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10 Do we understand the internal spaces of second quantized fermion and boson fields, with gravity included?

The relation with strings theories

N.S. Mankoč Borštnik

Department of Physics, University of Ljubljana
SI-1000 Ljubljana, Slovenia

Abstract. The article proposes the description of internal spaces of fermion (quarks and leptons and antiquarks and antileptons) and boson (photons, weak bosons, gluons, gravitons and scalars) second quantized fields in an unique way if they all are massless. The internal spaces are described by "basis vectors" which are the superposition of odd (for fermions) and even (for bosons) products of the operators γ^a. For an arbitrarily symmetry $SO(d-1,1)$ of the internal spaces, it is the number of fermion fields (they appear in families and have their Hermitian conjugated partners in a separate group) equal to the number of boson fields (they appear in two orthogonal groups), manifesting a kind of supersymmetry, which differ of the string supersymmetry. On the assumption that fermions and bosons are active (they have momenta different from zero) only in $d = (3+1)$ ordinary space-time, bosons present vectors if they carry the space index $\mu = (0,1,2,3)$, and present scalars if they carry the index $\sigma \geq 5$. The author discusses this theory's latest achievements, with a trial to understand whether the extension to strings or to odd-dimensional spaces can lead to new kind of supersymmetry. This model, named *spin-charge-family* theory, manifests in a long series of papers the phenomenological success of the theory in elementary particle physics and cosmology.

Povzetek: Članek predlaga enoten opis notranjih prostorov brezmasnih fermionov (kvarkov in leptonov ter antikvarkov in antileptonov) in bozonov (fotonov, šibkih bozonov, gluonov, gravitonov in skalarjev) v drugi kvantizaciji. Notranji prostori so opisani z "bazičnimi vektorji", ki so superpozicija lihih (za fermione) in sodih (za bozone) produktov operaterjev γ^a. Za izbrano simetrijo $SO(d-1,1)$ notranjih prostorov je število fermionskih "bazičnih vektorjev" (pojavijo se v družinah, njihovi hermitsko konjugirani partnerji pa tvorijo ločeno grupo) enako številu bozonskih "bazičnih vektorjev" (pojavijo se v dveh ortogonalnih grupah), kar manifestira neko vrsto supersimetrije, ki se razlikuje od supersimetrije pri strunah. Ob predpostavki, da so fermioni in bozoni aktivni (imajo od nič različne gibalne količine) samo v $d = (3+1)$ prostoru-času, so bozoni vektorji, če nosijo indeks $\mu = (0,1,2,3)$ in so skalarji, če nosijo indeks $\sigma \geq 5$. Avtor predstavi zadnje dosežke te teorije, v katerih poskuša ugotoviti, ali razširitev "bazičnih vektorjev" v strune ali v lihe dimenzionalne prostore lahko vodi do nove vrste supersimetrije. Fenomenološko uspešnost teorije *spina-nabojev-družin* potrjujejo številne objave v uglednih revijah fizike osnovnih delcev in kozmologije.

10.1 Introduction

The proposed article discusses how all the fermion and boson fields can be treated uniquely if they all start as massless fields. One can, namely, describe the internal spaces of fermions and bosons by "basis vectors" which are products of nilpotents and projectors, chosen to be the eigenvectors of the Cartan subalgebra members of the Lorentz algebra in internal spaces of fermions and bosons [1–11].

Nilpotents are the superposition of an odd number of operators γ^a's ($\overset{ab}{(k)} := \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)$, $(\overset{ab}{(k)})^2 = 0$); projectors are the superposition of an even number of operators γ^a's ($\overset{ab}{[k]} := \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)$, $(\overset{ab}{[k]})^2 = \overset{ab}{[k]}$), with the properties : $S^{ab}\ \overset{ab}{(k)} = \frac{k}{2}\ \overset{ab}{(k)}$, $S^{ab}\ \overset{ab}{[k]} = \frac{k}{2}\ \overset{ab}{[k]}$ with $k^2 = \eta^{aa}\eta^{bb}$.

"Basis vectors" of fermions, chosen to be the algebraic products of an odd number of nilpotents (at least one, the rest are projectors), and of bosons, chosen to be the algebraic products of an even number of nilpotents (or only of projectors) are correspondingly eigenvectors of all $\frac{d(d-1)}{2}$ Cartan subalgebra members.

Fermion "basis vectors" appear in $2^{\frac{d}{2}-1}$ irreducible representations — families — each family having $2^{\frac{d}{2}-1}$ members. All the fermion "basis vectors" are mutually orthogonal, while the "basis vectors" fulfil together with their Hermitian conjugated partners, appearing in a separate group, the Dirac second quantization postulates for fermions, therefore, explaining the Dirac's postulates for fermions. Fermion "basis vectors" and their Hermitian conjugated partners have together 2^{d-1} members.

Boson "basis vectors" appear in two orthogonal groups, each of the two groups with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members have their Hermitian conjugated partners within the same group.

Correspondingly, the number of fermion "basis vectors" is equal to the number of boson "basis vectors" manifesting a kind of supersymmetry, which differ from the one offered by string theories [1], requiring the doubling of so far observed fermions and bosons.

Following the Grassmann algebra, the theory proposes two kinds of operators γ^a, namely γ^a's and $\tilde{\gamma}^a$'s [1–4].

Operators γ^a are used to describe the internal spaces of fermions, $\tilde{\gamma}^a$ determine the quantum numbers of families. The operators γ^a describe the internal space of bosons as well [2].

[1] The supersymmetry usually requires that to each fermion there exists a superpartner with the same charges but with spin zero; and for each spin 1 boson there exist a superpartner with the same charges but with spin 1/2.

[2] It is not difficult to reproduce the Dirac matrices in any $d = 2n$ dimensional space, but it is no need for this; the "basis vectors" are much more appropriate to work with, what it will be demonstrated in what follows. Let us point out that Dirac matrices were designed for massive fermions; while "basis vectors" describing internal spaces for fermions anti-commute, the Dirac matrices do not [26]. The families do not appear in the Dirac formulations of internal spaces of fermions.

The operators γ^a and $\tilde{\gamma}^a$ fulfil the following commutation relations: $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ , \{\gamma^a, \tilde{\gamma}^b\}_+ = 0 , (a, b) = (0, 1, 2, 3, 5, \cdots , d) , (\gamma^a)^\dagger = \eta^{aa} \gamma^a , (\tilde{\gamma}^a)^\dagger = \eta^{aa} \tilde{\gamma}^a$, making fermions and bosons second quantized fields since fermion "basis vectors" including odd products of nilpotents correspondingly anti-commute and boson "basis vectors" including even products of nilpotents commute [1–4, 7–10]. Although the multiplication of the fermion "basis vectors" by a γ^a generates the boson "basis vectors", and the multiplication of the boson "basis vectors" by a γ^a generates the fermion "basis vectors", these two kinds of "basis vectors" have completely different properties: Not only that the fermion "basis vectors" anti-commute while the boson "basis vectors" commute; the fermion "basis vectors" appear in $2^{\frac{d}{2}-1}$ families, each family having $2^{\frac{d}{2}-1}$ members, together with their Hermitian conjugated partners, which appear in a separate group, they have $2\times 2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members; the boson "basis vectors" appear in two orthogonal groups, each of the two groups with $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members have their Hermitian conjugated partners within the same group; the algebraic multiplication of one of fermion "basis vectors" with one of the Hermitian conjugated partner generates one of the Clifford even "basis vectors"; the algebraic multiplication of one of boson "basis vector" algebraically applying on a fermion "basis vector" generates a fermion "basic vector", demonstrating that since fermion "basis vectors" carry a half-integer value of the Cartan subalgebra members ($S^{ab} = \pm\frac{i}{2}$ or $S^{ab} = \pm\frac{1}{2}$, the same is true for $\tilde{S}^{ab} = \pm\frac{i}{2}$ or $\tilde{S}^{ab} = \pm\frac{1}{2}$), the boson "basis vectors" carry an integer value of the Cartan subalgebra members, determined by $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$. These, explained above, are valid in even dimensional spaces. In odd dimensional spaces [9], $d = (2n + 1), n \geq 0$, there are two different kinds of "basis vectors": One kind behaves as they do in $2n$ even dimensional spaces; the second kind with the same number of "basis vectors" as the first one behaves completely different — the anti commuting "basis vectors" appear in two orthogonal groups, each with their Hermitian conjugated partners in their group; the commuting "basis vectors" appear $2^{\frac{2n}{2}-1}$ families, each family having $2^{\frac{2n}{2}-1}$ members, while their Hermitian conjugated partners appear in a separate group. The second kind offers the presence of the Fadeev-Popov ghosts [22].

The proposed "basis vectors", describing the internal spaces of fermions and bosons in a tensor product with the basis in ordinary space-time determine creation and annihilation operators, which explain the Dirac's second quantized postulates. Under the assumption that fermions and bosons are active (have non zero momenta) in $d = (3 + 1)$ while manifesting the internal space in $d = 2(2n + 1)$ (the experiments and observations require $n = 3$), the theory offers the explanations of all the assumptions of the Standard model for fermions (quarks and leptons and antiquarks and antileptons), requiring the existence of a right handed neutrinos and left handed anti-neutrinos and of families; the theory explains the existence of photons, weak bosons and gluons, requiring the existence of gravitons.

The theory explains the existence of scalar fields as it is Higgs boson, predicting new scalar fields, which caused the inflation at the Big Bang, explains several cosmological observations.

We briefly discuss the case that the point fermion and boson fields are extended to strings, describing the fermion and boson fields in space-time $d = (3 + 1)$ as

tensor products of the "basis vectors" and basis in ordinary space-time extended to strings [11]. This article presents shortly also a possibility that the internal space of a string, which has $d=(1+1)$, Sect. 11.2.1, is used. This part is new and yet to be discussed. The longer version will be discussed in a separate article in this proceedings.
Sect. 10.2 presents the properties of "basis vectors" of fermions and bosons. We show how to construct the internal spaces of fermions and bosons, and also demonstrate that the internal spaces of bosons (photons, gravitons, gauge fields, scalar fields) are expressible as algebraic products of the fermion "basis vectors" and their Hermitian conjugated partners, Sect. 10.6, [10], meaning that we do not have to know bosons' internal spaces since they can be presented by algebraic products from fermions' ones.
In Subsect. 10.2.1 "basis vectors" for fermions and bosons in $d=2(2n+1)$-dimensional internal spaces from the point of view of $d=(3+1)$ are discussed.
Subsect. 10.2.2 presents the relations among the Clifford odd and the Clifford even "basis vectors".
Subsect. 10.2.3 discusses relations among fermions and their vector and scalar gauge fields under the assumptions that all the gauge fields are active (have non zero momentum) only in $d=(3+1)$.
Subsect. 10.2.4 discusses the case with $d=(13+1)$, which offers the explanation of all the assumptions of the *Standard model*, with the families included, requires the existence of right handed neutrinos and the left handed antineutrinos, the dark matter, predicts the fourth family to the observed three, offers the explain for the inflation after Bing Bang and explains also several cosmological observations.
Sect. 10.3 presents creation and annihilation operators for fermion and boson fields in $d=(3+1)$.
Sect. 11.2.1 discusses the possibility to extend ordinary space-time to strings, as well as the offer of the extension of an even dimensional space to one dimension more.
In Sect. 11.3, we shortly overview the whole talk, pointing out the open questions.

10.2 "Basis vectors" describing internal spaces of anticommuting fermion and commuting boson second quantized fields

This section is a short overview of the reference [7] (and the references [1, 3]) presenting the "basis vectors" of fermion internal spaces, while the presentation of the "basis vectors" describing the internal spaces of boson internal spaces follows references [1, 8, 10].
We could start with the Grassmann algebra [1, 8] which offers 2×2^d anticommuting operators in d-dimensional space [13], θ^a and $p^{\theta a}=i\frac{\partial}{\partial\theta_a}$ [1], fulfilling the relations $\{\theta^a,\theta^b\}_+=0$, $\{\frac{\partial}{\partial\theta_a},\frac{\partial}{\partial\theta_b}\}_+=0$, $\{\theta_a,\frac{\partial}{\partial\theta_b}\}_+=\delta_{ab}$, $(a,b)=(0,1,2,3,5,\cdots,d)$.

We shall rather use two kinds of the Clifford algebra elements (operators), γ^a and $\tilde{\gamma}^a$, expressible with θ^a's and their Hermitian conjugate momenta $p^{\theta a} = i\frac{\partial}{\partial\theta_a}$ [1],

$$\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,,$$
$$\theta^a = \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,, \tag{10.1}$$

offering together $2\cdot 2^d$ operators: 2^d are superposition of products of γ^a and 2^d of $\tilde{\gamma}^a$, with the properties

$$\{\gamma^a,\gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a,\tilde{\gamma}^b\}_+\,,$$
$$\{\gamma^a,\tilde{\gamma}^b\}_+ = 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,,$$
$$(\gamma^a)^\dagger = \eta^{aa}\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a\,. \tag{10.2}$$

Both kinds offer the description of the internal spaces of fermions with the "basis vectors" which are superpositions of odd products of either γ^a's or $\tilde{\gamma}^a$'s and fulfil correspondingly, the anti-commuting postulates of second quantized fermion fields, as well as the description of the internal spaces of boson fields with the "basis vectors" which are superposition of even products of either γ^a's or $\tilde{\gamma}^a$'s and fulfil correspondingly the commuting postulates of second quantized boson fields. *Since there are not two kinds of anti-commuting fermions, and not two corresponding kinds of their gauge fields, the postulate of Eq. (11.3) gives the possibility that only one of the two kinds of operators are used to describe fermions and their gauge fields, namely* γ^a's.

Postulating how does $\tilde{\gamma}^a$ operate on γ^a, reduces the two Clifford subalgebras, γ^a and $\tilde{\gamma}^a$, to the one described by γ^a [1,3,12]

$$\{\tilde{\gamma}^a B = (-)^B\, i B\gamma^a\}|\psi_{oc}>\,, \tag{10.3}$$

with $(-)^B = -1$, if B is (a function of) odd products of γ^a's, otherwise $(-)^B = 1$ [3], the vacuum state $|\psi_{oc}>$ will be defined in Eq. (10.10).
The operators $\tilde{\gamma}^a$'s can after the postulate, Eq. (11.3), be used to describe the quantum numbers of each of the $2^{\frac{d}{2}-1}$ irreducible representations (with $2^{\frac{d}{2}-1}$ members each, representing "families" of fermions) of the Lorentz group with the infinitesimal generators S^{ab} [3] by $\tilde{S}^{ab}$ [4]. We shall see [8] that the quantum numbers of each irreducible representation of the Lorentz group in the internal space of bosons are equal to $\mathcal{S}^{ab}$ $(= S^{ab} + \tilde{S}^{ab})$ [5].
We shall arrange all the "basis vectors" describing internal spaces of fermion and boson second quantized fields to be the eigenstates of the Cartan subalgebra

[3] Lorentz group has in the internal space of fermions the generators $S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a))$
[4] $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$.
[5] One can prove (or read in App. I of Ref. [7]) that the relations of Eq. (11.2) remain valid also after the *postulate*, presented in Eq. (11.3).

members of the Lorentz algebra, chosen to be

$$
\begin{aligned}
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\,d}\,,\\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\,d}\,,\\
&\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}\,.
\end{aligned}
\tag{10.4}
$$

In even-dimensional spaces there are $\frac{d}{2}$ members of the Cartan subalgebra, Eq. (11.4) [6].

