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14 Fermionization, Number of Families

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Abstract. We investigate bosonization/fermionization for free massless fermions being equivalent to free massless bosons with the purpose of checking and correcting the old rule by Aratyn and one of us (H.B.F.N.) for the number of boson species relative to the number of fermion species which is required to have bosonization possible. An important application of such a counting of degrees of freedom relation would be to invoke restrictions on the number of families that could be possible under the assumption, that all the fermions in nature are the result of fermionizing a system of boson species. Since a theory of fundamental fermions can be accused for not being properly local because of having anticommutativity at space like distances rather than commutation as is more physically reasonable to require, it is in fact called for to have all fermions arising from fermionization of bosons. To make a realistic scenario with the fermions all coming from fermionizing some bosons we should still have at least some not fermionized bosons and we are driven towards that being a gravitational field, that is not fermionized. Essentially we reach the spin-charge-families theory by one of us (N.S.M.B.) with the detail that the number of fermion components and therefore of families get determined from what possibilities for fermionization will finally turn out to exist. The spin-charge-family theory has long be plagued by predicting 4 families rather than the phenomenologically more favoured 3. Unfortunately we do not yet understand well enough the unphysical negative norm square components in the system of bosons that can fermionize in higher dimensions because we have no working high dimensional case of fermionization. But suspecting they involve gauge fields with complicated unphysical state systems the corrections from such states could putatively improve the family number prediction.

Povzetek. Avtorja diskutirata bozonizacijo/fermionizacijo za proste brezmasne fermione, ki jih obravnavata kot ekvivalentne prostim brezmasnim bozonom. Namen je preveriti in popraviti staro pravilo Aratyna in H.B.F.N. za število vrst bozonov glede na število vrst fermionov kot pogoj za obstoj fermionizacije bozonov. Pomembna uporaba takega pravila bi bil pogoj na število možnih družin, če privzamemo, da so vsi fermioni v naravi rezultat fermionizacije vrst bozonov. Teoriji fermionov kot fundamentalnih delcev lahko očitamo, da nima pravilne lokalnosti, ker zahtevamo za fermione antikomutativnost, ne pa komutativnosti. Temu očitku bi se lahko izognili, če vsi fermioni izhajajo iz fermionizacije bozonov. Za realističen opis v modelu, v katerem fermione dobimo s fermionizacijo bozonov, mora vsaj nekaj bozonov ostati nefermioniziranih. Avtorja predlagata, da so ti nefermionizirani bozoni gravitacijska polja. Želita na ta način reproducirati teorijo Spina-nabojev-družin enega od avtorjev (S.N.M.B.), kjer bi število fermionskih komponent in

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število družin določale možnosti za fermionizacijo. Teorija spina-nabojev-družin napove 4 družine, namesto doslej opaženih 3. Avtorja še ne razumeta dovolj dobro nefizikalnih komponent z negativnim kvadratom norm v sistemu bozonov, ki se fermionizira v višjih dimenzijah, ker jima fermionizacije v višjih dimenzijah še ni uspelo zares izpeljati. Domnevata, da bodo spoznanja o vlogi nefizikalnih stanj pomagala pri napovedi števila družin.

Keywords: Fermionization, Bosonization, Number of families

14.1 Introduction

One of the general requirements for quantum field theories is microcausality [1,2], the requirement of causality, which in its form as suggested from tensor product deduction says, that for two relative to each other spacelikely placed events x_1 and x_2 in Minkowski space-time a couple of quantum field operators $\mathcal{O}_1$ and $\mathcal{O}_2$ taken at these events will **commute**

$$\{\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)\}_- = 0 \text{ for spacelike } x_1 - x_2. \tag{14.1}$$

This is so for the $\mathcal{O}_1$ and $\mathcal{O}_2$ being boson fields, but if they are both fermion fields, one would have instead to let them **anticommute**

$$\{\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)\}_+ = 0 \text{ for spacelike } x_1 - x_2. \tag{14.2}$$

Such **anti**commutation is, however, from the tensor product way of arguing for the relation completely wrong. We could therefore claim that it is not truly allowed to have fermions in the usual way, because it leads to a "crazy" locality axiom. It is one of the purposes of the present proceedings article to suggest to investigate the consequences of such an attitude, that fermions as fundamental particles are not good, but that one should rather seek to obtain fermions, not as fundamental, but rather only by fermionization of some boson fields instead. But then it becomes very important what combinations - what systems - of fermions can be obtained from appropriate bosonic models. For the existence of quite nontrivial restrictions on the number of fermions, we can expect to be obtainable by fermionization from a system bosons, the theorem [3] by Aratyn and one of us (H.B.F.N.) is quite suggestive. In fact this theorem tells, that the ratio between the number of fermion spin components for all the species (families) counted together and the corresponding number of boson spin components counted together must be $\frac{2^{d_{spatial}}}{2^{d_{spatial}-1}}$. A priori this theorem seems to enforce that in say the experimental number of dimensions, $d_{spatial} = 3$ and 1 time, the collective number of fermions components must be divisible by $2^3 = 8$. If we count the components as real fields a Weyl fermion has 2*2 = 4 such real components, and thus the number of Weyl fermions must be divisible by $8/4$ =2.

Let us immediately include the remark, that although we shall below mainly go for the presumably simplest case of non-interacting massless bosons - presumably Kalb-Ramond fields - being fermionized into also free massless fermions, that does NOT mean that we seriously suggest Nature to have no interactions. Rather the hope is that gravitational degrees of freedom couple in a way specified alone by the flow of energy and momentum, so that we can hope that having a free theory it should be very easy and almost unique how to add a gravitational interaction. Let us say, that by the *spin-charge-family* theory by one of us, the interacting fields are the gravitational ones (vielbeins and spin connections) [5–7] only, but in $d > (1 + 3)$, fermions manifesting in $(1 + 3)$ as spins and all the observed charges, as well as families, gravity manifesting all the observed gauge fields as well as the scalar fields, explaining higgs and the Yukawa couplings.

In analogy to, how one sometimes says that the electromagnetic interaction is added to a system of particles or fields with a global charge is "minimally coupled", if one essentially just replace the derivatives by the corresponding covariant ones, we shall imagine that our free theory, which has energy and momentum as global charges could be made to contain gravity by some sort of "minimal coupling". To introduce other extra interactions than just gravity is, however, expected to be much more complicated: Especially higher order Kalb-Ramond fields couple naturally to strings and branes, which in any case would tend to have disappeared in the present status of the universe. So effectively to day the Kalb-Ramond fields [13] should be free except for their "minimal coupling to gravity". This would mean that allowing such a "later" rather trivial inclusion of gravity, which should be relatively easy, would make our at first free model be precisely the since long beloved model of one of us, the-spin-charge-family theory [5–7]. Fundamentally we have thus in our picture some series of Kalb-Ramond fields together with gravity coupling to them in the minimal way. Then we fermionize only this series of Kalb-Ramond fields, but keep bosonic the gravitational field, which probably cannot be fermionized even, if we wanted to. The resulting theory thus becomes precisely of similar type as the one by one of us, the spin-charge-family theory.

Now, however, the Kalb-Ramond fields are plagued by a lot of gauge symmetry and "unphysical" degrees of freedom, some of which even show up with even negative norm squared inner products. In principle these unphysical degrees of freedom must also somehow be treated in the fermionization procedure. Especially, if we want to use our theorem of counting degrees of freedom under bosonization [3], we should have such a theorem allowed to be used also when the "unphysical" d.o.f. are present.

In fact it is the main new point in the present article, that we put forward a slightly more complicated Aratyn-Nielsen-theorem - an extended Aratyn-Nielsen theorem -, allowing for theories with negative norm squared normalizations.

It is the true motivation of the present work, that once when we shall find some genuinely working case(s) of theories that bosonize/fermionize into each other in high dimensions, they will almost certainly turn out to involve gauge theories on the bosonic side. That is to say it will be combinations of various Kalb-Ramond fields [13] (among which we can formally count also electromagnetic fields and even a scalar field), and such Kalb-Ramond fields often have lots of negative norm square components. Thus once we know what is the boson theory that can be fermionized we need an extended Aratyn-Nielsen theorem to calculate the correct number of fermion components matching the fermionization correspondance.

