Small cycles in the Pancake graph

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Abstract

The Pancake graph is well known because of the open Pancake problem. It has the structure that any l–cycle, $6 \leqslant l \leqslant n!$, can be embedded in the Pancake graph $P_n, n \geqslant 3$. Recently it was shown that there are exactly $\frac{n!}{6}$ independent 6–cycles and $n!(n-3)$ distinct 7–cycles in the graph. In this paper we characterize all distinct 8–cycles by giving their canonical forms as products of generating elements. It is shown that there are exactly $\frac{n!(n^3+12n^2-103n+176)}{16}$ distinct 8–cycles in $P_n, n \geqslant 4$. A maximal set of independent 8–cycles contains $\frac{n!}{8}$ of these.

Keywords: Cayley graphs, Pancake graph, cycle embedding, small cycles.

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1 Introduction

The Pancake graph $P_n = (Sym_n, PR), n \geqslant 2$, is the Cayley graph on the symmetric group Sym_n of permutations $\pi = [\pi_1 \pi_2 \ldots \pi_n]$, where $\pi_i = \pi(i)$ for any $1 \leqslant i \leqslant n$, with the generating set $PR = \{r_i \in Sym_n : 2 \leqslant i \leqslant n\}$ of all prefix–reversals r_i reversing the order of any substring $[1, i], 2 \leqslant i \leqslant n$, of a permutation π when multiplied on the right, i.e. $[\pi_1 \ldots \pi_i \pi_{i+1} \ldots \pi_n]r_i = [\pi_i \ldots \pi_1 \pi_{i+1} \ldots \pi_n]$. It is a connected vertex–transitive $(n-1)$-regular graph of order $n!$. This graph is well known because of the combinatorial *Pancake problem* which was posed in [4] as the problem of finding its diameter. The problem is still

open. Some upper and lower bounds [5, 6] as well as exact values for $2 \leqslant n \leqslant 19$ [1, 2] are known. One of the main difficulties in solving this problem is the complicated cycle structure of the Pancake graph. This graph is hamiltonian [11] and the following result is also known.

Theorem 1.1. *[7, 10] All cycles of length l, where $6 \leqslant l \leqslant n!$, can be embedded in the Pancake graph $P_n, n \geqslant 3$, but there are no cycles of length $3, 4$ or 5.*

An explicit description of cycles is gradually being developed. The first results concerning cycle characterization in the Pancake graph were obtained in [8] where the following cycle representation via a product of generating elements was used. A sequence of prefix–reversals $C_l = r_{i_0} \ldots r_{i_{l-1}}$, where $2 \leqslant i_j \leqslant n$, and $i_j \neq i_{j+1}$ for any $0 \leqslant j \leqslant l-1$, such that $\pi r_{i_0} \ldots r_{i_{l-1}} = \pi$, where $\pi \in Sym_n$, is called *a form of a cycle C_l of length l*. A cycle C_l of length l is also called an *l–cycle*, and a vertex of P_n is identified with the permutation which corresponds to this vertex. It is evident that any l–cycle can be presented by $2l$ forms (not necessarily different) with respect to a vertex and a direction. *The canonical form C_l of an l–cycle* is called a form with a lexicographically maximal sequence of indices $i_0 \ldots i_{l-1}$. By using this description, the results characterizing 6– and 7–cycles were obtained in [8].

Theorem 1.2. *[8] The Pancake graph $P_n, n \geqslant 3$, has $\frac{n!}{6}$ independent 6–cycles of the canonical form $C_6 = r_3 r_2 r_3 r_2 r_3 r_2$ and $n!(n-3)$ distinct 7–cycles of the canonical form $C_7 = r_k r_{k-1} r_k r_{k-1} r_{k-2} r_k r_2$, where $4 \leqslant k \leqslant n$. Each of the vertices of P_n belongs to exactly one 6–cycle and $7(n-3)$ distinct 7–cycles.*

The main result of this paper is the following theorem.

Theorem 1.3. *Each of vertices of $P_n, n \geqslant 4$, belongs to $N = \frac{n^3 + 12n^2 - 103n + 176}{2}$ distinct 8–cycles of the following canonical forms:*

$$C_8^1 = r_k \, r_j \, r_i \, r_j \, r_k \, r_{k-j+i} \, r_i \, r_{k-j+i}, \qquad 2 \leqslant i < j \leqslant k-1, \ 4 \leqslant k \leqslant n, \quad (1.1)$$

$$C_8^2 = r_k \, r_{k-1} \, r_2 \, r_{k-1} \, r_k \, r_2 \, r_3 \, r_2, \qquad\qquad\qquad\qquad 4 \leqslant k \leqslant n, \quad (1.2)$$

$$C_8^3 = r_k \, r_{k-i} \, r_{k-1} \, r_i \, r_k \, r_{k-i} \, r_{k-1} \, r_i, \qquad 2 \leqslant i \leqslant k-2, \ 4 \leqslant k \leqslant n, \quad (1.3)$$

$$C_8^4 = r_k \, r_{k-i+1} \, r_k \, r_i \, r_k \, r_{k-i} \, r_{k-1} \, r_{i-1}, \qquad 3 \leqslant i \leqslant k-2, \ 5 \leqslant k \leqslant n, \quad (1.4)$$

