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Plan S – A Golden Addition to the Diamond Open Access Publishing Model

We wish to tell readers about 'Plan S', which is a very recent initiative in Europe, begun in September 2018 with the aim of speeding up a transition from the traditional, subscription-based publishing model to the Gold Open Access model.

This plan envisages that all research supported by public money in Europe should be published in Gold OA journals from 2020. Although not all details have been decided, it appears that Plan S could involve imposing sanctions against authors whose research was supported by public research agencies but publish their work subscription-based journals, even in leading journals such as *Annals of Mathematics*, *Acta Mathematica*, *Inventiones Mathematicae*, *JAMS* and *JEMS*.

An obvious possible consequence of this is that some high quality research will be sponsored by private funding (to avoid such sanctions), and that public funding will support research of lesser importance. Accordingly, it seems to us that Plan S is not about research, or about authors or readers, but is driven by financial considerations.

Plan S was put up by cOAlition S, which is a consortium of 16 European National Research Agencies. These agencies intend to pay the costs of APC (article processing charges).

For example, the Slovenian Research Agency ARRS (which is a founding member of cOAlition S) issued a call in October 2018 and set aside 300 000 € for paying back APC for gold OA for articles published in 2018. At the end of November 2018 the results were published and can be found here: `http://www.arrs.gov.si/sl/progproj/rproj/rezultati/18/inc/rezultati-odprti-dostop-18.pdf`

About 80 Gold OA papers are listed there as winners. The APC for each of them will be paid by the ARRS. The costs of publication vary from about 500 € to over 4000 € per paper, with the majority of APC sitting between 1000 and 2000 €, and one can easily check that most of this public money goes to private publishing houses.

There were three criteria for the eligibility of a paper for this payment:

(a) acknowledgement of support by the ARRS,
(b) payment for the APC in 2018, and
(c) ranking of the journal in which the paper appears as Q1 in WoS or Scopus.

We note that the third criterion is not compliant with the 2013 Declaration on Research Assessment (DORA), because it involves the use of impact factors. We also note that to date, no authors of mathematical papers have applied for this support by the ARRS.

The pertinent question for *Ars Mathematica Contemporanea* is how should Diamond OA journals adapt to Plan S.

First, we plan to ensure that *AMC* is compliant with Plan S. For any article in which support is to be acknowledged from a Plan S compliant agency (such as the ARRS), we will offer the authors the option of either removing the acknowledgement, or paying APC at some reasonable rate (such as 2000 €). Before we start implementing this policy, we will negotiate with the ARRS (and through the ARRS with other Plan S agencies and other institutions) to make sure they will indeed reimburse the APC.

Strictly speaking, this means we will change our publishing model from Diamond OA to Gold OA, but on the other hand, we will waive APC for most papers. We believe other Diamond Open Access journals should consider similar adjustments, to deal with Plan S.

An interesting question remains. The supporters of Plan S are being encouraged to sign DORA. But when this happens, it is not at all clear how the 'excellence' of eligible journals will be determined.

Klavdija Kutnar, Dragan Marušič and Tomaž Pisanski
Editors in Chief

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Splittable and unsplittable graphs and configurations*

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Abstract

We prove that there exist infinitely many splittable and also infinitely many unsplittable cyclic (n_3) configurations. We also present a complete study of trivalent cyclic Haar graphs on at most 60 vertices with respect to splittability. Finally, we show that all cyclic flag-transitive configurations with the exception of the Fano plane and the Möbius-Kantor configuration are splittable.

Keywords: Configuration of points and lines, unsplittable configuration, unsplittable graph, independent set, Levi graph, Grünbaum graph, splitting type, cyclic Haar graph.

Math. Subj. Class.: 51A20, 05B30

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1 Introduction and preliminaries

The idea of *unsplittable configuration* was conceived in 2004 and formally introduced in the monograph [8] by Grünbaum. Later, it was also used in [19]. In [20], the notion was generalized to graphs. In this paper we present some constructions for splittable and unsplittable cyclic configurations. In [9], the notion of *cyclic Haar graph* was introduced. It was shown that cyclic Haar graphs are closely related to cyclic configurations. Namely, each cyclic Haar graph of girth 6 is a Levi graph of a cyclic combinatorial configuration; see also [18]. For the definition of the *Levi graph* (also called *incidence graph*) of a configuration the reader is referred to [4]. The classification of configurations with respect to splittability is a purely combinatorial problem and can be interpreted purely in terms of Levi graphs.

Let n be a positive integer, let $\mathbb{Z}_n$ be the cyclic group of integers modulo n and let $S \subseteq \mathbb{Z}_n$ be a set, called the *symbol*. The graph $H(n, S)$ with the vertex set $\{u_i \mid i \in \mathbb{Z}_n\} \cup \{v_i \mid i \in \mathbb{Z}_n\}$ and edges joining u_i to v_{i+k} for each $i \in \mathbb{Z}_n$ and each $k \in S$ is called a *cyclic Haar graph* over $\mathbb{Z}_n$ with symbol S [9]. In practice, we will simplify the notation by denoting u_i by i^+ and v_i by i^-.

Definition 1.1. A *combinatorial* (v_k) *configuration* is an incidence structure $\mathcal{C} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, where $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$, $\mathcal{P} \cap \mathcal{B} = \emptyset$ and $|\mathcal{P}| = |\mathcal{B}| = v$. The elements of $\mathcal{P}$ are called *points*, the elements of $\mathcal{B}$ are called *lines* and the relation $\mathcal{I}$ is called the *incidence* relation. Furthermore, each line is incident with k points, each point is incident with k lines and two distinct points are incident with at most one common line, i.e.,

$$\{(p_1, b_1), (p_2, b_1), (p_1, b_2), (p_2, b_2)\} \subseteq \mathcal{I}, p_1 \neq p_2 \implies b_1 = b_2. \tag{1.1}$$

If $(p, b) \in \mathcal{I}$ then we say that the line b passes through point p or that the point p lies on line b. An element of $\mathcal{P} \cup \mathcal{B}$ is called an *element of configuration* $\mathcal{C}$.

A combinatorial (v_k) configuration $\mathcal{C} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is *geometrically realisable* if the elements of $\mathcal{P}$ can be mapped to different points in the Euclidean plane and the elements of $\mathcal{B}$ can be mapped to different lines in the Euclidean plane, such that $(p, b) \in \mathcal{I}$ if and only if the point that corresponds to p lies on the line that corresponds to b. A geometric realisation of a combinatorial (v_k) configuration is called a *geometric* (v_k) *configuration*. Note that examples in Figures 2, 3 and 4 are all geometric configurations. The Fano plane (7_3) is an example of a geometrically non-realizable configuration.

An *isomorphism* between configurations $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ and $(\mathcal{P}', \mathcal{B}', \mathcal{I}')$ is a pair of bijections $\psi\colon \mathcal{P} \to \mathcal{P}'$ and $\varphi\colon \mathcal{B} \to \mathcal{B}'$, such that

$$(p, b) \in \mathcal{I} \text{ if and only if } (\psi(p), \varphi(b)) \in \mathcal{I}'. \tag{1.2}$$

The configuration $\mathcal{C}^* = (\mathcal{B}, \mathcal{P}, \mathcal{I}^*)$, where $\mathcal{I}^* = \{(b, p) \in \mathcal{B} \times \mathcal{P} \mid (p, b) \in \mathcal{I}\}$, is called the *dual configuration* of $\mathcal{C}$. A configuration that is isomorphic to its dual is called a *self-dual* configuration.

The *Levi graph* of a configuration $\mathcal{C}$ is the bipartite graph on the vertex set $\mathcal{P} \cup \mathcal{B}$ having an edge between $p \in \mathcal{P}$ and $b \in \mathcal{B}$ if and only if the elements p and b are incident in $\mathcal{C}$, i.e., if $(p, b) \in \mathcal{I}$. It is denoted $L(\mathcal{C})$. Condition (1.1) in Definition 1.1 implies that the girth of $L(\mathcal{C})$ is at least 6. Moreover, any combinatorial (v_k) configuration is completely determined by a k-regular bipartite graph of girth at least 6 with a given black-and-white vertex coloring, where black vertices correspond to points and white vertices correspond to lines. Such a graph will be called a *colored Levi graph*. Note that the reverse coloring determines the dual configuration $\mathcal{C}^* = (\mathcal{B}, \mathcal{P}, \mathcal{I}^*)$. Also, an isomorphism between configurations corresponds to color-preserving isomorphism between their respective colored Levi graphs.

A configuration $\mathcal{C}$ is said to be *connected* if its Levi graph $L(\mathcal{C})$ is connected. Similarly, a configuration $\mathcal{C}$ is said to be *k-connected* if its Levi graph $L(\mathcal{C})$ is k-connected.

Definition 1.2 ([19])**.** A combinatorial (v_k) configuration $\mathcal{C}$ is *cyclic* if it admits an automorphism of order v that cyclically permutes the points and lines, respectively.

In [9] the following was proved:

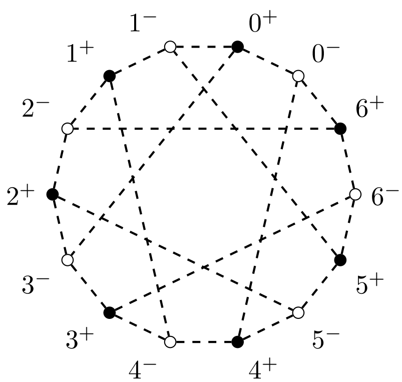
Proposition 1.3 ([9])**.** *A configuration $\mathcal{C}$ is cyclic if and only if its Levi graph is isomorphic to a cyclic Haar graph of girth* 6.

It can be shown that each cyclic configuration is self-dual, see for instance [9].

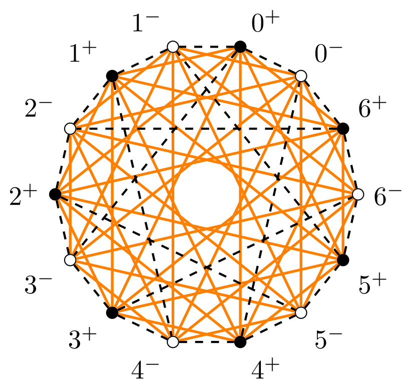
2 Splittable and unsplittable configurations (and graphs)

Let G be any graph. The *square* of G, denoted G^2, is a graph with the same vertex set as G, where two vertices are adjacent if and only if their distance in G is at most 2. In other words, $V(G^2) = V(G)$ and $E(G^2) = \{uv \mid d_G(u, v) \leq 2\}$. The square of the Levi graph $L(\mathcal{C})$ of a configuration $\mathcal{C}$ is called the *Grünbaum graph* of $\mathcal{C}$ in [19] and [20]. In [8], it is called the *independence graph*. Two elements of a configuration $\mathcal{C}$ are said to be *independent* if they correspond to independent vertices of the Grünbaum graph.

Example 2.1. The Grünbaum graph of the Heawood graph is shown in Figure 1. Its complement is the Möbius ladder M_{14}.



(a)



(b)

Figure 1: The Heawood graph $H = H(7, \{0, 1, 3\}) \cong \text{LCF}[5, -5]^7$ (on the left) is the Levi graph of the Fano plane. Its Grünbaum graph G is on the right. Note that there is an orange solid edge between two vertices of G if and only if they are at distance 2 in H.

It is easy to see that two elements of $\mathcal{C}$ are independent if and only if one of the following hold:

(i) two points of $\mathcal{C}$ that do not lie on a common line of $\mathcal{C}$;

(ii) two lines of $\mathcal{C}$ that do not intersect in a common point of $\mathcal{C}$;

(iii) a point of $\mathcal{C}$ and a line of $\mathcal{C}$ that are not incident.

The definition of unsplittable configuration was introduced in [8] and is equivalent to the following:

Definition 2.2. A configuration $\mathcal{C}$ is *splittable* if there exists an independent set of vertices Σ in the Grünbaum graph $(L(\mathcal{C}))^2$ such that $L(\mathcal{C}) - \Sigma$, i.e., the graph obtained by removing the set of vertices Σ from the Levi graph $L(\mathcal{C})$, is disconnected. In this case the set Σ is called a *splitting set of elements*. A configuration that is not splittable is called *unsplittable*.

This definition carries over to graphs:

Definition 2.3. A connected graph G is *splittable* if there exists an independent set Σ in G^2 such that $G - \Sigma$ is disconnected.

Example 2.4. Every cycle of length at least 6 is splittable (there exists a pair of vertices at distance 3 in G).

Every graph of diameter 2 without a cut vertex is unsplittable. The square of such a graph on n vertices is the complete graph K_n. This implies that $|S| = 1$. Since there are no cut vertices, a splitting set does not exist. The Petersen graph is an example of unsplittable graph.

In [8], refinements of the above definition are also considered. Configuration $\mathcal{C}$ is *point-splittable* if it is splittable and there exists a splitting set of elements that consists of points only (i.e., only black vertices in the corresponding colored Levi graph). In a similar way *line-splittable* configurations are defined. Note that these refinements can be defined for any bipartite graph with a given black-and-white coloring. There are four possibilities, that we call *splitting types*. Any configuration may be:

(T1) point-splittable, line-splittable,

(T2) point-splittable, line-unsplittable,

(T3) point-unsplittable, line-splittable,

(T4) point-unsplittable, line-unsplittable.

Any configuration of splitting type T1, T2 or T3 is splittable. A configuration of splitting type T4 may be splittable or unsplittable. For an example of a point-splittable (T2) configuration see Figure 2. The configuration on Figure 2 is isomorphic to a configuration on Figure 5.1.11 from [8]. For an example of a line-splittable (T3) configuration see Figure 3.

Note the following:

Proposition 2.5. *If $\mathcal{C}$ is of type T1 then its dual is also of type T1. If it is of type T2 then its dual is of type T3 (and vice versa). If it is of type T4 then its dual is also of type T4.*

Since types are mutually disjoint, this has a straightforward consequence for cyclic configurations:

Corollary 2.6. *Any self-dual configuration, in particular any cyclic configuration, is either of type T1 or T4.*

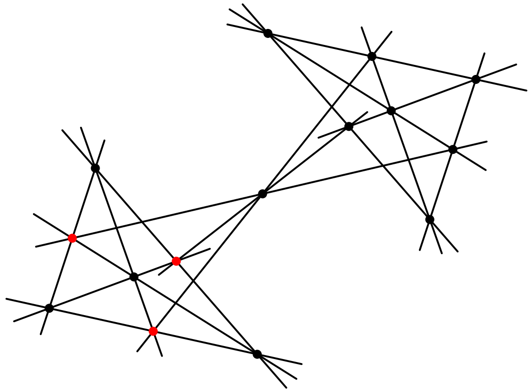


Figure 2: A point-splittable (15_3) configuration of type T2. Points that belong to a splitting set are colored red. Its dual is of type T3 (see Figure 3).

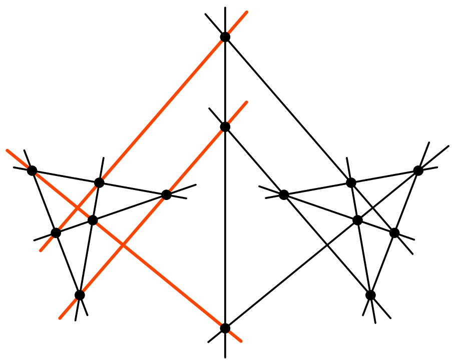


Figure 3: A line-splittable (15_3) configuration of type T3. Lines that belong to a splitting set are colored orange. Its dual is depicted in Figure 2.

Obviously, unsplittable configurations are of type T4. However, the converse is not true:

Proposition 2.7. *Any unsplittable configuration is point-unsplittable and line-unsplittable. There exist splittable configurations that are both point-unsplittable and line-unsplittable.*

Proof. The first statement of Proposition 2.7 is obviously true. An example that provides the proof of the second statement is shown in Figure 4. The splitting set is

$$\{0, 8, 10, (1, 9, 11), (6, 7, 14)\}. \qquad \square$$

Note that configuration in Figure 4 is not cyclic, but it is 3-connected. In [8], the following theorem is proven:

Theorem 2.8 ([8, Theorem 5.1.5])**.** *Any unsplittable (n_3) configuration is 3-connected.*

Our computational results show that the converse to Theorem 2.8 is not true. There exist 3-connected splittable configurations. See, for instance, the configuration in Figure 4.

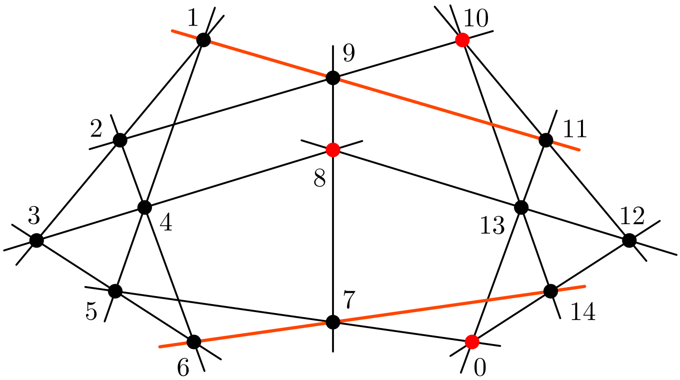


Figure 4: A splittable (15_3) configuration of type T4. Elements of a splitting set are points 0, 8 and 10 (colored red) and lines $(1, 9, 11)$ and $(6, 7, 14)$ (colored orange).

3 Splittable and unsplittable cyclic (n_3) configurations

We used a computer program to analyse all cyclic (n_3) configurations for $7 \leq n \leq 30$ (see Table 1, Table 2 and Table 3). In [9] it was shown that cyclic Haar graphs contain all information about cyclic combinatorial configurations. In trivalent case combinatorial isomorphisms of cyclic configurations are well-understood; see [11]. Namely, it is known how to obtain all sets of parameters of isomorphic cyclic Haar graphs. We would like to draw the reader's attention to the manuscript [10], where the main result of [11] is extended to cyclic (n_k) configurations for all $k > 3$. One would expect that large sparse graphs are splittable. In this sense the following result is not a surprise:

Theorem 3.1. *Let $H(n, \{0, a, b\})$ be a cyclic Haar graph, where $0 < a < b$. Let*

$$\mathcal{W} = \{0, a, b, 2b, b + a, b - a, 2b - a, 2b - 2a, 3b - a, 3b - 2a, 2b + a, 3b\} \text{ and}$$
$$\mathcal{B} = \{0, a, b, 2b, b + a, b - a, 2b - a, 2b - 2a, 3b - a, 3b - 2a, -a, b - 2a\}$$

be multisets with elements from $\mathbb{Z}_n$. If all elements of $\mathcal{W}$ are distinct and all elements of $\mathcal{B}$ are distinct (i.e. $\mathcal{W}$ and $\mathcal{B}$ are ordinary sets, $|\mathcal{W}| = |\mathcal{B}| = 12$) then $H(n, \{0, a, b\})$ is splittable and

$$\Sigma = \{0^+, 2b^+, (2b - 2a)^+, (b - a)^-, (b + a)^-, (3b - a)^-\}$$

is a splitting set for $H(n, \{0, a, b\})$.

Proof. See Figure 5. If $\mathcal{W}$ and $\mathcal{B}$ are ordinary sets then the graph in Figure 5 is a subgraph of $H(n, \{0, a, b\})$. It is easy to see that Σ is a splitting set. The set Σ is indeed an independent set in the square of the graph $H(n, \{0, a, b\})$ since no two vertices of Σ are adjacent to the same vertex. In order to see that the subgraph obtained by removing the vertices of Σ is disconnected, note that one of the connected components is the cycle determined by vertices $\{b^+, (b - a)^+, (2b - a)^+, b^-, 2b^-, (2b - a)^-\}$. □

Corollary 3.2. *Under conditions of Theorem 3.1, the girth of the graph $H(n, \{0, a, b\})$ is* 6.

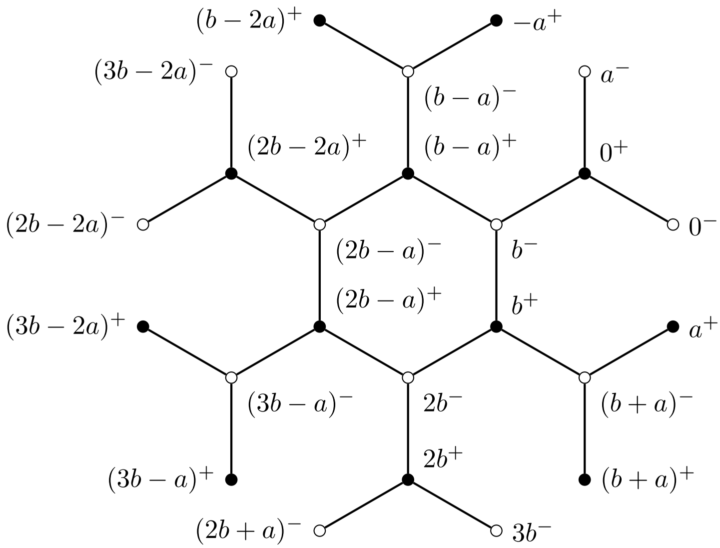


Figure 5: The set $\Sigma = \{0^+, 2b^+, (2b-2a)^+, (b-a)^-, (b+a)^-, (3b-a)^-\}$ is a splitting set for $H(n, \{0, a, b\})$.

Proof. The girth of such a graph is at most 6 because it contains a 6-cycle (see Figure 5). It is easy to see that the girth cannot be 4. Because the graph $H(n, \{0, a, b\})$ is bipartite, each 4-cycle must contain a black vertex. Consider vertex b^+ in Figure 5. Its neighborhood is $\{b^-, 2b^-, (b+a)^-\}$. None of those vertices have a common neighbor, so b^+ does not belong to any 4-cycle. Because of symmetry this argument holds for all black vertices. □

Corollary 3.3. *There exist infinitely many cyclic (n_3) configurations that are splittable. For example, the following three families of cyclic Haar graphs are splittable:*

(a) $H(n, \{0, 1, 4\})$ *for* $n \geq 13$,

(b) $H(n, \{0, 1, 5\})$ *for* $n \geq 16$, *and*

(c) $H(n, \{0, 2, 5\})$ *for* $n \geq 16$.

Proof. Corollary 3.2 implies that each graph from any of the three families has girth 6. From Theorem 3.1 it follows that $\Sigma = \{0^+, 6^+, 8^+, 3^-, 5^-, 11^-\}$ is a splitting set for $H(n, \{0, 1, 4\})$ if $n \geq 13$ (see Figure 6), $\{0^+, 8^+, 10^+, 4^-, 6^-, 14^-\}$ is a splitting set for $H(n, \{0, 1, 5\})$ if $n \geq 16$, and $\{0^+, 6^+, 10^+, 3^-, 7^-, 13^-\}$ is a splitting set for $H(n, \{0, 2, 5\})$ if $n \geq 16$. □

If $n < 13$ then conditions of Theorem 3.1 are not fulfilled. If $n = 12$ then $(n-1)^+ = 11^+$ which means that the vertices of the graph in Figure 6 are not all distinct. If $n = 9$ then $9^- = 0^-$ since we work with $\mathbb{Z}_9$. Similar arguments can be made if $n < 16$ in the case of the other two families from Corollary 3.3.

We investigated the first 100 graphs from the $H(n, \{0, 1, 4\})$ family. All but two are zero symmetric, nowadays called graphical regular representation or GRR for short (see [5]). The exceptions are for $n = 13$ and $n = 15$.

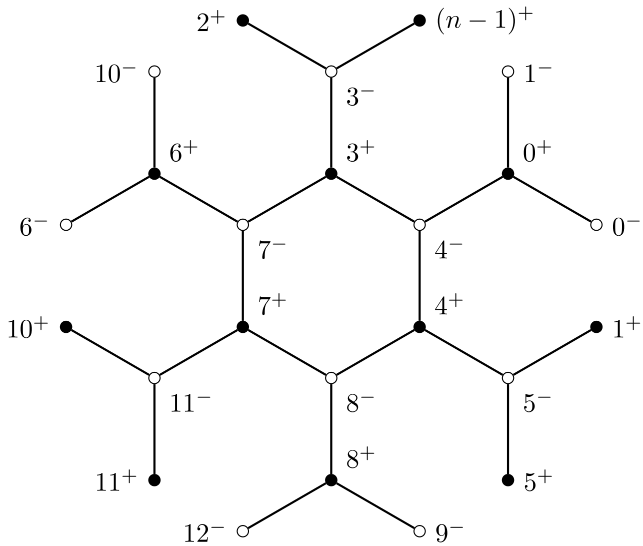


Figure 6: The set $\Sigma = \{0^+, 6^+, 8^+, 3^-, 5^-, 11^-\}$ is a splitting set for $H(n, \{0, 1, 4\})$ where $n \geq 13$.

By Corollary 3.3, there are infinitely many splittable (n_3) configurations. However, we are also able to show that there is no upper bound on the number of vertices of unsplittable (n_3) configurations:

Theorem 3.4. *There exist infinitely many cyclic* (n_3) *configurations that are unsplittable.*

Proof. We use the cyclic Haar graphs $X = H(n, \{0, 1, 3\})$, where $n \geq 7$. Clearly, each of them has girth 6. The graph can be written as $\mathrm{LCF}[5, -5]^n$. (For the LCF notation see [19].) This means that the edges determined by symbols 0 and 1 form a Hamiltonian cycle while the edges arising from the symbol 3 form chords of length 5. See Figure 1 for an example.

Let us assume the result does not hold. This means there exists a splitting set Σ. By removing Σ from the graph the Hamiltonial cycle breaks into paths. Each path must contain at least two vertices. Let the sequence $\Pi = (p_1, p_2, \ldots, p_k)$ denote the lengths of the consecutive paths along the Hamiltonial cycle. The rest of the proof is in two steps:

Step 1. If there are no two consecutive numbers of Π equal to 2, then the corresponding segments are connected in $X - \Sigma$ since there is a chord of length 5 joining these two segments. But this means that all paths are connected by chords, so Σ is not a splitting set.

Step 2. We can show that no two consecutive segments are of length 2. In case of two adjacent segments of length 2 we would have vertices $\{i - 3, i, i + 3\} \subseteq \Sigma$. But that is impossible, since $i - 3$ is adjacent to $i + 3$ in X^2. □

Note that this is not the only such family. Here is another one:

Theorem 3.5. *Cyclic configurations defined by* $H(3n, \{0, 1, n\})$*, where* $n \geq 2$*, are unsplittable.*

Proof. The technique used here is similar to the technique used in proof of Theorem 3.4. Let $X = H(3n, \{0, 1, n\})$. The graph X can be written as $\mathrm{LCF}[2n-1, -(2n-1)]^{3n}$. Suppose that there exists a splitting set Σ. The edges determined by symbols 0 and 1 form a Hamiltonian cycle which breaks into paths when the splitting set Σ is removed.

We show that any two consecutive paths are connected in $X - \Sigma$. Without loss of generality (because of symmetry), we may assume that $0^+ \in \Sigma$ is the vertex adjacent to the two paths under consideration. If $0^+ \in \Sigma$ then $1^-, 0^-, n^-, n^+, 1^+, 2n^+, (2n+1)^+ \notin \Sigma$. We show that vertices 1^- and 0^- (which belong to the two paths under consideration) are connected in $X - \Sigma$.

If $(2n+1)^- \notin \Sigma$ then $2n^+$ and $(2n+1)^+$ are connected in $X-\Sigma$. Since 0^- is adjacent to $2n^+$ and 1^- is adjacent to $(2n+1)^+$, vertices 0^- and 1^- are also connected in $X - \Sigma$. Now, suppose that $(2n+1)^- \in \Sigma$. This implies that $2n^-, (n+1)^+(n+1)^- \notin \Sigma$. Then $2n^+$ is adjacent to $2n^-$, $2n^-$ is adjacent to n^+, n^+ is adjacent to $(n+1)^-$, $(n+1)^-$ is adjacent to 1^+, and 1^+ is adjacent to 1^- in $X - \Sigma$. Therefore, 1^- and 0^- are connected in $X - \Sigma$. □

Cubic symmetric bicirculants were classified in [13] and [16]. These results can be summarised as follows:

Theorem 3.6 ([13, 16])**.** *A connected cubic symmetric graph is a bicirculant if and only if it is isomorphic to one of the following graphs:*

(1) the complete graph K_4,

(2) the complete bipartite graph $K_{3,3}$,

(3) the seven symmetric generalized Petersen graphs $GP(4,1)$, $GP(5,2)$, $GP(8,3)$, $GP(10,2)$, $GP(10,3)$, $GP(12,5)$ and $GP(24,5)$,

(4) the Heawood graph $H(7, \{0, 1, 3\})$, and

(5) the cyclic Haar graph $H(n, \{0, 1, r+1\})$, where $n \geq 11$ is odd and $r \in \mathbb{Z}_n^*$ such that $r^2 + r + 1 \equiv 0 \pmod{n}$.

