Image Anal Stereol 2007;26:63-71
Original Research Paper

LIMIT DISTRIBUTIONS OF SOME STEREOLOGICAL ESTIMATORS IN WICKSELL'S CORPUSCLE PROBLEM

LOTHAR HEINRICH

University of Augsburg, Institute of Mathematics, 86135 Augsburg, Germany
e-mail: heinrich@math.uni-augsburg.de
(Accepted April 19, 2007)

ABSTRACT

Suppose that a homogeneous system of spherical particles (d-spheres) with independent identically distributed radii is contained in some opaque d-dimensional body, and one is interested to estimate the common radius distribution. The only information one can get is by making a cross-section of that body with an s-flat $(1 \le s \le d-1)$ and measuring the radii of the s-spheres and their midpoints. The analytical solution of (the hyper-stereological version of) Wicksell's corpuscle problem is used to construct an empirical radius distribution of the d-spheres. In this paper we study the asymptotic behaviour of this empirical radius distribution for $s = d-1$ and $s = d-2$ under the assumption that the s-dimensional intersection volume becomes unboundedly large and the point process of the midpoints of the d-spheres is Brillinger-mixing. Of course, in stereological practice the only relevant cases are $d = 3\,, s = 2$ or $s = 1$ and $d = 2\,, s = 1$. Among others we generalize and extend some results obtained in Franklin (1981) and Groeneboom and Jongbloed (1995) under the Poisson assumption for the special case $d = 3\,, s = 2$.

Keywords: asymptotic normality, Brillinger-mixing point processes, shot-noise processes, α-stable distribution functions..

INTRODUCTION

In 1925 the Swedish statistician Sven D. Wicksell (1890-1939) studied the following problem which belongs meanwhile to the classical toolbox of stereologists. Suppose that a system of three-dimensional random spheres $\{B_3(X_i,R_i) : i \ge 1\}$ with midpoints $\{X_i : i \ge 1\}$ forming a homogeneous point field in $\mathbb{R}^3$, and with identically distributed radii $\{R_i : i \ge 1\}$ having an unkown common distribution function (briefly df) $F_3(r)\,, r \ge 0$, is embedded in an opaque medium. Since the medium is opaque, one cannot observe the sphere radii directly. What can be observed is a bounded part of a planar cross-section through the medium, showing circular sections of some spheres. It has been shown in Wicksell (1925) that the observable circle radii have a common probability density function $\overline{f}_2(r)\,, r \ge 0$, depending on F_3 as follows:

$$\overline{f}_2(r) = r \int_r^\infty \frac{\mathrm{d}F_3(\rho)}{\sqrt{\rho^2 - r^2}} \left(\int_0^\infty \big(1 - F_3(\rho)\big)\,\mathrm{d}\rho \right)^{-1} . \quad (1)$$

There exists a vast and widely scattered literature dealing with the numerical and statistical inversion of Eq. 1, *i.e.*, the approximative determination of the df F_3. Actually the solution of this problem is the essential point in numerous applications of *Wicksell's corpuslce problem* in various fields such as material science, biology, medicine and so on (see, *e.g.*, Stoyan *et al.*, 1995; Ohser and Mücklich, 2000 and references therein).

In Mecke and Stoyan (1980) the reader can find a rigorous derivation of Eq. 1 based on the assumption that the sequence of pairs $\{[X_i,R_i] : i \ge 1\}$ constitutes a stationary marked point process in $\mathbb{R}^3$. The given proof extends straightforwardly to higher dimensions. For our purposes we presuppose in addition that the radii of distinct spheres are independent of each other and that they are also independent of the locations of the sphere centres. For the sake of generality we consider Wicksell's corpuslce problem in the d-dimensional Euclidean space $\mathbb{R}^d$ equipped with the Euclidean norm $\|\cdot\|_d$ and the corresponding Borel $\sigma-$field $\mathscr{B}(\mathbb{R}^d)$.

Let $\Psi_d = \{[X_i,R_i] : i \ge 1\}$ be a stationary, independently marked point process in $\mathbb{R}^d$ with generic non-negative mark R_0 having the df $F_d(r)\,, r \ge 0$. The intensity measure $\Lambda_d(\cdot)$ of Ψ_d is then given by $\Lambda_d(B \times (0,r]) = \lambda_d\, \nu_d(B)\, F_d(r)$, where ν_d denotes the d-volume and $\lambda_d = \mathsf{E}\#\{\Psi_d^* \cap [0,1]^d\}$ denotes the intensity of the corresponding stationary non-marked point process $\Psi_d^* = \{X_i : i \ge 1\}$ (see Stoyan *et al.*, 1995 for details). In order to impose an appropriate (mixing) condition on Ψ_d^* we need the kth-order cumulant measures $\gamma_k(\cdot)$ defined on the Borel $\sigma-$field $\mathscr{B}(\mathbb{R}^{dk})$ of any $k \ge 2$ (see, *e.g.*, Heinrich and Schmidt, 1985 for a precise definition). The stationarity of Ψ_d^* enables us to define an associated (signed) measure—the reduced

kth-order cumulant measure—$\gamma_k^{(red)}(\cdot)$ on $\mathscr{B}(\mathbb{R}^{d(k-1)})$ by disintegration w.r.t. ν_d, *i.e.*,

$$\gamma_k\big(\mathop{\times}_{i=1}^{k} B_i\big) = \lambda_d \int_{B_k} \gamma_k^{(red)}\big(\mathop{\times}_{i=1}^{k-1}(B_i - x)\big)\, \nu_d(\mathrm{d}x) \,.$$

To facilitate the interpretation of the below Conditions 1–3 we point out that, for disjoint bounded $B_1, B_2 \in \mathscr{B}(\mathbb{R}^d)$, the second-order cumulant measure $\gamma_2(\cdot)$ of the Cartesian product $B_1 \times B_2$ is just given by the covariance of the numbers $\#\{\Psi_d^* \cap B_1\}$ and $\#\{\Psi_d^* \cap B_2\}$. The rate of decay of $\gamma_2(B_1 \times B_2)$ to zero, if the distance of B_1 and B_2 grows unboundedly, expresses some kind of asymptotic independence between distant parts of the point process Ψ_d^*. For a complete description of such weak dependences we have to put restrictions on all higher-order (reduced) cumulant measures of Ψ_d^*. Finally, note that $\gamma_k^{(red)}(\cdot) \equiv 0$ for any $k \geq 2$ is necessary and sufficient for Ψ_d^* to be a stationary Poisson point process. Further, let $B_d(x,r)$ denote the closed sphere in $\mathbb{R}^d$ with radius $r > 0$ centered at x and let us put $\omega_d = \nu_d(B_d(o,1)) = \pi^{d/2}/\Gamma(\frac{d}{2}+1)$, where $\Gamma(p) = \int_0^\infty e^{-x} x^{p-1}\,\mathrm{d}x$ for $p > 0$.