We define [1,3] in even dimensional spaces $\frac{d}{2}$ nilpotents, $\overset{ab}{(k)}$, each nilpotent is a superposition of an odd number of γ^a's, and projectors, $\overset{ab}{[k]}$, each is a superposition of an even number of γ^a's,

$$
\begin{aligned}
\overset{ab}{(k)} &:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad (\overset{ab}{(k)})^2 = 0\,,\\
\overset{ab}{[k]} &:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad (\overset{ab}{[k]})^2 = \overset{ab}{[k]},
\end{aligned}
\tag{10.5}
$$

each nilpotent and projector is chosen to be the eigenvector of one of $\frac{d-1}{2}$ members of the Cartan subalgebra

$$
\begin{aligned}
S^{ab}\ \overset{ab}{(k)} &= \frac{k}{2}\ \overset{ab}{(k)}\,, \qquad \tilde{S}^{ab}\ \overset{ab}{(k)} = \frac{k}{2}\ \overset{ab}{(k)}\,,\\
S^{ab}\ \overset{ab}{[k]} &= \frac{k}{2}\ \overset{ab}{[k]}\,, \qquad \tilde{S}^{ab}\ \overset{ab}{[k]} = -\frac{k}{2}\ \overset{ab}{[k]}\,,
\end{aligned}
\tag{10.6}
$$

with $k^2 = \eta^{aa}\eta^{bb}$, pointing out that the eigenvalues of S^{ab} on projectors expressed with γ^a differ from the eigenvalues of $\tilde{S}^{ab}$ on projectors expressed with γ^a. Taking into account Eq. (11.2) one finds the relations

$$
\begin{aligned}
&\gamma^a\ \overset{ab}{(k)} = \eta^{aa}\ \overset{ab}{[-k]},\ \ \gamma^b\ \overset{ab}{(k)} = -ik\ \overset{ab}{[-k]},\ \ \gamma^a\ \overset{ab}{[k]} = \overset{ab}{(-k)},\ \ \gamma^b\ \overset{ab}{[k]} = -ik\eta^{aa}\ \overset{ab}{(-k)}\,,\\
&\tilde{\gamma^a}\ \overset{ab}{(k)} = -i\eta^{aa}\ \overset{ab}{[k]},\ \ \tilde{\gamma^b}\ \overset{ab}{(k)} = -k\ \overset{ab}{[k]},\ \ \tilde{\gamma^a}\ \overset{ab}{[k]} = i\ \overset{ab}{(k)},\ \ \tilde{\gamma^b}\ \overset{ab}{[k]} = -k\eta^{aa}\ \overset{ab}{(k)}\,,\\
&\overset{ab}{(k)}\overset{ab}{(-k)} = \eta^{aa}\ \overset{ab}{[k]}\,, \quad \overset{ab}{(-k)}\overset{ab}{(k)} = \eta^{aa}\ \overset{ab}{[-k]}\,, \quad \overset{ab}{(k)}\overset{ab}{[k]} = 0\,, \quad \overset{ab}{(k)}\overset{ab}{[-k]} = \overset{ab}{(k)}\,,\\
&\overset{ab}{(-k)}\overset{ab}{[k]} = \overset{ab}{(-k)}\,, \quad \overset{ab}{[k]}\overset{ab}{(k)} = \overset{ab}{(k)}\,, \quad \overset{ab}{[k]}\overset{ab}{(-k)} = 0\,, \quad \overset{ab}{[k]}\overset{ab}{[-k]} = 0\,,\\
&\overset{ab}{(k)}^{\dagger} = \eta^{aa}\ \overset{ab}{(-k)}\,, \quad (\overset{ab}{(k)})^2 = 0\,, \quad \overset{ab}{(k)}\overset{ab}{(-k)} = \eta^{aa}\ \overset{ab}{[k]}\,,\\
&\overset{ab}{[k]}^{\dagger} = \overset{ab}{[k]}\,, \quad (\overset{ab}{[k]})^2 = \overset{ab}{[k]}\,, \quad \overset{ab}{[k]}\overset{ab}{[-k]} = 0\,.
\end{aligned}
\tag{10.7}
$$

More relations are presented in App. A of Ref. [11].

We define [1,3,8] *in even dimensional spaces the "basis vectors" of fermion and boson second quantized fields as algebraic,* $*_A$*, products of nilpotents and projectors so that each product is an eigenvector of all* $\frac{d}{2}$ *Cartan subalgebra members,* Eq. (11.4).

Fermion"basis vectors" must be products of an odd number of nilpotents, at least one, the rest are projectors; boson "basis vectors" must be products of an even

[6] In odd-dimensional spaces there are $\frac{d-1}{2}$ members of the Cartan subalgebra.

number of nilpotents, the rest are projectors [7].

We recognize: Half of 2^{d} different products of γ^{a}'s are odd, and half of them are even. The Clifford odd "basis vectors" appear in $2^{\frac{d}{2}-1}$ irreducible representations, we call them families, each family having $2^{\frac{d}{2}-1}$ members (obtainable from any other member by S^{ab}; the family member of any other family is obtainable by $\tilde{S}^{ab}$). Since the Hermitian conjugated partner of a nilpotent $\overset{ab}{(k)}{}^{\dagger}$ is $\eta^{aa}\ \overset{ab}{(-k)}$, it follows that the Hermitian conjugated partners of the Clifford odd "basis vectors" with an odd number of nilpotents must belong to a different group of $2^{\frac{d}{2}-1}$ members in $2^{\frac{d}{2}-1}$ families.

The Clifford even "basis vectors" with an even number of nilpotents have their Hermitian conjugated partners within the same group; projectors are self adjoint, S^{ac} transforms $\overset{ab}{(k)} *_{A} \overset{cd}{(k')}$ into $\overset{ab}{[-k]} *_{A} \overset{cd}{[-k']}$, while $\tilde{S}^{ac}$ transforms $\overset{ab}{(k)} *_{A} \overset{cd}{(k')}$ into $\overset{ab}{[k]} *_{A} \overset{cd}{[k']}$, what follows if taking into account Eq. (10.7). Since the number of the Clifford odd products of γ^{a}'s is equal to the number of the Clifford even products of γ^{a}'s, there must be another group of the Clifford even "basis vectors" with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members.
We learn [8, 10] that one group of the Clifford even "basis vectors" transforms the family members of the Clifford odd "basis vectors" among themselves, while the second group of the Clifford even "basis vectors" transform any member of a family to the same member of another families.
Let us clear up that the algebraic application, $*_{A}$, of the Clifford even "basis vectors", we name them either ${}^{I}\hat{\mathcal{A}}_{f}^{m\dagger}$ if they are of the first kind or ${}^{II}\hat{\mathcal{A}}_{f}^{m\dagger}$ if they are of the second kind, on the Clifford odd "basis vectors", we name them $\hat{b}_{f'}^{m'\dagger}$, transforms the Clifford odd "basis vectors" into the Clifford odd "basis vectors", meaning that while the Clifford odd "basis vectors" carry the half integer values of the Cartan subalgebra members eigenvalues, $\pm\frac{i}{2}$ or $\pm\frac{1}{2}$, carry the Clifford even "basis vectors" the eigenvalues of the Cartan subalgebra members $(\pm i, 0)$ or $(\pm 1, 0)$, in agreement with $\mathcal{S}^{ab} = (S^{ab} + \tilde{S}^{ab})$ [8, 10].

10.2.1 "Basis vectors" in $d = 2(2n+1)$-dimensional internal spaces from the point of view of $d = (3+1)$

This part overviews several papers with the same topic ([7–10] and references therein).

i. *The Clifford odd "basis vectors"*

[7] An odd product of nilpotents anti-commute with another product of an odd number of nilpotents, explaining the anti-commutation postulates for fermions. To the creation operators, which are tensor products of the "basis vectors" and the basis in ordinary space, determine anti-commutativity the "basis vectors". Correspondingly the even products of nilpotents explain the commutation postulates for boson fields.

Let us start in $d = 2(2n+1)$ with the "basis vector" $\hat{b}^{1\dagger}_1$ which is the product of only nilpotents, all the rest members belonging to the $f = 1$ family follow by the application of S^{ab}, presented on the left-hand side of Eq. (10.8). Their Hermitian conjugated partners, $\hat{b}^m_f = (\hat{b}^{m\dagger}_f)^\dagger$, are presented on the right-hand side [8].

$$
\begin{aligned}
\hat{b}^{1\dagger}_1 &= (+i)^{03}(+)^{12}(+)^{56}\cdots(+)^{d-1\,d}\,, & \hat{b}^{1}_1 &= (-i)^{03}(-)^{12}\cdots(-)^{d-1\,d}\,,\\
\hat{b}^{2\dagger}_1 &= [-i]^{03}[-]^{12}(+)^{56}\cdots(+)^{d-1\,d}\,, & \hat{b}^{2}_1 &= [-i]^{03}[-]^{12}(-)^{56}\cdots(-)^{d-1\,d}\,,\\
&\cdots & &\cdots\\
\hat{b}^{2^{\frac{d}{2}-1}\,\dagger}_1 &= (+i)^{03}[-]^{12}[-]^{56}\ldots[-]^{d-3\,d-2}[-]^{d-1\,d}\,, & \hat{b}^{2^{\frac{d}{2}-1}}_1 &= (-i)^{03}[-]^{12}[-]^{56}\ldots[-]^{d-3\,d-2}[-]^{d-1\,d}\,,\\
&\cdots\,, & &\cdots\,.
\end{aligned}
\tag{10.8}
$$

All the members on the left hand side are orthogonal among themselves, and all the members of the right hand side are orthogonal among themselves, due to Eq. (10.7) [9]. The anti-commutation relations among the "basis vectors" and their Hermitian conjugated partners fulfil the anti-commutation relations postulated by Dirac for the second quantized fermion fields, Eq. (10.11).

It is easy to reproduce all the matrices postulated by Dirac for all S^{ab}, although it is not needed; the application of any S^{ab} on any of the members of any of the families is very simple. [10]

The application of $\tilde{S}^{ab}$, they do not appear among the Dirac operators, are equally simple. The application of γ^a's, as well as of $\tilde{\gamma}^a$'s, on these states needs presentation of the Clifford even "basis vectors", since they change the Clifford odd "basis

[8] The algebraic product mark, $*_A$, among nilpotents and projectors is skipped.

[9]
$$
\hat{b}^{m\dagger}_f *_A \hat{b}^{m'\dagger}_{f'} = 0\,, \quad \hat{b}^{m}_f *_A \hat{b}^{m'}_{f'} = 0\,, \quad \forall m, m', f, f'\,. \tag{10.9}
$$
Any member of $2^{\frac{d}{2}-1}$ families follow by the application of $\tilde{S}^{ab}$.

Choosing the vacuum state equal to
$$
|\psi_{oc}> = \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^m_f *_A \hat{b}^{m\dagger}_f\,|1>\,, \tag{10.10}
$$
for one of the members m, which can be anyone of the odd irreducible representations f it follows that the Clifford odd "basis vectors" obey the relations
$$
\begin{aligned}
\hat{b}^m_f *_A |\psi_{oc}> &= 0.|\psi_{oc}>\,,\\
\hat{b}^{m\dagger}_f *_A |\psi_{oc}> &= |\psi^m_f>\,,\\
\{\hat{b}^m_f, \hat{b}^{m'}_{f'}\}_{*_A +}|\psi_{oc}> &= 0.|\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_f, \hat{b}^{m'\dagger}_{f'}\}_{*_A +}|\psi_{oc}> &= 0.|\psi_{oc}>\,,\\
\{\hat{b}^m_f, \hat{b}^{m'\dagger}_{f'}\}_{*_A +}|\psi_{oc}> &= \delta^{mm'}\delta_{ff'}|\psi_{oc}>\,.
\end{aligned}
\tag{10.11}
$$

[10] Let be pointed out that the Dirac's matrices were constructed for the massive fermions, correspondingly they do not anticommute [26] .

vectors" into the Clifford even "basis vectors" [8, 10].

ii. *The Clifford even "basis vectors"*

The Clifford even "basis vectors" appear in two orthogonal groups, named ${}^{\mathrm{I}}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{\mathrm{II}}\hat{\mathcal{A}}^{m\dagger}_{f}$. Each group has $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members [11].
The generators S^{ab} and $\tilde{S}^{ab}$ generate from the starting "basis vector" of each group all the $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members. Each group contains the Hermitian conjugated partners within the same group; $2^{\frac{d}{2}-1}$ members of each group are products of only (self-adjoint) projectors, with all the eigenvalues of the Cartan subalgebra members, $\mathcal{S}^{ab}=(S^{ab}+\tilde{S}^{ab})$, presented in Eq. (11.4), equal zero.

$$
\begin{aligned}
{}^{\mathrm{I}}\hat{\mathcal{A}}^{1\dagger}_{1} &= \overset{03}{(+i)}\overset{12}{(+)}\cdots\overset{d-1\,d}{[+]}\,, & {}^{\mathrm{II}}\hat{\mathcal{A}}^{1\dagger}_{1} &= \overset{03}{(-i)}\overset{12}{(+)}\cdots\overset{d-1\,d}{[+]}\,,\\
{}^{\mathrm{I}}\hat{\mathcal{A}}^{2\dagger}_{1} &= \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)}\cdots\overset{d-1\,d}{[+]}\,, & {}^{\mathrm{II}}\hat{\mathcal{A}}^{2\dagger}_{1} &= \overset{03}{[+i]}\overset{12}{[-]}\overset{56}{(+)}\cdots\overset{d-1\,d}{[+]}\,,\\
{}^{\mathrm{I}}\hat{\mathcal{A}}^{3\dagger}_{1} &= \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-3\,d-2\;d-1\,d}{[-]\quad(-)}\,, & {}^{\mathrm{II}}\hat{\mathcal{A}}^{3\dagger}_{1} &= \overset{03}{(-i)}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-3\,d-2\;d-1\,d}{[-]\quad(-)}\,,\\
&\cdots & &\cdots
\end{aligned}
\tag{10.12}
$$

The Clifford even "basis vectors" belonging to two different groups are orthogonal due to the fact that they differ in the sign of one nilpotent or one projector or the algebraic product of a member of one group with a member of another group gives zero according to the third and fourth lines of Eq. (10.7): $\overset{ab\;ab}{(k)[k]}=0$, $\overset{ab\;ab}{[k](-k)}=0$, $\overset{ab\;ab}{[k][-k]}=0$.

$$
{}^{\mathrm{I}}\hat{\mathcal{A}}^{m\dagger}_{f}\ast_{A}\,{}^{\mathrm{II}}\hat{\mathcal{A}}^{m\dagger}_{f}=0={}^{\mathrm{II}}\hat{\mathcal{A}}^{m\dagger}{}_{.f}\ast_{A}\,{}^{\mathrm{I}}\hat{\mathcal{A}}^{m\dagger}_{f}\,. \tag{10.13}
$$

The members of each of these two groups have the property.

$$
{}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}\ \ast_{A}\ {}^{i}\hat{\mathcal{A}}^{m'\dagger}_{f'}\rightarrow\begin{cases}{}^{i}\hat{\mathcal{A}}^{m\dagger}_{f'}\,, i=(\mathrm{I},\mathrm{II})\\ \text{or zero}\,.\end{cases} \tag{10.14}
$$

For a chosen (m, f, f') there is only one m' (out of $2^{\frac{d}{2}-1}$) which gives a nonzero contribution.
Two "basis vectors", ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{i}\hat{\mathcal{A}}^{m'\dagger}_{f'}$, the algebraic product, $\ast_{A}$, of which gives non zero contribution, "scatter" into the third one ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f'}$, for $i=(\mathrm{I},\mathrm{II})$.