It is well known that an (n_3) configuration is flag-transitive if and only if its Levi graph is cubic symmetric graph of girth at least 6. From Theorem 3.6 it follows that the girth of any connected cubic symmetric bicirculant is at most 6. If the girth of such a graph is 6 or more then it is a Levi graph of a flag-transitive configuration. This enables us to characterise splittability of such configurations:

Theorem 3.7. *The Fano plane (7_3), the Möbius-Kantor configuration (8_3), and the Desargues configuration (10_3) are unsplittable. Their Levi graphs are*

$$H(7, \{0, 1, 3\}), \quad H(8, \{0, 1, 3\}) \cong GP(8, 3) \quad \text{and} \quad GP(10, 3),$$

respectively.

If $n \geq 9$, all flag-transitive (n_3) configurations, except the Desargues configuration, are splittable.

Proof. We start with the classification given in Theorem 3.6. Only bipartite graphs of girth 6 have to be considered. This rules out the complete graph K_4, the complete bipartite graph $K_{3,3}$, and the generalised Petersen graphs $GP(5,2)$, $GP(10,2)$ and $GP(4,1)$. Note that $GP(4,1)$ is isomorphic to the cube graph Q_3.

Table 1: Overview of splittable and unsplittable connected cyclic Haar graphs.

n	(a)	(b)	(c)	(d)	(e)	(f)
3	1	0	0	1	0	0
4	1	0	0	1	0	0
5	1	0	0	1	0	0
6	2	0	0	2	0	0
7	2	1	0	2	0	1
8	3	1	1	2	0	1
9	2	1	0	2	0	1
10	3	1	1	2	0	1
11	2	1	0	2	0	1
12	5	3	1	4	0	3
13	3	2	1	2	1	1
14	4	2	2	2	1	1
15	5	4	1	4	1	3
16	5	3	3	2	2	1
17	3	2	1	2	1	1
18	6	4	3	3	2	2
19	4	3	2	2	2	1
20	7	5	5	2	4	1
21	7	6	3	4	3	3
22	6	4	4	2	3	1
23	4	3	2	2	2	1
24	11	9	7	4	6	3
25	5	4	3	2	3	1
26	7	5	5	2	4	1
27	6	5	3	3	3	2
28	9	7	7	2	6	1
29	5	4	3	2	3	1
30	13	11	9	4	8	3

(a) Number of non-isomorphic connected cubic cyclic Haar graphs on $2n$ vertices.
(b) Those that have girth 6. (c) Those that are splittable. (d) Those that are unsplittable.
(e) Those that are splittable of girth 6. (f) Those that are unsplittable of girth 6.

It is well known, but one may check by computer that $GP(8,3) \cong H(8,\{0,1,3\})$. See for instance [9, Table 2].

One may also check by computer that $GP(8,3)$, $GP(10,3)$ and the Heawood graph $H(7,\{0,1,3\})$ are unsplittable.

Let

$$V(GP(n,k)) = \{0,1,\ldots,n-1,0',1',\ldots,(n-1)'\} \text{ and}$$
$$E(GP(n,k)) = \{\{i',((i+1) \bmod n)'\},\{i,i'\},\{i,(i+k) \bmod n\} \mid i=0,\ldots,n-1\}.$$

Note that $\Sigma = \{0',4',8',2,6,10\}$ is a splitting set for $GP(12,5)$ as shown in Figure 7. Also, $GP(12,5) - S \cong 3C_6$, i.e., a disjoint union of three copies of C_6. The splitting

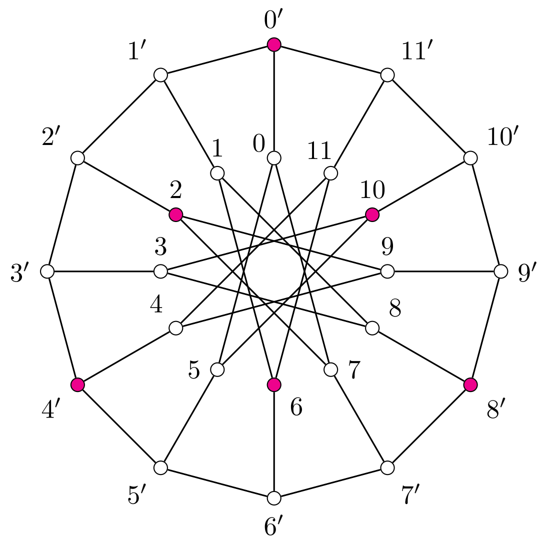


Figure 7: The magenta vertices form a splitting set for the Nauru graph $GP(12,5)$ [6, 21].

set for $GP(24,5)$ is $\Sigma = \{0',4',8',12',16',20',2,6,10,14,18,22\}$ as shown in Figure 8. Note that $GP(24,5) - S \cong 3C_{12}$. Also, note that $GP(24,5)$ is not isomorphic to a cyclic Haar graph since its girth is 8.

Using Theorem 3.1, one may verify that all graphs in item (5) of Theorem 3.6 have girth 6 and for each of them the splitting set is $\{0^+, 2r^+, (2r+2)^+, r^-, (r+2)^-, (3r+2)^-\}$. We have

$$\mathcal{W} = \{0,1,r,r+1,r+2,2r,2r+1,2r+2,2r+3,3r+1,3r+2,3r+3\},$$
$$\mathcal{B} = \{0,1,n-1,r-1,r,r+1,r+2,2r,2r+1,2r+2,3r+1,3r+2\}.$$

It is easy to verify that all elements of $\mathcal{W}$ are distinct and that all elements of $\mathcal{B}$ are distinct. For example, suppose that $r \equiv 3r+3 \pmod{n}$. This means that

$$2r \equiv -3 \pmod{n}. \tag{3.1}$$

From condition $r^2+r+1 \equiv 0 \pmod{n}$ we obtain

$$4r^2+4r+4 = (2r)^2 + 2\cdot 2r + 4 \equiv 0 \pmod{n}. \tag{3.2}$$

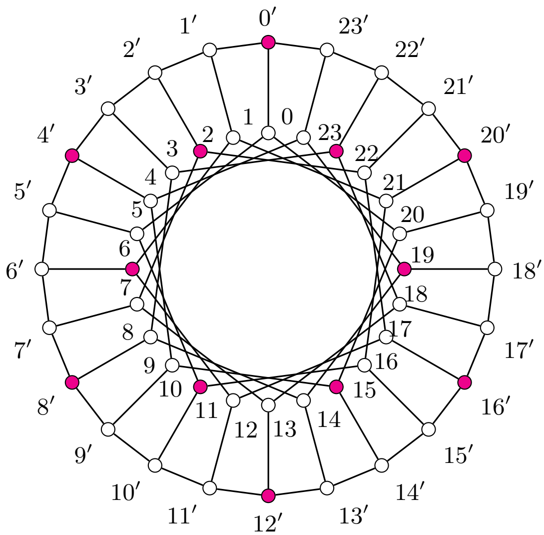


Figure 8: The magenta vertices form a splitting set for $GP(24, 5)$ which was recently named the ADAM graph [14].

Equations (3.1) and (3.2) together imply that $(-3)^2 + 2 \cdot (-3) + 4 = 7 \equiv 0 \pmod{n}$, which is a contradiction since $n > 11$. All other cases can be checked in a similar way. □

From Theorem 3.7 we directly obtain the following corollary.

Corollary 3.8. *A cyclic flag-transitive* (n_3) *configuration is splittable if and only if* $n > 8$. *The only two exceptions are:*

(1) $H(7, \{0, 1, 3\})$*, i.e. the Fano plane, and*

(2) $H(8, \{0, 1, 3\})$*, i.e. the Möbius-Kantor configuration.*

4 Splittable geometric (n_k) configurations

We will now show that for any k there exist a geometric, triangle-free, (n_k) configuration which is of type T1, i.e., it is point-splittable and line-splittable.

Let us first provide a construction to obtain a geometric (n_k) configuration for any k. We start with an unbalanced $(k_1, 1_k)$ configuration, denoted $\mathcal{G}_k^{(1)}$, that consists of a single line containing k points. Let $\mathcal{G}_k^{(i)}$ be a configuration that is obtained from $\mathcal{G}_k^{(i-1)}$ by the *k-fold parallel replication* (see [19, p. 245]). The configuration $\mathcal{G}_k^{(k)}$ is a balanced $({k^k}_k)$ configuration, called a *generalised Gray configuration*; see [17].

Lemma 4.1. *Let* $\mathcal{C}$ *be an arbitrary geometric* (n_k) *configuration. There exists a geometric* (kn_k) *configuration* $\mathcal{D}$ *that is point- and line-splittable. Moreover, if* $\mathcal{C}$ *is triangle-free then* $\mathcal{D}$ *is also triangle-free.*

Proof. Let $\mathcal{C}$ be as stated. Select an arbitrary line L of $\mathcal{C}$ passing through points $p^{(1)}, p^{(2)}, \ldots, p^{(k)}$ of $\mathcal{C}$ as shown in Figure 9(a). Remove line L and call the resulting structure

Table 2: List of non-isomorphic connected trivalent cyclic Haar graphs $H(n, S)$ with $n \leq 25$ and some of their properties.

n	S	(a)	(b)	(c)	(d)
3	$\{0,1,2\}$	$\bot$	4	2	$\top$
4	$\{0,1,2\}$	$\bot$	4	3	$\top$
5	$\{0,1,2\}$	$\bot$	4	3	$\bot$
6	$\{0,1,2\}$	$\bot$	4	4	$\bot$
6	$\{0,1,3\}$	$\bot$	4	3	$\bot$
7	$\{0,1,2\}$	$\bot$	4	4	$\bot$
7	$\{0,1,3\}$	$\bot$	6	3	$\top$
8	$\{0,1,2\}$	$\bot$	4	5	$\bot$
8	$\{0,1,3\}$	$\bot$	6	4	$\top$
8	$\{0,1,4\}$	$\top$	4	4	$\bot$
9	$\{0,1,2\}$	$\bot$	4	5	$\bot$
9	$\{0,1,3\}$	$\bot$	6	4	$\bot$
10	$\{0,1,2\}$	$\bot$	4	6	$\bot$
10	$\{0,1,3\}$	$\bot$	6	4	$\bot$
10	$\{0,1,5\}$	$\top$	4	5	$\bot$
11	$\{0,1,2\}$	$\bot$	4	6	$\bot$
11	$\{0,1,3\}$	$\bot$	6	5	$\bot$
12	$\{0,1,2\}$	$\bot$	4	7	$\bot$
12	$\{0,1,3\}$	$\bot$	6	5	$\bot$
12	$\{0,1,4\}$	$\bot$	6	5	$\bot$
12	$\{0,1,5\}$	$\bot$	6	5	$\bot$
12	$\{0,1,6\}$	$\top$	4	6	$\bot$
13	$\{0,1,2\}$	$\bot$	4	7	$\bot$
13	$\{0,1,3\}$	$\bot$	6	5	$\bot$
13	$\{0,1,4\}$	$\top$	6	5	$\top$
14	$\{0,1,2\}$	$\bot$	4	8	$\bot$
14	$\{0,1,3\}$	$\bot$	6	6	$\bot$
14	$\{0,1,4\}$	$\top$	6	5	$\bot$
14	$\{0,1,7\}$	$\top$	4	7	$\bot$
15	$\{0,1,2\}$	$\bot$	4	8	$\bot$
15	$\{0,1,3\}$	$\bot$	6	6	$\bot$
15	$\{0,1,4\}$	$\top$	6	5	$\bot$
15	$\{0,1,5\}$	$\bot$	6	5	$\bot$
15	$\{0,1,6\}$	$\bot$	6	5	$\bot$
16	$\{0,1,2\}$	$\bot$	4	9	$\bot$
16	$\{0,1,3\}$	$\bot$	6	6	$\bot$
16	$\{0,1,4\}$	$\top$	6	5	$\bot$
16	$\{0,1,7\}$	$\top$	6	5	$\bot$
16	$\{0,1,8\}$	$\top$	4	8	$\bot$
17	$\{0,1,2\}$	$\bot$	4	9	$\bot$
17	$\{0,1,3\}$	$\bot$	6	7	$\bot$
17	$\{0,1,4\}$	$\top$	6	5	$\bot$
18	$\{0,1,2\}$	$\bot$	4	10	$\bot$
18	$\{0,1,3\}$	$\bot$	6	7	$\bot$
18	$\{0,1,4\}$	$\top$	6	6	$\bot$
18	$\{0,1,5\}$	$\top$	6	6	$\bot$
18	$\{0,1,6\}$	$\bot$	6	6	$\bot$
18	$\{0,1,9\}$	$\top$	4	9	$\bot$
19	$\{0,1,2\}$	$\bot$	4	10	$\bot$
19	$\{0,1,3\}$	$\bot$	6	7	$\bot$
19	$\{0,1,4\}$	$\top$	6	6	$\bot$
19	$\{0,1,8\}$	$\top$	6	5	$\top$
20	$\{0,1,2\}$	$\bot$	4	11	$\bot$
20	$\{0,1,3\}$	$\bot$	6	8	$\bot$
20	$\{0,1,4\}$	$\top$	6	6	$\bot$
20	$\{0,1,5\}$	$\top$	6	6	$\bot$
20	$\{0,1,6\}$	$\top$	6	6	$\bot$
20	$\{0,1,9\}$	$\top$	6	7	$\bot$
20	$\{0,1,10\}$	$\top$	4	10	$\bot$
21	$\{0,1,2\}$	$\bot$	4	11	$\bot$
21	$\{0,1,3\}$	$\bot$	6	8	$\bot$
21	$\{0,1,4\}$	$\top$	6	6	$\bot$
21	$\{0,1,5\}$	$\top$	6	6	$\top$
21	$\{0,1,7\}$	$\bot$	6	7	$\bot$
21	$\{0,1,8\}$	$\bot$	6	7	$\bot$
21	$\{0,1,9\}$	$\top$	6	6	$\bot$
22	$\{0,1,2\}$	$\bot$	4	12	$\bot$
22	$\{0,1,3\}$	$\bot$	6	8	$\bot$
22	$\{0,1,4\}$	$\top$	6	7	$\bot$
22	$\{0,1,5\}$	$\top$	6	6	$\bot$
22	$\{0,1,6\}$	$\top$	6	7	$\bot$
22	$\{0,1,11\}$	$\top$	4	11	$\bot$
23	$\{0,1,2\}$	$\bot$	4	12	$\bot$
23	$\{0,1,3\}$	$\bot$	6	9	$\bot$
23	$\{0,1,4\}$	$\top$	6	7	$\bot$
23	$\{0,1,5\}$	$\top$	6	7	$\bot$
24	$\{0,1,2\}$	$\bot$	4	13	$\bot$
24	$\{0,1,3\}$	$\bot$	6	9	$\bot$
24	$\{0,1,4\}$	$\top$	6	7	$\bot$
24	$\{0,1,5\}$	$\top$	6	7	$\bot$
24	$\{0,1,6\}$	$\top$	6	7	$\bot$
24	$\{0,1,7\}$	$\top$	6	7	$\bot$
24	$\{0,1,8\}$	$\bot$	6	8	$\bot$
24	$\{0,1,9\}$	$\bot$	6	8	$\bot$
24	$\{0,1,10\}$	$\top$	6	6	$\bot$
24	$\{0,1,11\}$	$\top$	6	7	$\bot$
24	$\{0,1,12\}$	$\top$	4	12	$\bot$
25	$\{0,1,2\}$	$\bot$	4	13	$\bot$
25	$\{0,1,3\}$	$\bot$	6	9	$\bot$
25	$\{0,1,4\}$	$\top$	6	7	$\bot$
25	$\{0,1,5\}$	$\top$	6	7	$\bot$
25	$\{0,1,10\}$	$\top$	6	7	$\bot$

(a) splittable? (b) girth (c) diameter (d) arc-transitive?

Table 3: List of non-isomorphic connected trivalent cyclic Haar graphs $H(n, S)$ with $26 \leq n \leq 30$ and some of their properties.

n	S	(a)	(b)	(c)	(d)
26	$\{0,1,2\}$	$\bot$	4	14	$\bot$
26	$\{0,1,3\}$	$\bot$	6	10	$\bot$
26	$\{0,1,4\}$	$\top$	6	8	$\bot$
26	$\{0,1,5\}$	$\top$	6	7	$\bot$
26	$\{0,1,7\}$	$\top$	6	8	$\bot$
26	$\{0,1,8\}$	$\top$	6	7	$\bot$
26	$\{0,1,13\}$	$\top$	4	13	$\bot$
27	$\{0,1,2\}$	$\bot$	4	14	$\bot$
27	$\{0,1,3\}$	$\bot$	6	10	$\bot$
27	$\{0,1,4\}$	$\top$	6	8	$\bot$
27	$\{0,1,5\}$	$\top$	6	7	$\bot$
27	$\{0,1,6\}$	$\top$	6	7	$\bot$
27	$\{0,1,9\}$	$\bot$	6	9	$\bot$
28	$\{0,1,2\}$	$\bot$	4	15	$\bot$
28	$\{0,1,3\}$	$\bot$	6	10	$\bot$
28	$\{0,1,4\}$	$\top$	6	8	$\bot$
28	$\{0,1,5\}$	$\top$	6	7	$\bot$
28	$\{0,1,6\}$	$\top$	6	8	$\bot$
28	$\{0,1,7\}$	$\top$	6	7	$\bot$
28	$\{0,1,8\}$	$\top$	6	7	$\bot$
28	$\{0,1,13\}$	$\top$	6	9	$\bot$
28	$\{0,1,14\}$	$\top$	4	14	$\bot$
29	$\{0,1,2\}$	$\bot$	4	15	$\bot$
29	$\{0,1,3\}$	$\bot$	6	11	$\bot$
29	$\{0,1,4\}$	$\top$	6	8	$\bot$
29	$\{0,1,5\}$	$\top$	6	7	$\bot$
29	$\{0,1,9\}$	$\top$	6	7	$\bot$
30	$\{0,1,2\}$	$\bot$	4	16	$\bot$
30	$\{0,1,3\}$	$\bot$	6	11	$\bot$
30	$\{0,1,4\}$	$\top$	6	9	$\bot$
30	$\{0,1,5\}$	$\top$	6	7	$\bot$
30	$\{0,1,6\}$	$\top$	6	7	$\bot$
30	$\{0,1,7\}$	$\top$	6	7	$\bot$
30	$\{0,1,8\}$	$\top$	6	9	$\bot$
30	$\{0,1,9\}$	$\top$	6	7	$\bot$
30	$\{0,1,10\}$	$\bot$	6	10	$\bot$
30	$\{0,1,11\}$	$\bot$	6	10	$\bot$
30	$\{0,1,12\}$	$\top$	6	8	$\bot$
30	$\{0,1,15\}$	$\top$	4	15	$\bot$
30	$\{0,2,5\}$	$\top$	6	8	$\bot$

(a) splittable? (b) girth (c) diameter (d) arc-transitive?

$\mathcal{C}'$. Make k copies of $\mathcal{C}'$: $\mathcal{C}'_1, \mathcal{C}'_2, \ldots, \mathcal{C}'_k$ and place them equally spaced in any direction $\vec{v}$ that is non-parallel to the direction of any line of $\mathcal{C}'$ (see Figure 9(b)). Point of $\mathcal{C}'_i$ that

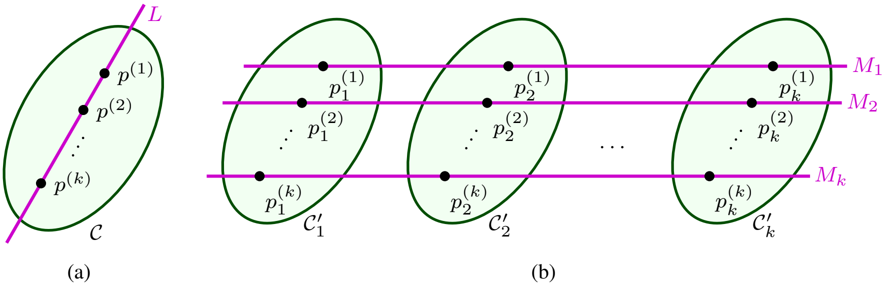


Figure 9: Construction provided by Lemma 4.1.

correspond to $p^{(j)}$ in $\mathcal{C}'$ is denoted $p_i^{(j)}$. Now add lines $M_1, M_2, \ldots, M_k$, such that M_i passes through points $p_1^{(i)}, p_2^{(i)}, \ldots, p_k^{(i)}$. The resulting structure, denoted $\mathcal{D}$, is clearly a (kn_k) configuration.

The set of lines $\{M_1, M_2, \ldots, M_k\}$ is a splitting set of $\mathcal{D}$ which proves that $\mathcal{D}$ is line-splittable. The set of points $\{p_i^{(1)}, p_i^{(2)}, \ldots, p_i^{(k)}\}$ is a splitting set for an arbitrary $1 \leq i \leq k$ which proves that $\mathcal{D}$ is also point-splittable.

It is easy to see that the resulting structure $\mathcal{D}$ is triangle-free. □

Now we can state the main result of this section.

Theorem 4.2. *For any $k > 1$ and any n_0 there exist a number $n > n_0$, such that there exists a splittable (n_k) configuration.*

Proof. Let $\mathcal{C}_0 = \mathcal{G}_k^{(k)}$, i.e. the generalised Gray $(k^k{}_k)$ configuration. Let $\mathcal{C}_i$ be a configuration obtained from $\mathcal{C}_{i-1}$ by an application of Lemma 4.1. Note that the obtained configuration $\mathcal{C}_i$ is not uniquely defined – it depends on the choice of the line L.

From Lemma 4.1 it follows that each $\mathcal{C}_i$, $i \geq 1$, is a point- and line-splittable configuration. Each configuration $\mathcal{C}_i$ is balanced and the number of points of $\mathcal{C}_{i+1}$ is strictly greater than the number of points of $\mathcal{C}_i$. Therefore, for increasing values of i, the number of points will eventually exceed any given number n_0. □

Since configurations $\mathcal{C}_1, \mathcal{C}_2, \ldots$ constructed in the proof of Theorem 4.2 are all of type T1, their duals are also of type T1.

Example 4.3. The generalised Gray $(k^k{}_k)$ configuration for $k = 3$ is simply called the *Gray configuration* (see Figure 10(a) and [17]). Let $\mathcal{C}_0$ be the Gray configuration. By one application of Lemma 4.1 we obtain a configuration $\mathcal{C}_1$ (see Figure 10(b)) which is point- and line-splittable.

5 Conclusion

Theorems 3.4 and 3.5, Corollary 3.3, and our experimental investigations (see periodic behaviour of the last column of Table 1 past $n = 9$) of splittability of cyclic Haar graphs led us to the following conjecture.

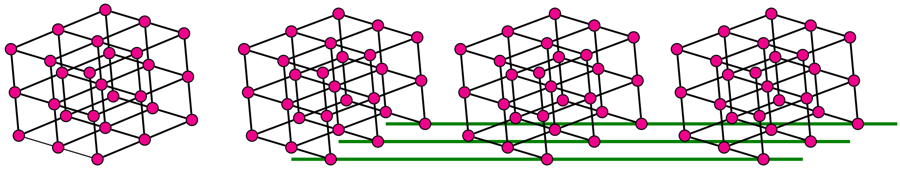

(a) (b)

Figure 10: The Gray (27_3) configuration $\mathcal{C}_0$ and the corresponding $\mathcal{C}_1$.

Conjecture 5.1. *A cyclic (n_3) configuration is unsplittable if and only if its Levi graph belongs to one of the following three infinite families:*

(1) $H(n, \{0, 1, 3\})$ *for* $n \geq 7$*;*

(2) $H(3n, \{0, 1, n\})$ *for* $n \geq 2$*;*

(3) $H(3n, \{0, 1, n + 1\})$ *for* $n \geq 4$ *where* $n \not\equiv 0 \pmod 3$.

To show that all other cyclic (n_3) configurations are splittable, we expect that the method used in the proof of Theorem 3.1, Corollary 3.2 and Corollary 3.3 can be extended. Nedela and Škoviera [15] showed a nice property of cubic graphs with respect to the cyclic connectivity. Their result is likely to have applications in splittablity.

In Section 4 we have shown how to construct geometric point- and line-splittable (n_k) configuration for any k. However, we were not able to obtain any splittable cyclic (n_k) configuration for $k \geq 4$ so far. Therefore, we pose the following claim.

Conjecture 5.2. *All cyclic (n_k) configurations for $k \geq 4$ are unsplittable.*

Notions of splittable and unsplittable configurations have been defined via associated graphs. Since splittability is a property of combinatorial configurations, it can be extended from bipartite graphs of girth at least 6 to more general graphs. We expect that results concerning cyclic connectivity such as those presented in [15] will play an important role in such investigations.

Note that cyclic Haar graphs have girth at most 6 and form a special class of bicirculants [16]. However, there exist other bicirculants with girth greater than 6. The corresponding configurations have been investigated in [3, 1]. One way of extending this study is on the one hand to consider splittability of these more general bicirculants and on the other hand to study tricirculants [12], tetracirculants and beyond [7]. In the language of configurations, they can be described as special classes of polycyclic configurations [2].

References

[1] M. Boben, B. Grünbaum, T. Pisanski and A. Žitnik, Small triangle-free configurations of points and lines, *Discrete Comput. Geom.* **35** (2006), 405–427, doi:10.1007/s00454-005-1224-9.

[2] M. Boben and T. Pisanski, Polycyclic configurations, *European J. Combin.* **24** (2003), 431–457, doi:10.1016/s0195-6698(03)00031-3.

[3] M. Boben, T. Pisanski and A. Žitnik, I-graphs and the corresponding configurations, *J. Combin. Des.* **13** (2005), 406–424, doi:10.1002/jcd.20054.

[4] H. S. M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.* **56** (1950), 413–455, doi:10.1090/s0002-9904-1950-09407-5.

[5] H. S. M. Coxeter, R. Frucht and D. L. Powers, *Zero-Symmetric Graphs*, Academic Press, London, 1981, doi:10.1016/c2013-0-10543-0.

[6] D. Eppstein, The many faces of the Nauru graph, 2007, `https://11011110.github.io/blog/2007/12/12/many-faces-of.html`.

[7] B. Frelih and K. Kutnar, Classification of cubic symmetric tetracirculants and pentacirculants, *European J. Combin.* **34** (2013), 169–194, doi:10.1016/j.ejc.2012.08.005.

[8] B. Grünbaum, *Configurations of Points and Lines*, volume 103 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, Rhode Island, 2009, doi:10.1090/gsm/103.

[9] M. Hladnik, D. Marušič and T. Pisanski, Cyclic Haar graphs, *Discrete Math.* **244** (2002), 137–152, doi:10.1016/s0012-365x(01)00064-4.

[10] H. Koike, I. Kovács, D. Marušič and M. Muzychuk, Cyclic groups are CI-groups for balanced configurations, *Des. Codes Cryptogr.* (2018), doi:10.1007/s10623-018-0517-y.

[11] H. Koike, I. Kovács and T. Pisanski, The number of cyclic configurations of type (v_3) and the isomorphism problem, *J. Combin. Des.* **22** (2014), 216–229, doi:10.1002/jcd.21387.

[12] I. Kovács, K. Kutnar, D. Marušič and S. Wilson, Classification of cubic symmetric tricirculants, *Electron. J. Combin.* **19** (2012), #P24, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v19i2p24`.

[13] D. Marušič and T. Pisanski, Symmetries of hexagonal molecular graphs on the torus, *Croat. Chem. Acta* **73** (2000), 969–981, `http://hrcak.srce.hr/131972`.

[14] D. Marušič and T. Pisanski, The ADAM graph and its configuration, *Art Discrete Appl. Math.* **1** (2018), #E1.01, doi:10.26493/2590-9770.1215.1e8.

