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Cyclic and symmetric hamiltonian cycle systems of the complete multipartite graph: even number of parts

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Abstract

In this paper, we present a complete solution to the existence problem for a cyclic hamiltonian cycle system for the complete multipartite graph with an even number of parts all of the same cardinality. We also give necessary and sufficient conditions for the system to be symmetric as well.

Keywords: Hamiltonian cycle, cyclic cycle system, symmetric hamiltonian cycle system, complete multipartite graph.

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1 Introduction

Throughout this paper, K_v will denote the complete graph on v vertices and, if v is even, $K_v - I$ will denote the cocktail party graph of order v, namely the graph obtained from K_v by removing a 1-factor I, that is, a set of $\frac{v}{2}$ pairwise disjoint edges. Also $K_{m\times n}$ will denote the complete multipartite graph with m parts of same cardinality n; if $n = 1$, we

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may identify $K_{m\times 1}$ with K_m, while if $n = 2$, $K_{m\times 2}$ is nothing but the cocktail party graph $K_{2m} - I$.

For any graph Γ we write $V(\Gamma)$ for the set of its vertices and $E(\Gamma)$ for the set of its edges. We denote by $C = (c_0, c_1, \dots, c_{\ell-1})$ the cycle of length ℓ whose edges are $[c_0, c_1], [c_1, c_2], \dots, [c_{\ell-1}, c_0]$. An *$\ell$-cycle system* of a graph Γ is a set $\mathcal{B}$ of cycles of length ℓ whose edges partition $E(\Gamma)$; clearly a graph may admit a cycle system only if the degree of each vertex is even. For general background on cycle systems we refer to the surveys [7, 8]. An ℓ-cycle system $\mathcal{B}$ of Γ is said to be *hamiltonian* if $\ell = |V(\Gamma)|$, and it is said to be *cyclic* if we may identify $V(\Gamma)$ with the cyclic group $\mathbb{Z}_v$, and if for any $C = (c_0, c_1, \dots, c_{\ell-1}) \in \mathcal{B}$, we have also $C + 1 = (c_0 + 1, c_1 + 1, \dots, c_{\ell-1} + 1) \in \mathcal{B}$. The existence problem for cyclic cycle systems of K_v has generated a considerable amount of interest. Many authors have contributed to give a complete answer in the case $v \equiv 1$ or $\ell \pmod{2\ell}$ (see [10, 11, 19, 20, 21, 22, 26]). We point out in particular that the existence problem of a cyclic hamiltonian cycle system (HCS, for short) for K_v has been solved by Buratti and Del Fra in [11], and that for $K_v - I$ it has been solved by Jordon and Morris [17].

The existence problem for cycle systems of the complete multipartite graph has not been solved yet, but we have many interesting recent results on this topic (see for instance [4, 5, 24, 25]). Still, very little is known about the same problem with the additional constraint that the system be cyclic. We have a complete solution in the following very special cases: the length of the cycles is equal to the cardinality of the parts [12]; the cycles are hamiltonian and the parts have cardinality two [14, 17]. We have also some partial results in [3].

Hamiltonian cycle systems of $K_{m\times n}$ have been shown to exist ([18]) whenever the degree of each vertex of the graph, that is $(m-1)n$, is even; in this paper we start investigating the existence of cyclic hamiltonian cycle systems of $K_{m\times n}$, and completely solve the problem when m is even.

We also consider the existence of a *symmetric* HCS for $K_{m\times n}$ with $n > 1$, a concept recently introduced by Schroeder in [23] generalizing the notion of symmetry given in [6] for cocktail party graphs: in this definition, an HCS for $K_{m\times n}$ is *n-symmetric* if each cycle in the system is invariant under a fixed-point-free automorphism of order n. We will show that the cycle systems we shall construct in will turn out to be symmetric in this sense.

The paper is organized as follows: after some preliminary notes in Section 2 on the methods we shall use, in Section 3 we establish a necessary condition in the case n even for the existence of a cyclic cycle system (not necessarily hamiltonian) of $K_{m\times n}$ from which we derive a necessary condition for the existence of a cyclic HCS of $K_{m\times n}$. Then in Section 4 we give a complete solution to the existence problem of a cyclic HCS with an even number of parts, proving that in this case the necessary condition we found is also sufficient. The main result of this paper is the following.

Theorem 1.1. *Let m be even; a cyclic and n-symmetric HCS for $K_{m\times n}$ exists if and only if*

(a) $n \equiv 0 \pmod{4}$*, or*

(b) $n \equiv m \equiv 2 \pmod{4}$.

The proof of Theorem 1.1 will follow from the various results proved in Sections 3 and 4. First, in Corollary 3.4 we give the necessary condition for the existence of a cyclic HCS of $K_{m\times n}$. Then, in Proposition 4.2 we study the bipartite case, finally in Theorem 4.3 and in Theorem 4.7 we deal with the case $n \equiv 0 \pmod 4$, and $n \equiv 2 \pmod 4$, respectively.

2 Preliminaries

The main results of this paper will be obtained by using the method of *partial differences* introduced by Marco Buratti and used in many papers, see for instance [2, 9, 10, 11, 13, 14, 27]. Here we recall some definitions and results useful in the rest of the paper.

Definition 2.1. Let $C = (c_0, c_1, \dots, c_{\ell-1})$ be an ℓ-cycle with vertices in an abelian group G and let d be the order of the stabilizer of C under the natural action of G, that is, $d = |\{g \in G : C + g = C\}|$. The multisets

$$\begin{aligned} \Delta C &= \{\pm(c_{h+1} - c_h) \mid 0 \le h < \ell\}, \\ \partial C &= \{\pm(c_{h+1} - c_h) \mid 0 \le h < \ell/d\}, \end{aligned}$$

where the subscripts are taken modulo ℓ, are called the *list of differences* from C and the *list of partial differences* from C, respectively.

More generally, given a set $\mathcal{B}$ of ℓ-cycles with vertices in G, by $\Delta\mathcal{B}$ and $\partial\mathcal{B}$ one means the union (counting multiplicities) of all multisets ΔC and ∂C respectively, where $C \in \mathcal{B}$.