10.2.2 The relations among the Clifford odd and the Clifford even "basis vectors"

The algebraic application, $\ast_{A}$, of the Clifford even "basis vectors" ${}^{\mathrm{I}}\hat{\mathcal{A}}^{m\dagger}_{f}$ on the Clifford odd "basis vectors" $\hat{b}^{m'\dagger}_{f'}$ and the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$ on

[11] The members of one group can not be reached by the members of another group by either S^{ab}'s or $\tilde{S}^{ab}$'s or both.

${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ gives

$$ {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_{A} \hat{b}^{m'\dagger}_{f`} \to \begin{cases} \hat{b}^{m\dagger}_{f`}, \\ \text{or zero}, \end{cases} \tag{10.15} $$

$$ \hat{b}^{m\dagger}_{f} *_{A} {}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`} \to \begin{cases} \hat{b}^{m\dagger}_{f``}, \\ \text{or zero}, \end{cases} \tag{10.16} $$

while

$$ \hat{b}^{m\dagger}_{f} *_{A} {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`} = 0\,, \quad {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} *_{A} \hat{b}^{m'\dagger}_{f`} = 0\,, \ \forall (m, m`, f, f`)\,. \tag{10.17} $$

Eq. (11.10) demonstrates that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, applying on $\hat{b}^{m'\dagger}_{f`}$, transforms the Clifford odd "basis vector" into another Clifford odd "basis vector" of the same family, transferring to the Clifford odd "basis vector" integer spins or gives zero.
Scattering of the Clifford odd "basis vector" $\hat{b}^{m\dagger}_{f}$ on ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`}$ transforms the Clifford odd "basis vector" into another Clifford odd "basis vector" $\hat{b}^{m\dagger}_{f``}$ belonging to the same family member $m$ of a different family $f``$.

Both groups of Clifford even "basis vectors" manifest as the gauge fields of the corresponding fermion fields: One concerning the family members quantum numbers, the other concerning the family quantum numbers.

The Clifford even "basis vectors" can be represented as algebraic products of the Clifford odd "basis vectors" and their Hermitian conjugated partners [10]:
Knowing the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$ of one family (any one) we are able to generate all the Clifford even ${}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f`}$ "basis vectors"

$$ {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} = \hat{b}^{m'\dagger}_{f`} *_{A} (\hat{b}^{m''\dagger}_{f`})^{\dagger}\,. \tag{10.18} $$

Knowing the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$ of one family member (any one) of all families we are able to generate all the Clifford even ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`}$ "basis vectors"

$$ {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} = (\hat{b}^{m'\dagger}_{f`})^{\dagger} *_{A} \hat{b}^{m'\dagger}_{f``}\,. \tag{10.19} $$

10.2.3 Relations among fermions and their vector and scalar gauge fields under the assumptions that all the gauge fields are active (have non zero momentum) only in $d = (3+1)$

We will learn in Sect. 10.3 that all the Clifford even "basis vectors" have to carry the space index α which is equal to $\mu = (0,1,2,3)$ if they describe the vector component of the "basis vectors", and they are equal to $\sigma = (5,6,...)$ if describing the scalar components of the "basis vectors".

Let us start with few examples.

a. The $d = (1+1)$ is very simple. We present it since we shall use it when trying to extend point fermion and boson fields into strings.

We have, in this case, two Clifford odd and two Clifford even eigenvectors of the Cartan subalgebra members

$$\begin{aligned} &\text{Clifford odd}\\ &\hat{b}_1^{1\dagger} = \overset{01}{(+i)}\,, \quad \hat{b}_1^{1} = \overset{01}{(-i)}\,,\\ &\text{Clifford even}\\ &{}^{I}\mathcal{A}_1^{1\dagger} = \overset{01}{[+i]}\,, \quad {}^{II}\mathcal{A}_1^{1\dagger} = \overset{01}{[-i]}\ . \end{aligned} \tag{10.20}$$

The two Clifford odd "basis vectors" are Hermitian conjugated to each other. The choice is made that $\hat{b}_1^{1\dagger} = \overset{01}{(+i)}$ is the "basis vector", and the second Clifford odd object is its Hermitian conjugated partner. There is only one family ($2^{\frac{d}{2}-1} = 1$) with one member. The vacuum state, Eq.(10.10), is for this choice equal to $|\psi_{oc}> = \overset{01}{[-i]}$.
The eigenvalue S^{01} of $\hat{b}_1^{1\dagger}(= \overset{01}{(+i)})$ is $\frac{i}{2}$.
Each of the two Clifford even "basis vectors" is self adjoint ($({}^{I,II}\mathcal{A}_1^{1\dagger})^\dagger = {}^{I,II}\mathcal{A}_1^{1\dagger}$), with the eigenvalues $\mathcal{S}^{01}$ equal to 0.
We find, according to Eqs. (11.13, 11.14, 10.7), that

$${}^{I}\mathcal{A}_1^{1\dagger} = \hat{b}_1^{1\dagger} *_A (\hat{b}_1^{1\dagger})^\dagger\,, \qquad {}^{II}\mathcal{A}_1^{1\dagger} = (\hat{b}_1^{1\dagger})^\dagger *_A \hat{b}_1^{1\dagger}.$$

b. The case with $d = 2(2n+1)$, with $n = 1$, is more illustrative. App. 10.6 presents the Clifford odd and even "basis vectors" in Table 10.1.
We have in this case 16 odd "basis vectors"; 4 families with 4 members each, and 16 their Hermitian conjugated partners.
Each family can represent the internal spaces of "positron" with positive "charge", $S^{56} = \frac{1}{2}$, and of an "electron" with negative "charge", $S^{56} = -\frac{1}{2}$, as can be seen in Table 10.1. The "basis vectors" of the "positron" and "electron" are related by the charge conjugation operator [12], which includes in $d = (5+1)$ operators $\gamma^0\gamma^5$, transforming $\hat{b}_f^{1\dagger}$ into $\hat{b}_f^{1\dagger}$, of each family f.
The Clifford even "basis vectors" appear in Table 10.1 in two orthogonal groups, each group has 16 members, their Hermitian conjugated partners appear within the same group. The eigenvalues of the Cartan subalgebra members $\mathcal{S}^{ab} = (S^{ab} + \tilde{S}^{ab})$ are equal to either $(\pm i, 0)$ or $(\pm 1, 0)$.
In Tables (2, 3, 4, 5) of Ref. [10], all the 32 Clifford even "basis vectors" are expressed as the algebraic products of the Clifford odd "basis vectors" and their Hermitian conjugated partners. Two of these tables, Tables (10.2, 10.3), are presented also in App. 10.6 of this contribution.
We will learn in Sect. 10.3 that all the Clifford even "basis vectors" have to carry the space index α which is equal to $\mu = (0,1,2,3)$ if they describe the vector component of the "basis vectors", and they are equal to $\sigma = (5,6)$ if describing the scalar components of the "basis vectors".

Let us illustrate these two groups of the Clifford even "basis vectors":

[12] In Ref. [23], the discrete symmetry operators for fermion fields in $d = 2(2n+1)$ with the desired properties in $d = (3+1)$ are discussed.

Sixteen ${}^{\mathrm{I}}\mathcal{A}_{\mathrm{f}}^{\mathrm{m}\dagger}$ transform family members of any of 4 families among themselves. One can check that the Clifford even "basis vectors" can be written as the algebraic products of the Clifford odd "basis vectors" and their Hermitian conjugated partners, Eqs. (11.13, 11.14); ${}^{\mathrm{I}}\hat{\mathcal{A}}_{\mathrm{f}}^{\mathrm{m}\dagger} = \hat{b}_{\mathrm{f}'}^{\mathrm{m}'\dagger} *_{\mathrm{A}} (\hat{b}_{\mathrm{f}'}^{\mathrm{m}''\dagger})^{\dagger}$.
To describe "photons", the "basis vectors" ${}^{\mathrm{I}}\hat{\mathcal{A}}_{\mathrm{f}}^{\mathrm{m}\dagger}$ must not carry any non zero eigenvalues of the Cartan subalgebra members, Eq. (11.4), since the "photons" can give to "fermions" (to "electrons" and "positrons" in our case) only the spin offered by their space index μ. Table 10.2 presents "basis vectors" for four "photons", marked in Table 10.2 with $\bigcirc$. All are selfadjoint operators, offering to "basis vectors" of "electrons" and "positrons" no spin and no charge.
The remaining four Clifford even "basis vectors", appearing in the Hermitian conjugated pairs (marked either by $\triangle$ or by $\bullet$), would be allowed only if "fermions" and "bosons" have the non zero momenta in all six dimensions [13].
To describe "gravitons", the "basis vectors" ${}^{\mathrm{I}}\hat{\mathcal{A}}_{\mathrm{f}}^{\mathrm{m}\dagger}$ must be able to offer the internal spin $\mathcal{S}^{12}$, (± 1), (with $\mathcal{S}^{03}$, $(\pm \mathrm{i})$), together with the spin offered by their space index μ, while $\mathcal{S}^{56}$ eigenvalue must be equal to zero (must be described by a projector, not to be able to change the charge of "electrons" and "positrons"). Table 10.3 presents "basis vectors" for four "gravitons", appearing in two Hermitian conjugated pairs, marked in Table 10.3 either by $\ddagger$ or by $\odot\odot$.
The remaining four Clifford even "basis vectors", ${}^{\mathrm{I}}\hat{\mathcal{A}}_{\mathrm{f}}^{\mathrm{m}\dagger}$, appearing again in the Hermitian conjugated pairs (marked either by $\star\star$ or by $\otimes$), would be allowed only if "fermions" and "bosons" have the non zero momenta in all six dimensions [14].

Sixteen ${}^{\mathrm{II}}\mathcal{A}_{\mathrm{f}}^{\mathrm{m}\dagger}$ transform a family member m (any member) of a family f to the same family member m of all 4 families. One can check that the Clifford even "basis vectors" can be written as the algebraic products of the Clifford odd "basis vectors" and their Hermitian conjugated partners, Eq. (11.14); ${}^{\mathrm{II}}\hat{\mathcal{A}}_{\mathrm{f}}^{\mathrm{m}\dagger} = (\hat{b}_{\mathrm{f}'}^{\mathrm{m}'\dagger})^{\dagger} *_{\mathrm{A}} \hat{b}_{\mathrm{f}'}^{\mathrm{m}''\dagger}$.
In Tables (4,5) of Ref. [10] all the sixteen members are presented.
Also ${}^{\mathrm{II}}\mathcal{A}_{\mathrm{f}}^{\mathrm{m}\dagger}$ carry the space index α when taking part in creation operators. The space index α must be $\mu = (0, 1, 2, 3)$ when representing vector fields and $\sigma = (5, 6)$ when representing scalars [15].

If fermions and bosons are active (have non zero momenta) only in $\mathrm{d} = (3+1)$ "gravitons" and "photons" ${}^{\mathrm{II}}\mathcal{A}_{\mathrm{f}}^{\mathrm{m}\dagger}$ are only possible.

Let us present here one case among the 32 cases, presented in Table 5 of Ref. [10].

$$ {}^{\mathrm{II}}\hat{\mathcal{A}}_{1}^{4\dagger}(\equiv [\overset{03}{-\mathrm{i}}]\,[\overset{12}{-}][\overset{56}{-}]) = (\hat{b}_{4}^{1\dagger})^{\dagger} *_{\mathrm{A}} \hat{b}_{4}^{1\dagger}(\equiv (\overset{03}{-\mathrm{i}})\,(\overset{12}{-})(\overset{56}{-}) *_{\mathrm{A}} (\overset{03}{+\mathrm{i}})\,(\overset{12}{+})(\overset{56}{+})). $$

This can easily be checked if taking into account Eq. (11.2) or Eq. (10.7).

---

[13] The Clifford even "basis vectors" including two nilpotents, $(\overset{03}{\pm \mathrm{i}})(\overset{56}{\pm 1})$, would transform "electrons" into "positrons" and back, changing spin $\mathcal{S}^{03}$ and charge.

[14] These Clifford even "basis vectors", they include two nilpotents $(\overset{12}{\pm 1})(\overset{56}{\pm 1})$, would transform "electrons" into "positrons" and back, changing the spin $\mathcal{S}^{12}$ and charge.

[15] In Ref. ([7], and reference therein) the role of the scalars in the realistic case $\mathrm{d} = (13+1)$, suggested by experiments while representing Higgs's scalars, is explained.

One can check also that $\hat{b}_4^{1\dagger} *_A {}^{II}\hat{\mathcal{A}}_1^{4\dagger} = \hat{b}_4^{1\dagger}$. The "basis vector" ${}^{II}\hat{\mathcal{A}}_1^{4\dagger}$ with $\mathcal{S}^{03} = \mathcal{S}^{12} = \mathcal{S}^{56} = 0$ can transfer to an "electron" or "positron" only its space momentum while it cannot change their internal properties.

10.2.4 The case with $d = 2(2n + 1)$, with $n = 3$, is what the experiments suggest

We have learned so far that to know all the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members of the Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}_f^{m\dagger}$ we need to know all the members of one (any one) family (one irreducible representation) of the Clifford odd "basis vectors" $\hat{b}_f^{m\dagger}$, Eq. (11.13) (and their Hermitian conjugated partners).
To know all the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members of the Clifford even "basis vectors" ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ we need to know one family member (any one) of all the families, Eq. (11.14) (and their Hermitian conjugated partners).
Before generating the Clifford even "basis vectors" as the algebraic products, $*_A$, of members of the Clifford odd "basis vectors", representing quarks and leptons and antiquarks and antileptons, let us shortly overview the properties of one irreducible representation of quarks and leptons appearing together with the antiquark and antileptons in Table 10.4 of App. 10.7.

In the even dimensional space $d = (13 + 1)$ ([14], App. A), one irreducible representation of the Clifford odd "basis vectors" if analysed from the point of view of the subgroups of the group $SO(13,1)$ (including $SO(7,1) \times SO(6)$, while $SO(7,1)$ breaks into $SO(3,1) \times SO(4)$, and $SO(6)$ breaks into $SU(3) \times U(1)$) contains the Clifford odd "basis vectors" describing internal spaces of quarks and leptons and antiquarks and antileptons, manifesting at low energies the quantum numbers assumed by the *standard model* before the electroweak break. Since $SO(4)$ contains two $SU(2)$ subgroups, $SU(2)_I$ and $SU(2)_{II}$, with the hypercharge of the *standard model* $Y = \tau^{23} + \tau^4$ (with τ^{23} belonging to $SU(2)_{II}$ and τ^4 originating in $SO(6)$, breaking to $SU(3) \times U(1)$), one irreducible representation includes the right-handed neutrinos and the left-handed antineutrinos, which are not in the *standard model* scheme [16].

