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# On the degrees of regular nut graphs and Cayley nut graphs

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**Abstract**

A *nut graph* is a simple graph for which the adjacency matrix has a single zero eigenvalue such that all non-trivial kernel eigenvectors have no zero entry. It is known that infinitely many $d$-regular nut graphs exist for $3 \leq d \leq 12$ and for $d \geq 4$ such that $d \equiv 0 \pmod 4$. Here it is shown that infinitely many $d$-regular nut graphs exist for each degree $d \geq 3$. Likewise, it is known that infinitely many $d$-regular Cayley nut graphs exist for $d \geq 4$ such that $d \equiv 0 \pmod 4$. Here we prove that there are infinitely many $d$-regular

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Cayley nut graphs for each even $d \geq 4$. This implies that we have identified all feasible degrees $d$ for which a $d$-regular Cayley nut graph exists.

*Keywords: Nut graph, regular graph, Cayley graph, circulant graph, nullity, graph spectra.*

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## 1 Introduction

*Nut graphs* are graphs that have a one-dimensional nullspace (i.e. $\eta(G) = 1$), where the non-trivial kernel eigenvector $\mathbf{x} = [x_1 \ \cdots \ x_n]^\intercal \in \ker \mathbf{A}(G)$ is full (i.e. it has no zero entries). Early work established that *non-trivial* nut graphs are connected, non-bipartite and have no leaves, and showed that the smallest non-trivial nut graphs have seven or more vertices [34].

If $G$ is a non-trivial *$d$-regular* nut graph, then the degree $d \geq 3$. Regular nut graphs with $d = 2$ do not exist, as the nullity of a cycle is either 0 or 2. The case $d = 3$ is of special interest in applications in chemistry and physics, where the simplest models for the electronic structure and properties of unsaturated carbon and hydrocarbon frameworks are essentially graph-theoretical [30]. In this context, the *chemical graphs* are connected subcubic graphs, with $v_1$, $v_2$ and $v_3$ vertices of degree 1, 2 and 3, respectively. Nut graphs have $v_1 = 0$. The pairs $(v_2, v_3)$ for the *chemical nut graphs* have been characterised [23]. Cubic nut graphs are relevant in applications to, for example, the polyhedral carbon cages known as fullerenes [22]. The Frucht graph [24, 25] is a small example of a cubic polyhedral nut graph that would serve as a model of reactivity and conduction of a hypothetical $C_{12}$ cluster [32, 33].

From the mathematical perspective, there has been interest in the question of the possible degrees and orders of regular nut graphs of degree higher than three. The main results of this paper are Theorems 1.2 and 1.6 presented below where they are set in the context of the order–degree existence problem. These deal with, respectively, regular nut graphs in general and Cayley nut graphs.

### 1.1 Regular nut graphs

The order–degree existence problem for regular nut graphs was initiated by Gauci, Pisanski and Sciriha [27]. Let $\mathfrak{N}_d^{\mathrm{reg}}$ denote the set of all the orders attainable by a $d$-regular nut graph. In these terms, Problem 12 of [27] is to determine the set $\mathfrak{N}_d^{\mathrm{reg}}$ for each degree $d$.

**Theorem 1.1** ([27, Theorems 2 and 3], [21, Theorem 7], and [4, Theorem 1.3])**.** *The following statements hold:*

(i) $\mathfrak{N}_3^{\mathrm{reg}} = \{12\} \cup \{n \in \mathbb{N}\colon n \text{ is even and } n \geq 18\}$;

(ii) $\mathfrak{N}_4^{\mathrm{reg}} = \{8, 10, 12\} \cup \{n \in \mathbb{N}\colon n \geq 14\}$;

(iii) $\mathfrak{N}_5^{\mathrm{reg}} = \{n \in \mathbb{N}\colon n \text{ is even and } n \geq 10\}$;

(iv) $\mathfrak{N}_6^{\mathrm{reg}} = \{n \in \mathbb{N}\colon n \geq 12\}$;

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(v) $\mathfrak{N}_7^{\mathrm{reg}} = \{n \in \mathbb{N}\colon n \textit{ is even and } n \geq 12\}$;

(vi) $\mathfrak{N}_8^{\mathrm{reg}} = \{12\} \cup \{n \in \mathbb{N}\colon n \geq 14\}$;

(vii) $\mathfrak{N}_9^{\mathrm{reg}} = \{n \in \mathbb{N}\colon n \textit{ is even and } n \geq 16\}$;

(viii) $\mathfrak{N}_{10}^{\mathrm{reg}} = \{n \in \mathbb{N}\colon n \geq 15\}$;

(ix) $\mathfrak{N}_{11}^{\mathrm{reg}} = \{n \in \mathbb{N}\colon n \textit{ is even and } n \geq 16\}$;

(x) $\mathfrak{N}_{12}^{\mathrm{reg}} = \{n \in \mathbb{N}\colon n \geq 16\}$.

The proof of this theorem relied on finding examples of smallest order and using a construction to extend them to all higher orders. In the initial paper [27] the sets $\mathfrak{N}_3^{\mathrm{reg}}$ and $\mathfrak{N}_4^{\mathrm{reg}}$ were determined. These results were then extended up to degree $d = 11$ [21]. The set $\mathfrak{N}_{12}^{\mathrm{reg}}$ was determined in [4], where consideration of circulant graphs furnished examples of such nut graphs of all even orders, while odd orders were resolved by a combination of computer search and construction.

Although the sets $\mathfrak{N}_d^{\mathrm{reg}}$ for $d > 12$ will not be completely determined here, by providing families of regular nut graphs of high degree, we will show that $\mathfrak{N}_d^{\mathrm{reg}}$ is infinite for every $d > 12$. Again, circulant graphs play an essential role, as the families are defined by cartesian products, where one factor is a 'small' regular nut graph and the other a circulant graph with carefully chosen parameters. Our first main result is:

**Theorem 1.2.** *For each $d \geq 3$, there exist infinitely many $d$-regular nut graphs.*

### 1.2 Cayley nut graphs

In parallel with the treatment of $d$-regular nut graphs in general, progress has been made on the order–degree existence problem for nut graphs of high symmetry, in particular, those that belong to the classes of vertex-transitive and Cayley graphs.

*Vertex-transitive graphs* are graphs whose automorphism group acts transitively on the vertex set. Given a group $\Gamma$ with identity $e$ and a subset $C \subseteq \Gamma \setminus \{e\}$ closed under inversion, by $\mathrm{Cay}(\Gamma, C)$ we denote the graph with vertex set $\Gamma$ such that any two vertices $u$ and $v$ are adjacent if and only if $vu^{-1} \in C$. The set $C$ is called the connection set. A *Cayley graph* is any graph that is isomorphic to $\mathrm{Cay}(\Gamma, C)$ for some finite group $\Gamma$ and some connection set $C$. Every Cayley graph is vertex-transitive, while there exist vertex-transitive graphs that are not Cayley, perhaps the most famous example being the Petersen graph. A *circulant graph* is a Cayley graph where $\Gamma \cong \mathbb{Z}_n$ for some $n \in \mathbb{N}$, i.e. the group $\Gamma$ is a cyclic group. For more details on algebraic graph theoretic terminology, the reader is referred to [5, 19, 28]. For more results on highly symmetric nut graphs, see [2, 3, 17] and references therein.