$$C_8^5 = r_k \, r_{k-1} \, r_{i-1} \, r_k \, r_{k-i+1} \, r_{k-i} \, r_k \, r_i, \qquad 3 \leqslant i \leqslant k-2, \ 5 \leqslant k \leqslant n, \quad (1.5)$$

$$C_8^6 = r_k \, r_{k-1} \, r_k \, r_{k-i} \, r_{k-i-1} \, r_k \, r_i \, r_{i+1}, \qquad 2 \leqslant i \leqslant k-3, \ 5 \leqslant k \leqslant n, \quad (1.6)$$

$$C_8^7 = r_k \, r_{k-j+1} \, r_k \, r_i \, r_k \, r_{k-j+1} \, r_k \, r_i, \qquad 2 \leqslant i < j \leqslant k-1, \ 4 \leqslant k \leqslant n, \quad (1.7)$$

$$C_8^8 = r_4 \, r_3 \, r_4 \, r_3 \, r_4 \, r_3 \, r_4 \, r_3. \qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.8)$$

There are two corollaries of Theorem 1.3, which will be proven in the final section of this paper.

Corollary 1.4. *There are $\frac{n!(n^3 + 12n^2 - 103n + 176)}{16}$ distinct 8–cycles in $P_n, n \geqslant 4$.*

Corollary 1.5. *A maximal set of independent 8–cycles in $P_n, n \geqslant 4$, contains $\frac{n!}{8}$ of these.*

The proof of Theorem 1.3 is based on the hierarchical (recursive) structure of the Pancake graph which can be presented as follows. The graph $P_n, n \geqslant 3$, is constructed

from n copies of $P_{n-1}(i), 1 \leqslant i \leqslant n$, such that each $P_{n-1}(i)$ has the vertex set $V_i = \{[\pi_1 \ldots \pi_{n-1}i], \text{ where } \pi_k \in \{1, \ldots, n\} \backslash \{i\} : 1 \leqslant k \leqslant n-1\}, |V_i| = (n-1)!$, and the edge set $E_i = \{\{[\pi_1 \ldots \pi_{n-1}i], [\pi_1 \ldots \pi_{n-1}i] r_j\} : 2 \leqslant j \leqslant n-1\}, |E_i| = \frac{(n-1)!(n-2)}{2}$. Any two copies $P_{n-1}(i), P_{n-1}(j), i \neq j$, are connected by $(n-2)!$ edges presented as $\{[i\pi_2 \ldots \pi_{n-1}j], [j\pi_{n-1} \ldots \pi_2 i]\}$, where $[i\pi_2 \ldots \pi_{n-1}j]r_n = [j\pi_{n-1} \ldots \pi_2 i]$. Prefix–reversals $r_j, 2 \leqslant j \leqslant n-1$, define *internal edges* in all n copies $P_{n-1}(i), 1 \leqslant i \leqslant n$, and the prefix–reversal r_n defines *external edges* between copies. Copies $P_{n-1}(i)$, or just P_{n-1} when it is not important to specify the last element of permutations belonging to the copy, are also called $(n-1)$–*copies*.

Since $P_3 \cong C_6$ and due to the hierarchical structure, P_4 has four copies of P_3, each of which obviously cannot contain 8–cycles. However, P_4 has 8–cycles consisting of paths within copies of P_3 together with external edges between these copies. In general, any 8–cycle of $P_n, n \geqslant 4$, must consist of paths within subgraphs that are isomorphic to P_{k-1} for some $4 \leqslant k \leqslant n$, joined by external edges between these subgraphs. Hence, all 8–cycles of $P_n, n \geqslant 4$, could be found recursively by considering 8–cycles within each $P_k, 4 \leqslant k \leqslant n$, consisting of vertices from some copies of P_{k-1}. This approach is used in the proof of Theorem 1.3. To get the main result, we also need some technical lemmas concerning paths of length three between vertices of a given form. So, in the next section we introduce additional notations and prove two small lemmas. In Section 3 we prove Theorem 1.3 and its corollaries.

2 Technical lemmas

A segment $[\pi_i \ldots \pi_j]$ of a permutation $\pi = [\pi_1 \ldots \pi_i \ldots \pi_j \ldots \pi_n]$ consists of all elements that lie between π_i and π_j inclusive. Any permutation can be written as a sequence of singleton and multiple segments. We use characters from $\{p, q, s, t\}$ to denote singletons and characters from $\{\alpha, \beta, \gamma, A, B, C\}$ to denote multiple segments. If $\pi = [\alpha\beta]$, where $\alpha = [\pi_1\pi_2 \ldots \pi_i]$ and $\beta = [\pi_{i+1} \ldots \pi_n]$, then $\pi r_{|\alpha|} = [\overline{\alpha}\beta]$, where $|\alpha|$ is the number of elements in a segment α, and $\overline{\alpha}$ is the inversion of a segment α. It is obvious that $\overline{\overline{\alpha}} = \alpha$. Note that we allow empty segments where this does not contradict the initial definitions.