Well really, if we know it well, we can just read off how many fermion components there are. It is namely this number of fermion components, that translates into the number of families, on which they are to be distributed. It means that knowing the detailed form of the boson system and the rule - the extended Aratyn-Nielsen theorem - for translating the number of boson components into the number of fermion components is crucial for obtaining the correct number of families. Will so to speak the number of fermion-families remain 4 as claimed by one of us in her model, which has reminiscent of being a fermionization, or will it be corrected somehow from the true bosonization requirement including the negative norm square components for the bosons? The reliability of the model would of course - according to the judgement of one of us (H.B.F.N.) - be much bigger, if it turned out that the true prediction were 3 families rather than the 4 as usually claimed, except, of course, if the fourth family, predicted by the spin-charge-family theory, will be measured.

With the old Aratyn-Nielsen theorem (the unextended version) it does crudely not look promising to get the number 3 rather than 4 as H.B.F.N would hope phenomenologically in as far this version implies that the number of fermion components is divisible by a rather high power of the number 2. Such a number-theoretic property of the number of families seems a priori to favor 4 much over 3.

The works of major importance for the present talk are:

- Aratyn & Nielsen We made a theorem [3] about the ratio of the number of bosons needed to represent a number of fermions based on statistical mechanics in the free case, under the provision that a bosonization exists.
- Kovner & Kurzepa They[8,9] present an explicit bosonization of two complex fermion fields in 2+1 dimensions being equivalent to QED_3 meaning 2+1 dimensional quantum electrodynamics.
- Mankoč-Borštnik [5–7] The spin-charge-family unification theory explains the number of families from the number of fermion components appeared in this theory.

In the next section 14.2 we put forward the main hope or point of view of our application of bosonization to make prediction of the number of families. In section 14.3 we give a loose argument for what we think should our picture for nature to cope with the investigations in the present article. Then we shall in section 14.4 and 14.5 review both Kalb Ramond fields and and our old Aratyn-Nielsen theorem about the number fermion components needed to make an equivalent theory with a number of boson components. In section 14.6 we look at the problem, that the components of a Kalb-Ramond field with an index being 0 are on the one hand to be a conjugate momentum to the other components and on the the other hand, if we use Lorentz invariance, have to lead to states with negative norm square. The latter is of course simply a reflection of the signature of the Minkowski metric tensor. It is for the application on such negative norm square components - the components with an index 0 - that our extension of our Aratyn-Nielsen theorem to negative norm square components become relevant.

In section 14.8 we review the work by Kovner and Kurzepa[8], who proposed a concrete bosonization including explicit expressions for the fermion fields in terms of the boson fields - actually simply electrodynamics - in the case of 2 space dimensions and one time, 1+ 2. Next in 14.9 we seek to check our Aratyn-Nielsen theorem on this special case of 1+2 both by counting the particle species including spin states 14.10 and by counting the fields 14.11.

Towards the end, section 14.12, we seek to reduce away some of the degrees of freedom from the Kovner and Kurzepa model to obtain a reduced case with fewer particles on which we - if it is also a case of bosonization - would be again able to check our counting theorem (Aratyn-Nielsen).

14.2 Hope

Use of Bosonization/Fermionization Justifying Number of Families

The governing philosophy and motivation for the present study is:

- Fermions do NOT exist fundamentally (because they do not have proper causal/local property) .
- Some boson degrees of freedom are rewritten by bosonization (better fermionization) to fermionic ones, which then make up the fermions in the world, we see. (but some other boson degrees of freedom, hopefully gravity, are not bosonized).
- We work here only with an at first free theory - for our presentation, it might be best if only bosonization worked for FREE theories in higher dimensions - i.e. free bosons can be rewritten as free fermions.
- We though suggest - hope- that exterior to both bosons and/or fermions, we can add a GRAVITATIONAL theory. So fundamentally: gravity with matter bosons. It gets rewritten to fermions in a gravitational field, just similar to the theory [5–7] of one of us called spin-charge-family unification theory.

Let us be more specific about the dream or hope behind the present project:

By using say ideas from the below discussed paper by Kovner and Kurzepa [8] or by our own earlier article in last years Bled Proceedings about bosonization, we hope to find at least a case of fermionizing some series of Kalb-Ramond fields (i.e. Boson fields) - and electrodynamics is of course considered here a special Kalb-Ramond one - into some system of fermions. Presumably it is easiest - and perhaps only possible - for free theories or only theories interacting in a very special way. We therefore are most eagerly going for such a free and even massless case.

But now if indeed we can find such a case, or if exists, then it is very likely that we can extend it to interact with gravity in a minimal way. In fact we all the time require our hoped for fermionization cases to have the same energy and momentum for the bosonic and the fermionic theories that shall be equivalent. Thus the fermionization procedure, if it exist at all, is compatible with energy and momentum.

If we therefore let our boson-theory interact with gravity, that couples to the energy and momentum - specifically to the energy momentum tensor $T_{\mu\nu}(x)$ - we have some hope that this coupling of the boson fields to gravity will simply transfer to a coupling of the fermionized theory, too.

As procedure we might have in mind writing the free massless fermionization procedure in arbitrary coordinates. That should of course be possible, but although the theory would now look as a gravitational theory, it would only have been derived for the case of the gravitational fields having zero curvature, i.e. for the Riemann tensor being zero all over. However, if the fermionization procedure could be described by a local expression for the Kalb-Ramond fields - or other boson fields - expressed in terms of fermion currents or the like, then the correspondence would in that formulation be local and lead to the energy momentum tensor being also related in such simple local way. I.e. we would have in this speculation

$$T_{\mu\nu}(x)|_{boson} = T_{\mu\nu}(x)|_{fermion}. \tag{14.3}$$

Here of course the two energy momentum tensors are the ones in respectively the fermion and the boson theory being equivalent by the dream for fermionization.

It is further our hope for further calculation that we may argue that in general it is very difficult to have interaction with Kalb-Ramond fields except for

- The appropriate branes,
- Some general gauge-theory coupling to the charges (think of global ones) conserved by the Kalb-Ramond- theory in question. But since the always conserved global charges are the energy and momentum this suggests the coupling to gravitational field.

We thus want to say that this starting form fundamental Klab-Ramond fields supposedly difficult to make interact points towards a theory at the end with gravity as the only interaction. Gravity namely is suggested to be hard to exclude as possibility even for otherwise difficult to make interact Kalb-Ramond fields.

If we manage to fermionize the Kalb-Ramond fields as just suggested, we therefore tend to end up with the spin-charge-family unification model of one of us in the sense that we get ONLY gravity interaction, and otherwise a free theory.

But it shall of course be understood here that we only fermionize some of the boson fields in as far as we leave the assumed fundamental gravity field non fermionized.

14.3 Guiding and Motivation

The reader might ask why we choose - and suppose Nature to choose - these Kalb-Ramond-type fields which are to be explained a bit more below in section (14.5). Let us therefore put forward a few wish-thinking arguments for our bosonic fundamental model:

- We have no way to make fermionization/bosonization conserving angular-momentum truly (at the same time keeping the spin statistics theorem): The bosons namely necessarily can only produce Fock space states with integer angular momentum, but the fermion sates should for an odd number of fermions in the the Fock state have half integer angular momentum. So clearly fermionization/bosonization conserving angular momentum is *impossible*!