Let $\mathfrak{N}_d^{\mathrm{VT}}$ (resp. $\mathfrak{N}_d^{\mathrm{Cay}}$, $\mathfrak{N}_d^{\mathrm{circ}}$) be the set comprising the orders attainable by a $d$-regular vertex-transitive (resp. Cayley, circulant) nut graph. Trivially,

$$\mathfrak{N}_d^{\mathrm{circ}} \subseteq \mathfrak{N}_d^{\mathrm{Cay}} \subseteq \mathfrak{N}_d^{\mathrm{VT}} \subseteq \mathfrak{N}_d^{\mathrm{reg}}.$$

The study of vertex-transitive nut graphs was initiated in [21] with the next result.

**Theorem 1.3** ([21, Theorem 10])**.** *Let $G$ be a non-trivial vertex-transitive nut graph on $n$ vertices, of degree $d$. Then $n$ and $d$ satisfy the following conditions. Either $d \equiv 0 \pmod 4$, and $n \equiv 0 \pmod 2$ and $n \geq d + 4$; or $d \equiv 2 \pmod 4$, and $n \equiv 0 \pmod 4$ and $n \geq d + 6$.*

Circulant nut graphs have been extensively studied [13, 15, 16, 18]. The paper [13] determined all the order–degree pairs for which circulant nut graphs exist.

**Theorem 1.4** ([13, Theorem 1.8]). *For each $d \in \mathbb{N}$, the set $\mathfrak{N}_d^{\mathrm{circ}}$ is given by*

$$\mathfrak{N}_d^{\mathrm{circ}} = \begin{cases} \varnothing, & \text{if } d \not\equiv 0 \pmod 4, \\ \{n \in \mathbb{N}\colon n \text{ is even and } n \geq d+4\}, & \text{if } d \equiv 4 \pmod 8, \\ \{14\} \cup \{n \in \mathbb{N}\colon n \text{ is even and } n \geq 18\}, & \text{if } d = 8, \\ \{n \in \mathbb{N}\colon n \text{ is even and } n \geq d+6\}, & \text{if } d \equiv 0 \pmod 8 \text{ and } d \geq 16. \end{cases}$$

The order–degree existence problems for vertex-transitive and Cayley nut graphs were solved for degrees divisible by four.

**Theorem 1.5** ([14, Corollaries 8 and 9]). *For each $d \in \mathbb{N}$ such that $d \equiv 0 \pmod 4$, the sets $\mathfrak{N}_d^{\mathrm{VT}}$ and $\mathfrak{N}_d^{\mathrm{Cay}}$ are given by*

$$\mathfrak{N}_d^{\mathrm{VT}} = \mathfrak{N}_d^{\mathrm{Cay}} = \{n \in \mathbb{N}\colon n \text{ is even and } n \geq d+4\}.$$

Here, we give a counterpart to Theorem 1.2, tailored to the class of Cayley nut graphs.

**Theorem 1.6.** *For each even $d \geq 4$, there exist infinitely many $d$-regular Cayley nut graphs.*

Note that Cayley nut graphs of odd degrees $d$ are already ruled out by Theorem 1.3. Earlier results of that kind relied on an algorithm by Filaseta and Schinzel [20] for finding a cyclotomic factor in a polynomial with integer coefficients. In contrast, the proof of Theorem 1.6 makes extensive use of trigonometric identities.

## 2 Useful facts about graph spectra

The *spectrum* of a graph, denoted by $\sigma(G)$, is the spectrum of its adjacency matrix $A(G)$. For an in-depth treatment of graph spectra, see standard references [7, 8, 11, 12]. The following lemma determines the spectrum (in closed form) of any circulant matrix.

**Lemma 2.1.** *For any $n \in \mathbb{N}$, the eigenvalues of a circulant matrix*

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$$

*are $P_A(\zeta)$, where*

$$P_A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

*and $\zeta \in \mathbb{C}$ ranges over the $n$-th roots of unity. Moreover, for each $n$-th root of unity $\zeta$,*

$$\begin{bmatrix} 1 & \zeta & \zeta^2 & \cdots & \zeta^{n-1} \end{bmatrix}^{\mathsf{T}}$$

*is an eigenvector of $A$ for the eigenvalue $P_A(\zeta)$.*

A proof of the above lemma can be found, e.g., in [29, Section 3.1]. Note that the vertices of a circulant graph can be arranged so that the corresponding adjacency matrix is circulant. Therefore, Lemma 2.1 can be used to determine $\sigma(G)$ when $G$ is a *circulant graph*. By $\mathrm{Circ}(n, S)$, where $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, we will denote the graph on the vertex set $V(G) = \mathbb{Z}_n$, where vertices $u, v \in V(G)$ are adjacent if and only if $u - v \in S$ or $v - u \in S$ (where addition is done in $\mathbb{Z}_n$). Note that the adjacency matrix of $\mathrm{Circ}(n, S)$ is a circulant matrix with $a_i = 1$ if $i \in S$ or $-i \in S$, and $a_i = 0$ otherwise. Thus, graph $\mathrm{Circ}(n, S)$ is a circulant graph on $n$ vertices with connection set $S \cup -S$, i.e. $\mathrm{Circ}(n, S) \cong \mathrm{Cay}(\mathbb{Z}_n, S \cup -S)$.

The *cartesian product* of graphs $G$ and $H$ will be denoted by $G \,\square\, H$. The vertex set of $G \,\square\, H$ is $V(G) \times V(H)$; vertices $(g, h)$ and $(g', h')$ are adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. For a comprehensive treatment of product graphs, see [31], and for a survey on their spectra, see [1].

**Lemma 2.2** ([7, Section 1.4.6])**.** *For any two graphs $G$ and $H$, the spectrum of the cartesian product $G \,\square\, H$ is given by*

$$\sigma(G \,\square\, H) = \{\lambda + \mu \colon \lambda \in \sigma(G),\, \mu \in \sigma(H)\}.$$

*Moreover, if $u \in \mathbb{R}^{V(G)}$ is an eigenvector of $G$ for the eigenvalue $\lambda$ and $v \in \mathbb{R}^{V(H)}$ is an eigenvector of $H$ for the eigenvalue $\mu$, then the vector $w \in \mathbb{R}^{V(G \,\square\, H)}$ defined as*

$$w_{(g,h)} = u_g\, v_h \quad (g \in V(G),\ h \in V(H))$$

*is an eigenvector of $G \,\square\, H$ for the eigenvalue $\lambda + \mu$.*

The following proposition is a direct consequence of Lemma 2.2.