Wicksell's corpuscle problem in its hyper-stereological version can be described as follows: The system of d-spheres $\Xi_d = \{B_d(X_i,R_i) : i \geq 1\}$ is intersected by the $s-$flat $H_s = \{x = (x_1,...,x_d) \in \mathbb{R}^d : x_{s+1} = \cdots = x_d = 0\}$ (which can be identified with $\mathbb{R}^s$). We assume that the collection of non-empty s-spheres $\overline{\Xi}_s := \Xi_d \cap H_s = \{B_s(\overline{X}_i,\overline{R}_i) : i \geq 1\}$ in the linear subspace H_s can be observed (all radii and midpoints are visible, without considering overlappings and edge-effects) in an expanding sampling window $W_n^{(s)} := nW^{(s)}$, where $W^{(s)}$ is a fixed convex set in $\mathbb{R}^s$ with unit s-volume, *i.e.*, $\nu_s(W^{(s)}) = 1$, and n runs through $\mathbb{N} = \{1,2,...\}$. Note that $B_s(\overline{X}_i,\overline{R}_i) \neq \emptyset$ iff $\overline{R}_i := (R_i^2 - \|\underline{X}_i\|_{d-s}^2)^{1/2} > 0$. Here and in what follows, write $\overline{x}$ (resp. $\underline{x}$) to indicate the projection of $x \in \mathbb{R}^d$ onto H_s) (resp. onto the orthogonal complement of H_s which can be identified with $\mathbb{R}^{d-s}$). The system of non-empty s-spheres $B_s(\overline{X}_i,\overline{R}_i)$ is completely described by the stationary marked point process $\overline{\Psi}_s = \{[\overline{X}_i,\overline{R}_i] : i \geq 1\}$ in $\mathbb{R}^s$ with intensity measure $\overline{\Lambda}_s(A \times (0,r]) = \overline{\lambda}_s\,\nu_s(A)\,\overline{F}_s(r)$, where $\overline{F}_s$ denotes the df of the 'typical radius' $\overline{R}_0$.

In the next section we restate the well-known explicit expressions of the df $\overline{F}_s$ and the intensity $\overline{\lambda}_s$ in terms of F_d and λ_d together with the corresponding inversion formulae. After that we present our results on the asymptotic behaviour (as $n \to \infty$) of appropriate empirical counterparts of the radius df F_d which are obtained from a single observation of all s-spheres whose midpoints lie in $W_n^{(s)}$. In particular, we state asymptotic normality (Theorem 1) and weak consistency (Theorem 4) in the cases $s = d-1$ and $s = d-2$, respectively. Using the terminology developed for limit theorems for sums of independent random variables we are in the situation of a non-normal domain of attraction of the Gaussian resp. degenerate law (see Ibragimov and Linnik, 1971). In other words, we are faced with (weakly dependent) random variables having infinite variance resp. infinite expectation, but nevertheless, after suitable centering and overnorming, their sums satisfy a central limit theorem resp. a weak law of large numbers. By $\underset{n\to\infty}{\Longrightarrow}$ and $\underset{n\to\infty}{\overset{\mathsf{P}}{\longrightarrow}}$ we denote *weak convergence* (*i.e.*, convergence in distribution) and *convergence in probability* P, respectively.

The Poisson framework, as presupposed in Franklin (1981), Groeneboom and Jongbloed (1995) and Golubev and Levit (1998), is replaced in the present paper by the assumption that the point process Ψ_d^* is *Brillinger-mixing*. This special mixing condition implies that numbers of points of $\Psi_d^* = \{X_i : i \geq 1\}$ in distant regions become asymptotically uncorrelated. For a precise formulation of this condition the existence of all higher-order moment measures is needed. It should be mentioned that under milder moment assumptions similar asymptotic results can be obtained for absolutely regular point processes Ψ_d^* (Heinrich, 1994), as well as for Poisson cluster processes Ψ_d^* (Heinrich, 1988).

However, it seems that the Poisson assumption can hardly be dropped in our Theorems 5 and 6 to derive α-stable limits (with $\alpha = (d-s)/2$) for the fluctuation of the corresponding empirical df's of F_d when $d - s \geq 2$. In the final section we put together the essential steps of the proofs of our results. All details of the proofs can be found in the paper *http://www.math.uni-augsburg.de/stochastik/heinrich/papers/asymwick.pdf.*

RELATIONSHIPS BETWEEN THE RADIUS DF'S

By means of Campbell's theorem (see, *e.g.*, Stoyan *et al.*, 1995) and the relation $\overline{R}_i^{\,2} = R_i^2 - \|\underline{X}_i\|_{d-s}^2 > 0$ the intensity measures $\overline{\Lambda}_s$ and Λ_d are connected by the identity

$$\overline{\Lambda}_s(A \times (a,b)) = \int_{\mathbb{R}^s \times \mathbb{R}^{d-s} \times [0,\infty)} \mathbf{1}\big(\overline{x} \in A\big) \times$$
$$\mathbf{1}\big(a^2 < \max\{0, \rho^2 - \|\underline{x}\|_{d-s}^2\} < b^2\big)\Lambda_d(\mathrm{d}(\overline{x},\underline{x},\rho)) \,,$$

for any $A \in \mathscr{B}(\mathbb{R}^s)$ and $0 \le a < b \le \infty$, where the indicator function $\mathbf{1}(S)$ takes the values 0 or 1 depending on whether the set S is empty or not. Setting $A = [0,1]^d$ and $a = r\,, b = \infty$ leads to the following Abel-type integral equation:

$$\overline{\lambda}_s\,(1-\overline{F}_s(r)) = \lambda_d\;\omega_{d-s}\int_r^\infty (\rho^2-r^2)^{(d-s)/2}\mathrm{d}F_d(\rho)$$

$$= \lambda_d\,(d-s)\,\omega_{d-s}\int_0^\infty \big(1-F_d(\sqrt{r^2+\rho^2})\big)\,\rho^{d-s-1}\mathrm{d}\rho\;.$$