[15] R. Nedela and M. Škoviera, Atoms of cyclic connectivity in cubic graphs, *Math. Slovaca* **45** (1995), 481–499, `http://maslo.mat.savba.sk/paper.php?id_paper=157`.

[16] T. Pisanski, A classification of cubic bicirculants, *Discrete Math.* **307** (2007), 567–578, doi:10.1016/j.disc.2005.09.053.

[17] T. Pisanski, Yet another look at the Gray graph, *New Zealand J. Math.* **36** (2007), 85–92, `http://nzjm.math.auckland.ac.nz/index.php/Yet_Another_Look_at_the_Gray_Graph`.

[18] T. Pisanski and M. Randić, Bridges between geometry and graph theory, in: C. A. Gorini (ed.), *Geometry at Work*, Math. Assoc. America, Washington, D.C., volume 53 of *MAA Notes*, pp. 174–194, 2000.

[19] T. Pisanski and B. Servatius, *Configurations from a Graphical Viewpoint*, Birkhäuser Advanced Texts, Birkhäuser, New York, 2013, doi:10.1007/978-0-8176-8364-1.

[20] T. Pisanski and T. W. Tucker, On the maximum number of independent elements in configurations of points and lines, *Discrete Comput. Geom.* **52** (2014), 361–365, doi:10.1007/s00454-014-9618-1.

[21] A. Žitnik, B. Horvat and T. Pisanski, All generalized Petersen graphs are unit-distance graphs, *J. Korean Math. Soc.* **49** (2012), 475–491, doi:10.4134/jkms.2012.49.3.475.

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A note on the 4-girth-thickness of $K_{n,n,n}$*

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Abstract

The 4-girth-thickness $\theta(4, G)$ of a graph G is the minimum number of planar subgraphs of girth at least four whose union is G. In this paper, we obtain that the 4-girth-thickness of complete tripartite graph $K_{n,n,n}$ is $\lceil \frac{n+1}{2} \rceil$ except for $\theta(4, K_{1,1,1}) = 2$. And we also show that the 4-girth-thickness of the complete graph K_{10} is three which disprove the conjecture posed by Rubio-Montiel concerning to $\theta(4, K_{10})$.

Keywords: Thickness, 4-girth-thickness, complete tripartite graph.

Math. Subj. Class.: 05C10

1 Introduction

The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G. It was defined by W. T. Tutte [10] in 1963. Then, the thicknesses of some graphs have been obtained when the graphs are hypercube [7], complete graph [1, 2, 11], complete bipartite graph [3] and some complete multipartite graphs [6, 12, 13].

In 2017, Rubio-Montiel [9] defined the g-girth-thickness $\theta(g, G)$ of a graph G as the minimum number of planar subgraphs whose union is G with the girth of each subgraph is at least g. It is a generalization of the usual thickness in which the 3-girth-thickness $\theta(3, G)$ is the usual thickness $\theta(G)$. He also determined the 4-girth-thickness of the complete graph K_n except K_{10} and he conjectured that $\theta(4, K_{10}) = 4$. Let $K_{n,n,n}$ denote a complete tripartite graph in which each part contains n ($n \geq 1$) vertices. In [13], Yang obtained $\theta(K_{n,n,n}) = \lceil \frac{n+1}{3} \rceil$ when $n \equiv 3 \pmod 6$.

In this paper, we determine $\theta(4, K_{n,n,n})$ for all values of n and we also give a decomposition of K_{10} with three planar subgraphs of girth at least four, which shows $\theta(4, K_{10}) = 3$.

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2 The 4-girth-thickness of $K_{n,n,n}$

Lemma 2.1 ([4]). *A planar graph with n vertices and girth g has at most $\frac{g}{g-2}(n-2)$ edges.*

Theorem 2.2. *The 4-girth-thickness of $K_{n,n,n}$ is*

$$\theta(4, K_{n,n,n}) = \left\lceil \frac{n+1}{2} \right\rceil$$

except for $\theta(4, K_{1,1,1}) = 2$.

Proof. It is trivial for $n = 1$, $\theta(4, K_{1,1,1}) = 2$. When $n > 1$, because $|E(K_{n,n,n})| = 3n^2$, $|V(K_{n,n,n})| = 3n$, from Lemma 2.1, we have

$$\theta(4, K_{n,n,n}) \geq \left\lceil \frac{3n^2}{2(3n-2)} \right\rceil = \left\lceil \frac{n}{2} + \frac{1}{3} + \frac{2}{3(3n-2)} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil.$$

In the following, we give a decomposition of $K_{n,n,n}$ into $\lceil \frac{n+1}{2} \rceil$ planar subgraphs of girth at least four to complete the proof. Let the vertex partition of $K_{n,n,n}$ be (U, V, W), where $U = \{u_1, \dots, u_n\}$, $V = \{v_1, \dots, v_n\}$ and $W = \{w_1, \dots, w_n\}$. In this proof, all the subscripts of vertices are taken modulo $2p$.

Case 1: When $n = 2p$ $(p \geq 1)$. Let $G_1, \dots, G_{p+1}$ be the graphs whose edge set is empty and vertex set is the same as $V(K_{2p,2p,2p})$.

Step 1: For each G_i $(1 \leq i \leq p)$, arrange all the vertices $u_1, v_{3-2i}, u_2, v_{4-2i}, u_3, v_{5-2i}, \dots, u_{2p}, v_{2p-2i+2}$ on a circle and join u_j to v_{j+2-2i} and v_{j+1-2i}, $1 \leq j \leq 2p$. Then we get a cycle of length $4p$, denote it by G_i^1 $(1 \leq i \leq p)$.

Step 2: For each G_i^1 $(1 \leq i \leq p)$, place the vertex w_{2i-1} inside the cycle and join it to $u_1, \dots, u_{2p}$, place the vertex w_{2i} outside the cycle and join it to $v_1, \dots, v_{2p}$. Then we get a planar graph G_i^2 $(1 \leq i \leq p)$.

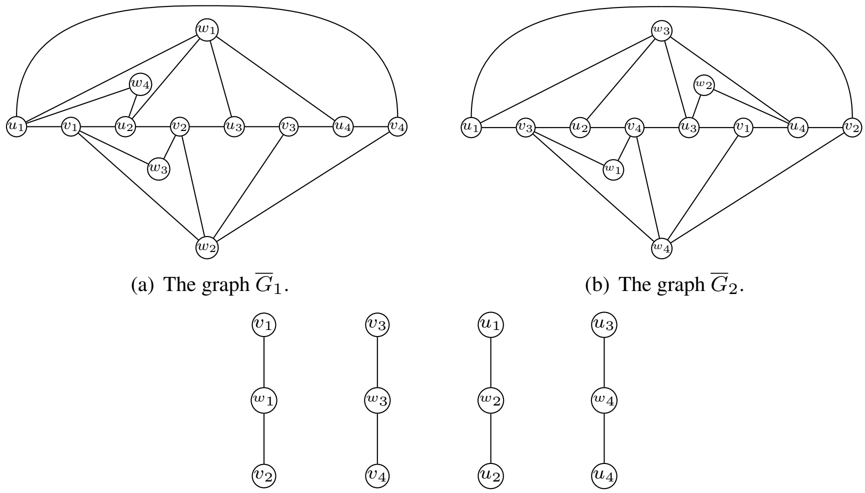
Step 3: For each G_i^2 $(1 \leq i \leq p)$, place vertices w_{2j} for $1 \leq j \leq p$ and $j \neq i$, inside of the quadrilateral $w_{2i-1}u_{2i-1}v_1u_{2i}$ and join each of them to vertices u_{2i-1} and u_{2i}. Place vertices w_{2j-1}, for $1 \leq j \leq p$ and $j \neq i$, inside of the quadrilateral $w_{2i}v_{2i-1}u_kv_{2i}$, in which u_k is some vertex from U. Join each of them to vertices v_{2i-1} and v_{2i}. Then we get a planar graph $\overline{G}_i$ $(1 \leq i \leq p)$.

Step 4: For G_{p+1}, join w_{2i-1} to both v_{2i-1} and v_{2i}, join w_{2i} to both u_{2i-1} and u_{2i}, for $1 \leq i \leq p$, then we get a planar graph $\overline{G}_{p+1}$.

For $\overline{G}_1 \cup \dots \cup \overline{G}_{p+1} = K_{n,n,n}$, and the girth of $\overline{G}_i$ $(1 \leq i \leq p+1)$ is at least four, we obtain a 4-girth planar decomposition of $K_{2p,2p,2p}$ with $p+1$ planar subgraphs. Figure 1 shows a 4-girth planar decomposition of $K_{4,4,4}$ with three planar subgraphs.

Case 2: When $n = 2p+1$ $(p > 1)$. Base on the 4-girth planar decomposition $\{\overline{G}_1, \dots, \overline{G}_{p+1}\}$ of $K_{2p,2p,2p}$, by adding vertices and edges to each $\overline{G}_i$ $(1 \leq i \leq p+1)$ and some other modifications on it, we will get a 4-girth planar decomposition of $K_{2p+1,2p+1,2p+1}$ with $p+1$ subgraphs.

Step 1: (Add u to $\overline{G}_i, 1 \leq i \leq p$.) For each $\overline{G}_i$ $(1 \leq i \leq p)$, we notice that the order of the $p-1$ interior vertices w_{2j}, $1 \leq j \leq p$, and $j \neq i$ in the quadrilateral



(a) The graph $\overline{G}_1$.

(b) The graph $\overline{G}_2$.

(c) The graph $\overline{G}_3$.

Figure 1: A 4-girth planar decomposition of $K_{4,4,4}$.

$w_{2i-1}u_{2i-1}v_1u_{2i}$ of $\overline{G}_i$ has no effect on the planarity of $\overline{G}_i$. We adjust the order of them, such that $w_{2i-1}u_{2i-1}w_{2p-2i+2}u_{2i}$ is a face of a plane embedding of $\overline{G}_i$. Place the vertex u in this face and join it to both w_{2i-1} and $w_{2p-2i+2}$. We denote the planar graph we obtain by $\widehat{G}_i$ $(1 \le i \le p)$.

Step 2: (Add v and w to $\widehat{G}_1$.) Delete the edge v_1u_2 in $\widehat{G}_1$, put both v and w in the face $w_ku_1v_1w_tv_2u_2$ in which w_k is some vertex from $\{w_{2j} \mid 1 < j \le p\}$ and w_t is some vertex from $\{w_{2j-1} \mid 1 < j \le p\}$. Join v to w, join v to u_1, u_2, and join w to v_1, v_2, we get a planar graph $\widetilde{G}_1$.

Step 3: (Add v and w to $\widehat{G}_i, 2 \le i \le p$.) For each $\widehat{G}_i$ $(2 \le i \le p)$, place the vertex v in the face $w_ku_{2i-1}v_1u_{2i}$ in which w_k is some vertex from $\{w_{2j} \mid 1 \le j \le p \text{ and } j \ne i\}$, and join it to u_{2i-1} and u_{2i}. Place the vertex w in the face $w_kv_{2i-1}u_tv_{2i}$ in which w_k is some vertex from $\{w_{2j-1} \mid 1 \le j \le p \text{ and } j \ne i\}$ and u_t is some vertex from U. Join w to both v_{2i-1} and v_{2i}, we get a planar graph $\widetilde{G}_i$ $(2 \le i \le p)$.

Step 4: (Add u, v and w to $\overline{G}_{p+1}$.) We add u, v and w to $\overline{G}_{p+1}$. For $1 \le i \le 2p$, join u to each v_i, join v to each w_i, join w to each u_i, join u to both v and w, and join v_1 to u_2, then we get a planar graph $\widetilde{G}_{p+1}$. Figure 2 shows a plane embedding of $\widetilde{G}_{p+1}$.

For $\widetilde{G}_1 \cup \cdots \cup \widetilde{G}_{p+1} = K_{n,n,n}$, and the girth of $\widetilde{G}_i$ $(1 \le i \le p+1)$ is at least four, we obtain a 4-girth planar decomposition of $K_{2p+1,2p+1,2p+1}$ with $p+1$ planar subgraphs. Figure 3 shows a 4-girth planar decomposition of $K_{5,5,5}$ with three planar subgraphs.

Case 3: When $n = 3$, Figure 4 shows a 4-girth planar decomposition of $K_{3,3,3}$ with two planar subgraphs.

Summarizing the above, the theorem is obtained. □

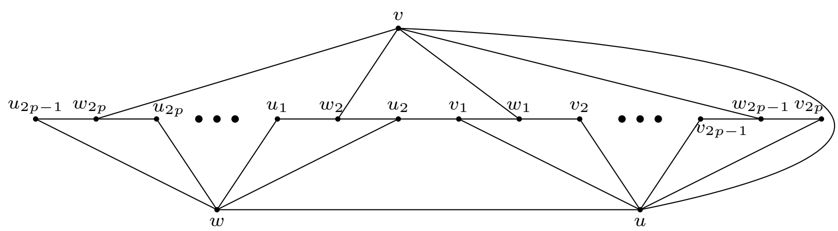
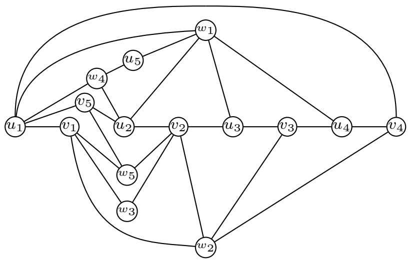
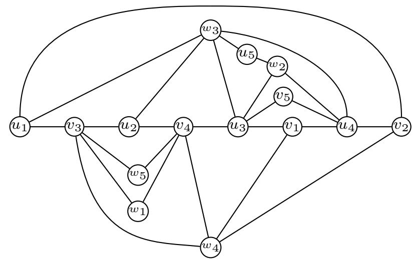


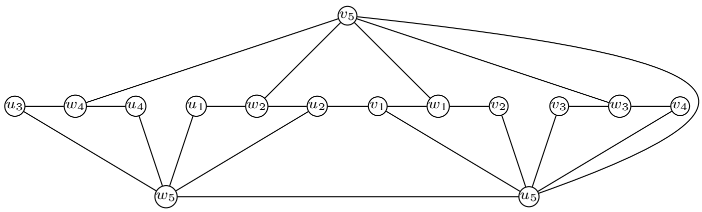
Figure 2: The graph $\widetilde{G}_{p+1}$.



(a) The graph $\widetilde{G}_1$.



(b) The graph $\widetilde{G}_2$.



(c) The graph $\widetilde{G}_3$.

Figure 3: A 4-girth planar decomposition of $K_{5,5,5}$.

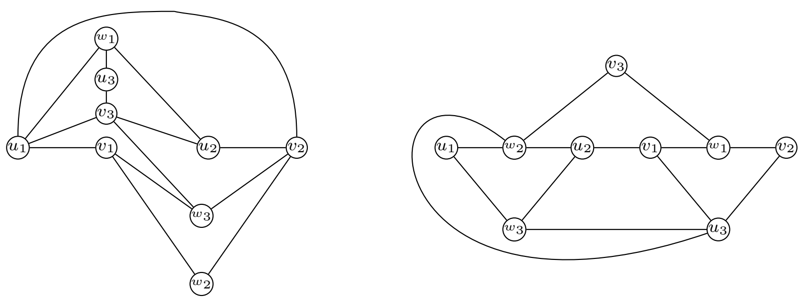


Figure 4: A 4-girth planar decomposition of $K_{3,3,3}$.

3 The 4-girth-thickness of K_{10}

In [9], the author posed the question whether $\theta(4, K_{10}) = 3$ or 4, and conjectured that it is four. We disprove his conjecture by showing $\theta(4, K_{10}) = 3$.

Theorem 3.1. *The 4-girth-thickness of K_{10} is three.*

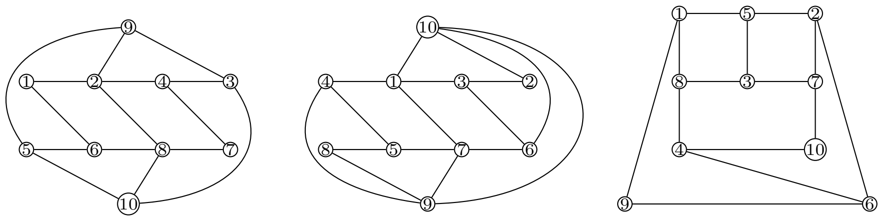


Figure 5: A 4-girth planar decomposition of K_{10}.

Proof. From [9], we have $\theta(4, K_{10}) \geq 3$. We draw a 4-girth planar decomposition of K_{10} with three planar subgraphs in Figure 5, which shows $\theta(4, K_{10}) \leq 3$. The theorem follows. □

We would like to state that after submitting this paper for review, we notice that there exist two results regarding the 4-girth-thickness of $K_{2p,2p,2p}$ and K_{10}. Rubio-Montiel [8] obtained the exact value of the 4-girth-thickness of the complete multipartite graph when each part has an even number of vertices. And by computer, Castañeda-López et al. [5] found the other two decompositions of K_{10} into three planar subgraphs of girth at least four. In this paper, we give these results in a constructive way.

References

[1] V. B. Alekseev and V. S. Gončakov, The thickness of an arbitrary complete graph, *Math. USSR Sbornik* **30** (1976), 187–202, http://stacks.iop.org/0025-5734/30/i=2/a=A04.

[2] L. W. Beineke and F. Harary, The thickness of the complete graph, *Canad. J. Math.* **17** (1965), 850–859, doi:10.4153/cjm-1965-084-2.

[3] L. W. Beineke, F. Harary and J. W. Moon, On the thickness of the complete bipartite graph, *Math. Proc. Cambridge Philos. Soc.* **60** (1964), 1–5, doi:10.1017/s0305004100037385.

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*, Springer-Verlag, London, 2008, doi:10.1007/978-1-84628-970-5.

[5] H. Castañeda-López, P. C. Palomino, A. B. Ramos-Tort, C. Rubio-Montiel and C. Silva-Ruíz, The 6-girth-thickness of the complete graph, 2017, arXiv:1709.07466 [math.CO].

[6] Y. C. Chen and Y. Yang, The thickness of the complete multipartite graphs and the join of graphs, *J. Comb. Optim.* **34** (2017), 194–202, doi:10.1007/s10878-016-0057-1.

[7] M. Kleinert, Die dicke des n-dimensionalen Würfel-graphen, *J. Comb. Theory* **3** (1967), 10–15, doi:10.1016/s0021-9800(67)80010-3.

[8] C. Rubio-Montiel, The 4-girth-thickness of the complete multipartite graph, 2017, arXiv:1709.03932 [math.CO].

[9] C. Rubio-Montiel, The 4-girth-thickness of the complete graph, *Ars Math. Contemp.* **14** (2018), 319–327, doi:10.26493/1855-3974.1349.b67.

[10] W. T. Tutte, The thickness of a graph, *Indag. Math. (Proceedings)* **66** (1963), 567–577, doi:10.1016/s1385-7258(63)50055-9.

[11] J. M. Vasak, The thickness of the complete graph, *Notices Amer. Math. Soc.* **23** (1976), A–479.

[12] Y. Yang, A note on the thickness of $K_{l,m,n}$, *Ars Combin.* **117** (2014), 349–351.

[13] Y. Yang, Remarks on the thickness of $K_{n,n,n}$, *Ars Math. Contemp.* **12** (2017), 135–144, doi:10.26493/1855-3974.823.068.

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On the k-metric dimension of metric spaces

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Abstract

The metric dimension of a general metric space was defined in 1953, applied to the set of vertices of a graph metric in 1975, and developed further for metric spaces in 2013. It was then generalised in 2015 to the k-metric dimension of a graph for each positive integer k, where $k = 1$ corresponds to the original definition. Here, we discuss the k-metric dimension of general metric spaces.

Keywords: Metric spaces, metric dimension, k-metric dimension.

Math. Subj. Class.: 54E35, 05C12

1 Introduction

The metric dimension of a general metric space was introduced in 1953 in [4, p. 95] but attracted little attention until, about twenty years later, it was applied to the distances between vertices of a graph [12, 14, 15, 18]. Since then it has been frequently used in graph theory, chemistry, biology, robotics and many other disciplines. The theory was developed further in 2013 for general metric spaces [1]. More recently, the theory of metric dimension has been generalised, again in the context of graph theory, to the notion of a k-metric dimension, where k is any positive integer, and where the case $k = 1$ corresponds to the original theory [7, 8, 9, 10, 11]. Here we develop the idea of the k-metric dimension both in graph theory and in metric spaces. As the theory is trivial when the space has at most two points, we shall assume that any space we are considering has *at least three points*.

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Finally, whenever we discuss a connected graph G, we shall always consider the metric space (X, d), where X is the vertex set of G, and d is the usual graph metric in which the distance between two vertices is the smallest number of edges that connect them.

Let (X, d) be a metric space. If X is a finite set, we denote its cardinality by $|X|$; if X is an infinite set, we put $|X| = +\infty$. In fact, it is possible to develop the theory with $|X|$ any cardinal number, but we shall not do this. The distances from a point x in X to the points a in a subset A of X are given by the function $a \mapsto d(x, a)$, and the subset A is said to *resolve* X if each point x is uniquely determined by this function. Thus A resolves X if and only if $d(x, a) = d(y, a)$ for all a in A implies that $x = y$; informally, if an object in X knows its distance from each point of A, then it knows exactly where it is located in X. The class $\mathcal{R}(X)$ of subsets of X that resolve X is non-empty since X resolves X. The *metric dimension* $\dim(X)$ of (X, d) is the minimum value of $|S|$ taken over all S in $\mathcal{R}(X)$. The sets in $\mathcal{R}(X)$ are called the *metric generators*, or *resolving subsets*, of X, and S is a *metric basis* of X if $S \in \mathcal{R}(X)$ and $|S| = \dim(X)$. A metric generator of a metric space (X, d) is, in effect, a *global co-ordinate system* on X. For example, if $(x_1, \dots, x_m)$ is an ordered metric generator of X, then the map $\Delta\colon X \to \mathbb{R}^m$ given by

$$\Delta(x) = \Big(d(x, x_1), \dots, d(x, x_m)\Big) \tag{1.1}$$

is injective (for this vector determines x), so that Δ is a bijection from X to a subset of $\mathbb{R}^m$, and X inherits its co-ordinates from this subset.

Now let k be a positive integer, and (X, d) a metric space. A subset S of X is a *k-metric generator* for X (see [8]) if and only if any pair of points in X is distinguished by at least k elements of S: that is, for any pair of distinct points u and v in X, there exist k points $w_1, w_2, \dots, w_k$ in S such that

$$d(u, w_i) \neq d(v, w_i), \quad i = 1, \dots, k.$$

A k-metric generator of minimum cardinality in X is called a *k-metric basis*, and its cardinality, which is denoted by $\dim_k(X)$, is called the *k-metric dimension* of X. Let $\mathcal{R}_k(X)$ be the set of k-metric generators for X. Since $\mathcal{R}_1(X) = \mathcal{R}(X)$, we see that $\dim_1(X) = \dim(X)$. Also, as $\inf \varnothing = +\infty$, this means that $\dim_k(X) = +\infty$ if and only if no finite subset of X is a k-metric generator for X.

Given a metric space (X, d), we define the *dimension sequence* of X to be the sequence

$$\big(\dim_1(X), \dim_2(X), \dots, \dim_k(X), \dots\big),$$

and we address the following two problems.

- Can we find necessary and sufficient conditions for a sequence $(d_1, d_2, d_3, \dots)$ to be the dimension sequence of some metric space?
- How does the dimension sequence of (X, d) relate to the properties of (X, d)?

In Sections 2, 3 and 4 we provide some basic results on the k-metric dimension, and in Section 5 we calculate the dimension sequences of some metric spaces. We then apply these ideas to the join of two metric spaces, and to the Cayley graph of a finitely generated group.

2 Bisectors

As shown in [1], the ideas about metric dimension are best described in terms of bisectors. For *distinct* u and v in X, the *bisector* $B(u|v)$ of u and v is given by

$$B(u|v) = \{x \in X : d(x,u) = d(x,v)\}.$$

The complement of $B(u|v)$ is denoted by $B^c(u|v)$; thus

$$B^c(u|v) = \{x \in X : d(x,u) \neq d(x,v)\},$$

and this contains both u and v. Whenever we speak of a bisector B, we shall assume that it is some bisector $B(u|v)$, where $u \neq v$, so that its complement B^c is not empty.

Let us now consider the k-metric dimension from the perspective of bisectors. A subset A of X *fails to resolve* X if and only if there are distinct points u and v in X such that $d(u,a) = d(v,a)$ for all a in A. Thus A resolves X if and only if A is not contained in any bisector or, equivalently, *if and only if for every bisector B, we have $|B^c \cap A| \geq 1$.* This leads to an alternative (but equivalent) definition of the metric dimension $\dim(X)$, namely

$$\dim(X) = \inf\{|A| : A \subset X \text{ and, for all bisectors } B,\ |B^c \cap A| \geq 1\}.$$

Again, this infimum may be $+\infty$. The extension to the k-metric dimension $\dim_k(X)$ of X is straightforward:

$$\dim_k(X) = \inf\{|A| : A \subset X \text{ and, for all bisectors } B,\ |B^c \cap A| \geq k\}. \tag{2.1}$$

Note that if X is a finite set then $\dim_{|X|+1}(X) = +\infty$.

Clearly, the values $\dim_k(X)$ depend only on the class $\mathcal{B}$ of bisectors in X; for example, $\dim_1(X) = 1$ if and only if there is some point in X that is not in any bisector. More generally, in all cases, $\dim_k(X) \geq k$, and equality holds here if and only if there are k points of X that do not lie in any bisector. For example, if X is the real, closed interval $[0,1]$ with the Euclidean metric, then $\dim_k(X) = k$ for $k = 1, 2$. For a more general example of this type, let $X = \{\sqrt{p} : p \text{ a prime number}\}$ with the Euclidean metric d. If p, q and r are primes, with $p \neq q$, then $\sqrt{r} \in B(\sqrt{p}|\sqrt{q})$ implies $\sqrt{r} = \frac{1}{2}(\sqrt{p} + \sqrt{q})$; hence $4r = p + q + 2\sqrt{pq}$. Since $\sqrt{pq}$ is irrational, this is false; hence every bisector is empty. It follows that $\dim_k(X) = k$ for $k = 1, 2, \ldots$; thus the dimension sequence of (X,d) is $(1, 2, 3, \ldots)$.

3 The monotonicity of dimensions

Let (X,d) be a metric space. Then, from (2.1), we have $\dim_k(X) \leq \dim_{k+1}(X)$, but we shall now establish the stronger inequality $\dim_k(X) + 1 \leq \dim_{k+1}(X)$ (which is $\dim_k(X) < \dim_{k+1}(X)$ when the dimensions are finite, but not when they are $+\infty$). This inequality is known for graphs; see [8, 10]) where it is an important tool.

Theorem 3.1. *Let (X,d) be a metric space. Then, for $k = 1, 2, \ldots$,*

(i) if $\dim_k(X) < +\infty$ then $\dim_k(X) < \dim_{k+1}(X)$;

(ii) if $\dim_k(X) = +\infty$ then $\dim_{k+1}(X) = +\infty$.

In particular, $\dim_k(X) + 1 \geq \dim_1(X) + k$.

Proof. First, (ii) follows immediately from (2.1). Next, (i) is true if $\dim_{k+1}(X) = +\infty$, so we may assume that $\dim_{k+1}(X) = p < +\infty$. Thus there is a subset $\{x_1, \ldots, x_p\}$ (with the x_i distinct) of X such that for every bisector B, $|B^c \cap \{x_1, \ldots, x_p\}| \geq k + 1$. As $k \geq 1$ we see that $p \geq 2$. Clearly, $|B^c \cap \{x_1, \ldots, x_{p-1}\}| \geq k$ for every bisector B; hence $\dim_k(X) \leq p - 1 < \dim_{k+1}(X)$. The last inequality follows by induction. □

4 The 1-metric dimension

Theorem 3.1 shows that if $+\infty$ occurs as a term in the dimension sequence of (X, d), then all subsequent terms are also $+\infty$. Thus $\dim_1(X) = +\infty$ if and only if (X, d) has dimension sequence $(+\infty, +\infty, +\infty, \ldots)$. The next result shows when this is so.