**Proposition 2.3.** *Let $G_1$ and $G_2$ be nut graphs. Then $G_1 \,\square\, G_2$ is a nut graph if and only if there does not exist $\lambda \neq 0$ such that $\lambda \in \sigma(G_1)$ and $-\lambda \in \sigma(G_2)$.*

For any $b \in \mathbb{N}$, the *cyclotomic polynomial* $\Phi_b(x)$ is given by

$$\Phi_b(x) = \prod_{\zeta} (x - \zeta),$$

where $\zeta$ ranges over the primitive $b$-th roots of unity. It is well known that for each $b \in \mathbb{N}$, the polynomial $\Phi_b(x)$ has integer coefficients and is irreducible in $\mathbb{Q}[x]$ (see, e.g., [26, Chapter 33]). Therefore, any $P(x) \in \mathbb{Q}[x]$ has a root that is a primitive $b$-th root of unity if and only if $\Phi_b(x) \mid P(x)$. Moreover, $\deg \Phi_b(x) = \varphi(b)$ for each $b \in \mathbb{N}$, where $\varphi$ denotes Euler's totient function. We end the section with the following number-theoretic result.

**Lemma 2.4.** *For any $n \in \mathbb{N}$, $\varphi(n) \geq \sqrt{n/2}$.*

*Proof.* Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorisation of $n$, where $p_1, p_2, \ldots, p_k$ are the distinct prime factors of $n$. Since $\varphi(n) = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_k^{\alpha_k - 1}(p_1 - 1)(p_2 - 1) \cdots (p_k - 1)$, it follows that

$$\frac{\varphi(n)}{\sqrt{n}} = \prod_{j=1}^{k} p_j^{\alpha_j/2 - 1}(p_j - 1) \geq \prod_{j=1}^{k} p_j^{1/2 - 1}(p_j - 1) = \prod_{j=1}^{k} \frac{p_j - 1}{\sqrt{p_j}}.$$

Observe that $q-1>\sqrt{q}$ for any prime $q\geq 3$, hence

$$\prod_{j=1}^{k}\frac{p_j-1}{\sqrt{p_j}}\geq \min\left\{1,\frac{2-1}{\sqrt{2}}\right\}=\frac{1}{\sqrt{2}},$$

which implies $\varphi(n)/\sqrt{n}\geq 1/\sqrt{2}$. □

## 3 Degrees of regular nut graphs

In this section we will prove Theorem 1.2. The following lemma will be useful.

**Lemma 3.1.** *For some $d\geq 3$, let $G$ be a $d$-regular nut graph, and for some $t\in\mathbb{N}$, let $S$ be a set consisting of $t$ odd and $t$ even positive integers. If $d<4t$, then there exists $\ell\in\mathbb{N}$ such that for every prime number $p\geq\ell$, we have that $G\,\square\,\mathrm{Circ}(2p,S)$ is a $(d+4t)$-regular nut graph.*

*Proof.* Let $S=\{s_1,s_2,\ldots,s_{2t}\}$ and suppose that $n\in\mathbb{N}$ is even. For every $n\in\mathbb{N}$, $n\geq 2\max S+2$, let $H_n=\mathrm{Circ}(n,S)$. By Lemmas 2.1 and 2.2, the eigenvalues of $G\,\square\,H_n$ are $\nu(\lambda,\zeta)$, where

$$\nu(z,x)=z+\sum_{j=1}^{2t}\left(x^{s_j}+x^{-s_j}\right),$$

and $\lambda$ ranges over $\sigma(G)$, while $\zeta$ ranges over the $n$-th roots of unity. Furthermore, we have

$$\nu(\lambda,1)=\lambda+4t\quad\text{and}\quad\nu(\lambda,-1)=\lambda.$$

From the Perron-Frobenius theorem (see, e.g., [7, Section 2.2]) it follows that $|\lambda|\leq d$ for $\lambda\in\sigma(G)$, hence $\nu(\lambda,1)>0$. As the graph $G$ is a nut graph, it follows that $\nu(\lambda,-1)=0$ is satisfied only for the simple eigenvalue $\lambda=0$ of $G$. Since Lemmas 2.1 and 2.2 guarantee that the corresponding eigenvector is full, it follows that $G\,\square\,H_n$ is a nut graph if and only if $\nu(\lambda,\zeta)\neq 0$ for every $\lambda\in\sigma(G)$ and every non-real $n$-th root of unity $\zeta\in\mathbb{C}$, $\zeta\notin\{1,-1\}$.

Now, let $\alpha=\max S$. For each $\lambda\in\sigma(G)$, we see that $\nu(\lambda,\zeta)=0$ holds if and only if $\zeta$ is a root of the $\mathbb{Z}[x]$-polynomial

$$P_\lambda(x)=\lambda x^\alpha+\sum_{j=1}^{2t}\left(x^{\alpha+s_j}+x^{\alpha-s_j}\right).$$

Furthermore, there are finitely many roots of unity that represent a root of $P_\lambda(x)$ for some $\lambda\in\sigma(G)$, and so we may take $\beta\in\mathbb{N}$ to be the greatest of all of their orders. Note that $\beta$ is well-defined, as $-1$ certainly is a root of $P_0(x)$. If we let $\ell=1+\max(\alpha,\beta)$, then for each prime number $p\geq\ell$, every $2p$-th root of unity apart from 1 and $-1$ has an order of either $p$ or $2p$, which means that it cannot be a root of any polynomial $P_\lambda(x)$, $\lambda\in\sigma(G)$. Therefore, $G\,\square\,H_{2p}$ is a nut graph for any $p\geq\ell$, and it is clear that the given graph is $(d+4t)$-regular. □

To illustrate the steps in the above proof, we provide an example.

**Example 3.2.** We show how to find examples of 17-regular nut graphs using Lemma 3.1. The values $d = 5$ and $t = 3$ fulfil the requirements of the lemma and will lead us to nut graphs of the desired degree. We require a 5-regular nut graph. There are nine such graphs of order 10 and four of order 12 (see, e.g., [6, 9, 10, 21, 27]) and we could choose any one of them. For example, let $G$ be the graph in Figure 1. In the House of Graphs [6, 9], graph $G$ is listed last of the 5-regular nut graphs on 10 vertices.

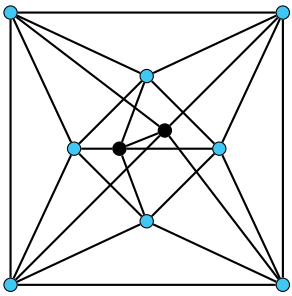


Figure 1: The graph $G$ used in Example 3.2. Note that we obtain a square antiprism on deleting the two black vertices.