Letting $r \to 0$, the previous formula yields

$$\overline{\lambda}_s = \lambda_d\;\omega_{d-s}\,\mathsf{E}R_0^{d-s}$$

provided that $\mathsf{E}R_0^{d-s} < \infty$, whence it follows that

$$1-\overline{F}_s(r) = \frac{1}{\mathsf{E}R_0^{d-s}}\int_r^\infty (\rho^2-r^2)^{(d-s)/2}\mathrm{d}F_d(\rho)$$

and probability density function $\overline{f}_s$ of $\overline{R}_0$ (which always exists!) takes the form

$$\overline{f}_s(r) = \frac{r\,(d-s)}{\mathsf{E}R_0^{d-s}}\int_r^\infty (\rho^2-r^2)^{(d-s-2)/2}\mathrm{d}F_d(\rho)\,. \qquad (2)$$

Here and throughout the integral $\int_r^\infty$ stretches over interval (r,∞). To express the radius df F_d in terms of the radius df $\overline{F}_s$ for any $s \in \{1,\ldots,d-1\}$ one has to solve the above Abel-type integral equation by *unfolding*. For doing this we distinguish between the cases $d-s$ is even and $d-s$ is odd, respectively. Put $q = \lfloor (d-s-1)/2 \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x and $n!! = n(n-2)\cdot\ldots\cdot 4\cdot 2$ or $3\cdot 1$. Then the df F_d can be expressed in terms of the probability density function $\overline{f}_s$ in the following way:

$$1-F_d(r) = (-1)^q\,\frac{\mathsf{E}R_0^{d-s}}{(d-s)!!} \times \begin{cases} \frac{1}{r}g_s(r)\,, & d-s \text{ even} \\ \frac{2}{\pi}\int_r^\infty g_s(\rho)(\rho^2-r^2)^{-1/2}\mathrm{d}\rho\,, & d-s \text{ odd} \end{cases},$$

with

$$g_s(r) = \begin{cases} \overline{f}_s(r) & \text{if } d-s = 1\,,2 \\ \Big(\frac{1}{r}\Big(\frac{1}{r}\cdots\Big(\frac{\overline{f}_s(r)}{r}\Big)'\cdots\Big)'\Big)' & \text{if } d-s \ge 3\,, \end{cases}$$

where in the last line q derivatives occur.

However, the statistical solution of the integral equation Eq. 2 leads to an inverse estimation problem which is rather unstable from both the computational and statistical view point (see Watson, 1971; Franklin, 1981; Van Es and Hoogendoorn, 1990; Groeneboom and Jongbloed, 1995; Stoyan *et al.*, 1995; Mair *et al.*, 2000 for further details).

In the most important case $s = d-1$ it is rapidly verified by a straightforward application of Campbell's theorem (see, *e.g.*, Stoyan *et al.*, 1995) that

$$\widehat{U}_n(r) = \frac{1}{\pi n^{d-1}}\sum_{i\ge 1}\mathbf{1}(\overline{X}_i \in W_n^{(d-1)})\,\frac{\mathbf{1}(\overline{R}_i > r)}{\sqrt{\overline{R}_i^{\,2}-r^2}}\,,$$

is an unbiased estimation of $\lambda_d\,(1-F_d(r))$ for any $r \ge 0$ and all $n \in \mathbb{N}$. On the other hand, the same calculation reveals that the variance of $\widehat{U}_n(r)$ is infinite (which has been first noticed in Franklin, 1981).

We refer to the fact that, for fixed $n \in \mathbb{N}$, the empirical process $\widehat{U}_n(r)$ regarded as random function of the argument $r \ge 0$ is by no means monotonically decreasing. It possesses downward jumps at the random points $r = \overline{R}_i$, however, between two such jumps $\widehat{U}_n(r)$ is strictly increasing. Such strange behaviour of this stereological estimator of $\lambda_d\,(1-F_d(r))$ gave rise to consider several modified and smoothed versions of $\widehat{U}_n(r)$ (see, *e.g.*, Groeneboom and Jongbloed, 1995 for an isotonic estimation and its asymptotic analysis).

ASYMPTOTIC RESULTS

THE CASE $s = d - 1$

We first put together some mixing-type conditions for the point process $\Psi_d^* = \{X_i : i \ge 1\}$ of the midpoints of the d-spheres.

Condition 1 Assume that Ψ_d^* is *Brillinger-mixing*, *i.e.*,

$$\int_{(\mathbb{R}^d)^{k-1}} |\gamma_k^{(red)}(\mathrm{d}(x_1,\ldots,x_{k-1}))| < \infty \quad \text{for} \quad k \ge 2\,.$$

Condition 2 Assume that the reduced second-order cumulant measure $\gamma_2^{(red)}(\cdot)$ satisfies

$$\int_{\mathbb{R}^{d-1}\times A} \big|\gamma_2^{(red)}(\mathrm{d}x)\big| \le \text{const}\;\nu_1(A)\,,$$

for any bounded Borel set $A \subset \mathbb{R}^1$.

Condition 3 Assume that the reduced second-order cumulant measure $\gamma_2^{(red)}(\cdot)$ has finite total variation, *i.e.*,

$$\int_{\mathbb{R}^d} \left|\gamma_2^{(red)}(\mathrm{d}x)\right| < \infty .$$

A Poisson point process Ψ_d^* trivially satisfies the Conditions 1-3 since all of its higher-order cumulant measures disappear. Sufficient conditions for some other classes of point processes to be Brillinger-mixing are discussed in Heinrich (1988). For example, Poisson cluster processes are Brillinger-mixing iff the number of points in the typical cluster has moments of any order. Also, several types of dependently thinned Poisson processes such as Matérn's hard (and soft)-core point processes possess this mixing property. In Ruelle (1988), conditions are derived that imply Brillinger-mixing of Gibbsian point processes with pair interactions. If Ψ_d^* is additionally isotropic with *pair correlation function* $g(r)$ (see Stoyan *et al.*, 1995), then Condition 2 is satisfied if

$$\sup_{a\geq 0} \int_0^\infty |\, g(\sqrt{r^2+a}) - 1\,|\; r^{d-2}\,\mathrm{d}r < \infty \ .$$

This as well as Condition 3 are rather mild restrictions on the point process Ψ_d^*.

Theorem 1 *Let the Conditions 1 and 2 be satisfied. If*

$$\sigma^2(r) := \lambda_d \int_r^\infty \frac{\mathrm{d}F_d(\rho)}{\sqrt{\rho^2 - r^2}} < \infty \qquad (3)$$

for some fixed $r \geq 0$ *and* $\mathsf{E}R_0 < \infty$*, then*

$$\sqrt{\frac{\pi^2\, n^{d-1}}{\log n^{d-1}}}\Big(\widehat{U}_n(r) - \lambda_d(1-F_d(r))\Big) \underset{n\to\infty}{\Longrightarrow} N(0,\sigma^2(r)) \ ,$$

where $N(0,\sigma^2)$ *denotes a zero mean Gaussian random variable with variance* σ^2. *For* $r > 0$, *condition Eq. 3 is equivalent to* $\overline{f}_{d-1}(r) < \infty$, *where the probability density function* $\overline{f}_{d-1}$ *is given by Eq. 2 for* $s = d-1$.