Theorem 4.1. *Let (X, d) be a metric space. Then $\dim_1(X) = +\infty$ if and only if every finite subset of X lies in some bisector. In particular, if X is the union of an increasing sequence of bisectors, then $\dim_1(X) = +\infty$.*

Proof. First, the definition of $\dim(X)$ implies that $\dim_1(X) = +\infty$ if and only if every finite subset of X lies in some bisector. The second statement holds because if $X = \cup_n B_n$, where $B_1, B_2, \ldots$ is an increasing sequence of bisectors, then, given any finite subset $\{x_1, \ldots, x_m\}$ of X, each x_j lies in some B_{i_j}, and $\{x_1, \ldots, x_m\} \subset B_r$, where $r = \max\{i_1, \ldots, i_m\}$. □

What can be said if $\dim_1(X) < +\infty$? It seems that we can obtain very little information from the *single* assumption that $\dim_1(X) < +\infty$; for example, for each $r \geq 0$ choose a point x_r in $\mathbb{R}^n$ with $\|x_r\| = r$, and let $X = \{x_r : r \geq 0\}$. Then $\{0\}$ is a 1-metric basis for X, and $\dim_1(X) = 1$ but we can say almost nothing about the topological structure of X. However, we can say more if we know that X is compact.

Theorem 4.2. *Let (X, d) be a compact metric space with $\dim_1(X) = m < +\infty$. Then (X, d) is homeomorphic to a compact subset of $\mathbb{R}^m$.*

Proof. Suppose that X is compact, and that $\dim_1(X) = m < +\infty$. Then there is a 1-metric basis $\{x_1, \ldots, x_m\}$, and the corresponding bijection Δ in (1.1) that maps X onto some subset of $\mathbb{R}^m$. Now Δ is continuous on X since

$$|\Delta(x) - \Delta(y)| \leq \sum_{j=1}^{m} |d(x, x_j) - d(y, x_j)| \leq md(x, y).$$

As Δ is a continuous, injective map from a compact space to the Hausdorff space $\mathbb{R}^m$ it follows (by a well known result in topology) that it is a homeomorphism. □

This result is related to the following result in [1] (see also [16]).

Theorem 4.3. *If (X, d) is a compact, connected metric space with $\dim_1(X) = 1$ then X is homeomorphic to $[0, 1]$.*

The compactness is essential here as there is an example in [1] of a connected, but not arcwise connected, metric space X with $\dim_1(X) = 1$. As X is not arcwise connected, it is not homeomorphic to $[0, 1]$. It is conjectured in [1] that if X is arcwise connected, and $\dim_1(X) = 1$ then X is a Jordan arc (this means that X is homeomorphic to one of the real intervals $[0, 1]$ and $[0, +\infty)$), and we can now show that this is so.

Theorem 4.4. *If X is an arcwise connected metric space with $\dim_1(X) = 1$, then X is a Jordan arc.*

Proof. As $\dim_1(X) = 1$, there is a metric basis, say $\{x_0\}$ for X, and every point x of X is uniquely determined by its distance $d(x, x_0)$ from x_0. Consider the map $\Delta\colon x \mapsto d(x, x_0)$ of X into $[0, +\infty)$. This map is (uniformly) continuous because

$$|\Delta(x) - \Delta(y)| = |d(x, x_0) - d(y, x_0)| \leq d(x, y),$$

and as X is arcwise connected (and therefore connected), so $\Delta(X)$ is connected. This means that Δ is an interval of the form $[0, a]$, where $a > 0$, or $[0, b)$, where $0 < b \leq +\infty$.

Let us consider the case when $\Delta(X) = [0, a]$. As Δ is injective, we see that for every r in the interval $[0, a]$ there is some unique x_r in X with $d(x_r, x_0) = r$. Thus $X = \{x_r : 0 \leq r \leq a\}$. However, as X is arcwise connected, there is a curve, say $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_a$. Now as γ is continuous, the set $\{d\big(\gamma(t), x_0\big) : t \in [0, 1]\}$ must contain every real number in the interval $[0, a]$, and it cannot contain any other numbers; thus $X = \gamma([0, 1])$. Now $\gamma([0, 1])$ is compact for it is the continuous image of the compact interval $[0, 1]$; thus X is compact and so, by Theorem 4.3, X is a Jordan arc.

The argument in the case when $\Delta(X) = [0, b)$ is similar. Indeed, the argument above holds for every a with $0 < a < b$, and it is easy to see that this implies that Δ is a homeomorphism from X to $[0, b)$. □

5 Some examples

In order to calculate the k-metric dimension of a metric space we need to understand the geometric structure of its bisectors, and we now illustrate this with several examples. In order to maintain the flow of ideas, the details of these examples will be given later.

Example 5.1. Let (X, d) be any one of the Euclidean, spherical and hyperbolic spaces $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$, respectively, each with the standard metric of constant curvature 0, 1 and -1, respectively. The bisectors are well understood in these spaces, and we shall show that any non-empty open subset of X has k-metric dimension $n + k$. In particular, each of these spaces has dimension sequence $(n + 1, n + 2, n + 3, \ldots)$. See [1, 13] for the 1-metric dimensions of these spaces.

Example 5.2. Let X be any finite set with the discrete metric d (equivalently, X is the vertex set of a complete, finite graph). For distinct u and v in X we have $B(u|v) = X\backslash\{u, v\}$, so that for any subset S of X, we have $B(u|v)^c \cap S = \{u, v\} \cap S$. Thus if $|S \cap B^c| \geq 1$ for all bisectors B, then S can omit at most one point of X. We conclude that $\dim_1(X) = |X| - 1$. If $|B^c \cap S| \geq 2$ for all bisectors B then $S = X$, and $\dim_2(X) = |X|$. We conclude that (X, d) has dimension sequence $(|X| - 1, |X|, +\infty, +\infty, \ldots)$.

Example 5.3. Let X be the real interval $[0, 1]$, with the Euclidean metric. Then B is a bisectors if and only if $B = \{x\}$ for some x in $(0, 1)$. Thus $\{0\}$ is a 1-metric basis, and $\{0, 1\}$ is a 2-metric basis, of $[0, 1]$. We leave the reader to show that if $k \geq 3$ then $\{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1\}$ is a k-metric basis, so that $[0, 1]$ has dimension sequence $(1, 2, 4, 5, 6, \ldots)$. A similar argument shows that $[0, +\infty)$ has dimension sequence $(1, 3, 4, 5, \ldots)$, and that $(-\infty, +\infty)$, which is $\mathbb{R}$, has dimension sequence $(2, 3, 4, \ldots)$.

Example 5.4. The Petersen graph, which is illustrated in Figure 1, has dimension sequence $(3, 4, 7, 8, 9, 10, +\infty, \ldots)$. The (finite) values $\dim_k(X)$ for $k = 1, \ldots, 6$ come from a computer search, and as $\dim_6(X) = 10 = |X|$, we have $\dim_7(X) = +\infty$.

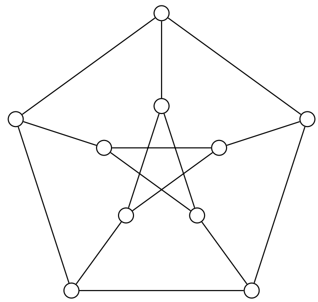

Figure 1: The Petersen graph.

Example 5.5. Let G be a group with a given set of generators, let V be the vertex set of the associated Cayley graph of G, and let d be its graph metric.

(i) If G is an infinite cyclic group then (V, d) has dimension sequence $(2, 3, 4, \dots)$.

(ii) If G is a free group on p generators, where $p \geq 2$, then (V, d) has dimension sequence $(+\infty, +\infty, +\infty, \dots)$.

(iii) Let G be an abelian group on p generators, where $p \geq 2$, and where each generator has infinite order. Then (V, d) has dimension sequence $(+\infty, +\infty, +\infty, \dots)$.

6 Three geometries of constant curvature

In this section we give the details of Example 5.1. It is shown in [1] that if U is any non-empty, open subset of any one of the three classical geometries $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$, then $\dim_1(U) = n + 1$. Here we show that if X is any of these spaces then $\dim_k(X) = n + k$ for $k = 1, 2, \dots$. The same result holds for non-empty *open* subsets of these spaces, and we leave the reader to make the appropriate changes to the proofs.

The proof that $\dim_k(X) = n+k$ when X is one of the three geometries $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$, is largely independent of the choice of X, and depends only on the nature of the bisectors in these geometries. Each of these three geometries has the following properties:

(P1) $\dim_1(X) = n + 1$;

(P2) there exists $x_1, x_2, \dots$ in X such that if $j_1 < j_2 < \cdots < j_n$ then $\{x_{j_1}, \dots, x_{j_n}\}$ lies on a unique bisector B, and no other x_i lies on B.

Now (P1) and (P2) imply that $\dim_k(X) = n+k$ for $k = 1, 2, \dots$. Indeed, (P2) implies that for any bisector B, $|B \cap \{x_1, \dots, x_{n+k}\}| \leq n$, so that $|B^c \cap \{x_1, \dots, x_{n+k}\}| \geq k$. This implies that $\dim_k(X) \leq n + k$. However, (P1) and Theorem 3.1 show that $\dim_k(X) \geq n+k$. Since we know that each of $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$ has the property (P1), it remains to show that they have the property (P2), and this depends on the nature of the bisectors in these geometries. We consider each in turn.

6.1 Euclidean space $\mathbb{R}^n$

Each bisector in $\mathbb{R}^n$ is a hyperplane (that is, the translation of an $(n-1)$-dimensional subspace of $\mathbb{R}^n$), and each hyperplane is a bisector. Any set of n points lies on a bisector, and there exists sets of $n+1$ points that do not lie on any single bisector. The appropriate geometry here is the affine geometry of $\mathbb{R}^n$, but we shall take a more informal view. First, we choose n points $x_1, \ldots, x_n$ that lie on a unique hyperplane H. Next, we select a point x_{n+1} not on H. Then any n points chosen from $\{x_1, \ldots, x_{n+1}\}$ lie on some hyperplane H', and the remaining point does not lie on H'. Now suppose that we have constructed the set $\{x_1, \ldots, x_{n+p}\}$ with the property that any set of n points chosen from this lie on a unique hyperplane, say H_α, and that no other x_i lies on H_α. Then we can choose a point x_{n+p+1} that is not on any of the $\binom{n+p}{n}$ hyperplanes H_α, and it is then easy to check that the sequence $x_1, x_2, \ldots$ has the property (P2).

Although we have not used it, we mention that there is a formula for the n-dimensional volume V of the Euclidean simplex whose vertices are the $n+1$ points $x_1, \ldots, x_{n+1}$ in $\mathbb{R}^n$, namely

$$V^2 = \frac{(-1)^{n+1}}{2^n (n\,!)^2}\,\Delta,$$

where Δ is the *Cayley-Menger determinant* given by

$$\Delta = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & d_{1,1}^2 & \cdots & d_{1,n+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & d_{n+1,1}^2 & \cdots & d_{n+1,n+1}^2 \end{vmatrix},$$

and $d_{i,j} = \|x_i - x_j\|$. As $V = 0$ precisely when the points x_j lie on a hyperplane, we see that this condition could be used to provide an algebraic background to the discussion above. For more details, see [3, 4] and [5]. We also mention that there are versions of the Cayley-Menger determinant that are applicable to spherical, and to hyperbolic, spaces.

6.2 Spherical space $\mathbb{S}^n$

Spherical space $(\mathbb{S}^n, d)$ is the space $\{x \in \mathbb{R}^{n+1} : \ \|x\| = 1\}$ with the path metric d induced on $\mathbb{S}^n$ by the Euclidean metric on $\mathbb{R}^{n+1}$. Explicitly, $\cos d(x, y) = x{\cdot}y$, where $x{\cdot}y$ is the usual scalar product in $\mathbb{R}^{n+1}$. If u and v are distinct points of $\mathbb{S}^n$, we let $B^{\mathcal{E}}(u|v)$ be the *Euclidean* bisector (in $\mathbb{R}^{n+1}$) of u and v, and $B^{\mathcal{S}}(u|v)$ the spherical bisector in the space $(\mathbb{S}^n, d)$. Then $B^{\mathcal{E}}(u|v)$ is a hyperplane that passes through the origin in $\mathbb{R}^{n+1}$, and

$$B^{\mathcal{S}}(u|v) = \mathbb{S}^n \cap B^{\mathcal{E}}(u|v). \tag{6.1}$$

The bisectors $B^{\mathcal{S}}(u|v)$ are the *great circles* (of the appropriate dimension) on $\mathbb{S}^n$.

The equation (6.1) implies that the k-metric dimension of the spherical spaces is the same as for Euclidean spaces. Indeed, our proof for Euclidean spaces depended on constructing a sequence $x_1, x_2, \ldots$ with the property (P2), and it is clear that this construction could be carried out in such a way that each x_j lies on $\mathbb{S}^n$.

6.3 Hyperbolic space $\mathbb{H}^n$

Our model of hyperbolic n-dimensional space is Poincare's half-space model

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_n > 0\}$$

equipped with the hyperbolic distance d which is derived from Riemannian metric $|dx|/x_n$. For more details, see for example, [2, 17]. Our argument for $\mathbb{H}^n$ is essentially the same as for $\mathbb{R}^n$ and $\mathbb{S}^n$ because if u and v are distinct points in $\mathbb{H}^n$, then the hyperbolic bisector $B(u|v)$ is the set $S \cap \mathbb{H}^n$, where S is some Euclidean sphere whose centre lies on the hyperplane $x_n = 0$. We omit the details.

7 The metric dimensions of graphs

The vertex set V of a graph G supports a natural *graph metric* d, where $d(u, v)$ is the smallest number of edges that can be used to join u to v. Some basic results on the k-metric dimension of a graph have recently been obtained in [7, 8, 9, 10, 11]. Moreover, it was shown in [19] that the problem of computing the k-metric dimension of a graph is NP-hard. A natural problem in the study of the k-metric dimension of a metric space (X, d) consists of finding the largest integer k such that there exists a k-metric generator for X. For instance, for the graph shown in Figure 2 the maximum value of k is four. It was shown in [7, 10] that for any graph of order n this problem has time complexity of order $O(n^3)$. If we consider the discrete metric space (X, d_0) (equivalently, a compete graph), then $\dim_1(X) = |X| - 1$ and $\dim_2(X) = |X|$. Furthermore, for $k \geq 3$ there are no k-metric generators for X. In general, for any metric space (X, d), the whole space X is a 2-metric generator, as two vertices are distinguished by themselves. As we have already seen, there are metric spaces, like the Euclidean space $\mathbb{R}^n$, where for any positive integer k, there exist at least one k-metric generator.

We shall now discuss the dimension sequences of the simplest connected graphs, that is paths and cycles (and we omit the elementary details).

A finite *path* P_n is a graph with vertices $v_1, \dots, v_n$, edges $\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}$, and bisectors $\{v_2\}, \dots, \{v_{n-1}\}$. We leave the reader to show that P_n has dimension sequence

$$\begin{cases} (1, 2, +\infty, \dots) & \text{if } n = 2, 3; \\ (1, 2, 4, 5, \dots, n, +\infty, \dots) & \text{if } n \geq 4. \end{cases}$$

A *semi-infinite path* $P_{\mathbb{N}}$ is a graph with vertices $v_1, v_2 \dots$, edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots$, and bisectors $\{v_2\}, \dots$. Thus $P_{\mathbb{N}}$ has dimension sequence $(1, 3, 4, 5, \dots)$. A *doubly-infinite path* $P_{\mathbb{Z}}$ is the graph with vertices $\dots, v_{-1}, v_0, v_1, \dots$, edges $\dots, \{v_{-1}, v_0\}, \{v_0, v_1\}, \dots$, and bisectors $\dots, \{v_{-1}\}, \{v_0\}, \{v_1\}, \dots$. Thus $P_{\mathbb{Z}}$ has dimension sequence $(2, 3, 4, 5, \dots)$. We note that a graph G has 1-metric dimension 1 if and only it is P_n or $P_{\mathbb{N}}$ [6, 14]. This, together with the results just stated, show that if G is a graph of order two or more, and $k \geq 2$, then $\dim_k(G) = k$ if and only if G is P_n and $k = 2$ (see also [8]).

We now consider cycles. A cycle C_n is a graph with vertices $v_1, \dots, v_n$, and edges $\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$. We must distinguish between the cases where n is even, and where n is odd (which is the easier of the two cases) and, as typical examples, we mention that C_7 has dimension sequence $(2, 3, \dots, 7, +\infty, \dots)$, and C_8 has dimension sequence $(2, 3, 4, 6, 7, 8, +\infty, \dots)$. Suppose that n is odd; then the bisectors are the singletons $\{v\}$. Thus if S is a set of $k+1$ vertices, where $k+1 \leq n$, then $|B^c \cap S| \geq k$ for every bisector B. Thus if n is odd, then $\dim_k(C_n) = k+1$, and C_n has dimension sequence $(2, 3, \dots, n, +\infty, \dots)$.

We now show that C_{2q} has dimension sequence

$$(2, 3, \dots, q, q+2, q+3, \dots, q+q, +\infty, \dots).$$

To see this, label the vertices as v_j, where $j \in \mathbb{Z}$, and where $v_i = v_j$ if and only if $i \equiv j \pmod{n}$. The vertices v_i and v_j are *antipodal vertices* if and only if $i - j \equiv q \pmod{2q}$;

thus v_j and v_{j+q} are antipodal vertices. The class of bisectors is the class of sets $\{v, v^*\}$, where v is a vertex, and v^* is the vertex that is antipodal to v. For $k = 1, \ldots, q-1$ we can take a set of $k+1$ points, no two of which are antipodal, as a k-metric basis, so that $\dim_k(C_{2q}) = k+1$ for $k = 1, \ldots, q-1$. To find $\dim_q(C_{2q})$, we need to take (for a q-metric basis) a set S which contains two pairs of antipodal points, and one more point from each pair of the remining antipodal pairs. We leave the details to the reader.

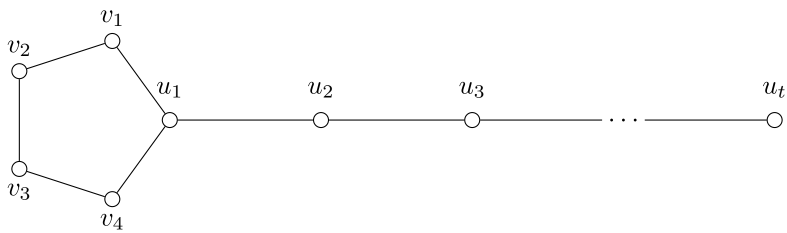

Figure 2: For $k \in \{1, 2, 3, 4\}$, $\dim_k(G) = k+1$.

As an example which joins a path to a cycle, consider the graph G illustrated in Figure 2 which is obtained from the cycle graph C_5 and the path P_t, by identifying one of the vertices of the cycle, say u_1, and one of the end vertices of P_t. Let $S_1 = \{v_1, v_2\}$, $S_2 = \{v_1, v_2, u_t\}$, $S_3 = \{v_1, v_2, v_3, u_t\}$ and $S_4 = \{v_1, v_2, v_3, v_4, u_t\}$. Then, for $k = 1, 2, 3, 4$, the set S_k is k-metric basis of G.

The following lemma is useful when discussing examples in graph theory.

Lemma 7.1. *Suppose that a graph G does not have any cycles of odd length. Then $B(u|v) = \varnothing$ when $d(u, v)$ is odd.*

The proof is trivial for if $x \in B(u|v)$ then there is a cycle of odd length (from u to x, then to v, and then back to u). This lemma applies, for example, to the usual grid (or graph) in $\mathbb{R}^n$ whose vertex set is $\mathbb{Z}^n$. A *bipartite graph* is a graph G whose vertex set V splits into complementary sets V_1 and V_2 such that each of the edges of G join a point of V_1 to a point of V_2. As a graph is bipartite if and only if it has no cycles of an odd length, this lemma is about bipartite graphs.

Example 7.2. Let us now consider a graph G that is an infinite tree in which every vertex has degree at least three. Now let v be any vertex, select three edges from v, say $\{v, a\}$, $\{v, b\}$ and $\{v, c\}$. As G is a tree, if we remove one edge the remaining graph is disconnected. Now let G_c be the subgraph of G that would be the component containing c if we were to remove the edge $\{v, c\}$ from G. It is clear that if u is a vertex in G_c, then $d(a, u) = d(b, u)$ since any path from a (or b) to u must pass through the edge $\{v, c\}$. We conclude that $G_c \subset B(a|b)$. It is now clear from Theorem 4.1 that G has dimension sequence $(+\infty, +\infty, \ldots)$.

For the rest of this section we shall consider the Cayley graph of a group with a given set of generators as a metric space. Let G be a group and let G_0 a set of generators of G. We shall always assume that if $g \in G_0$ then $g^{-1} \in G_0$ also. Then the Cayley graph of the pair (G, G_0) is a graph whose vertex set is G, and such that the pair (g_1, g_2) is an edge if and only if $g_2 = g_0 g_1$ for some g_0 in G_0. Thus, for example, $P_{\mathbb{Z}}$ is the Cayley graph of an infinite cyclic group (on one generator), and C_n is the Cayley graph of an finite cyclic group (on one generator). We shall always assume that the set G_0 of generators of G is finite; then the Cayley graph is locally finite (that is, each vertex is the endpoint of only finitely many edges). Note also that if a generator g_0 has order two then $g_0^{-1} = g_0$ so this only provides one edge (not two edges) from each vertex. The following result, which characterises Cayley graphs within the class of all graphs, is well known.

Theorem 7.3. *A graph is a Cayley graph of a group G if and only if it admits a simply transitive action of G by graph automorphisms.*

Theorem 7.3 suggests that if we use the homogeneity implied by this result there is a reasonable chance of finding the dimension sequence of a Cayley graph. However, for a graph that is not the Cayley graph of a group, it seems that we are reduced to finding its metric dimensions by a case by case analysis.

We shall now verify the claims made in Example 5.5. First, suppose that G is a free group on p generators. Then the Cayley graph of G is a tree in which every vertex has degree $2p$; thus, using Example 7.2, we see that G has dimension sequence $(+\infty, +\infty, +\infty, \dots)$.

Next, we consider an abelian group G on two generators of infinite order (the proof for p generators is entirely similar). The Cayley graph of G has $\mathbb{Z}^2$ as its vertex set and (if we identify the lattice point (m, n) with the Gaussian integer $m + in$) edges $\{m + in, m + 1 + in\}$ and $\{m + in, m + i(n + 1)\}$, where $m, n \in \mathbb{Z}$. It is (geometrically) clear that for any $m \in \mathbb{Z}$ we have, with $\zeta = m + im$,

$$B(\zeta + 1|\zeta + i) \supset \{p + iq : p \geq m + 1, q \geq m + 1\}.$$

It now follows from Theorem 4.1 (by taking $|m|$ large and m negative) that G has dimension sequence $(+\infty, +\infty, \dots)$.

In contrast to Example 5.5 we have the following result for the infinite dihedral group whose Cayley graph is an infinite ladder; for example we can take the group generated by the two Euclidean isometries which, in complex terms, are $z \mapsto z + 1$ and $z \mapsto \bar{z}$.

Theorem 7.4. *The infinite dihedral group has dimension sequence* $(3, 4, 6, 8, \dots)$.

Proof. We may assume that (in complex terms) the vertices of the ladder graph are the points $m + in$, where $m \in \mathbb{Z}$ and $n = 0, 1$. The key to computing the metric dimensions of the ladder graph is the observation that

$$B(0|1 + i) = \{1, 2, 3, \dots\} \cup \{i, i - 1, i - 2, \dots\}.$$

Of course, similar bisectors arise at other pairs of similarly located points; equivalently, each automorphism of the graph maps a bisector to a bisector. All other bisectors are either empty or of cardinality two. We claim that $\{0, 1, i\}$ is a 1-metric basis for the graph so that $\dim_1(G) = 3$. Next, it is easy to see that $\{0, 1, i, 1 + i\}$ is a 2-metric basis for X so that $\dim_2(X) = 4$. The set $\{0, 1, 2, i, 1 + i, 2 + i\}$ is a 3-metric basis so that $\dim_3(X) = 6$. We leave the details, and the remainder of the proof to the reader. □

8 The join of metric spaces

The k-metric dimension of the *join* $G_1 + G_2$ of two finite graphs G_1 and G_2 was studied in [7]. Let us briefly recall the notion of the join of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2, respectively. The *join* $G_1 + G_2$ of G_1 and G_2 is the graph whose vertex set is $V_1 \cup V_2$, and whose edges are the edges in G_1, the edges in G_2, together with all edges obtained by joining each point in V_1 to each point in V_2. Let d_1, d_2 and d be the graph metrics of G_1, G_2 and $G_1 + G_2$, respectively; then

$$d(u,v) = \begin{cases} \min\{d_1(u,v), 2\} & \text{if } u, v \in V_1; \\ \min\{d_2(u,v), 2\} & \text{if } u, v \in V_2; \\ 1 & \text{if } u \in V_i,\, v \in V_j, \text{ where } i \neq j, \end{cases}$$

because if $u, v \in V_1$, say, then for w in V_2, we have $d(u,v) \leq d(u,w) + d(w,v) = 2$.

The join of two metric spaces is defined in a similar way, but before we do this we recall that if (X, d) is a metric space, and $t > 0$, then d^t, defined by

$$d^t(x,y) = \min\{d(x,y), 2t\},$$

is a metric on X. If $d(x,y) < 2t$ then $d^t(x,y) = d(x,y)$, so that the d^t-metric topology coincides with the d-metric topology on X. As the metric d^t will appear in our definition of the join, we first show how the metric dimension of a single metric space varies when we distort the metric from d to d^t as above. From now on, the k-metric dimension of (X, d^t) will be denoted by $\dim_k^t(X)$.

Theorem 8.1. *Let (X, d) be a metric space, and k a positive integer, and suppose that $0 < s < t$. Then $\dim_k^s(X) \geq \dim_k^t(X) \geq \dim_k(X)$. However, it can happen that*

$$\lim_{t \to +\infty} \dim_k^t(X) > \dim_k(X). \tag{8.1}$$

The join of two metric spaces is defined in a similar way to the join of two graphs, and to motivate this, suppose that (X, d) is a metric space, and that X_1 and X_2 are bounded subsets X whose distance apart is very large compared with their diameters. Then, in some sense, we can approximate the metric space $(X_1 \cup X_2, d)$ by replacing all values $d(x_1, x_2)$, where $x_j \in X_j$, by t, where t is some sort of average of the values $d(x_1, x_2)$. We shall now define the join, so suppose that (X_1, d_1) and (X_2, d_2) are metric spaces, with $X_1 \cap X_2 = \varnothing$, and $t > 0$. Then the *join* of (X_1, d_1) and (X_2, d_2) (relative to the parameter t) is the metric space $(X_1 \cup X_2, d^t)$, where

$$d^t(u,v) = \begin{cases} d_1^t(u,v) & \text{if } u, v \in X_1; \\ d_2^t(u,v) & \text{if } u, v \in X_2; \\ t & \text{if } u \in X_i \text{ and } v \in X_j, \text{ where } i \neq j. \end{cases}$$

As with graphs, $X_1 + X_2$ always represents the metric space $(X_1 \cup X_2, d^t)$, where in this case t will be understood from the context.

We might hope that the metric dimension is additive with respect to the join, but unfortunately it is not. Let $X_1 = \{1, 3\}$ and $X_2 = \{2, 4\}$, each with the Euclidean metric, and let $t = 1$. Then $X_1 \cup X_2 = \{1, 2, 3, 4\}$ with the metric d^1, where $d^1(1,3) = d^1(2,4) = 2$ and, for all other x and y, $d^1(x,y) = 1$. The bisectors in $X_1 + X_2$ are X_1, X_2 and $\varnothing$, and from this we conclude that $\dim_1^1(X_1 + X_2) = 3$. Obviously, $\dim_1(X_1) = \dim_1(X_2) = 1$, so that in this case, $\dim_1(X_1) + \dim_1(X_2) < \dim_1^1(X_1 + X_2)$.

We now give some inequalities which hold for the join of two metric spaces.