The spectrum of $G$ is $\sigma(G) = \{5, (\sqrt{17}-1)/2, \sqrt{2}, \sqrt{2}, 0, -\sqrt{2}, -\sqrt{2}, -2, -2, -(\sqrt{17}+1)/2\}$. Now we must choose a suitable set $S$. Any choice for which $S$ contains 3 odd and 3 even integers is valid. Let us take, for example, $S = \{1, 2, 3, 6, 7, 10\}$. The corresponding value of the parameter $\alpha$ defined in the proof is $\alpha = 10$, and the polynomial $P_\lambda(x)$ is then

$$x^{20} + x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + \lambda x^{10} + x^9 + x^8 + x^7 + x^4 + x^3 + 1.$$

Let $\omega_n = e^{2\pi i/n}$ be a primitive $n$-th root of unity. By using a computer algebra system such as `SageMath` [35], we can verify that roots of unity appear as roots of the polynomial $P_\lambda(x)$ for certain values of $\lambda \in \sigma(G)$. For $\lambda = \sqrt{2}$, such roots are $\omega_8^3$ and $\omega_8^5$, both of order 8; for $\lambda = -\sqrt{2}$, they are $\omega_8$ and $\omega_8^7$, also both of order 8. Finally, for $\lambda = 0$, the roots of unity that are roots of the polynomial $P_\lambda(x)$ are

$$\omega_2 = -1,\ \omega_3,\ \omega_3^2,\ \omega_6,\ \omega_6^5,\ \omega_{12},\ \omega_{12}^5,\ \omega_{12}^7,\ \omega_{12}^{11},\ \omega_{18},\ \omega_{18}^5,\ \omega_{18}^7,\ \omega_{18}^{11},\ \omega_{18}^{13} \text{ and } \omega_{18}^{17}.$$

The orders of the roots in this list are all between 2 and 18, leading to $\beta = 18$. Hence, $\ell = 1 + \max(10, 18) = 19$. For every prime number $p \geq 19$, the graph $G \,\square\, \mathrm{Circ}(2p, \{1, 2, 3, 6, 7, 10\})$ is a 17-regular nut graph. In particular, $G \,\square\, \mathrm{Circ}(38, \{1, 2, 3, 6, 7, 10\})$ is a 17-regular nut graph of order 380. ♢

Note that we have extensive freedom in choosing suitable $(d, t)$ pairs, choosing the graph $G$ and choosing the set $S$. In fact, in our example, as we were looking for a 17-regular nut graph, the only possible choice for $(d, t)$ was $(5, 3)$. However, if we had been seeking, say, a 37-regular nut graph, then we could have picked any $(d, t) \in \{(5, 8), (9, 7), (13, 6), (17, 5)\}$. There are always infinitely many choices for the graph $G$. For the example, we chose the graph with the highest symmetry among those of the smallest order. Finally, there are infinitely many feasible candidates for the set $S$. Needless to say, the value of $\ell$ depends on all of these choices.

Note also that Lemma 3.1 gives us an infinite family of $(d+4t)$-regular nut graphs since we may pick any prime number $p \geq \ell$. However, it would be sufficient to provide only one such graph, since a construction described in [27], which replaces a vertex of degree $d$ of a nut graph by a gadget of $2d+1$ vertices of degree $d$ while retaining the nut property, could have been used iteratively to produce an infinite family.

With the aid of Lemma 3.1, we can now prove Theorem 1.2.

*Proof of Theorem 1.2.* Theorems 1.1 and 1.4 show that there exist infinitely many $d$-regular nut graphs for each $d \geq 3$ such that $d \leq 12$ or $d \equiv 0 \pmod 4$. Thus, it is sufficient to show the existence of infinitely many $d$-regular nut graphs for every $d \geq 13$ such that $d \not\equiv 0 \pmod 4$.

Let $G_3$, $G_5$, and $G_6$ be arbitrary 3-regular, 5-regular, and 6-regular nut graphs, respectively. Such graphs exist in abundance (see, e.g., [21]). Let $S$ be any set consisting of $t$ odd and $t$ even positive integers, $t \geq 2$. Furthermore, for every $n \geq 2\max S + 2$, let $H_n = \mathrm{Circ}(n, S)$. From Lemma 3.1, we obtain immediately that each of $G_3 \,\square\, H_n$, $G_5 \,\square\, H_n$, and $G_6 \,\square\, H_n$ is a nut graph for infinitely many values of $n \in \mathbb{N}$. Therefore, there exist infinitely many $(4t+3)$-regular, $(4t+5)$-regular and $(4t+6)$-regular nut graphs, for any $t \geq 2$. □

## 4 Degrees of Cayley nut graphs

In the present section we prove Theorem 1.6. We begin with the following lemma.

**Lemma 4.1.** *Let $\psi \in \mathbb{C}$ be a root of unity of order $b$ such that $b \geq 3$ and $b \not\equiv 0 \pmod 4$. Then $\psi$ is not a root of either of the polynomials*

$$P_1(x) := 2x^{4t+3} + x^{2t+8} - x^{2t} - 2x^5 \quad \textit{and} \quad P_2(x) := 2x^{4t+3} - x^{2t+8} + x^{2t} - 2x^5$$

*for any $t \in \mathbb{N}$.*

*Proof.* To obtain a contradiction, suppose that $\psi$ is a root of $P_1(x)$ or $P_2(x)$ for some $t \in \mathbb{N}$. In this case, we have $\Phi_b(x) \mid P_1(x)$ or $\Phi_b(x) \mid P_2(x)$, hence $\omega := e^{2\pi i/b}$ is a root of $P_1(x)$ or $P_2(x)$. Therefore,

$$2\omega^{2t-1} - 2\omega^{-(2t-1)} = \omega^{-4} - \omega^4 \quad \text{or} \quad 2\omega^{2t-1} - 2\omega^{-(2t-1)} = \omega^4 - \omega^{-4}. \tag{4.1}$$

By applying the identity $\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$ to (4.1), we obtain

$$2\sin\tfrac{2(2t-1)\pi}{b} = -\sin\tfrac{8\pi}{b} \quad \text{or} \quad 2\sin\tfrac{2(2t-1)\pi}{b} = \sin\tfrac{8\pi}{b}.$$

Since $\sin\frac{8\pi}{b} = 2\sin\frac{4\pi}{b}\cos\frac{4\pi}{b}$, it follows that

$$\left|\sin\tfrac{2(2t-1)\pi}{b}\right| = \left|\sin\tfrac{4\pi}{b}\right| \cdot \left|\cos\tfrac{4\pi}{b}\right|. \tag{4.2}$$

It is easy to verify, e.g., by using `SageMath`, that (4.2) does not hold for any $b \in \{3, 4, \ldots, 8\}$ regardless of the value of $(2t-1) \bmod b$. Therefore, $b \geq 9$. With this in mind, we have $0 < \frac{4\pi}{b} < \frac{\pi}{2}$, which implies $0 < \sin\frac{4\pi}{b} < 1$ and $0 < \cos\frac{4\pi}{b} < 1$. Hence, (4.2) gives

$$\left|\sin\tfrac{2(2t-1)\pi}{b}\right| < \left|\sin\tfrac{4\pi}{b}\right|. \tag{4.3}$$

We proceed by splitting the argument into two cases depending on the parity of $b$.

**Case 1:** $b$ is even.

Observe that if

$$(2t-1) \bmod b \in \{3, 5, \ldots, \tfrac{b}{2}-2\} \quad \text{or} \quad (2t-1) \bmod b \in \{\tfrac{b}{2}+2, \tfrac{b}{2}+4, \ldots, b-3\},$$

then $\left|\sin \frac{2(2t-1)\pi}{b}\right| \geq \left|\sin \frac{4\pi}{b}\right|$, which contradicts (4.3). Also, if $(2t-1) \bmod b = \frac{b}{2}$, then $\sin \frac{2(2t-1)\pi}{b} = 0$, which is impossible because $0 < \sin \frac{4\pi}{b} < 1$ and $0 < \cos \frac{4\pi}{b} < 1$. Therefore, $(2t-1) \bmod b \in \{1, b-1\}$. With this in mind, (4.2) becomes

$$\sin \tfrac{2\pi}{b} = \sin \tfrac{4\pi}{b} \cos \tfrac{4\pi}{b}, \quad \text{i.e.} \quad 2 \cos \tfrac{2\pi}{b} \cos \tfrac{4\pi}{b} = 1.$$

If $b = 10$, then

$$2 \cos \tfrac{2\pi}{b} \cos \tfrac{4\pi}{b} = 2 \cos \tfrac{\pi}{5} \cos \tfrac{2\pi}{5} = \frac{\sin \frac{4\pi}{5}}{2 \sin \frac{\pi}{5}} = \tfrac{1}{2},$$

while for $b \geq 14$, we get

$$2 \cos \tfrac{2\pi}{b} \cos \tfrac{4\pi}{b} \geq 2 \cos \tfrac{\pi}{7} \cos \tfrac{2\pi}{7} > 1,$$

yielding a contradiction either way.