Remark 1 Provided that $F_d(0) = 0$, Theorem 1 (for $r = 0$) yields a central limit theorem for the unbiased estimator $\widehat{U}_n(0)$ of the intensity λ_d.

Note that without assuming Brillinger-mixing – merely under Condition 3 – $\widehat{U}_n(r)$ turns out to be weakly consistent (as $n \to \infty$) for $\lambda_d\,(1-F_d(r))$. Hence, we get that

$$\frac{\widehat{U}_n(r)}{\widehat{U}_n(0)} \quad \underset{n\to\infty}{\overset{\mathsf{P}}{\longrightarrow}} 1 - F_d(r) \quad \text{for any} \quad r \geq 0 .$$

It should be noted that, in case Ψ_d^* is a stationary ergodic point process, the latter relation holds P−a.s..

Remark 2 For $r > 0$, the assumption Eq. 3 is satisfied if the df F_d is α-Hölder continuous for some $\alpha > 1/2$ in $[r, r+\delta]$, *i.e.*, there exists a positive number $H_{\alpha,\delta}$ depending on α and δ such that

$$F_d(\rho) - F_d(r) \ \leq \ H_{\alpha,\delta}\,(\rho - r)^\alpha$$

for $r \leq \rho \leq r + \delta$ and some $\delta > 0$ (see Golubev and Levit, 1998 for an analogous smoothness condition).

The multivariate extension of Theorem 1 (by employing the well-known method of Cramér-Wold) shows that the finite-dimensional distributions of the sequence of standardized empirical processes in Theorem 1 tend to those of a Gaussian 'white noise' process as $n \to \infty$.

Theorem 2 *Let the Conditions 1 and 2 and* Eq. 3 *for* $r \in \{r_1,\ldots,r_k\}$, $0 \leq r_1 < \cdots < r_k < \infty$, *be satisfied. Then,*

$$\sqrt{\frac{\pi^2\, n^{d-1}}{\log n^{d-1}}}\left(\frac{\widehat{U}_n(r_j) - \lambda_d(1-F_d(r_j))}{\sqrt{\sigma^2(r_j)}}\right)_{j=1}^{k} \underset{n\to\infty}{\Longrightarrow} N_k\big(\mathbf{o}, I_k\big)$$

where $N_k\big(\mathbf{o}, I_k\big)$ *denotes a* k*-dimensional Gaussian random vector having zero mean components and a covariance matrix being equal to the unit matrix* I_k.

A simple application of Theorem 2 for $k = 2$, $r_1 = 0$, $r_2 = r$ (using the asymptotic independence of the components) and Slutski's theorem (see, *e.g.*, Ibragimov and Linnik, 1971) leads to

Corollary 1 *Let the Conditions 1 and 2,* $F_d(0) = 0$ *and* Eq. 3 *for* $r = 0$ *and some* $r > 0$ *be satisfied. Then*

$$\sqrt{\frac{\pi^2\, n^{d-1}}{\log n^{d-1}}}\Big(\frac{\widehat{U}_n(r)}{\widehat{U}_n(0)} - (1-F_d(r))\Big) \underset{n\to\infty}{\Longrightarrow} N(0, s^2(r)) \ ,$$

where $s^2(r) := \big(\sigma^2(r) + \sigma^2(0)\,(1-F_d(r))^2\big)/\lambda_d^2$.

There exists indeed a weakly consistent estimator of the asymptotic variance $\sigma^2(r)$ (although its expectation does not exist) which is given by the following 'overnormed' random sum

$$\widehat{\sigma}_n^2(r) := \frac{1}{n^{d-1}\log n^{d-1}} \sum_{i\geq 1} \mathbf{1}(\overline{X}_i \in W_n^{(d-1)})\, \frac{\mathbf{1}(\overline{R}_i > r)}{\overline{R}_i^{\,2} - r^2} \ .$$

Theorem 3 *Under Condition 3 and* $\mathsf{E}R_0 < \infty$ *it holds*

$$\widehat{\sigma}_n^2(r) \xrightarrow[n\to\infty]{\mathsf{P}} \sigma^2(r) \quad \text{for each } r \geq 0 \text{ satisfying Eq. 3} \; .$$

As an immediate consequence of Eq. 2 for $s = d-1$, the ratios $2r\,\widehat{\sigma}_n^2(r)/(\widehat{\lambda}_{d-1})_n$ are weakly consistent estimators of $\overline{f}_{d-1}(r)$ for each $r \geq 0$ satisfying Eq. 3, where

$$(\widehat{\lambda}_{d-1})_n = \frac{1}{n^{d-1}} \sum_{i\geq 1} \mathbf{1}(\overline{X}_i \in W_n^{(d-1)})$$

is an unbiased and weakly consistent estimator of $\lambda_{d-1} = 2\lambda_d\,\mathsf{E}R_0$ (see Van Es and Hoogendoorn, 1990 or Golubev and Levit, 1998, for alternative kernel-type estimators of $\overline{f}_{d-1}(r)$).

Combining Theorem 1 with Theorem 3 together with Slutski's theorem provides

Corollary 2 *Let the Conditions 1 and 2,* $\mathsf{E}R_0 < \infty$ *and* Eq. 3 *for some fixed* $r \geq 0$ *be satisfied. Then*

$$\sqrt{\frac{\pi^2\, n^{d-1}}{\widehat{\sigma}_n^2(r)\,\log n^{d-1}}}\Big(\widehat{U}_n(r) - \lambda_d(1-F_d(r))\Big) \underset{n\to\infty}{\Longrightarrow} N(0,1)\,.$$

Remark 3 By means of Corollary 2 (applied to $r = 0$ provided $F_d(0) = 0$) we are able to construct an asymptotically exact confidence interval for the unknown intensity λ_d of the midpoints of d-spheres.