Theorem 8.2. *Let (X_j, d_j), $j = 1, 2$, be metric spaces with $X_1 \cap X_2 = \varnothing$, and consider the join $(X_1 \cup X_2, d^t)$. Then, for any positive integer k, we have*

$$\dim_k(X_1) + \dim_k(X_2) \leq \dim_k^t(X_1) + \dim_k^t(X_2) \leq \dim_k^t(X_1 + X_2). \qquad (8.2)$$

We shall now give an example which shows that (8.1) can hold; then we end with the proofs of Theorems 8.1 and 8.2, and stating a consequence of Theorem 8.2.

Example 8.3. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$, so that $\dim_1(X) = 2$. We shall now show that if $t > 0$ then $\dim_1^t(X) = +\infty$, so that (8.1) can hold. Suppose that $a < b$, and consider the bisector $B^t(a|b)$. If $x \leq a - 2t$, then $d^t(x, a) = d^t(x, b) = 2t$ so that $x \in B^t(a|b)$. Thus $B^t(a|b) \supset (-\infty, a - 2t]$. Now let S be any finite set, and let s be the largest element in S. Then $B^t(s + 2t, s + 3t) \supset (-\infty, s] \supset S$, so that $\dim_1^t(X) = +\infty$.

This is a convenient place to describe the notation that will be used in the following two proofs. We have metric spaces (X_1, d_1) and (X_2, d_2) with $X_1 \cap X_2 = \varnothing$. For $j = 1, 2$ we use $B_j(u|v)$ for the bisectors in X_j, and $\dim_k(X_j)$ for their metric dimensions. Now consider the join $(X_1 \cup X_2, d^t)$, and its metric subspaces (X_j, d^t). We use $B^t(u|v)$ and $B_j^t(u|v)$ for the bisectors in these spaces, and $\dim_k^t(X_1 + X_2)$ and $\dim_k^t(X_j)$ for their metric dimensions. In general, we write $[B]^c$ for the complement of a bisector B of any type.

We shall need the following lemma in our proof of Theorem 8.1.

Lemma 8.4. *Let (X, d) be a metric space, and suppose that $0 < s < t$. Then $B(u|v) \subset B^t(u|v) \subset B^s(u|v)$.*

Proof. First, observe that for all real r, and all real, distinct α and β, we have $\min\{\alpha, r\} = \min\{\beta, r\}$ if and only if (i) $r \leq \min\{\alpha, \beta\}$ or (ii) $\alpha = \beta$. Now suppose that $x \in B^t(u|v)$. Then $d^t(x, u) = d^t(x, v)$ so that $\min\{d(x, u), t\} = \min\{d(x, v), t\}$. This implies that $t \leq \min\{d(x, u), d(x, v)\}$ or $d(x, u) = d(x, v)$, and (since $s < t$) in both cases we have $d^s(x, u) = d^s(x, v)$. Thus $B^t(u|v) \subset B^s(u|v)$. The proof that $B(u|v) \subset B^t(u|v)$ is trivial: if $x \in B(u|v)$ then $d(x, u) = d(x, v)$ so that $d^t(x, u) = d^t(x, v)$; hence $x \in B^t(u|v)$. □

The proof of Theorem 8.1. Let A be any finite subset of X. Then, by Lemma 8.4, for all u and v in X with $u \neq v$, we have

$$|A \cap [B(u|v)]^c| \geq |A \cap [B^t(u|v)]^c| \geq |A \cap [B^s(u|v)]^c|.$$

It follows that if A is a k-metric generator for (X, d^s) (that is, if, for all u and v, $|A \cap [B^s(u|v)]^c| \geq k$), then it is also a k-metric generator for (X, d^t). Thus the minimum of $|S|$ taken over all k-metric generators S of (X, d^t) is less than or equal to the minimum over all k-metric generators of (X, d^s); hence $\dim_k^s(X) \geq \dim_k^t(X)$. The proof that $\dim_k^t(X) \geq \dim_k(X)$ is entirely similar. □

The proof of Theorem 8.2. The first inequality follows from Theorem 8.1. The inequality is trivially true if $\dim_k^t(X_1 + X_2) = +\infty$, so we may assume that there is a k-metric basis, say W, of $X_1 + X_2$. Thus $|W| = \dim_k^t(X_1 + X_2)$. Now take any u and v in X_1; then

$$B^t(u|v) = \{x \in X_1 \cup X_2 : d^t(x, u) = d^t(x, v)\} = B_1^t(u|v) \cup X_2,$$

so that, from Lemma 8.4, $[B^t(u|v)]^c = [B_1^t(u|v)]^c \subset X_1$. We put $W_j = W \cap X_j$, $j = 1, 2$. Then, if we let u and v vary over X_1, with $u \neq v$, we find that

$$k \leq |[B^t(u|v)]^c \cap W| = |[B_1^t(u|v)]^c \cap X_1 \cap W| = |[B_1^t(u|v)]^c \cap W_1|,$$

so that $\dim_k^t(X_1) \leq |W_1|$. Similarly, $\dim_k^t(X_2) \leq |W_2|$, so that

$$\dim_k^t(X_1) + \dim_k^t(X_2) \leq |W_1| + |W_2| = |W| = \dim_k^t(X_1 + X_2)$$

as required. □

If (X_j, d_j), $j = 1, 2$, are metric spaces, each with diameter less than t, such that $X_1 \cap X_2 = \varnothing$, the for any k-metric basis A_i of (X_j, d_j), $A_1 \cup A_2$ is a k-metric generator for the join $(X_1 \cup X_2, d^t)$. This shows that $\dim_k^t(X_1 + X_2) \leq \dim_k(X_1) + \dim_k(X_2)$, and so Theorem 8.2 leads to the following corollary.

Corollary 8.5. *Let (X_j, d_j), $j = 1, 2$, be metric spaces, each with diameter less than t, such that $X_1 \cap X_2 = \varnothing$. Then, for $k = 1, 2, \ldots$,*

$$\dim_k^t(X_1 + X_2) = \dim_k(X_1) + \dim_k(X_2).$$

References

[1] S. Bau and A. F. Beardon, The metric dimension of metric spaces, *Comput. Methods Funct. Theory* **13** (2013), 295–305, doi:10.1007/s40315-013-0024-0.

[2] A. F. Beardon, *The Geometry of Discrete Groups*, volume 91 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1983, doi:10.1007/978-1-4612-1146-4.

[3] M. Berger, *Geometry I*, Universitext, Springer-Verlag, New York, 1987, doi:10.1007/978-3-540-93815-6, translated from the French by M. Cole and S. Levy.

[4] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford, 1953.

[5] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Chelsea Publishing, New York, 2nd edition, 1970.

[6] G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.* **105** (2000), 99–113, doi:10.1016/s0166-218x(00)00198-0.

[7] A. Estrada-Moreno, Y. Ramírez-Cruz and J. A. Rodríguez-Velázquez, On the adjacency dimension of graphs, *Appl. Anal. Discrete Math.* **10** (2016), 102–127, doi:10.2298/aadm151109022e.

[8] A. Estrada-Moreno, J. A. Rodríguez-Velázquez and I. G. Yero, The k-metric dimension of a graph, *Appl. Math. Inf. Sci.* **9** (2015), 2829–2840, `http://www.naturalspublishing.com/Article.asp?ArtcID=10013`.

[9] A. Estrada-Moreno, I. G. Yero and J. A. Rodríguez-Velázquez, k-metric resolvability in graphs, in: J. A. Rodríguez-Velázquez and I. G. Yero (eds.), *Jornadas de Matemática Discreta y Algorítmica*, Elsevier, Amsterdam, volume 46 of *Electronic Notes in Discrete Mathematics*, 2014 pp. 121–128, doi:10.1016/j.endm.2014.08.017, papers from the 9th Conference held in Tarragona, July 7 – 9, 2014.

[10] A. Estrada-Moreno, I. G. Yero and J. A. Rodríguez-Velázquez, The k-metric dimension of corona product graphs, *Bull. Malays. Math. Sci. Soc.* **39** (2016), S135–S156, doi:10.1007/s40840-015-0282-2.

[11] A. Estrada-Moreno, I. G. Yero and J. A. Rodríguez-Velázquez, The k-metric dimension of the lexicographic product of graphs, *Discrete Math.* **339** (2016), 1924–1934, doi:10.1016/j.disc.2015.12.024.

[12] F. Harary and R. A. Melter, On the metric dimension of a graph, *Ars Combinatoria* **2** (1976), 191–195.

[13] M. Heydarpour and S. Maghsoudi, The metric dimension of geometric spaces, *Topology Appl.* **178** (2014), 230–235, doi:10.1016/j.topol.2014.09.012.

[14] S. Khuller, B. Raghavachari and A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.* **70** (1996), 217–229, doi:10.1016/0166-218x(95)00106-2.

[15] R. A. Melter and I. Tomescu, Metric bases in digital geometry, *Comput. Vis. Graph. Image Process.* **25** (1984), 113–121, doi:10.1016/0734-189x(84)90051-3.

[16] G. P. Murphy, A metric basis characterization of Euclidean space, *Pacific J. Math.* **60** (1975), 159–163, `http://projecteuclid.org/euclid.pjm/1102868444`.

[17] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, volume 149 of *Graduate Texts in Mathematics*, Springer, New York, 2nd edition, 2006.

[18] P. J. Slater, Leaves of trees, in: R. S. D. Thomas (ed.), *Proceedings of the Sixth Southeastern Conference on Combinatorics, Graph Theory, and Computing*, Utilitas Mathematica Publishing, Winnipeg, Manitoba, volume 14 of *Congressus Numerantium*, 1975 pp. 549–559, held at Florida Atlantic University, Boca Raton, Florida, February 17 – 20, 1975.

[19] I. G. Yero, A. Estrada-Moreno and J. A. Rodríguez-Velázquez, Computing the k-metric dimension of graphs, *Appl. Math. Comput.* **300** (2017), 60–69, doi:10.1016/j.amc.2016.12.005.

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Transversals in generalized Latin squares*

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Abstract

We are seeking a sufficient condition that forces a transversal in a generalized Latin square. A generalized Latin square of order n is equivalent to a proper edge-coloring of $K_{n,n}$. A transversal corresponds to a multicolored perfect matching. Akbari and Alipour defined $l(n)$ as the least integer such that every properly edge-colored $K_{n,n}$, which contains at least $l(n)$ different colors, admits a multicolored perfect matching. They conjectured that $l(n) \leq n^2/2$ if n is large enough. In this note we prove that $l(n)$ is bounded from above by $0.75n^2$ if $n > 1$. We point out a connection to anti-Ramsey problems. We propose a conjecture related to a well-known result by Woolbright and Fu, that every proper edge-coloring of K_{2n} admits a multicolored 1-factor.

Keywords: Latin squares, transversals, anti-Ramsey problems, Lovász local lemma.

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1 Multicolored matchings and generalized Latin squares

A subgraph H of an edge-colored host graph G is *multicolored* if the edges of H have different colors. The study of *multicolored* (also called *rainbow, heterochromatic*) subgraphs dates back to the 1960's. However, the special case of finding multicolored perfect matchings in complete bipartite graphs was first studied much earlier by Euler in the language of Latin squares. Since then this branch of combinatorics, especially the mentioned special case, has been flourishing. Several excellent surveys were dedicated to the subject, see [8, 9, 10, 13].

In this paper we mainly focus on the case when the host graph is a complete bipartite graph $K_{n,n}$, and the multicolored subgraph in view is a perfect matching (1-*factor*). There is a natural constraint on the coloring: it has to be proper.

These conditions can be reformulated in the language of Latin squares. A *Latin square* of order n is an $n \times n$ matrix, which has n different symbols as entries, and each symbol appears exactly once in each row and in each column. A *generalized* Latin square of order n is an $n \times n$ matrix, in which each symbol appears at most once in each row and in each column. A *diagonal* of a generalized Latin square of order n is a set of entries, which contains exactly one representative from each row and column. If the symbols are all different in a diagonal, then we call it a *transversal*.

Generalized Latin squares correspond to properly edge-colored complete bipartite graphs, while transversals correspond to multicolored 1-factors (perfect matchings). The so called *partial transversals* correspond to multicolored matchings. This intimate relation allows us to use the concept of symbols and colors interchangeably.

It is known that there exist Latin squares without a transversal. One might think that using more symbols should help finding a transversal. Therefore, it is natural to seek the sufficient number of symbols. We recall the following

Definition 1.1 (Akbari and Alipour [1])**.** Let $l(n)$ be the least number of symbols satisfying $l(n) \geq n$ that forces a transversal in any generalized Latin square of order n that contains at least $l(n)$ symbols.

In the terminology of matchings, they asked the threshold for the number $l(n)$ of colors such that any proper l-coloring of $K_{n,n}$ contains a multicolored perfect matching if $l \geq l(n)$. Notice that the function $l(n)$ is not obviously monotone increasing.

Akbari and Alipour determined $l(n)$ for small n: $l(1) = 1$, $l(2) = l(3) = 3$, $l(4) = 6$. They also proved that $l(n) \geq n + 3$ for $n = 2^a - 2$ ($2 < a \in \mathbb{N}$). They posed the following

Conjecture 1.2 (Akbari and Alipour [1])**.** *The difference $l(n) - n$ is not bounded if $n \to \infty$, while $l(n) \leq n^2/2$ if $n > 2$.*

Our main contribution is the following

Theorem 1.3. *$l(n) \leq 0.75n^2$ if $n > 1$.*

Although we conjecture that $l(n) = o(n^2)$, we must mention that if we relax the settings by allowing symbols to appear more than once in the columns, then for all n, there exist $n \times n$ transversal-free matrices, which contain $n^2/2 + O(n)$ symbols [2].

The paper is built up as follows. In Section 2 we show the connection of the problem to a classical Erdős–Spencer result. We prove an upper bound on $l(n)$ using a refined variant of the Lovász local lemma. We present the proof of Theorem 1.3, which is mainly built on König's theorem. Finally in Section 3, we propose the study of a function similar to $l(n)$, and investigate the relation to certain anti-Ramsey problems.

2 Two approaches to bound the number of symbols

2.1 Lovász local lemma

It is a classical application of the Lovász local lemma (LLL) that there exists a transversal in an $n \times n$ matrix if no color appears more than $\frac{1}{4e}n$ times. In fact, Erdős and Spencer [7] weakened the conditions of LLL by introducing the so called lopsided dependency graph G of the events, on which the following holds for every event E_i and every subfamily $\mathcal{F}$ of events $\{E_j : j \notin N_G[i]\}$:

$$P(E_i \mid \cap_{j\in\mathcal{F}} \overline{E_j}) \leq P(E_i),$$

where $N_G[i]$ denotes the closed neighborhood of vertex i in graph G. Under this assumption, it is enough to show the existence of an assignment $i \longmapsto (\mu_i > 0)$ which fulfills

$$P(E_i) \leq \frac{\mu_i}{\sum_{S\subseteq N_G[i]} \prod_{j\in S} \mu_j} \tag{2.1}$$

to obtain $P(\cap_i \overline{E_i}) > 0$.

Applying the ideas of Scott and Sokal [11]; Bissacot, Fernández, Procacci and Scoppola [4] observed that LLL remains valid if the summation in Inequality (2.1) is restricted to those sets S which are independent in G.

Let $c(a_{ij})$ denote the number of occurrences of the symbol a_{ij} in an $n \times n$ array A $(n > 1)$. Let $c_{i*}(A)$ and $c_{*j}(A)$ measure the average occurrence in row i and column j as

$$c_{i*}(A) = \left(\sum_t c(a_{it})\right) - n \quad \text{and} \quad c_{*j}(A) = \left(\sum_t c(a_{tj})\right) - n.$$

It can be viewed as some kind of weight-function on the rows and columns, where the weight is zero only if all entries admit uniquely occurring colors.

We follow the proof of the improvement on the Erdős-Spencer result in [4]. We show that $P(\cap_v \overline{E_v}) > 0$ holds for the set of events $\{E_v\}$ that a random diagonal contains a particular pair v of monochromatic entries. Here $|N_G[v]|$ in the lopsided dependency graph G depends only on the number of monochromatic pairs (v, v') of entries, which shares (at least) one row or column with an entry from both v and v'. Thus if v consists of a_{ij} and a_{kl}, then $|N_G[v]| \leq c_{i*}(A) + c_{*j}(A) + c_{k*}(A) + c_{*l}(A)$. Also if $w, w' \in N_G[v]$ covers the same row from $\{i, k\}$ or column from $\{j, l\}$ then w and w' are adjacent in G.

If we set $\mu_v := \mu \; \forall v$, then it is enough to provide a μ such that

$$P(E_v) = \frac{1}{n(n-1)} \leq \frac{\mu_v}{\sum_{S\subseteq N_G[v],\ S \text{ indep.}} \prod_{j\in S} \mu_j} = \frac{\mu}{\sum_{S\subseteq N_G[v],\ S \text{ indep.}} \mu^{|S|}}$$

Consequently, it is enough to set μ in such a way that

$$\frac{\mu}{\sum_{S\subseteq N_G[v],\ S \text{ indep.}} \mu^{|S|}} >$$

$$\frac{\mu}{(1 + c_{i*}(A)\mu)(1 + c_{*j}(A)\mu)(1 + c_{k*}(A)\mu)(1 + c_{*l}(A)\mu)} \geq \frac{1}{n(n-1)}$$

holds.

It is easy to see that $(1+U\mu)(1+V\mu) \leq (1+\frac{U+V}{2}\mu)^2$ for all $U, V \in \mathbb{R}$, hence

$$\frac{\mu}{(1+\overline{c_v}\mu)^4} \geq \frac{1}{n(n-1)}$$

implies the required condition, where

$$\overline{c_v} := \frac{c_{i*}(A) + c_{*j}(A) + c_{k*}(A) + c_{*l}(A)}{4}.$$

Thus if we set $\mu := \frac{1}{3\overline{c_v}}$, we obtain the following

Proposition 2.1. *There always exists a transversal in a generalized Latin square unless*

$$\left(\frac{4}{3}\right)^3 (c_{i*}(A) + c_{*j}(A) + c_{k*}(A) + c_{*l}(A)) > n(n-1) \tag{2.2}$$

holds for a pair of monochromatic entries a_{ij} and a_{kl}.

Corollary 2.2. $l(n) \leq (1 - \frac{27}{256})n^2 + \frac{27}{256}n \approx 0.895n^2$ *if* $n > 1$.

Proof. Observe that $n^2 - c_{i*}(A)$ or $n^2 - c_{*j}(A)$ bounds from above the number of colors in A for all $i, j \in [1, n]$. Consequently, if the number of colors is at least $(1 - \frac{27}{256})n^2 + \frac{27}{256}n$, then

$$\left(\frac{4}{3}\right)^3 c_{i*}(A) \leq \frac{1}{4}(n^2 - n) \quad \text{and} \quad \left(\frac{4}{3}\right)^3 c_{*j}(A) \leq \frac{1}{4}(n^2 - n)$$

for every row i and column j, which in turn provides the existence of a transversal according to Proposition 2.1. □

Remark 2.3. Note that while the proof of Erdős and Spencer points out the existence of one frequently occurring symbol, the proof above reveals that in fact many symbols must occur frequently to avoid a transversal.

2.2 Using König's theorem

We start with a lemma on the structure of partial transversals, which is essentially the consequence of the greedy algorithm. The following easy observation is due to Stein [12].

Result 2.4. *Consider r rows in a generalized Latin square A of order n. If $\frac{n+1}{2} \geq r$, then there exists a partial transversal of order r in A covering the r rows in view.*

We need the following consequence:

Lemma 2.5. *Consider p rows and q columns in an $n \times n$ generalized Latin square. If $q \leq p \leq (n+1)/2$, then either*

Case (a) $q \leq p/2$ *and there exists a partial transversal of size p covering the p rows and q columns, or*

Case (b) $q > p/2$ *and there exists a partial transversal of size $\lfloor p/2 \rfloor + q$ covering the p rows and q columns.*

Proof. Both parts follow from the fact that we can build a partial transversal by choosing first $\min\{q, \lceil p/2 \rceil\}$ different symbols in the array formed by the intersection of the p rows and q columns, and then we can extend this greedily by entries in the uncovered rows and columns of the array (essentially using Result 2.4). □

We proceed by recalling a variant of König's theorem, see Brualdi, Ryser [5].

Lemma 2.6. *There exists an all-*1 *diagonal in a* $0/1$ *square matrix of order* n *if and only if there does not exist an all-*0 *submatrix of size* $x \times y$*, where* $x + y \geq n + 1$.

Now we prove another upper bound on $l(n)$.

Theorem 2.7. *If a generalized Latin square of order* n *contains at least* $0.75n^2$ *symbols, then it has a transversal.*

Proof. First notice that the statement holds for $n = 1, 2$. We proceed by induction. Consider a generalized Latin square A of order n, which contains at least $0.75n^2$ symbols. A symbol is a *singleton* if it appears exactly once in A. We refer to the other symbols as *repetitions*. A submatrix is called a *singleton-*, resp. *repetition-submatrix* if every entry of the matrix is a singleton, resp. repetition.

Let p be the number of rows consisting only of repetitions and q be the number of columns consisting only of repetitions. We refer to these as full rows and columns, and assume that $q \leq p$. Notice that $p \leq n/2$, since the number of symbols is at least $0.75n^2$. Our aim is to choose a partial transversal that covers all full rows and columns, and then we complete this to a transversal by adding only singletons. First we apply Lemma 2.5 to get a partial transversal that covers the full rows and columns. Next, we omit the rows and columns that are covered by the chosen partial transversal. We obtain a generalized Latin square A' of order $n - p$ in Case (a) or of order $n - \lfloor p/2 \rfloor - q$ in Case (b). Now we are done by Lemma 2.6, if there are not too large repetition-submatrices in A'.

Suppose to the contrary that such a repetition-submatrix of size $x \times y$ exists in one of the cases, where $x + y$ is larger than the order of A'. Note first that in either case, A' does not contain full rows and columns. Therefore, we can choose a singleton σ_1 in A' such that at least x repetitions appear in its row. Similarly, we can choose a singleton σ_2 in A' such that at least y repetitions appear in its column.

Claim 2.8. *There exists a singleton* σ *such that the row of* σ *or the column of* σ *contains more than* $n/2$ *repetitions in the original Latin square* A.

Proof. In Case (a) of Lemma 2.5: $q \leq p/2$. The number of repetitions in the row of σ_1 is at least $q + x$ and number of repetitions in the column of σ_2 is at least $p + y$. Thus the statement holds since $p + q + x + y > p + q + (n - p) \geq n$.

In Case (b) of Lemma 2.5: $q > p/2$. The number of repetitions in the row of σ_1 is at least $q + x$ and number of repetitions in the column of σ_2 is at least $p + y$. Thus the statement holds since $p + q + x + y > p + q + (n - \lfloor p/2 \rfloor - q) \geq n$. □

In view of Claim 2.8, if we omit the row and column of the singleton σ, we obtain a generalized Latin square B of order $n - 1$, which admits more than $0.75n^2 - (2n - 1) + n/2 > 0.75(n - 1)^2$ symbols. By the induction hypothesis, there exists a transversal in B, hence it can be completed to a transversal of A by adding σ. □

3 Discussion

At the time of submission, we learned that Best, Hendrey, Wanless, Wilson and Wood [3] achieved results similar to ours. As the best upper bound, they proved $l(n) < (2-\sqrt{2})n^2$.

Nevertheless, not only the conjecture of Akbari and Alipour remained open, but it is plausible that it can be strengthened in the order of magnitude as well. In fact, the bound $\frac{1}{2}n^2$ is intimately related to the number of singletons, which took a crucial part in both our proof and the proof in [3]. If the number of colors does not exceed $\frac{1}{2}n^2$, then there might be no singletons at all. However, our first probabilistic proof implies also that either there exists a transversal in a generalized Latin square of order n with Cn^2 colors ($C > 0.45$), or the number of singletons is large. This fact points out that the constant $1/2$ in Conjecture 1.2 is highly unlikely to be sharp. More precisely, we show the following

Proposition 3.1. *If the number of singletons is less than* $(2C + 0.25\left(\frac{3}{4}\right)^3 - 1 + o(1))n^2$ *in a generalized Latin square of order* n *with* Cn^2 *symbols, then there exists a transversal.*

Proof. Suppose first that in every row and column, the sum $c_{i*}(A)$ and $c_{*j}(A)$ are below $0.25\left(\frac{3}{4}\right)^3(n^2-n)$. This in turn implies the existence of a transversal by Proposition 2.1. On the other hand, if for example $c_{i*}(A)$ exceeds that bound, then consider only the symbols not appearing in row i, and let us denote by n_k the number of symbols which occur exactly k times overall, with none of those occurrences being in row i. Clearly $\sum_k n_k = Cn^2 - n$ and $\sum_k kn_k = (n(n-1) - c_{i*}(A)) \leq (1 - 0.25\left(\frac{3}{4}\right)^3)(n^2-n)$. Consequently, for the number of singletons not appearing in the ith row,

$$n_1 \geq 2\sum_k n_k - \sum_k kn_k \geq (2C + 0.25\left(\frac{3}{4}\right)^3 - 1 + o(1))n^2,$$

which makes this case impossible. □

A special case, that appears as a bottleneck in some arguments concerns generalised Latin squares, where each repeated symbol has maximum multiplicity. We show that also in this special case, one can find a transversal.

Lemma 3.2. *If* A *is a generalised Latin square of order* n*, where each symbol has multiplicity* 1 *or* n *(and both multiplicities occur), then* A *has a transversal.*

Proof. We associate an edge-colored complete bipartite graph G_A to A such that vertices on one side correspond to rows the other side to columns and the colors of the edges to the symbols. Our goal is to find a multicolored matching.

Notice that the Latin property implies that a symbol with multiplicity n corresponds to a perfect matching. Let us remove all edges corresponding to symbols with multiplicity n. If there are r such colors, then the remaining bipartite graph is $(n-r)$-regular. As an easy corollary of Hall's theorem, any regular bipartite graph contains a perfect matching. In our case there are only singleton colors on the edges, so the perfect matching is multicolored. □

It seems likely that if the number of colors is large, then we not only obtain one transversal, but also a set of disjoint transversals. This motivates the study of the following function.

Definition 3.3. Let $l^*(n)$ be the least integer satisfying $l^*(n) \geq n$ such that for any proper edge-coloring of $K_{n,n}$ by at least $l^*(n)$ colors, the colored graph can be decomposed into the disjoint union of n multicolored perfect matchings.

Conjecture 3.4. *$l^*(n) \leq n^2/2$ if n is large enough.*

We remark that the difference of $l(n)$ and $l^*(n)$ is at least linear in n if $l(n) \neq n$.

Proposition 3.5. $l^*(n) - l(n) \geq n - 1$.

Proof. Suppose first that there exists a transversal-free generalised Latin square of order n, i.e., $l(n) > n$.

For $n \leq 2$ the claim is straightforward. Suppose $n \geq 3$. By definition, there exists a transversal-free generalized Latin square A of order n with $l(n) - 1$ symbols. Since $l(n) \leq 0.75n^2$, we can find a set S of $n - 1$ different repetitions, where $n - 1 \leq 0.25n^2$. We assign new symbols to the entries of S to create a new generalized Latin square A' of the same order. Since S cannot cover n disjoint transversals, and there were no transversals disjoint to S, matrix A' cannot be decomposed to n transversals, but contains $l(n) + n - 2$ symbols. Now consider the case when $l(n) = n$, which implies that n must be odd. Take the cyclic Latin square of order $n-1$ (which has no transversal, since $n-1$ is even) and add one row and column of singletons. The resulting matrix has $n-1+2n-1 = 3n-2$ symbols in it. However, it cannot be decomposed into transversals because such a decomposition would need to include a transversal of the embedded cyclic group table. □

Remark 3.6. Observe that the above result implies $l^*(n) \geq 2n - 1$ for all n. Notice that there are some orders n, for which $l(n) = n$, e.g. $n \in \{1, 3, 7\}$, see also [3].

The question we studied concerning $l(n)$ clearly has an anti-Ramsey flavor. The anti-Ramsey number $AR(n, \mathcal{G})$ for a graph family $\mathcal{G}$, introduced by Erdős, Simonovits and Sós [6], is the maximum number of colors in an edge coloring of K_n that has no multicolored (rainbow) copy of any graph in $\mathcal{G}$. To emphasize this connection, we propose the following problem.

Problem 3.7. What is the least number of colors $t(n, 2)$, which guarantees a rainbow 2-factor subgraph on at least $n - 1$ vertices in a properly edge-colored complete graph K_n colored by at least $t(n, 2)$ colors?