- The trick to overcome this angular-momentum-problem is to *reinterprete a spin 1/2 index on the fermions as a family index instead*. That is to say we accept at first that the fermions come out of the fermionization with bosonic integer spin index combination, and then seek to reinterprete part of the spin polarisation information as instead being a family information.
- In fact we shall be inspired by the spin-charge-family unification model to go for that the fermions come out from the fermionization at first with *two* spinor indices, so that they have indeed formally at this tage *integer* spin. Then we make the interpretation that one of these spinor indices is indeed a *family index*. That of course means, that we let one of the two indices be taken as a scalar index i.e. being not transformed under Lorentz transformations.
- So we decide to go for a system of fermions at the "first interpretation" being a two-spinor-indexed field. But now such a field $B_{\alpha\beta}$, where α and β are the spinor indices, is indeed a Clifford algebra element, or we could say a Dirac matrix (or a Weyl matrix only if we use only the Weyl components). In any case we can expand it on antisymmetrized products of gamma-matrices:

$$B_{\alpha\beta} = \big(a\mathbf{1} + a_\mu\gamma^\mu + \cdots + a_{\mu\nu\ldots\rho}\gamma^\mu\gamma^\nu\cdots\gamma^\rho + \ldots \\ + a_{0,1,\ldots,(d-1)}\gamma^0\gamma^1\cdots\gamma^{(d-1)}\big)_{\alpha\beta}, \quad (14.4)$$

and thus the boson fields suggested to by fermionization leading to such fermion fields should be a series of antisymmetric tensor fields of all the different orders from the scalar a and the d-vector a_μ all the way up to the maximal antisymmetric order tensor $a_{0,1,\ldots,(d-1)}$.
- With random coefficients on a Lagrange-density expansion for a theory with boson fields, which have d-vectorial indices one unavoidably loose the bottom in the Hamiltonian as one can see from e.g. just a term like

$$c * (\partial_\mu\cdots\partial_\nu a_{\rho\ldots\tau})^2 \,. \quad (14.5)$$

Think for instance on the terms for which the series of the derivative indices are spatial so that we have to do with a potential energy term. If the coefficient c is adjusted to let the contribution with the indices on $a_{\rho\ldots\tau}$ being spatial to the Hamiltonian be positive, then the contributions with a 0 among these indices will from Lorentz invariance have to be of the wrong sign. So it is at best exceedingly hard to organize a positive definite Hamiltonian density.
Consider only the free part - meaning bilinear part in the field $a_{\rho\ldots\tau}$ - in the Lagrangian. For simplicity let us consider the situation of a field $a_{\rho\ldots\tau}$ being constant as function of the time coordinate x^0, and that the number of derivatives acting on the field is so low that some of the indices- say ρ - on the $a_{\rho\ldots\tau}$ has to be contracted with another one or the same index on this field in order to cope with Lorentz invariance. Then if this (sum of) squares of the field in some combination shall get a for the hamiltonian positive contribution from a spatial value of the index ρ, it will get the opposite sign for $\rho = 0$. So it looks that we cannot avoid the Hamiltonian having both signs for a "free term" in the Lagrangian, unless all the indices on $a_{\rho\ldots\tau}$ are in the term contracted with derivatives. But with the antisymmetry this would be zero for more than

one index on $a_{\rho\ldots\tau}$. So indeed it seems that unless one gets the fields restricted in some way, so that these fields or their conjugate variables are somehow not allowed to take independent values, then the Hamiltonian will loose its bottom and (presumably infinite) negative energy values will be allowed.

- We are thus driven towards theories with constraints!
- Such constraints are typically obtained by means of some gauge symmetry, and thus we are driven towards theories with gauge symmetry, if we want to uphold a positive definite Hamiltonian for the by the constraints allowed states of the field and its conjugate momenta.
- The obvious candidate for such a gauge theory with antisymmetrised tensor fields is of course the Kalb-Ramond fields. (Personally we suspect, that we can show that ONLY Kalb-Ramond-fields will solve this problem of positivity by providing enough constraints.)
- Thus it seems that it is very hard to hope for our to be used fermionization unless we make use of presumably a whole series of Kalb-Ramond fields!

14.4 Review

In theoretical condensed matter physics and particle physics, Bosonization/fermionization is a mathematical procedure by which a system of possibly interacting fermions in (1+1) dimensions can be transformed to a system of massless, non-interacting bosons. In the present article we shall dream about extending such bosonization to higher dimensions, and we shall be most interested in the case when even the fermions do not interact. The method of bosonization was conceived independently by particle physicists Sidney Coleman and Stanley Mandelstam; and condensed matter physicists Daniel Mattis and Alan Luther in 1975. [4] The progress to higher dimensions has been less developed [11] than the 1+1 dimensional case, but there has been some works also on higher dimensions. especially we shall below review a bit a work[8] by Kovner and Kurzepa for the next to simplest case, namely 2+1 dimensions. There has also been developments based on Chern-Simon type action[11], but we suspect that the type of bosonization we are hoping for in the present article should rather be of the Kovner Kurzepa type than of the Chern-Simon one, although we have difficulty in explaining rationally why we believe so.

When we have such transformation and thus two equivalent theories, one with fermions and one with bosons, one will of course expect that the number of degrees of freedom should in some way be the same for the boson and for the fermion theory. Otherwise of course they could not be equivalent. In the most studied case of 1+1 dimensions it has turned out in the cases known that there are two fermion components per boson component. This ratio is in accord with the theorem by Aratyn and one of us [3] - which we call the Aratyn-Nielsen theorem - in as far as this theorem predicts the ratio to be $\frac{2^{d_{spatial}}}{2^{d_{spatial}-1}}$ where $d_{spatial}$ is the dimension of space (not including time) so that we talk about the dimension $d_{spatial}+1$. In fact of course for the case 1+1 we have thus $d_{spatial} = 1$ and the fraction predicted becomes $\frac{2^1}{2^1-1} = 2$ times as many fermion components as boson components. This is really assuming that a "component" corresponds to a polarization state of a particle. What we - one of us and Aratyn - really derived was that for a theory with massless interacting there ahd to be the mentioned ratio between the number of polarization states for the fermion(s) relative to that for the bosons. It were namely the contributions of such polarization states to the average energy in a Boltzmann distribution calculation that was used to derive the theorem. Although derived for this non-interacting massless case there could be reasons to believe that by taking a couple of limits in an imagined case of interacting and perhaps massive bosonization it could be argued, that the theorem of ours would have to hold anyway. For instance going to a very small distance scale approximation an approximately massless theory would arrive and the theorem should be applicable even if there is a mass. Since we are concerned in this theorem really with a counting of degrees of freedom a very general validity is in fact, what would be expected. As already said, we are, however, in the present article more concentrating on the generalization to include some unphysical degrees of freedom with possibly wrong signature,

14.5 AratynN

Aratyn-Nielsen Theorem for massless free Bosonization

If there exist two free massless quantum field theories respectively with Boson, and Fermion particles and they are equivalent w.r.t. to the number of states of given momenta and energies, **then** the two theories must have the same average energy densities for a given temperature T, or simply same average energies, if we take them with the same infrared cut off(a quantisation volume V):

$$< U_{boson} > = < U_{fermion} > \text{ where} \tag{14.6}$$

$$< U_{boson} > = \sum_{\vec{p}} \frac{E(\vec{p})}{1 - \exp\left(E(\vec{p})/T\right)} \tag{14.7}$$

$$< U_{fermion} > = \sum_{\vec{p}} \frac{E(\vec{p})}{1 + \exp\left(E(\vec{p})/T\right)}. \tag{14.8}$$

$$\tag{14.9}$$

Here $\vec{p}$ runs through the by the infrared cut off allowed momentum eigenstates, and $E(\vec{p})$ are the corresponding single particle energies. Of course the single particle energy for a mass-less theory is

$$E(\vec{p}) = |\vec{p}|, \tag{14.10}$$

when c=1, and in $d_{spatial}$ dimensions and with an infrared cut off spatial volume V the sum gets replaced in the continuum limit by the integral

$$\sum_{\vec{p}} ... \to \int \sum_{components} ... \frac{V}{(2\pi)^{d_{spatial}}}, \tag{14.11}$$

where $\sum_{components}$ ... stands for the sum over the different polarization components of the particles in question. So effectively in the simplest case of all the particles having the same "spin"/the same set of components we have the replacement

$$\sum_{components} ... \rightarrow N_{families} * N_c ... \tag{14.12}$$

where N_c is the number of components for each particle and $N_{families}$ is the number of families. **Some formulas for deriving Aratyn-Nielsen**

$$< U_{boson} > = \sum_{\vec{p}} \frac{E(\vec{p})}{1 - \exp\left(E(\vec{p})/T\right)} \tag{14.13}$$

$$= \text{“}N_{families} * N_c'' * V/(2\pi)^{d_{spatial}} * \tag{14.14}$$

$$\int O(d_{spatial}) |\vec{p}|^{d_{spatial}} E(\vec{p}) \sum_{n=0,1,...} \exp\left(nE(\vec{p})/Td|\vec{p}|\right. \tag{14.15}$$

Simple Aratyn-Nielsen Relation For a given temperature must the average energies of respectively the boson and the with it equivalent fermion theories

Our Realization Suggestion

- Fermions
  For the fermions we shall use the needed number of say Weyl fermions, i.e. we must adjust the number of families hoping that we get an integer number.
- Bosons
  For the bosons we let the number $2^{d_{spatial}} - 1$ suggest that we take a series of all Kalb-Ramond fields, one combination of fields for each value of the number p of indices on the "potential field" $A_{ab...k}$ (where then there are just p symbols in the chain $ab...k$). At first we take these symbols $a, b, ..., k$ to be only spatial coordinate numbers.

