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Facial parity edge coloring of outerplane graphs

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Abstract

A *facial parity edge coloring* of a 2-edge-connected plane graph is such an edge coloring in which no two face-adjacent edges (consecutive edges of a facial walk of some face) receive the same color, in addition, for each face f and each color c, either no edge or an odd number of edges incident with f is colored with c. It is known that any 2-edge-connected plane graph has a facial parity edge coloring with at most 92 colors. In this paper we prove that any 2-edge-connected outerplane graph has a facial parity edge coloring with at most 15 colors. If a 2-edge-connected outerplane graph does not contain any inner edge, then 10 colors are sufficient. Moreover, this bound is tight.

Keywords: Plane graph, facial walk, edge coloring.

Math. Subj. Class.: 05C10, 05C15

1 Introduction

The facial parity edge coloring concept was introduced in [4]. The motivation has come from the papers of Bunde et al. [1, 2]. They introduced parity edge colorings of graphs. A *parity walk* in an edge coloring of a simple graph is a walk along which each color is used an even number of times. Let $p(G)$ be the minimum number of colors in an edge coloring of G having no parity path (*parity edge coloring*). Let $\widehat{p}(G)$ be the minimum number of colors in an edge coloring of G in which every parity walk is closed (*strong parity edge coloring*). Clearly, every parity edge coloring is a proper edge coloring. Although there are graphs G with $\widehat{p}(G) > p(G)$ [1], it remains unknown how large $\widehat{p}(G)$ can be when $p(G) = k$. In [1] it is mentioned that computing $p(G)$ or $\widehat{p}(G)$ is NP-hard even when G is a tree.

The facial parity edge coloring can be considered as a relaxation of the parity edge coloring. We focus on facial cycles of plane graphs. This coloring has to satisfy the following two conditions:

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1. face-adjacent edges receive different colors,
2. for every color c and every face f the total number of occurrences of edges colored with c on a facial walk of f is odd or zero.

The authors of [4] proved that every 2-edge-connected plane graph has a facial parity edge coloring with at most 92 colors.

In this paper we substantially improve this bound for the class of 2-edge-connected outerplane graphs.

Note that the vertex version of this problem was investigated in [5]. The authors proved that every 2-connected plane graph admits a parity vertex coloring using at most 118 colors. Kaiser et al. [8] improved this bound to 97. Czap [3] proved that any 2-connected outerplane graph has such a coloring with at most 12 colors. The generalization of the parity coloring for graphs and set systems can be found in [6].

2 Notation

Let us introduce the notation used in this paper. A graph which can be embedded in the plane is called *planar graph*; a fixed embedding of a planar graph is called *plane graph*. *Outerplane graphs* are plane graphs such that every vertex lies on the outer face.

A *bridge* is an edge whose removal increases the number of components. A graph which contains no bridge is said to be *bridgeless* or *2-edge-connected*. In this paper we consider connected bridgeless plane graphs, multiple edges and loops are allowed.

Given a graph G and one of its edges $e = uv$ (the vertices u and v do not have to be different), the *contraction* of e consists of replacing u and v by a new vertex adjacent to all the former neighbors of u and v, and removing the loop corresponding to the edge e. (We keep multiple edges if they arise.)

Two (distinct) edges are *face-adjacent* if they are consecutive edges of a facial walk of some face f.

A *k-edge coloring* of a graph $G = (V, E)$ is a mapping $\varphi : E(G) \to \{1, \ldots, k\}$. We say that an edge coloring of a plane graph G is *facially proper* if no two face-adjacent edges of G receive the same color. The *facial parity edge coloring* of a 2-edge-connected plane graph is a facially proper edge coloring such that for each face f and each color c, either no edge or an odd number of edges incident with f is colored with c.

Question 2.1. What is the minimum number of colors $\chi'_p(G)$ such that a 2-edge-connected plane graph G has a facial parity edge coloring with at most $\chi'_p(G)$ colors?

3 Results

Lemma 3.1. *Let C_n be a cycle on n edges, $n \geq 1$. Then $\chi'_p(C_n) \leq 5$.*

Proof. If $n = 1$, then we use one color. Let $n = 4k + z$, where k is a non-negative integer and $z \in \{2, 3, 4, 5\}$. We repeat k times the pattern $1, 2, 1, 2$ and then use colors $1, 2, \ldots, z$. The colors 1 and 2 are thus used $2k + 1$ times, the remaining three colors are used at most once. □

An edge of a plane graph not incident with the outer face is called *inner edge*.

Theorem 3.2. *Let G be a 2-edge-connected outerplane graph with no inner edge (bridgeless cactus graph). Then $\chi'_p(G) \leq 10$. Moreover, this bound is tight.*

Proof. First we prove that the edges of G can be colored with at most 5 colors, say $1, 2, 3, 4, 5$, in such a way that for every inner face f and every color $c \in \{1, 2, 3, 4, 5\}$, either no edge or an odd number of edges incident with f is colored with c, in addition, face-adjacent edges receive different colors.

