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12 Relations Between Clifford Algebra and Dirac Matrices ⋆

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Abstract. In the *spin-charge-family* theory [2–7] there are $\forall n \in \mathbb{N}$, 2^d Clifford operators, forming the vector space. Space can have for given $n \in \mathbb{N}$ dimension $d = 2(2n+1)$ or $4n$. Half of them are Clifford odd operators with the properties of fermion creation and annihilation operators for $2^{\frac{d}{2}-1}$ family members of $2^{\frac{d}{2}-1}$ families, fulfilling for each momentum p_k the anticommutation relations for the second quantized fermions [8]. Families in Clifford space are reachable by $\tilde{S}^{ab} = \frac{1}{2}\tilde{\gamma}^a\tilde{\gamma}^b$, $a \neq b$ and family members by $S^{ab} = \frac{i}{2}\gamma^a\gamma^b$, $a \neq b$. In this paper the basis in $d = (3+1)$ Clifford space is discussed, chosen in a way that the matrix representation of γ^a and of generators of the Lorentz transformations in internal space, S^{ab}, coincide for each family quantum number, determined by $\tilde{S}^{ab}$, with Dirac matrices. The appearance of charges in Clifford space is discussed by embedding $d = (3+1)$ space into $d = (5+1)$-dimensional space.

Povzetek. V teoriji *spina-naboja-družin* [2–7] je v d dimenzionalnem prostoru 2^d Cliffordovih operatorjev, ki določajo vektorski prostor. Teorija izbere $d \geq (13+1)$. Če uredimo vektorski prostor tako, da so vektorji lastni vektorji Cartanove podalgebre Lorentzove grupe, izpolnjujejo lihi Cliffordovi vektorji $2^{\frac{d}{2}-1}$ družin s po $2^{\frac{d}{2}-1}$ člani vse Diracove pogoje za fermione v drugi kvantizaciji. Družinske člane določajo generatorji Lorentzove grupe $\tilde{S}^{ab}$ $(= \frac{1}{2}\tilde{\gamma}^a\tilde{\gamma}^b$, $a \neq b)$, družine pa $S^{ab} = \frac{i}{2}\gamma^a\gamma^b$, $a \neq b$.

V tem prispevku predstavijo avtorji bazo v $d = (3+1)$ razsežnem Cliffordovem prostoru ter matrično upodobitev za operatorje γ^a, S^{ab}, $\tilde{S}^{ab}$, $\tilde{\gamma}^a$ ter $\tilde{S}^{ab}$. $d = (3+1)$ razsežni Cliffordov prostor vgradijo v prostor $d = (5+1)$ ter komentirajo pojav naboja fermionov v $d = (3+1)$.

12.1 Introduction

In the Grassmann graded algebra of anticommuting coordinates θ^a there are in d-dimensional space 2^d vectors, which define, together with the corresponding derivatives $\frac{\partial}{\partial\theta_a}$, two kinds of the Clifford algebra objects: γ^a and $\tilde{\gamma}^a$ [2,6–8], both with the anticommutation properties of the Dirac γ^a matrices, while the

⋆ Talk presented by N.S. Mankoč Borštnik anticommutators among γ^a and $\tilde{\gamma}^b$ are equal to zero.

$$\begin{aligned}
\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \,, \quad \{\gamma^a, \tilde{\gamma}^b\}_+ = 0 \,, \\
(\gamma^a)^\dagger &= \eta^{aa}\gamma^a \,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a \,, \\
S^{ab} &= \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a) \,, \quad \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a) \,, \\
\{S^{ab}, \tilde{S}^{ab}\}_+ &= 0 \,, \\
(a, b) &= (0, 1, 2, 3, 5, \cdots, d) \,.
\end{aligned} \tag{12.1}$$

The two Clifford algebras, γ^a's and $\tilde{\gamma}^a$'s, are obviously completely independent and form two independent spaces, each with 2^d vectors [9].

Sacrificing the space of $\tilde{\gamma}^a$'s by defining

$$\tilde{\gamma}^a B(\gamma^a) = (-)^B \, i \, B\gamma^a \,, \tag{12.2}$$

with $(-)^B = -1$, if B is an odd product of γ^a's, otherwise $(-)^B = 1$ [7], we end up with vector space of 2^d degrees of freedom, defined by γ^a's only.

A general vector can correspondingly be written as

$$\mathbf{B} = a_0 + \sum_{k=1}^{d} a_{a_1 a_2 \ldots a_k} \gamma^{a_1}\gamma^{a_2} \ldots \gamma^{a_k} |\psi_o> , a_i < a_{i+1} \,, k = 1, \ldots, d \tag{12.3}$$

where $|\psi_o>$ is the vacuum state.