Perhaps the size $n - 1$ of the 2-factor subgraph seems artificial in some sense at first, or at least it could be generalized to any given function $f(n)$. We recall that for the function $t(n, 1)$ corresponding to 1-factors, Woolbright and Fu provided the following related result. In Problem 3.7, we have to allow two values $n - 1$ and n to avoid parity issues.

Proposition 3.8 ([14])**.** *Every properly colored K_{2n} has a multicolored 1-factor if the number of colors is at least $2n - 1$ and $n > 2$. That is, $t(n, 1) = n - 1$.*

In another formulation, the necessary number of colors for a proper edge-coloring is already sufficient to guarantee a multicolored perfect matching. It might happen that it also forces a much larger structure as required in Problem 3.7. We propose the following

Conjecture 3.9. *Any proper edge-coloring of K_{2n} by at least $2n - 1$ colors contains a multicolored 2-factor on $2n - 1$ or $2n$ vertices.*

If the above conjecture fails, then possibly there are proper edge-colorings of K_n without multicolored 2-factors of size n or $n-1$. In that case, we can use a connection between $t(n,2)$ and $l(n)$ to show a lower bound.

Proposition 3.10. $l(n) \geq t(n,2)+1$.

Proof. Consider an edge-coloring C of the complete graph K_n on vertex set V without multicolored 2-factors of size n or $n-1$. We associate to C a coloring of the complete bipartite graph $K_{n,n}$ on partite classes U and W as follows: let us assign the color of $v_iv_j \in E(K_n)$ $(i,j \in [1,n])$ to the edge $u_iw_j \in E(K_{n,n})$ if $i \neq j$, and color the set of independent edges u_iw_i $(i \in [1,n])$ by a separate color. Suppose that we found a multicolored 1-factor M in the complete bipartite graph. We omit at most one edge of M if we delete the edges u_iw_i and M' remains. Consider the edges v_kv_l in K_n, for which u_kw_l is contained in the multicolored M' of edges. This edge set is multicolored too, and each vertex has degree 2. □

References

[1] S. Akbari and A. Alipour, Transversals and multicolored matchings, *J. Combin. Des.* **12** (2004), 325–332, doi:10.1002/jcd.20014.

[2] J. Barát and I. M. Wanless, Rainbow matchings and transversals, *Australas. J. Combin.* **59** (2014), 211–217, https://ajc.maths.uq.edu.au/pdf/59/ajc_v59_p211.pdf.

[3] D. Best, K. Hendrey, I. M. Wanless, T. E. Wilson and D. R. Wood, Transversals in Latin arrays with many distinct symbols, *J. Combin. Des.* **26** (2018), 84–96, doi:10.1002/jcd.21566.

[4] R. Bissacot, R. Fernández, A. Procacci and B. Scoppola, An improvement of the Lovász local lemma via cluster expansion, *Combin. Probab. Comput.* **20** (2011), 709–719, doi:10.1017/s0963548311000253.

[5] R. A. Brualdi and H. J. Ryser, *Combinatorial Matrix Theory*, volume 39 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1991, doi:10.1017/cbo9781107325708.

[6] P. Erdős, M. Simonovits and V. T. Sós, Anti-Ramsey theorems, in: A. Hajnal, R. Rado and V. T. Sós (eds.), *Infinite and Finite Sets, Volume II*, North-Holland, Amsterdam, volume 10 of *Colloquia Mathematica Societatis János Bolyai*, pp. 633–643, 1975, proceedings of a Colloquium held at Keszthely, June 25 – July 1, 1973, dedicated to Paul Erdős on his 60th birthday.

[7] P. Erdős and J. Spencer, Lopsided Lovász local lemma and Latin transversals, *Discrete Appl. Math.* **30** (1991), 151–154, doi:10.1016/0166-218x(91)90040-4.

[8] S. Fujita, C. Magnant and K. Ozeki, Rainbow generalizations of Ramsey theory: a survey, *Graphs Combin.* **26** (2010), 1–30, doi:10.1007/s00373-010-0891-3.

[9] M. Kano and X. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphs – a survey, *Graphs Combin.* **24** (2008), 237–263, doi:10.1007/s00373-008-0789-5.

[10] C. F. Laywine and G. L. Mullen, *Discrete Mathematics Using Latin Squares*, volume 49 of *Wiley-Interscience Series in Discrete Mathematics and Optimization*, John Wiley & Sons, New York, 1998.

[11] A. D. Scott and A. D. Sokal, The repulsive lattice gas, the independent-set polynomial, and the Lovász local lemma, *J. Stat. Phys.* **118** (2005), 1151–1261, doi:10.1007/s10955-004-2055-4.

[12] S. K. Stein, Transversals of Latin squares and their generalizations, *Pacific J. Math.* **59** (1975), 567–575, http://projecteuclid.org/euclid.pjm/1102905365.

[13] I. M. Wanless, Transversals in Latin squares: a survey, in: R. Chapman (ed.), *Surveys in Combinatorics 2011*, Cambridge University Press, Cambridge, volume 392 of *London Mathematical Society Lecture Note Series*, pp. 403–437, 2011, doi:10.1017/cbo9781139004114.010, papers from the 23rd British Combinatorial Conference held at the University of Exeter, Exeter, July 3 – July 8, 2011.

[14] D. E. Woolbright and H.-L. Fu, On the existence of rainbows in 1-factorizations of K_{2n}, *J. Combin. Des.* **6** (1998), 1–20, doi:10.1002/(sici)1520-6610(1998)6:1⟨1::aid-jcd1⟩3.0.co;2-j.

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On the parameters of intertwining codes*

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Abstract

Let F be a field and let $F^{r\times s}$ denote the space of $r \times s$ matrices over F. Given equinumerous subsets $\mathcal{A} = \{A_i \mid i \in I\} \subseteq F^{r\times r}$ and $\mathcal{B} = \{B_i \mid i \in I\} \subseteq F^{s\times s}$ we call the subspace $C(\mathcal{A},\mathcal{B}) := \{X \in F^{r\times s} \mid A_iX = XB_i \text{ for } i \in I\}$ an *intertwining code*. We show that if $C(\mathcal{A},\mathcal{B}) \neq \{0\}$, then for each $i \in I$, the characteristic polynomials of A_i and B_i and share a nontrivial factor. We give an exact formula for $k = \dim(C(\mathcal{A},\mathcal{B}))$ and give upper and lower bounds. This generalizes previous work. Finally we construct intertwining codes with large minimum distance when the field is not 'too small'. We give examples of codes where $d = rs/k = 1/R$ is large where the minimum distance, dimension, and rate of the linear code $C(\mathcal{A},\mathcal{B})$ are denoted by d, k, and $R = k/rs$, respectively.

Keywords: Linear code, dimension, distance.

Math. Subj. Class.: 94B65, 60C05

1 Introduction

Let F be a field and let $F^{r\times s}$ denote the space of $r \times s$ matrices over F. Given equinumerous subsets $\mathcal{A} = \{A_i \mid i \in I\} \subseteq F^{r\times r}$ and $\mathcal{B} = \{B_i \mid i \in I\} \subseteq F^{s\times s}$ we call the subspace $C(\mathcal{A},\mathcal{B}) := \{X \in F^{r\times s} \mid A_iX = XB_i \text{ for } i \in I\}$ an *intertwining code*. The parameters of this linear code are denoted $[n, k, d]$ where $n = rs$, $k := \dim(C(\mathcal{A}, B))$ and d is the *minimum distance* of $C(\mathcal{A},\mathcal{B})$. Given $u, v \in F^n$ the *Hamming distance* $d(u, v) = |\{i \mid u_i \neq v_i\}|$ is the number of different coordinate entries, and a subspace $C \leqslant F^n$ has minimal (Hamming) distance $d(C) := \min\{d(u, v) \mid u \neq v\}$ which equals $\min\{d(0, w) \mid w \in V \text{ where } w \neq 0\}$. If $|I| = 1$ we write $C(A, B)$ instead

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of $C(\{A\},\{B\})$. Centralizer codes [1] have the form $C(A,A)$ and twisted centralizer codes [2, 3] have the form $C(A,\alpha A)$ where $A \in F^{r\times s}$ and $\alpha \in F$. Intertwining codes $C(A,B)$ are more general still, so our dimension formula (Theorem 2.8) has particularly wide applicability. Furthermore, the greater abundance of intertwining codes turns out to help us construct intertwining codes with large minimum distance, cf. Theorem 4.3 and [3, Theorem 3.2]. Intertwining codes have the advantage of a short description, and fast matrix multiplication algorithms give rise to efficient syndrome computations which, in turn, may be used for decoding as described in [3, §3].

Given representations $g_i \mapsto A_i$ and $g_i \mapsto B_i$ a group algebra $F\langle g_i \mid i \in I\rangle$, elements of $C(\mathcal{A},\mathcal{B})$ are homomorphisms between the associated modules. Hence Lemma 2.2 generalizes the fact that irreducible representations with distinct characters are inequivalent.

An exact formula for $k := \dim(C(A,B))$ is given in Theorems 2.9 and 2.8 of Section 2. The formula for k is simplified by an identity involving partitions proved in Section 4. Simpler upper and lower bounds for k are given in Section 5. In Theorem 4.3 in Section 4, we give an algorithm to construct A,B for which the minimum distance is $d(C(A,B)) = \lfloor r/k \rfloor s$. These examples have $dR \leqslant 1$ where $R = \frac{k}{rs}$ is the rate of $C(A,B)$.

Corollary 4.4 to Theorem 4.3 shows that there exist matrices $A \in F^{r\times r}$ and $B \in F^{s\times s}$ such that the intertwining code $C(A,B)$ has dimension $\min\{r,s\}$ and minimum distance $\max\{r,s\}$. We wonder how much this result can be improved.

2 A formula for $\dim_F(C(\mathcal{A},\mathcal{B}))$

Throughout this section $\mathcal{A} = \{A_i \mid i \in I\} \subset F^{r\times r}$ and $\mathcal{B} = \{B_i \mid i \in I\} \subset F^{s\times s}$ for a field F. The idea underlying this section is to use the Jordan form over the algebraic closure $\overline{F}$ of F to compute $\dim_F(C(\mathcal{A},\mathcal{B}))$. To implement this idea we must simultaneously conjugate each $A_i \in \mathcal{A}$, and each $B_i \in \mathcal{B}$, into Jordan form. This is always possible when $|I| = 1$.

Let $\mathrm{GL}(r,F)$ denote the general linear group of $r \times r$ invertible matrices over F. An ordered pair $(R,S) \in \mathrm{GL}(r,F) \times \mathrm{GL}(s,F)$ acts on $F^{r\times s}$ via $X^{(R,S)} = R^{-1}XS$. Clearly

$$\begin{aligned}(X^{(R_1,S_1)})^{(R_2,S_2)} &= X^{(R_1R_2,S_1S_2)},\\ (XS_1)^{(R,S)} &= X^{(R,S)}S_1^S, \quad \text{and}\\ (R_1X)^{(R,S)} &= R_1^RX^{(R,S)}.\end{aligned}$$

Lemma 2.1. *If* $(R,S) \in \mathrm{GL}(r,F) \times \mathrm{GL}(s,F)$*, then*

$$C(\mathcal{A},\mathcal{B})^{(R,S)} = R^{-1}C(\mathcal{A},\mathcal{B})S = C(\mathcal{A}^R,\mathcal{B}^S)$$

where $\mathcal{A}^R := \{R^{-1}A_iR \mid i \in I\}$ *and* $\mathcal{B}^S := \{S^{-1}B_iS \mid i \in I\}$.

Proof. For each $i \in I$, the equation $A_iX = XB_i$ is equivalent to

$$A_i^RX^{(R,S)} = (A_iX)^{(R,S)} = (XB_i)^{(R,S)} = X^{(R,S)}B_i^S. \qquad \square$$

Let $c_A(t) = \det(tI - A)$ be the characteristic polynomial of A.

Lemma 2.2. *If* $C(\mathcal{A},\mathcal{B}) \neq \{0\}$*, then* $\gcd(c_{A_i}(t), c_{B_i}(t)) \neq 1$ *for all* $i \in I$.

Proof. Suppose that for some $i \in I$ we have $\gcd(c_{A_i}(t), c_{B_i}(t)) = 1$. Then there exist polynomials $f(t), g(t)$ such that $f(t)c_{A_i}(t) + g(t)c_{B_i}(t) = 1$. Evaluating this equation at $t = B_i$, and noting that $c_{B_i}(B_i) = 0$, shows $f(B_i)c_{A_i}(B_i) = I$. Hence $c_{A_i}(B_i)$ is invertible. For $X \in C(\mathcal{A}, \mathcal{B})$, we have $A_i X = X B_i$. Thus

$$\left(\sum_{k\geqslant 0} \alpha_k A_i^k\right) X = X \left(\sum_{k\geqslant 0} \alpha_k B_i^k\right),$$

for all $\alpha_k \in F$, and therefore $c_{A_i}(A_i)X = X c_{A_i}(B_i)$. Since $c_{A_i}(A_i) = 0$, post-multiplying by $c_{A_i}(B_i)^{-1}$ shows that $X = 0$, and hence $C(\mathcal{A}, \mathcal{B}) = \{0\}$. □

Henceforth when we wish to emphasize the field F, we write $C_F(\mathcal{A}, \mathcal{B})$. Lemma 3.1 of [2], in essence, says $C_{\overline{F}}(\mathcal{A}, \mathcal{B}) = C_F(\mathcal{A}, \mathcal{B}) \otimes \overline{F}$. This immediately yields Lemma 2.3.

Lemma 2.3. *If K is an extension field of F, then* $\dim_F(C_F(\mathcal{A}, \mathcal{B})) = \dim_K(C_K(\mathcal{A}, \mathcal{B}))$. *In particular,* $\dim_F(C_F(\mathcal{A}, \mathcal{B})) = \dim_{\overline{F}}(C_{\overline{F}}(\mathcal{A}, \mathcal{B}))$ *where $\overline{F}$ is the algebraic closure of F.*

Lemma 2.3 allows us to assume that F is algebraically closed, which we shall do for the rest of this section. Given $A \in F^{r\times r}$ and $B \in F^{s\times s}$ define $A \oplus B$ to be the block diagonal matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & B\end{smallmatrix}\right)$, and define $\mathcal{A} \oplus \mathcal{B}$ to be $\{A_i \oplus B_i \mid i \in I\} \subseteq F^{(r+s)\times(r+s)}$.

Lemma 2.4. *If $\mathcal{A}_1 \subseteq F^{r_1\times r_1}, \dots, \mathcal{A}_m \subseteq F^{r_m\times r_m}$ and $\mathcal{B}_1 \subseteq F^{s_1\times s_1}, \dots, \mathcal{B}_n \subseteq F^{s_n\times s_n}$ are subsets, all with the same finite cardinality, then*

$$C\left(\bigoplus_{i=1}^m \mathcal{A}_i, \bigoplus_{j=1}^n \mathcal{B}_j\right) \cong \bigoplus_{i=1}^m \bigoplus_{j=1}^n C(\mathcal{A}_i, \mathcal{B}_j).$$

Proof. Write $X = (X_{ij})$ as a block matrix where X_{ij} has size $r_i \times s_j$. The condition $X \in C\left(\bigoplus_{i=1}^m \mathcal{A}_i, \bigoplus_{j=1}^n \mathcal{B}_j\right)$ is equivalent to $X_{ij} \in C(\mathcal{A}_i, \mathcal{B}_j)$ for each i, j. □

Corollary 2.5. *Suppose that $\mathcal{A}_1, \dots, \mathcal{A}_m$ and $\mathcal{B}_1, \dots, \mathcal{B}_n$ are as in Lemma 2.4, and suppose that for $i \neq j$, the characteristic polynomials of matrices in $\mathcal{A}_i$ are coprime to the characteristic polynomials of matrices in $\mathcal{B}_j$. Then*

$$C\left(\bigoplus_{i=1}^m \mathcal{A}_i, \bigoplus_{j=1}^n \mathcal{B}_j\right) \cong \bigoplus_{i=1}^{\min\{m,n\}} C(\mathcal{A}_i, \mathcal{B}_i).$$

Proof. Use Lemma 2.4, and note that $C(\mathcal{A}_i, \mathcal{B}_j) = \{0\}$ for $i \neq j$ by Lemma 2.2. □

Remark 2.6. Let F be a finite field. Standard arguments, for example [6, p. 168], can be used to relate $\dim_{\overline{F}}(C_{\overline{F}}(\mathcal{A}, \mathcal{B}))$ to data computed over F. This remark and Remark 2.10 explain the details. Let $p_1(t), p_2(t), \dots$ enumerate the (monic) irreducible polynomials over F and write $c_A(t) = \prod_{i\geqslant 1} p_i(t)^{k_i}$ and $c_B(t) = \prod_{i\geqslant 1} p_i(t)^{\ell_i}$, respectively. This gives rise to the A-invariant primary decomposition $F^r = \bigoplus_{i\geqslant 1} \ker(p_i(A)^{k_i})$, and the B-invariant decomposition $F^s = \bigoplus_{i\geqslant 1} \ker(p_i(A)^{\ell_i})$. Let A_i be the restriction of A to $\ker(p_i(A)^{k_i})$ and B_i the restriction of B to $\ker(p_i(A)^{\ell_i})$. Corollary 2.5 shows that $\dim(C(A, B)) = \sum_{i\geqslant 1} \dim(C(A_i, B_i))$. The second ingredient involves partitions and is described in Remark 2.10.

It is straightforward to see that $C(\mathcal{A},\mathcal{B}) = \bigcap_{i\in I} C(A_i, B_i)$ where $C(A_i, B_i)$ means $C(\{A_i\},\{B_i\})$. Recall that a matrix $A \in F^{r\times r}$ is *nilpotent* if and only if $A^r = 0$. We say that A is *α-potent*, where $\alpha \in F$, if $(A-\alpha I)^r = 0$. The following lemma and theorem reduce our deliberations from α-potent matrices to nilpotent matrices. For $\mathcal{A} = \{A_i \mid i \in I\} \subseteq F^{r\times r}$, let $\mathcal{A} - \alpha I_r$ denote the set $\{A_i - \alpha I_r \mid i \in I\}$.

Lemma 2.7. *If $\mathcal{A} \subseteq F^{r\times r}$, $\mathcal{B} \subseteq F^{s\times s}$ and $\alpha \in F$, then $C(\mathcal{A},\mathcal{B}) = C(\mathcal{A}-\alpha I_r, \mathcal{B}-\alpha I_s)$.*

Proof. For $i \in I$, $A_iX = XB_i$ holds if and only if $(A_i - \alpha I_r)X = X(B_i - \alpha I_s)$. □

A *partition* λ of r, written $\lambda \vdash r$, is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of integers satisfying

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0 \quad \text{and} \quad \lambda_1 + \lambda_2 + \cdots = r.$$

We call λ_i the ith part of λ, and we usually omit parts of size zero. Let N_r be the $r \times r$ nilpotent matrix with all entries 0 except for an entry 1 in position $(i, i+1)$ for $1 \leqslant i < r$. Let $N_\lambda = \bigoplus N_{\lambda_i}$ where $\lambda \vdash r$. Every nilpotent $r \times r$ matrix is conjugate to some N_λ for a unique $\lambda \vdash r$. Furthermore, if an $r \times r$ matrix R has eigenvalues $\rho_1, \dots, \rho_m$ and associated generalized eigenspaces of dimensions $r_1, \dots, r_m$ where $r_1 + \cdots + r_m = r$, then R has Jordan form $\bigoplus_{i=1}^m (\rho_i I_{r_i} + N_{\lambda_i})$ where λ_i is a *partition* of r_i (not a part of a partition).

Theorem 2.8. *Suppose $A \in F^{r\times r}$, $B \in F^{s\times s}$ and $\gcd(c_A(t), c_B(t))$ has distinct roots $\zeta_1, \dots, \zeta_m$ in $\overline{F}$. Suppose that the sizes of the Jordan blocks of A associated with the generalized ζ_i-eigenspace of A determine a partition α_i, and the sizes of the Jordan blocks of B associated with the generalized ζ_i-eigenspace of B determine a partition β_i. Then*

$$\dim(C(A,B)) = \sum_{i=1}^m \dim(C(N_{\alpha_i}, N_{\beta_i})).$$

Proof. By Lemma 2.3 we may assume that $F = \overline{F}$. Let A_i be the restriction of A to its generalized ζ_i-eigenspace $\{v \mid v(A-\zeta_i I)^k \text{ for some } k \geqslant 0\}$. Then A_i is ζ_i-potent, and so determines a partition α_i. Similarly, the restriction B_i of B to the ζ_i-eigenspace determines a partition β_i. By Corollary 2.5 and Lemma 2.7, we have

$$\dim(C(A,B)) = \sum_{i=1}^m \dim(C(A_i,B_i)) = \sum_{i=1}^m \dim(C(N_{\alpha_i}, N_{\beta_i})).$$ □

Theorem 2.9. *Given partitions λ of r and μ of s, the dimension of $C(N_\lambda, N_\mu)$ equals*

$$\dim(C(N_\lambda, N_\mu)) = \sum_{i\geqslant 1}\sum_{j\geqslant 1} \min\{\lambda_i, \mu_j\}.$$

Proof. As $\lambda \vdash r$ and $\mu \vdash s$, we have $\sum_{i\geqslant 1} \lambda_i = r$ and $\sum_{j\geqslant 1} \mu_j = s$. Lemma 2.4 shows that $C(N_\lambda, N_\mu) \cong \bigoplus_{i\geqslant 1}\bigoplus_{j\geqslant 1} C(N_{\lambda_i}, N_{\mu_j})$. Taking dimensions, it suffices to show $\dim(C(N_{\lambda_i}, N_{\mu_j})) = \min\{\lambda_i, \mu_j\}$. This can be shown by solving $N_{\lambda_i}X = XN_{\mu_j}$ for X and counting the number of free variables. Alternatively, F^{λ_i} is a uniserial $\langle N_{\lambda_i}\rangle$-module with 1-dimensional quotient modules, and similarly for F^{λ_j}. As their largest common quotient module is $F^{\min\{\lambda_i,\lambda_j\}}$, we have $\dim(C(N_{\lambda_i}, N_{\lambda_j})) = \min\{\lambda_i, \lambda_j\}$. □

Remark 2.10. Suppose $|F| = q$ is finite. Following on from Remark 2.6 it suffices to consider the case where $c_A(t) = p(t)^{r/d}$, $c_B(t) = p(t)^{s/d}$, where $p(t)$ is irreducible over F of degree d. The field $K := F[t]/(p(t))$ has order q^d. In this case the structure of the modules $F^r = K^{r/d}$ and $F^s = K^{s/d}$ is determined by partitions $\lambda \vdash r/d$ and $\mu \vdash s/d$. It turns out that A is conjugate (see below) to $N_{\lambda,p} := \mathrm{diag}(N_{\lambda_1,p}, N_{\lambda_2,p}, \dots) \in F^{r\times r}$ where

$$N_{m,p} = \begin{pmatrix} C(p) & I & & \\ & \ddots & \ddots & \\ & & C(p) & I \\ & & & C(p) \end{pmatrix} \in F^{dm\times dm}$$

and $C(p) \in F^{d\times d}$ is the companion matrix of $p(t)$. Now $C(p)$ is conjugate in $\mathrm{GL}(d, K)$ to $\mathrm{diag}(\zeta_1, \dots, \zeta_d)$ where $\zeta_1, \dots, \zeta_d$ are the (distinct) roots of $p(t)$ in K. It follows from Theorems 2.9 and 2.8 that

$$\dim(C(A, B)) = \dim(C(N_{\lambda,p}, N_{\mu,p})) = d \sum_{i\geqslant 1} \sum_{j\geqslant 1} \min\{\lambda_i, \mu_j\}. \tag{2.1}$$

As an example, suppose A is cyclic and $c_A(t) = p(t)^3$ where $d = \deg(p) = 3$. In this case $r = 9$ and $\lambda = (3)$. Write $p(t) = t^3 + p_2t^2 + p_1t + p_0 = (t - \zeta_1)(t - \zeta_2)(t - \zeta_3)$. Then A is conjugate in $\mathrm{GL}(9, F)$ by [5] to

$$\begin{pmatrix} C(p) & N & 0 \\ 0 & C(p) & N \\ 0 & 0 & C(p) \end{pmatrix}$$

where

$$C(p) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As $p(t)$ is separable, [5, Theorem 1] implies that A is conjugate in $\mathrm{GL}(9, F)$ to

$$\begin{pmatrix} C(p) & I & 0 \\ 0 & C(p) & I \\ 0 & 0 & C(p) \end{pmatrix}.$$

Hence A is conjugate in $\mathrm{GL}(9, K)$ to

$$\begin{pmatrix} D(\zeta_1) & 0 & 0 \\ 0 & D(\zeta_2) & 0 \\ 0 & 0 & D(\zeta_3) \end{pmatrix}$$

where

$$D(\zeta) = \begin{pmatrix} \zeta & 1 & 0 \\ 0 & \zeta & 1 \\ 0 & 0 & \zeta \end{pmatrix}.$$

This explains the factor of $d = \deg(p(t))$ in equation (2.1) and relates the generalized Jordan form of A over F to the Jordan form of A over K.

3 Conjugate partitions

In this section we simplify the formula in Theorem 2.9 for $\dim(C(N_\lambda, N_\mu))$. We prove an identity in Lemma 3.2 involving partitions which replaces multiple sums by a single sum. In order to state the simpler dimension formula we need to define 'conjugate partitions'. The *conjugate* of $\lambda \vdash r$ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ of r whose parts satisfy $\lambda'_i = |\{j \mid \lambda_j \geqslant i\}|$, for each i. The Young diagram of λ', is obtained from that of λ by swapping rows and columns as shown in Figure 1.

Figure 1: Young diagrams for $\lambda = (5, 3, 3, 1)$ and $\lambda' = (4, 3, 3, 1, 1)$.

For the following result, note that the number of nonzero λ_i is λ'_1, and $r = \sum_{i=1}^{\lambda'_1} \lambda_i$.

Theorem 3.1. *Given partitions λ of r and μ of s, the dimension of $C(N_\lambda, N_\mu)$ equals*

$$\dim(C(N_\lambda, N_\mu)) = \sum_{i\geqslant 1} \lambda'_i \mu'_i = \sum_{i=1}^{\min\{\lambda_1,\mu_1\}} \lambda'_i \mu'_i.$$

To prove Theorem 3.1 we need a technical lemma which we have not been able to find in the literature, see [4]. Lemma 3.2 below says $\sum_{i\geqslant 1} \lambda_i = \sum_{i\geqslant 1} \lambda'_i$ when $k = 1$. We only need the case $k = 2$ for the proof of Theorem 3.1, however, the argument for $k > 2$ is not much harder.