In Table 10.4, one can read the quantum numbers of the Clifford odd "basis vectors" representing quarks and leptons *and antiquarks and antileptons* if taking into account that all the nilpotents and projectors are eigenvectors of one of the Cartan subalgebra members, $(S^{03}, S^{12}, S^{56}, \ldots, S^{13\,14})$, with the eigenvalues $\pm\frac{i}{2}$ or

[16] For an overview of the properties of the vector and scalar gauge fields in the *spin-charge-family* theory, the reader is invited to reed Refs. ([5,7,8,24,25] and the references therein). The reader can find in Table 10.4 that "basis vectors" of quarks have identical content of $SO(7,1)$ as "basis vectors" of leptons (and antiquarks as antiletons); they differ only in the $SU(3) \times U(1)$.

$\pm\frac{1}{2}$, while for $(\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56}, \cdots, \mathcal{S}^{13\,14})$ the eigenvalues for $\overset{ab}{(\pm i)}$ are $\pm i$, and for $\overset{ab}{(\pm)}$ are ± 1, and for $\overset{ab}{[\pm i]}$ and $\overset{ab}{[\pm]}$ the eigenvalues are zero [17].
Taking into account that the third component of the *standard model* weak charge, $\tau^{13} = \frac{1}{2}(S^{56} - S^{78})$, of the third component of the second $SU(2)_{II}$ charge (not appearing in the *standard model*) $\tau^{23} = \frac{1}{2}(S^{56} + S^{78})$, of the colour charge $[\tau^{33} = \frac{1}{2}(S^{9\,10} - S^{11\,12})$ and $\tau^{38} = \frac{1}{2\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})]$, of the "fermion charge" $\tau^4 = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})$, of the hyper charge $Y = \tau^{23} + \tau^4$, and of the electromagnetic charge $Q = Y + \tau^{13}$, one reproduces all the quantum numbers of quarks, leptons, and antiquarks, and antileptons.

Let us represent internal spaces (that is the "basis vectors") of photons, gravitons, weak bosons and coloured bosons in the way we learned in the cases **iv.b**, that is as algebraic products of the Clifford odd "basis vectors" (represented in Table 10.4 as $u^{ci}_R, d^{ci}_R, u^{ci}_L, d^{ci}_L$ for quarks of colour ci, as $\bar{u}^{\bar{ci}}_L, \bar{d}^{\bar{ci}}_L, \bar{u}^{\bar{ci}}_R, \bar{d}^{\bar{ci}}_R$ for antiquarks of the opposite colour, as ν_R, e_R, ν_L, e_L for leptons, $\bar{\nu}_L, \bar{e}_L, \bar{\nu}_R, \bar{e}_R$ for the corresponding antileptons, their Hermitian conjugated partners will be written as $(\ )^\dagger$).

Starting from photons we need to have in mind that photons can give to quarks and leptons only momentum in ordinary space, but cannot change internal spaces of quarks and leptons.
Let us look in Table 10.4 for $u^{c\bar{1}}_R$, first line ([10], Eq. (46)). The photon ${}^{I}\hat{\mathcal{A}}^{\dagger}_{ph\,\bar{u}^{c\bar{1}}_R \to \bar{u}^{c\bar{1}}_R}$ interacts with $\bar{u}^{c\bar{1}}_{R\,39^{th}}$ as follows

$$
\begin{aligned}
&{}^{I}\hat{\mathcal{A}}^{\dagger}_{ph\,\bar{u}^{c\bar{1}}_R \to \bar{u}^{c\bar{1}}_R}(\equiv \overset{03\ \ 12\ \ 56\ \ 78\ 9\,10\,11\,12\,13\,14}{[+i][+][-][+][-]\ [+]\ \ [+]}\ ) \ast_A \bar{u}^{c\bar{1}}_{R\,39^{th}}(\equiv \overset{03\ \ 12\ \ 56\ \ 78\ 9\,10\,11\,12\,13\,14}{(+i)[+](-)(+)[-]\,(+)\ (+)}\ )\\
&\quad \to \bar{u}^{c\bar{1}}_{R\,39^{th}}(\equiv \overset{03\ \ 12\ \ 56\ \ 78\ 9\,10\,11\,12\,13\,14}{(+i)[+](-)(+)[-]\,(+)\ (+)}\ ),\ {}^{I}\hat{\mathcal{A}}^{\dagger}_{ph\,\bar{u}^{c\bar{1}}_R \to \bar{u}^{c\bar{1}}_R} = u^{c\bar{1}}_{R\,39^{th}} \ast_A (u^{c\bar{1}}_{R\,39^{th}})^{\dagger}.
\end{aligned}
\tag{10.21}
$$

Similarly, one finds photons interacting with the rest of quarks and electrons and antiquarks and positrons [18].

Having in mind that bosons have an even number of nilpotents, and taking $u^{c\bar{1}}_R$ from the first and the second line of Table 10.4, we find that the "basis vector" of

[17] Let us remind the reader the difference between eigenvectors of S^{ab} and $\tilde{S}^{ab}$: Applying on a nilpotent they both give the same eigenvalue, while they give opposite eigenvalue if applying on a projector, Eq. (11.7).

[18] The break of symmetry, discussed in Ref. [7], but not yet really understood in this new way of presenting the internal space of boson fields, prevents neutrinos to interact with photons. Let us add that photons can not directly interact with another photons since the algebraic products of different selfadjoint operators are zero, but can interact with other boson fields, for example, with gravitons [10]. However, algebraic product of the "basis vector" which is a product of projectors by itself is the same "basis vector" back.

the "graviton" ${}^{\mathrm{I}}\hat{\mathcal{A}}^{\dagger}_{\mathrm{gr}\, u^{c1}_{R\uparrow}\to u^{c1}_{R\downarrow}}$ applies on $u^{c1}_{R\,1^{st}}$ as follows

$$
\begin{aligned}
{}^{\mathrm{I}}\hat{\mathcal{A}}^{\dagger}_{\mathrm{gr}\, u^{c1}_{R\uparrow}\to u^{c1}_{R\downarrow}}(\equiv\overset{03\ \ 12\ \ 56\ \ 78\ \ 9\,10\,11\ 12\,13\,14}{(-\mathrm{i})(-)[+][+][+]\,[-]\ \ [-]\,})\ *_{\mathrm{A}}\ u^{c1}_{R\,1^{st}}(\equiv\overset{03\ \ 12\ \ 56\ \ 78\ \ 9\,10\,11\ 12\,13\,14}{(+\mathrm{i})[+][+](+)(+)\,[-]\ \ [-]\,})\\
\to u^{c1}_{R\,2^{nd}}(\equiv\overset{03\ \ 12\ \ 56\ \ 78\ \ 9\,10\,11\ 12\,13\,14}{[-\mathrm{i}](-)[+](+)(+)\,[-]\ \ [-]\,}),\ {}^{\mathrm{I}}\hat{\mathcal{A}}^{\dagger}_{\mathrm{gr}\, u^{c1}_{R\uparrow}\to u^{c1}_{R\downarrow}} = u^{c1}_{R\,2^{nd}}\ *_{\mathrm{A}}\ (u^{c1}_{R\,1^{st}})^{\dagger},
\end{aligned}
\tag{10.22}
$$

Similarly we can find the "basis vectors" of all gravitons transforming the spins S^{03} and S^{12} in internal spaces of quarks and leptons. The gravitons carry in addition the momentum in ordinary space time.

Looking for the weak bosons, which transform $d^{c1}_{L\,5th}$ (presented in Table 10.4 in 5^{th} line) to $u^{c1}_{L\,7^{th}}$ (presented in Table 10.4 in 7^{th} line), we find

$$
\begin{aligned}
{}^{\mathrm{I}}\hat{\mathcal{A}}^{\dagger}_{w1\, d^{c1}_{L}\to u^{c1}_{L}}(\equiv\overset{03\ \ 12\ \ 56\ \ 78\ \ 9\,10\,11\ 12\,13\,14}{[-\mathrm{i}][+](+)(-)[+]\,[-]\ \ [-]\,})\ *_{\mathrm{A}}\ d^{c1}_{L\,5th}(\equiv\overset{03\ \ 12\ \ 56\ \ 78\ \ 9\,10\,11\ 12\,13\,14}{[-\mathrm{i}][+](-)(+)(+)\,[-]\ \ [-]\,})\\
\to u^{c1}_{L\,7th}(\equiv\overset{03\ \ 12\ \ 56\ \ 78\ \ 9\,10\,11\ 12\,13\,14}{[-\mathrm{i}][+][+][-](+)\,[-]\ \ [-]\,})(\equiv\overset{03\ \ 12}{[-\mathrm{i}][+]},\ {}^{\mathrm{I}}\hat{\mathcal{A}}^{\dagger}_{w1\, d^{c1}_{L}\to u^{c1}_{L}} = u^{c1}_{L\,7th}\ *_{\mathrm{A}}\ (d^{c1}_{L\,5th})^{\dagger}.
\end{aligned}
\tag{10.23}
$$

We can find equivalently the weak bosons "basis vectors" which cause transformations among other weak pairs.

Let us look for the 'basis vectors" of gluons, transforming colour charges of quarks, for example, of $d^{c1}_{L\,5^{th}}$ (presented in Table 10.4 in 5^{th} line) to $d^{c3}_{L\,21^{st}}$ (presented in Table 10.4 in 21^{st} line)

$$
\begin{aligned}
{}^{\mathrm{I}}\hat{\mathcal{A}}^{\dagger}_{gl\, d^{c1}_{L}\to d^{c3}_{L}}(\equiv\overset{03\ \ 12\ \ 56\ \ 78\ \ 9\,10\,11\ 12\,13\,14}{[-\mathrm{i}][+][-][+](-)\,[-]\ \ (+)})\ *_{\mathrm{A}}\ d^{c1}_{L\,5^{th}}(\equiv\overset{03\ \ 12\ \ 56\ \ 78\ \ 9\,10\,11\ 12\,13\,14}{[-\mathrm{i}][+](-)(+)(+)\,[-]\ \ [-]\,})\\
\to d^{c3}_{L\,21^{st}}(\equiv\overset{03\ \ 12\ \ 56\ \ 78\ \ 9\,10\,11\ 12\,13\,14}{[-\mathrm{i}][+](-)(+)[-]\,[-]\ \ (+)}),\quad {}^{\mathrm{I}}\hat{\mathcal{A}}^{\dagger}_{gl\, d^{c1}_{L}\to d^{c3}_{L}} = d^{c3}_{L\,21^{st}}\ *_{\mathrm{A}}\ (d^{c1}_{L\,5^{th}})^{\dagger}.
\end{aligned}
\tag{10.24}
$$

Let us conclude this section by repeating what we have learned in it: This section presents the internal degrees of freedom as the Clifford odd "basis vectors" for fermion fields and as the Clifford even "basis vectors" for boson fields for internal spaces $d = 2(2n+1), n = 0, 1$ and for $n = 3$. The creation operators for both fields are tensor products of "basis vectors" and basis in ordinary space-time. The Clifford even "basis vectors" of boson fields carry in addition the space index α, Sect. 10.3.

There is the same number of the "basis vectors" of fermion and boson fields, manifesting a kind of supersymmetry. While fermion fields appear in families and have their Hermitian conjugated partners in a separate groups, appear boson "basis vectors" in two orthogonal groups, with their Hermitian conjugated partners within each group.

There is the break of symmetries which makes the number of observed families of fermions and their observed gauge fields reduced [7]. But the assumption that the

fermion and boson fields have non zero momenta only in $d=(3+1)$ breaks the Lorentz symmetry; the rotations $M^{ms}=L^{ms}+S^{ms}$, $m=(0,1,2,3$, $s=(5,6,...)$, (S^{ms} can be replaced by $\tilde{S}^{ms}$ and $\mathcal{S}^{ms}$), for example, can not go.

To extend point fields to strings, we can use the "basis vectors" for the extended part, presented in Eq. (11.18). For the extension of the fermion fields we can have $\hat{b}^{1\dagger}_1 = \overset{01}{(+i)}$ and for its Hermitian conjugated partner $\hat{b}^1_1 = (\hat{b}^{1\dagger}_1)^\dagger = \overset{01}{(-i)}$. For the two boson fields we have ${}^{I}\mathcal{A}^{1\dagger}_1 = \overset{01}{[+i]}$ and ${}^{II}\mathcal{A}^{1\dagger}_1 = \overset{01}{[-i]}$ [19].

Sect. 11.2.1 discusses a possible extension of point fields to strings.

10.3 Creation and annihilation operators for fermion and boson fields in $d=(3+1)$

To define the creation operators for fermion or boson fields, besides the "basis vectors" defining the internal spaces of fermions and bosons, the basis in ordinary space in momentum or coordinate representation is needed. Let us shortly overview Ref. [10], Sect. 3. (The extended version is presented in Ref. [7] in Subsect. 3.3 and App. J.)
The momentum basis is continuously infinite

$$
\begin{aligned}
|\vec{p}> = \hat{b}^\dagger_{\vec{p}}\,|0_p>\,, \quad <\vec{p}\,| =< 0_p\,|\,\hat{b}_{\vec{p}}\,,\\
<\vec{p}\,|\,\vec{p}\,'> = \delta(\vec{p}-\vec{p}\,') =< 0_p\,|\hat{b}_{\vec{p}}\;\hat{b}^\dagger_{\vec{p}\,'}|\,0_p>\,,\\
\text{pointing out}\\
< 0_p\,|\hat{b}_{\vec{p'}},\,\hat{b}^\dagger_{\vec{p}}\,|\,0_p> = \delta(\vec{p'}-\vec{p})\,,
\end{aligned}
\tag{10.25}
$$

with the normalization $< 0_p\,|\,0_p> = 1$. While the quantized operators $\hat{\vec{p}}$ and $\hat{\vec{x}}$ commute $\{\hat{p}^i\,,\hat{p}^j\}_- = 0$, $\{\hat{x}^k\,,\hat{x}^l\}_- = 0$, it follows for $\{\hat{p}^i\,,\hat{x}^j\}_- = i\eta^{ij}$. One correspondingly finds

$$
\begin{aligned}
&< \vec{p}\,|\,\vec{x}> =< 0_{\vec{p}}\,|\,\hat{b}_{\vec{p}}\;\hat{b}^\dagger_{\vec{x}}|0_{\vec{x}}> = (< 0_{\vec{x}}\,|\,\hat{b}_{\vec{x}}\;\hat{b}^\dagger_{\vec{p}}\;|0_{\vec{p}}>)^\dagger\\
&< 0_{\vec{p}}\,|\{\hat{b}^\dagger_{\vec{p}}\,,\,\hat{b}^\dagger_{\vec{p}\,'}\}_-|0_{\vec{p}}> = 0\,, < 0_{\vec{p}}\,|\{\hat{b}_{\vec{p}},\,\hat{b}_{\vec{p}\,'}\}_-|0_{\vec{p}}> = 0\,, < 0_{\vec{p}}\,|\{\hat{b}_{\vec{p}},\,\hat{b}^\dagger_{\vec{p}\,'}\}_-|0_{\vec{p}}> = 0\,,\\
&< 0_{\vec{x}}\,|\{\hat{b}^\dagger_{\vec{x}},\,\hat{b}^\dagger_{\vec{x}\,'}\}_-|0_{\vec{x}}> = 0\,, < 0_{\vec{x}}\,|\{\hat{b}_{\vec{x}},\,\hat{b}_{\vec{x}\,'}\}_-|0_{\vec{x}}> = 0\,, < 0_{\vec{x}}\,|\{\hat{b}_{\vec{x}},\,\hat{b}^\dagger_{\vec{x}\,'}\}_-|0_{\vec{x}}> = 0\,,\\
&< 0_{\vec{p}}\,|\{\hat{b}_{\vec{p}},\,\hat{b}^\dagger_{\vec{x}}\}_-|0_{\vec{x}}> = e^{i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,, < 0_{\vec{x}}\,|\{\hat{b}_{\vec{x}},\,\hat{b}^\dagger_{\vec{p}}\}_-|0_{\vec{p}}> = e^{-i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,.
\end{aligned}
\tag{10.26}
$$

The creation operators for either fermion or boson fields must be tensor products, $*_T$, of both contributions, the "basis vectors" describing the internal space of fermions or bosons and the basis in ordinary momentum or coordinate space.

