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The strongly distance–balanced property of the generalized Petersen graphs*

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Abstract

A graph X is said to be *strongly distance–balanced* whenever for any edge uv of X and any positive integer i, the number of vertices at distance i from u and at distance $i+1$ from v is equal to the number of vertices at distance $i+1$ from u and at distance i from v. It is proven that for any integers $k \geq 2$ and $n \geq k^2 + 4k + 1$, the generalized Petersen graph GP(n, k) is not strongly distance–balanced.

Keywords: Graph, strongy distance–balanced, generalized Petersen graph.

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1 Introduction

Let X be a graph with diameter d, and let $V(X)$ and $E(X)$ denote the vertex set and the edge set of X, respectively. For $u, v \in V(X)$, we let $d(u, v)$ denote the minimal path-length distance between u and v. We say that X is *distance–balanced* whenever for an arbitrary pair of adjacent vertices u and v of X

$$|\{x \in V(X) \mid d(x, u) < d(x, v)\}| = |\{x \in V(X) \mid d(x, v) < d(x, u)\}|$$

holds. These graphs were, at least implicitly, first studied by Handa [1] who considered distance–balanced partial cubes. The term itself, however, is due to Jerebic, Klavžar and Rall [3] who studied distance–balanced graphs in the framework of various kinds of graph products.

Let uv be an arbitrary edge of X. For any two nonnegative integers i, j, we let

$$D^i_j(u, v) = \{x \in V(X) \mid d(u, x) = i \quad \text{and} \quad d(v, x) = j\}.$$

The triangle inequality implies that only the sets $D^{i-1}_i(u, v)$, $D^i_i(u, v)$ and $D^i_{i-1}(u, v)$ $(1 \le i \le d)$ can be nonempty. One can easily see that X is distance–balanced if and only if for every edge $uv \in E(X)$

$$\sum_{i=1}^{d} |D^i_{i-1}(u, v)| = \sum_{i=1}^{d} |D^{i-1}_i(u, v)| \tag{1.1}$$

holds.

Obviously, if $|D^i_{i-1}(u, v)| = |D^{i-1}_i(u, v)|$ holds for $1 \le i \le d$ and for every edge $uv \in E(X)$, then X is distance–balanced. The converse, however, is not necessarily true. For instance, in the generalized Petersen graphs $\mathrm{GP}(24, 4)$, $\mathrm{GP}(35, 8)$ and $\mathrm{GP}(35, 13)$ (see Section 2 for the definition of generalized Petersen graphs), we can find two adjacent vertices u, v and an integer i, such that $|D^i_{i-1}(u, v)| \ne |D^{i-1}_i(u, v)|$. But it is easy to see that these graphs are distance–balanced.

We therefore say that X is *strongly distance–balanced*, if $|D^i_{i-1}(u, v)| = |D^{i-1}_i(u, v)|$ for every positive integer i and every edge $uv \in E(X)$. Let us remark that graphs with this property are also called *distance–degree regular*. Distance–degree regular graphs were studied in [2].

For a graph X, a vertex u of X and an integer i, let $S_i(u) = \{x \in V(X) \mid d(x, u) = i\}$ denote the set of vertices of X which are at distance i from u. The following result was proven in [4].

Proposition 1.1. [4, Proposition 2.1] *Let X be a graph with diameter d. Then X is strongly distance–balanced if and only if $|S_i(u)| = |S_i(v)|$ holds for every edge $uv \in E(X)$ and every $i \in \{0, \ldots, d\}$.*

In [3], the following conjecture was stated.

Conjecture 1.2. [3, Conjecture 2.5] *For any integer $k \ge 2$ there exists a positive integer n_0 such that the generalized Petersen graph $\mathrm{GP}(n, k)$ is not distance–balanced for every integer $n \ge n_0$.*

In this short note we prove the following slightly weaker result.