In order to find an asymptotic confidence interval for $1 - F_d(r)$ we combine Corollary 1, Theorem 2 and Slutski's theorem and obtain

Corollary 3 *Assume that the Conditions 1 and 2,* $\mathsf{E}R_0 < \infty$*,* $F_d(0) = 0$*, and* Eq. 3 *for* $r = 0$ *and some fixed* $r > 0$ *are satisfied. Then*

$$\frac{\pi}{\widehat{s}_n(r)}\sqrt{\frac{n^{d-1}}{\log n^{d-1}}}\Big(\frac{\widehat{U}_n(r)}{\widehat{U}_n(0)} - (1-F_d(r))\Big) \underset{n\to\infty}{\Longrightarrow} N(0,1)\,,$$

where

$$\widehat{s}_n(r) = \frac{1}{(\widehat{U}_n(0))^2}\sqrt{\widehat{\sigma}_n^2(r)\,(\widehat{U}_n(0))^2 + \widehat{\sigma}_n^2(0)\,(\widehat{U}_n(r))^2}\,.$$

In other words, for any $0 < \alpha < 1$ *and large enough observation window* $W_n^{(d-1)}$*, the interval* $[b_n^-(\alpha,r), b_n^+(\alpha,r)]$ *contains the value* $1 - F_d(r)$ *approximately with probability* $1-\alpha$*, where*

$$b_n^{\pm}(\alpha,r) = \frac{\widehat{U}_n(r)}{\widehat{U}_n(0)} \pm z_{\alpha/2}\,\frac{\widehat{s}_n(r)}{\pi}\sqrt{\frac{\log \nu_{d-1}(W_n^{(d-1)})}{\nu_{d-1}(W_n^{(d-1)})}}\,.$$

Here, $z_{\alpha/2}$ *denotes the* $(1-\alpha/2)$*-quantile of the* $N(0,1)$*-distribution.*

A further immediate consequence of Theorem 2 and Slutski's theorem is

Corollary 4 *Let the assumptions of Theorem 2 and* $\mathsf{E}R_0 < \infty$ *be satisfied. Then*

$$\frac{\pi^2\, n^{d-1}}{\log n^{d-1}} \sum_{j=1}^{k} \frac{\Big(\widehat{U}_n(r_j) - \lambda_d(1-F_d(r_j))\Big)^2}{\widehat{\sigma}_n^2(r_j)} \underset{n\to\infty}{\Longrightarrow} \chi_k^2\,,$$

where the random variable χ_k^2 *is* χ^2*-distributed with k degrees of freedom.*

The latter result can be used to test the goodness-of-fit of certain hypothesised radius df F_d (if λ_d is known).

THE CASE $s = d-2$

For fixed $n \in \mathbb{N}$, define the empirical process

$$\widehat{V}_n(r) = \frac{1}{\pi\, n^{d-2}\,\log n^{d-2}} \sum_{i\geq 1} \mathbf{1}(\overline{X}_i \in W_n^{(d-2)})\,\frac{\mathbf{1}(\overline{R}_i > r)}{\overline{R}_i^{\,2} - r^2}$$

which has an infinite mean for any $r \geq 0$. Nevertheless, $\widehat{V}_n(r)$ is weakly consistent for $\lambda_d\,(1 - F_d(r))$ under slight additional assumptions.

Theorem 4 *Under Condition 3 and* $\mathsf{E}R_0^2 < \infty$ *it holds*

$$\widehat{V}_n(r) \xrightarrow[n\to\infty]{\mathsf{P}} \lambda_d(1-F_d(r)) \qquad \text{for any} \quad r \geq 0\,,$$

and therefore, together with $F_d(0) = 0$*,*

$$\frac{\hat{V}_n(r)}{\hat{V}_n(0)} \xrightarrow[n\to\infty]{\mathsf{P}} 1 - F_d(r) \qquad \text{for any} \quad r \geq 0\,.$$

Theorem 5 *Let* $\Psi_d^* = \{X_i : i \geq 1\}$ *be a stationary Poisson process with intensity* λ_d*. If, in addition,*

$$\int_r^\infty |\log(\rho^2 - r^2)|\,\mathrm{d}F_d(\rho) < \infty \tag{4}$$

for some fixed $r \geq 0$ *with* $F_d(r) < 1$*, then*

$$\log n^{d-2}\Big(\frac{\widehat{V}_n(r)}{\lambda_d\,(1-F_d(r))} - 1\Big) - \log\Big(\pi\,\lambda_d\,(1-F_d(r))\Big)$$

$$-\frac{\int_r^\infty \log(\rho^2 - r^2)\,\mathrm{d}F_d(\rho)}{1-F_d(r)} - 1 + \gamma \underset{n\to\infty}{\Longrightarrow} S_1\,,$$

where $\gamma := \lim_{n\to\infty}(1 + 1/2 + \cdots + 1/n - \log n) \simeq 0.5772$ *denotes the Euler-Mascheroni constant and the random variable* S_1 *possesses an* α*-stable df with characteristic exponent* $\alpha = 1$ *and skewness parameter* $\beta = 1$ *having the Fourier-Stieltjes transform*

$$\mathsf{E}\exp\{it\,S_1\} = \exp\Big\{ -\frac{\pi}{2}\,|t| - it\,\log|t| \Big\}\,,$$

for $t \in \mathbb{R}^1$.

Remark 4 Nolan (1997) provides tables and numerical procedures for calculating the density of S_1 (and other stable densities). This gives at least in principle the possibility for testing the null hypothesis $H_0 : F_d = F_d^{(0)}\,,\ \lambda_d = \lambda_d^{(0)}$.

THE CASE $d - s > 2$

Of course, the previous cases are of particular interest in stereological practice for $d = 3,\ s = 2$, $d = 2,\ s = 1$ and $d = 3,\ s = 1$. To be complete we also investigate the asymptotic behaviour of a simple generalization of $\widehat{U}_n(r)$ resp. $\widehat{V}_n(r)$ to the case $d - s > 2$. The below result seems to be of interest for its own right (from the view point of pure asymptotics) and it gives insight how the instability increases when $d - s$ becomes greater than two.

Let $p := d - s$ and define

$$\widehat{Y}_n^{(p)}(r) = \frac{1}{n^{s\,p/2}}\ \sum_{i\geq 1} \mathbf{1}(\overline{X}_i \in W_n^{(s)})\ \frac{\mathbf{1}(\overline{R}_i > r)}{(\overline{R}_i^{\,2} - r^2)^{p/2}}\ .$$

Theorem 6 *Let* $\Psi_d^* = \{X_i : i \geq 1\}$ *be a stationary Poisson process with intensity* λ_d *and* $\mathsf{E}R_0^{p-2} < \infty$. *Then, for any fixed* $r \geq 0$ *with* $F_d(r) < 1$*, it holds*

$$\frac{\widehat{Y}_n^{(p)}(r)}{\Big(c_p\,\lambda_d\,\int_r^\infty(\rho^2 - r^2)^{(p-2)/2}\,\mathrm{d}F_d(\rho)\Big)^{p/2}} \underset{n\to\infty}{\Longrightarrow} S_{2/p}\ ,$$

where $c_p = \omega_p\ \frac{p}{2}\ \Gamma(1-\frac{2}{p})\ \cos(\frac{\pi}{p})$ *and the random variable* $S_{2/p}$ *possesses an* α*-stable df with characteristic exponent* $\alpha = 2/p \in (0,1)$ *and skewness parameter* $\beta = 1$ *having the Fourier-Stieltjes transform*

$$\mathsf{E}\exp\{it\,S_{2/p}\} = \exp\Big\{ -|t|^{2/p}\,\Big(1 - i\,\mathrm{sgn}(t)\,\tan\Big(\frac{\pi}{p}\Big)\Big)\Big\}$$

for $t \in \mathbb{R}^1$.