[19] Let us repeat that knowing the fermion "basis vectors" allows us to find both kinds of "boson fields", ${}^{I}\mathcal{A}^{1\dagger}_1 = \hat{b}^{1\dagger}_1 *_A (\hat{b}^{1\dagger}_1)^\dagger = \overset{01}{(+i)} *_A \overset{01}{(-i)} = \overset{01}{[+i]}$, and ${}^{II}\mathcal{A}^{1\dagger}_1 = (\hat{b}^{1\dagger}_1)^\dagger *_A \hat{b}^{1\dagger}_1 = \overset{01}{(-i)} *_A \overset{01}{(+i)} = \overset{01}{[-i]}$.

The creation operators for a free massless fermion field of the energy $p^0 = |\vec{p}|$, belonging to a family f and to a superposition of family members m applying on the extended vacuum state including both spaces, $|\psi_{oc}> *_T |0_{\vec{p}}>$, can be written as

$$\hat{b}^{s\dagger}_f(\vec{p}) = \sum_m c^{sm}{}_f(\vec{p})\, \hat{b}^{\dagger}_{\vec{p}} *_T \hat{b}^{m\dagger}_f \,. \tag{10.27}$$

The creation operators $\hat{b}^{s\dagger}_f(\vec{p})$ and their Hermitian conjugated partners annihilation operators $\hat{b}^{s}_f(\vec{p})$, creating and annihilating the single fermion states, respectively, fulfil when applying the vacuum state, $|\psi_{oc}> *_T |0_{\vec{p}}>$, the anti-commutation relations for the second quantized fermions, postulated by Dirac (Ref. [8], Sect.3), explaining the Dirac's second quantization postulates for fermions.

$$\begin{aligned}
< 0_{\vec{p}} |\{\hat{b}^{s'}_{f'}(\vec{p'}) , \hat{b}^{s\dagger}_f(\vec{p})\}_+ |\psi_{oc}> |0_{\vec{p}}> &= \delta^{ss'}\delta_{ff'}\,\delta(\vec{p}' - \vec{p}) \cdot |\psi_{oc}>, \\
\{\hat{b}^{s'}_{f'}(\vec{p'}) , \hat{b}^{s}_f(\vec{p})\}_+ |\psi_{oc}> |0_{\vec{p}}> &= 0 \cdot |\psi_{oc}> |0_{\vec{p}}>, \\
\{\hat{b}^{s'\dagger}_{f'}(\vec{p'}) , \hat{b}^{s\dagger}_f(\vec{p})\}_+ |\psi_{oc}> |0_{\vec{p}}> &= 0 \cdot |\psi_{oc}> |0_{\vec{p}}>, \\
\hat{b}^{s\dagger}_f(\vec{p}) |\psi_{oc}> |0_{\vec{p}}> &= |\psi^s_f(\vec{p})>, \\
\hat{b}^{s}_f(\vec{p}) |\psi_{oc}> |0_{\vec{p}}> &= 0 \cdot |\psi_{oc}> |0_{\vec{p}}>, \\
|p^0| &= |\vec{p}|\,.
\end{aligned} \tag{10.28}$$

The creation operators for boson gauge fields must carry the space index α, describing the α component of the boson field in the ordinary space [10], Eq. (24)). We, therefore, add the space index α as follows

$${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f\alpha}(\vec{p}) = {}^{i}\hat{\mathcal{C}}^{m}{}_{f\alpha}(\vec{p}) *_T {}^{i}\hat{\mathcal{A}}^{m\dagger}_f \,, i = (I, II)\,, \tag{10.29}$$

with ${}^{i}\hat{\mathcal{C}}^{m}{}_{f\alpha}(\vec{p}) = {}^{i}\mathcal{C}^{m}{}_{f\alpha}\, \hat{b}^{\dagger}_{\vec{p}}$, with $\hat{b}^{\dagger}_{\vec{p}}$ defined in Eqs. (11.15, 10.26). We treat free massless bosons of momentum $\vec{p}$ and energy $p^0 = |\vec{p}|$ and of particular "basis vectors" ${}^{i}\hat{\mathcal{A}}^{m\dagger}_f$'s which are eigenvectors of all the Cartan subalgebra members. The creation operators for boson gauge fields commute.

10.4 Points in ordinary space-time extended to strings; representations of fermions and bosons in odd dimensional spaces

One possibility to achieve renormalizability of the proposed theory might be, learning from string theories [18, 19], by extending the points in ordinary space-time to strings. A second possibility might be to find out what can offer the internal odd-dimensional spaces, manifesting two groups of anti-commuting "basis vectors" and two groups of commuting "basis vectors", as mentioned in Sect. 11.1 and explained in Ref. [9, 22]. One of the two groups of either anti-commuting or commuting "basis vectors" manifest a kind of Fadeev-Popov ghost.

Let us start with the first possibility: extending the points in ordinary space-time to strings.

Let us try to extend the fermion and boson fields, expressed as tensor products, $*_T$, of the "basis vectors" describing the internal spaces of fermion and boson fields and the basis in ordinary space-time to strings, in the hope of achieving renormalizability for the theory of free massless fermion and boson fields. As we discussed in Sect. 10.2, the number of "basis vectors" which represent internal spaces of fermions and the number of their Hermitian conjugated partners (both together have $2 \times 2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$), equals to the number of "basis vectors" representing the internal spaces of bosons, appearing in two orthogonal groups, 2^{d-1} [20], manifesting a kind of supersymmetry. (This is, however, not the usual kind of supersymmetry requiring that each fermion has a partner with the same charge among bosons, and vice versa.

We shall see that a trial to extend point particles in ordinary space-time to strings in a "stringy way" requires to introduce the strings time, besides the ordinary space-time and correspondingly also the corresponding "basis vectors" in $d = (1+1)$ in a string.

We assume that fermion and boson fields are active (having non-zero momenta) only in $d = (3+1)$ [21], while we choose $d = 2(2n+1)$ in the internal space. The observations suggest $n = 3$.

Let us first assume a simple starting action, simplifying the action in Ref ([7] and in the references therein) for free massless fermion fields, and the corresponding free massless boson fields in $d = 2(2n+1)$-dimensional space, having non zero momenta only in $d = (3+1)$, while taking into account the creation operators for fermion and boson fields expressed by the corresponding "basic vectors", Eqs. (11.16, 11.17) [22].

$$
\begin{aligned}
\mathcal{A} &= \int d^4x \, \frac{1}{2} (\bar{\psi} \gamma^\alpha p_{0\alpha} \psi) + h.c. + \\
&\quad \int d^4x \sum_{i=(I,II)} {}^i\hat{F}^{mf}_{\alpha\beta} \, {}^i\hat{F}^{mf\alpha\beta} \,, \\
p_{0\alpha} &= p_\alpha - \sum_{mf} {}^I\hat{\mathcal{A}}^{m\dagger}_f \, {}^I\mathcal{C}^m_{f\alpha} - \sum_{mf} {}^{II}\hat{\mathcal{A}}^{m\dagger}_f \, {}^{II}\mathcal{C}^m_{f\alpha} \,, \\
{}^i\hat{F}^{mf}_{\alpha\beta} &= \partial_\alpha \, {}^i\hat{\mathcal{A}}^{m\dagger}_{f\beta} - \partial_\beta \, {}^i\hat{\mathcal{A}}^{m\dagger}_{f\alpha} + \varepsilon f^{m''f''mfm'f'i} \, {}^i\hat{\mathcal{A}}^{m\dagger}_{f\alpha} \, {}^i\hat{\mathcal{A}}^{m'\dagger}_{f'\beta} \,, \\
&\quad i = (I, II) \,. \qquad (10.30)
\end{aligned}
$$

[20] One group of boson "basis vectors", while applying to any of members of fermion families, forms a member of the same family. The second group of boson "basis vectors", when applying to a particular member of a family transforms this member into the same member of another family, Subsect. 10.2.4.

[21] This assumption is not needed, but it seems meaningful.

[22] The fermion states ψ are defined by $\hat{b}^{s\dagger}_f(\vec{x}, x^0) = \sum_m \hat{b}^{m\dagger}_f \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} c^{ms}{}_f \, (\vec{p}) \, \hat{b}^\dagger_{\vec{p}} \, e^{-i(p^0 x^0 - \varepsilon \vec{p}\cdot\vec{x})}$, applying on the vacuum state $|\psi_{oc}> \, *_T |0_{\vec{p}}>$. The boson states are defined by ${}^I\hat{\mathcal{A}}^{m\dagger}_{f\mu}(\vec{x}, x^0) = \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} \, {}^I\hat{\mathcal{A}}^{m\dagger}_{f\mu}(\vec{p}) \, e^{-i(p^0 x^0 - \varepsilon \vec{p}\cdot\vec{x})}|_{p^0=|\vec{p}|}$, with the vacuum state equal to $|\, 1>$.

ψ represent $2^{\frac{d}{2}-1}$ members of one family of all the $2^{\frac{d}{2}-1}$ families, and $\bar{\psi} = (\psi)^\dagger\gamma^0$ their $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Hermitian conjugated partners (multiplied by γ^0). ${}^i\hat{\mathcal{A}}_f^{m\dagger}$ ${}^i\mathcal{C}^m_{f\alpha}, i = (I, II)$, represent two orthogonal groups of "basis vectors" in a tensor product with basis in ordinary space-time, extended by the space index $\alpha = \mu$ for vectors and $\alpha \geq 5$ for scalars in ordinary space-time, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members [23]. The assumed action, Eq. (10.30), needs further studies, since the studies so far were mainly done by Eqs. (100, 101) of the reference [7].

Let us repeat: The description of the internal spaces of fermion and boson fields with the odd and even Clifford algebra objects, respectively, offers an equal number of fermions and bosons, demonstrating a (kind of) supersymmetry. However, none of the "basis vectors" of the boson fields, obtained by the application of γ^a on the fermion "basis vectors" (the application from the left-hand side generates ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$, the application from the right-hand side generates ${}^{I}\hat{\mathcal{A}}_f^{m\dagger}$) do carry the same internal charges as the fermion "basis vectors" [24].

We represent in Sect. 10.2 the "basis vectors" of fermion and boson fields for any $d = 2(2n+1)$, in particular also for $d = (13+1)$ [25].

We demonstrated the "basis vectors" for the corresponding vector and scalar gauge fields observed so far, Subsect. 10.2.4: Photons, gravitons, weak bosons and gluons. [26]

---

[23] The assumption that all the fermion and boson fields are active (have non zero momentum) only in $d = (3+1)$ ordinary space-time mean that the derivative with respect to x_α gives zero.

[24] The quantum numbers of $u^{c\,1}_L$, presented in Table 10.4 in the seventh line $u^{c\,1}_L (\equiv \overset{03}{[-i]}\,\overset{12}{[+]} \mid \overset{56}{[+]}\,\overset{78}{[-]} \parallel \overset{9\,10}{(+)}\quad \overset{11\,12}{[-]}\quad \overset{13\,14}{[-]}$, are: $S^{12} = \frac{1}{2}$, $S^{03} = -\frac{i}{2}$, $\tau^{13} = \frac{1}{2}(S^{56} - S^{78}) = \frac{1}{2}$, $\tau^{23} = \frac{1}{2}(S^{56} + S^{78}) = 0$, $\tau^{33} = \frac{1}{2}(S^{9\,10} - S^{11\,12}) = \frac{1}{2}$, $\tau^{38} = \frac{1}{2\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14}) = \frac{1}{2\sqrt{3}}$. For the corresponding boson "basic vector" obtained by multiplication of $u^{c\,1}_L$ from the left-hand side by γ^9 one obtains: $\gamma^9\ u^{c\,1}_L\ \overset{03}{[-i]}\,\overset{12}{[+]} \mid \overset{56}{[+]}\,\overset{78}{[-]} \parallel \overset{9\,10}{(+)}\quad \overset{11\,12}{[-]}\quad \overset{13\,14}{[-]}$ leading to $\overset{03}{[-i]}\,\overset{12}{[+]} \mid \overset{56}{[+]}\,\overset{78}{[-]} \parallel \overset{9\,10}{[-]}\quad \overset{11\,12}{[-]}\quad \overset{13\,14}{[-]}$. Having all the quantum numbers equal zero, this boson represents a photon. Having all the members of the algebraic product equal to projectors, and taking into account that for Clifford even "basis vectors" $\mathcal{S}^{ab} = (S^{ab} + \tilde{S}^{ab})$, which gives zero for the projectors, the product of projectors only means that such an object can not change the internal space of Clifford odd "basis vectors", what photons do not. Photons can give to fermions only the momentum in the ordinary space time, Eq. (10.21).

[25] Let us repeat that the choice $d = (13+1)$ offers to describe all the quarks and antiquarks and leptons and antileptons observed so far, predicts the existence of the right-handed neutrinos and left-handed antineutrinos and another weak field; requires the existence of families of quarks and leptons and the existence of the dark matter; announces the existence of a fourth family to the observed three; explains the existence of the scalar fields as the Higgs boson; predicts new scalar fields that gave rise to inflation of the universe after the Big Bang, offers the explanation for many a cosmological observation.

[26] There are additional gauge fields which can not be observed since this would break Lorentz invariance if requiring Lorentz transformations of the kind M^{as}, $a = (0,1,2,3)$ and $s \geq 5$ due to the assumption that all the fields have non zero momentum only in $d = (3+1)$. Without additional break of symmetry there could exist gauge fields, which change more than only one kind of charge at the same time.

We can obtain gravitino if we extend points in ordinary space-time to strings. Describing strings with "basis vectors" being the eigenvectors of the Cartan subalgebra members $S^{01}, \tilde{S}^{01}, \mathcal{S}^{01} = (S^{01} + \tilde{S}^{01})$, on a string (σ, τ), we have, in this case, two Clifford odd — one"basis vector" and one Hermitian conjugated partner — and two Clifford even objects, Eq. (11.18),

$$\text{Clifford odd}$$
$$\hat{b}^{1\dagger}_{1_s} = \overset{01}{(+i)}_s\,, \quad \hat{b}^{1}_{1_s} = \overset{01}{(-i)}\,,$$
$$\text{Clifford even}$$
$$^{I}\mathcal{A}^{1\dagger}_{1_s} = \overset{01}{[+i]}_s\,, \quad {}^{II}\mathcal{A}^{1\dagger}_{1_s} = \overset{01}{[-i]}_s\ .$$

Index s points out that these "basis vectors" belong to the strings extensions and not to the "basis vectors" representing internal spaces of fermion and boson fields. The two Clifford odd "basis vectors" in the above equation are Hermitian conjugated to each other. Let us make a choice that $\hat{b}^{1\dagger}_{1_s} \equiv \overset{01}{(+i)}_s$ is the "basis vector" (applying on the vacuum state, Eq. (10.10), $|\psi_{ocs}> = \overset{01}{[-i]}_s$). There is only one family ($2^{\frac{d}{2}-1} = 1$) with one member. The eigenvalue S^{01} of $\hat{b}^{1\dagger}_{1_s}(= \overset{01}{(+i)}_s)$ is $\frac{i}{2}$.
Each of the two Clifford even "basis vectors" is self adjoint ($(^{I,II}\mathcal{A}^{1\dagger}_{1_s})^\dagger = {}^{I,II}\mathcal{A}^{1\dagger}_{1_s}$), with the eigenvalues $\mathcal{S}^{01}$ equal to 0.