We recall the definition of a *Cayley graph on a group G with connection set* Ω, denoted by $Cay[G : \Omega]$. Let G be an additive group and let $\Omega \subseteq G \setminus \{0\}$ such that for every $\omega \in \Omega$ we also have $-\omega \in \Omega$. The Cayley graph $Cay[G : \Omega]$ is the graph whose vertices are the elements of G and in which two vertices are adjacent if and only if their difference is an element of Ω (an analogous definition can be given in multiplicative notation). Note that $K_{m\times n}$ can be interpreted as the Cayley graph $Cay[\mathbb{Z}_{mn} : \mathbb{Z}_{mn} \setminus m\mathbb{Z}_{mn}]$, where by $m\mathbb{Z}_{mn}$ we mean the subgroup of order n of $\mathbb{Z}_{mn}$. The vertices of $K_{m\times n}$ will be always understood as elements of $\mathbb{Z}_{mn}$ and the parts of $K_{m\times n}$ are the cosets of $m\mathbb{Z}_{mn}$ in $\mathbb{Z}_{mn}$. We consider the natural action of $\mathbb{Z}_{mn}$ on the cycles of $K_{m\times n}$: given a cycle $C = (c_0, c_1, \dots, c_{\ell-1})$ of $K_{m\times n}$ we define $C+t$ as the cycle $(c_0+t, c_1+t, \dots, c_{\ell-1}+t)$, where $c_0, c_1, \dots, c_{\ell-1}, t$ are elements of $\mathbb{Z}_{mn}$. The stabilizer and the orbit of any cycle C of $K_{m\times n}$ will be understood with respect to this action and will be denoted by $Stab(C)$ and $Orb(C)$, respectively. A cyclic HCS of $K_{m\times n}$ is completely determined by a set of *base cycles*, namely, a complete system of representatives for the orbits of its cycles under the action of $\mathbb{Z}_{mn}$. The next theorem, which is a consequence of the theory of partial differences, will play a fundamental role in this paper.

Theorem 2.2. *A set $\mathcal{B}$ of mn-cycles is a set of base cycles of a cyclic HCS of $K_{m\times n}$ if and only if $\partial\mathcal{B} = \mathbb{Z}_{mn} \setminus m\mathbb{Z}_{mn}$.*

In Example 2.4 we will show how to construct a cyclic HCS of $K_{10\times 6}$ applying Theorem 2.2.

For our purposes the following notation will be useful. Let $c_0, c_1, \dots, c_{r-1}, x$ be elements of an additive group G, with x of order d. The closed trail

$$[c_0, c_1, c_2, \dots, c_{r-1},$$

$$c_0 + x, c_1 + x, c_2 + x, \dots, c_{r-1} + x, \dots,$$

$$c_0 + (d-1)x, c_1 + (d-1)x, c_2 + (d-1)x, \dots, c_{r-1} + (d-1)x]$$

will be denoted by

$$[c_0, c_1, \dots, c_{r-1}]_x.$$

For brevity, given $P = [c_0, c_1, \dots, c_{r-1}]$, we write $[P]_x$ for the closed trail $[c_0, c_1, \dots, c_{r-1}]_x$. For instance in $\mathbb{Z}_{12}$ $[0,5,1]_9$ represents the closed trail (a cycle in this case) $(0,5,1,9,2,10,6,11,7,3,8,4)$.

Remark 2.3. Note that $[c_0, c_1, \dots, c_{r-1}]_x$ is a (dr)-cycle if and only if the elements c_i, for $i = 0, \dots, r-1$, belong to pairwise distinct cosets of the subgroup $\langle x \rangle$ in G. Also, if $C = [c_0, c_1, \dots, c_{r-1}]_x$ is a (dr)-cycle, then

$$\partial C = \{\pm(c_i - c_{i-1}) \mid i = 1, \dots, r-1\} \cup \{\pm(c_0 + x - c_{r-1})\}.$$

We point out that in the case of cyclic HCS of $K_{m\times n}$ we have that $dr = mn$. Hence, if the list ∂C has no repeated elements, the order of $Stab(C)$ is d, and the length of the $\mathbb{Z}_{mn}$-orbit of C is r.

Example 2.4. Here we present the construction of a cyclic HCS of $K_{10\times 6}$. Consider the following cycles with vertices in $\mathbb{Z}_{60}$:

$$C_1 = [0,19,1,17,3,15,6,14,8,12]_{10}, \quad C_2 = [0,29,1,28,2,27,3,26,4,25]_{10},$$

$$C_3 = [0,3]_2, \quad C_4 = [0,7]_2, \quad C_5 = [0,13]_2, \quad C_6 = [0]_{17}.$$

One can easily check that $\mathcal{B} = \{C_1, \dots, C_6\}$ is a set of hamiltonian cycles of $K_{10\times 6}$ and that:

$$\begin{aligned} \partial C_1 &= \pm\{19,18,16,14,12,9,8,6,4,2\}, \\ \partial C_2 &= \pm\{29,28,27,26,25,24,23,22,21,15\}, \end{aligned}$$

$$\partial C_3 = \pm\{3,1\}, \quad \partial C_4 = \pm\{7,5\}, \quad \partial C_5 = \pm\{13,11\}, \quad \partial C_6 = \pm\{17\}.$$

Hence $\partial\mathcal{B} = \mathbb{Z}_{60} \setminus 10\mathbb{Z}_{60}$. So, in view of Theorem 2.2, we can conclude that $\mathcal{B}$ is a set of base cycles of a cyclic HCS of $K_{10\times 6}$.
Explicitly the required system consists of the following 27 cycles:

$$\{C_1 + i, C_2 + i \mid i = 0, \dots, 9\} \cup \{C_3 + i, C_4 + i, C_5 + i \mid i = 0, 1\} \cup \{C_6\}.$$

An HCS of the complete graph K_v, v odd, is said to be *symmetric* if there is an involutory permutation ϕ of the vertices of K_v fixing all its cycles; in the case v is even, an HCS of the cocktail party graph $K_v - I$ is *symmetric* if all its cycles are fixed by the involution switching all pairs of endpoints of the edges of I. This definition is due to Akiyama, Kobayashi and Nakamura [1] in the case v odd, and to Brualdi and Schroeder [6] in the case v even. Symmetric hamiltonian cycle systems always exist in the odd case: an example is the well-known Walecki construction, and more generally, any 1-rotational HCS is clearly symmetric (an HCS is called 1-rotational if it has an automorphism group G acting sharply transitively on all but one vertex). It was recently proved that the number of nonisomorphic 1-rotational HCSs of order $v = 2n+1 > 9$ is bounded below by $2^{\lceil 3n/4 \rceil}$ ([16]), so that in the case v odd symmetric HCSs are quite common.

In the case v even we have the following result.

Theorem 2.5 (Brualdi and Schroeder [6]). *A symmetric HCS of $K_v - I$ exists if and only if $\frac{v}{2} \equiv 1$ or $2 \pmod 4$.*

In [14], the authors study the case of an HCS of K_v which is *both* cyclic and symmetric; their result in the case v even is that there exists a cyclic and symmetric HCS of K_v for all values for which a cyclic HCS exists, that is, for $\frac{v}{2} \equiv 1$ or $2 \pmod 4$ and $\frac{v}{2}$ not a prime power.

Very recently Michael Schroeder [23] studied hamiltonian cycle systems for a graph Γ in which each cycle is fixed by a fixed-point-free automorphism ϕ of Γ of order $n > 2$, so that $V(\Gamma) = mn$ for some m; we shall call such an HCS n*-symmetric*.
To admit an n-symmetric HCS, Γ must be a subgraph of $K_{m\times n}$, and in [23] the existence problem of an n-symmetric HCS for $K_{m\times n}$ is completely solved in the following result.