Free Kalb-Ramond A Kalb-Ramond field[13] with p indices on the "potential" and p+1 indices on the strength

$$F_{\mu\nu\rho...\tau}(x) = \partial_{[\mu}A_{\nu\rho...\tau]}(x), \tag{14.16}$$

where [...] means antisymmetrizing, and the "potential" $A_{\nu\rho...\tau}$ is antisymmetric in its p indices $\nu\rho...\tau$, is defined to have an action invariant under the gauge transformation:

$$A_{\nu\rho...\tau}(x) \rightarrow A_{\nu\rho...\tau}(x) + \partial_{[\nu}\lambda_{\rho...\tau]}(x) \tag{14.17}$$

for any arbitrary antisymmetric gauge function $\lambda_{\rho...\tau}(x)$ with $p-1$ indices.

Free Kalb-Ramond Action:

Note that the strength $F_{\mu\nu\rho...\tau} = \partial_{[\mu}A_{\nu\rho...\tau]}$ is gauge invariant, and that thus we could have a gauge invariant Lagrangian density as a square of this field strength

$$\mathcal{L}(x) = F_{\mu\nu...\tau}F_{\mu'\nu'...\tau'}g^{\mu\mu'} * g^{\nu\nu'} * ... * g^{\tau\tau'}. \tag{14.18}$$

Then the conjugate momentum of the potential becomes(formally):

$$\Pi_{\nu\rho\ldots\tau} = \Pi_{A_{\nu\mu\ldots\tau}} = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_{\nu\rho\ldots\tau})} = F_{0\nu\rho\ldots\tau}. \tag{14.19}$$

A Lorentz gauge choice:

$$\partial_\mu A\nu\rho\ldots\tau g^{\mu\nu} = 0, \tag{14.20}$$

allows to write the Lagrange density instead as

$$\mathcal{L}_{modified}(x) = 1/2 * \partial_\mu A_{\mu\nu\ldots\tau}\partial_{\mu'} A_{\mu'\nu'\ldots\tau'} * g^{\mu\mu'} g^{\nu\nu'} \cdots g^{\tau\tau'}, \tag{14.21}$$

which leads to the very simple equations of motion letting each component of the "potential" $A_{\nu\rho\ldots\tau}$ independently obey the Dalambertian equation of motion

$$g^{\mu\mu'}\partial_\mu\partial_{\mu'} A_{\nu\rho\ldots\tau} = 0. \tag{14.22}$$

Lorentz Invariance Requires Indefinite Inner Product!:

Lorentz invariant norm square for the states generated by the creation operators $a^\dagger_{\nu\rho\ldots\tau}(p)$, i.e. $a^\dagger_{\nu\rho\ldots\tau}(p)|0>$, must have different sign of the norm square depending on whether there is an even (i.e. no) 0's among the indices or whether there is an odd number (i.e. 1). A priori we are tempted to take

$$\begin{aligned} &< 0|a_{\nu\rho\ldots\tau}(p) a^\dagger_{\nu\rho\ldots\tau}(p)|0> > 0 \text{ for no } 0 \text{ among the indices,} \\ &< 0|a_{\nu\rho\ldots\tau}(p) a^\dagger_{\nu\rho\ldots\tau}(p)|0> < 0 \text{ for one } 0 \text{ among the indices,} \end{aligned} \tag{14.23}$$

14.6 Time-index

Problem with Components with the time index 0:

But full Kalb-Ramond fields require also components a 0 among the indices.(This is the main new thing in the present article to treat this problem of the components with one 0 among the indices.)

Remember about these components with a 0 index:

- Using a usual Minkowskian metric tensor $g^{\mu\nu}$ in constructing an inner product between Kalb-Ramond fields, say

$$g^{\mu\nu} g^{\rho\sigma} \cdots g^{\tau\kappa} A_{\mu\rho\ldots\tau}(\text{potentially an } \partial^0) A_{\nu\sigma\ldots\kappa}, \tag{14.24}$$

  we get the opposite signature (=sign of the square norm) depending on whether there is a 0 or not!

  This means that if particles produced by the components without the 0 index have normal positive norm square, then those produced by the ones with the 0 have negative norm-square!

Good Luck We Removed the Kalb-Ramond A with $p = 0$ Indices! We could namely not have replaced on A one among its indices by a 0 because it has no indices. So we would not have known what to do for the fields A with 0 indices.

We correspondingly also have to leave out the Kalb-Ramond-field with $p = d_{spatial} + 1$ indices, because for that there would be no components without an index 0.

For the unexceptional index numbers $p = 1, 2, ..., d_{spatial}$ there are some components both with and without the 0.

For the two exceptions $p = 0$ and $d = d_{spatial} + 1$ we have chosen not to have a Kalb-Ramond-field in our scheme, using it to get the -1 in the from Aratyn-Nielsen required $2^{d_{spatial}} - 1$.

Simplest (Naive) Norm Square Assignment

Note that for each Kalb-Ramond-field we can choose an overall extra sign on the inner product, because we simply can define the overall inner product with an extra minus sign, if we so choose. But the simplest choice is to just let the particles corresponding to fields with only spatial indices (i.e. all p indices different from 0) to have positive norm square, while then those with one 0 have negative norm square.

This simple rule would lead to equally many components/particles with positive as with negative norm square, so that dreaming about imposing a constraint that removes equally many negative and positive norm square at a time would leave us with nothing.

Numbers of Components with and without 0. An of course totally antisymmetric field $A_{\mu\nu\ldots\tau}$ with p indices has

$$\begin{aligned}
\#\,\text{components}_{KR\ p\text{indices}} &= \binom{d}{p} = \binom{d_{spatial}+1}{p}\\
\#\,\text{no } 0 \text{ components}_{KR\ p\text{indices}} &= \binom{d_{spatial}}{p} = \binom{d-1}{p}\\
\#\,\text{cmps. with } 0 \,\&\, \text{p-1 non-}0_{KR\ p\text{indices}} &= \binom{d_{spatial}}{p-1} = \binom{d-1}{p-1}.
\end{aligned}$$

and so one must have as is easily checked

$$\binom{d}{p} = \binom{d-1}{p} + \binom{d-1}{p-1}$$

corresponding to

$$\text{“All components”} = \text{“Without 0”} + \text{“With 0”}$$

Using ONLY the Components WITHOUT 0 would fit $2^{d_{spatial}}$ Nicely ! Having decided to leave out the number of indices p values $p = 0$ and $p = d$ the number of components **without** any component indices being 0 just makes up

$$\#\,\text{without } 0_{\text{ for all } p = 1, 2, ..., d-1} = \sum_{p=1,2,\ldots,d-1} \binom{d-1}{p} = 2^{d-1} - 1$$

so these “only with spatial indices components” could elegantly correspond to $2^{d-1} = 2^{d_{spatial}}$ fermion components.

But problem: Kalb- Ramond fields need also the component with an index being 0.

Using ONLY the Components WITH 0 **could also fit** $2^{d_{spatial}}$ **Nicely !** Having decided to leave out the number of indices p values $p = 0$ and $p = d$ the number of components **with** the 0 just makes up

$$\# \text{ with } 0 \text{ for all } p = 1, 2, ..., d-1 = \sum_{p=1,2,...,d-1} \binom{d-1}{p-1} = 2^{d-1} - 1$$

also, so these "only with 0 index components" could elegantly correspond to 2^{d-1} = $2^{d_{spatial}}$ fermion components, also!