An independent set D of vertices in a graph is called an *efficient dominating set* if each vertex not in D is adjacent to exactly one vertex in D [3]. It is known [9] that $D_p = \{[p\pi_2 \ldots \pi_n] : \pi_j \in \{1, \ldots, n\} \backslash \{p\}, 2 \leqslant j \leqslant n\}, |D_p| = (n-1)!, 1 \leqslant p \leqslant n$, are efficient dominating sets in $P_n, n \geqslant 3$. Let us note that external edges of P_n are incident to vertices from different efficient dominating sets of P_n. The *distance* $d = d(\pi, \tau)$ between two vertices π, τ in P_n is defined as the least number of prefix–reversals transforming π into τ, i.e. $\pi r_{i_1} r_{i_2} \ldots r_{i_d} = \tau$.

The next lemma gives a full list of paths of length three between two vertices of the same efficient dominating set.

Lemma 2.1. *Two permutations $\pi, \tau \in D_p, 1 \leqslant p \leqslant n$, are at distance three from each other in $P_n, n \geqslant 3$, if and only if:*
1) either $\tau = \pi r_j r_i r_j, 2 \leqslant i < j \leqslant n$, where $\pi = [AB\gamma]$, $\tau = [A\overline{B}\gamma]$;
2) or $\tau = \pi r_j r_i r_{i-j+1}, 2 \leqslant j < i \leqslant n$, where $\pi = [pAB\gamma]$, $\tau = [pBA\gamma]$.

Proof. We consider $\pi \in D_p$ such that $\pi = [p\alpha q\beta k], \pi_j = q$. Let us find other vertices from D_p being at the distance three from π. Let $\pi^1 = \pi r_j = [q\overline{\alpha}p\beta k]$, where $\pi_j^1 = p, 2 \leqslant j \leqslant n$. An application of a prefix–reversal $r_i, 2 \leqslant i \leqslant n, i \neq j$, to the permutation π^1 gives us two cases: either $i < j$ or $i > j$.

1) If $i < j$ then $\pi^2 = \pi^1 r_i = [\alpha_2 q \overline{\alpha_1} p \beta k]$, where $\pi_j^2 = p$, $\alpha = \alpha_1 \alpha_2$ and $|\alpha_2| = i - 1$, and then $\tau = \pi^2 r_j = [p\alpha_1 q \overline{\alpha_2} \beta k]$. Hence $\tau = \pi r_j r_i r_j$ and we get $\pi = [AB\gamma]$, $\tau = [A\overline{B}\gamma]$ by setting $A = p\alpha_1$, $B = \alpha_2 q$, $\gamma = \beta k$. Note, that using r_j is the only way to restore p to the first position and thus to end at an element of D_p after reaching π^2.

2) If $i > j$ then $\pi^2 = \pi^1 r_i = [\overline{\beta_1} p \alpha q \beta_2 k]$, where $\pi_{i-j+1}^2 = p$, $\beta = \beta_1 \beta_2$ and $|\beta_1| = i - j$, and then $\tau = \pi^2 r_{i-j+1} = [p\beta_1 \alpha q \beta_2 k]$. Hence $\tau = \pi r_j r_i r_{i-j+1}$ and we get $\pi = [pAB\gamma]$, $\tau = [pBA\gamma]$ by setting $A = \alpha q$, $B = \beta_1$, $\gamma = \beta_2 k$. Note, that using r_{i-j+1} is the only way to restore p to the first position and thus to end at an element of D_p after reaching π^2. $\square$

The next lemma gives a description of paths of length three defined on internal edges of the graph between vertices of given forms.

Lemma 2.2. *Let permutations π and τ be presented as:*
1) $\pi = [\gamma_1 AB\gamma_2]$ and $\tau = [\gamma_1 \overline{A}B\gamma_2]$, where $|\gamma_1| \geqslant 1$, $|A| \geqslant 2$. Then:
*   a) there exists a unique path of length three:*

$$\tau = \pi r_{|\gamma_1|+|A|} r_{|A|} r_{|\gamma_1|+|A|}, \tag{2.1}$$

provided that either $|\gamma_1| \geqslant 2$ and $|A| \geqslant 2$, or $|\gamma_1| = 1$ and $|A| \geqslant 3$;
*   b) there are two paths of length three:*

$$\tau = \pi r_2 r_3 r_2, \quad \tau = \pi r_3 r_2 r_3, \tag{2.2}$$

provided that $|\gamma_1| = 1$ and $|A| = 2$;
2) $\pi = [\gamma_1 AB\gamma_2]$ and $\tau = [\gamma_1 BA\gamma_2]$, where $|\gamma_1| \geqslant 0$, $|A| \geqslant 1$, $|B| \geqslant 1$. Then:
*   a) there is a unique path of length three:*

$$\tau = \pi r_{|\gamma_1|+2} r_2 r_{|\gamma_1|+2}, \tag{2.3}$$

provided that $|\gamma_1| \geqslant 2$, and $|A| = |B| = 1$;
*   b) there is a unique path of length three:*

$$\tau = \pi r_{|\gamma_1|+|A|} r_{|\gamma_1|+|A|+|B|} r_{|\gamma_1|+|B|}, \tag{2.4}$$

provided that $|\gamma_1| = 1$, and $|A| \neq 1$ or $|B| \neq 1$;
*   c) there are two paths of length three:*

$$\tau = \pi r_2 r_3 r_2 = \pi r_3 r_2 r_3, \tag{2.5}$$

provided that $|\gamma_1| = |A| = |B| = 1$;
*   d) there is a unique path of length three:*

$$\tau = \pi r_{|A|} r_{|A|+|B|} r_{|B|}, \tag{2.6}$$

provided that $|\gamma_1| = 0$ and $|A| \geqslant 2$, $|B| \geqslant 2$.
There are no other paths of length three between π and τ of these types.