We arrange these vectors as products of nilpotents and projectors

$$\begin{aligned}
\overset{ab}{(k)} &:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) \,, \quad (\overset{ab}{(k)})^2 = 0 \,. \\
\overset{ab}{[k]} &:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) \,, \quad (\overset{ab}{[k]})^2 = \overset{ab}{[k]} \,,
\end{aligned} \tag{12.4}$$

where $k^2 = \eta^{aa}\eta^{bb}$. Their Hermitian conjugated values follow from Eq. (12.1).

$$\overset{ab}{(k)}{}^\dagger = \eta^{aa} \overset{ab}{(-k)}, \quad \overset{ab}{[k]}{}^\dagger = \overset{ab}{[k]} \,. \tag{12.5}$$

Vectors in Clifford space are chosen to be eigenstates of the Cartan subalgebra, Eq. (12.6), of the generators of the Lorentz transformations S^{ab} in the internal space of γ^a's.

$$\begin{aligned}
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\; d} \,, \\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\; d} \,,
\end{aligned} \tag{12.6}$$

with the eigenvalues $S^{ab} \overset{ab}{(k)} = \frac{1}{2}k \overset{ab}{(k)}$, $S^{ab} \overset{ab}{[k]} = \frac{1}{2}k \overset{ab}{[k]}$. All the relations of Eq. (12.1) remain unchanged after the assumption of Eq. (12.3), while each irreducible representation of the Lorentz algebra S^{ab} receives the additional quantum number f, defined by $\tilde{S}^{ab}$.

$$\begin{aligned}
S^{ab} \overset{ab}{(k)} &= \frac{k}{2} \overset{ab}{(k)} \,, \qquad \tilde{S}^{ab} \overset{ab}{(k)} = \frac{k}{2} \overset{ab}{(k)} \,, \\
S^{ab} \overset{ab}{[k]} &= \frac{k}{2} \overset{ab}{[k]} \,, \qquad \tilde{S}^{ab} \overset{ab}{[k]} = -\frac{k}{2} \overset{ab}{[k]} \,.
\end{aligned} \tag{12.7}$$

Eq. (12.7) demonstrates that the eigenvalues of S^{ab} on nilpotents and projectors generated by γ^a's differ from the eigenvalues of $\tilde{S}^{ab}$.

States, which are products of projectors and nilpotents, have well defined handedness of both kinds, $\Gamma^{(d)}$ and $\tilde{\Gamma}^{(d)}$.

$$\Gamma^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}}\gamma^a), \quad \text{if} \quad d = 2n,$$
$$\tilde{\Gamma}^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}}\tilde{\gamma}^a), \quad \text{if} \quad d = 2n. \tag{12.8}$$

The *spin-charge-family* theory [2–7] of N.S. Mankoč Borštnik uses products of nilpotents, $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)$, and projectors, $\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)$, to define 2^d vectors in this space of the Clifford graded algebra [3–5]. In this theory S^{ab} determine in $d = (3+1)$ space, which is a part of $d = (13+1)$-dimensional space, spins and charges of quarks and leptons, while $\tilde{S}^{ab}$ determine families of quarks and leptons.

It is interesting to notice ([9,8] and references therein): *Vectors, which are superposition of odd products of nilpotents and projectors, anticommute fulfilling the anticommutation relations postulated by Dirac [1] for second quantized fermions, explaining correspondingly Dirac's postulate* [9,8].

In Sect. 12.2 the properties of products of nilpotents and projectors are discussed, arranged in eigenvectors of the Cartan subalgebra, defining the internal vector space of fermions in d-dimensional space when $d = (3+1)$-dimensional space is embedded into $d = (5+1)$-dimensional space, so that the spin in $d = (5,6)$ determines the charge of fermions in $d = (3+1)$.

In Sect. 12.2.3 the matrix representation of vectors are presented.

12.2 Properties of vectors in Clifford space

In Refs. [9,8] the fact that the Clifford vectors, spanned by products of an odd number of γ^a's, fulfill the anticommutation relations postulated by Dirac for the second quantized fermions, explains these Dirac's anticommutation relations. Let us see on the case that $d = (5+1)$ how this happens.

Let us denote vectors in $d = (5+1)$, presented in Table 12.1 as products of three nilpotents or projectors or both, by $\hat{b}^{f\dagger}_m$, $m = (ch, s)$, the member quantum number m includes the charge, ch and the spin s. The corresponding Hermitian conjugated partner is denoted by $(\hat{b}^{f\dagger}_m)^\dagger = \hat{b}^f_m$.