Lemma 3.2. *If $\lambda, \mu, \dots, \omega$ are partitions and $\lambda', \mu', \dots, \omega'$ are their conjugates, then*

$$\sum_{i=1}^{\lambda'_1} \sum_{j=1}^{\mu'_1} \cdots \sum_{k=1}^{\omega'_1} \min\{\lambda_i, \mu_j, \dots, \omega_k\} = \sum_{i=1}^{\min\{\lambda_1,\mu_1,\dots,\omega_1\}} \lambda'_i \mu'_i \cdots \omega'_i. \tag{3.1}$$

Proof. By permuting the partitions $\lambda, \mu, \dots, \omega$ if necessary, we can assume that

$$\lambda_1 \leqslant \mu_1 \leqslant \cdots \leqslant \omega_1.$$

If $\lambda_1 = 0$, then $\lambda \vdash 0$ and both sides of (3.1) are zero. If $\lambda_1 = 1$, then

$$\mathrm{LHS}(1) = \sum_{i=1}^{\lambda'_1} \sum_{j=1}^{\mu'_1} \cdots \sum_{k=1}^{\omega'_1} 1 = \lambda'_1 \mu'_1 \cdots \omega'_1 = \sum_{i=1}^{\min\{\lambda_1,\mu_1,\dots,\omega_1\}} \lambda'_i \mu'_i \cdots \omega'_i = \mathrm{RHS}(1).$$

Suppose now that $\lambda_1 > 1$. We use induction on λ_1. Let $\widehat{\lambda}$ be the partition of $(\sum_{i\geqslant 1} \lambda_i) - \lambda'_1$ obtained by deleting the first column of the Young diagram of λ. Since $1 < \mu_1 \leqslant \cdots \leqslant \omega_1$, we define $\widehat{\mu}, \dots, \widehat{\omega}$ similarly. It is clear that $\widehat{\lambda}_i = \lambda_i - 1$ for $1 \leqslant i \leqslant \lambda'_1$ and $\widehat{\lambda}'_i = \lambda'_{i+1}$ for $i \geqslant 1$, and similarly for $\widehat{\mu}, \dots, \widehat{\omega}$. As $\widehat{\lambda}_1 < \lambda_1$, induction shows

$$\sum_{i=1}^{\widehat{\lambda}'_1} \sum_{j=1}^{\widehat{\mu}'_1} \cdots \sum_{k=1}^{\widehat{\omega}'_1} \min\{\widehat{\lambda}_i, \widehat{\mu}_j, \dots, \widehat{\omega}_k\} = \sum_{i=1}^{\min\{\widehat{\lambda}_1,\widehat{\mu_1},\dots,\widehat{\omega}_1\}} \widehat{\lambda}'_i \widehat{\mu}'_i \cdots \widehat{\omega}'_i.$$

Note that $\widehat{\lambda}_i = 0$ for each $i \in [\widehat{\lambda}'_1 + 1, \lambda'_1]$ since $\widehat{\lambda}'_1 = \lambda'_2$, so the upper limit $\widehat{\lambda}'_1$ of the sum $\sum_{i=1}^{\widehat{\lambda}'_1}$ can be replaced by λ'_1. Similarly, the upper limits $\widehat{\mu}'_1, \dots, \widehat{\omega}'_1$ can be replaced by $\mu'_1, \dots, \omega'_1$. Hence, since $\widehat{\lambda}_i = \lambda'_i - 1, \dots, \widehat{\omega}_i = \omega'_i - 1$, we have

$$\sum_{i=1}^{\lambda'_1}\sum_{j=1}^{\mu'_1}\cdots\sum_{k=1}^{\omega'_1}\min\{\lambda_i - 1, \mu_j - 1, \dots, \omega_k - 1\} = \sum_{i=1}^{\min\{\lambda_1-1,\mu_1-1,\dots,\omega_1-1\}} \lambda'_{i+1}\mu'_{i+1}\cdots\omega'_{i+1}.$$

Re-indexing the right sum, and using $\sum_{i=1}^{\lambda'_1}\sum_{j=1}^{\mu'_1}\cdots\sum_{k=1}^{\omega'_1}(-1) = -\lambda'_1\mu'_1\cdots\omega'_1$ gives

$$-\lambda'_1\mu'_1\cdots\omega'_1 + \sum_{i=1}^{\lambda'_1}\sum_{j=1}^{\mu'_1}\cdots\sum_{k=1}^{\omega'_1}\min\{\lambda_i, \mu_j, \dots, \omega_k\} = \sum_{i=2}^{\min\{\lambda_1,\mu_1,\dots,\omega_1\}} \lambda'_i\mu'_i\cdots\omega'_i.$$

Adding $\lambda'_1\mu'_1\cdots\omega'_1$ to both sides completes the inductive proof of (3.1). □

Proof of Theorem 3.1. Apply Theorem 2.9 and Lemma 3.2 with $k = 2$. □

4 Minimum distances

In Section 2 a formula is given for $k := \dim(C(\mathcal{A}, B))$; where we suppress mention of the field F in our notation. In this section we choose $\mathcal{A}$ and $\mathcal{B}$ to maximize the value of the minimum distance $d := d(C(\mathcal{A}, \mathcal{B}))$ as a function of k. We focus on the case when $|\mathcal{A}| = |\mathcal{B}| = 1$. The action of $\mathrm{GL}(r, F) \times \mathrm{GL}(s, F)$ of $C(A, B)$ fixes $k = \dim(C(A, B))$ but can change d wildly, e.g. from 1 to rs as setting $k = 1$ in Theorem 4.3 illustrates.

Let E_{ij} denote the $r \times s$ matrix with all entries 0, except the (i, j) entry which is 1.

Lemma 4.1. *Suppose $r, s, k \in \mathbb{Z}$ where $1 \leqslant k \leqslant \min\{r, s\}$, and suppose F is a field with $|F| \geqslant k+\min\{1, r-k\}+\min\{1, s-k\}$. Fix pairwise distinct scalars $\zeta_1, \dots, \zeta_k, \alpha, \beta \in F$ and set*

$$A_0 := \mathrm{diag}(\zeta_1, \dots, \zeta_k, \alpha, \dots, \alpha) \in F^{r\times r} \quad \textit{and}$$
$$B_0 := \mathrm{diag}(\zeta_1, \dots, \zeta_k, \beta, \dots, \beta) \in F^{s\times s}.$$

Then $C(A_0, B_0) = \langle E_{11}, \dots, E_{kk}\rangle$ has dimension k and minimum distance 1.

Proof. Note first that if $k = \min\{r, s\}$, then A_0 has no αs, or B_0 has no βs. Thus the assumption $|F| \geqslant k + \min\{1, r - k\} + \min\{1, s - k\}$ ensures that distinct scalars $\zeta_1, \dots, \zeta_k, \alpha, \beta$ in F exist. Using a direct calculation of $C(A_0, B_0)$, or Corollary 2.5, shows that $C(A_0, B_0) = \langle E_{11}, \dots, E_{kk}\rangle$. Since $d(0, E_{11}) = 1$, we have $d(C(A_0, B_0)) = 1$. □

We now seek $R \in \mathrm{GL}(r, F)$ and $S \in \mathrm{GL}(s, F)$ such that $R^{-1}\langle E_{11}, \dots, E_{kk}\rangle S$ has large minimum distance. For brevity, we write $T := R^{-1}$.

Denote the ith row of a matrix A by A_{i*} and its jth column by A_{*j}.

Lemma 4.2. *Suppose $r, s, k \in \mathbb{Z}$ where $k \leqslant \min\{r, s\}$. Fix $S \in F^{s\times s}$ and $T \in F^{r\times r}$ and define $X^{(1)}, \dots, X^{(k)} \in F^{r\times s}$ by $X^{(\ell)} = T_{*\ell}S_{\ell*}$ for $1 \leqslant \ell \leqslant k$. Then $TE_{\ell\ell}S = X^{(\ell)}$ for $1 \leqslant \ell \leqslant k$.*

Proof. Suppose δ_{ij} is 1 if $i = j$ and 0 otherwise. Then the (i, j) entry of $E_{\ell\ell}$ is $\delta_{i\ell}\delta_{\ell j}$. The (i', j') entry of $T_{*\ell}S_{\ell *}$ is $t_{i'\ell}s_{\ell j'}$. This agrees with the (i', j') entry of $TE_{\ell\ell}S$, namely

$$\sum_{i=1}^{r}\sum_{j=1}^{s} t_{i'i}\delta_{i\ell}\delta_{\ell j}s_{jj'} = t_{i'\ell}s_{\ell j'}. \qquad \square$$

Theorem 4.3. *Suppose $r, s, k \in \mathbb{Z}$ where $1 \leqslant k \leqslant \min\{r, s\}$, and suppose F is a field with $|F| \geqslant k + 2$. Then there exist $A \in F^{r\times r}$ and $B \in F^{s\times s}$ such that the linear code $C(A, B)$ has dimension k and minimum distance $d = \lfloor r/k\rfloor s$.*

Proof. By Lemma 4.1 there exist diagonal matrices $A_0 \in F^{r\times r}$ and $B_0 \in F^{s\times s}$ such that $C(A_0, B_0) = \langle E_{11}, \dots, E_{kk}\rangle$ has dimension k. We seek invertible matrices $R \in F^{r\times r}$ and $S \in F^{s\times s}$ such that $A = A_0^R$ and $B = B_0^S$ give $C(A, B) = \langle E_{11}, \dots, E_{kk}\rangle^{(R,S)}$ with minimum distance $d = \lfloor r/k\rfloor s$. Let $X^{(\ell)} = R^{-1}E_{\ell\ell}S$ for $1 \leqslant \ell \leqslant k$. The $r \times s$ matrices $X^{(\ell)}$, $1 \leqslant \ell \leqslant k$, will have a form which makes it clear that $d = \lfloor r/k\rfloor s$.

First, we partition the set $\{1, \dots, r\}$ of rows into the following k subsets:

$$I_1 = \left\{1, \dots, \left\lfloor \frac{r}{k}\right\rfloor\right\}, I_2 = \left\{\left\lfloor \frac{r}{k}\right\rfloor + 1, \dots, 2\left\lfloor \frac{r}{k}\right\rfloor\right\}, \dots,$$
$$I_k = \left\{(k-1)\left\lfloor \frac{r}{k}\right\rfloor + 1, \dots, r\right\}.$$

Choose the ith row of the matrix $X^{(\ell)}$ to be zero if $i \notin I_\ell$, and to be a vector with all s entries nonzero otherwise. Since $\left\lfloor \frac{r}{k}\right\rfloor = |I_\ell| \leqslant |I_k|$ for $\ell < k$, it follows that

$$d(0, X^{(\ell)}) = \sum_{i\in I_\ell} s = |I_\ell|s \geqslant \left\lfloor \frac{r}{k}\right\rfloor s \qquad \text{for } 1 \leqslant \ell \leqslant k$$

with equality if $\ell < k$. The choice of these matrices is such that for each nonzero X in the span $\langle X^{(1)}, \dots, X^{(k)}\rangle$ we also have $d(0, X) \geqslant d(0, X^{(\ell)})$ for some ℓ, and hence $\langle X^{(1)}, \dots, X^{(k)}\rangle$ has minimum distance $d = \lfloor r/k\rfloor s$.

It is well known that if the first few rows of a square matrix are linearly independent, then the remaining rows can be chosen so that the matrix is invertible. A similar remark holds if the first few columns are linearly independent. Our construction uses k linearly independent $1 \times s$ row vectors $u_1, \dots, u_k$ which give the first k rows of $S \in \mathrm{GL}(s, F)$, and k linearly independent $r \times 1$ column vectors $v^{(1)}, \dots, v^{(k)}$ which give the first k columns of $R^{-1} \in \mathrm{GL}(r, F)$. The pair (R, S) will be used to construct A and B.

Henceforth suppose that $1 \leqslant \ell \leqslant k$. Since $|F| \geqslant 3$, we may choose $\gamma \in F \setminus \{1, 1 - s\}$. Let J be the $s \times s$ matrix with all entries 1. Then the $s \times s$ matrix $S' = (\gamma - 1)I + J$ is invertible as $\det(S') = (\gamma - 1)^{s-1}(\gamma + s - 1)$ is nonzero. Let $u_\ell = (1, \dots, 1, \gamma, 1, \dots, 1)$ be the ℓth row of S'. Since $u_1, \dots, u_k$ are linearly independent, there exists an invertible matrix $S \in \mathrm{GL}(s, F)$ with $S_{\ell *} = u_\ell$. Of course $S = S'$ is one possibility. Similarly, let $v^{(\ell)}$ be the $r \times 1$ column vector

$$v_i^{(\ell)} = \begin{cases} 1 & \text{if } i \in I_\ell, \\ 0 & \text{if } i \notin I_\ell. \end{cases}$$

As $v^{(1)}, \dots, v^{(k)}$ are linearly independent, there exists an $r \times r$ invertible matrix, which we call R^{-1}, whose first k columns are $v^{(1)}, \dots, v^{(k)}$. Lemma 4.2 shows that $R^{-1}E_{\ell\ell}S = X^{(\ell)}$ for $1 \leqslant \ell \leqslant k$. Hence $C(A_0^R, B_0^S) = C(A_0, B_0)^{(R,S)} = \langle X^{(1)}, \dots, X^{(k)}\rangle$ has minimum distance $\lfloor r/k\rfloor\, s$ as desired. $\square$

Corollary 4.4. *If* $|F| \geqslant \min\{r,s\}+2$*, then there exist matrices* $A \in F^{r\times r}$ *and* $B \in F^{s\times s}$ *such that* $C(A,B)$ *has dimension* $\min\{r,s\}$ *and minimum distance* $\max\{r,s\}$.

Proof. Since $AX = XB$ if and only if $X^T A^T = B^T X^T$ we see that $C(B^T, A^T)$ equals $C(A,B)^T$. Because $C(A,B)$ and $C(A,B)^T$ have the same dimension and minimum distance, we may assume that $r \leqslant s$. If $|F| \geqslant r+2$, then applying Theorem 4.3 with $k = r$ gives the desired result. □

Remark 4.5. Suitable matrices A and B in Theorem 4.3 are constructed by first choosing the diagonal matrices A_0 and B_0 in Lemma 4.1, and then taking $A = R^{-1}A_0R$ and $B = S^{-1}B_0S$ where R and S are constructed in the proof of Theorem 4.3.

It is desirable for a code to have both a high rate, *viz.* $R = k/n$, and a high distance d. Can the product Rd be a constant for intertwining codes? By setting $r = s = k$ in Theorem 4.3 we obtain a rate of $R = 1/r$ and a distance of $d = r$, so the answer is affirmative. It is natural to ask how the maximum value of Rd for an intertwining code depends on (r,s,F)? *We wonder whether there is a sequence* $C_1, C_2, \dots$ *of intertwining codes over a field* F *with parameters* $[r_is_i, k_i, d_i]$ *for which* $R_id_i = \frac{k_id_i}{r_is_i}$ *approaches infinity.*

5 Upper and lower bounds for $\dim_F(C(\mathcal{A},\mathcal{B}))$

Denote that rank and nullity of $A \in F^{r\times r}$ by $\mathrm{Rk}(A)$ and $\mathrm{Null}(A)$, respectively. Note that $\mathrm{Rk}(A) + \mathrm{Null}(A) = r$ and $\mathrm{Null}(N_\lambda) = \lambda'_1$. In this section we bound $k = \dim(C(A,B))$ in terms of the rank and nullity of A and B. If $\lambda \vdash r$ and $\mu \vdash s$, Theorem 2.9 implies that

$$\lambda'_1\mu'_1 \leqslant \sum_{i\geqslant 1}\lambda'_i\mu'_i = \dim(C(N_\lambda, N_\mu)) \leqslant \left(\sum_{i\geqslant 1}\lambda'_i\right)\left(\sum_{j\geqslant 1}\mu'_j\right) = rs. \tag{5.1}$$

View $A \in F^{r\times r}$ as acting on an r-dimensional vector space over the algebraic closure $\overline{F}$. Let the α-eigenspace, and the generalized α-eigenspace, of A have dimensions $k_{A,\alpha}$ and $m_{A,\alpha}$, respectively. Then $c_A(t) = \prod(t-\alpha)^{m_{A,\alpha}}$ where $m_{A,\alpha} \neq 0$ for finitely many $\alpha \in \overline{F}$ and $0 \leqslant k_{A,\alpha} \leqslant m_{A,\alpha}$. The following result generalizes [2, Theorems 2.8 and 4.7].

Theorem 5.1. *If* $A \in F^{r\times r}$ *and* $B \in F^{s\times s}$*, then*

(a)

$$\sum k_{A,\alpha}k_{B,\alpha} \leqslant \dim(C(A,B)) \leqslant \sum m_{A,\alpha}m_{B,\alpha}, \quad \textit{and}$$

(b)

$$(r - \mathrm{Rk}(A))(s - \mathrm{Rk}(B)) \leqslant \dim(C(A,B)) \leqslant \\ (r - \mathrm{Rk}(A))(s - \mathrm{Rk}(B)) + \mathrm{Rk}(A)\,\mathrm{Rk}(B).$$

Proof. Part (a) follows immediately from Theorem 2.8 and (5.1).

(b) The lower bound follows from part (a) since $r - \mathrm{Rk}(A) = \mathrm{Null}(A) = k_{A,0}$. For the upper bound, note that A is similar to a diagonal direct sum $N_\lambda \oplus A'$ where N_λ is nilpotent of size $m_{0,A}$ and A' is invertible of size $r - m_{0,A}$. Similarly, B is similar to

$N_\mu \oplus B'$ where N_μ is nilpotent of size $m_{0,B}$ and B' is invertible of size $s - m_{0,B}$. It follows from Theorem 2.8 that $\dim(C(A,B)) = \dim(C(N_\lambda, N_\mu)) + \dim(C(A', B'))$. Further by Theorem 2.9 $\dim(C(N_\lambda, N_\mu)) = \sum_{i\geqslant 1} \lambda'_i \mu'_i$ where, as usual, λ' and μ' denote conjugate partitions. We use the observation:

$$\text{if} \quad 0 \leqslant x \leqslant a \quad \text{and} \quad 0 \leqslant y \leqslant b, \quad \text{then} \quad (a-x)(b-y) + xy \leqslant ab \tag{5.2}$$

to show that

$$\begin{aligned}\dim(C(A,B)) &= \lambda'_1\mu'_1 + \sum_{i\geqslant 2} \lambda'_i\mu'_i + \dim(C(A',B')) \\ &\leqslant \lambda'_1\mu'_1 + (m_{0,A} - \lambda'_1)(m_{0,B} - \mu'_1) + (r - m_{0,A})(s - m_{0,B}) \\ &\leqslant \lambda'_1\mu'_1 + (r - \lambda'_1)(s - \mu'_1).\end{aligned}$$

The result follows since

$$\lambda'_1 = \mathrm{Null}(N_\lambda) = \mathrm{Null}(A) = r - \mathrm{Rk}(A) \quad \text{and} \quad \mu'_1 = s - \mathrm{Rk}(B). \qquad \square$$

The Singleton bound $d + k \leqslant n + 1$ implies that if d is close to $n = rs$, then k is small, and the lower bound of Theorem 5.1(b) implies that A or B has high rank. Setting $k = 1$ in Theorem 4.3, shows that this bound is attained for intertwining codes.

The code $C(A,B)$ is the row nullspace of $A^T \otimes I_s + I_r \otimes B$ and the column nullspace of $A \otimes I_s + I_r \otimes B^T$ where ^{T} denotes transpose.

References

[1] A. Alahmadi, S. Alamoudi, S. Karadeniz, B. Yildiz, C. Praeger and P. Solé, Centraliser codes, *Linear Algebra Appl.* **463** (2014), 68–77, doi:10.1016/j.laa.2014.08.024.

[2] A. Alahmadi, S. P. Glasby and C. E. Praeger, On the dimension of twisted centralizer codes, *Finite Fields Appl.* **48** (2017), 43–59, doi:10.1016/j.ffa.2017.07.005.

[3] A. Alahmadi, S. P. Glasby, C. E. Praeger, P. Solé and B. Yildiz, Twisted centralizer codes, *Linear Algebra Appl.* **524** (2017), 235–249, doi:10.1016/j.laa.2017.03.011.

[4] S. P. Glasby, Lemmas involving two partitions of integers, MathOverflow, 2017, `https://mathoverflow.net/q/258722`.

[5] D. W. Robinson, Classroom notes: the generalized jordan canonical form, *Amer. Math. Monthly* **77** (1970), 392–395, doi:10.2307/2316152.

[6] R. Stong, Some asymptotic results on finite vector spaces, *Adv. in Appl. Math.* **9** (1988), 167–199, doi:10.1016/0196-8858(88)90012-7.

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On the size of maximally non-hamiltonian digraphs*

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In memory of Nicolas Lichiardopol.

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Abstract

A graph is called *maximally non-hamiltonian* if it is non-hamiltonian, yet for any two non-adjacent vertices there exists a hamiltonian path between them. In this paper, we naturally extend the concept to directed graphs and bound their size from below and above. Our results on the lower bound constitute our main contribution, while the upper bound can be obtained using a result of Lewin, but we give here a different proof. We describe digraphs attaining the upper bound, but whether our lower bound can be improved remains open.

Keywords: Maximally non-hamiltonian digraphs.

Math. Subj. Class.: 05C45, 05C20

1 Introduction

Throughout this paper all graphs will be simple, finite, connected, and will not admit multiple edges or loops. In a *digraph*, each edge between two adjacent vertices u and v may be oriented from u to v, from v to u, or both ways. We call a digraph an *oriented graph* if no edge has both orientations.

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We were led to the study of the titular subject from a related concept: homogeneous traceability, a notion introduced by Skupień. A digraph is called *homogeneously traceable* if every vertex is the start-vertex of a hamiltonian path. If additionally every vertex is also the end-vertex of some hamiltonian path, the digraph is called *bihomogeneously traceable*. A graph or digraph D is called *hypohamiltonian* if D is non-hamiltonian, but for any vertex v in D, the graph $D - v$ is hamiltonian. Obviously, any digraph that is hamiltonian or hypohamiltonian is bihomogeneously traceable. But not every homogeneously traceable digraph is hamiltonian [4]. Not even bihomogeneous traceability implies hamiltonicity. At a meeting in Kalamazoo (in 1980) Skupień showed that for all $n \geq 7$ there exists a 2-diregular bihomogeneously traceable non-hamiltonian oriented digraph of order n, see [16], which appeared in 1981.

Moreover, Skupień [17] later constructed exponentially many bihomogeneously traceable non-hamiltonian oriented graphs. Independently, in another paper which also appeared in 1981, Hahn and T. Zamfirescu [12] also constructed an infinite sequence of bihomogeneously traceable non-hamiltonian oriented graphs, and gave three special examples: the first is arc-minimal (i.e. with the smallest possible number of arcs for a given number of vertices) and order 7, the second is planar and has 8 vertices (it is proven in [12] that there are no smaller examples), and the third is both arc-minimal and planar, and has 9 vertices. Note that arc-minimality amounts in this context to 2-diregularity.

Hahn and T. Zamfirescu asked in [12] the natural question whether infinitely many planar bihomogeneously traceable non-hamiltonian oriented graphs exist, as very few were known. Infinite families of planar hypohamiltonian digraphs containing opposite arcs have been found by Fouquet and Jolivet [10]. In [20], Thomassen proved that a planar hypohamiltonian digraph with n vertices (and many edges with both orientations—in fact, all but six) exists for each $n \geq 6$. It was shown by the second author [21] that, indeed, there exist infinitely many planar bihomogeneously traceable non-hamiltonian oriented graphs. A stronger result was recently obtained by van Aardt, Burger, and Frick [1], who showed that there exist infinitely many planar hypohamiltonian oriented graphs, thereby solving a problem of Thomassen [20].

If one now asks for an even larger set of spanning paths, one may be led to the case demanding that between any two non-adjacent vertices there exists a hamiltonian path. Such graphs have been studied in the non-oriented case, see for instance [7] and [2], and are called *maximally non-hamiltonian* (which, in the following, will often be abbreviated to MNH). For a digraph D, we write $V(D)$ and $A(D)$ for its set of vertices and arcs, respectively. Mirroring the non-directed definition, a digraph D is *maximally non-hamiltonian* if D is non-hamiltonian, but for every $x, y \in V(D)$ with $yx \notin A(D)$ there is a hamiltonian path from x to y.

A few words concerning the notation. For a set X, we denote with $|X|$ the cardinality of X. In a graph G, for adjacent vertices x and y in G we denote by xy the edge between x and y. If G is a digraph, xy will be the arc from x to y. A digraph D is called *strong* if for any pair $x, y \in V(D)$ there exists a (directed) path from x to y. Denote with $\delta^+(D)$ ($\delta^-(D)$) the minimum out-degree (minimum in-degree) and with $\delta^0(D)$ the *minimum semi-degree* of D, which is the minimum of $\delta^+(D)$ and $\delta^-(D)$. $N^+(x)$ ($N^-(x)$) shall be the set of out-neighbours (in-neighbours) of a vertex x. For a set of vertices S of D we denote the digraph induced by S in D as $D[S]$. Further definitions follow when needed.

2 Results

An important direction of research on non-directed MNH graphs has been determining the smallest size of an MNH graph of order n, which we shall denote by $f(n)$. This study was initiated by Bondy [6], who showed that for $n \geq 7$ we have $f(n) \geq \lceil \frac{3n}{2} \rceil$. Bollobás [5] conjectured that there exist infinitely many graphs for which this lower bound is in fact attained. This was proven to be correct for all even $n \geq 36$ by Clark and Entringer [7]. In the same paper graphs of order n and size $\lceil \frac{3n}{2} \rceil + 1$ were constructed for all odd $n \geq 55$. Horák and Širáň [13] also found such almost extremal graphs by using a construction of Thomassen [19] (which was originally intended to construct hypohamiltonian graphs). They also proved that for every $n \geq 48$ there exists a triangle-free MNH graph of order n.

Clark and Entringer asked in [7] whether for infinitely many n the MNH graph on n vertices of smallest size is unique or not. Combining the results from [7, 8, 9, 15], one obtains that for infinitely many n there exist two non-isomorphic MNH graphs of order n and smallest size. Still, there were infinitely many orders for which only one MNH graph of smallest size was known. This was investigated by Stacho [18]; he proved that for any $n \geq 88$ there exist three pairwise non-isomorphic MNH graphs of order n and smallest size.

In the following we extend the study of the size of MNH graphs to directed graphs. First, we characterise the non-strong MNH digraphs. A digraph is *symmetric* if for every arc in D the corresponding inverted arc also lies in D. For integers $m \geq 1$ and $p \geq 1$, let $D_{m,p}$ denote the digraph whose vertices are the vertices of two vertex-disjoint complete symmetric digraphs K_m^* and K_p^* and whose arcs are those of the digraphs K_m^*, K_p^*, and additionally the arcs xy with x in K_m^* and y in K_p^*. We claim:

Lemma 2.1. *The non-strong MNH digraphs are the digraphs $D_{m,p}$.*

Proof. Clearly, a digraph $D_{m,p}$ is a non-strong MNH digraph. Conversely, let D be a non-strong MNH digraph. There exists a partition V_1, V_2 of $V(D)$ such that there are no arcs from V_2 to V_1. Let x and y be two distinct vertices of D. It is easy to see that if x, y are both in V_1 or in V_2 there is no hamiltonian path from x to y. Furthermore, if x is in V_2 and y is in V_1, there is also no hamiltonian path from x to y. Then, since D is MNH, it follows that $D[V_1]$ and $D[V_2]$ are complete symmetric digraphs and that every ordered pair xy with $x \in V_1$ and $y \in V_2$ is an arc of D. This means that D is a digraph $D_{m,p}$, and so we are done. □

The upper bound contained in the following theorem can be obtained by using a result of Lewin [14], but we choose to give here a different proof. (In fact, our proof of the upper bound is a new proof of Lewin's [14, Corollary 1].) In upcoming arguments, we require the following.

Theorem 2.2 (Ghouila-Houri [11])**.** *A strong digraph D with $\delta^0(D) \geq |V(D)|/2$ is hamiltonian.*

Theorem 2.3. *For an MNH digraph D of order $n \geq 4$ we have $|A(D)| \leq (n-1)^2$ and this upper bound is attained.*

Proof. There exists a vertex $x \in V(D)$ with $d^+(x) + d^-(x) < n$ (for otherwise, by Theorem 2.2, D would be hamiltonian). Let us put

$$B = N^+(x) \cap N^-(x), \quad A = N^+(x) \setminus B, \quad \text{and} \quad C = N^-(x) \setminus B.$$

Furthermore, let a, b, and c be the respective cardinalities of A, B, and C. Clearly, for a vertex y in A or in C we have $d^+(y) + d^-(y) \leq 2n - 3$. For a vertex y in B, we have $d^+(y)+d^-(y) \leq 2n-2$, and for a vertex y not adjacent with x, we have $d^+(y)+d^-(y) \leq 2n - 4$. By addition, we get

$$2 \times |A(D)| \leq \\ n - 1 + a(2n - 3) + b(2n - 2) + c(2n - 3) + (n - 1 - a - b - c)(2n - 4),$$

hence

$$2 \times |A(D)| \leq \\ n - 1 + (n - 1)(2n - 4) + a + c + 2b = (n - 1)(2n - 3) + a + c + 2b.$$

But $d^+(x) + d^-(x) < n$ means that we have $a + c + 2b \leq n - 1$. It follows that

$$2 \times |A(D)| \leq (n - 1)(2n - 3) + n - 1 \leq 2(n - 1)^2,$$

and the upper bound is proved; it is attained since $D_{1,n-1}$ and $D_{n-1,1}$ are MNH digraphs of size $(n - 1)^2$. □

For integers $r \geq 1$ and n with $n \geq 2r + 1$ define the digraph $H_{n,r}$ of order n as follows. The vertices of $H_{n,r}$ are those of a complete symmetric digraph K^*_{n-r-1} and $r + 1$ additional vertices $y_1, \dots, y_{r+1}$. Let $x_1, \dots, x_r$ be r vertices of K^*_{n-r-1}. The arcs of $H_{n,r}$ are the arcs of K^*_{n-r-1}, $y_i x$ where $1 \leq i \leq r + 1$, $x \in V(K^*_{n-r-1})$, and $x_i y_j$ where $1 \leq i \leq r$ and $1 \leq j \leq r + 1$.