Theorem 1.3. *For any integers $k \geq 2$ and $n \geq k^2 + 4k + 1$, the generalized Petersen graph* $\mathrm{GP}(n, k)$ *is not strongly distance–balanced.*

We will prove Theorem 1.3 in two steps. In the first step we prove that the graph $\mathrm{GP}(k^2 + 4k + 1, k)$ is not strongly distance–balanced. In the second step we use the result from the first step to prove that $\mathrm{GP}(n, k)$ is not strongly distance-balanced if $n \geq k^2 + 4k + 1$.

2 Proof of Theorem 1.3

Let $n \geq 3$ be a positive integer, and let $k \in \{1, \dots, n-1\} \setminus \{n/2\}$. The generalized Petersen graph $\mathrm{GP}(n, k)$ is defined to have the following vertex set and edge set:

$$\begin{aligned} V(\mathrm{GP}(n,k)) &= \{u_i \mid i \in \mathbb{Z}_n\} \cup \{v_i \mid i \in \mathbb{Z}_n\}, \\ E(\mathrm{GP}(n,k)) &= \{u_i u_{i+1} \mid i \in \mathbb{Z}_n\} \cup \{v_i v_{i+k} \mid i \in \mathbb{Z}_n\} \cup \{u_i v_i \mid i \in \mathbb{Z}_n\}. \end{aligned} \tag{2.1}$$

Note that $\mathrm{GP}(n, k)$ is cubic, and that it is bipartite precisely when n is even and k is odd. It is easy to see that $\mathrm{GP}(n, k) \cong \mathrm{GP}(n, n-k)$. Furthermore, if the multiplicative inverse k^{-1} of k exists in $\mathbb{Z}_n$, then the mapping $f : V(\mathrm{GP}(n, k)) \to V(\mathrm{GP}(n, k^{-1}))$ defined by the rule

$$f(u_i) = v_{k^{-1}i}, \qquad f(v_i) = u_{k^{-1}i} \tag{2.2}$$

gives rise to an isomorphism of graphs $\mathrm{GP}(n, k)$ and $\mathrm{GP}(n, k^{-1})$, where the use of the same symbols for vertices in $\mathrm{GP}(n, k)$ and $\mathrm{GP}(n, k^{-1})$ should cause no confusion.

We first investigate the sets $S_i(u_0)$ and $S_i(v_0)$ of the graph $\mathrm{GP}(k^2 + 4k + 1, k)$.

Lemma 2.1. *Let $k \geq 9$ be an integer, let $n = k^2 + 4k + 1$ and let $u_0 \in V(GP(n, k))$. Then the following statements hold:*

(i) $S_1(u_0) = \{u_{\pm 1}, v_0\}$, $S_2(u_0) = \{u_{\pm 2}, v_{\pm 1}, v_{\pm k}\}$,
$S_3(u_0) = \{u_{\pm 3}, u_{\pm k}, v_{\pm 2}, v_{\pm(k+1)}, v_{\pm(k-1)}, v_{\pm 2k}\}$;

(ii) if $i \in \{4, \dots, \lfloor k/2 \rfloor + 1\}$, *then*

$$\begin{aligned} S_i(u_0) = {} & \{u_{\pm i}, u_{\pm(i-2)k}\} \cup \{v_{\pm(i-1)}, v_{\pm(i-1)k}\} \cup \\ & \{u_{\pm(lk+i-l-2)}, u_{\pm(lk-i+l+2)} \mid 1 \leq l \leq i-3\} \cup \\ & \{v_{\pm(lk+i-l-1)}, v_{\pm(lk-i+l+1)} \mid 1 \leq l \leq i-2\}; \end{aligned}$$

(iii) if k is odd, then

$$\begin{aligned} S_{(k+3)/2}(u_0) = {} & \{u_{\pm(k+3)/2}, u_{\pm(k-1)k/2}, u_{\pm(3k-3)/2}\} \cup \\ & \{u_{\pm(lk+(k-1)/2-l)}, u_{\pm(lk-(k-1)/2+l)} \mid 2 \leq l \leq (k-3)/2\} \cup \\ & \{v_{\pm(k+1)/2}, v_{\pm(k+1)k/2}, v_{\pm(3k-1)/2}\} \cup \\ & \{v_{\pm(lk+(k+1)/2-l)}, v_{\pm(lk-(k+1)/2+l)} \mid 2 \leq l \leq (k-1)/2\}; \end{aligned}$$