The first member $m = (\frac{1}{2}, \frac{1}{2})$ of the first family a, which is the product of three nilpotents, is correspondingly denoted by $\hat{b}^{a\dagger}_{(\frac{1}{2},\frac{1}{2})} = \overset{03}{(+i)}\overset{12}{(+)} \mid \overset{56}{(+)}$. All the rest vectors of the family $f = a$ follow by the application of S^{ab}. The families $f = (b, c, d)$ follow from $f = a$ by the application of $\tilde{S}^{ab}$. The Hermitian conjugated partners follow by the application of Eq. (12.1).

Table 12.1, taken from Table IV of Ref. [8], represents four families of Clifford odd vectors and their Hermitian conjugated partners. All the families have the same quantum numbers m of the corresponding members, (S^{03}, S^{12}, S^{56}), each family carries its own family quantum number f.

f(amily) m	(ch, s)	$\hat{b}^{f\dagger}_m$	$\hat{b}^{f}_m$	S^{03}	S^{12}	S^{56}	Γ^{3+1}	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$
a 1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\mid\stackrel{56}{(+)}$	$(-)\stackrel{56}{(-)}\mid(-)\stackrel{12}{(-)}\stackrel{03}{(-i)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
a 2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\mid\stackrel{56}{(+)}$	$(-)\stackrel{56}{(-)}\mid\stackrel{12}{[-]}\stackrel{03}{[-i]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
a 3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\mid\stackrel{56}{[-]}$	$\stackrel{56}{[-]}\mid(-)\stackrel{12}{(-)}\stackrel{03}{[-i]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
a 4	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\mid\stackrel{56}{[-]}$	$\stackrel{56}{[-]}\mid\stackrel{12}{[-]}\stackrel{03}{(-i)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
b 1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\mid\stackrel{56}{(+)}$	$(-)\stackrel{56}{(-)}\mid\stackrel{12}{[+]}\stackrel{03}{[+i]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
b 2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{(-)}\mid\stackrel{56}{(+)}$	$(-)\stackrel{56}{(-)}\mid(-)\stackrel{12}{(+)}\stackrel{03}{(+i)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
b 3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{[+]}\mid\stackrel{56}{[-]}$	$\stackrel{56}{[-]}\mid\stackrel{12}{[+]}\stackrel{03}{(+i)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
b 4	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{(-)}\mid\stackrel{56}{[-]}$	$\stackrel{56}{[-]}\mid(-)\stackrel{12}{(+)}\stackrel{03}{[+i]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
c 1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\mid\stackrel{56}{[+]}$	$\stackrel{56}{[+]}\mid(-)\stackrel{12}{(-)}\stackrel{03}{[+i]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
c 2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{[-]}\mid\stackrel{56}{[+]}$	$\stackrel{56}{[+]}\mid\stackrel{12}{[-]}\stackrel{03}{(+i)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
c 3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(-i)}\stackrel{12}{(+)}\mid\stackrel{56}{(-)}$	$(-)\stackrel{56}{(+)}\mid(-)\stackrel{12}{(-)}\stackrel{03}{(+i)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
c 4	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[+i]}\stackrel{12}{[-]}\mid\stackrel{56}{(-)}$	$(-)\stackrel{56}{(+)}\mid\stackrel{12}{[-]}\stackrel{03}{[+i]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
d 1	$(\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}$	$\stackrel{56}{[+]}\mid\stackrel{12}{[+]}\stackrel{03}{(-i)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
d 2	$(\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}$	$\stackrel{56}{[+]}\mid(-)\stackrel{12}{(+)}\stackrel{03}{[-]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
d 3	$(-\frac{1}{2},\frac{1}{2})$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}$	$(-)\stackrel{56}{(+)}\mid\stackrel{12}{[+]}\stackrel{03}{[-i]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
d 4	$(-\frac{1}{2},-\frac{1}{2})$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}$	$(-)\stackrel{56}{(+)}\mid(-)\stackrel{12}{(+)}\stackrel{03}{(-i)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Table 12.1. The basic creation operators, which are sums of odd products of γ^a's — $\hat{b}^{f\dagger}_m$, $m = (ch, s)$, ch represents the spin in $d = (5, 6)$, manifesting in $d = (3+1)$ as the charge, and s represents the spin in d=(1,2), according to the choice of the Cartan subalgebra, Eq. (12.6) — and their annihilation partners — $\hat{b}^{f}_m$ — are presented for the $d = (5+1)$-dimensional case. The basic creation operators are the products of nilpotents and projectors, which are the "eigenstates" of the Cartan subalgebra generators, (S^{03}, S^{12}, S^{56}) and ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$), presented in Eq. (12.6). The Clifford odd parts of creation operators, belonging to $d = (3+1)$ space, are marked.