But problem: Kalb- Ramond fields need also the components without an index being 0**, and these with** 0 **usually come with wrong norm square.**

The Trick Suggested is to use for Some KR-fields Opposite Hilbert Norm Square

In other words we shall look along the chain of all the allowed p-values $p = 1, 2, ..., d-1$; and for each of these p-values we can choose whether

- Normal: The states associated with the polarization components **without** the 0 among the indices shall be of **positive norm square**, as usual, and then from Lorentz invariance essentially the ones **with the** 0 **shall have negative norm square**, or
- Opposite The states **with** 0 **shall have positive** norm square, while the components **without** 0 **negative** norma square.

Our proposal: Choose so that we get the largest number of positive norm square components. **How to get Maximal Number of Positive over Negative Norm Square Single Boson States**

For each value of p (=the number of indices on the Kalb Ramond "potential") $p = 1, 2, ..., d_{space}$ decided to be used in the bosonization ansatz a priori, we investigate whether the number of (independent) components **with** or **without** a 0 is the bigger:

$$\# \text{ no } 0 \text{ components}_{KR\ p \text{indices}} = \binom{d_{spatial}}{p} = \binom{d-1}{p}$$

$$\# \text{ cmps. with } 0 \text{ \& p-1 non-}0_{KR\ p \text{indices}} = \binom{d_{spatial}}{p-1} = \binom{d-1}{p-1}.$$

So if there are **most components without** 0, i.e. if $\binom{d_{spatial}}{p-1} < \binom{d_{spatial}}{p}$, then we give the particle states corresponding to the **without** 0 "potentials" have **positive norm square**. And opposite if $\binom{d_{spatial}}{p-1} > \binom{d_{spatial}}{p}$.

But if there are **most components with** 0, i.e. if $\binom{d_{spatial}}{p-1} > \binom{d_{spatial}}{p}$, then we give the particle states corresponding to the **with** 0 "potentials" have **positive norm square**.

To Maximize Positive Norm Square we Choose:

- When $p < \frac{d}{2}$, choose **without** 0 **positive norm** squared, while "with 0" negative;
- but when $p > \frac{d}{2}$, choose **with** 0 **positive norm** squared, while "without 0" negative;

For e.g. $p < d/2$ the excess of positive norm square "components " over the negative norm ones becomes:

$$\begin{aligned}
&\binom{d_{space}}{p} - \binom{d_{space}}{p-1} = \binom{d_{space}}{p}(1 - \frac{p}{d_{space} - p + 1}) \\
&= \frac{d_{space}!(d_{space} + 1 - 2p)}{(d_{space} - p + 1)!p!} = \frac{(d-1)!(d-2p)}{(d-p)!p!},
\end{aligned} \tag{14.25}$$

However, for $p > d/2$ the excess is

$$\begin{aligned}
&\binom{d_{space}}{p-1} - \binom{d_{space}}{p} = \binom{d_{space}}{p-1}(1 - \frac{d_{space} - p + 1}{p}) \\
&= \frac{d_{space}!(2p - d_{space} - 1)}{(d_{space} - p + 1)!p!} = \frac{(d-1)!(2p-d)}{(d-p)!p!},
\end{aligned} \tag{14.26}$$

Adding up Positive Norm Square over Negative Excess:

The sums over p " telescopes" from each of the two cases of p bigger or smaller than $d/2$, and gives by symmetry the same excess of positive over negative norm square states, namely for each for say d even (i.e. d_{space} odd)

$$\binom{d-1}{d/2-1} - 1 = \frac{(d-1)!}{(d/2-1)!(d/2+1)!} - 1, \tag{14.27}$$

where we used that the middle value $p = d/2$ contribution vanishes. Including as we shall both "sides" smaller than $d/2$ and also bigger than $d/2$ we get the double of this.

Example Excesses States for even d for Bosons

$$\begin{aligned}
\text{Excess}(d=2) &= 2(\binom{2-1}{2/2-1} - 1) = 0 \\
\text{Excess}(d=4) &= 2(\binom{4-1}{4/2-1} - 1) = 2 \\
\text{Excess}(d=6) &= 2(\binom{6-1}{6/3-1} - 1) = 18 \\
\text{Excess}(d=14) &= 2(\binom{14-1}{14/2-1} - 1)
\end{aligned} \tag{14.28}$$

Contribution from a Negative Norm square Component

One shall count the Hilbert space states with the negative norm square into the Boltzmann weighted averaging with a minus extra.

This extra minus for a negative norm square boson functions accidentally just like the fermi-statistics versus bose statistics. And thus e.g. a small p timelike polarization contributes to the average energy just like a fermion, though with an over all minus sign.

14.7 Extension of Our Theorem on Counting

It is a major purpose of the present talk to present an extension of the Aratyn-Nielsen theorem[3] on the numbers of bosons versus fermions in a bosonization to include the just above discussed negative norm square states associated with the Kalb-Ramond components having an index 0. Since such states obtaining at first negative norm squares are seemingly enforced by Lorentz invariance, it seems to be important to extend our Aratyn-Nielsen theorem to the case, where some of the components of the fields are quantized with a negative norm square.

We take such a negative norm square mode to mean, that whenever there in a Fock space state is an *odd* number of particles with the component in question, then such a Fock-space basis vector is in the "Hilbert norm" given a negative norm square. Of course that means that strictly speaking our Fock space is no longer a genuine Hilbert space, but rather just an (infinite dimensional) space with an *indefinite* inner product, $|$, giving the inner product between two Focks, $|a>$ and $|b>$ say, as $< b|a >$. But now the point is just that *we have no sign restriction on* $< a|a >$*; it can easily be negative.*

The in usual Hilbert spaces used expansion on an orthonormal basis

$$\mathbf{1} = \sum_a |a >< a| \text{ (usual)}, \tag{14.29}$$

cannot now be applied. Now we rather have to use

$$\mathbf{1} = \sum_a (-1)^{N_{neg}(a)} |a >< a| \text{ (with negative norm square also)}, \tag{14.30}$$

where $N_{neg}(a)$ denotes the number of particles in the various negative norm square single particle states together. If for instance a basis state $|a>$ for the Fock space has 3 particle in the states with 0 index all together (and we have used the choice of letting the components with a 0-index be the ones with negative norm, rather than the more complicated possibilities discussed above), $N_{neg}(a) = 3$ and thus such a state would come with a minus sign in the expansion of the unit operator $\mathbf{1}$.

Let us now calculate the average energy for a system described by a Fock space with only one single particle state present, so it really is the system with only one single particle state, that may be filled or empty according the rule for it being bosonic or fermionic and having negative or positive norm square. For this purpose we have to think about how one shall define the concept of a trace - which goes into the average procedure to provide us with such a an average of the energy, and we claim that we must indeed in the case with negative norm square states take the trace definition:

$$\mathrm{Tr}(\mathbf{O}) = \sum_a (-1)^{N_{neg}(a)} < a|\mathbf{O}|a > . \tag{14.31}$$