Theorem 2.6 (Schroeder [23]). *Let $m \geq 2$ and $n \geq 1$ be integers such that $(m-1)n$ is even. An n-symmetric HCS for $K_{m\times n}$ always exists except when we have, simultaneously, $n \equiv 2 \pmod 4$ and $m \equiv 0$ or $3 \pmod 4$.*

Note that we shall see the same non-existence condition later on in Corollary 3.4. It makes sense therefore to study, as done in [14] for the cocktail party graph, hamiltonian cycle systems for the complete multipartite graph that are *both* cyclic and symmetric. As noted above, $K_{m\times n}$ is the Cayley graph $Cay[\mathbb{Z}_{mn} : \mathbb{Z}_{mn} \backslash m\mathbb{Z}_{mn}]$. Let γ be the morphism $x \mapsto x + 1 \pmod{mn}$ and set $\phi = \gamma^m$. We have the following condition for a cycle in a cyclic cycle system to be ϕ-invariant.

Lemma 2.7. *A cycle C in a cyclic HCS of $K_{m\times n}$ is ϕ-invariant if and only if n divides $|Stab(C)|$ - or equivalently, if $|Orb(C)|$ divides $\frac{mn}{n} = m$.*

Example 2.8. Let us consider once more the cycles we used in Example 2.4; we can easily see that the cycle system is also 6-symmetric, since the length of the orbit is 10 for cycles C_1 and C_2, 2 for cycles C_3, C_4, C_5 and 1 for C_6.

3 Non-existence results

In this section we shall present some non-existence results for cycle systems of the complete multipartite graph $K_{m\times n}$; the methods used here will be closely related to those used in [15], where the case of the cocktail party graph is considered. The results will concern general cycle systems; we will then apply these results to the hamiltonian case.

The following lemma is an immediate generalization of Lemma 2.1 of [15], hence we omit the proof.

Lemma 3.1. *Let $C = (c_0, c_1, \ldots, c_{\ell-1})$ be a cycle belonging to a cyclic cycle system of $K_{m\times n}$ and let d be the order of $Stab(C)$. Then $Orb(C)$ is an ℓ-cycle system of $Cay[\mathbb{Z}_{mn} : \{\pm(c_{i-1} - c_i) \mid 1 \leq i \leq \frac{\ell}{d}\}]$.*

The next result generalizes Theorem 2.2 of [15].

Proposition 3.2. *Let n be an even integer. The number of cycle orbits of odd length in a cyclic cycle decomposition of $K_{m\times n}$ has the same parity of $\frac{m(m-1)n^2}{8}$.*

Proof. Let $\mathcal{B}$ be a cyclic cycle system of $K_{m\times n}$. For every ℓ-cycle $C = (c_0, c_1, \ldots, c_{\ell-1})$ of $\mathcal{B}$ set

$$\sigma(C) = \sum_{i=1}^{\ell/d}(c_{i-1} - c_i) = (c_0 - c_1) + (c_1 - c_2) + \ldots + (c_{\ell/d-1} - c_{\ell/d}) = c_0 - c_{\ell/d},$$

where d is the order of $Stab(C)$. It is easy to see that $c_{\ell/d} = c_0 + \rho$ where ρ is an element of $\mathbb{Z}_{mn}$ of order d and hence we have

$$\sigma(C) = \frac{mnx}{d} \quad \text{with } \gcd(x, d) = 1.$$

Since n is even, we have that $\sigma(C)$ is even if and only if d is a divisor of $\frac{mn}{2}$; on the other hand, since the length of $Orb(C)$ is $\frac{mn}{d}$, also $|Orb(C)|$ is even if and only if d is a divisor of $\frac{mn}{2}$. For any cycle $C \in \mathcal{B}$, we thus have that

$$\sigma(C) \equiv |Orb(C)| \pmod{2}. \tag{3.1}$$

Let $\mathcal{S} = \{C_1, \ldots, C_s\}$ be a set of base cycles of $\mathcal{B}$, that is, a complete system of representatives for the orbits of the cycles of $\mathcal{B}$, so that we have

$$\mathcal{B} = Orb(C_1) \cup Orb(C_2) \cup \ldots \cup Orb(C_s).$$

By Lemma 3.1, the cycles of $Orb(C_i)$ form a cycle system of $Cay[\mathbb{Z}_{mn} : \partial C_i]$. Hence it follows that

$$Cay[\mathbb{Z}_{mn} : \mathbb{Z}_{mn} \setminus m\mathbb{Z}_{mn}] = \bigcup_{i=1}^{s} Cay[\mathbb{Z}_{mn} : \partial C_i] = Cay\,[\mathbb{Z}_{mn} : \partial\mathcal{S}]$$

so that we obtain

$$\partial\mathcal{S} = \mathbb{Z}_{mn} \setminus m\mathbb{Z}_{mn}. \tag{3.2}$$

Note that ∂C_i is a disjoint union of the set of summands of $\sigma(C_i)$ and the set of their additive inverses. Hence, by (3.2), it follows that $\mathbb{Z}_{mn} \setminus m\mathbb{Z}_{mn}$ is a disjoint union of the set of all summands of the sum $\sum_{i=1}^{s}\sigma(C_i)$ and the set of their additive inverses. Then, considering that additive inverses elements have the same parity and that $\mathbb{Z}_{mn} \setminus m\mathbb{Z}_{mn} = \pm(\{1, 2, \ldots, \frac{mn}{2} - 1\} \setminus \{m, 2m, \ldots, (\frac{n}{2} - 1)m\})$ we can write:

$$\sum_{i=1}^{s}\sigma(C_i) \equiv \sum_{i=1}^{\frac{mn}{2}-1} i - m\sum_{i=1}^{\frac{n}{2}-1} i \pmod{2}$$

and then

$$\sum_{i=1}^{s}\sigma(C_i) \equiv \frac{m(m-1)n^2}{8} \pmod{2}.$$

From (3.1) we have

$$\sum_{i=1}^{s}|Orb(C_i)| \equiv \frac{m(m-1)n^2}{8} \pmod{2}.$$

Hence the number of cycles C_i of $\mathcal{S}$ whose orbit has odd length has the same parity as $\frac{m(m-1)n^2}{8}$, and the assertion follows. □

Now we are ready to prove the main non-existence result. In the following given a positive integer x by $|x|_2$ we will denote the largest e for which 2^e divides x.