Proof. 1) If $\pi = [\gamma_1 A B \gamma_2]$ and $\tau = [\gamma_1 \overline{A} B \gamma_2]$, then (2.1) is checked by a direct verification: $[\gamma_1 A B \gamma_2] \overset{r_{|\gamma_1|+|A|}}{\to} [\overline{A \gamma_1} B \gamma_2] \overset{r_{|A|}}{\to} [A \overline{\gamma_1} B \gamma_2] \overset{r_{|\gamma_1|+|A|}}{\to} [\gamma_1 \overline{A} B \gamma_2]$. Suppose that there is one more path of length three. Then these two paths should form a 6–cycle. In part (a), either $|\gamma_1| \geqslant 2$ and $|A| \geqslant 2$, or $|\gamma_1| = 1$ and $|A| \geqslant 3$, so $r_{|\gamma_1|+|A|} = r_m$ for some $m \geqslant 4$, but by Theorem 1.2, no 6–cycle includes r_m with $m \geqslant 4$ in its form. Therefore, the given path is the only one in this case. In part (b), $|\gamma_1| = 1$ and $|A| = 2$, so $m = 3$ and the condition of Theorem 1.2 holds, hence we obtain two distinct paths of stated forms (2.2).

2) If $\pi = [\gamma_1 A B \gamma_2]$ and $\tau = [\gamma_1 B A \gamma_2]$, and $|\gamma_1| \geqslant 2$, $|A| \geqslant 1$, $|B| \geqslant 1$, then there is the following path of length four between these vertices:

$$\pi = [\gamma_1 A B \gamma_2] \overset{r_{|\gamma_1|+|A|}}{\to} [\overline{A \gamma_1} B \gamma_2] \overset{r_{|\gamma_1|+|A|+|B|}}{\to} [\overline{B} \gamma_1 A \gamma_2] \overset{r_{|\gamma_1|+|B|}}{\to}$$

$$[\overline{\gamma_1} B A \gamma_2] \overset{r_{|\gamma_1|}}{\to} [\gamma_1 B A \gamma_2] = \tau. \tag{2.7}$$

Suppose there is also a path of length three between π and τ. By Theorem 1.1, there are no cycles of length 3 or 5, and hence no paths of lengths 3 and 4 exist between two fixed vertices unless the paths are disjoint. This means that these two paths should form a 7–cycle, including the sequence $r_{m+a} r_{m+a+b} r_{m+b} r_m$, where $|\gamma_1| = m$, $|A| = a$ and $|B| = b$. By Theorem 1.2, this is possible only in the case when $m = k - 2$, $k \geqslant 4$, and $a = b = 1$, which implies that a unique path of length three has the form $r_{m+2} r_2 r_{m+2}$ that corresponds to (2.3).

Putting $|\gamma_1| = 1$ in (2.7), a path $\tau = \pi r_{|\gamma_1|+|A|} r_{|\gamma_1|+|A|+|B|} r_{|\gamma_1|+|B|}$, corresponding to (2.4), is obtained. Taking $|\gamma_1| = m$, $|A| = a$ and $|B| = b$, the obtained path is presented as $r_{m+a} r_{m+a+b} r_{m+b}$. Suppose that there is one more path of length three between π and τ. Then these two paths should form a 6–cycle. By Theorem 1.2, this is possible only in the case when $m = a = b = 1$, which gives us the paths $\tau = \pi r_2 r_3 r_2$ and $\tau = \pi r_3 r_2 r_3$, corresponding to (2.5).

Putting $|\gamma_1| = 0$ in (2.7), a path $\tau = \pi r_{|A|} r_{|A|+|B|} r_{|B|}$, corresponding to (2.6) with $|A| \geqslant 2$, $|B| \geqslant 2$, is obtained. Suppose there is one more path of length three between π and τ. Then these two paths should form a 6–cycle. By the conditions of Lemma, $|A| + |B| \geqslant 4$, hence $r_{|A|+|B|} = r_m$ for some $m \geqslant 4$, but by Theorem 1.2, no 6–cycle includes r_m with $m \geqslant 4$ in its form. Therefore, the given path is the only one in this case. If $|A| = 1$ or $|B| = 1$, then the path above is transformed into a 2–path or an edge. This completes the proof of the lemma. $\square$

3 Proof of Theorem 1.3

To find all 8–cycles passing through the same vertex in P_n, $n \geqslant 4$, we use its hierarchical structure by considering recursively 8–cycles within each copy P_k, $4 \leqslant k \leqslant n$, consisting of vertices from copies of P_{k-1}. It is assumed that any copy of P_{k-1} has at least two vertices, since each vertex has a unique external edge. We obtain canonical forms of 8–cycles and count their numbers.

