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Gluons in Point Form QCD

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Abstract. Point form quantum field theory is used to analyze the QCD gluon vacuum and bound state problems. An algebra of operators formed from gluon creation and annihilation operators is used to generate a total four momentum operator from the gluon self coupling terms. The vacuum is then the Lorentz invariant state which is annihilated by the four-momentum operator. Such a state is obtained from the generalization of the coupled-cluster technique, familiar from nuclear physics. An example in which the color symmetry is SU(2) is given.

1 Point Form Quantum Field Theory

In point form field theory [1] all interactions are in the four-momentum operator and Lorentz transformations are kinematic. Interactions are introduced via vertices, products of local free fields, which are integrated over the forward hyperboloid to give the interacting four-momentum operator.

The four-momentum operator P^μ is written as the sum of free and interacting four-momentum operators, $P^\mu = P^\mu(fr) + P^\mu(I)$. To guarantee the relativistic covariance of the theory, it is required that

$$[P^\mu, P^\nu] = 0, \tag{1}$$

$$U_\Lambda P^\mu U_\Lambda^{-1} = (\Lambda^{-1})^\mu_\nu P^\nu, \tag{2}$$

where U_Λ is the unitary operator representing the Lorentz transformation Λ. These "point form" equations [1] lead to the eigenvalue problem

$$P^\mu|\Psi_p> = p^\mu|\Psi_p>, \tag{3}$$

where p^μ is the four-momentum eigenvalue and $|\Psi_p>$ the eigenvector of the four-momentum operator, which acts in generalized fermion-antifermion-boson Fock spaces. Then the physical vacuum and physical bound and scattering states should all arise as the appropriate solutions of the eigenvalue Eq.(3). What is unusual in Eq.(3) is that the momentum operator has interaction terms. But since the momentum and energy operators commute and can be simultaneously diagonalized, they have common eigenvectors. One of the important properties of the point form is that since the Lorentz generators have no interactions, the action of global Lorentz transformations on operators and states is simple.

2 Gluons

Gluons are massless vector particles that transform as representations of the little group E(2), the euclidean group in two dimensions [2]; a four dimensional nonunitary irrep of E(2) generates four polarization degrees of freedom, labeled by α. A standard four-vector $k^{st} = (1, 0, 0, 1)$ leaves E(2) invariant and the helicity boost, $B(k)$, which gives the four-momentum k, generates a gluon state with transformation properties

$$\begin{aligned}
|k, \alpha, a >: &= U_{B(k)}|k^{st}, \alpha, a >, \\
U_\Lambda|k, \alpha, a > &= U_\Lambda U_{B(k)}|k^{st}, \alpha, a > \\
&= \sum |\Lambda k, \alpha', a > \Lambda_{\alpha'\alpha}(e_W), \\
U_g|k, \alpha, a > &= \sum |k, \alpha, a' > D_{a'a}(g),
\end{aligned}$$

where $\Lambda(e_W) = B^{-1}(\Lambda k)\Lambda B(k)$ is a euclidean Wigner "rotation", g is an element of the internal symmetry (color) group and a, a' are color indices.

Many gluon states are most simply obtained from gluon creation and annihilation operators:

$$\begin{aligned}
|k, \alpha, a > &= g^\dagger(k, \alpha, a)|0 > \\
g(k, \alpha, a)|0 > &= 0, \forall k, \alpha, a \\
[g(k, \alpha, a), g^\dagger(k', \alpha', a')] &= -g_{\alpha,\alpha'} k_0 \delta^3(\mathbf{k} - \mathbf{k}')\delta_{aa'} \\
U_\Lambda g(k, \alpha, a)U_\Lambda^{-1} &= \sum g(\Lambda k, \alpha', a)\Lambda_{\alpha,\alpha'}(e_W) \\
U_g g(k, \alpha, a)U_g^{-1} &= \sum g(k, \alpha, a')D_{a'a}(g) \\
P^\mu_{free} &= -\sum \int dk k^\mu g^\dagger(k, \alpha, a) g_{\alpha,\alpha} g(k, \alpha, a);
\end{aligned}$$

the auxiliary condition eliminating the 0 and 3 components is the annihilation operator condition, $\sum k^{st}_\alpha g_{\alpha\alpha} g(k, \alpha, a) = 0$. $dk := \frac{d^3k}{2k}$.