(iv) if k is even, then

$$\begin{aligned} S_{(k+4)/2}(u_0) = {} & \{u_{\pm k^2/2}, u_{\pm(3k-2)/2}\} \cup \\ & \{u_{\pm(lk+k/2-l)}, u_{\pm(lk-k/2+l)} \mid 2 \leq l \leq (k-2)/2\} \cup \\ & \{v_{\pm 3k/2}, v_{\pm(k+2)k/2}\} \cup \\ & \{v_{\pm(lk+3k/2-l)}, v_{\pm(lk+k/2+l)} \mid 1 \leq l \leq (k-2)/2\}. \end{aligned}$$

Proof. Using the fact that by assumption $k \geq 9$, a careful inspection of the neighbors' sets of vertices u_i and v_i, we see that (i) holds.

We now prove part (ii) by induction. Similarly as above we see that (ii) holds for $i \in \{4, 5\}$.

Let us now assume that (ii) holds for $i-1$ and i, where $i \in \{5, \dots, \lfloor k/2 \rfloor\}$. Hence we have

$$S_{i-1}(u_0) = \{u_{\pm(i-1)}, u_{\pm(i-3)k}\} \cup \{u_{\pm(lk+i-l-3)}, u_{\pm(lk-i+l+3)} \mid 1 \leq l \leq i-4\} \cup \{v_{\pm(i-2)}, v_{\pm(i-2)k}\} \cup \{v_{\pm(lk+i-l-2)}, v_{\pm(lk-i+l+2)} \mid 1 \leq l \leq i-3\}$$

and

$$S_i(u_0) = \{u_{\pm i}, u_{\pm(i-2)k}\} \cup \{u_{\pm(lk+i-l-2)}, u_{\pm(lk-i+l+2)} \mid 1 \leq l \leq i-3\} \cup \{v_{\pm(i-1)}, v_{\pm(i-1)k}\} \cup \{v_{\pm(lk+i-l-1)}, v_{\pm(lk-i+l+1)} \mid 1 \leq l \leq i-2\}.$$

Now we compute the neighbors of the vertices belonging to the set $S_i(u_0)$. Since

$$S_1(u_{-r}) = \{u_{-q}, v_{-q} \mid u_q, v_q \in S_1(u_r)\} \quad \text{and}$$
$$S_1(v_{-r}) = \{u_{-q}, v_{-q} \mid u_q, v_q \in S_1(v_r)\},$$

we will only list the following sets:

- $S_1(u_i) = \{u_{i+1}, u_{i-1}, v_i\}$,
- $S_1(u_{(i-2)k}) = \{u_{(i-2)k+(i+1)-(i-2)-2}, u_{(i-2)k-(i+1)+(i-2)+2}, v_{(i-2)k}\}$,
- $S_1(u_{lk+i-l-2}) = \{u_{lk+(i+1)-l-2}, u_{lk+(i-1)-l-2}, v_{lk+(i-1)-l-1}\}$,
- $S_1(u_{lk-i+l+2}) = \{u_{lk-(i-1)+l+2}, u_{lk-(i+1)+l+2}, v_{lk-(i-1)+l+1}\}$,
- $S_1(v_{i-1}) = \{u_{i-1}, v_{k+(i+1)-2}, v_{-(k-(i+1)+2)}\}$,
- $S_1(v_{(i-1)k}) = \{u_{(i-1)k}, v_{ik}, v_{(i-2)k}\}$,
- $S_1(v_{lk+i-l-1}) = \{u_{lk+(i+1)-l-2}, v_{(l+1)k+(i+1)-(l+1)-1}, v_{(l-1)k+(i-1)-(l-1)-1}\}$,
- $S_1(v_{lk-i+l+1}) = \{u_{lk-(i+1)+l+2}, v_{(l+1)k-(i+1)+(l+1)+1}, v_{(l-1)k-(i-1)+(l-1)+1}\}$.