Case 1: an 8–cycle within P_k has vertices from two copies of P_{k-1}

Suppose that an 8–cycle is formed on vertices from copies $P_{k-1}(p)$ and $P_{k-1}(s)$, $1 \leqslant p \neq s \leqslant k$. It was shown in [8] that if two vertices π and τ, belonging the same $(k-1)$–copy, are at the distance at most two, then their external neighbours $\overline{\pi}$ and $\overline{\tau}$ should belong to distinct $(k-1)$–copies. Hence, an 8–cycle cannot occur in situations when its two (three)
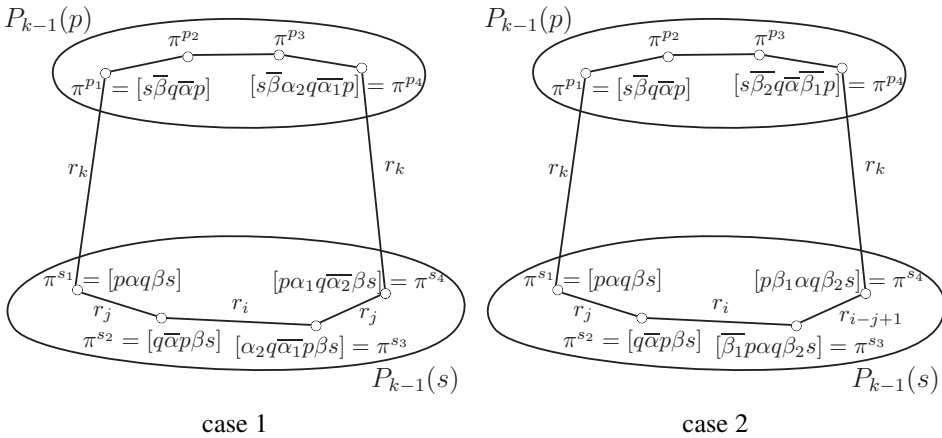
Figure 1: $(4 + 4)$–situation.

vertices belong to one copy and six (five) vertices belong to another one. Therefore, such an 8–cycle must have four vertices in each of the two copies.

$(4 + 4)$–situation. Suppose that four vertices of such an 8–cycle belong to a copy $P_{k-1}(s)$, and other four vertices belong to a copy $P_{k-1}(p)$. Herewith, four vertices of $P_{k-1}(s)$ should form a path of length three whose endpoints should be adjacent to vertices from $P_{k-1}(p)$, which means both vertices should belong to the efficient dominating set D_p. So, one vertex of $P_{k-1}(s)$ that is adjacent to a vertex of $P_{k-1}(p)$ must have the form $[p\alpha q\beta s]$. By Lemma 2.1, it is not hard to see that this gives rise to two possible forms for the remaining vertices of $P_{k-1}(s)$. These are given in Figure 1, where $|\alpha| = j - 2$ and so $|\beta| = k - j - 1$. In the first case we also have $\alpha = \alpha_1\alpha_2$ and $|\alpha_2| = i - 1 \geqslant 1$, while in the second case we also have $|\beta_1| = i - j \geqslant 1$ and $|\beta_2| = k - i - 1$.

Denote $\gamma_1 = s\overline{\beta}$, $A = q\overline{\alpha_2}$, $B = \alpha_1$, $\gamma_2 = p$, where $|\gamma_1| = |\beta| + 1 \geqslant 1$, $|A| \geqslant 2$, $|B| \geqslant 0$, then in the first case π^{p_1}, π^{p_4} have the forms $[\gamma_1 A B \gamma_2]$ and $[\gamma_1 \overline{A} B \gamma_2]$. By Lemma 2.2 (case 1a), there is a unique path of length three between these permutations if $|\gamma_1| = |\beta| + 1 = k - j \geqslant 1$ and $|A| = |\alpha_2| + 1 = i \geqslant 3$, or $k - j \geqslant 2$ and $i \geqslant 2$, and by Lemma 2.2 (case 1b), there are two distinct paths if $k - j = 1$ and $i = 2$. Hence, such an 8–cycle has the form $C_8^1 = r_{k-j+i} r_i r_{k-j+i} r_k r_j r_i r_j r_k$, with $2 \leqslant i < j \leqslant k - 1$, $4 \leqslant k \leqslant n$, the canonical form of which corresponds to (1.1). The case of $k - j = 1$ and $i = 2$ by symmetry gives one additional form $C_8^2 = r_{k-1} r_2 r_{k-1} r_k r_2 r_3 r_2 r_k$, the canonical form of which corresponds to (1.2).

Denote $\gamma_1 = s\overline{\beta_2}$, $A = \overline{\beta_1}$, $B = q\overline{\alpha}$, $\gamma_2 = p$, where $|\gamma_1| = |\beta_2| + 1 \geqslant 1$, $|A| = |\beta_1| \geqslant 1$, $|B| = |\alpha| + 1 \geqslant 1$, then in the second case we have $\pi^{p_1} = [\gamma_1 A B \gamma_2]$, $\pi^{p_4} = [\gamma_1 B A \gamma_2]$. By Lemma 2.2 (case 2a), there is a unique path of length three between π^{p_1} and π^{p_4} if $|\gamma_1| = k - i \geqslant 2$, $|A| = |\beta_1| = i - j = 1$ and $|B| = |\alpha| + 1 = j - 1 = 1$. Hence, $j = 2$, $i = 3$, and for $k \geqslant 5$ an 8–cycle has the form $r_{k-1} r_2 r_{k-1} r_k r_2 r_3 r_2 r_k$, corresponding again to the canonical form (1.2).