Half of vectors, the eigenvectors of the Cartan subalgebra, Eq. (12.6), which are products of nilpotents and projectors, are odd products of γ^a's and half of them are even products of γ^a's. On Table 12.1 only Clifford odd vectors are presented.

Let us make a choice of the vacuum state [6–9]. (In the case of a general even d the normalization factor is $\frac{1}{\sqrt{2^{\frac{d}{2}-1}}}$, since the vacuum states, generated by projectors only, follows from the starting products of $\frac{d}{2}$ projectors, let say $\stackrel{03}{[-i]}\stackrel{12}{[-]}\mid\stackrel{56}{[-]}\stackrel{d-1\,d}{[-]}$), by changing all possible pairs of $[-]\ldots[-]$, with $[-i]$ included, to $[+]\ldots[+]$, leading therefore to $2^{\frac{d}{2}-1}$ summands.

$$|\psi_o> = (\frac{1}{\sqrt{2}})^2\,(\stackrel{03}{[-i]}\stackrel{12}{[-]}\mid\stackrel{56}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\mid\stackrel{56}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[-]}\mid\stackrel{56}{[+]} + \stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]})\,|1> \;. \tag{12.9}$$

It then follows that

$$
\begin{aligned}
&\hat{b}^{f\dagger}_{m}\,|\psi_o> = |\psi^{f}_{m}>,\\
&\hat{b}^{f}_{m}\,|\psi_o> = 0\,|\psi_o>,\\
&\{\hat{b}^{f\dagger}_{m}, \hat{b}^{f'}_{m'}\}_{+} = \delta^{ff'}\delta_{mm'}\,|\psi_o>,\\
&\{\hat{b}^{f\dagger}_{m}, \hat{b}^{f'\dagger}_{m'}\}_{+} = 0\,|\psi_o>,\\
&\{\hat{b}^{f}_{m}, \hat{b}^{f'}_{m'}\}_{+} = 0\,|\psi_o>,\\
&\forall\, m \,\text{and}\, \forall\, f\,.
\end{aligned}
\tag{12.10}
$$

Eq. (12.10) represents all the requirements for the second quantized fermions.

12.2.1 Action

The action for a free massless fermion is needed and the corresponding equations of motion to take into account the ordinary space as well.

The Lorentz invariant action for a free massless fermion in Clifford space is well known

$$\mathcal{A} = \int d^d x\, \frac{1}{2}\,(\psi^{\dagger}\gamma^0\,\gamma^a p_a\psi) + h.c.\,, \tag{12.11}$$

$p_a = i\,\frac{\partial}{\partial x^a}$, leading to the equations of motion

$$\gamma^a p_a|\psi> = 0\,, \tag{12.12}$$

which fulfill also the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b|\psi> = p^a p_a|\psi> = 0\,, \tag{12.13}$$

for each of the basic vectors $|\psi^m_f> = \hat{b}^{f\dagger}_{m}\,|\psi_o>$. ($\gamma^0$ appears in the action to take care of the Lorentz invariance of the action.)

Solutions of equtions of motion, Eq. (12.12), for a free massless fermions with momentum $p^a = (p^0, p^1, p^2, p^3, 0, 0)$ and a particular charge $\pm\frac{1}{2}$, are superposition of vectors with spin $\frac{1}{2}$ and $-\frac{1}{2}$, multiplied by the plane wave $e^{-ip_a x^a}$. Coefficients in superposition depend on the momentum p^a.

12.2.2 Creation and annihilation operators in $d = (3+1)$ space embedded in $d = (5+1)$

The creation and annihilation operators of Table 12.1 are all of an odd Clifford character (they are superposition of odd products of γ^a's). The rest of 2^4 creation operators of an even Clifford character can be found in Refs. [9,8].

Taking into account Eq. (12.1) one recognizes that γ^a transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[-k]}$, never to $\stackrel{ab}{[k]}$, while $\tilde{\gamma}^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[k]}$, never to $\stackrel{ab}{[-k]}$

$$
\begin{aligned}
&\gamma^a \stackrel{ab}{(k)} = \eta^{aa} \stackrel{ab}{[-k]},\; \gamma^b \stackrel{ab}{(k)} = -ik \stackrel{ab}{[-k]},\; \gamma^a \stackrel{ab}{[k]} = \stackrel{ab}{(-k)},\; \gamma^b \stackrel{ab}{[k]} = -ik\eta^{aa} \stackrel{ab}{(-k)}\,,\\
&\tilde{\gamma^a} \stackrel{ab}{(k)} = -i\eta^{aa} \stackrel{ab}{[k]},\; \tilde{\gamma^b} \stackrel{ab}{(k)} = -k \stackrel{ab}{[k]},\; \tilde{\gamma^a} \stackrel{ab}{[k]} = \; i \stackrel{ab}{(k)},\; \tilde{\gamma^b} \stackrel{ab}{[k]} = -k\eta^{aa} \stackrel{ab}{(k)}
\end{aligned}
\tag{12.14}
$$