**Case 2:** $b$ is odd.

Similarly, we observe that if

$$(2t-1) \bmod b \in \{2, 3, 4, \ldots, \tfrac{b-5}{2}\} \quad \text{or} \quad (2t-1) \bmod b \in \{\tfrac{b+5}{2}, \tfrac{b+7}{2}, \ldots, b-2\},$$

then $\left|\sin \frac{2(2t-1)\pi}{b}\right| \geq \left|\sin \frac{4\pi}{b}\right|$, which contradicts (4.3). Furthermore, if $(2t-1) \bmod b = 0$, then $\sin \frac{2(2t-1)\pi}{b} = 0$, which is not possible. Therefore, $(2t-1) \bmod b \in \{1, \frac{b-3}{2}, \frac{b-1}{2}, \frac{b+1}{2}, \frac{b+3}{2}, b-1\}$. If $(2t-1) \bmod b \in \{1, b-1\}$, then the proof can be completed analogously to Case 1.

Now, suppose that $(2t-1) \bmod b \in \{\frac{b-1}{2}, \frac{b+1}{2}\}$. In this case, (4.2) transforms to

$$\sin \tfrac{\pi}{b} = \sin \tfrac{4\pi}{b} \cos \tfrac{4\pi}{b}. \tag{4.4}$$

By applying the identity $\sin 4\theta = 4 \sin \theta \cos \theta \cos 2\theta$ to (4.4), we get

$$4 \cos \tfrac{\pi}{b} \cos \tfrac{2\pi}{b} \cos \tfrac{4\pi}{b} = 1.$$

If $b = 9$, then

$$4 \cos \tfrac{\pi}{b} \cos \tfrac{2\pi}{b} \cos \tfrac{4\pi}{b} = 4 \cos \tfrac{\pi}{9} \cos \tfrac{2\pi}{9} \cos \tfrac{4\pi}{9} = \frac{\sin \frac{8\pi}{9}}{2 \sin \frac{\pi}{9}} = \tfrac{1}{2},$$

while for $b \geq 11$, we have

$$4 \cos \tfrac{\pi}{b} \cos \tfrac{2\pi}{b} \cos \tfrac{4\pi}{b} \geq 4 \cos \tfrac{\pi}{11} \cos \tfrac{2\pi}{11} \cos \tfrac{4\pi}{11} > 1,$$

which yields a contradiction either way.

Finally, suppose that $(2t-1) \bmod b \in \{\frac{b-3}{2}, \frac{b+3}{2}\}$. In this case, (4.2) becomes

$$\sin \tfrac{3\pi}{b} = \sin \tfrac{4\pi}{b} \cos \tfrac{4\pi}{b}. \tag{4.5}$$

By plugging the identities $\sin 3\theta = \sin\theta\,(2\cos 2\theta + 1)$ and $\sin 4\theta = 4\sin\theta\cos\theta\cos 2\theta$ into (4.5), we get

$$2\cos\tfrac{2\pi}{b} + 1 = 4\cos\tfrac{\pi}{b}\cos\tfrac{2\pi}{b}\cos\tfrac{4\pi}{b}, \quad \text{i.e.} \quad 2\cos\tfrac{2\pi}{b}\left(2\cos\tfrac{\pi}{b}\cos\tfrac{4\pi}{b} - 1\right) = 1.$$

It is easy to verify, e.g., by using `SageMath`, that

$$2\cos\tfrac{2\pi}{b}\left(2\cos\tfrac{\pi}{b}\cos\tfrac{4\pi}{b} - 1\right) < 1$$

holds when $b \le 17$. Since $2\cos\frac{\pi}{19}\cos\frac{4\pi}{19} > 1$, this means that for any $b \ge 19$, we have

$$2\cos\tfrac{2\pi}{b}\left(2\cos\tfrac{\pi}{b}\cos\tfrac{4\pi}{b} - 1\right) \ge 2\cos\tfrac{2\pi}{19}\left(2\cos\tfrac{\pi}{19}\cos\tfrac{4\pi}{19} - 1\right) > 1,$$

yielding a contradiction. □

We are now in a position to complete the proof of Theorem 1.6 by relying on the next lemma.

**Lemma 4.2.** *Let $G$ be the graph*

$$\mathrm{Circ}(m, \{1, 2, \ldots, t-1\} \cup \{\tfrac{m+2}{4}, \tfrac{m+6}{4}\} \cup \{\tfrac{m}{2} - (t-1), \ldots, \tfrac{m}{2} - 2, \tfrac{m}{2} - 1\} \cup \{\tfrac{m}{2}\}).$$

*For any $t \in \mathbb{N}$ and $m \ge 4t+6$ such that $m \equiv 2 \pmod 4$, the graph $G \,\square\, K_2$ is a $(4t+2)$-regular Cayley nut graph of order $2m$.*

*Proof.* Note that $t - 1 < \frac{m+2}{4}$ and $\frac{m+6}{4} < \frac{m}{2} - (t-1)$, hence $G$ and $G \,\square\, K_2$ are $(4t+1)$- and $(4t+2)$-regular, respectively. We also observe that $G \,\square\, K_2$ is a Cayley graph for the group $\mathbb{Z}_m \times \mathbb{Z}_2$. By Lemmas 2.1 and 2.2, the spectrum of $G \,\square\, K_2$ comprises the values $\nu(\lambda, \zeta)$, where

$$\begin{aligned}\nu(\lambda,\zeta) = \lambda &+ \left(\zeta^{\frac{m+2}{4}} + \zeta^{-\frac{m+2}{4}}\right) + \left(\zeta^{\frac{m+6}{4}} + \zeta^{-\frac{m+6}{4}}\right) + \zeta^{\frac{m}{2}} \\ &+ \sum_{j=1}^{t-1}\left(\zeta^j + \zeta^{-j}\right) + \sum_{j=\frac{m}{2}-t+1}^{\frac{m}{2}-1}\left(\zeta^j + \zeta^{-j}\right),\end{aligned}$$

and $\lambda \in \{1, -1\}$, while $\zeta$ ranges over the $m$-th roots of unity. A straightforward computation gives

$$\nu(1,1) = 4t+2, \quad \nu(-1,1) = 4t, \quad \nu(1,-1) = 0 \quad \text{and} \quad \nu(-1,-1) = -2.$$

Furthermore, Lemma 2.2 implies that the eigenvector for the eigenvalue $\nu(1,-1) = 0$ is full. Therefore, $G \,\square\, K_2$ is a nut graph if and only if $\nu(\lambda,\zeta) \ne 0$ for any $\lambda \in \{1,-1\}$ and $m$-th root of unity $\zeta \notin \{1,-1\}$. Observe that

$$\begin{aligned}\nu(\lambda,\zeta) = \lambda &+ \left(\zeta^{\frac{m+6}{4}} + \zeta^{\frac{m+2}{4}} + \zeta^{\frac{m}{2}} + \zeta^{-\frac{m+2}{4}} + \zeta^{-\frac{m+6}{4}}\right) \\ &+ \sum_{j=1}^{t-1}\left(\zeta^j + \zeta^{-j} + \zeta^{\frac{m}{2}-j} + \zeta^{j-\frac{m}{2}}\right).\end{aligned} \tag{4.6}$$

We now split the argument into two cases depending on the value of $\zeta^{\frac{m}{2}}$.