By Lemma 2.2 (case 2b), there also exists a unique path of length three between π^{p_1} and π^{p_4} if $|\gamma_1| = k - i = 1$, $|A| = |\beta_1| = i - j \geqslant 1$, $|B| = |\alpha| + 1 = j - 1 \geqslant 1$. This means that $i = k - 1$, and such an 8–cycle has the form $C_8^4 = r_k r_{k-j} r_{k-1} r_j r_k r_{k-j} r_{k-1} r_j$, $2 \leqslant j \leqslant k - 2$, the canonical form of which corresponds to (1.3), if we set $j = i$. So, there
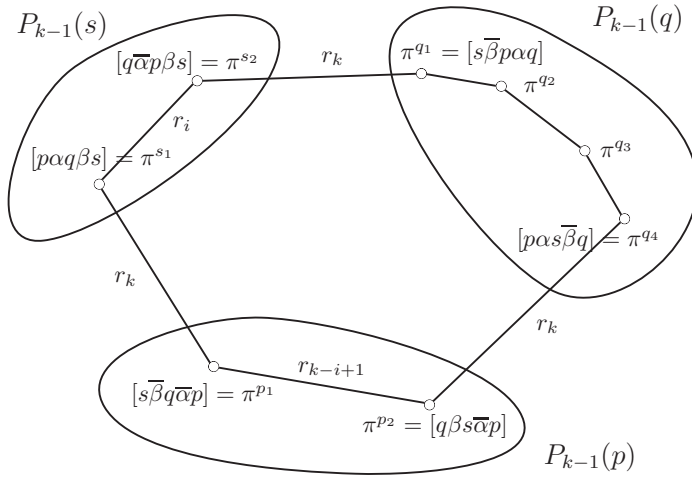
Figure 2: $(2 + 2 + 4)$–situation.

is a unique path of length three under the conditions listed, unless $|A| = |B| = 1$ when by Lemma 2.2 (case 2c) this path is not unique. So, $k = 4, j = 2, i = 3$ and 8–cycles take forms $r_2 r_3 r_2 r_4 r_3 r_2 r_3 r_4$ and $r_3 r_2 r_3 r_4 r_3 r_2 r_3 r_4$, corresponding to forms (1.2) and (1.1).

Thus, all 8–cycles occurring in the case of two copies are found.

Case 2: an 8–cycle within P_k has vertices from three copies of P_{k-1}

Suppose an 8–cycle is formed on vertices from copies $P_{k-1}(p)$, $P_{k-1}(q)$, $P_{k-1}(s)$, where $1 \leqslant p \neq q \neq s \leqslant k$. There are following possible situations in this case.

$(2 + 2 + 4)$–situation. The distribution of vertices among the copies is presented by Figure 2. Let $\pi^{s_1} = [p\alpha q\beta s]$ where $\pi_i^{s_1} = q$ with $|\alpha| = i - 2$, $|\beta| = k - i - 1$. Then $\pi^{s_2}, \pi^{p_1}, \pi^{p_2}, \pi^{q_1}$ and π^{q_4} are straightforward to define. Vertices π^{q_1} and π^{q_4} differ in the order of segments $s\overline{\beta}$ and $p\alpha$, hence they have the forms $[\gamma_1 A B \gamma_2]$ and $[\gamma_1 B A \gamma_2]$, where γ_1 is empty, $A = s\overline{\beta}$, $B = p\alpha$, $\gamma_2 = q$ and $|A| = |\beta| + 1 \geqslant 1$, $|B| = |\alpha| + 1 \geqslant 1$. By Lemma 2.2 (case 2d), between π^{q_1} and π^{q_4} there exists a unique path of length three provided that $|A| = k - i \geqslant 2$, $|B| = i - 1 \geqslant 2$, and no path of this length if $|A| = 1$ or $|B| = 1$. Thus, an 8–cycle has the form $C_8^4 = r_k r_{k-i+1} r_k r_{i-1} r_{k-1} r_{k-i} r_k r_i$, where $3 \leqslant i \leqslant k - 2$, $k \geqslant 5$, the canonical form of which corresponds to (1.4).

$(2 + 3 + 3)$–situation. The distribution of vertices among the copies is presented by Figure 3. Let $\pi^{s_1} = [p\alpha q\beta s]$, where $|\alpha| = i - 2$ and $|\beta| = k - i - 1$. Then π^{s_2}, π^{p_1} and π^{q_1} are straightforward to define. Since π^{p_3} and π^{q_3} are joined by an external edge, $\pi_1^{p_3} = q$. Moreover, π^{p_1} and π^{p_3} should be joined by a path of length two that can be obtained by two ways:

$$\pi^{p_1} = [s\overline{\beta} q\overline{\alpha} p] \rightarrow \begin{cases} [\beta_2 s\overline{\beta_1} q\overline{\alpha} p] \rightarrow [q\beta_1 s\overline{\beta_2}\overline{\alpha} p] = \pi^{p_3^1}, & \text{where } |\beta_2| \neq 0. \\ [\alpha_2 q\beta s\overline{\alpha_1} p] \rightarrow [q\overline{\alpha_2}\beta s\overline{\alpha_1} p] = \pi^{p_3^2}, & \text{where } |\alpha_2| \neq 0. \end{cases}$$