Theorem 3.3. *Let n be an even integer. A cyclic ℓ-cycle system of $K_{m\times n}$ cannot exist in each of the following cases:*

(a) $m \equiv 0 \pmod 4$ *and* $|\ell|_2 = |m|_2 + 2|n|_2 - 1$*;*

(b) $m \equiv 1 \pmod 4$ *and* $|\ell|_2 = |m-1|_2 + 2|n|_2 - 1$*;*

(c) $m \equiv 2, 3 \pmod 4$*,* $n \equiv 2 \pmod 4$ *and* $\ell \not\equiv 0 \pmod 4$*; or*

(d) $m \equiv 2, 3 \pmod 4$*,* $n \equiv 0 \pmod 4$ *and* $|\ell|_2 = 2|n|_2$*.*

Proof. If $\mathcal{B}$ is an ℓ-cycle system of $K_{m\times n}$, then $|\mathcal{B}| = |E(K_{m\times n})|/\ell = mn^2(m-1)/2\ell$. Hence the number of cycle orbits of odd length of a cyclic ℓ-cycle system of $K_{m\times n}$ has the same parity as $mn^2(m-1)/2\ell$. By Proposition 3.2, we have that $mn^2(m-1)/2\ell \equiv mn^2(m-1)/8 \pmod 2$. Now the conclusion can be easily proved distinguishing four cases according to the congruence class of m modulo 4. □

If the cycles of the system are hamiltonian, that is if $\ell = mn$, we obtain the following corollary.

Corollary 3.4. *Let n be an even integer. A cyclic HCS of $K_{m\times n}$ cannot exist if both* $m \equiv 0, 3 \pmod 4$ *and* $n \equiv 2 \pmod 4$*.*

4 Existence of a cyclic and symmetric HCS of $K_{m\times n}$, m even

In this section we present direct constructions of a cyclic and symmetric HCS of the complete multipartite graph with an even number of parts. Since $(m-1)n$ must be even, if m is even then n is even too; the condition in Corollary 3.4 tells us that when $n \equiv 2 \pmod 4$, m should also be congruent to 2 modulo 4. If these two requirements are met, we will show that a cyclic and symmetric HCS of $K_{m\times n}$ always exists, and therefore we prove Theorem 1.1.

As observed in the Introduction, $K_{m\times 2} = K_{2m} - I$ is the cocktail party graph; thus we can suppose $n > 2$, since for $n = 2$ we can rely on the following result.

Theorem 4.1 (Jordon, Morris [17]; Buratti, Merola [14])**.** *For an even integer $v \geq 4$ there exists a cyclic and symmetric HCS of $K_v - I$ if and only if* $v \equiv 2, 4 \pmod 8$ *and $v \neq 2p^\alpha$, where p is an odd prime and $\alpha \geq 1$.*

We start by considering the complete bipartite graph.

Proposition 4.2. *For any even integer n there exists a cyclic and n-symmetric HCS of $K_{2\times n}$.*

Proof. For $n = 2\ell$ we need a set $\mathcal{B}$ of base cycles such that $\partial\mathcal{B} = \pm\{1, 3, \ldots, 2\ell-1\}$. Let us first assume ℓ even. For $i = 0, 1, \ldots, \ell/2 - 1$ consider the cycle $C_i = [0, 4i+3]_2$. We have $\partial C_i = \pm\{4i+1, 4i+3\}$, and thus $\mathcal{B} = \{C_0, C_1, \ldots, C_{\ell/2-1}\}$ is a set of hamiltonian cycles of $K_{2\times n}$ such that $\partial\mathcal{B} = \mathbb{Z}_{2n} \setminus 2\mathbb{Z}_{2n}$. Now assume that ℓ is odd. For $i = 0, 1, \ldots, \lfloor \ell/2 \rfloor - 1$ take $C_i = [0, 4i+3]_2$ as above, and add the cycle $C' = [0]_{2\ell-1}$. Now $\mathcal{B} = \{C_0, C_1, \ldots, C_{\lfloor \ell/2 \rfloor - 1}, C'\}$ is a set of base cycles for a cyclic HCS of $K_{2\times n}$. This cycle system is also n-symmetric by Lemma 2.7 since each cycle belongs to an orbit of length 1 or 2. □

Now we tackle the case $n \equiv 0 \pmod 4$.

Theorem 4.3. *Let m be an even integer and $n \equiv 0 \pmod 4$. Then there exists a cyclic and n-symmetric HCS of $K_{m\times n}$.*

Proof. We may assume $m \geq 4$, since if $m = 2$, the statement follows from Proposition 4.2. We shall first give a construction for m a power of 2. Let $m = 2^a$ and $n = 4t$ with $a > 1$ and $t \geq 1$. We will build a set of $a \cdot t$ base cycles. For all $b = 1, \ldots, a$ and $i = 0, \ldots, t-1$ consider the following path:

$$\begin{aligned} P_{i,b} &= [0, 2mi + (2^{b+1} - 1), 1, 2mi + (2^{b+1} - 2), 2, 2mi + (2^{b+1} - 3), \ldots, \\ &\quad (2^{b-1} - 1), 2mi + (2^{b+1} - 2^{b-1})]. \end{aligned}$$

Note that the elements of $P_{i,b}$ are pairwise distinct modulo 2^b: hence $A_{i,b} = [P_{i,b}]_{2^b}$ is a hamiltonian cycle of $K_{m\times n}$. It is straightforward to check that

$$\partial A_{i,b} = \pm(\{2mi + 2^{b-1}\} \cup \{2mi + (2^b + 1), 2mi + (2^b + 2), \ldots, 2mi + (2^{b+1} - 1)\}).$$

Thus $\cup(\partial A_{i,b}) = \mathbb{Z}_{mn} \setminus m\mathbb{Z}_{mn}$, and the existence of a cyclic HCS of $K_{m\times n}$ follows from Theorem 2.2.

Now assume $m = 2^a\overline{m}$ with $a \geq 1$ and $\overline{m} > 1$ odd. Take $n = 4t$ with $t \geq 1$. We start constructing for all $i = 0, \ldots, t-1$ the following paths:

$$P_{i,j} = \begin{cases} [0, 2mi + (4j - 1)] & \text{if } j = 1, \ldots, \frac{\overline{m}-1}{2} \\ [0, 2mi + (4j + 1)] & \text{if } j = \frac{\overline{m}+1}{2}, \ldots, \overline{m} - 1 \end{cases}. \tag{4.1}$$

Since $2mi + (4j - 1)$ and $2mi + (4j + 1)$ are odd, $A_{i,j} = [P_{i,j}]_2$ is a hamiltonian cycle of $K_{m\times n}$ for any i, j. Clearly $\partial A_{i,j} = \pm\{2mi+(4j-3), 2mi+(4j-1)\}$ for $j = 1, \ldots, \frac{\overline{m}-1}{2}$ and $\partial A_{i,j} = \pm\{2mi + (4j-1), 2mi + (4j+1)\}$ for $j = \frac{\overline{m}+1}{2}, \ldots, \overline{m} - 1$. Hence for any fixed i we have

$$\begin{aligned} \bigcup_{j=1}^{\overline{m}-1} \partial A_{i,j} &= \pm\big(\{2mi + 1, 2mi + 3, 2mi + 5, \ldots, 2mi + (2\overline{m} - 3)\} \cup \\ &\{2mi + (2\overline{m} + 1), 2mi + (2\overline{m} + 3), 2mi + (2\overline{m} + 5), \ldots, 2mi + (4\overline{m} - 3)\}\big). \end{aligned}$$