**Case 1:** $\zeta^{\frac{m}{2}} = -1$.

In this case we have $\zeta^{\frac{m}{2}-j} = -\zeta^{-j}$ and $\zeta^{j-\frac{m}{2}} = -\zeta^{j}$, hence (4.6) simplifies to

$$\nu(\lambda,\zeta) = \lambda - 1 + \zeta^{\frac{m+6}{4}} + \zeta^{\frac{m+2}{4}} + \zeta^{-\frac{m+2}{4}} + \zeta^{-\frac{m+6}{4}}.$$

If $\lambda = 1$, then

$$\begin{aligned}\nu(1,\zeta) &= \zeta^{\frac{m+6}{4}} + \zeta^{\frac{m+2}{4}} + \zeta^{-\frac{m+2}{4}} + \zeta^{-\frac{m+6}{4}} = \zeta^{-\frac{m+6}{4}}\left(\zeta^{\frac{m}{2}+3} + \zeta^{\frac{m}{2}+2} + \zeta + 1\right)\\ &= \zeta^{-\frac{m+6}{4}}\left(-\zeta^3 - \zeta^2 + \zeta + 1\right) = -\zeta^{-\frac{m+6}{4}}(\zeta-1)(\zeta+1)^2,\end{aligned}$$

which implies $\nu(1,\zeta) \neq 0$ provided $\zeta \notin \{1,-1\}$. To obtain a contradiction, suppose that $\nu(-1,\zeta) = 0$ holds for some $\zeta \notin \{1,-1\}$. Since

$$\nu(-1,\zeta) = -2 + \zeta^{\frac{m+6}{4}} + \zeta^{\frac{m+2}{4}} + \zeta^{-\frac{m+2}{4}} + \zeta^{-\frac{m+6}{4}} = -2 - \zeta^{-\frac{m+6}{4}}(\zeta-1)(\zeta+1)^2,$$

we obtain

$$(\zeta-1)^2(\zeta+1)^4 = \left(-2\zeta^{\frac{m+6}{4}}\right)^2 = 4\zeta^{\frac{m}{2}+3} = -4\zeta^3.$$

Therefore, $\zeta$ is a root of the polynomial

$$(x-1)^2(x+1)^4 + 4x^3 = x^6 + 2x^5 - x^4 - x^2 + 2x + 1. \tag{4.7}$$

We obtain a contradiction by showing that (4.7) is not divisible by $\Phi_b(x)$ for any $b \in \mathbb{N}$. If $b > 72$, then Lemma 2.4 gives $\deg \Phi_b(x) = \varphi(b) \geq \sqrt{b/2} > 6$, hence $\Phi_b(x)$ does not divide (4.7). For each $b \in \{1,2,\ldots,72\}$, the required non-divisibility can be verified using, e.g., `SageMath`.

**Case 2:** $\zeta^{\frac{m}{2}} = 1$.

In this case, $\zeta^{\frac{m}{2}-j} = \zeta^{-j}$ and $\zeta^{j-\frac{m}{2}} = \zeta^{j}$, hence (4.6) transforms to

$$\nu(\lambda,\zeta) = \lambda + 1 + \zeta^{\frac{m+6}{4}} + \zeta^{\frac{m+2}{4}} + \zeta^{-\frac{m+2}{4}} + \zeta^{-\frac{m+6}{4}} + 2\sum_{j=1}^{t-1}\left(\zeta^j + \zeta^{-j}\right). \tag{4.8}$$

As shown in [15, Theorem 4], the graph

$$G_1 \coloneqq \mathrm{Circ}(m, \{1,2,\ldots,t-1\} \cup \{\tfrac{m+2}{4}, \tfrac{m+6}{4}\} \cup \{\tfrac{m}{2}-(t-1), \ldots, \tfrac{m}{2}-2, \tfrac{m}{2}-1\})$$

is a nut graph. By Lemma 2.1, the spectrum of $G_1$ comprises the values $\kappa(\zeta)$, where

$$\kappa(\zeta) = \left(\zeta^{\frac{m+2}{4}} + \zeta^{-\frac{m+2}{4}}\right) + \left(\zeta^{\frac{m+6}{4}} + \zeta^{-\frac{m+6}{4}}\right) + \sum_{j=1}^{t-1}\left(\zeta^j + \zeta^{-j}\right) + \sum_{j=\frac{m}{2}-t+1}^{\frac{m}{2}-1}\left(\zeta^j + \zeta^{-j}\right),$$

and $\zeta$ ranges over the $m$-th roots of unity. Since the nullspace of $G_1$ is one-dimensional, and $\kappa(\bar{\zeta}) = \overline{\kappa(\zeta)}$, it follows that $\kappa(\zeta) \neq 0$ for any $m$-th root of unity $\zeta \notin \{1,-1\}$. A straightforward computation gives $\nu(-1,\zeta) = \kappa(\zeta)$ provided $\zeta^{\frac{m}{2}} = 1$, which implies that $\nu(-1,\zeta) \neq 0$ for any $\frac{m}{2}$-th root of unity $\zeta \notin \{1,-1\}$.

To obtain a contradiction, suppose that $\nu(1,\zeta) = 0$ holds for some $\zeta \notin \{1,-1\}$, and note that (4.8) becomes

$$\nu(1,\zeta) = \zeta^{\frac{m+6}{4}} + \zeta^{\frac{m+2}{4}} + \zeta^{-\frac{m+2}{4}} + \zeta^{-\frac{m+6}{4}} + 2\sum_{j=-t+1}^{t-1} \zeta^j.$$

A routine computation gives

$$\zeta^{t-1}(\zeta-1)\,\nu(1,\zeta) = \zeta^{t+\frac{m+6}{4}} - \zeta^{t+\frac{m-2}{4}} + \zeta^{t-\frac{m+2}{4}} - \zeta^{t-\frac{m+10}{4}} + 2\zeta^{2t-1} - 2. \quad (4.9)$$

Let $\psi \in \mathbb{C}$ be such that $\psi^2 = \zeta$, so that (4.9) becomes

$$\begin{aligned}\zeta^{t-1}(\zeta-1)\,\nu(1,\zeta) &= 2\psi^{4t-2} + \psi^{2t+3+\frac{m}{2}} - \psi^{2t-1+\frac{m}{2}} + \psi^{2t-1-\frac{m}{2}} - \psi^{2t-5-\frac{m}{2}} - 2\\ &= 2\psi^{4t-2} + \psi^{2t+3+\frac{m}{2}} - \psi^{2t-5-\frac{m}{2}} - 2.\end{aligned}$$