PROOFS OF THE THEOREMS

First observe that the empirical processes $\widehat{U}_n(r)$, $\widehat{V}_n(r)$, $\widehat{\sigma}_n^2(r)$, and $\widehat{Y}_n^{(p)}(r)$ can be regarded as so-called shot-noise processes $\sum_{i\geq 1} f(\overline{X}_i, \underline{X}_i, R_i)$ with different 'response functions' $f\,|\,\mathbb{R}^s \times \mathbb{R}^{d-s} \times (0,\infty) \mapsto [0,\infty)$, see Heinrich and Schmidt (1985) and references therein. However, only $\widehat{U}_n(r)$ has a finite first moment. In fact, applying Campbell's theorem gives $\mathsf{E}\widehat{U}_n(r) = \lambda_d\,(1 - F_d(r))$ and further that $\mathsf{E}(\widehat{U}_n(r))^m < \infty$ for $1 < m < 2$, but $\mathsf{E}(\widehat{U}_n(r))^2 = \infty$. In order to prove Theorem 1 we have to replace the terms $(\overline{R}_i^{\,2} - r^2)^{-1/2}$ (which are responsible for the large fluctuations of the sum) by truncated terms. More precisely, for any $\varepsilon > 0$, we introduce the 'truncated' shot-noise process

$$\widehat{U}_{n,\varepsilon}(r) = \frac{1}{\pi n^{d-1}}\ \sum_{i\geq 1} \frac{\mathbf{1}(\overline{X}_i \in W_n^{(d-1)})}{\sqrt{\overline{R}_i^{\,2} - r^2}} \times \mathbf{1}\left(\overline{R}_i^{\,2} - r^2 > \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-1}\log n^{d-1}}\right)\,,$$

and the nonnegative random integer

$$N_{n,\varepsilon}(r) = \sum_{i\geq 1} \mathbf{1}\big(\overline{X}_i \in W_n^{(d-1)}\big) \times \mathbf{1}\left(0 < \overline{R}_i^{\,2} - r^2 \leq \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-1}\log n^{d-1}}\right)\,.$$

First step: For any Borel set $B \subseteq \mathbb{R}^1$ we have the identity $\{\widehat{U}_{n,\varepsilon}(r) \in B\} \cap \{N_{n,\varepsilon}(r) = 0\} = \{\widehat{U}_n(r) \in B\} \cap \{N_{n,\varepsilon}(r) = 0\}$ which in turn implies the estimate

$$\Big|\mathsf{P}(\widehat{U}_{n,\varepsilon}(r) \in B)\ -\ \mathsf{P}(\widehat{U}_n(r) \in B)\Big| \leq \mathsf{P}(N_{n,\varepsilon}(r) \geq 1)\ \leq\ \mathsf{E}N_{n,\varepsilon}(r)\,. \quad (5)$$

Applying Campbell's theorem to the shot-noise process $N_{n,\varepsilon}(r)$ we obtain after a short calculation using Eq. 3 and $\mathsf{E}R_0 < \infty$ that

$$\mathsf{E}N_{n,\varepsilon}(r) = 2\,\lambda_d\,n^{d-1}\int_r^\infty \int_0^{\sqrt{\rho^2 - r^2}} \mathbf{1}\Big(\rho^2 - x^2 - r^2 \leq \frac{\max\{\varepsilon, \rho^2 - r^2\}}{\varepsilon^2 n^{d-1}\log n^{d-1}}\Big)\,\mathrm{d}x\,\mathrm{d}F_d(\rho) \underset{n\to\infty}{\longrightarrow} 0\,,$$

for any $\varepsilon > 0$.

Second step: Using once more Eq. 3 and the formula $\int_0^1(1 - w^2)^{-1/2}\mathrm{d}w = \pi/2$ we may show that

$$\sqrt{\frac{\pi^2\,n^{d-1}}{\log n^{d-1}}}\Big(\mathsf{E}\widehat{U}_{n,\varepsilon}(r) - \lambda_d\,(1 - F_d(r))\Big) \underset{n\to\infty}{\longrightarrow} 0\,,$$

and Condition 2 enables us to prove that

$$\overline{\lim_{n\to\infty}} \left| \frac{\pi^2 n^{d-1}}{\log n^{d-1}} \mathsf{Var}\big(\widehat{U}_{n,\varepsilon}(r)\big) - \sigma^2(r) \right| \leq \lambda_d \int\limits_r^{\sqrt{r^2+\varepsilon}} \frac{\mathrm{d}F_d(\rho)}{\sqrt{\rho^2 - r^2}} \quad \text{for any } \varepsilon > 0 \,.$$

In the third step we make use of Condition 1 and show that the cumulants of order $m \geq 3$ (abbreviated by the symbol Cum_m) of $(\pi^2 n^{d-1}/\log n^{d-1})^{1/2}\, \hat{U}_{n,\varepsilon}(r)$ become arbitrarily small.

More precisely, using some relationships and estimates for general shot-noise processes derived in Heinrich and Schmidt (1985) we arrive at the estimates

$$\overline{\lim_{n\to\infty}} \left(\frac{\pi^2 n^{d-1}}{\log n^{d-1}} \right)^{m/2} \left| \mathrm{Cum}_m\{\widehat{U}_{n,\varepsilon}(r)\} \right| \leq \varepsilon^{(m-2)/2}\, C_m\, \sigma^2(r) \quad \text{for } m \geq 3 \,,$$

where the constant C_m depends on the total variations of the signed measures $\gamma_{red}^{(k)}(\cdot)$, $k = 2,\ldots,m$.

This last estimate confirms the asymptotic normality of the truncated shot-noise process $\widehat{U}_{n,\varepsilon}(r)$ by applying the classical 'method of moments'.