Now for $i = 0, \ldots, t-1$ consider the paths

$$\begin{aligned} Q_{i,1} &= [0, 2mi + (4\overline{m} - 1), 1, 2mi + (4\overline{m} - 3), 3, \ldots, \overline{m} - 2, 2mi + 3\overline{m}, \\ &\quad \overline{m} + 1, 2mi + (3\overline{m} - 1), \overline{m} + 3, 2mi + (3\overline{m} - 3), \ldots, 2\overline{m} - 2, \\ &\quad 2mi + (2\overline{m} + 2)]; \end{aligned} \tag{4.2}$$

and finally, if $a \geq 2$, for all $b = 2, \ldots, a$ consider also

$$\begin{aligned} Q_{i,b} &= [0, 2mi + (2^{b+1}\overline{m} - 1), 1, 2mi + (2^{b+1}\overline{m} - 2), 2, \ldots, (2^{b-1}\overline{m} - 1), \\ &\quad 2mi + (2^{b+1}\overline{m} - 2^{b-1}\overline{m})]. \end{aligned}$$

Notice that the elements of $Q_{i,b}$ are pairwise distinct modulo $2^b\overline{m}$ and hence $B_{i,b} = [Q_{i,b}]_{2^b\overline{m}}$ is a hamiltonian cycle of $K_{m\times n}$ for any $i = 0, \ldots, t-1$ and $b = 1, \ldots, a$. Also,

$$\begin{aligned} \partial B_{i,1} &= \pm(\{2mi + 2, 2mi + 4, 2mi + 6, \ldots, 2mi + 2\overline{m} - 2\} \cup \\ &\quad \{2mi + 2\overline{m} + 2, 2mi + 2\overline{m} + 4, 2mi + 2\overline{m} + 6, \ldots, 2mi + 4\overline{m} - 2\} \cup \\ &\quad \{2mi + 2\overline{m} - 1, 2mi + 4\overline{m} - 1\}) \end{aligned}$$

and for $b = 2, \dots, a$

$$\partial B_{i,b} = \pm(\{2mi + 2^{b-1}\overline{m}\} \cup \{2mi + (2^b\overline{m} + 1), 2mi + (2^b\overline{m} + 2), \dots, 2mi + (2^{b+1}\overline{m} - 1)\}).$$

It turns out that for every fixed i we have

$$\bigcup_{b=1}^{a} \partial B_{i,b} = \pm\big(\{2mi + 2, 2mi + 4, 2mi + 6, \dots, 2mi + 4\overline{m}\} \cup$$

$$\{2mi + (4\overline{m} + 1), 2mi + (4\overline{m} + 2), 2mi + (4\overline{m} + 3), \dots, 2mi + (m - 1)\} \cup$$

$$\{2mi + (m + 1), 2mi + (m + 2), 2mi + (m + 3), \dots, 2mi + (2m - 1)\}\big).$$

Let $\mathcal{B} = \{A_{i,j} \mid 0 \le i < t, 1 \le j < \overline{m}\} \cup \{B_{i,b} \mid 0 \le i < t, 1 \le b \le a\}$. From what we have seen above, for every fixed i we have

$$\left(\bigcup_{j=1}^{\overline{m}-1} \partial A_{i,j}\right) \cup \left(\bigcup_{b=1}^{a} \partial B_{i,b}\right) = \pm\big(\{2mi + 1, 2mi + 2, 2mi + 3, \dots, 2mi + (m - 1)\} \cup$$

$$\{2mi + (m + 1), 2mi + (m + 2), 2mi + (m + 3), \dots, 2mi + (2m - 1)\}\big)$$

and so $\partial\mathcal{B} = \mathbb{Z}_{mn} \setminus m\mathbb{Z}_{mn}$. We conclude that $\mathcal{B}$ is a set of base cycles of a cyclic HCS of $K_{m\times n}$.

It is easily seen from Lemma 2.7 that these cycle systems are also n-symmetric, since in all cases the length of the orbit of each cycle divides m. □

Example 4.4. Following the proof of Theorem 4.3 we give here the construction of a set of base cycles of a cyclic and 4-symmetric HCS of $K_{18\times 4}$. In the notation of the Theorem, $a = 1$, $\overline{m} = 9$ and $t = 1$. Take the following cycles:

$$A_{0,1} = [0, 3]_2, \quad A_{0,2} = [0, 7]_2, \quad A_{0,3} = [0, 11]_2, \quad A_{0,4} = [0, 15]_2,$$

$$A_{0,5} = [0, 21]_2, \quad A_{0,6} = [0, 25]_2, \quad A_{0,7} = [0, 29]_2, \quad A_{0,8} = [0, 33]_2,$$

$$B_{0,1} = [0, 35, 1, 33, 3, 31, 5, 29, 7, 27, 10, 26, 12, 24, 14, 22, 16, 20]_{18}.$$

We have

$$\bigcup_{j=1}^{8} \partial A_{0,j} = \pm(\{1, 3, 5, \dots, 15\} \cup \{19, 21, 23, \dots, 33\})$$

and

$$\partial B_{0,1} = \pm(\{2, 4, 6, \dots, 16\} \cup \{20, 22, 24, \dots, 34\} \cup \{17, 35\}).$$

So, given $\mathcal{B} = \{A_{0,1}, A_{0,2}, \dots, A_{0,8}, B_{0,1}\}$, we have $\partial\mathcal{B} = \mathbb{Z}_{72} \setminus 18\mathbb{Z}_{72}$.

Now, we give the construction of a set of base cycles of a cyclic and 8-symmetric HCS of $K_{72\times 8}$. Notice that $\overline{m} = 9$ as before, but $a = 3$ and $t = 1$, so we need to construct a larger number of cycles. For $i = 0$ we take

$$A_{0,1} = [0, 3]_2, \quad A_{0,2} = [0, 7]_2, \quad A_{0,3} = [0, 11]_2, \quad A_{0,4} = [0, 15]_2,$$

$$A_{0,5} = [0, 21]_2, \quad A_{0,6} = [0, 25]_2, \quad A_{0,7} = [0, 29]_2, \quad A_{0,8} = [0, 33]_2,$$

$$\begin{aligned} B_{0,1} &= [0,35,1,33,3,31,5,29,7,27,10,26,12,24,14,22,16,20]_{18}, \\ B_{0,2} &= [0,71,1,70,2,69,3,68,\ldots,17,54]_{36}, \\ B_{0,3} &= [0,143,1,142,2,141,3,140,\ldots,35,108]_{72}. \end{aligned}$$

