept of a covering graph. Definitions of *graphs*, *covering graphs* and *voltages* are in detail explained in [15, pp. 96–97]. Covering graphs were first introduced in [9] for *directed* graphs, but they may be applied also to *undirected* graphs, as in [16]. Lifting graph automorphisms from the underlying graph is explained in [14].

7 Ganna Kudryavtseva

Some problems on skew Boolean algebras are proposed below. For a concise background on skew Boolean algebras, please refer to [13].

Problem 7.1. Study objects obtained by relaxations of the definition of an SBA S. What if normality is dropped and one requires only S is symmetric, has 0 and $S/\mathcal{D}$ is a Boolean lattice? What if symmetricity is dropped and one requires only S is normal, has 0 and $S/\mathcal{D}$ is a Boolean lattice? These, in each case, are generalizations of SBAs. Study their structure. Do they form a variety (in each case)? How are they related to SBAs?

Problem 7.2. Let S be a LSBIA (left handed skew Boolean intersection algebra). Then S^* is a (locally compact) Boolean space so that there is its dual GBA, B. Is it possible to extend the assignment $S \mapsto B$ to a functor? What are adjoints, if exist? What can be said about the relationship between $S/\mathcal{D}$ and B?

Problem 7.3. There is a forgetful functor from Boolean inverse semigroups to LSBAs (indeed, domain and range maps of an étale groupoids are étale maps). Does it have a left adjoint? Then how is it constructed? A similar question can be asked also about Boolean inverse $\wedge$-semigroups and LSBIAs.

8 Tomaž Pisanski

Problem 8.1. Given a finite lattice K. Find a test that will tell whether K is isomorphic to a congruence lattice $\mathrm{Con}(N)$ of some antilattice.

Problem 8.2. Any automorphism α of N induces an automorphism $\hat{\alpha}$ of $\mathrm{Con}(N)$. Hence, there is an obvious group homomorphism $h\colon \ \mathbf{Aut}\, N \to \mathbf{Aut}\, \mathrm{Con}(N)$ mapping α to $\hat{\alpha}$. Classify those antilattices N for which h is an isomorphism.

Problem 8.3. Let us call a lattice L *transitive* if for each pair of maximal chains a and b of L there exists an automorphism of L that maps one to the other one, i.e. it maps a into b. We call an antilattice N *transitive*, if its congruence lattice $\mathrm{Con}(N)$ is transitive. Classify transitive antilattices.

Background

For an antilattice N, a congruence θ is an equivalence relation on N that is compatible with both $\wedge$ and $\vee$. Congruence relations form a lattice, $\mathrm{Con}(N)$.

9 Joao Pita Costa

Problem 9.1. Can we build on the coset structure of a skew distributive lattice to characterize skew Heyting algebras through their coset structure. And what can be said about the coset structure of skew Boolean algebras? Will it give us some insights on the respective dual spaces?

Problem 9.2. Can we always describe coset law-like identities that can determine a variety of skew lattices? Or are there varieties of skew lattices that do not permit such characterization?

Problem 9.3. The index theorems presented for some varieties of skew lattices in [3, 17] and [19] show a combinatorial perspective on these algebras as a direct consequence of their coset structure. Can we always obtain such results? What can they bring to the study of skew lattices?

For a concise background on the coset structure of several varieties of skew lattices and how they relate to each other, and to their index theorems, please refer to [18].

10 Andreja Tepavčević

Problem 10.1. *Representation of algebraic lattices by a weak congruence lattice*. All weak congruences form an algebraic lattice under inclusion. The diagonal relation (equality) has a particular role within weak congruence lattice. A 30-years unsolved problem [22] is as follows: if we take an algebraic lattice L and pick an element a, under which conditions there is an algebra such that its weak congruence lattice is isomorphic to L and the diagonal element corresponds to the fixed element a under this isomorphism. This is not possible for every element in a lattice, since this element should be codistributive and should satisfy some particular conditions. For further background please read [4] and [23].

Problem 10.2. *Congruence lattice characterization*. Characterize congruence lattices (and later weak congruence lattices) of skew lattices. It is well-known [8] that the congruence lattice of a lattice is distributive since there is the corresponding Mal'cev term. Is there a similar result for skew lattices?

Background

Congruences are compatible equivalence relations. Weak congruences are compatible relations satisfying the same as congruences except they are not reflexive. Weak congruences are congruences on subalgebras of an algebra A (equivalently - symmetric, transitive and compatible relations of A).

11 Russ Woodroofe

Problem 11.1. What is the correct universal algebra generalization of order congruence lattices? More broadly: find examples of comodernistic lattices!

Background

An element m of a lattice L is *left-modular* if for every pair $x \leq y$ of elements in L, we can write $x \vee m \wedge y$ without parentheses. That is, m is left-modular if

$$\forall x \leq y, \quad (x \vee m) \wedge y = x \wedge (m \wedge y)\,.$$

Left-modularity is a lattice-theoretic generalization of the Dedekind modular identity of group theory.

A lattice L is *comodernistic* if every nontrivial interval $[a, b]$ of the lattice has a coatom that is left-modular in the interval. That is, for every $a < b$ in L, there is an m covered by b, so that m satisfies the left-modular relation with respect to every pair $a \leq x \leq y \leq b$. Jay Schweig and myself introduced comodernistic lattices in [20].

Comodernistic lattices generalize dual semimodular lattices, which can be defined as those lattices where *every* coatom of every interval is left-modular. Comodernistic lattices also generalize supersolvable lattices, which are those graded lattices which have a maximal chain consisting of left-modular elements. There are additional examples:

Example 11.2. The subgroup lattice of any finite solvable group is comodernistic. This follows from the fact that normal subgroups are left-modular, by the Dedekind modular identity.

Example 11.3. Similarly, the subalgebra lattice of any finite Lie algebra is comodernistic.

Example 11.4. The *order congruence lattice* $\mathcal{O}(P)$ of a finite poset P consists of all the level sets of all the order preserving maps with domain P, ordered by refinement. Order congruence lattices are comodernistic, as can be seen in two steps:

(1) Let μ be the partition with a maximal element in a singleton block, and all other elements together in a big block. Then μ is left-modular.

(2) Intervals in $\mathcal{O}(P)$ are direct sums of order congruence lattices of quotients of subposets of P. Existence of modularity in coatoms of quotients follows from (1). Existence of left-modular coatoms is preserved by direct sum.

Back to the Problem

I'm interested in finding comodernistic lattices (or dually, *modernistic* lattices) in other settings. Perhaps there is a wider class of congruence lattices from universal algebra or similar which yields comodernistic lattices?

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