Obviously, $S_{i+1}(u_0)$ consists of all the neighbors of vertices in $S_i(u_0)$, which are not in $S_{i-1}(u_0)$ or $S_i(u_0)$. Thus

$$S_{i+1}(u_0) = \{u_{\pm(i+1)}, u_{\pm(i-1)k}\} \cup \{u_{\pm(lk+(i+1)-l-2)}, u_{\pm(lk-(i+1)+l+2)} \mid 1 \leq l \leq i-2\} \cup \{v_{\pm i}, v_{\pm ik}\} \cup \{v_{\pm(lk+(i+1)-l-1)}, v_{\pm(lk-(i+1)+l+1)} \mid 1 \leq l \leq i-1\}$$

and the result follows.

Let us now prove (iii). Assume first k is odd, and abbreviate $b = (k+1)/2$. By (ii),

$$S_{b-1}(u_0) = \{u_{\pm(b-1)}, u_{\pm(b-3)k}\} \cup \{u_{\pm(lk+b-l-3)}, u_{\pm(lk-b+l+3)} \mid 1 \leq l \leq b-4\} \cup \{v_{\pm(b-2)}, v_{\pm(b-2)k}\} \cup \{v_{\pm(lk+b-l-2)}, v_{\pm(lk-b+l+2)} \mid 1 \leq l \leq b-3\}$$

and

$$S_b(u_0) = \{u_{\pm b}, u_{\pm(b-2)k}\} \cup \{u_{\pm(lk+b-l-2)}, u_{\pm(lk-b+l+2)} \mid 1 \le l \le b-3\} \cup \{v_{\pm(b-1)}, v_{\pm(b-1)k}\} \cup \{v_{\pm(lk+b-l-1)}, v_{\pm(lk-b+l+1)} \mid 1 \le l \le b-2\}.$$

Let us now compute the neighbors of the vertices in $S_b(u_0)$. Since $S_1(u_{-r}) = \{u_{-q}, v_{-q} \mid u_q, v_q \in S_1(u_r)\}$ and $S_1(v_{-r}) = \{u_{-q}, v_{-q} \mid u_q, v_q \in S_1(v_r)\}$, we will only list the following sets:

- $S_1(u_b) = \{u_{b+1}, u_{b-1}, v_b\}$,
- $S_1(u_{(b-2)k}) = \{u_{(b-2)k+(b+1)-(b-2)-2}, u_{(b-2)k-(b+1)+(b-2)+2}, v_{(b-2)k}\}$,
- $S_1(u_{lk+b-l-2}) = \{u_{lk+b-l-1}, u_{lk+b-l-3}, v_{lk+b-l-2}\}$,
- $S_1(u_{lk-b+l+2}) = \{u_{lk-b+l+3}, u_{lk-b+l+1}, v_{lk-b+l+2}\}$,
- $S_1(v_{b-1}) = \{u_{b-1}, v_{k+b-1}, v_{-(k-b+1)}\} = \{u_{b-1}, v_{k+b-1}, v_{-b}\}$,
- $S_1(v_{(b-1)k}) = \{u_{(b-1)k}, v_{bk}, v_{(b-2)k}\}$,
- $S_1(v_{lk+b-l-1}) = \{u_{lk+b-l-1}, v_{(l+1)k+b-l-1}, v_{(l-1)k+b-l-1}\}$,
- $S_1(v_{lk-b+l+1}) = \{u_{lk-b+l+1}, v_{(l+1)k-b+l+1}, v_{(l-1)k-b+l+1}\}$.

Observe that $u_{\pm(k-b+2)} = u_{\pm(b+1)}$. Therefore, sorting out those neigbors of the vertices in $S_b(u_0)$ which are either in $S_{b-1}(u_0)$ or $S_b(u_0)$, we obtain that

$$S_{b+1}(u_0) = \{u_{\pm(b+1)}, u_{\pm(b-1)k}, u_{\pm(k+b-2)}\} \cup \{u_{\pm(lk+b-l-1)}, u_{\pm(lk-b+l+1)} \mid 2 \le l \le b-2\} \cup \{v_{\pm b}, v_{\pm bk}, v_{\pm(k+b-1)}\} \cup \{v_{\pm(lk+b-l)}, v_{\pm(lk-b+l)} \mid 2 \le l \le b-1\}$$

and hence the result follows.