The internal space of photons
(like it is $^{I}\hat{\mathcal{A}}^{\dagger}_{ph\,\tilde{u}^{c_1}_R \to \tilde{u}^{c_1}_R}(\equiv \overset{03\ \ 12\ \ 56\ \ 78\ 9\,10\,11\,12\,13\,14}{[+i][+][-][+][-]\ [+]\ \ [+]}$), Eq.(10.21)), extended by a tensor product, $*_{T'}$, with a string $\hat{b}^{1\dagger}_{1_s}(\equiv \overset{01}{(+i)}_s)$, can represent anticommuting gravitinos since the photon carries the space-time index μ.
The extensions of all the other "basis vectors" of space-time — either the Clifford odd ones describing the internal spaces of fermions, or Clifford even ones describing the internal spaces of bosons — by the tensor product, $*_{T'}$ with the two self adjoint "basis vectors" describing the internal space on the string, $^{i}\mathcal{A}^{1\dagger}_{1_s}, i = (I, II)$, do not change commutation properties. The extended "basis vectors" remain commuting or anti-commuting.
The extensions of "basis vectors" of space-time by the tensor product, $*_{T'}$, with $\hat{b}^{1\dagger}_{1_s}$ do change the commutation relations: The commuting ones become anticommuting, the anti-commuting become commuting.
The meaning of the $2^{\frac{14}{2}-1}$ "basis vectors" in tensor extension by $\hat{b}^{1\dagger}_{1_s}$ needs further study.
The extension of point fields in $d = (3+1)$ ordinary space-time (with "basis vectors" determined in $d = (13+1)$) to strings needs the corresponding change of the action, presented in Eq. (10.30), which is under consideration.
In a separate paper of this proceedings the author discusses the content of the "basis vectors" after the "basis vectors" describing the internal spaces of fermions ($\hat{b}^{m\dagger}_{f}$) and bosons ($^{I,II}\mathcal{A}^{m\dagger}_{f}$) in $d = (13+1)$ are extended by a tensor products with the string's "basis vectors" of fermions ($\hat{b}^{1\dagger}_{1_s}$) and bosons ($^{I,II}\mathcal{A}^{1\dagger}_{1_s}$) presented in Eq. (11.18).

It follows:

$$\hat{b}^{m\dagger}_{f} \; *_{T'} \; {}^{I,II}\mathcal{A}^{1\dagger}_{1s}$$

represent the anti-commuting "basis vectors" of fermions in the internal space $SO(d-1,1)$, having the Hermitian conjugated partners in a separate group; the projectors $[\overset{01}{\pm i}]_s$, namely, do not change anti-commutativity of fermion "basis vectors".

$$ {}^{I,II}\mathcal{A}^{m\dagger}_{f} \; *_{T'} \; {}^{I,II}\mathcal{A}^{1\dagger}_{1s}$$

represent the commuting "basis vectors" of bosons in the internal space $SO(d-1,1)$, appearing in two orthogonal groups; the projectors $[\overset{01}{\pm i}]_s$, namely, do not change commutativity of boson "basis vectors".

$$ {}^{I,II}\mathcal{A}^{m\dagger}_{f} \; *_{T'} \; \hat{b}^{1\dagger}_{1s}$$

represent the anticommuting "basis vectors" of new fermions, with the quantum numbers of bosons in the internal space $SO(d-1,1)$; the nilpotents $\hat{b}^{1\dagger}_{1s} \equiv (\overset{01}{+i})_s$, namely, changes commutativity of boson "basis vectors" keeping their quantum numbers unchanged.

$$\hat{b}^{m\dagger}_{f} \; *_{T'} \; \hat{b}^{1\dagger}_{1s}$$

represent the commuting "basis vectors" of new bosons with the quantum numbers of fermions in the internal space $SO(d-1,1)$, $\hat{b}^{1\dagger}_{1s} \equiv (\overset{01}{+i})$ change the anticommutativity of $\hat{b}^{m\dagger}_{f}$.

This does not look as the usually desired supersymmetry, requiring that each fermion (in our case one of $2^{\frac{d}{2}-1}$ members appearing in $2^{\frac{d}{2}-1}$ families, having their Hermitian conjugated partner in a separate group) has a supersymmetric partner with the same charges and with the spin 0. It does not look either that any boson field (in our case any one of two groups each with $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members) has a supersymmetric partner with the same charges and with the spin $\frac{1}{2}$.

There is another possibility to achieve the supersymmetric partners to the "basis vectors" presented fermions and bosons in $2(2n+1)$ dimensional internal spaces of fermions and bosons. It will be discussed in a separate contribution to the Bled Proceeding 2024. One can namely go to an odd dimensional space $d = 2(2n+1)+1$. As discussed in the article [9] there are two groups of "basis vectors" in odd dimensional spaces;

One group determines the anti-commuting "basis vectors" of $2^{\frac{d-1}{2}-1}$ fermions appearing in $2^{\frac{d-1}{2}-1}$ families with their $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ Hermitian conjugated partners appearing in a separate group, as well as two orthogonal groups each with $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ of "basis vectors", with their Hermitian conjugated partners within the same group.

The second group determines anti-commuting "basis vectors" appearing in two separate orthogonal groups each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ of "basis vectors", with their Hermitian conjugated partners within the same group, as well as the commuting "basis vectors" of "fermions" appearing in families with their Hermitian conjugated partners in a separate group.

Also these two groups might demonstrate a kind of supersymmetry.
Let us finish this section by pointing out that the proposed theory offers:
Photons with "basis vectors" including only projectors; gravitons with "basis vectors" having two nilpotents only in $SO(3,1)$ of $SO(13,1)$, all the rest are projectors; weak bosons with "basis vectors" which have two nilpotents only in $SO(4)$ manifesting as $SU(2) \times SU(2)$ of $SO(13,1)$; and gluons "basis vectors" with two nilpotents in $SU(3) \times U(1)$ of $SO(13,1)$, all the rest are projectors.
All the observed gauge fields, either vectors or scalars, are presented by only two nilpotents all the rest are projectors. There are breaks of symmetries [7] which make choices.

The extension of the "basis vectors" to strings in the context of renormalizability, as well as the offer of odd-dimensional spaces needs further studies.

10.5 Conclusions

This contribution presents the "basis vectors" of internal spaces of fermion and boson second quantized fields, Sect. 10.2, manifesting that fermion "basis vectors", written as products of an odd number of nilpotents (at least one, the rest of projectors), anti-commute, while boson "basis vectors", written as products of an even number of nilpotents (or of only projectors), commute, explaining the second quantization postulates of Dirac for fermions and bosons.
Each nilpotent and projector is chosen to be the eigenvector of one of the members of the Cartan subalgebra, Eq. 11.4; correspondingly are the "basis vectors" eigenvectors of all the Cartan subalgebra members; they offer a transparent and elegant way to find the Dirac matrices in any even dimensional space and the corresponding matrices for the boson fields [27]. However, knowing the states and operators, $\mathcal{S}^{ab}, \tilde{S}^{ab}, \mathcal{S}^{ab} = (\tilde{S}^{ab} + S^{ab})$, ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, the matrix representation is not needed.
We demonstrated that the number of fermion "basis vectors" and their Hermitian conjugated partners are equal to the number of "basis vectors" of bosons, appearing in two orthogonal groups, manifesting a (kind of) supersymmetry.
Representing the corresponding creation and annihilation operators for fermion and boson fields as tensor products, $*_T$, of "basis vectors" and basis in ordinary

[27] The usual presentation of Dirac matrices in higher dimensional spaces is much more complicated and much less transparent. The same is true for the usual group presentations for either fermions or bosons. The Dirac matrices in $d = (3+1)$ are designed for massive fermions, they do not anticommute. Dirac, having no $\tilde{\gamma}^a$, does not include families of fermions; although he could, since he has 2^d products of odd and even possibilities of products of γ^a.

space-time, Eqs. (11.16, 11.17) — the boson fields have to obtain the space index — the creation and annihilation operators manifest properties of the second quantized anti-commuting fermion fields and commuting boson fields, Sect. 10.3.

We assumed in this talk that fermions and bosons have non zero momenta only in $d = (3+1)$, while the involved internal space needs to be $d = (13+1)$ to be able to offer the observed properties of quarks and leptons and antiquarks and antileptons, appearing in families, as well as the corresponding vector (if the space index α is $(0,1,2,3)$) and scalar (if the space index α is ≥ 5) boson fields.

We presented "basis vectors" for fermion and boson fields in any $d = 2(2n+1)$, Sect. 10.2, in particular in $d = (1+1)$ while searching for the "basis vectors" for the strings, for $d = (5+1)$ while manifesting how do the "basis vectors" work in a simple case with fermions representing only "positrons" and "electrons" appearing in families and with bosons representing "photons" and "gravitons", and in $d = (13+1)$ manifesting "basis vectors" which explain the appearance of the observed quarks and leptons and antiquarks and antileptons appearing in families and of photons, weak bosons, gluons, and a second kind of weak bosons (not yet observed) and gravitons (not yet observed), Subsect. 10.2.1. There are two kinds of boson "basis vectors"; one kind transforms the family members of any family among themselves, the second kind transforms any of the family members into the same family member of all the families.

Does the description of the second quantized fermion and boson fields with the "basis vectors", bringing a new understanding of the second quantized fermion and boson fields, bring also a new understanding of cosmology? It is at least a very promising suggestion. Let us assume that at Big Bang, all the fermion, vector and scalar boson fields were massless, with the "basis vectors" in internal space determined by $SO(13,1)$; fermion, vector and scalar gauge fields have non zero momenta only in $d = (3+1)$ of ordinary space-time; all the scalar gauge fields with the space index $\alpha \geq 5$ (with non-zero momentum only in $d = (3+1)$) contribute to inflation. The Lorentz transformations of the kind $M^{ms} = L^{ms} + S^{ms}$ (or $M^{ms} = L^{ms} + \mathcal{S}^{ms}$), $m = (0,1,2,3)$, $s \geq 5$ are not possible (since L^{ms} can not be performed).

As discussed in ([7] and the references therein), the condensate breaks symmetry, so that the second weak $SU(2)_{II}$ bosons, interacting with the condensate, become at low energies massive; all the rest gauge fields ($SU(2)_I$, $SU(3)$, $U(1)$, gravity) [7], do not interact with the condensate, remaining therefore massless up to the electroweak break, which is caused by scalar fields with the space index $\alpha = (5,6,7,8)$. After the electroweak break there are photons, gluons and gravitons which keep their masslessness [10].

We demonstrate that the "basis vectors" for boson second quantized fields can also be expressed as algebraic products of fermion "basis vectors" and their Hermitian conjugated partners; consequently, we need only to know the fermion "basis vectors".

In this contribution, the trial to extend the points in ordinary space-time to strings is presented in Sect. 11.2.1. The internal space of strings, that is their "basis vectors", were added in a tensor product, $*_{T'}$, to the "basis vectors" of fermions and bosons describing the $d = (13+1)$ internal space. The tensor products of the fermion and

boson "basis vectors" with the bosons self adjoint "basis vectors" on the string leads to the anti-commuting creation and annihilation operators for fermions and commutation operators for bosons, suggesting that at low energies only the internal spaces of the second quantized fields with the "basis vectors" describing the $d=(13+1)$ internal space are important. The tensor product of the "basis vectors" of fermions and bosons describing the $d=(13+1)$ internal space with the Clifford odd basis vectors on a string gives to the commuting "basis vectors" in $d=(13+1)$ the anti-commuting partners with the same charges, and to the anti-commuting "basis vectors" in $d=(13+1)$ the commuting partners with the same charges. This, might be an interesting part, needs further studies.
There is another possibility to achieve the supersymmetric partners to the "basis vectors" presented fermions and bosons in $2(2n+1)$-dimensional internal spaces of fermions and bosons, Sect. 11.2.1.
In an odd dimensional space $d=2(2n+1)+1$ there are two groups of "basis vectors" [9]: One group determines the anti-commuting "basis vectors" of $2^{\frac{d-1}{2}-1}$ fermions appearing in $2^{\frac{d-1}{2}-1}$ families, with their $2^{\frac{d-1}{2}-1}\times 2^{\frac{d-1}{2}-1}$ Hermitian conjugated partners appearing in a separate group, as well as two orthogonal groups each with $2^{\frac{d-1}{2}-1}\times 2^{\frac{d-1}{2}-1}$ of "basis vectors", with their Hermitian conjugated partners within the same group.
The second group determines anti-commuting "basis vectors" appearing in two separate orthogonal groups each with $2^{\frac{d-1}{2}-1}\times 2^{\frac{d-1}{2}-1}$ of "basis vectors", with their Hermitian conjugated partners within the same group, as well as the commuting "basis vectors" of "fermions" appearing in families with their Hermitian conjugated partners in a separate group.
Also these two groups might demonstrate a kind of supersymmetry, suggesting to be used to achieve renormalizability.
We want to understand whether the elegant and simple description of the internal degrees of freedom of fermions and bosons with the Clifford odd and even "basis vectors" manifesting a kind of supersymmetry — offering explanations for so many observed properties of fermion and boson second quantized fields, explaining as well the second quantization postulates for fermion and boson fields, offering expressions for boson "basis vectors" (their internal spaces) as algebraic products of fermion "basis vectors" and their Hermitian conjugated partners, offering explanations for cosmological observations with inflation included — can be related to strings, or it offers a new way to understand renormalizability on all levels of energy.

10.6 Useful tables

These Tables are mainly taken from Refs. [8, 10] and are meant to illustrate Sects. (10.2,10.2.3 10.2.1,10.2.2).
The case with $d=(5+1)$ is meant to learn how Clifford odd and Clifford even "basis vectors" illustrate internal spaces of fermions and bosons, Table 10.1.
Fermions, in this case "electrons" with a negative "charge", $S^{56}=-\frac{1}{2}$, and "positrons" with a positive "charge", $S^{56}=\frac{1}{2}$, appear in 16 odd "basis vectors"

(with one or three nilpotents), in 4 families with 4 members in each family, having 16 Hermitian conjugated partners.
Bosons, in this case "photons" and "gravitons", appear in two orthogonal groups; each group has 16 members, and their Hermitian conjugated partners appear within the same group. The eigenvalues of the Cartan subalgebra members $\mathcal{S}^{ab} = (S^{ab} + \tilde{S}^{ab})$ are equal to either $(\pm i, 0)$ or $(\pm 1, 0)$. "Photons" are products of only projectors, "gravitons" have two nilpotents with $\mathcal{S}^{03}$ equal to $(\pm i)$ and $\mathcal{S}^{12}$ equal to (± 1).
Tables (10.2, 10.3), manifest that the Clifford even "basis vectors" can be expressed as products of the Clifford odd "basis vectors" and their Hermitian conjugated partners. Let us remind the reader that projectors are self adjoint operators, while the Hermitian conjugated partner to $(\overset{ab}{\pm i})$ are $(\overset{ab}{\mp i})$ and to $(\overset{ab}{\pm 1})$ are $(\overset{ab}{\mp 1})$, Eq. (10.7).