The proof is by induction on the number of inner faces. If G has one inner face, then the statement follows from Lemma 3.1. Let G have k inner faces $f_1, \ldots, f_k$ and assume that the face f_k is such a face of G that the corresponding vertex in a block graph of G is a leaf. Let H be a subgraph of G consisting of faces $f_1, \ldots, f_{k-1}$. The graph H has fewer inner faces than G, hence it has a required coloring. It is easy to extend the coloring of H to a coloring of G. Assume that the boundary of f_k is a cycle C. Clearly, C and H have exactly one vertex v in common. There are at most two forbidden colors for the edges of C incident with v. We have five colors, hence there is such a facial parity edge coloring of the cycle C that no two face-adjacent edges incident with v receive the same color in G.

In the next step we recolor some edges. Assume that a color $i \in \{1, 2, 3, 4, 5\}$ appears an even number of times in G. Let f be an arbitrary inner face which is incident with an edge of color i. We recolor all the edges of color i incident with f with a new color $i + 5$. Now the total number of occurrences of edges colored with i and $i + 5$ is odd in G.

This recoloring uses at most 10 colors.

To see that the upper bound is tight it is sufficient to consider the graph in Figure 1. □

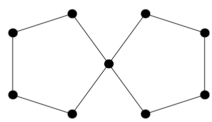

Figure 1: An example of a graph with no facial parity edge coloring using less than 10 colors.

Corollary 3.3. *Let G be a bridgeless cactus graph with no C_5. Then $\chi'_p(G) \leq 8$. Moreover, this bound is tight.*

Proof. First we show that among all cycles only the cycle on five edges requires 5 colors for a facial parity edge coloring. Let $n = 4k + z$, where k is a non-negative integer and $z \in \{2, 3, 4, 5\}$. If $z \neq 5$, then $\chi'_p(C_n) \leq 4$ (see the proof of Lemma 3.1). If $n = 4k + 5, k \geq 1$, then $\chi'_p(C_n) = 3$ (we repeat the pattern $1, 2, 3$ three times and then repeat $k - 1$ times the pattern $1, 2, 1, 2$).

Now we can proceed as in the proof of Theorem 3.2. □

Corollary 3.4. *Let G be a bridgeless cactus graph with no C_5 and no C_{4k}, $k \geq 1$. Then $\chi'_p(G) \leq 6$. Moreover, this bound is tight.*

The *dual* G^* of a plane graph G can be obtained as follows: Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*; two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G (an edge separates the faces incident with it). The *weak dual* of a plane graph G is the subgraph of the dual graph G^* whose vertices correspond to the bounded faces of G.

Lemma 3.5. *[7] The weak dual of an outerplane graph is a forest.*

We say that an edge coloring of a graph is *odd*, if each color class induces an odd subgraph (each vertex has an odd degree).

Lemma 3.6. *Let S_n be a star on n edges, $n \geq 1$. Then it has a facially proper odd edge coloring using at most 5 colors.*

Proof. We can use the coloring defined in the proof of Lemma 3.1. □

Corollary 3.7. *Let T be a tree. Then it has a facially proper odd edge coloring using at most 5 colors.*

Proof. Pick any vertex of T to be the root. We color the edges of T starting from the root to the leaves. In each step it is sufficient to find a facially proper odd edge coloring of a star with (at most) one precolored edge. □

Corollary 3.8. *Let F be a forest. Then it has a facially proper odd edge coloring using at most 5 colors.*

Theorem 3.9. *Let G be a 2-edge-connected outerplane graph. Then $\chi'_p(G) \leq 15$.*

Proof. First we color all the edges on the outer face with yellow color. The other edges let be green.

We successively contract the green edges and we obtain a graph H. The graph H is outerplane with no inner edge, hence, from Theorem 3.2 it follows that there exists a facial parity edge coloring of H with at most 10 colors.

In the following we extend the coloring of H to a coloring of G. Let F be the weak dual of G. Lemma 3.5 implies that F is a forest. By Corollary 3.8, F has a facially proper odd edge coloring which uses at most 5 colors. This coloring induces a coloring of the green edges of G in a natural way. The coloring of the yellow edges with at most 10 colors and the coloring of the green edges with at most 5 colors (these five colors are different than the previous ten ones) induce a required coloring of G. □

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References

[1] D. P. Bunde, K. Milans, D. B. West and H. Wu, Parity and strong parity edge-coloring of graphs, *Congressus Numerantium* **187** (2007), 193–213.

[2] D. P. Bunde, K. Milans, D. B. West and H. Wu, Optimal strong parity edge-coloring of complete graphs, *Combinatorica* **28** (2008), 625–632.

[3] J. Czap, Parity vertex coloring of outerplane graphs, *Discrete Math.* **311** (2011), 2570–2573.

[4] J. Czap, S. Jendroľ and F. Kardoš, Facial parity edge colouring, *Ars Math. Contemp.* **4** (2011), 255–269.

[5] J. Czap, S. Jendroľ and M. Voigt, Parity vertex colouring of plane graphs, *Discrete Math.* **311** (2011), 512–520.

[6] J. Czap and Zs. Tuza, Partitions of graphs and set systems under parity constraints, preprint, 2011.

[7] H. J. Fleischner, D. P. Geller and F. Harary, Outerplanar graphs and weak duals, *J. Indian Math. Soc.* **38** (1974), 215–219.

[8] T. Kaiser, O. Rucký, M. Stehlík and R. Škrekovski, Strong parity vertex coloring of plane graphs, *IMFM Preprint series* **49** (2011), #1144.