The proof of (iv) is done in a similar way to that of (iii) above and is omitted. □

We have the following immediate corollary of Lemma 2.1.

Corollary 2.2. *Let $k \ge 9$ be an integer, let $n = k^2 + 4k + 1$ and let $u_0 \in V(GP(n, k))$. Then the following statements hold:*

(i) $|S_1(u_0)| = 3$, $|S_2(u_0)| = 6$, $|S_3(u_0)| = 12$;

(ii) $|S_i(u_0)| = 8i - 12$ *for* $i \in \{4, \dots, \lfloor k/2 \rfloor + 1\}$;

(iii) if k is odd, then $|S_{(k+3)/2}(u_0)| = 4k - 4$;

(iv) if k is even, then $|S_{(k+4)/2}(u_0)| = 4k - 4$.

The proofs of the next lemma and corollary are omitted as they can be carried out using the same arguments as in the proof of Lemma 2.1. (Note that $-(k+4)$ is the multiplicative inverse of k in $\mathbb{Z}_{k^2+4k+1}$.)

Lemma 2.3. *Let $k \ge 9$ be an integer, let $n = k^2 + 4k + 1$, and let $u_0 \in V(GP(n, k+4))$. Then the following statements hold:*

(i) $S_1(u_0) = \{u_{\pm 1}, v_0\}$, $S_2(u_0) = \{u_{\pm 2}, v_{\pm 1}, v_{\pm(k+4)}\}$,
$S_3(u_0) = \{u_{\pm 3}, u_{\pm(k+4)}, v_{\pm 2}, v_{\pm(k+5)}, v_{\pm(k+3)}, v_{\pm 2(k+4)}\}$;

(ii) *if* $i \in \{4, \ldots, \lfloor k/2 \rfloor + 1\}$, *then*

$$S_i(u_0) = \{u_{\pm i}, u_{\pm(i-2)(k+4)}\} \cup \{v_{\pm(i-1)}, v_{\pm(i-1)(k+4)}\} \cup \{u_{\pm(lk+i+3l-2)}, u_{\pm(lk-i+5l+2)} \mid 1 \le l \le i-3\} \cup \{v_{\pm(lk+i+3l-1)}, v_{\pm(lk-i+5l+1)} \mid 1 \le l \le i-2\};$$

(iii) *if* k *is odd, then*

$$S_{(k+3)/2}(u_0) = \{u_{\pm(k+3)/2}, u_{\pm(k-1)(k+4)/2}\} \cup \{u_{\pm(lk+(k-1)/2+3l)}, u_{\pm(lk-(k-1)/2+5l)} \mid 1 \le l \le (k-3)/2\} \cup \{v_{\pm(k+1)/2}, v_{\pm(k+1)(k+4)/2}, v_{\pm(k^2+3k-6)/2}\} \cup \{v_{\pm(lk+(k+1)/2+3l)}, v_{\pm(lk-(k+1)/2+5l)} \mid 1 \le l \le (k-3)/2\};$$

(iv) *if* k *is even, then*

$$S_{(k+4)/2}(u_0) = \{u_{\pm(k+4)/2}, u_{\pm k(k+4)/2}\} \cup \{u_{\pm(lk+k/2+3l)}, u_{\pm(lk-k/2+5l)} \mid 1 \le l \le (k-2)/2\} \cup \{v_{\pm(k+2)/2}, v_{\pm(k+2)^2/2}\} \cup \{v_{\pm(lk+(k+2)/2+3l)}, v_{\pm(lk-(k+2)/2+5l)} \mid 1 \le l \le (k-2)/2\}.$$

Corollary 2.4. *Let* $k \ge 9$ *be an integer, let* $n = k^2+4k+1$ *and let* $u_0 \in V(GP(n, k+4))$. *Then the following statements hold:*

(i) $|S_1(u_0)| = 3$, $|S_2(u_0)| = 6$, $|S_3(u_0)| = 12$;

(ii) $|S_i(u_0)| = 8i - 12$ *for* $i \in \{4, \ldots, \lfloor k/2 \rfloor + 1\}$;

(iii) *if* k *is odd, then* $|S_{(k+3)/2}(u_0)| = 4k - 2$;

(iv) *if* k *is even, then* $|S_{(k+4)/2}(u_0)| = 4k$.