10.7 One family representation of Clifford odd "basis vectors" in $d = (13 + 1)$

This appendix, is following App. D of Ref. [10]. In the even dimensional space $d = (13 + 1)$ ([14], App. A), one irreducible representation of the Clifford odd "basis vectors" if analysed from the point of view of the subgroups of the group $SO(13, 1)$ (including $SO(7, 1) \times SO(6)$, while $SO(7, 1)$ breaks into $SO(3, 1) \times SO(4)$, and $SO(6)$ breaks into $SU(3) \times U(1)$) contains the Clifford odd "basis vectors" describing internal spaces of quarks and leptons and antiquarks and antileptons, manifesting at low energies the quantum numbers assumed by the *standard model* before the electroweak break. Since $SO(4)$ contains two $SU(2)$ subgroups, $SU(2)_I$ and $SU(2)_{II}$, with the hypercharge of the *standard model* $Y = \tau^{23} + \tau^4$ (with τ^{23} belonging to $SU(2)_{II}$ and τ^4 originating in $SO(6)$, breaking to $SU(3) \times U(1)$), one irreducible representation includes the right-handed neutrinos and the left-handed antineutrinos, which are not in the *standard model* scheme [28]. An overview of the properties of the vector and scalar gauge fields in the *spin-charge-family* theory can be found in Refs. ([5,7,8,24,25] and the references therein). The reader can find in Table 10.4 that "basis vectors" of quarks have identical content of $SO(7, 1)$ as "basis vectors" of leptons (and antiquarks as antiletons); they differ only in the $SU(3) \times U(1)$ part.

[28] The handedness is defined as follows

$$\Gamma^{(d)} = \prod_a (\sqrt{\eta^{aa}}\gamma^a) \cdot \begin{cases} (i)^{\frac{d}{2}}, & \text{for d even}, \\ (i)^{\frac{d-1}{2}}, & \text{for d odd}. \end{cases} \tag{10.31}$$

Table 10.1: This table, taken from [8], represents $2^d = 64$ "basis vectors", which are the eigenstates of the Cartan subalgebra members, Eq. (11.4). Half of them are the Clifford odd "basis vectors" with the odd number of nilpotents (one or three), the other half are the Clifford even "basis vectors" with the even number of nilpotents (two or none). They are divided into four groups. The first group, odd I, represents $\hat{b}^{m\dagger}_f$, appearing in $2^{\frac{d}{2}-1} = 4$ families ($f = 1,2,3,4$), each family having $2^{\frac{d}{2}-1} = 4$ family members ($m = 1,2,3,4$). The second group, odd II, contains Hermitian conjugated partners of the first group (for each family separately), $\hat{b}^{m}_f = (\hat{b}^{m\dagger}_f)^\dagger$. Either odd I or odd II are products of an odd number of nilpotents (one or three) and of projectors (two or none). The family quantum numbers of $\hat{b}^{m\dagger}_f$ (the eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$) appear for the first *odd I* group above each family, the quantum numbers of the "family" members (S^{03}, S^{12}, S^{56}) are written in the last three columns. For the Hermitian conjugated partners of *odd I*, presented in the group *odd II*, the quantum numbers (S^{03}, S^{12}, S^{56}) are presented above each group of the Hermitian conjugated partners, the last three columns tell eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$. The two groups with the even number of nilpotents (two or none), *even I* and *even II*, have their Hermitian conjugated partners within its groups. The quantum numbers f, that is the eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, are written above column of four members, the quantum numbers of the members, (S^{03}, S^{12}, S^{56}), are written in the last three columns. The quantum numbers of the even "basi vectors" are $(\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56})$, determined by $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$.

"basis vectors" $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	m $\rightarrow$	f = 1 $(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$	f = 2 $(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$	f = 3 $(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$	f = 4 $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$	S^{03}	S^{12}	S^{56}
odd I $\hat{b}^{m\dagger}_f$	1	03 12 56 (+i)[+][+]	03 12 56 [+i][+](+)	03 12 56 [+i](+)[+]	03 12 56 (+i)(+)(+)	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	[−i](−)[+]	(−i)(−)(+)	(−i)[−][+]	[−i][−](+)	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	[−i][+](−)	(−i)[+][−]	(−i)(+)(−)	[−i](+)[−]	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	(+i)(−)(−)	[+i](−)[−]	[+i][−](−)	(+i)[−][−]	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
(S^{03}, S^{12}, S^{56})	$\rightarrow$	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$
odd II $\hat{b}^{m}_f$	1	(−i)[+][+]	[+i][+](−)	[+i](−)[+]	(−i)(−)(−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	2	[−i](+)[+]	(+i)(+)(−)	(+i)[−][+]	[−i][−](−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	3	[−i][+](+)	(+i)[+][−]	(+i)(−)(+)	[−i](−)[−]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	4	(−i)(+)(+)	[+i](+)[−]	[+i][−](+)	(−i)[−][−]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	$\rightarrow$	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}
even I ${}^{I}\mathcal{A}^{m}_f$	1	[+i](+)(+)	(+i)[+](+)	[+i][+][+]	(+i)(+)[+]	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	(−i)[−](+)	[−i](−)(+)	(−i)(−)[+]	[−i][−][+]	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	(−i)(+)[−]	[−i][+][−]	(−i)[+](−)	[−i](+)(−)	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	[+i][−][−]	(+i)(−)[−]	[+i](−)(−)	(+i)[−](−)	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	$\rightarrow$	$(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}
even II ${}^{II}\mathcal{A}^{m}_f$	1	[−i](+)(+)	(−i)[+](+)	[−i][+][+]	(−i)(+)[+]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	(+i)[−](+)	[+i](−)(+)	(+i)(−)[+]	[+i][−][+]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	(+i)(+)[−]	[+i][+][−]	(+i)[+](−)	[+i](+)(−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	[−i][−][−]	(−i)(−)[−]	[−i](−)(−)	(−i)[−](−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Table 10.2: This table is taken from Ref. [10]. The Clifford even "basis vectors", ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, belonging to zero momentum in internal space, $\mathcal{S}^{12}=0$, for $d=(5+1)$, are presented as algebraic products of the $f=1$ family "basis vectors" $\hat{b}^{m'\dagger}_{1}$ and their Hermitian conjugated partners $(\hat{b}^{m''\dagger}_{1})^{\dagger}$, presented in Table 10.1: $\hat{b}^{m'\dagger}_{1} *_{A} (\hat{b}^{m''\dagger}_{1})^{\dagger}$. The two ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ which are Hermitian conjugated partners, are marked with the same symbol (either $\triangle$ or $\bullet$). The symbol $\bigcirc$ presents selfadjoint members. The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are products of one projector and two nilpotents or three projectors (they are self-adjoint), the Clifford odd "basis vectors" and their Hermitian conjugated partners are products of one nilpotent and two projectors or of three nilpotents. The Clifford even and Clifford odd objects are eigenvectors of all the corresponding Cartan subalgebra members, Eq. (11.4). There are $\frac{1}{2}\times 2^{\frac{6}{2}-1}\times 2^{\frac{6}{2}-1}$ algebraic products of $\hat{b}^{m'\dagger}_{1} *_{A} (\hat{b}^{m''\dagger}_{1})^{\dagger}$. The rest 8 of 16 members have ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\mathcal{S}^{12}=+1$ (four) and with $\mathcal{S}^{12}=-1$ (four), presented in Table 10.3. The members $\hat{b}^{m'\dagger}_{f}$ together with their Hermitian conjugated partners of each of the four families, $f=(1,2,3,4)$, offer the same ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\mathcal{S}^{12}=0$ as the ones presented in this table. The table is taken from Ref. [10,11].

$\mathcal{S}^{12}$	symbol	${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}=$	$\hat{b}^{m'\dagger}_{f'} *_{A} (\hat{b}^{m''\dagger}_{f'})^{\dagger}$
0	$\triangle$	${}^{I}\hat{\mathcal{A}}^{2\dagger}_{1}=$	$\hat{b}^{2\dagger}_{1} *_{A} (\hat{b}^{4\dagger}_{1})^{\dagger}$
		$\overset{03}{(-i)}\ \overset{12}{[-]}\ \overset{56}{(+)}$	$\overset{03}{[-i]}\ \overset{12}{(-)}\ \overset{56}{[+]} *_{A} \overset{03}{(-i)}\ \overset{12}{(+)}\ \overset{56}{(+)}$
0	$\bigcirc$	${}^{I}\hat{\mathcal{A}}^{4\dagger}_{1}=$	$\hat{b}^{4\dagger}_{1} *_{A} (\hat{b}^{4\dagger}_{1})^{\dagger}$
		$\overset{03}{[+i]}\ \overset{12}{[-]}\ \overset{56}{[-]}$	$\overset{03}{(+i)}\ \overset{12}{(-)}\ \overset{56}{(-)} *_{A} \overset{03}{(-i)}\ \overset{12}{(+)}\ \overset{56}{(+)}$
0	$\bullet$	${}^{I}\hat{\mathcal{A}}^{1\dagger}_{2}=$	$\hat{b}^{1\dagger}_{1} *_{A} (\hat{b}^{3\dagger}_{1})^{\dagger}$
		$\overset{03}{(+i)}\ \overset{12}{[+]}\ \overset{56}{(+)}$	$\overset{03}{(+i)}\ \overset{12}{[+]}\ \overset{56}{[+]} *_{A} \overset{03}{[-i]}\ \overset{12}{[+]}\ \overset{56}{(+)}$
0	$\bigcirc$	${}^{I}\hat{\mathcal{A}}^{3\dagger}_{2}=$	$\hat{b}^{3\dagger}_{1} *_{A} (\hat{b}^{3\dagger}_{1})^{\dagger}$
		$\overset{03}{[-i]}\ \overset{12}{[+]}\ \overset{56}{[-]}$	$\overset{03}{[-i]}\ \overset{12}{[+]}\ \overset{56}{(-)} *_{A} \overset{03}{[-i]}\ \overset{12}{[+]}\ \overset{56}{(+)}$
0	$\bigcirc$	${}^{I}\hat{\mathcal{A}}^{1\dagger}_{3}=$	$\hat{b}^{1\dagger}_{1} *_{A} (\hat{b}^{1\dagger}_{1})^{\dagger}$
		$\overset{03}{[+i]}\ \overset{12}{[+]}\ \overset{56}{[+]}$	$\overset{03}{(+i)}\ \overset{12}{[+]}\ \overset{56}{[+]} *_{A} \overset{03}{(-i)}\ \overset{12}{[+]}\ \overset{56}{[+]}$
0	$\bullet$	${}^{I}\hat{\mathcal{A}}^{3\dagger}_{3}=$	$\hat{b}^{3\dagger}_{1} *_{A} (\hat{b}^{1\dagger}_{1})^{\dagger}$
		$\overset{03}{(-i)}\ \overset{12}{[+]}\ \overset{56}{(-)}$	$\overset{03}{[-i]}\ \overset{12}{[+]}\ \overset{56}{(-)} *_{A} \overset{03}{(-i)}\ \overset{12}{[+]}\ \overset{56}{[+]}$
0	$\bigcirc$	${}^{I}\hat{\mathcal{A}}^{2\dagger}_{4}=$	$\hat{b}^{2\dagger}_{1} *_{A} (\hat{b}^{2\dagger}_{1})^{\dagger}$
		$\overset{03}{[-i]}\ \overset{12}{[-]}\ \overset{56}{[+]}$	$\overset{03}{[-i]}\ \overset{12}{(-)}\ \overset{56}{[+]} *_{A} \overset{03}{[-i]}\ \overset{12}{(+)}\ \overset{56}{[+]}$
0	$\triangle$	${}^{I}\hat{\mathcal{A}}^{4\dagger}_{3}=$	$\hat{b}^{4\dagger}_{1} *_{A} (\hat{b}^{2\dagger}_{1})^{\dagger}$
		$\overset{03}{(+i)}\ \overset{12}{[-]}\ \overset{56}{(-)}$	$\overset{03}{(+i)}\ \overset{12}{(-)}\ \overset{56}{(-)} *_{A} \overset{03}{[-i]}\ \overset{12}{(+)}\ \overset{56}{[+]}$

Table 10.3: The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, belonging to transverse momentum in internal space, $\mathcal{S}^{12}=1$, the first half of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, and $\mathcal{S}^{12}=-1$, the second half of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, for $d=(5+1)$, are presented as algebraic products of the $f=1$ family "basis vectors" $\hat{b}^{m'\dagger}_{1}$ and their Hermitian conjugated partners $(\hat{b}^{m''\dagger}_{1})^{\dagger}$: $\hat{b}^{m'\dagger}_{1}*_{A}(\hat{b}^{m''\dagger}_{1})^{\dagger}$. Two ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ which are the Hermitian conjugated partners are marked with the same symbol ($\star\star$, ‡, $\otimes$, $\odot\odot$). The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are products of one projector and two nilpotents, the Clifford odd "basis vectors" and their Hermitian conjugated partners are products of one nilpotent and two projectors or of three nilpotents. The Clifford even and Clifford odd objects are eigenvectors of all the corresponding Cartan subalgebra members, Eq. (11.4). There are $\frac{1}{2}\times 2^{\frac{6}{2}-1}\times 2^{\frac{6}{2}-1}$ algebraic products of $\hat{b}^{m'\dagger}_{1}*_{A}(\hat{b}^{m''\dagger}_{1})^{\dagger}$ with $\mathcal{S}^{12}$ equal to ± 1. The rest 8 of 16 members present ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\mathcal{S}^{12}=0$. The members $\hat{b}^{m'\dagger}_{f}$ together with their Hermitian conjugated partners of each of the four families, $f=(1,2,3,4)$, offer the same ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\mathcal{S}^{12}=\pm 1$ as the ones presented in this table. (And equivalently for $\mathcal{S}^{12}=0$.)