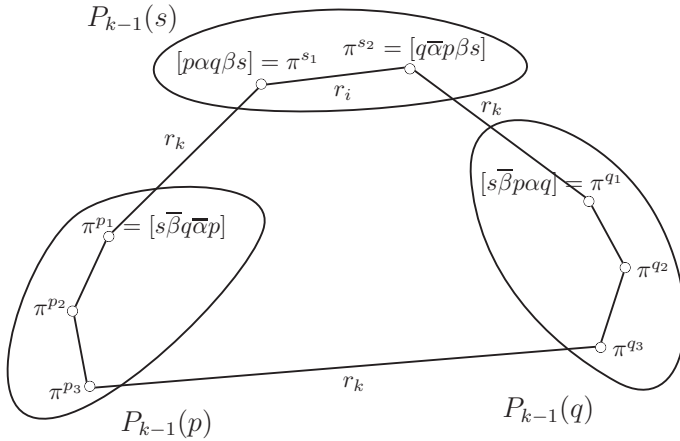From the other side, π^{q_3} and π^{p_3} are joined by an external edge, hence $\pi_1^{q_3} = p$, and there

Figure 3: $(2 + 3 + 3)$–situation.

should be a path of length two between π^{q_1} and π^{q_3} such that:

$$\pi^{q_1} = [s\overline{\beta}p\alpha q] \rightarrow \begin{cases} [\beta_2 s\overline{\beta_1}p\alpha q] \rightarrow [p\beta_1 s\overline{\beta_2}\alpha q] = \pi^{q_3^1}, & \text{where } |\beta_2| \neq 0. \\ [\overline{\alpha_1}p\beta s\alpha_2 q] \rightarrow [p\alpha_1\beta s\alpha_2 q] = \pi^{q_3^2}, & \text{where } |\alpha_1| \neq 0. \end{cases}$$

Analysis of non–empty segments in these permutations shows that external edges occur between: $\pi^{p_3^1}$ and $\pi^{q_3^2}$, if $|\alpha_2| = 0$, $|\beta_1| = 0$; $\pi^{p_3^2}$ and $\pi^{q_3^1}$, if $|\alpha_1| = 0$, $|\beta_1| = 0$; $\pi^{p_3^2}$ and $\pi^{q_3^2}$, if $|\beta| = 0$. There is no external edge between $\pi^{p_3^1}$ and $\pi^{q_3^1}$ since they have the same order of elements in the segment $s\overline{\beta_2}$.

Since $|\alpha| = i - 2$, $|\beta| = k - i - 1$, then using the edge between $\pi^{p_3^1}$ and $\pi^{q_3^2}$, where $|\alpha_2| = 0$, $|\beta_1| = 0$, we have $|\alpha| \geqslant 1$, $|\beta| \geqslant 1$, and such an 8–cycle has the form $C_8^5 = r_k r_{k-i} r_{k-i+1} r_k r_{i-1} r_{k-1} r_k r_i$, with $3 \leqslant i \leqslant k - 2$, $5 \leqslant k \leqslant n$, the canonical form of which corresponds to (1.5). Using the external edge between $\pi^{p_3^1}$ and $\pi^{q_3^1}$, where $|\alpha_1| = 0$, $|\beta_1| = 0$, we have $|\alpha| \geqslant 1$, $|\beta| \geqslant 1$, and such an 8–cycle has the form $r_k r_{k-1} r_{i-1} r_k r_{k-i+1} r_{k-i} r_k r_i$, where $3 \leqslant i \leqslant k - 2$, the canonical form of which also corresponds to the form (1.5). Finally, using the external edge between $\pi^{p_3^2}$ and $\pi^{q_3^2}$, where $|\beta| = 0$, we have $i = k - 1$, $|\alpha_1| = j \geqslant 1$, $|\alpha_2| = k - 3 - j \geqslant 1$, so there is one more 8–cycle of the form $C_8^6 = r_k r_{k-j-1} r_{k-j-2} r_k r_{j+1} r_{j+2} r_k r_{k-1}$, where $1 \leqslant j \leqslant k - 4$, $5 \leqslant k \leqslant n$, the canonical form of which corresponds to (1.6), if we put $j = i - 1$.

Thus, all 8–cycles occurring in the case of three copies are found.

Case 3: an 8–cycle within P_k has vertices from four copies of P_{k-1}

The distribution of vertices among the copies is presented by Figure 4. Let $\pi^{q_1} = [s\alpha t\beta p\gamma q]$, where $|\alpha| \geqslant 0$, $|\beta| \geqslant 0$, $|\gamma| \geqslant 0$. There are two cases.