With this definition we easily check some usual rule for traces:

$$\mathrm{Tr}(\mathbf{OP}) = \sum_a (-1)^{N_{neg}(a)} < a|\mathbf{OP}|a > \tag{14.32}$$

$$= \sum_a \sum_b (-1)^{N_{neg}(a)} < a|\mathbf{O}|b >< b|(-1)^{N_{neg}(b)}\mathbf{P}|a > \tag{14.33}$$

$$= \mathrm{Tr}(\mathbf{PO}). \tag{14.34}$$

Using this definition of the trace Tr we can then put in the quite analogous way to the usual case for Boltzmann distribution in quantum mechanics

$$< E > = \frac{\mathrm{Tr}(\exp(-H/T)H)}{\mathrm{Tr}(\exp(-H/T))}, \tag{14.35}$$

where the Boltzmann-Constant k has been absorbed into the temperature T, and where now we use in the case of negative norm square the expression (14.31). Let us enumerate the single particle states with the letter n and denote the single particle energy of the state n as E_n. Then the free Hamiltonian H is given by means of the number operators

$$N_n = a_n^\dagger a_n \tag{14.36}$$

as

$$H = \sum_n E_n N_n = \sum_n E_n a_n^\dagger a_n, \tag{14.37}$$

and we immediately see that

$$< E_n N_n > |_{boson\, pos.} = \frac{\sum_{N_n=0,1,\ldots} E_n N_n \exp(-E_n N_n/T)}{\sum_{N_n=0,1,2,\ldots} \exp(-E_n N_n/T)}$$

$$= \frac{-\frac{d\left(\frac{1}{1-\exp(-E_n/T)}\right)}{d(1/T)}}{\frac{1}{1-\exp(-E_n/T)}} = \frac{E_n}{\exp(E_n/T)-1} \text{(boson; pos. norm sq.)} \tag{14.38}$$

$$< E_n N_n > |_{boson\, neg.} = \frac{\sum_{N_n=0,1,\ldots} (-1)^{N_n} E_n N_n \exp(-E_n N_n/T)}{\sum_{N_n} (-1)^{N_n} \exp(-E_n N_n/T)}$$

$$= \frac{-\frac{d\left(\frac{1}{1+\exp(-E_n/T)}\right)}{d(1/T)}}{\frac{1}{1+\exp(-E_n/T)}} = -\frac{E_n}{\exp(E_n/T)+1} \text{(boson; neg. norm sq.)} \tag{14.39}$$

$$< E_n N_n > |_{fermion\, pos.} = \frac{\sum_{N_n=0,1} E_n N_n \exp(-E_n N_n/T)}{\sum_{N_n=0,1} \exp(-E_n N_n/T)}$$

$$= \frac{-\frac{d(1+\exp(-E_n/T))}{d(1/T)}}{1+\exp(-E_n/T)} = \frac{E_n}{\exp(E_n/T)+1} \text{(fermion; pos. norm sq.)} \tag{14.40}$$

$$< E_n N_n > |_{fermion\, neg.} = \frac{\sum_{N_n=0,1} (-1)^{N_n} E_n N_n \exp(-E_n N_n/T)}{\sum_{N_n=0,1} (-1)^{N_n} \exp(-E_n N_n/T)}$$

$$= \frac{-\frac{d(1-\exp(-E_n/T))}{d(1/T)}}{1-\exp(-E_n/T)} = -\frac{E_n}{\exp(E_n/T)-1} \text{(fermion; neg. norm sq.)} \tag{14.41}$$

We notice that - by accident - the contribution from a negative norm square fermion mode happens to be just the opposite of that of a positive norm square boson mode with the same energy E_n. And also the positive fermion mode contribution is just minus one time the negative boson contribution. Thus we can get the requirement for the theory of fermions and that of bosons to provide the same average energy:

$$\sum_{\substack{E_n' s \text{for (pos.)fermions} \\ \text{plus neg. bosons}}} \frac{E_n}{\exp(E_n/T)+1} = \sum_{\substack{E_n' s \text{for (pos.)bosons} \\ \text{plus neg. fermions}}} . \qquad (14.42)$$

14.7.1 Free Massless

The simplest case to consider is the one in which both the fermions and the bosons - on their respective sides of the identification of the theories - are supposed to be both free and massless relativistic particles. In this case - which is the one we shall keep to in the present article - we introduce for definiteness an infra red cut off so that we get discretized momentum eigenstates, and the above n now really becomes a pair of a discretized momentum $\vec{p}$ and an index denoting the component, which means typically the vector or spinor index including also the family index, all put together say to t, standing for the word "total component", meaning that both family and genuine component is included. The number of possible values for this total component enumeration is of course for what we are indeed obtaining restrictions for. Let us therefore immediately define the four numbers

$$N_{t\ ferm\ pos.} = N_{families\ ferm\ pos.} * N_{c\ ferm\ pos},$$

$$N_{t\ ferm\ neg.} = N_{families\ ferm\ neg.} * N_{c\ ferm\ neg.},$$

$$N_{t\ boson\ pos.} = N_{families\ boson\ pos.} * N_{c\ boson\ pos},$$

$$N_{t\ boson\ neg.} = N_{families\ boson\ neg.} * N_{c\ boson\ neg.},$$

to denote the total numbers of components of the respective types of particles w.r.t. statistics and normsquare sign.

One technique for calculating the integrals over the momentum space consists in first Taylor expanding the expressions to be integrated

$$\frac{E_n}{\exp(E_n/T)-1} = \frac{E_n}{\exp(E_n/T)} * (1+\exp(-E_n/T)+\exp(-2E_n/T)+...)$$
$$= E_n \left(\sum_{j=1,2,...} \exp(-jE_n/T) \right) \qquad (14.43)$$

$$\frac{E_n}{\exp(E_n/T)+1} = \frac{E_n}{\exp(E_n/T)} * (1-\exp(-E_n/T)+\exp(-2E_n/T)-...)$$
$$= E_n \left(\sum_{j=1,2,...} (-1)^{j-1} \exp(-jE_n/T) \right), \qquad (14.44)$$

and then using

$$\sum_{\vec{l} \in \text{integer lattice}} \exp(-j|\vec{l} * 2\pi/L|) = \int \exp(-j|\vec{x}2\pi/L|) d^{d_{spatial}}\vec{x} \quad (14.45)$$

$$= \left(\frac{L}{2\pi * j}\right)^{d_{spatial}} \int \exp(-|\vec{x}|) d^{d_{spatial}}\vec{x} \quad (14.46)$$

$$= \left(\frac{L}{2\pi * j}\right)^{d_{spatial}} \mathcal{O}(d_{spatial} - 1) \int_0^\infty \exp(-x) x^{d_{spatial}} dx \quad (14.47)$$

$$= \left(\frac{L}{2\pi * j}\right)^{d_{spatial}} \mathcal{O}(d_{spatial} - 1)/d_{spatial}!. \quad (14.48)$$

Here we denoted the surface area of the unit sphere in $d_{spatial}$ dimensions by $\mathcal{O}(d_{spatial} - 1)$ because this surface then has the dimension $d_{spatial} - 1$. In fact

$$\mathcal{O}(d_{surface}) = \frac{2\pi^{d_{surface}/2}}{\Gamma(d_{surface}/2)}. \quad (14.49)$$

We then finally shall use

$$\zeta(d_{spatial}) = \sum_{j=0,1,2,\ldots} \frac{1}{j^{d_{spatial}}}. \quad (14.50)$$

$$\zeta(d_{spatial}) \left(1 - \frac{1}{2^{d_{spatial}}}\right) = \sum_{j=0,1,2,\ldots} \frac{(-1)^j}{j^{d_{spatial}}}. \quad (14.51)$$

When we compare the different expressions for bosons versus for fermions, most factors drop out and the only important factor is the factor $\left(1 - \frac{1}{2^{d_{spatial}}}\right)$. It is then easy to see that we obtain the extended Aratyn-Nielsen theorem:

$$N_{t\ ferm\ pos.} + N_{t\ boson\ neg.} = \frac{2^{d_{spatial}}}{2^{d_{spatial}} - 1} * (N_{t\ boson\ pos.} + N_{t\ ferm\ neg.}). \quad (14.52)$$

14.7.2 Properties and Examples

Let us first of all call attention to that this extended Aratyn-Nielsen theorem like the original one has the property of "additivity" meaning that if we have two cases of functioning bosonization - i.e. two cases of a system of fermions being equivalent to a system of bosons - and thus by combining them formally a system with both sets of bosons making up its set of bosons and similarly construct a set of fermions by combing the fermions then the combined system will automatically - just algebraically - come to obey the requirement from our theorem.

Let us also remark that the old Aratyn-Nielsen theorem[3] just is the special case, in which there are no negative norm square components.