We have

$$\left(\bigcup_{j=1}^{8} \partial A_{0,j}\right) \cup \left(\bigcup_{b=1}^{3} \partial B_{0,b}\right) = \pm\left(\{1,2,3,\ldots,71\} \cup \{73,74,75,\ldots,143\}\right).$$

Furthermore, for $i = 1$:

$$A_{1,1} = [0,147]_2, \quad A_{1,2} = [0,151]_2, \quad A_{1,3} = [0,155]_2, \quad A_{1,4} = [0,159]_2,$$

$$A_{1,5} = [0,165]_2, \quad A_{1,6} = [0,169]_2, \quad A_{1,7} = [0,173]_2, \quad A_{1,8} = [0,177]_2,$$

$$\begin{aligned} B_{1,1} &= [0,179,1,177,3,175,5,173,7,171,10,170,12,168,14,166,16,164]_{18}, \\ B_{1,2} &= [0,215,1,214,2,213,3,212,\ldots,17,198]_{36}, \\ B_{1,3} &= [0,287,1,286,2,285,3,284,\ldots,35,252]_{72}. \end{aligned}$$

We have

$$\left(\bigcup_{j=1}^{8} \partial A_{1,j}\right) \cup \left(\bigcup_{b=1}^{3} \partial B_{1,b}\right) = \pm(\{145,146,147,\ldots,215\} \cup \{217,218,219,\ldots,287\}).$$

So, given $\mathcal{B} = \{A_{i,j} \mid i = 0,1,\ j = 1,\ldots,8\} \cup \{B_{i,b} \mid i = 0,1,\ b = 1,2,3\}$, we have $\partial\mathcal{B} = \mathbb{Z}_{576} \setminus 72\mathbb{Z}_{576}$.

The following definition and lemma are instrumental in proving Theorem 4.7, where we shall settle the case $n \equiv 2 \pmod 4$.

Definition 4.5. For all positive integers s, d and all odd integers $w \geq 3$, set

$$S(s,d,w) = \left\{ s + id \mid 0 \leq i \leq \frac{w-3}{2} \right\}$$

and

$$\varphi(s,d,w) = |\{x \in S(s,d,w) : \gcd(x,w) = 1\}|.$$

Lemma 4.6. *Assume* $\gcd(s,d,w) = 1$*. If* $3 \nmid s$ *when* $w = 3$*, then* $\varphi(s,d,w) > 0$*.*

Proof. If $w = 3$ then $\varphi(s,d,3) = 1$, since $S(s,d,3) = \{s\}$ and $3 \nmid s$. Suppose now $w \geq 5$. The assertion is trivial for $\gcd(s,w) = 1$, since $s \in S(s,d,w)$. If $\gcd(s,w) \neq 1$, consider the set $T = \{p \text{ prime} : p \mid w, p \nmid s\}$ and let $x = \prod_{p \in T} p$ (with the usual convention that $x = 1$ if $T = \emptyset$). Since $w \geq 5$ and $x < w$, we have that $s + dx \in S(s,d,w)$: we claim that that $\gcd(s + dx, w) = 1$. Note that no prime factor of $\gcd(s,w)$ divides d, otherwise we would have $\gcd(s,d,w) \neq 1$. Let p be any prime divisor of w. By definition of x, p divides either s or x, but not both. So, we have that p divides one summand of $s + dx$ but not both: thus $s + dx$ is coprime with w. □

Theorem 4.7. *Let m, n be integers with $m, n \equiv 2 \pmod 4$. Then there exists a cyclic and n-symmetric HCS of $K_{m\times n}$.*

Proof. In view of Propositions 4.2 and Theorem 4.1 we may assume $m = 2\overline{m} > 2$ and $n = 4t + 2 > 2$. Using the notation of Definition 4.5 take

$$s = \begin{cases} 3\overline{m} + 2 & \text{if } m \equiv 2 \pmod 8 \\ 3\overline{m} - 2 & \text{if } m \equiv 6 \pmod 8 \end{cases},$$

$d = 4\overline{m}$ and $w = \frac{n}{2}$. Now Lemma 4.6 guarantees that the set $S(3\overline{m}\pm 2, 4\overline{m}, \frac{n}{2})$ contains an element $\nu = s + 2m\kappa$ coprime with $\frac{n}{2}$, where $0 \le \kappa \le \frac{n-6}{4}$. It is useful for the following to observe that $\gcd(\nu, mn) = 1$, as $\gcd(3\overline{m} \pm 2, \overline{m}) = 1$.

For all $i = 0, \dots, \kappa$ consider the paths $Q_{i,1}$ as in (4.2) and, if $\kappa \ge 1$, for all $i = 0, \dots, \kappa - 1$ consider the paths $P_{i,j}$ as in (4.1). As we have seen in Theorem 4.3, $A_{i,j} = [P_{i,j}]_2$ and $B_i = [Q_{i,1}]_m$ are hamiltonian cycles of $K_{m\times n}$ for any i, j.

If $t \ge \kappa + 2$, for all $i = \kappa + 1, \dots, t - 1$, take also the following paths:

$$\begin{aligned} \widetilde{P}_{i,j} &= \begin{cases} [0, (2i+1)m + (4j-1)] & \text{if } j = 1, \dots, \frac{\overline{m}-1}{2} \\ [0, (2i+1)m + (4j+1)] & \text{if } j = \frac{\overline{m}+1}{2}, \dots, \overline{m} - 1 \end{cases}; \\ \widetilde{Q}_i &= [0, (2i+1)m + (4\overline{m}-1), 1, (2i+1)m + (4\overline{m}-3), 3, \dots, \overline{m}-2, \\ &\quad (2i+1)m + 3\overline{m}, \overline{m}+1, (2i+1)m + (3\overline{m}-1), \overline{m}+3, \\ &\quad (2i+1)m + (3\overline{m}-3), \dots, 2\overline{m}-2, (2i+1)m + (2\overline{m}+2)]. \end{aligned}$$

We define

$$u = \begin{cases} \frac{3\overline{m}+1}{4} & \text{if } m \equiv 2 \pmod 8 \\ \frac{3\overline{m}-1}{4} & \text{if } m \equiv 6 \pmod 8 \end{cases}$$

and take the paths:

$$\begin{aligned} \widetilde{R}_j &= \begin{cases} [0, 2m\kappa + (4j-1)] & \text{if } j = 1, \dots, \frac{\overline{m}-1}{2} \\ [0, 2m\kappa + (4j+1)] & \text{if } j = \frac{\overline{m}+1}{2}, \dots, \overline{m} - 1 \text{ and } j \ne u \end{cases}; \\ \widetilde{S} &= [0, (2\kappa+1)m + (4\overline{m}-1), 1, (2\kappa+1)m + (4\overline{m}-2), 2, \dots, \\ &\quad \overline{m}-1, (2\kappa+1)m + 3\overline{m}]. \end{aligned}$$