The free gluon field is then

$$\begin{aligned}
G^\mu_a(x) &= \int dk B^{\mu\alpha}(k)(e^{-ik\cdot x} g(k, \alpha, a) + e^{ik\cdot x} g^\dagger(k, \alpha, a)); \\
\partial G^+_{\mu,a}(x)/\partial x_\mu &= i \int dk k^\mu B_{\mu\alpha}(k) g_{\alpha,\alpha} e^{-ik\cdot x} g(k, \alpha, a) \\
&= i \sum \int dk e^{-ik\cdot x} k^{st}_\alpha g_{\alpha\alpha} g(k, \alpha, a) \\
&= 0;
\end{aligned}$$

this last relation shows the connection between the auxiliary condition and the Lorentz gauge condition. In fact, because Lorentz transformations are kinematic, the only gauge transformations allowed are those that leave the Lorentz gauge condition invariant.

Gauge invariance then fixes the field tensor to be

$$F^{\mu\nu}_a(x) = \frac{\partial G^\nu_a}{\partial x_\mu} - \frac{\partial G^\mu_a}{\partial x_\nu} - \alpha c_{a,b,c} G^\mu_b(x) G^\nu_c(x)$$

where $c_{a,b,c}$ are the color structure constants and α is the strong bare coupling constant.

By integrating the stress energy tensor generated by the field operators over the forward hyperboloid, the pure glue part of the four-momentum operator takes on the form

$$
\begin{aligned}
P^{\mu}_{glue} &= P^{\mu}_{ke} + P^{\mu}_{tri} + P^{\mu}_{quar} \\
P^{\mu}_{quar} &= \alpha^2 \sum \int dx^{\mu} dk_1 dk_2 dk_3 dk_4 c_{a,b,c} c_{a,b',c'} \\
&\quad B^{\mu\alpha_1}(k_1) B^{\nu\alpha_2}(k_2) B^{\alpha_3}_{\mu}(k_3) B^{\alpha_4}_{\nu}(k_4) \\
&\quad (e^{-ik_1\cdot x} g(k_1,\alpha_1,b) + e^{ik_1\cdot x} g^{\dagger}(k_1,\alpha_1,b)) \\
&\quad (e^{-ik_2\cdot x} g(k_2,\alpha_2,c) + e^{ik_2\cdot x} g^{\dagger}(k_2,\alpha_2,c)) \\
&\quad (e^{-ik_3\cdot x} g(k_3,\alpha_3,b^{'}) + e^{ik_3\cdot x} g^{\dagger}(k_3,\alpha_3,b^{'})) \\
&\quad (e^{-ik_4\cdot x} g(k_4,\alpha_4,c^{'}) + e^{ik_4\cdot x} g^{\dagger}(k_4,\alpha_4,c^{'})) \\
P^{\mu}_{tri} &= i\alpha \sum c_{a,b,c} \int dx^{\mu} dk_1 dk_2 dk_3 \\
&\quad (B^{\nu\alpha_1}(k_1) k^{\mu}_1 - B^{\mu\alpha_1}(k_1) k^{\nu}_1) B^{\alpha_2}_{\mu}(k_2) B^{\alpha_3}_{\nu}(k_3) \\
&\quad (e^{-ik_1\cdot x} g(k_1,\alpha_1,a) - e^{ik_1\cdot x} g^{\dagger}(k_1,\alpha_1,b)) \\
&\quad (e^{-ik_2\cdot x} g(k_2,\alpha_2,b) + e^{ik_2\cdot x} g^{\dagger}(k_2,\alpha_2,b)) \\
&\quad (e^{-ik_3\cdot x} g(k_3 \alpha_3,c) + e^{ik_3\cdot x} g^{\dagger}(k_3,\alpha_3,c)) \\
P^{\mu}_{ke} &= -\sum \int dk k^{\mu} g^{\dagger}(k,\alpha,a) g_{\alpha\alpha} g(k,\alpha,a)
\end{aligned}
$$