Since $m \equiv 2 \pmod 4$, we observe that $\psi$ is an $m$-th root of unity whose order $b$ satisfies $b \geq 3$ and $b \not\equiv 0 \pmod 4$. Depending on whether $\psi^{\frac{m}{2}} = 1$ or $\psi^{\frac{m}{2}} = -1$, we have

$$2\psi^{4t-2} + \psi^{2t+3} - \psi^{2t-5} - 2 = 0 \quad \text{or} \quad 2\psi^{4t-2} - \psi^{2t+3} + \psi^{2t-5} - 2 = 0.$$

In both cases, a contradiction follows from Lemma 4.1. □

The cases $d \equiv 0 \pmod 4$ and $d \equiv 2 \pmod 4$ of Theorem 1.6 follow directly from Theorem 1.5 and Lemma 4.2, respectively.

**Remark 4.3.** Note that Lemma 4.2 also shows that

$$\mathfrak{N}_d^{\mathrm{VT}} \supseteq \mathfrak{N}_d^{\mathrm{Cay}} \supseteq \{2d+8, 2d+16, 2d+24, 2d+32, \ldots\}$$

for any $d \geq 6$ such that $d \equiv 2 \pmod 4$. Therefore, for any even $d \geq 4$, the sets $\mathfrak{N}_d^{\mathrm{VT}}$ and $\mathfrak{N}_d^{\mathrm{Cay}}$ are not only infinite, but they also have a natural density of at least $1/8$.

## 5 Concluding remarks

We have shown that $d$-regular nut graphs exist for all $d \geq 3$ (Theorem 1.2). When $d \equiv 0 \pmod 2$, nut graphs exist even within the class of $d$-regular Cayley graphs (Theorem 1.6). In this sense, Theorem 1.6 is a stronger version of Theorem 1.2, but only for even degree.

The proof of Theorem 1.6 relies on the nut graphs constructed in Theorem 1.5 and Lemma 4.2, all of which are explicitly determined. However, the present proof of Theorem 1.2 is non-constructive in nature. That is to say, the proof does not give an explicit algorithmic construction of the nut graphs in question, as the intermediate quantity $\beta$ (and hence $\ell$) is not easy to determine. For even degree $d$, we could replace this branch of the proof by simple invocation of Theorem 1.6, but for the odd case this route is not available. In the present section, we suggest a strategy that would lead to such a construction, based on a conjecture supported by numerical computation.

Recall that existence of $d$-regular nut graphs is guaranteed for $3 \leq d \leq 12$ (by Theorem 1.1). Recall also the following theorem from previous work.

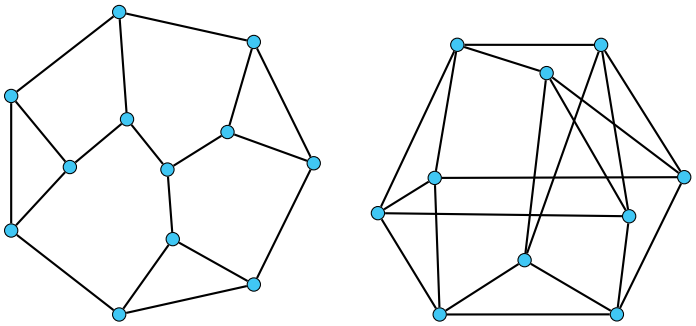
**Theorem 5.1** ([15, Theorem 4]). *For each $t \in \mathbb{N}$ and $n \geq 4t+6$ such that $n \equiv 2 \pmod 4$, the circulant graph*

$$\mathcal{D}(n,t) \coloneqq \mathrm{Circ}(n, \{1,2,\ldots,t-1\} \cup \{(n+2)/4, (n+6)/4\} \cup \{n/2-(t-1), \ldots, n/2-2, n/2-1\})$$

*is a $4t$-regular nut graph of order $n$.*

We can obtain nut graphs with odd degrees $d \geq 13$ using Proposition 2.3 by taking cartesian products of members of the above family of circulant graphs with certain 3- and 5-regular nut graphs. For example, let $F_3$ be the Frucht graph, and let $F_5$ be the graph whose complement is shown in Figure 2. Their characteristic polynomials are:

$$\Phi(F_3,x) = x(x-3)(x-2)(x+1)(x+2)(x^3+x^2-2x-1)(x^4+x^3-6x^2-5x+4),$$
$$\Phi(F_5,x) = x(x-5)(x+1)(x^2+x-1)(x^5+3x^4-6x^3-21x^2-x+16).$$



(a) The Frucht graph $F_3$. (b) Complement of the graph $F_5$.

Figure 2: The regular nut graphs used in Conjecture 5.2 (drawn as the graph or its complement).

We now propose the following conjecture.

**Conjecture 5.2.** *The following statements hold:*

(i) *$\mathcal{D}(4t+6,t) \,\square\, F_5$ is a nut graph of degree $4t+5$ for every $t \geq 1$;*

(ii) *$\mathcal{D}(4t+6,t) \,\square\, F_3$ is a nut graph of degree $4t+3$ for $t \geq 1$ such that $t \not\equiv 0 \pmod 3$;*

(iii) *$\mathcal{D}(4t+10,t) \,\square\, F_3$ is a nut graph of degree $4t+3$ for $t \geq 1$ such that $t \equiv 0 \pmod 3$.*

Statement (i) resolves the case $d \equiv 1 \pmod 4$. Statements (ii) and (iii) taken together resolve the case $d \equiv 3 \pmod 4$. Note that the graph $\mathcal{D}(4t+6,t)$ has an eigenvalue $-3$ when $t \equiv 0 \pmod 3$.

Computations show that the conjecture is true for all values of $t$ such that the degree of the obtained product graph is less than 1000. This implies that we have constructed a $d$-regular nut graph for each odd $3 \leq d < 1000$. Proof of Conjecture 5.2 would furnish a fully constructive proof of Theorem 1.2.

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## References

[1] S. Barik, D. Kalita, S. Pati and G. Sahoo, Spectra of graphs resulting from various graph operations and products: A survey, *Spec. Matrices* **6** (2018), 323–342, doi:10.1515/spma-2018-0027, https://doi.org/10.1515/spma-2018-0027.

[2] N. Bašić and P. W. Fowler, Nut graphs with a given automorphism group, *J. Algebraic Comb.* **61** (2025), Art. No. 17, doi:10.1007/s10801-025-01389-4, https://doi.org/10.1007/s10801-025-01389-4.

[3] N. Bašić, P. W. Fowler and T. Pisanski, Vertex and edge orbits in nut graphs, *Electron. J. Comb.* **31** (2024), #P2.38, 37 pp., doi:10.37236/12619, https://doi.org/10.37236/12619.

[4] N. Bašić, M. Knor and R. Škrekovski, On 12-regular nut graphs, *Art Discrete Appl. Math.* **5** (2022), #P2.01, 13 pp., doi:10.26493/2590-9770.1403.1b1, https://doi.org/10.26493/2590-9770.1403.1b1.