With the knowledge presented in Eq. (12.14) it is not difficult to reproduce Table 12.2, representing vectors that belong to $d = (3+1)$ space. Vectors carry no charge and have either an odd or an even Clifford character. Multiplying these vectors by the appropriate charge (that is by either the nilpotent— if the $d = (3+1)$ part has an even Clifford character — or the projector — if the $d = (3+1)$ part has an odd Clifford character — both must be the eigenfunction of S^{56}) we end up with the Clifford odd vectors from Table 12.1.

The properties of vectors of Table 12.2 are analyzed in details in order that the correspondence with the Dirac γ matrices in $d = (3+1)$ space would be easy to recognize. Superposition of vectors with the spin $\pm\frac{1}{2}$ (either Clifford even or odd) solve the equations of motion, Eq. (12.12), for free massless fermions.

As seen in Table 12.2 γ^a as well as $\tilde{\gamma}^a$ change the handedness of states. S^{ab}, which do not belong to Cartan subalgebra, generate all the states of one representation of particular handedness, Eq. (12.8), and particular family quantum number. $\tilde{S}^{ab}$, which do not belong to Cartan subalgebra, transform a family member of one family into the same family member of another family, $\tilde{\gamma}^a$ change the family quantum number as well as the handedness $\tilde{\Gamma}^{(3+1)}$, Eq. (12.8).

Dirac matrices γ^a and S^{ab} do not distinguish among the families: Corresponding family members of any family have the same properties with respect to S^{ab} and γ^a, manifesting for $d = (3+1)$ space four times twice 2×2 by diagonal matrices, which are, up to a phase, identical. The operators γ^a and S^{ab} are correspondingly four times 4×4 matrices.

One finds that half of vectors of Table 12.2 are Hermitian conjugated to each other. In the Clifford odd part of Table 12.2 one finds that $\hat{b}^{a\dagger}_{m=(3,4)}$ ($\stackrel{03}{[-i]}\stackrel{12}{(+)}$, $\stackrel{03}{(+i)}\stackrel{12}{[-]}$) have as the Hermitian conjugated partners $\hat{b}^{(d,c)}_{m=2}$ ($\stackrel{03}{[-i]}\stackrel{12}{(-)}$, $\stackrel{03}{(-i)}\stackrel{12}{[-]}$), respectively. And $\hat{b}^{b\dagger}_{m=(3,4)}$ ($\stackrel{03}{(-i)}\stackrel{12}{[+]}$, $\stackrel{03}{[+i]}\stackrel{12}{(-)}$) have as the Hermitian conjugated partners $\hat{b}^{(d,c)}_{m=1}$ ($\stackrel{03}{(+i)}\stackrel{12}{[+]}$, $\stackrel{03}{[+i]}\stackrel{12}{(+)}$), respectively.

The vacuum state for the $d = (3+1)$ case is correspondingly:
$(\frac{1}{\sqrt{2}})^2\, (\stackrel{03}{[-i]}\stackrel{12}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[-]} + \stackrel{03}{[-i]}\stackrel{12}{[+]})$.

Embedding $\hat{b}^{b\dagger}_{m=3}$ ($=\stackrel{03}{[-i]}\,\stackrel{12}{(+)}$) into odd part of Table 12.1 the creation operator extends into $\stackrel{03}{[-i]}\,\stackrel{12}{(+)}\,\stackrel{12}{[-]}$, manifesting in $d = (3+1)$ the charge $-\frac{1}{2}$.

12.2.3 γ^a matrices in $d = (3+1)$

There are $2^4 = 16$ basic states in $d = (3+1)$, presented in Table 12.2. They all can be found as well as a part of states in Table 12.1 with either nilpotent or projector, expressing the charge, added. We make a choice of products of nilpotents and projectors, which are eigenstates of the Cartan subalgebra operators, Eq. (12.6), as presented in Eqs. (12.7).