$\mathcal{S}^{12}$	symbol	${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}=\quad \hat{b}^{m'\dagger}_{f'}*_{A}(\hat{b}^{m''\dagger}_{f'})^{\dagger}$
1	$\star\star$	${}^{I}\hat{\mathcal{A}}^{1\dagger}_{1}=\quad \hat{b}^{1\dagger}_{1}*_{A}(\hat{b}^{4\dagger}_{1})^{\dagger}$ $\overset{03\ 12\ 56}{[+i]\ (+)(+)}\quad \overset{03\ 12\ 56}{(+i)\ [+][+]}\ *_{A}\ \overset{03\ 12\ 56}{(-i)\ (+)(+)}$
1	‡	${}^{I}\hat{\mathcal{A}}^{3\dagger}_{1}=\quad \hat{b}^{3\dagger}_{1}*_{A}(\hat{b}^{4\dagger}_{1})^{\dagger}$ $\overset{03\ 12\ 56}{(-i)\ (+)[-]}\quad \overset{03\ 12\ 56}{[-i]\ [+](-)}\ *_{A}\ \overset{03\ 12\ 56}{(-i)\ (+)(+)}$
1	$\odot\odot$	${}^{I}\hat{\mathcal{A}}^{1\dagger}_{4}=\quad \hat{b}^{1\dagger}_{1}*_{A}(\hat{b}^{2\dagger}_{1})^{\dagger}$ $\overset{03\ 12\ 56}{(+i)\ (+)[+]}\quad \overset{03\ 12\ 56}{(+i)\ [+][+]}\ *_{A}\ \overset{03\ 12\ 56}{[-i]\ (+)[+]}$
1	$\otimes$	${}^{I}\hat{\mathcal{A}}^{3\dagger}_{4}=\quad \hat{b}^{3\dagger}_{1}*_{A}(\hat{b}^{2\dagger}_{1})^{\dagger}$ $\overset{03\ 12\ 56}{[-i]\ (+)(-)}\quad \overset{03\ 12\ 56}{[-i]\ [+](-)}\ *_{A}\ \overset{03\ 12\ 56}{[-i]\ (+)[+]}$
-1	$\otimes$	${}^{I}\hat{\mathcal{A}}^{2\dagger}_{2}=\quad \hat{b}^{2\dagger}_{1}*_{A}(\hat{b}^{3\dagger}_{1})^{\dagger}$ $\overset{03\ 12\ 56}{[-i]\ (-)(+)}\quad \overset{03\ 12\ 56}{[-i]\ (-)[+]}\ *_{A}\ \overset{03\ 12\ 56}{[-i]\ [+](+)}$
-1	‡	${}^{I}\hat{\mathcal{A}}^{4\dagger}_{2}=\quad \hat{b}^{4\dagger}_{1}*_{A}(\hat{b}^{3\dagger}_{1})^{\dagger}$ $\overset{03\ 12\ 56}{(+i)\ (-)[-]}\quad \overset{03\ 12\ 56}{(+i)\ (-)(-)}\ *_{A}\ \overset{03\ 12\ 56}{[-i]\ [+](+)}$
-1	$\odot\odot$	${}^{I}\hat{\mathcal{A}}^{2\dagger}_{3}=\quad \hat{b}^{2\dagger}_{1}*_{A}(\hat{b}^{1\dagger}_{1})^{\dagger}$ $\overset{03\ 12\ 56}{(-i)\ (-)[+]}\quad \overset{03\ 12\ 56}{[-i]\ (-)[+]}\ *_{A}\ \overset{03\ 12\ 56}{(-i)\ [+][+]}$
-1	$\star\star$	${}^{I}\hat{\mathcal{A}}^{4\dagger}_{3}=\quad \hat{b}^{4\dagger}_{1}*_{A}(\hat{b}^{1\dagger}_{1})^{\dagger}$ $\overset{03\ 12\ 56}{[+i]\ (-)(-)}\quad \overset{03\ 12\ 56}{(+i)\ (-)(-)}\ *_{A}\ \overset{03\ 12\ 56}{(-i)\ [+][+]}$

i		$\vert^{a}\psi_{i}>$ (Anti)octet, $\Gamma^{(7,1)} = (-1)\,1$, $\Gamma^{(6)} = (1) - 1$ of (anti)quarks and (anti)leptons	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^{4}	Y	Q
1	$u^{c^1}_R$	03 12 56 78 9 10 11 12 13 14 (+i) [+] | [+] (+) || (+) [−] [−]	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
2	$u^{c^1}_R$	03 12 56 78 9 10 11 12 13 14 [−i] (−) | [+] (+) || (+) [−] [−]	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	$d^{c^1}_R$	03 12 56 78 9 10 11 12 13 14 (+i) [+] | (−) [−] || (+) [−] [−]	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	$d^{c^1}_R$	03 12 56 78 9 10 11 12 13 14 [−i] (−) | (−) [−] || (+) [−] [−]	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	$d^{c^1}_L$	03 12 56 78 9 10 11 12 13 14 [−i] [+] | (−) (+) || (+) [−] [−]	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	$d^{c^1}_L$	03 12 56 78 9 10 11 12 13 14 − (+i) (−) | (−) (+) || (+) [−] [−]	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	$u^{c^1}_L$	03 12 56 78 9 10 11 12 13 14 − [−i] [+] | [+] [−] || (+) [−] [−]	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
8	$u^{c^1}_L$	03 12 56 78 9 10 11 12 13 14 (+i) (−) | [+] [−] || (+) [−] [−]	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	$u^{c^2}_R$	03 12 56 78 9 10 11 12 13 14 (+i) [+] | [+] (+) || [−] (+) [−]	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	$u^{c^2}_R$	03 12 56 78 9 10 11 12 13 14 [−i] (−) | [+] (+) || [−] (+) [−]	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
11	$d^{c^2}_R$	03 12 56 78 9 10 11 12 13 14 (+i) [+] | (−) [−] || [−] (+) [−]	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
12	$d^{c^2}_R$	03 12 56 78 9 10 11 12 13 14 [−i] (−) | (−) [−] || [−] (+) [−]	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	$d^{c^2}_L$	03 12 56 78 9 10 11 12 13 14 [−i] [+] | (−) (+) || [−] (+) [−]	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
14	$d^{c^2}_L$	03 12 56 78 9 10 11 12 13 14 − (+i) (−) | (−) (+) || [−] (+) [−]	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
15	$u^{c^2}_L$	03 12 56 78 9 10 11 12 13 14 − [−i] [+] | [+] [−] || [−] (+) [−]	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
16	$u^{c^2}_L$	03 12 56 78 9 10 11 12 13 14 (+i) (−) | [+] [−] || [−] (+) [−]	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
17	$u^{c^3}_R$	03 12 56 78 9 10 11 12 13 14 (+i) [+] | [+] (+) || [−] [−] (+)	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	$u^{c^3}_R$	03 12 56 78 9 10 11 12 13 14 [−i] (−) | [+] (+) || [−] [−] (+)	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
19	$d^{c^3}_R$	03 12 56 78 9 10 11 12 13 14 (+i) [+] | (−) [−] || [−] [−] (+)	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
20	$d^{c^3}_R$	03 12 56 78 9 10 11 12 13 14 [−i] (−) | (−) [−] || [−] [−] (+)	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
21	$d^{c^3}_L$	03 12 56 78 9 10 11 12 13 14 [−i] [+] | (−) (+) || [−] [−] (+)	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
22	$d^{c^3}_L$	03 12 56 78 9 10 11 12 13 14 − (+i) (−) | (−) (+) || [−] [−] (+)	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
23	$u^{c^3}_L$	03 12 56 78 9 10 11 12 13 14 − [−i] [+] | [+] [−] || [−] [−] (+)	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
24	$u^{c^3}_L$	03 12 56 78 9 10 11 12 13 14 (+i) (−) | [+] [−] || [−] [−] (+)	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
25	ν_R	03 12 56 78 9 10 11 12 13 14 (+i) [+] | [+] (+) || (+) (+) (+)	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
26	ν_R	03 12 56 78 9 10 11 12 13 14 [−i] (−) | [+] (+) || (+) (+) (+)	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
27	e_R	03 12 56 78 9 10 11 12 13 14 (+i) [+] | (−) [−] || (+) (+) (+)	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	−1	−1
28	e_R	03 12 56 78 9 10 11 12 13 14 [−i] (−) | (−) [−] || (+) (+) (+)	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	−1	−1
29	e_L	03 12 56 78 9 10 11 12 13 14 [−i] [+] | (−) (+) || (+) (+) (+)	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	−1
30	e_L	03 12 56 78 9 10 11 12 13 14 − (+i) (−) | (−) (+) || (+) (+) (+)	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	−1
31	ν_L	03 12 56 78 9 10 11 12 13 14 − [−i] [+] | [+] [−] || (+) (+) (+)	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
32	ν_L	03 12 56 78 9 10 11 12 13 14 (+i) (−) | [+] [−] || (+) (+) (+)	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
33	$\bar{u}^{\bar{c}^1}_L$	03 12 56 78 9 10 11 12 13 14 [−i] [+] | [+] (+) || [−] (+) (+)	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\bar{d}^{\bar{c}^1}_L$	03 12 56 78 9 10 11 12 13 14 (+i) (−) | [+] (+) || [−] (+) (+)	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\bar{u}^{\bar{c}^1}_L$	03 12 56 78 9 10 11 12 13 14 − [−i] [+] | (−) [−] || [−] (+) (+)	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\bar{u}^{\bar{c}^1}_L$	03 12 56 78 9 10 11 12 13 14 − (+i) (−) | (−) [−] || [−] (+) (+)	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\bar{d}^{\bar{c}^1}_R$	03 12 56 78 9 10 11 12 13 14 (+i) [+] | [+] [−] || [−] (+) (+)	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\bar{d}^{\bar{c}^1}_R$	03 12 56 78 9 10 11 12 13 14 − [−i] (−) | [+] [−] || [−] (+) (+)	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\bar{u}^{\bar{c}^1}_R$	03 12 56 78 9 10 11 12 13 14 (+i) [+] | (−) (+) || [−] (+) (+)	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\bar{u}^{\bar{c}^1}_R$	03 12 56 78 9 10 11 12 13 14 [−i] (−) | (−) (+) || [−] (+) (+)	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\bar{d}^{\bar{c}^2}_L$	03 12 56 78 9 10 11 12 13 14 [−i] [+] | [+] (+) || (+) [−] (+)	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
42	$\bar{d}^{\bar{c}^2}_L$	03 12 56 78 9 10 11 12 13 14 (+i) (−) | [+] (+) || (+) [−] (+)	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\bar{u}^{\bar{c}^2}_L$	03 12 56 78 9 10 11 12 13 14 − [−i] [+] | (−) [−] || (+) [−] (+)	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$

i		$\vert^{a}\psi_{i}>$ (Anti)octet, $\Gamma^{(7,1)}=(-1)1$, $\Gamma^{(6)}=(1)-1$ of (anti)quarks and (anti)leptons	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^{4}	Y	Q
44	$\tilde{u}^{c^2}_L$	03 12 56 78 9 10 11 12 13 14 $-(+i)(-)\vert(-)[-]\vert\vert(+)\quad[-]\quad(+)$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\tilde{d}^{c^2}_R$	03 12 56 78 9 10 11 12 13 14 $(+i)[+]\vert[+][-]\vert\vert(+)\quad[-]\quad(+)$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\tilde{d}^{c^2}_R$	03 12 56 78 9 10 11 12 13 14 $-[-i](-)\vert[+][-]\vert\vert(+)\quad[-]\quad(+)$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
47	$\tilde{u}^{c^2}_R$	03 12 56 78 9 10 11 12 13 14 $(+i)[+]\vert(-)(+)\vert\vert(+)\quad[-]\quad(+)$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{2}{3}$
48	$\tilde{u}^{c^2}_R$	03 12 56 78 9 10 11 12 13 14 $[-i](-)\vert(-)(+)\vert\vert(+)\quad[-]\quad(+)$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\tilde{d}^{c^3}_L$	03 12 56 78 9 10 11 12 13 14 $[-i][+]\vert[+](+)\vert\vert(+)\quad(+)\quad[-]$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\tilde{d}^{c^3}_L$	03 12 56 78 9 10 11 12 13 14 $(+i)(-)\vert[+](+)\vert\vert(+)\quad(+)\quad[-]$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\tilde{u}^{c^3}_L$	03 12 56 78 9 10 11 12 13 14 $-[-i][+]\vert(-)[-]\vert\vert(+)\quad(+)\quad[-]$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\tilde{u}^{c^3}_L$	03 12 56 78 9 10 11 12 13 14 $-(+i)(-)\vert(-)[-]\vert\vert(+)\quad(+)\quad[-]$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\tilde{d}^{c^3}_R$	03 12 56 78 9 10 11 12 13 14 $(+i)[+]\vert[+][-]\vert\vert(+)\quad(+)\quad[-]$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\tilde{d}^{c^3}_R$	03 12 56 78 9 10 11 12 13 14 $-[-i](-)\vert[+][-]\vert\vert(+)\quad(+)\quad[-]$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\tilde{u}^{c^3}_R$	03 12 56 78 9 10 11 12 13 14 $(+i)[+]\vert(-)(+)\vert\vert(+)\quad(+)\quad[-]$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\tilde{u}^{c^3}_R$	03 12 56 78 9 10 11 12 13 14 $[-i](-)\vert(-)(+)\vert\vert(+)\quad(+)\quad[-]$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$\tilde{e}_L$	03 12 56 78 9 10 11 12 13 14 $[-i][+]\vert[+](+)\vert\vert[-]\quad[-]\quad[-]$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$\tilde{e}_L$	03 12 56 78 9 10 11 12 13 14 $(+i)(-)\vert[+](+)\vert\vert[-]\quad[-]\quad[-]$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\tilde{\nu}_L$	03 12 56 78 9 10 11 12 13 14 $-[-i][+]\vert(-)[-]\vert\vert[-]\quad[-]\quad[-]$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\tilde{\nu}_L$	03 12 56 78 9 10 11 12 13 14 $-(+i)(-)\vert(-)[-]\vert\vert[-]\quad[-]\quad[-]$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\tilde{\nu}_R$	03 12 56 78 9 10 11 12 13 14 $(+i)[+]\vert(-)(+)\vert\vert[-]\quad[-]\quad[-]$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\tilde{\nu}_R$	03 12 56 78 9 10 11 12 13 14 $-[-i](-)\vert(-)(+)\vert\vert[-]\quad[-]\quad[-]$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$\tilde{e}_R$	03 12 56 78 9 10 11 12 13 14 $(+i)[+]\vert[+][-]\vert\vert[-]\quad[-]\quad[-]$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$\tilde{e}_R$	03 12 56 78 9 10 11 12 13 14 $[-i](-)\vert[+][-]\vert\vert[-]\quad[-]\quad[-]$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Table 10.4: The left-handed ($\Gamma^{(13,1)}=-1$, Eq. (10.31)) irreducible representation of one family of spinors — the product of the odd number of nilpotents and of projectors, which are eigenvectors of the Cartan subalgebra of the $SO(13,1)$ group [3, 4], manifesting the subgroup $SO(7,1)$ of the colour charged quarks and antiquarks and the colourless leptons and antileptons — is presented. It contains the left-handed ($\Gamma^{(3,1)}=-1$) weak ($SU(2)_I$) charged ($\tau^{13}=\pm\frac{1}{2}$), and $SU(2)_{II}$ chargeless ($\tau^{23}=0$) quarks and leptons, and the right-handed ($\Gamma^{(3,1)}=1$) weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged ($\tau^{23}=\pm\frac{1}{2}$) quarks and leptons, both with the spin S^{12} up and down ($\pm\frac{1}{2}$, respectively). Quarks distinguish from leptons only in the $SU(3)\times U(1)$ part: Quarks are triplets of three colours ($c^i=(\tau^{33},\tau^{38})=[(\frac{1}{2},\frac{1}{2\sqrt{3}}),(-\frac{1}{2},\frac{1}{2\sqrt{3}}),(0,-\frac{1}{\sqrt{3}})$, carrying the "fermion charge" ($\tau^4=\frac{1}{6}$). The colourless leptons carry the "fermion charge" ($\tau^4=-\frac{1}{2}$). The same multiplet contains also the left handed weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged antiquarks and antileptons and the right handed weak ($SU(2)_I$) charged and $SU(2)_{II}$ chargeless antiquarks and antileptons. Antiquarks distinguish from antileptons again only in the $SU(3)\times U(1)$ part: Antiquarks are anti-triplets carrying the "fermion charge" ($\tau^4=-\frac{1}{6}$). The anti-colourless antileptons carry the "fermion charge" ($\tau^4=\frac{1}{2}$). $Y=(\tau^{23}+\tau^4)$ is the hyper charge, the electromagnetic charge is $Q=(\tau^{13}+Y)$.

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