[5] N. Biggs, *Algebraic Graph Theory*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2nd edition, 1993.

[6] G. Brinkmann, K. Coolsaet, J. Goedgebeur and H. Mélot, House of Graphs: A database of interesting graphs, *Discrete Appl. Math.* **161** (2013), 311–314, doi:10.1016/j.dam.2012.07.018, https://doi.org/10.1016/j.dam.2012.07.018, available at https://hog.grinvin.org/.

[7] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Universitext, Springer, New York, 2012, doi:10.1007/978-1-4614-1939-6, https://doi.org/10.1007/978-1-4614-1939-6.

[8] F. R. K. Chung, *Spectral Graph Theory*, volume 92 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, 1997.

[9] K. Coolsaet, S. D'hondt and J. Goedgebeur, House of graphs 2.0: A database of interesting graphs and more, *Discrete Appl. Math.* **325** (2023), 97–107, doi:10.1016/j.dam.2022.10.013, https://doi.org/10.1016/j.dam.2022.10.013.

[10] K. Coolsaet, P. W. Fowler and J. Goedgebeur, Generation and properties of nut graphs, *MATCH Commun. Math. Comput. Chem.* **80** (2018), 423–444, http://match.pmf.kg.ac.rs/electronic_versions/Match80/n2/match80n2_423-444.pdf.

[11] D. M. Cvetković, P. Rowlinson and S. Simić, *Eigenspaces of Graphs*, volume 66 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1997, doi:10.1017/cbo9781139086547, https://doi.org/10.1017/cbo9781139086547.

[12] D. M. Cvetković, P. Rowlinson and S. Simić, *An Introduction to the Theory of Graph Spectra*, volume 75 of *London Mathematical Society Student Texts*, Cambridge University Press, Cambridge, 2010.

[13] I. Damnjanović, Complete resolution of the circulant nut graph order–degree existence problem, *Ars Math. Contemp.* **24** (2024), Id/No P4.03, 28 pp., doi:10.26493/1855-3974.3009.6df, https://doi.org/10.26493/1855-3974.3009.6df.

[14] I. Damnjanović, A note on Cayley nut graphs whose degree is divisible by four, *Art Discrete Appl. Math.* **9** (2026), Id/No P2.02, 6 pp., doi:10.26493/2590-9770.1662.4e9, https://doi.org/10.26493/2590-9770.1662.4e9.

[15] I. Damnjanović, Two families of circulant nut graphs, *Filomat* **37** (2023), 8331–8360, doi:10.2298/fil2324331d, https://doi.org/10.2298/fil2324331d.

[16] I. Damnjanović, On the null spaces of quartic circulant graphs, *Discrete Math. Chem.* **1** (2025), #P1.04, 16 pp., doi:10.26493/2820-6657.6.5a1, https://doi.org/10.26493/2820-6657.6.5a1.

[17] I. Damnjanović, N. Bašić, T. Pisanski and A. Žitnik, Classification of cubic tricirculant nut graphs, *Electron. J. Comb.* **31** (2024), #P2.31, 32 pp., doi:10.37236/12668, https://doi.org/10.37236/12668.

[18] I. Damnjanović and D. Stevanović, On circulant nut graphs, *Linear Algebra Appl.* **633** (2022), 127–151, doi:10.1016/j.laa.2021.10.006, https://doi.org/10.1016/j.laa.2021.10.006.

[19] T. Dobson, A. Malnič and D. Marušič, *Symmetry in Graphs*, volume 198 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 2022, doi:10.1017/9781108553995, https://doi.org/10.1017/9781108553995.

[20] M. Filaseta and A. Schinzel, On testing the divisibility of lacunary polynomials by cyclotomic polynomials, *Math. Comput.* **73** (2004), 957–965, doi:10.1090/S0025-5718-03-01589-8, https://doi.org/10.1090/S0025-5718-03-01589-8.

[21] P. W. Fowler, J. B. Gauci, J. Goedgebeur, T. Pisanski and I. Sciriha, Existence of regular nut graphs for degree at most 11, *Discuss. Math. Graph Theory* **40** (2020), 533–557, doi:10.7151/dmgt.2283, https://doi.org/10.7151/dmgt.2283.

[22] P. W. Fowler, B. T. Pickup, T. Z. Todorova, R. De Los Reyes and I. Sciriha, Omni-conducting fullerenes, *Chem. Phys. Lett.* **568** (2013), 33–35, doi:10.1016/j.cplett.2013.03.022, https://doi.org/10.1016/j.cplett.2013.03.022.

[23] P. W. Fowler, T. Pisanski and N. Bašić, Charting the space of chemical nut graphs, *MATCH Commun. Math. Comput. Chem.* **86** (2021), 519–538, https://match.pmf.kg.ac.rs/electronic_versions/Match86/n3/match86n3_519-538.pdf.

[24] R. Frucht, Herstellung von Graphen mit vorgegebener abstrakter Gruppe, *Compositio Math.* **6** (1939), 239–250, http://www.numdam.org/item?id=CM_1939__6__239_0.

[25] R. Frucht, Graphs of degree three with a given abstract group, *Canad. J. Math.* **1** (1949), 365–378, doi:10.4153/cjm-1949-033-6, https://doi.org/10.4153/cjm-1949-033-6.

[26] J. A. Gallian, *Contemporary Abstract Algebra*, Cengage Learning, 9th edition, 2017.

[27] J. B. Gauci, T. Pisanski and I. Sciriha, Existence of regular nut graphs and the Fowler construction, *Appl. Anal. Discrete Math.* **17** (2023), 321–333, doi:10.2298/aadm190517028g, https://doi.org/10.2298/aadm190517028g.

[28] C. Godsil and G. Royle, *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2001, doi:10.1007/978-1-4613-0163-9, https://doi.org/10.1007/978-1-4613-0163-9.

[29] R. M. Gray, Toeplitz and circulant matrices: A review, *Found. Trends Commun. Inf. Theory* **2** (2006), 155–239, doi:10.1561/0100000006, https://doi.org/10.1561/0100000006.

[30] I. Gutman and O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer-Verlag, Berlin, 1986, doi:10.1007/978-3-642-70982-1, https://doi.org/10.1007/978-3-642-70982-1.

[31] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs*, Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 2nd edition, 2011.

[32] I. Sciriha and A. Farrugia, *From Nut Graphs to Molecular Structure and Conductivity*, volume 23 of *Mathematical Chemistry Monographs*, University of Kragujevac and Faculty of Science Kragujevac, Kragujevac, 2021.

[33] I. Sciriha and P. W. Fowler, Nonbonding orbitals in fullerenes: Nuts and cores in singular polyhedral graphs, *J. Chem. Inf. Model.* **47** (2007), 1763–1775, doi:10.1021/ci700097j, https://doi.org/10.1021/ci700097j.

[34] I. Sciriha and I. Gutman, Nut graphs: Maximally extending cores, *Util. Math.* **54** (1998), 257–272.

[35] The Sage Developers, SageMath, the Sage Mathematics Software System (Version 9.4), 2021, https://www.sagemath.org.