The family members of a family are reachable by either S^{ab} or by γ^a, and represent twice two vectors of definite handedness $\Gamma^{(d)}$ in $d = (3+1)$. Different families are reachable by either $\tilde{S}^{ab}$ or by $\tilde{\gamma}^a$. Each state carries correspondingly

	ψ^f_m	$\gamma_0\,\psi^f_m$	$\gamma_1\,\psi^f_m$	$\gamma_2\,\psi^f_m$	$\gamma_3\,\psi^f_m$	$\tilde\gamma_0\,\psi^f_m$	$\tilde\gamma_1\,\psi^f_m$	$\tilde\gamma_2\,\psi^f_m$	$\tilde\gamma_3\,\psi^f_m$	S^{03}	S^{12}	$\tilde S^{03}$	$\tilde S^{12}$	Γ^{3+1}	$\tilde\Gamma^{3+1}$
ψ^a_1	$(+i)(+)$	ψ^a_3	ψ^a_4	$i\psi^a_4$	ψ^a_3	$-i\psi^c_1$	$-i\psi^d_1$	ψ^d_1	$-i\psi^c_1$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	1	1
ψ^a_2	$[-i][-]$	ψ^a_4	ψ^a_3	$-i\psi^a_3$	$-\psi^a_4$	$i\psi^c_2$	$i\psi^d_2$	$-\psi^d_2$	$i\psi^c_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	1	1
ψ^a_3	$[-i](+)$	ψ^a_1	$-\psi^a_2$	$-i\psi^a_2$	$-\psi^a_1$	$i\psi^c_3$	$i\psi^d_3$	$-\psi^d_3$	$i\psi^c_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	-1	1
ψ^a_4	$(+i)[-]$	ψ^a_2	$-\psi^a_1$	$i\psi^a_1$	ψ^a_2	$-i\psi^c_4$	$-i\psi^d_4$	ψ^d_4	$-i\psi^c_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	-1	1
ψ^b_1	$[+i][+]$	ψ^b_3	ψ^b_4	$i\psi^b_4$	ψ^b_3	$-i\psi^d_1$	$i\psi^c_1$	ψ^c_1	$i\psi^d_1$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	1	1
ψ^b_2	$(-i)(-)$	ψ^b_4	ψ^b_3	$-i\psi^b_3$	$-\psi^b_4$	$i\psi^d_2$	$-i\psi^c_2$	$-\psi^c_2$	$-i\psi^d_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	1	1
ψ^b_3	$(-i)[+]$	ψ^b_1	$-\psi^b_2$	$-i\psi^b_2$	$-\psi^b_1$	$i\psi^d_3$	$-i\psi^c_3$	$-\psi^c_3$	$-i\psi^d_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	-1	1
ψ^b_4	$[+i](-)$	ψ^b_2	$-\psi^b_1$	$i\psi^b_1$	ψ^b_2	$-i\psi^d_4$	$i\psi^c_4$	ψ^c_4	$i\psi^d_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	-1	1
ψ^c_1	$[+i](+)$	ψ^c_3	$-\psi^c_4$	$-i\psi^c_4$	ψ^c_3	$i\psi^a_1$	$i\psi^b_1$	$-\psi^b_1$	$-i\psi^a_1$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	1	-1
ψ^c_2	$(-i)[-]$	ψ^c_4	$-\psi^c_3$	$i\psi^c_3$	$-\psi^c_4$	$-i\psi^a_2$	$-i\psi^b_2$	ψ^b_2	$i\psi^a_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	1	-1
ψ^c_3	$(-i)(+)$	ψ^c_1	ψ^c_2	$i\psi^c_2$	$-\psi^c_1$	$-i\psi^a_3$	$-i\psi^b_3$	ψ^b_3	$i\psi^a_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	-1	-1
ψ^c_4	$[+i][-]$	ψ^c_2	ψ^c_1	$-i\psi^c_1$	ψ^c_2	$i\psi^a_4$	$i\psi^b_4$	$-\psi^b_4$	$-i\psi^a_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	-1	-1
ψ^d_1	$(+i)[+]$	ψ^d_3	$-\psi^d_4$	$-i\psi^d_4$	ψ^d_3	$i\psi^b_1$	$-i\psi^a_1$	$-\psi^a_1$	$i\psi^b_1$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	1	-1
ψ^d_2	$[-i](-)$	ψ^d_4	$-\psi^d_3$	$i\psi^d_3$	$-\psi^d_4$	$-i\psi^b_2$	$i\psi^a_2$	ψ^a_2	$-i\psi^b_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	1	-1
ψ^d_3	$[-i][+]$	ψ^d_1	ψ^d_2	$i\psi^d_2$	$-\psi^d_1$	$-i\psi^b_3$	$i\psi^a_3$	ψ^a_3	$-i\psi^b_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	-1	-1
ψ^d_4	$(+i)(-)$	ψ^d_2	ψ^d_1	$-i\psi^d_1$	ψ^d_2	$i\psi^b_4$	$-i\psi^a_4$	$-\psi^a_4$	$i\psi^b_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	-1	-1