Now set $C_{i,j} = [\widetilde{P}_{i,j}]_2$, $D_i = [\widetilde{Q}_i]_m$, $E_j = [\widetilde{R}_j]_2$, $F = [\widetilde{S}]_m$ and $G = [0]_\nu$: these are all hamiltonian cycles of $K_{m\times n}$, and we have that for $j = 1, \dots, \frac{\overline{m}-1}{2}$

$$\partial C_{i,j} = \pm\{(2i+1)m + (4j-3), (2i+1)m + (4j-1)\}$$

and for $j = \frac{\overline{m}+1}{2}, \dots, \overline{m} - 1$

$$\partial C_{i,j} = \pm\{(2i+1)m + (4j-1), (2i+1)m + (4j+1)\}.$$

Also,

$$\begin{aligned} \partial D_i &= \pm(\{(2i+1)m + 2, (2i+1)m + 4, (2i+1)m + 6, \dots, (2i+1)m + \\ &\quad +2\overline{m} - 2\} \cup \{(2i+1)m + 2\overline{m} + 2, (2i+1)m + 2\overline{m} + 4, \dots, (2i+1)m + \\ &\quad +4\overline{m} - 2\} \cup \{(2i+1)m + 2\overline{m} - 1, (2i+1)m + 4\overline{m} - 1\}); \end{aligned}$$

moreover, for $j = 1, \ldots, \frac{\overline{m}-1}{2}$

$$\partial E_j = \pm\{2m\kappa + (4j-3), 2m\kappa + (4j-1)\}$$

and for $j = \frac{\overline{m}+1}{2}, \ldots, \overline{m}-1$ with $j \neq u$

$$\partial E_j = \pm\{2m\kappa + (4j-1), 2m\kappa + (4j+1)\}.$$

Finally,

$$\partial F = \pm(\{(2\kappa+2)m+1, (2\kappa+2)m+2, \ldots, (2\kappa+3)m-1\} \cup \{2m\kappa + 3\overline{m}\})$$

and $\partial G = \pm\{\nu\}$.

Let $\mathcal{B} = \{A_{i,j} \mid 0 \leq i < \kappa, 1 \leq j < \overline{m}\} \cup \{B_i \mid 0 \leq i \leq \kappa\} \cup \{C_{i,j} \mid \kappa < i < t, 1 \leq j < \overline{m}\} \cup \{D_i \mid \kappa < i < t\} \cup \{E_j \mid 1 \leq j < \overline{m}, j \neq u\} \cup \{F, G\}$. It is routine to check that $\partial\mathcal{B} = \mathbb{Z}_{mn} \setminus m\mathbb{Z}_{mn}$, hence we conclude that $\mathcal{B}$ is a set of base cycles of a cyclic HCS of $K_{m\times n}$. Once more, it is easily checked using Lemma 2.7 that this cycle system is also n-symmetric, since in all cases the length of the orbit of each cycle divides m. □

We point out that the base cycles used in Example 2.4 were constructed following the proof of Theorem 4.7. In particular, according to the notation of the theorem we have

$$C_1 = B_0, \quad C_2 = F, \quad C_3 = E_1, \quad C_4 = E_2, \quad C_5 = E_3, \quad C_6 = G.$$

Example 4.8. Here we present a set of base cycles of a cyclic and 14-symmetric HCS of $K_{6\times 14}$. In the notation of Theorem 4.7, $\overline{m} = t = 3$ and we choose $\kappa = 1$ and $\nu = 19$ which is coprime with $6 \cdot 14$. Following the proof of the theorem we have to take the following cycles:

$$A_{0,1} = [0,3]_2, \quad A_{0,2} = [0,9]_2, \quad B_0 = [0,11,1,9,4,8]_6, \quad B_1 = [0,23,1,21,4,20]_6,$$

$$C_{2,1} = [0,33]_2, \quad C_{2,2} = [0,39]_2, \quad D_2 = [0,41,1,39,4,38]_6, \quad E_1 = [0,15]_2,$$
$$F = [0,29,1,28,2,27]_6, \quad G = [0]_{19}.$$

It follows that

$$\partial\{A_{0,1}, A_{0,2}\} = \pm\{1,3,7,9\}, \; \partial\{B_0, B_1\} = \pm\{2,4,5,8,10,11,14,16,17,20,22,23\},$$

$$\partial\{C_{2,1}, C_{2,2}\} = \pm\{31,33,37,39\}, \quad \partial D_2 = \pm\{32,34,35,38,40,41\},$$
$$\partial E_1 = \pm\{13,15\}, \quad \partial F = \pm\{21,25,26,27,28,29\}, \quad \partial G = \pm\{19\}.$$

So, letting $\mathcal{B}$ be the set of the constructed cycles, we have $\partial\mathcal{B} = \mathbb{Z}_{84} \setminus 6\mathbb{Z}_{84}$.

Now, we give a set of base cycles of a cyclic and 10-symmetric HCS of $K_{10\times 10}$. In the notation of Theorem 4.7, $\overline{m} = 5$, $t = 2$ and we choose $\kappa = 1$ and $\nu = 37$ which is coprime with 100. We have to take the following cycles:

$$A_{0,1} = [0,3]_2, \quad A_{0,2} = [0,7]_2, \quad A_{0,3} = [0,13]_2, \quad A_{0,4} = [0,17]_2,$$

$$B_0 = [0,19,1,17,3,15,6,14,8,12]_{10}, \quad B_1 = [0,39,1,37,3,35,6,34,8,32]_{10},$$

$$E_1 = [0,23]_2, \quad E_2 = [0,27]_2, \quad E_3 = [0,33]_2,$$

$$F = [0, 49, 1, 48, 2, 47, 3, 46, 4, 45]_{10}, \quad G = [0]_{37}.$$

We have:

$$\bigcup_{i=1}^{4} \partial A_{0,i} = \pm\{1, 3, 5, 7, 11, 13, 15, 17\},$$

$$\partial\{B_0, B_1\} = \pm\{2, 4, 6, 8, 9, 12, 14, 16, 18, 19, 22, 24, 26, 28, 29, 32, 34, 36, 38, 39\},$$

$$\bigcup_{j=1}^{3} \partial E_j = \pm\{21, 23, 25, 27, 31, 33\},$$

$$\partial F = \pm\{35, 41, 42, 43, 44, 45, 46, 47, 48, 49\}, \quad \partial G = \pm\{37\}.$$

Hence, letting $\mathcal{B}$ be the set of the constructed cycles, we have $\partial\mathcal{B} = \mathbb{Z}_{100} \setminus 10\mathbb{Z}_{100}$.

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References

[1] J. Akiyama, M. Kobayashi and G. Nakamura, Symmetric Hamilton cycle decompositions of the complete graph, *J. Combin. Des.* **12** (2004), 39–45, doi:10.1002/jcd.10066, http://dx.doi.org/10.1002/jcd.10066.