3 The Gluon Vacuum Equations

Neglecting the quark sector, the gluon vacuum structure can be analyzed by writing the vacuum as $|\Omega >= F|0>$ so that $P^{\mu}_{glue}|\Omega >= P^{\mu}_{glue}F|0>= 0$. Since there are no quarks, the operator F will act only in the gluon space; it must satisfy the properties of being invariant under Lorentz and color transformations. So write

$$
\begin{aligned}
F &= f_0 I + \sum \int dk_1 dk_2 f_{k_1,\beta_1,a-1;k_2\beta_2,a_2} \\
&\quad g^{\dagger}(k_1,\beta_1,a_1) g^{\dagger}(k_2,\beta_2,a_2) + \ldots \\
f_{k_1,\beta_1,a_1;k_2,\beta_2,a_2} &= f_2((k_1+k_2)^2) B^{\mu\beta_1}(k_1) B^{\beta_2}_{\mu}(k_2) C^{1}_{1,a_1,a_2}
\end{aligned}
$$

where $f_2((k_1+k_2)^2)$ is a Lorentz invariant function and C^{1}_{1,a_1,a_2} is a Clebsch-Gordan coefficient coupling the adjoint representation to itself to give the identity representation. There are no odd powers of gluon creation operators because of invariance under the internal symmetry. When quarks are coupled to the gluon sector, this will no longer be the case.

As an example of the structure of the gluon vacuum equations, choose SU(2) as the internal symmetry. The tri interactions do not contribute when acting on F; the general form of the equations arising from the quartic interactions are

$$
(\alpha^2 \int dx^{\mu} (e^{-ik\cdot x} g + e^{ik\cdot x} g^{\dagger})^4 - \int dv v^{\mu} g^{\dagger} g) F|0> = 0.
$$

The lowest order equation resulting from these equations is

$$\alpha^2 \int dx^\mu [\int dk_1 dk_2 f_0 - 4 \int dk_1 dk_2 dk_3 f_2(k_2 + k_3) e^{-i(k_2+k_3)\cdot x}$$
$$+8 \int dk_1 dk_2 dk_3 dk_4 f_4(k_1 + k_2, k_3 + k_4) e^{-i(k_1+k_2+k_3+k_4)\cdot x}] = 0;$$

where the $f's$ are Lorentz invariant functions of their arguments. More generally there is a hierarchy of equations in even powers of the gluon creation operators. By factoring out the infinite Lorentz volume at each level of the hierarchy, a set of recursive equations results, which have no infinities.

4 Glueballs

The simplest glueballs are bound states of two gluons, bound by their self interactions [3]. A two gluon state can be written as $|\nu, |\mathbf{k}|, j, \sigma, \lambda_1, \lambda_2 >$, where

$$|\nu, |\mathbf{k}|, j, \sigma\lambda_1\lambda_2 > = \int dR \sum C^1_{1,a_1 a_2} D^j_{\sigma, \lambda_1 - \lambda_2}(R)\, U_{B(\nu)} U_R$$
$$g^\dagger(k_1 \alpha_1 a_1) g^\dagger(k_2 \alpha_2 a_2) F|0 >,$$

with $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}$, $C^1_{1,a_1 a_2}$ a color Clebsch Gordan coefficient, and R a rotation.

Again a set of (bound state) equations in powers of the gluon creation operators results, generated from

$$(P^\mu - \lambda^\mu_\nu)|\nu, k, j, \sigma\lambda_1\lambda_2 > = 0.$$

Setting $j = \sigma = 0$ gives a scalar glueball.

References

1. E. P. Biernat, et al, Ann. Phys. 323 (2008) 1361.
2. This formalism is worked out in W. H. Klink, Nucl. Phys. A716 (2003) 158.
3. Potential models for glueballs can be found in V. Mathieu, et al, Phys. Rev. D 77 (2008)114022 and references cited therein.