In the Bled workshop in 2015 [12] we presented speculations, that one could make a free massless case of bosonization/fermionization in an arbitrary number of dimensions. This attempt were indeed already strongly inspired from our theorem and counted just $2^{d_{spatial}} - 1$ boson particle components and $2^{d_{spatial}}$ fermionic components. There were no negative norm square components and the there suggested case of bosonization should thus be an example on the use of the "old" Aratyn-Nielsen theorem. The ratio of the number $2^{d_{spatial}}$ of fermion components equivalent to $2^{d_{spatial}} - 1$ bosonic components is namely of course just equal to $\frac{2^{d_{spatial}}}{2^{d_{spatial}-1}}$ as it should according to our theorem(s). The special feature of that proposal [12] was that we imagined having chosen such infrared cut off periodicity or antiperiodicity conditions, that these (anti)periodicity conditions specified the components of the fields. Indeed there were just one fermion component for each combination of a choice of periodicity versus antiperiodicity for each of the $d_{spatial}$ spatial dimensions. That makes up of course $2^{d_{spatial}}$ combinations of periodicity antiperiodicity choices and thus so many fermion components. Similarly almost all such combinations gave rise to a boson component, except that we deleted so to speak the boson components, that should have corresponded to being periodic in all $d_{spatial}$ coordinates (taken with infrared cut off). Thus there were just $2^{d_{spatial}} - 1$ boson components in the in this Bled proceeding speculated case of bosonization.

14.7.3 A speculative semi-trivial example

Starting from the example[12] we would now highly suggestively - but really a bit speculatively - construct a not completely trivial although not so very physically interesting at first example with negative norm square components. Since we have anyway broken in this model full rotational invariance, it is no longer a catastrophe to treat one of coordinate axis - say x^1 in a different way from the other ones.

We modify the model in the 2015 Bled proceedings by:

- On the fermionic side we take all the components specified by having odd momentum along say the x^1-axis or equivalently have antiperiodic boundary condition in x^1 to have **negative norm square**. They make up just half - and thus $2^{d_{spatial}-1}$ - of all the fermionic components.
- On the bosonic side we also change the norm-square to be negative for the components antiperiodic in the x^1-coordinate. This is for even more than half of the components in as far as it is again for $2^{d_{spatial}-1}$, but now only out of the $2^{d_{spatial}} - 1$ bosonic components.

Both of these two modifications have in the Fock-space the same effect in as far as they both just lead to shifting the norm square form positive to negative for all the states with the **total** p^1**-momentum odd**. So the two modifications suggested for respectively the bosons and the fermions seem to be the same one in the Fock space. At least speculatively then we expect, that the modified model will have functioning bosonization - provided we trust that the original model from the Bled 2015 proceeding were indeed consistently a case of bosonization.

Now we want to test, if this suggestive speculative case of bosonization will obey our extended Aratyn-Nielsen requirement(14.52):

We have in this modified model/case of bosonization:

- We are left with $2^{d_{spatial}-1} - 1$ bosonic positive norm square components, i.e. $N_{t\ boson\ pos.} = 2^{d_{spatial}-1} - 1$.
- While $2^{d_{spatial}-1}$ of the bosonic components were made to have negative norm squared. So $N_{t\ boson\ neg.} = 2^{d_{spatial}-1}$.
- Of the fermionic components $2^{d_{spatial}-1}$ remained of positive norm-square; so $N_{t\ ferm\ pos.} = 2^{d_{spatial}-1}$.
- Also $2^{d_{spatial}-1}$ components had the odd momentum in the x^1-direction and were made to have negative norm square. So $N_{t\ ferm\ neg.} = 2^{d_{spatial}-1}$.

Inserting these numbers of components into (14.52) is easily seen to make it satisfied. The point really is, that we made the same number of boson components and of fermion components negative norm square. This sign of norm square in our formula makes them move from one side to the other, but since the two groups were of the same number at the end nothing were changed and the formula still satisfied.

14.8 Kovner...

Kovner and Kurzepa made 2+1 The article by these authors [8] contains the expression

$$\psi_\alpha(x) = k\Lambda V_\alpha(x)\Phi(x)U_\alpha(x) \tag{14.53}$$

for the fermion fields expressed in terms of the boson fields in their fermionization in 2+1 dimensions. Here the expressions $V_\alpha(x)$, $\Phi(x)$, and $U_\alpha(x)$ are exponentials of integrals over the boson field, which are indeed electromagnetic fields in 2+1 dimensions. The variants of expressions are denoted by the index α, which takes two values. There are thus (a priori) two complex fermion fields defined here.

14.9 Match?

Does the Kovner Kurzepa Bosonization Match with the Aratyn-Nielsen Counting Rule?

First look at number of hermitean counted fields: Kovner and Kurzepa gets two complex meaning **4** real fermion fields $Re\psi_1(x)$, $Im\psi_1(x)$, $Re\psi_2(x)$, and $Im\psi_2(x)$ out of the for the construction relevant boson-fields $A_1(x)$, $A_2(x)$, $\partial_i E_i = \partial_1 E_1 + \partial_2 E_2$. This looks agreeing with the Aratyn Nielsen prediction that the ratio shall be

$$\frac{\#bosons}{\#fermions} = \frac{2^{d_s}-1}{2^{d_s}} = \frac{2^2-1}{2^2} \text{for the spatial dimension being} d_s = d-1 = 2 \tag{14.54}$$

Four real fermion fields bosonize to three real boson-fields! o.k.

What about the conjugate momenta to the fields? While the fermion fields are normally each others conjugate variables(fields) in as far as they anticommute with each other having only no-zero anticommutators with themselves, the boson-fields typically are taken each to have associated an extra field - its conjugate -

with which it does not commute, while of course any variable must commute with itself. But a field, that depends on an x-point or on a momentum, need NOT to commute with itself, though.

But then the question: Shall we for bosons somehow also count the conjugate momentum fields, when we shall compare the number of fermion and boson fields equivalent through bosonization ? For the fermions the conjugate fields are unavoidable already included into the set of fields describing the fermions, because the it is the field in question itself, but for bosons we could easily get the number of fields *doubled*, if we include for each field also its conjugate.

Conjugate Momentum Fields NOT to be Included in Counting.

Let us argue that it is enough in the counting to count the number of fields, from which you by Fourier resolution can extract the annihilation and creation operators needed to annihilate or create the particles, the species of which are to be counted:

- Normally we could extract the conjugate field by differentiating w.r.t. to time the field because usually you can replace the fields and their conjugate by the fields and their time derivatives.
- Using equations of motion these time derivatives can in turn be obtained by some way - also some sort of differentiation - from the field itself.
- Thus at the end the information on the conjugate is extractable from the field itself!

Further Support for NOT including also Conjugate Momentum Fields

We could very easily construct linear (or more complicated) combinations of boson fields and their conjugate fields. Such combinations would like the fermion fields typically not commute/anticommute with themselves.

So provided we can extra the particle creation and annihilation operators from the combined field we would have no rule to tell that we should include more. Thus we would need only the combined field, and with that rule have quite analogy to the fermion case.

Meaning of NOT Counting also the Conjugate Field

In QED_3 say $A_1(x)$ and $A_2(x)$ would be enough to represent both longitudinal and transversely polarized photons. It would NOT be needed also to have the essentially conjugate electric fields $E_1(x)$ and $E_2(x)$.

The field $\partial_i E_i$ is in fact the conjugate A_0 so that we - having the symmetry between a field and its conjugate, it being conjugate of its conjugate - can consider that timelike photons are described by this $\partial_i E_i$ field combination.

14.10 Particles

But in terms of Particles, How??

Usually one thinks of electrodynamics in 2+1 dimensions as having only one particle polarisation, since there is only *one* transversely polarisation for a photon. So seemingly only one component of boson. This transversely polarized photon is even its own antiparticle, so even the anti-particle is not new.

On the contrary the fermions after the fermionization counts two complex fields meaning two different fermion components (ψ_1 and ψ_2) each with an a priori different antiparticle in as far as the fields ψ_1 and ψ_2 both are complex(non-Hermitean). That seems NOT to match!

Where have the two missing photon-polarizations gone?

Suggestion for How 3 photons.

To count independently both A_i (i=1,2.) as real fields, we need to consider it that we have not only the transverse photon, but also a **longitudinal photon** !

The third of the real fields $\partial_i E_i = div\vec{E}$ is actually the conjugate variable to the time component $A_0(x)$ of the fourcomponent photon field. So if we take it that conjugate or not does not matter it could correspond to the **timelike polarized photon**.