Table 12.2. In this table $2^d = 16$ vectors, describing internal space of fermions in $d = (3+1)$, are presented. Each vector carries the family member quantum number $f = (a, b, c, d)$ — determined by S^{03} and S^{12}, Eqs. (12.7) — and the family quantum number m — determined by $\tilde S^{03}$ and $\tilde S^{12}$, Eq. (12.6). Vectors ψ^f_m are obtained by applying $\hat b^{f\dagger}_m$ on the vacuum state. Vectors — that is the family members of any family — split into even (they are sums of products of even number of γ^a's) and odd (they are sums of products of odd number of γ^a's). If these vectors are embedded into the vectors of $d = (5+1)$ (by gaining the appropriate nilpotent or projector), they gain charges. The Clifford odd parts of vectors are marked, entering into Table 12.1.

quantum numbers of the two kinds of the Cartan subalgebra. In Table 12.2 also $\Gamma^{(3+1)}$ ($= -4iS^{03}S^{12}$) and $\tilde{\Gamma}^{(3+1)}$ ($= -4i\tilde{S}^{03}\tilde{S}^{12}$) are presented.

When once the basic states are chosen and Table 12.2 is made it is not difficult to find the matrix representations for the operators (γ^a, S^{ab}, $\tilde{\gamma}^a$, $\tilde{S}^{ab}$, $\Gamma^{(3+1)}$, $\tilde{\Gamma}^{(3+1)}$). They are obviously 16×16 matrices with a 4×4 diagonal or off diagonal or partly diagonal and partly off diagonal substructure.

Let us define, to simplify the notation, the unit 4×4 submatrix and the submatrix with all the matrix elements equal to zero as follows

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^0, \qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{12.15}$$

We also use (2×2) Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12.16}$$

It is easy to find the matrix representations for γ^0, γ^1, γ^2 and γ^3 from Table 12.2

$$\gamma^0 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} \end{pmatrix}, \tag{12.17}$$

$$\gamma^1 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{smallmatrix} \end{pmatrix}, \tag{12.18}$$

$$\gamma^2 = \begin{pmatrix} \begin{smallmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{smallmatrix} & 0 & 0 & 0 \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{smallmatrix} \end{pmatrix}, \tag{12.19}$$

$$\gamma^3 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} \end{pmatrix}, \tag{12.20}$$

manifesting the 4×4 substructure along the diagonal of 16×16 matrices.

The representations of the $\tilde{\gamma}^a$ do not appear in the Dirac case. They manifest the off diagonal structure as follows

$$\tilde{\gamma}^0 = \begin{pmatrix} \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} \end{pmatrix} \tag{12.21}$$

$$\tilde{\gamma}^1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} \\ \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{12.22}$$

$$\tilde{\gamma}^2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} \\ \begin{smallmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{12.23}$$

$$\tilde{\gamma}^3 = \begin{pmatrix} \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} \end{pmatrix}. \tag{12.24}$$

Matrices S^{ab} have again along the diagonal the 4×4 substructure, as expected, manifesting the repetition of the Dirac 4×4 matrices, up to a phase, since the Dirac S^{ab} do not distinguish among families.

$$S^{01} = \begin{pmatrix} \begin{smallmatrix} \frac{i}{2}\sigma^1 & 0 \\ 0 & -\frac{i}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\frac{i}{2}\sigma^1 & 0 \\ 0 & \frac{i}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{i}{2}\sigma^1 & 0 \\ 0 & \frac{i}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^1 & 0 \\ 0 & -\frac{i}{2}\sigma^1 \end{smallmatrix} \end{pmatrix}, \tag{12.25}$$

$$S^{02} = \begin{pmatrix} \begin{smallmatrix} -\frac{i}{2}\sigma^2 & 0 \\ 0 & \frac{i}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^2 & 0 \\ 0 & -\frac{i}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^2 & 0 \\ 0 & -\frac{i}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{i}{2}\sigma^2 & 0 \\ 0 & \frac{i}{2}\sigma^2 \end{smallmatrix} \end{pmatrix}, \tag{12.26}$$

$$S^{03} = \begin{pmatrix} \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} \end{pmatrix}, \tag{12.27}$$

$$S^{12} = \begin{pmatrix} \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} \end{pmatrix}, \tag{12.28}$$

$$S^{13} = \begin{pmatrix} \begin{smallmatrix} \frac{1}{2}\sigma^2 & 0 \\ 0 & \frac{1}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^2 & 0 \\ 0 & -\frac{1}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^2 & 0 \\ 0 & -\frac{1}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^2 & 0 \\ 0 & \frac{1}{2}\sigma^2 \end{smallmatrix} \end{pmatrix}, \tag{12.29}$$

$$S^{23} = \begin{pmatrix} \begin{smallmatrix} \frac{1}{2}\sigma^1 & 0 \\ 0 & \frac{1}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^1 & 0 \\ 0 & -\frac{1}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^1 & 0 \\ 0 & -\frac{1}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^1 & 0 \\ 0 & \frac{1}{2}\sigma^1 \end{smallmatrix} \end{pmatrix}. \tag{12.30}$$

$$\Gamma^{3+1} = -4iS^{03}S^{12} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix}. \tag{12.31}$$