Readers interested in detailed proofs of the Theorems 1–4 are referred to an extended version of the paper being available under *http://www.math.uni-augsburg.de/stochastik/heinrich/papers/asymwick.pdf*.

The proof of Theorem 3 is quite similar to that of Theorem 4. For this reason we will outline the essential proving steps only in case of Theorem 4.

Let $\delta > 0$ be arbitrarily small, but fixed and $\varepsilon > 0$ be chosen small enough (in fact, $\varepsilon = \varepsilon_n$ can be thought of as a positive sufficiently slowly decreasing null sequence). Define in analogy to $\widehat{U}_{n,\varepsilon}(r)$ the truncated process

$$\widehat{V}_{n,\varepsilon}(r) = \frac{1}{\pi n^{d-2} \log n^{d-2}} \sum_{i\geq 1} \frac{\mathbf{1}(\overline{X}_i \in W_n^{(d-2)})}{\overline{R}_i^{\,2} - r^2} \times \mathbf{1}\Big(\overline{R}_i^{\,2} - r^2 > \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-2} \log n^{d-2}} \Big) \,,$$

and let $M_{n,\varepsilon}(r)$ denote the above random integer $N_{n,\varepsilon}(r)$ with $d-2$ instead of $d-1$.

Since the 'truncation inequality' (Eq. 5) remains valid for the shot-noise process $\widehat{V}_n(r)$ with $M_{n,\varepsilon}(r)$ instead of $N_{n,\varepsilon}(r)$, it follows together with Chebychev's inequality that

$$\begin{aligned} &\mathsf{P}\big(\big|\widehat{V}_n(r) - \lambda_d(1 - F_d(r))\big| \geq \delta \big) \leq \mathsf{P}(M_{n,\varepsilon}(r) \geq 1) \\ &+ \mathsf{P}\big(\big|\widehat{V}_{n,\varepsilon}(r) - \lambda_d(1 - F_d(r))\big| \geq \delta \big) \leq \mathsf{E}M_{n,\varepsilon}(r) \\ &+ \frac{\mathsf{Var}\big(\widehat{V}_{n,\varepsilon}(r)\big)}{\delta^2} + \frac{\big(\mathsf{E}\widehat{V}_{n,\varepsilon}(r) - \lambda_d(1 - F_d(r))\big)^2}{\delta^2} \,. \end{aligned}$$

The following relations can be proved for any $\varepsilon > 0$:

$$\mathsf{E}M_{n,\varepsilon}(r)) \underset{n\to\infty}{\longrightarrow} 0 \qquad (\text{since } \mathsf{E}R_0^2 < \infty) \,,$$

$$\overline{\lim_{n\to\infty}} \left| \mathsf{E}\widehat{V}_{n,\varepsilon}(r) - \lambda_d\,(1 - F_d(r)) \right| \leq \lambda_d \left(F_d(\sqrt{r^2+\varepsilon}) - F_d(r) \right) \,,$$

and

$$\overline{\lim_{n\to\infty}} \mathsf{Var}\big(\widehat{V}_{n,\varepsilon}(r)\big) \leq \varepsilon\, \frac{\lambda_d}{\pi} \int\limits_{\mathbb{R}^d} |\, \gamma_2^{(red)}(\mathrm{d}x) \,| \,.$$

Therefore, using Condition 3 and the right-continuity of F_d completes the proof of Theorem 4.

The proofs of the Theorems 5 and 6 rely on the exponential shape of the generating functional of the stationary, independently marked Poisson process Ψ_d, which is as follows (see, *e.g.*, Stoyan *et al.*, 1995 for details):

$$\mathsf{E}\prod_{i\geq 1} v(X_i, R_i) = \exp\Big\{ \lambda_d \int\limits_{\mathbb{R}^d} \int\limits_0^\infty \big(v(x,\rho) - 1 \big)\, \mathrm{d}F_d(\rho)\, \mathrm{d}x \Big\}$$

for any Borel-measurable, complex-valued function $v(\cdot)$ on $R^d \times [0,\infty)$ satisfying

$$\int_{\mathbb{R}^d} \int_0^\infty |\, v(x,\rho) - 1\,| \mathrm{d}F_d(\rho)\, \mathrm{d}x < \infty \,.$$

Choosing

$$v(x,\rho) = \exp\Big\{ \frac{it\, \mathbf{1}(\overline{x} \in W_n^{(d-2)})\, \mathbf{1}(\rho^2 - \|\underline{x}\|_2^2 > r^2)}{\pi\, n^{d-2}\, (\rho^2 - \|\underline{x}\|_2^2 - r^2)} \Big\}$$

yields the following expression for the logarithm of the characteristic function $\mathsf{E}\exp\{it\, \log n^{d-2}\, \widehat{V}_n(r)\}$:

$$\begin{aligned} &\lambda_d\, \pi\, n^{d-2} \int\limits_r^\infty \int\limits_0^{\rho^2 - r^2} \Big(\exp\Big\{ \frac{it}{\pi\, n^{d-2}\, y} \Big\} - 1 \Big)\, \mathrm{d}y\, \mathrm{d}F_d(\rho) \\ &= \lambda_d \int\limits_r^\infty \int\limits_{\left(\pi n^{d-2}(\rho^2 - r^2)\right)^{-1}}^\infty \frac{\exp\{itz\} - 1}{z^2}\, \mathrm{d}z\, \mathrm{d}F_d(\rho) \,. \end{aligned} \quad (6)$$

The inner integral in Eq. 6 can be approximated by elementary functions with explicit remainder term in the following way :

$$\int_A^\infty \frac{\exp\{itz\}-1}{z^2}\,\mathrm{d}z = -\frac{\pi}{2}\,|t| - it\,\log|t| + it\,\big(1-\gamma-\log A\big) + \frac{A\,t^2}{2}\,(1+A\,|t|)\,\theta\,,$$

where $A = \big(a_n\,(\rho^2-r^2)\big)^{-1}$, $a_n = \pi\,n^{d-2}$, and θ denotes some complex number satisfying $|\theta| \le 1$. Next, splitting the outer integral in Eq. 6 into two integrals over $(r_n(\varepsilon),\infty)$ and $(r, r_n(\varepsilon)]$ with $r_n(\varepsilon) = \sqrt{r^2+(\varepsilon\,a_n)^{-1}}$, we arrive at

$$\begin{aligned}\log \mathsf{E}\exp\{it\,\log n^{d-2}\,\widehat{V}_n(r)\} &= \lambda_d\,(1-F_d(r_n(\varepsilon)))\\ &\times \Big(-\frac{\pi}{2}\,|t| - it\,\log|t| + it\,\big(1-\gamma+\log a_n\big)\Big)\\ &+ it\,\lambda_d \int_{r_n(\varepsilon)}^\infty \log(\rho^2-r^2)\,\mathrm{d}F_d(\rho) + \frac{\varepsilon\,\lambda_d}{2}\,t^2\,(1+\varepsilon\,|t|)\,\theta\\ &+ 2\,\lambda_d\,\widetilde{\theta}\,a_n \int_r^{r_n(\varepsilon)} (\rho^2-r^2)\,\mathrm{d}F_d(\rho) \quad \text{with} \quad |\widetilde{\theta}| \le 1\,.\end{aligned}$$