1) Suppose that π^{q_1} is adjacent to π^{s_1}, and π^{q_2} is adjacent to π^{t_1}. Since there is only one cycle edge within each copy, hence this edge is uniquely defined and all vertices' labels are straightforward to obtain (see Figure 4, case 1). Thus, we end up with $\pi^{p_1} = [s\alpha t\beta q\overline{\gamma}p]$, $\pi^{p_2} = [t\overline{\alpha}s\beta q\overline{\gamma}p]$. If an 8–cycle does exist, then π^{p_1}, π^{p_2} should be incident to the same
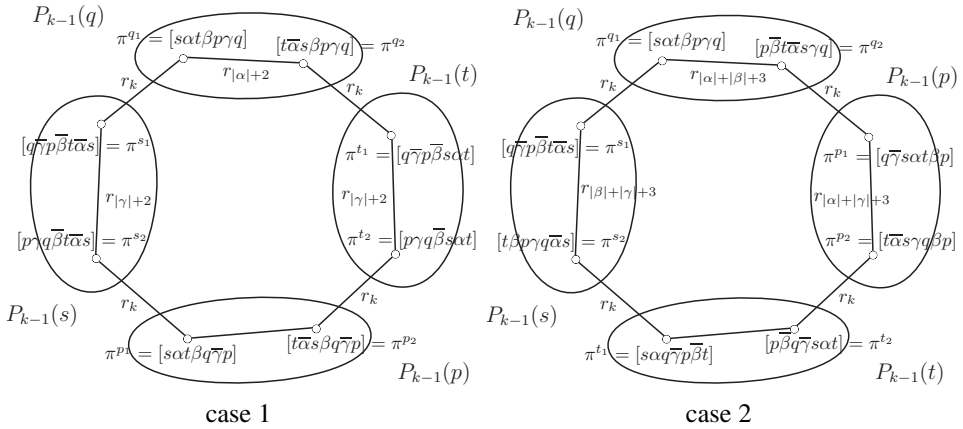
Figure 4: $(2 + 2 + 2 + 2)$–situation.

internal edge, and hence, there should exist a prefix–reversal transforming π^{p_1} into π^{p_2}, namely, $r_{|\alpha|+2}$. If we set $|\alpha| = i-2$, $|\beta| = j-i-1$, $|\gamma| = k-j-1$, where $2 \leqslant i < j < k$, then such an 8–cycle is presented by the form $C_8^7 = r_k r_{k-j+1} r_k r_i r_k r_{k-j+1} r_k r_i$, where $2 \leqslant i < j \leqslant k - 1$, $4 \leqslant k \leqslant n$, the canonical form of which corresponds to (1.7).

2) Suppose that π^{q_1} is adjacent to π^{s_1}, and π^{q_2} is adjacent to π^{p_1} (see Figure 4, case 2), then we end up with $\pi^{t_1} = [s\alpha q\overline{\gamma}p\overline{\beta}t]$, $\pi^{t_2} = [p\overline{\beta}q\overline{\gamma}s\alpha t]$. In this case, an internal edge between vertices π^{t_1} and π^{t_2} does exist only if $|\alpha| = |\beta| = |\gamma| = 0$, which means that $k = 4$ and such an 8–cycle takes the form (1.8).

Therefore, all canonical forms for 8–cycles in $P_n, n \geqslant 4$, are obtained.

Now we count the total number $N = \sum_{i=1}^{8} N_{C_8^i}$ of distinct 8–cycles passing through a given vertex, where $N_{C_8^i}$ corresponds to the number of distinct 8–cycles described by the canonical form $C_8^i, 1 \leqslant i \leqslant 8$. Let us note that any canonical form of an l–cycle describes l cycles (not necessarily distinct) for a given vertex. Among all canonical forms (1.1)–(1.8), there is the only one, namely the form (1.5), which describes eight distinct 8–cycles. In other cases, identical forms occur. For example, from the canonical form $C_8^8 = r_4 r_3 r_4 r_3 r_4 r_3 r_4 r_3$ one can get two forms, namely, $r_4 r_3 r_4 r_3 r_4 r_3 r_4 r_3$ and $r_3 r_4 r_3 r_4 r_3 r_4 r_3 r_4$ which are identical because they describe the same 8–cycle. Thus, the canonical form C_8^8 gives the only 8–cycle, hence, $N_{C_8^8} = 1$. In other cases, it can be shown in the same manner (by taking into account identical forms) that the numbers $N_{C_8^i}, 1 \leqslant i \leqslant 7$, are given as follows: $N_{C_8^1} = \frac{(n-3)(n-2)(n-1)}{3}$, $N_{C_8^2} = 4(n-3)$, $N_{C_8^3} = (n-2)(n-3)$, $N_{C_8^4} = N_{C_8^6} = 2(n-3)(n-4)$, $N_{C_8^5} = 4(n-3)(n-4)$, $N_{C_8^7} = \frac{(n-3)(n-2)(n-1)}{6}$. Thus, the total number is given by

$$N = \frac{n^3 + 12n^2 - 103n + 176}{2},$$

which completes the proof of the theorem. $\qquad\square$

The total number of distinct 8–cycles in $P_n, n \geqslant 4$, is given by $\frac{n!(n^3+12n^2-103n+176)}{16}$ since there are $n!$ vertices in the graph each of which belongs to N distinct 8–cycles. This proves Corollary 1.4.

A maximal set of independent 8–cycles in $P_n, n \geqslant 4$, contains $\frac{n!}{8}$ of these, since P_4 has three independent 8–cycles, and there are $\frac{n!}{24}$ copies of P_4, each of which consists of exactly three independent 8–cycles. This proves Corollary 1.5.

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