The operators $\tilde{S}^{ab}$ have again off diagonal 4×4 substructure, except $\tilde{S}^{03}$ and $\tilde{S}^{12}$, which are diagonal.

$$\tilde{S}^{01} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{12.32}$$

$$\tilde{S}^{02} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{12.33}$$

$$\tilde{S}^{03} = \begin{pmatrix} \frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{i}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} \end{pmatrix}, \tag{12.34}$$

$$\tilde{S}^{12} = \begin{pmatrix} \frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{2}\mathbf{1} \end{pmatrix}, \tag{12.35}$$

$$\tilde{S}^{13} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \frac{i}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{12.36}$$

$$\tilde{S}^{23} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{12.37}$$

$$\tilde{\Gamma}^{3+1} = -4i\tilde{S}^{03}\tilde{S}^{12} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} . \tag{12.38}$$

12.3 Conclusions

We present in this contribution the matrix representations of operators applying on the basis, defined by the creation and annihilation operators in d-dimensional Clifford space — $d = 2(2n+1)$, or $4n$, n is a positive integer. We make a choice of $d = (3+1)$ and $d = (5+1)$.

Creation and annihilation operators, which define the vector space, are in our case products of nilpotents and projectors (applying on the vacuum state, Eq. (12.9)), which are eigenvectors of the Cartan subalgebra, Eq. (12.6), of the Lorentz algebra of S^{ab}, as well as of the corresponding Cartan subalgebra, Eq. (12.6), of the Lorentz algebra of $\tilde{S}^{ab}$. Creation and annihilation operators are Hermitian conjugated to each other. We make a choice of the creation operators by choosing the vacuum state, Eq. (12.9), to be the sum of the Clifford odd (they are superposition of an odd number of γ^a's) annihilation operators multiplying their Hermitian conjugated partners from the left hand side.

S^{ab} generate $2^{\frac{d}{2}-1}$ family members of a particular family of an odd Clifford character, $\tilde{S}^{ab}$ generate the corresponding $2^{\frac{d}{2}-1}$ families. The Hermitian conjugation determines their $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ partners (which are reachable also by $\gamma^a\tilde{\gamma}^a$). The Clifford even representations follow from the odd 2^{d-1} vectors by the application of γ^a's or $\tilde{\gamma}^a$'s. There are correspondingly 2^d vectors in d-dimensional space ($d = 2(2n+1), 4n$).

The Clifford even operators keep the Clifford character unchanged. γ^a's and $\tilde{\gamma}^a$'s change the Clifford character of vectors — from odd to even or opposite.

Embedding $SO(3+1)$ into $SO(d)$, $d > (3+1)$, d even, spins in $d \geq (5+1)$ manifest in $d = (3+1)$ as charges.

One can check that the creation operators of an odd Clifford character and their Hermitian conjugated partners, applied on the vacuum state, Eq.(12.9), fulfill the anticomutation relations for the second quantized fermions, Eq. (12.10), postulated by Dirac, what explains the Dirac's second quantization postulates.

One can also observe the appearance of families, used in the *spin-charge-family* theory for the explanation of families of quarks and leptons [3–5].

In this contribution the matrix representations for operators (γ^a's, S^{ab}'s, $\tilde{\gamma}^a$'s, $\tilde{S}^{ab}$'s) are presented for the basis in which creation operators are eigenstates of the Cartan subalgebras of both kinds, Eq. (12.7). It is discussed how do Clifford odd and even products of nilloptents and projectors in $(3+1)$ become a part of creation and annihilation operators of an odd Clifford character in $d = (5+1)$, manifesting the spin in $a = (5,6)$ as the charge in $d = (5+1)$.

There are $2^4 = 16$ basic vectors in $d = (3+1)$ and correspondingly all the matrices have dimension 16×16, which are for the operators, determined by γ^a's, by diagonal and for the operators, determined by $\tilde{\gamma}^a$'s, off diagonal. We keep the

Clifford odd and the Clifford even vectors as the basic vectors. We treat in the Clifford odd part the creation and annihilation operators as they would all define the vector space, to point out, that if space of $d = (3 + 1)$ is embedded into $d \geq 6$, all the parts, even and odd contribute to the enlarged vector space as factors.

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