This would mean that we could hope for interpreting the three photon polarizations as being

- 1) The transverse photon.
- 2) The longitudinal photon.
- 3) The time-like photon.

But the time like photon has wrong signature ?!

Better Suggestion for the 3 particles ?

To avoid the problem with the ltime-like photon form Lorentz invariance having the signature with negative norm square states we can instead take a further scalar. If so we could have 3 bosons corresponding to the four (real) fermions.

In any case if we want a fermion system with positive definite Hilbert space we better have the bosons also give positive definite Hilbert space if they shall match in their Hilbert spaces.

14.11 Fields

How to count Hermitean Boson fields ?

To exercise we shall for the moment even begin with a 1+1 dimensional only right moving Hermitean field constructed as a superposition of momentum state creation $a^\dagger(p)$ and annihilation operators $a(p)$ for say a series discretized momentum values, which we for "elegance"(and later interest) shall take to be odd integers in some unit:

$$\begin{aligned}\phi(x) &= \sum_{p \text{ odd},p>0} \sqrt{p}a(p)\exp(ipx) + \sum_{p \text{ odd},p<0} \sqrt{|p|}a^\dagger(|p|)\exp(ipx) \\ &= \sum_{p \text{ odd}} \sqrt{|p|}a(p),\end{aligned} \tag{14.55}$$

where we have put

$$a(p) = a^\dagger(-p) \text{ for all the odd } p \tag{14.56}$$

Properties of the Hermitean field A Hermitean field of the form (in 1+1 dimension say)

$$\begin{aligned}\phi(x) &= \sum_{p \text{ odd}, p>0} \sqrt{p}\, a(p) \exp{(ipx)} + \sum_{p \text{ odd}, p<0} \sqrt{|p|}\, a^{\dagger}(|p|) \exp{(ipx)} \\ &= \sum_{p \text{ odd}} \sqrt{|p|}\, a(p) \end{aligned} \tag{14.57}$$

obeys

$$\phi(x)^{\dagger} = \phi(x) \text{ (Hermiticity) } and \tag{14.58}$$

$$\begin{aligned}[\phi(x), \phi(y)] &= \sum_{p \text{ odd}} \sum_{p' \text{ odd}} \sqrt{|p|}\sqrt{|p'|}[a(p), a(p')] \exp{(ipx + ip'y)} && (14.59)\\ &= \sum_{p \text{ odd}} p \exp{(ip(x-y))} = 2\pi \frac{d}{id(x-y)} \delta(x-y) && (14.60)\\ &= -i2\pi\partial\delta(x-y) \text{ (local commutation rule).} && (14.61)\end{aligned}$$

14.12 New

New, Reduce the Kovner Kurzepa model.

We claim, that in a way the Kovner and Kurzepa bosonization in 2 + 1 dimensions has included a kind of "funny extra bosonic degree of freedom" the charge density compared to our own plan of doing a completely free model.

Really we want to say: In a truly free electrodynamics "free QED_3" (in 2 +1 dimensions) the divergence of the electric field is zero:

$$\partial_i E_i \approx 0 \text{ (on physical states).} \tag{14.62}$$

When we use $\approx$ instead of $=$ it is because we may need the divergence $\partial_i E_i$ as an operator even though we may take it to be zero on the "physical states".

Reduction of Kovner Kurzepa model w.r.t. degrees of freedom

Inserting formally our claim of a constraint equation

$$\partial_i E_i \approx 0 \text{ (on physical states).} \tag{14.63}$$

into the expressions of Kovner and Kurzepa

$$V_1(x) = -i \exp{\left(\frac{i}{2e} \int (\theta(x-y) - \pi)\partial_i E_i\right)} \tag{14.64}$$

$$U_1(x) = \exp{\left(-\frac{i}{2e} \theta(y-x)\partial_i E_i\right)} \tag{14.65}$$

we get

$$V_1(x) \approx -i \tag{14.66}$$

$$U_1(x) \approx 1. \tag{14.67}$$

Using the constraint equation formally on Kovner and Kurzepa In Kovner and Kurzepa one finds

$$\psi_\alpha(x) = k\Lambda V_\alpha(x)\Phi(x)U_\alpha(x) \tag{14.68}$$

$$\Phi(x) = \exp(ie\int e_i(y-x)A_i(y)d^2y); e_i(y-x) = \frac{y_i - x_i}{(y-x)^2} \tag{14.69}$$

$$V_1(x) = -i\exp(\frac{i}{2e}\int(\theta(x-y)-\pi)\partial_i E_i); V_2(x) = -iV_1^\dagger(x) \tag{14.70}$$

$$U_1(x) = \exp(-\frac{i}{2e}\theta(y-x)\partial_i E_i); U_2(x) = V_1^\dagger(x) \tag{14.71}$$

and thus with the constraint formally included

$$\psi_2(x) \approx i\psi_1(x) \tag{14.72}$$

Our Constraint would Spoil Rotation Symmetry A constraint equation

$$\psi_2(x) \approx i\psi_1(x) \tag{14.73}$$

would *not* be consistent with the rotation symmetry and the transformation property for the fermion field suggested in Kovner and Kurzepa

$$\psi_1 \to \exp(i\phi/2)\psi_1; \psi_2 \to \exp(-i\phi/2)\psi_2. \tag{14.74}$$

So including the constraint would make the bosonization/fermionization become *non-rotational invariant*. But it is our philosophy not to take that as a so serious problem, because it is in any case *impossible* to get in a rotational invariant way spin 1/2 fermions from a purely bosonic theory with only integer spin!

Rotation symmetry broken in reduced model!

14.13 Conclusion

We have extended the previous "Aratyn-Nielsen-thorem" relating the number of degrees of freedom / number of components / number of particle (orthogonal) polarizations for a set of bosons that by bosonization/fermionization is in correspondance with each other. The extension consists in also allowing negative norm square single particle states. We only considered yet the case of massless noninteracting both bosons and fermions, but expect that by thinking of the limit of small distances the relation of the theorem would also have to hold for massive particles. If there existed a common for both bosons and fermions weak interaction limit you would also expect that the noninteraction assumption could be avoided.

The main result is the relation (14.52):

$$N_{t\ ferm\ pos.} + N_{t\ boson\ neg.} = \frac{2^{d_{spatial}}}{2^{d_{spatial}} - 1} * (N_{t\ boson\ pos.} + N_{t\ ferm\ neg.}),$$

where the "normal" boson and fermion component numbers are denoted with $N_{t\ boson\ pos.}$ and $N_{t\ ferm\ pos.}$ respectively for bosons and for fermions, and where the corresponding numbers of components with negative norm square are $N_{t\ boson\ neg.}$ and $N_{t\ ferm\ neg.}$.

We have also looked at some examples where one might apply and test our theorem, but the problem is that we do not know the higher dimensional examples so well. Basically the dimension limit where the examples basically stop is not high. Googling you find mainly at most 2+1. The case 3+1 is very rare.

14.13.1 Outlook Dream

Our motivation, which has not quite ran out to be realized yet is that we shall find in literature or develop bosonization case(s) for the dimensions of interest as dimension of the space time, such as the experimental dimension 3+1 or the in the spin-charge-family theory practical starting dimension 13 +1. That is to say we hope to find a set of boson fields that is equivalent to a set of fermion fields in the bosonization way. If we have a valid theorem as the one we just extended we strictly speaking only need to know one side, i.e. either the bosons or the fermions, because then we can calculate the number of components for the other side. Without the "extension " of our theorem it looks that the number of fermion components must always be a number divisible by $2^{d_{spatial}}$, which e.g. for the case of the experimental dimension is $2^3 = 8$. It makes it especially difficult to avoid the number of families being even, because if we think of Weyl fermions at least and even count real components so that we get twice as many as if we used complex components, we still need a multiplum of 2 families of Weyl particle. With Dirac fermions we could use up a factor 2 more and we would get no prediction than just the number of families being integer. But in the Standard model we know that we have the weak interactions and the components put together to Dirac fermions have separate gauge quantum numbers are are hardly suitable for coming from the same fermionization.

With an extended theorem relating the two sides fermions and bosons, however, the situation gets less clear and the hope for even getting somehow a phenomenologically good number is not excluded yet.

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