[2] A. Benini and A. Pasotti, On the existence of elementary abelian cycle systems, *Graphs Combin.* **25** (2009), 1–14, doi:10.1007/s00373-008-0826-4, http://dx.doi.org/10.1007/s00373-008-0826-4.

[3] A. Benini and A. Pasotti, Decompositions of complete multipartite graphs via generalized graceful labelings, *Australas. J. Combin.* **59** (2014), 120–143.

[4] E. J. Billington, N. J. Cavenagh and B. R. Smith, Path and cycle decompositions of complete equipartite graphs: four parts, *Discrete Math.* **309** (2009), 3061–3073, doi:10.1016/j.disc.2008.08.009, http://dx.doi.org/10.1016/j.disc.2008.08.009.

[5] E. J. Billington, N. J. Cavenagh and B. R. Smith, Path and cycle decompositions of complete equipartite graphs: 3 and 5 parts, *Discrete Math.* **310** (2010), 241–254, doi:10.1016/j.disc.2008.09.003, http://dx.doi.org/10.1016/j.disc.2008.09.003.

[6] R. A. Brualdi and M. W. Schroeder, Symmetric Hamilton cycle decompositions of complete graphs minus a 1-factor, *J. Combin. Des.* **19** (2011), 1–15, doi:10.1002/jcd.20257, http://dx.doi.org/10.1002/jcd.20257.

[7] D. Bryant, Cycle decompositions of complete graphs, in: *Surveys in combinatorics 2007*, Cambridge Univ. Press, Cambridge, volume 346 of *London Math. Soc. Lecture Note Ser.*, pp. 67–97, 2007, doi:10.1017/CBO9780511666209.004, http://dx.doi.org/10.1017/CBO9780511666209.004.

[8] D. Bryant and C. A. Rodger, Cycle decompositions, in: C. J. Colbourn and J. H. Dinitz (eds.), *Handbook of combinatorial designs*, Chapman & Hall/CRC, Boca Raton, FL, pp. 373–382, 2nd edition, 2007.

[9] M. Buratti, Rotational k-cycle systems of order $v < 3k$; another proof of the existence of odd cycle systems, *J. Combin. Des.* **11** (2003), 433–441, doi:10.1002/jcd.10061, http://dx.doi.org/10.1002/jcd.10061.

[10] M. Buratti and A. Del Fra, Existence of cyclic k-cycle systems of the complete graph, *Discrete Math.* **261** (2003), 113–125, doi:10.1016/S0012-365X(02)00463-6, papers on the occasion of the 65th birthday of Alex Rosa, http://dx.doi.org/10.1016/S0012-365X(02)00463-6.

[11] M. Buratti and A. Del Fra, Cyclic Hamiltonian cycle systems of the complete graph, *Discrete Math.* **279** (2004), 107–119, doi:10.1016/S0012-365X(03)00267-X, in honour of Zhu Lie, http://dx.doi.org/10.1016/S0012-365X(03)00267-X.

[12] M. Buratti and L. Gionfriddo, Strong difference families over arbitrary graphs, *J. Combin. Des.* **16** (2008), 443–461, doi:10.1002/jcd.20201, http://dx.doi.org/10.1002/jcd.20201.

[13] M. Buratti and F. Merola, Dihedral Hamiltonian cycle systems of the cocktail party graph, *J. Combin. Des.* **21** (2013), 1–23, doi:10.1002/jcd.21311, http://dx.doi.org/10.1002/jcd.21311.

[14] M. Buratti and F. Merola, Hamiltonian cycle systems which are both cyclic and symmetric, *J. Combin. Des.* **22** (2014), 367–390, doi:10.1002/jcd.21351, http://dx.doi.org/10.1002/jcd.21351.

[15] M. Buratti and G. Rinaldi, A non-existence result on cyclic cycle-decompositions of the cocktail party graph, *Discrete Math.* **309** (2009), 4722–4726, doi:10.1016/j.disc.2008.05.042, http://dx.doi.org/10.1016/j.disc.2008.05.042.

[16] M. Buratti, G. Rinaldi and T. Traetta, Some results on 1-rotational Hamiltonian cycle systems, *J. Combin. Des.* **22** (2014), 231–251, doi:10.1002/jcd.21352, http://dx.doi.org/10.1002/jcd.21352.

[17] H. Jordon and J. Morris, Cyclic Hamiltonian cycle systems of the complete graph minus a 1-factor, *Discrete Math.* **308** (2008), 2440–2449, doi:10.1016/j.disc.2007.05.009, http://dx.doi.org/10.1016/j.disc.2007.05.009.

[18] R. Laskar and B. Auerbach, On decomposition of r-partite graphs into edge-disjoint Hamilton circuits, *Discrete Math.* **14** (1976), 265–268.

[19] R. Peltesohn, Eine Lösung der beiden Heffterschen Differenzenprobleme, *Compositio Math.* **6** (1939), 251–257, http://www.numdam.org/item?id=CM_1939__6__251_0.

[20] A. Rosa, On cyclic decompositions of the complete graph into $(4m + 2)$-gons, *Matematicko-fyzikálny časopis* **16** (1966), 349–352.

[21] A. Rosa, On cyclic decompositions of the complete graph into polygons with odd number of edges (slovak), *Casopis Pest. Mat.* **91** (1966), 53–63.

[22] A. Rosa, On decompositions of a complete graph into $4n$-gons, *Mat. Casopis Sloven. Akad. Vied* **17** (1967), 242–246.

[23] M. W. Schroeder, ϕ-symmetric Hamilton cycle decompositions of graphs, *Discrete Math.* **338** (2015), 1586–1594, doi:10.1016/j.disc.2015.04.008, http://dx.doi.org/10.1016/j.disc.2015.04.008.

[24] B. R. Smith and N. J. Cavenagh, Decomposing complete equipartite graphs into short odd cycles, *Electron. J. Combin.* **17** (2010), Research Paper 130, 21, http://www.combinatorics.org/Volume_17/Abstracts/v17i1r130.html.

[25] B. R. Smith and N. J. Cavenagh, Decomposing complete equipartite graphs into short even cycles, *J. Combin. Des.* **19** (2011), 131–143, doi:10.1002/jcd.20265, http://dx.doi.org/10.1002/jcd.20265.

[26] A. Vietri, Cyclic k-cycle systems of order $2kn + k$: a solution of the last open cases, *J. Combin. Des.* **12** (2004), 299–310, doi:10.1002/jcd.20002, http://dx.doi.org/10.1002/jcd.20002.

[27] S.-L. Wu and M. Buratti, A complete solution to the existence problem for 1-rotational k-cycle systems of k_v, *Journal of Combinatorial Designs* **17** (2009), 283–293, doi:10.1002/jcd.20217, http://dx.doi.org/10.1002/jcd.20217.