Since, in view of Eq. 4, the term in the last line vanishes as $n \to \infty$ for any $\varepsilon > 0$ and also

$$\log n^{d-2}\,\big(F_d(r_n(\varepsilon)) - F_d(r)\big) \le \frac{\log n^{d-2}}{\log(\varepsilon\,a_n)} \int_r^{r_n(\varepsilon)} |\log(\rho^2-r^2)|\,\mathrm{d}F_d(\rho) \underset{n\to\infty}{\longrightarrow} 0\,,$$

it follows from the foregoing equation (after replacing t by $t/\lambda_d\,(1-F_d(r))$ and some further rearrangements) that

$$\begin{aligned}\log \mathsf{E}\exp\Big\{it\,\log n^{d-2}\Big(\frac{\widehat{V}_n(r)}{\lambda_d\,(1-F_d(r))} - 1\Big)\Big\}\\ \underset{n\to\infty}{\longrightarrow} \log \mathsf{E}\exp\{it\,S_1\} + it\,\frac{\int_r^\infty \log(\rho^2-r^2)\,\mathrm{d}F_d(\rho)}{1-F_d(r)}\\ + it\,\log\Big(\pi\lambda_d\,\big(1-F_d(r)\big)\Big) + it\,\big(1-\gamma\big)\,,\end{aligned}$$

which is nothing else but the assertion of Theorem 5.

To prove Theorem 6 we make use of the subsequent representation of $L_n^{(p)}(t) := \log \mathsf{E}\exp\{it\,\widehat{Y}_n^{(p)}(r)\}$ which can be derived in analogy to Eq. 6 by using the generating functional of the Poisson process $\Psi_d^* = \{X_i : i \ge 1\}$:

$$\begin{aligned}L_n^{(p)}(t) = \lambda_d\,\omega_p \int_r^\infty \int_{\left(n^s\,(\rho^2-r^2)\right)^{-p/2}}^\infty \frac{\exp\{itz\}-1}{z^{1+2/p}}\\ \times\,\big(\rho^2-r^2-z^{-2/p}\,n^{-s}\big)^{-1+p/2}\,\mathrm{d}z\,\mathrm{d}F_d(\rho)\,.\end{aligned}$$

The following formula goes back to L. Euler and can be found in any 'Table of Integrals' for $0 < \alpha < 1$:

$$\int_0^\infty \frac{\exp\{itz\}-1}{z^{1+\alpha}}\,\mathrm{d}z = \frac{\Gamma(1-\alpha)}{\alpha}\,\cos\Big(\frac{\alpha\,\pi}{2}\Big) \times |t|^\alpha \Big(-1 + i\,\mathrm{sgn}(t)\,\tan\Big(\frac{\pi\,\alpha}{2}\Big)\Big)\,, \quad (7)$$

where $\Gamma(1-\alpha) = \int_0^\infty e^{-x}\,x^{-\alpha}\,\mathrm{d}x$.

Therefore, applying Eq. 7 for $\alpha = \frac{2}{p}$ we obtain after a simple calculation that

$$\begin{aligned}L_n^{(p)}(t) &\underset{n\to\infty}{\longrightarrow} c_p\,\lambda_d\,I_p(r)\,\log \mathsf{E}\exp\{it\,S_{2/p}\}\\ &= \log \mathsf{E}\exp\{it\,(c_p\,\lambda_d\,I_p(r))^{p/2}\,S_{2/p}\}\,,\end{aligned}$$

where $I_p(r) = \int_r^\infty (\rho^2-r^2)^{(p-2)/2}\,\mathrm{d}F_d(\rho)$ and c_p is as defined in Theorem 6. Thus, replacing t by $t/(c_p\,\lambda_d\,I_p(r))^{p/2}$ completes the proof of Theorem 6.

ACKNOWLEDGEMENT

The author is grateful to the referees for their valuable comments leading to a considerably improved new version of the paper.

REFERENCES

Franklin JN (1981). Confidence intervals for stereological estimators with infinite variance. SIAM J Appl Math 40:179–90.

Golubev GK, Levit BY (1998). Asymptotically efficient estimation in the Wicksell problem. Ann Statist 26:2407–19.

Groeneboom P, Jongbloed G (1995). Isotonic estimation and rates of convergence in Wicksell's problem. Ann Statist 23:1518–42.

Heinrich L, Schmidt V (1985). Normal convergence of multidimensional shot noise and rates of this convergence. Adv Appl Prob 17:709–30.

Heinrich L (1988). Asymptotic Gaussianity of some estimators for reduced factorial moment measures and product densities of stationary Poisson cluster processes. Statistics 19:87–106.

Heinrich L (1994). Normal approximation for some mean-value estimates of absolutely regular tessellations. Meth Math Statist 3:1–24.

Ibragimov IA, Linnik YuV (1971). Stationary and Independent Sequences of Random Variables. Groningen: Wolters-Noordhoff.

Mair BA, Ruymgaart FH, Urrabazo T (2000). Some comments on Wicksell's problem. J Stat Plan Infer 87:31–42.

Mecke J, Stoyan D (1980). Stereological formulas for spherical particles. Math Nachr 96:311–17.

Nolan JP (1997). Numerical calculation of stable densities. Stoch Models 13:759–74.

Ohser J, Mücklich F (2000). Statistical analysis of microstructures in material science. Chichester: Wiley & Sons.

Ruelle D (1988). Statistical mechanics: Rigorous results. Redwood City: Addison & Wesley, 86–93.

Stoyan D, Mecke J, Kendall WS (1995). Stochastic Geometry and Its Applications, 2nd ed. Chichester: Wiley & Sons.

Van Es B, Hoogendoorn AW (1990). Kernel estimation in Wicksell's corpuscle problem. Biometrika 77:139–45.

Watson GS (1971). Estimating functionals of particle size distributions. Biometrika 58:483–90.

Wicksell SD (1925). The corpuscle problem, Part I. Biometrika 17:84–99.