Corollary 2.5. *Let* $k \ge 2$ *be an integer, let* $n = k^2 + 4k + 1$, *let* $b = \lfloor k/2 \rfloor + 2$ *and let* $u_0, v_0 \in V(GP(n, k))$. *Then* $|S_b(u_0)| \ne |S_b(v_0)|$. *In particular,* $GP(n, k)$ *is not strongly distance–balanced.*

Proof. If $k \le 8$, then a direct check shows that $|S_b(u_0)| \ne |S_b(v_0)|$. Assume now $k \ge 9$. Note that $-(k+4) = n-(k+4) \in \mathbb{Z}_n$ is the multiplicative inverse of $k \in \mathbb{Z}_n$. Therefore, by (2.2), we have

$$\mathrm{GP}(n, (k+4)) \cong \mathrm{GP}(n, -(k+4)) \cong \mathrm{GP}(n, k).$$

Under this isomorphism, the vertex $u_0 \in V(\mathrm{GP}(n, (k+4)))$ maps to the vertex $v_0 \in V(\mathrm{GP}(n, k))$. (Recall that the same symbols are used for vertices in $\mathrm{GP}(n, k)$ and in $\mathrm{GP}(n, (k+4))$.) The result now follows from Corollaries 2.2 and 2.4. □

We are now ready to prove our main result.

Proof of Theorem 1.3. Let $k \ge 2$ be an integer, let $n_0 = k^2 + 4k + 1$, let $n \ge n_0$, and let $b = \lfloor k/2 \rfloor + 2$. We now show that $\mathrm{GP}(n, k)$ is not strongly distance–balanced. In what follows, the same symbols are used for vertices in $\mathrm{GP}(n_0, k)$ and those in $\mathrm{GP}(n, k)$.

Observe that $kb < n_0/2$. By (2.1), for $i \in \{1, 2, \ldots, b\}$ we have that $u_j \in V(\mathrm{GP}(n, k))$ ($v_j \in V(\mathrm{GP}(n, k))$, respectively) is at distance i from $u_0 \in V(\mathrm{GP}(n, k))$ if and only if $u_j \in V(\mathrm{GP}(n_0, k))$ ($v_j \in V(\mathrm{GP}(n_0, k))$, respectively) is at distance i from $u_0 \in V(\mathrm{GP}(n_0, k))$. Therefore, the number of vertices which are at distance i from $u_0 \in$

$V(\mathrm{GP}(n,k))$ is the same as the number of vertices which are at distance i from $u_0 \in V(\mathrm{GP}(n_0,k))$. Similarly, for $i \in \{1,2,\ldots,b\}$, we have that $u_j \in V(\mathrm{GP}(n,k))$ ($v_j \in V(\mathrm{GP}(n,k))$, respectively) is at distance i from $v_0 \in V(\mathrm{GP}(n,k))$ if and only if $u_j \in V(\mathrm{GP}(n_0,k))$ ($v_j \in V(\mathrm{GP}(n_0,k))$, respectively) is at distance i from $v_0 \in V(\mathrm{GP}(n_0,k))$. Hence the number of vertices which are at distance i from the vertex $v_0 \in V(\mathrm{GP}(n,k))$ is the same as the number of vertices which are at distance i from the vertex $v_0 \in V(\mathrm{GP}(n_0, k))$. Therefore, by Corollary 2.5, $|S_b(u_0)| \neq |S_b(v_0)|$ for $u_0, v_0 \in V(\mathrm{GP}(n,k))$. By Proposition 1.1, $\mathrm{GP}(n,k)$ is not strongly distance–balanced. □

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