It is easy to see that a digraph $H_{n,r}$, like its converse, is MNH, of minimum semi-degree r and of strong connectivity r.

By Theorem 2.3, the maximum size of a non-strong MNH digraph is at most $(n - 1)^2$ and this bound is reached. It was proved in [3] that a digraph D with minimum semi-degree r and with more than

$$a_{n,r} = n^2 - (r + 2)n + (r + 1)^2$$

arcs is hamiltonian. When $r > (n - 1)/2$, by Theorem 2.2, D is hamiltonian and therefore cannot be MNH. But when $r \leq (n - 1)/2$, the result of [3] shows that the maximum size of an MNH digraph D of order n with $\delta^0(D) = r$ is at most $a_{n,r}$, and since $H_{n,r}$ is MNH, of minimum semi-degree r and of size $a_{n,r}$, this upper bound is reached.

If $1 \leq r \leq s \leq (n - 1)/2$, then $a_{n,r} \geq a_{n,s}$, so the maximum size of an MNH digraph D of order n and of strong connectivity $\kappa(D) = r$ is at most $a_{n,r}$. As $\kappa(H_{n,r}) = r$, the bound is sharp.

We now establish a lower bound on the size of an MNH digraph.

Lemma 2.4. *For an MNH digraph D on at least four vertices, either $\delta^0(D) \geq 2$ or $|A(D)| \geq 3n - 4$.*

Proof. We proceed by induction on n. It is easy to verify that the assertion is true for $n = 4$. Now let $n \geq 5$ and suppose the assertion is true for all $k \leq n - 1$. Consider a digraph D of order n having the required property.

Let us put $V(D) = \{x_1, \dots, x_n\}$. Let there exist a vertex of D which is of out-degree at most 1. W.l.o.g., we may assume that this vertex is x_1. Suppose first that $d^+(x_1) = 0$.

In this case D is a non-strong MNH digraph, and by Lemma 2.1, D is in fact $D_{n-1,1}$. We then have $|A(D)| = (n-1)^2 \geq 3n-4$, and so we are done.

Suppose now that $d^+(x_1) = 1$. W.l.o.g., we may assume that the unique out-neighbour of x_1 is x_2. We claim that $x_2x_1 \in A(D)$. Suppose the opposite. Then there exists a hamiltonian path P from x_1 to x_2. But then x_1 has at least two out-neighbours, a contradiction. We also claim that all of the vertices of $V(D) \setminus \{x_1, x_2\}$ are in-neighbours of x_2. Suppose the opposite. Then there exists a vertex x_i, $i \geq 3$, which is not an in-neighbour of x_2. Thus, there exists a hamiltonian path from x_2 to x_i, whence, x_1 has an out-neighbour in this hamiltonian path distinct from x_2, a contradiction.

We claim that $D' = D - x_1$ is MNH, i.e. that for any two vertices y, z of D' such that $zy \notin A(D)$, there exists a hamiltonian path in D' from y to z. Observe first that $y \neq x_2$. There exists in D a hamiltonian path P from y to z. P contains the arc x_1x_2, and since x_1 is not the first vertex of P it admits an in-neighbour u in P. Then $P' = P - x_1 + ux_2$ (i.e. the path P from which we delete the vertex x_1 and add the arc ux_2) is a hamiltonian path in D' from y to z. So, the claim is proved.

By induction hypothesis, either every vertex of D' is of in-degree at least 2 in D', or $|A(D')| \geq 3(n-1) - 4 = 3n-7$. In the first case, since x_2 is of in-degree $n-2$ in D', we have $|A(D')| \geq n-2+2(n-2)$, hence $|A(D')| \geq 3n-6$. Since x_1x_2 and x_2x_1 are arcs of D but not of D', we get $|A(D)| \geq 3n-4$, and the theorem is proved in this case.

Suppose now that $|A(D')| \geq 3n-7$. Assume first that there are no distinct vertices x_i, x_j, where $3 \leq i, j \leq n$, such that x_jx_i is not an arc of D'. Then we have $|A(D')| \geq (n-2)^2 \geq 3n-6$. Thus $|A(D)| \geq 3n-4$, and again we are done. Suppose now that there exist distinct vertices x_i, x_j, where $3 \leq i, j \leq n$, such that $x_jx_i \notin A(D')$. Then there exists a hamiltonian path of D from x_i to x_j, and then necessarily x_1 has an in-neighbour x_k with $3 \leq k \leq n$. It follows that $|A(D)| \geq 3n-7+3 = 3n-4$, and we are done. □

As a corollary, we can state:

Theorem 2.5. *For an MNH digraph D of order $n \geq 4$ we have $|A(D)| \geq 2n$.*

Proof. If $|A(D)| \geq 3n-4$, we are done. If $|A(D)| < 3n-4$, by Lemma 2.4, each vertex of D is of out-degree at least 2, and we get $|A(D)| \geq 2n$. □

Observe that for $n = 3$, this lower bound inequality is untrue: consider the digraph $D = (\{x, y, z\}, \{xy, yx, xz, yz\})$. D is an MNH digraph of order 3 and size 4. Also note that for $n = 4$, the lower bound is tight due to the digraph $D_{2,2}$, which is not regular.

From [6] we know that for $n \geq 7$ the size of an MNH graph of order n is at least $\lceil \frac{3n}{2} \rceil$. For every $n \geq 19$, this lower bound is reached; consult [18] and [15] for details. Thus, for $n \geq 19$, there exists an MNH graph G of order n and size $\lceil \frac{3n}{2} \rceil$. It is easy to prove that the symmetric digraph obtained by doubly orienting each edge of G is an MNH digraph of order n and size $2 \times \lceil \frac{3n}{2} \rceil$. So the minimum size of an MNH digraph of order $n \geq 19$ is at least $2n$ and at most $2 \times \lceil \frac{3n}{2} \rceil$.

We now give a lower bound on the size of an MNH non-strong digraph:

Theorem 2.6. *Let D be an MNH non-strong digraph of order $n \geq 2$. Then*

$$|A(D)| \geq \left\lceil \frac{3}{4}n^2 \right\rceil - n.$$

Proof. We know that D is of the form $D_{\alpha n,(1-\alpha)n}$, where $0 < \alpha < 1$ and αn is an integer. Then we have

$$\begin{aligned}|A(D)| &= \alpha n(\alpha n - 1) + (1-\alpha)n((1-\alpha)n - 1) + \alpha n(1-\alpha)n \\ &= n^2(\alpha^2 - \alpha + 1) - n \geq \tfrac{3}{4}n^2 - n,\end{aligned}$$

since $\alpha^2 - \alpha + 1 \geq \frac{3}{4}$. □

If the strong connectivity is 1, we have the following result.

Theorem 2.7. *Let D be an MNH digraph of order $n \geq 4$ and of strong connectivity 1. Then*

$$|A(D)| \geq \min\left\{3n-4, \left\lceil \frac{5}{2}n \right\rceil - 1\right\}.$$

Proof. If $\delta^+(D) \leq 1$ or $\delta^-(D) \leq 1$, by Lemma 2.4, we have $|A(D)| \geq 3n-4$, and the result is proved.

Suppose now that $\delta^+(D) \geq 2$ and $\delta^-(D) \geq 2$, i.e. each vertex of D is of out-degree at least 2 and of in-degree at least 2. There is a vertex x such that the digraph $D - x$ is not strong, so there exists a partition of $V(D) \setminus \{x\}$ into two non-empty sets A and B such that there are no arcs from B to A. W.l.o.g., we suppose that $|B| \leq |A|$, so $|A| \geq 2$. We claim that all the vertices of B are out-neighbours of x. Let us suppose this not to be the case. Then there exists a vertex y of B such that $xy \notin A(D)$. Since D is MNH, there exists a hamiltonian path P from y to x. Necessarily, P contains an arc uv with $u \in B$ and $v \in A$, which is impossible. Similarly, all the vertices of A are in-neighbours of x. Since x has at least one out-neighbour in A and at least one in-neighbour in B, we get $d^+(x) + d^-(x) \geq n+1$.

Suppose now that x has exactly one out-neighbour y in A. It is easy to see that $A \setminus \{y\}$, $B \cup \{x\}$ is a partition of $V(D) \setminus \{y\}$ into non-empty sets such that there are no arcs from $B \cup \{x\}$ to $A \setminus \{y\}$. So, $D - y$ is not strong, and from the preceding arguments, we have $d^+(y) + d^-(y) \geq n+1$. Since $d^+(z) + d^-(z) \geq 4$ for z distinct from x and y, by addition we get $2 \times |A(D)| \geq 2(n+1) + 4(n-2)$, hence $2 \times |A(D)| \geq 6n-6$, and $|A(D)| \geq 3n-3 > 3n-4$, and the result is proved.

Suppose now that x has at least two out-neighbours in A. We then get $d^+(x)+d^-(x) \geq n+2$, and for $z \neq x$ we have $d^+(z) + d^-(z) \geq 4$. By addition this yields $2 \times |A(D)| \geq n+2+4(n-1)$, hence $|A(D)| \geq \lceil \frac{5}{2}n \rceil - 1$, as stated. □

With above conclusions in mind, an MNH digraph D with

$$|A(D)| < \min\left\{3n-4, \left\lceil \frac{5}{2}n \right\rceil - 1, \left\lceil \frac{3}{4}n^2 \right\rceil - n\right\}$$

is necessarily of strong connectivity at least 2, and thus, of semi-degree at least 2. Now it is easy to see that for $n \geq 5$, if the lower bound $2n$ is attained by an MNH digraph D of order n, then necessarily D is a 2-diregular digraph of order n and of strong connectivity 2. We were not able to show the existence of such digraphs, but an advance in this direction is perhaps the following:

Theorem 2.8. *Let D be a 2-diregular MNH digraph of order n. Then D is hypohamiltonian.*

Proof. We know that D is non-hamiltonian. Suppose for the sake of a contradiction that there exists a vertex z such that $D - z$ is non-hamiltonian. Let x be an in-neighbour of z, and let y be the other out-neighbour of x. We claim that $zy \in A(D)$. Suppose the opposite. Then there exists a hamiltonian path from y to z, and it is easy to see that the in-neighbour of z in P is x. But then $P - z + xy$ is a hamiltonian cycle of $D - z$, a contradiction.

Assume that y is an in-neighbour of z. Then y and z are of in-degree 2 in the induced sub-digraph $D[\{x, y, z\}]$. Since D is 2-strong and $n \geq 5$, this is not possible. Therefore y is not an in-neighbour of z. Suppose that x is an out-neighbour of z. Then x and z are of out-degree 2 in the induced sub-digraph $D[\{x, y, z\}]$. Since D is 2-strong and $n \geq 5$, this is not possible.

So x is not an out-neighbour of z. Then z has an in-neighbour u distinct from x, y, and z. The vertex y is not an out-neighbour of u (for otherwise y would be of in-degree at least 3, which is impossible), and x is not an out-neighbour of u (for otherwise, with a previous argument, zx would be an arc of D, which is false). Thus u has an out-neighbour v distinct from x, y, and z. Then, by previous arguments, zv is an arc of D, and v is not an in-neighbour of z. The last statement implies that there exists a hamiltonian path P' from z to v. The second vertex of P' is necessarily y, and then it is easy to see that x has an out-neighbour w distinct from y and z. Since x is of out-degree 2, this is impossible. Thus $D - z$ cannot be non-hamiltonian, and so the theorem is proved. □

The above discussions concerning the size of MNH digraphs led us to the following question.

Problem. *Does every non-hamiltonian oriented graph of order at least* 3 *contain an arc* xy *for which there is no hamiltonian path from* x *to* y*?*

Lastly, we would like to point out the connection between maximally non-hamiltonian graphs and so-called *platypus*, which are non-hamiltonian graphs in which every vertex-deleted subgraph is traceable. They contain the family of all hypohamiltonian and hypotraceable graphs. The second author showed [22] that a maximally non-hamiltonian graph G is a platypus if and only if $\Delta(G) < |V(G)| - 1$, where $\Delta(G)$ denotes the maximum degree of G. Directed platypus have not yet been investigated, but the above results may provide a good starting point.

References

[1] S. A. van Aardt, A. P. Burger and M. Frick, An Infinite Family of Planar Hypohamiltonian Oriented Graphs, *Graphs Combin.* **29** (2013), 729–733, doi:10.1007/s00373-012-1165-z.

[2] A. Ainouche and N. Christofides, Conditions for the existence of hamiltonian circuits in graphs based on vertex degrees, *J. London Math. Soc.* **32** (1985), 385–391, doi:10.1112/jlms/s2-32.3.385.

[3] D. Amar, I. Fournier and A. Germa, Some conditions for digraphs to be Hamiltonian, in: M. Rosenfeld and J. Zaks (eds.), *Convexity and Graph Theory*, North-Holland, Amsterdam, volume 20 of *Annals of Discrete Mathematics*, 1984 pp. 37–41, doi:10.1016/s0304-0208(08)72804-4, proceedings of the conference held in Jerusalem, March 16 – 20, 1981.

[4] J.-C. Bermond, J. M. S. Simões-Pereira and C. M. Zamfirescu, On nonhamiltonian homogeneously traceable digraphs, *Math. Japonica* **24** (1979), 423–426.

[5] B. Bollobás, *Extremal Graph Theory*, volume 11 of *London Mathematical Society Monographs*, Academic Press, London, 1978.

[6] J. A. Bondy, Variations on the Hamiltonian theme, *Canad. Math. Bull.* **15** (1972), 57–62, doi:10.4153/cmb-1972-012-3.

[7] L. Clark and R. Entringer, Smallest maximally nonhamiltonian graphs, *Period. Math. Hungar.* **14** (1983), 57–68, doi:10.1007/bf02023582.

[8] L. H. Clark, R. P. Crane, R. C. Entringer and H. D. Shapiro, On smallest maximally nonhamiltonian graphs, in: F. Hoffman, R. C. Mullin, R. G. Stanton and K. B. Reid (eds.), *Proceedings of the Seventeenth Southeastern International Conference on Combinatorics, Graph Theory, and Computing*, Utilitas Mathematica Publishing, Winnipeg, Manitoba, volume 53 of *Congressus Numerantium*, 1986 pp. 215–220, held at Florida Atlantic University, Boca Raton, Florida, February 10 – 14, 1986.

[9] L. H. Clark, R. C. Entringer and H. D. Shapiro, Smallest maximally nonhamiltonian graphs II, *Graphs Combin.* **8** (1992), 225–231, doi:10.1007/bf02349959.

[10] J.-L. Fouquet and J.-L. Jolivet, Hypohamiltonian oriented graphs, *Cahiers Centre Études Rech. Opér.* **20** (1978), 171–181.

[11] A. Ghouila-Houri, Une condition suffisante d'existence d'un circuit hamiltonien, *C. R. Acad. Sci. Paris* **251** (1960), 495–497.

[12] S. Hahn and T. Zamfirescu, Bihomogeneously traceable oriented graphs, *Rend. Sem. Mat. Univ. Politec. Torino* **39** (1981), 137–145.

[13] P. Horák and J. Širáň, On a construction of Thomassen, *Graphs Combin.* **2** (1986), 347–350, doi:10.1007/bf01788108.

[14] M. Lewin, On maximal circuits in directed graphs, *J. Comb. Theory Ser. B* **18** (1975), 175–179, doi:10.1016/0095-8956(75)90045-3.

[15] X. Lin, W. Jiang, C. Zhang and Y. Yang, On Smallest Maximally Non-Hamiltonian Graphs, *Ars Combin.* **45** (1997), 263–270.

[16] Z. Skupień, On homogeneously traceable non-Hamiltonian digraphs and oriented graphs, in: G. Chartrand, Y. Alavi, D. L. Goldsmith, L. Lesniak-Foster and D. R. Lick (eds.), *The Theory and Applications of Graphs*, John Wiley & Sons, New York, 1981 pp. 517–527, proceedings of the Fourth International Conference held at Western Michigan University, Kalamazoo, Michigan, May 6 – 9, 1980.

[17] Z. Skupień, Exponential constructions of some non-Hamiltonian minima, in: J. Nešetřil and M. Fiedler (eds.), *Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity*, North-Holland, Amsterdam, volume 51 of *Annals of Discrete Mathematics*, 1992 pp. 321–328, doi:10.1016/s0167-5060(08)70649-6, proceedings of the symposium held in Prachatice, 1990.

[18] L. Stacho, Non-Isomorphic Smallest Maximally Non-Hamiltonian Graphs, *Ars Combin.* **48** (1998), 307–317.

[19] C. Thomassen, Hypohamiltonian and hypotraceable graphs, *Discrete Math.* **9** (1974), 91–96, doi:10.1016/0012-365x(74)90074-0.

[20] C. Thomassen, Hypohamiltonian graphs and digraphs, in: Y. Alavi and D. R. Lick (eds.), *Theory and Applications of Graphs*, Springer-Verlag, Berlin, volume 642 of *Lecture Notes in Mathematics*, 1978 pp. 557–571, proceedings of the International Conference held at Western Michigan University, Kalamazoo, Michigan, May 11 – 15, 1976.

[21] C. T. Zamfirescu, An Infinite Family of Planar Non-Hamiltonian Bihomogeneously Traceable Oriented Graphs, *Graphs Combin.* **26** (2010), 141–146, doi:10.1007/s00373-010-0900-6.

[22] C. T. Zamfirescu, On Non-Hamiltonian Graphs for which every Vertex-Deleted Subgraph Is Traceable, *J. Graph Theory* **86** (2017), 223–243, doi:10.1002/jgt.22122.

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On chromatic indices of finite affine spaces*

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Abstract

A line-coloring of the finite affine space $\mathrm{AG}(n,q)$ is *proper* if any two lines from the same color class have no point in common, and it is *complete* if for any two different colors i and j there exist two intersecting lines, one is colored by i and the other is colored by j. The pseudoachromatic index of $\mathrm{AG}(n,q)$, denoted by $\psi'(\mathrm{AG}(n,q))$, is the maximum number of colors in any complete line-coloring of $\mathrm{AG}(n,q)$. When the coloring is also proper, the maximum number of colors is called the achromatic index of $\mathrm{AG}(n,q)$. We prove that $\psi'(\mathrm{AG}(n,q)) \sim q^{1.5n-1}$ for even n, and that $q^{1.5(n-1)} < \psi'(\mathrm{AG}(n,q)) < q^{1.5n-1}$ for odd n. Moreover, we prove that the achromatic index of $\mathrm{AG}(n,q)$ is $q^{1.5n-1}$ for even n, and we provide the exact values of both indices in the planar case.

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1 Introduction

This paper is motivated by the well-known combinatorial conjecture about colorings of finite linear spaces stated by Erdős, Faber and Lovász in 1972. As a starting point, we briefly recall some definitions and state the conjecture. Let $\mathbf{S}$ be a finite linear space. A *line-coloring* of $\mathbf{S}$ with k colors is a surjective function ς from the lines of $\mathbf{S}$ to the set of colors $[k] = \{1, \ldots, k\}$. For short, a line-coloring with k colors is called k*-coloring*. If $\varsigma \colon \mathbf{S} \to [k]$ is a k-coloring and $i \in [k]$ then the subset of lines $\varsigma^{-1}(i)$ is called the i-th *color class* of ς. A k-coloring of $\mathbf{S}$ is *proper* if any two lines from the same color class have no point in common. The *chromatic index* $\chi'(\mathbf{S})$ of $\mathbf{S}$ is the smallest k for which there exists a proper k-coloring of $\mathbf{S}$. The *Erdős-Faber-Lovász conjecture* (1972) states that if a finite linear space $\mathbf{S}$ contains v points then $\chi'(\mathbf{S}) \leq v$, see [12, 13].

Several papers have investigated the conjecture for particular classes of linear spaces. For instance, if each line of $\mathbf{S}$ has the same number κ of points then $\mathbf{S}$ is called a *block design* or a (v, κ)*-design*. The conjecture is still open for designs even for $\kappa = 3$, however, it was proved for finite projective spaces by Beutelspacher, Jungnickel and Vanstone [8]. It is not hard to see that the conjecture is also true for the n-dimensional affine space $\mathrm{AG}(n, q)$ of order q defined over the Galois field $\mathrm{GF}(q)$. Indeed,

$$\chi'(\mathrm{AG}(n, q)) = \frac{q^n - 1}{q - 1}.$$

For some related results, see for instance [6, 7].

A natural question is to determine similar, but slightly different color parameters in finite linear spaces. A k-coloring of $\mathbf{S}$ is *complete* if for each pair of different colors i and j there exist two intersecting lines of $\mathbf{S}$, such that one of them belongs to the i-th and the other one to the j-th color class. Observe that any proper coloring of $\mathbf{S}$ with $\chi'(\mathbf{S})$ colors is a complete coloring. The *pseudoachromatic index* $\psi'(\mathbf{S})$ of $\mathbf{S}$ is the largest k such that there exists a complete k-coloring (not necessarily proper) of $\mathbf{S}$. When the k-coloring is required to be complete and proper, the parameter is called the *achromatic index* and it is denoted by $\alpha'(\mathbf{S})$. Therefore, we have that

$$\chi'(\mathbf{S}) \leq \alpha'(\mathbf{S}) \leq \psi'(\mathbf{S}).$$

Several authors studied the pseudoachromatic index, see [2, 3, 4, 5, 9, 14, 15, 17]. Moreover, in [1, 10, 18] the achromatic indices of some block designs were also estimated.

In this paper we study the pseudoachromatic and achromatic indices of finite affine spaces. In the proofs we will often use the notion of the projective closure of $\mathrm{AG}(n, q)$. This is the finite projective space $\mathrm{PG}(n, q) = \mathrm{AG}(n, q) \cup \mathcal{H}_\infty$, where the points of $\mathcal{H}_\infty$ correspond to the parallel classes of lines in $\mathrm{AG}(n, q)$. The space $\mathcal{H}_\infty$ is isomorphic to $\mathrm{PG}(n - 1, q)$, and it is called the *hyperplane at infinity*. We assume that the reader is

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familiar with the most important properties of affine and projective geometries. For the detailed description of these spaces we refer to [16].

The main results in the paper are proved in Sections 2 and 3. They are stated in Theorems 1.1, 1.2 and 1.3. In these theorems $v = q^n$ always denotes the number of points of the finite affine space $\mathrm{AG}(n, q)$.

Theorem 1.1. *For all n:*

$$\psi'(\mathrm{AG}(n,q)) \leq \frac{\sqrt{v}(v-1)}{q-1} - \Theta(q\sqrt{v}/2).$$

Theorem 1.2. *If n is even:*

$$\frac{1}{2} \cdot \frac{\sqrt{v}(v-1)}{q-1} - \Theta(\sqrt{v}/2) \leq \psi'(\mathrm{AG}(n,q)).$$

If n is odd:

$$\frac{1}{\sqrt{q}} \cdot \frac{\sqrt{v}(v-1)}{q-1} - \Theta(v\sqrt{v/q^5}) \leq \psi'(\mathrm{AG}(n,q)).$$

Theorem 1.3. *If n is even:*

$$\frac{1}{3} \cdot \frac{\sqrt{v}(v-1)}{q-1} + \Theta(v/q) \leq \alpha'(\mathrm{AG}(n,q)).$$

Note that when n is even Theorems 1.1 and 1.2 show that $\psi'(\mathrm{AG}(n,q))$ grows asymptotically as $\Theta(v^{1.5}/q)$, while Theorems 1.2 and 1.3 show that $\alpha'(\mathrm{AG}(n,q))$ grows asymptotically as $\Theta(v^{1.5}/q)$. Let us remark that no similar estimates regarding the asymptotic behavior of these indices have appeared so far in the literature.

Finally, in Section 4 we determine the exact values of pseudoachromatic and achromatic indices of arbitrary (not necessarily Desarguesian) finite affine planes and we improve the previous lower bounds in dimension 3.

2 Upper bounds

In this section, upper bounds for the pseudoachromatic index of $\mathrm{AG}(n,q)$ are presented when $n > 2$. The following lemma is pivotal in the proof.

Lemma 2.1. *Let $\mathcal{L}$ be a set of s lines in $\mathrm{AG}(n,q)$, $n > 2$. Then the number of lines of $\mathrm{AG}(n,q)$ intersecting at least one element of $\mathcal{L}$ is at most*

$$q^2 \left(s\frac{q^{n-1}-1}{q-1} - (s-1) \right).$$

Proof. In $\mathrm{AG}(n,q)$ there are $q\left(\frac{q^n-1}{q-1} - 1\right) = q^2\left(\frac{q^{n-1}-1}{q-1}\right)$ lines intersecting any fixed line. The number of lines intersecting two lines, say ℓ_1 and ℓ_2, is at least q^2, because if $\ell_1 \cap \ell_2 = \emptyset$ then the q^2 lines joining a point of ℓ_1 and a point of ℓ_2 intersect both ℓ_1 and ℓ_2, while, if $\ell_1 \cap \ell_2 = \{P\}$ then the other $\frac{q^n-1}{q-1} - 2 > q^2$ lines through P intersect both ℓ_1 and ℓ_2. Consequently, the number of lines intersecting at least one element of $\mathcal{L}$ is at most

$$sq^2\left(\frac{q^{n-1}-1}{q-1}\right) - (s-1)q^2.$$

Notice that the previous inequality is tight, since if $\mathcal{L}$ consists of s parallel lines in a plane then there are exactly $q^2\left(s\frac{q^{n-1}-1}{q-1}-(s-1)\right)$ lines intersecting at least one element of $\mathcal{L}$. □

Lemma 2.2. *Let $n > 2$ be an integer. Then the colorings of the finite affine space* $\mathrm{AG}(n, q)$ *satisfy the inequality*

$$\psi'(\mathrm{AG}(n,q)) \leq \frac{\sqrt{4q^n(q^n-1)(q^n-q^2)+(q^2+1)^2(q-1)^2}}{2(q-1)} + \frac{q^2+1}{2}. \qquad (2.1)$$

Proof. Consider a complete coloring which contains $\psi'(\mathrm{AG}(n,q))$ color classes. Then the number of lines in the smallest color class is at most

$$s = \frac{q^{n-1}(q^n-1)}{(q-1)\psi'(\mathrm{AG}(n,q))}.$$

Each of the other $\psi'(\mathrm{AG}(n,q)) - 1$ color classes must contain at least one line which intersects a line from the smallest color class. Hence, by Lemma 2.1, we obtain

$$\psi'(\mathrm{AG}(n,q)) - 1 \leq q^2\left(s\frac{q^{n-1}-1}{q-1}-(s-1)\right).$$

Multiplying it by $\psi'(\mathrm{AG}(n,q))$, we get a quadratic inequality on $\psi'(\mathrm{AG}(n,q))$, whence the assertion follows. □

We are in a position to prove our first main theorem.

Proof of Theorem 1.1. For $n > 2$ a straightforward computation shows

$$\begin{aligned}
&4q^n(q^n-1)(q^n-q^2)+(q^2+1)^2(q-1)^2 \\
&\quad = \left(2q^{\frac{n}{2}}(q^n-1)-q^{\frac{n}{2}}(q^2-1)\right)^2 - q^n(q^2-1)^2+(q^2+1)^2(q-1)^2 \\
&\quad\quad < \left(2q^{\frac{n}{2}}(q^n-1)-q^{\frac{n}{2}}(q^2-1)\right)^2,
\end{aligned}$$

because $n > 2$ implies that $q^n(q^2-1)^2 > (q^2+1)^2(q-1)^2$. This together with Inequality (2.1) give

$$\psi'(\mathrm{AG}(n,q)) \leq q^{\frac{n}{2}}\left(\frac{q^n-1}{q-1}\right) - q^{\frac{n}{2}}\left(\frac{q+1}{2}\right) + \frac{q^2+1}{2},$$

which proves the theorem for $n > 2$. For $n = 2$ the statement is clear. □

3 Lower bounds

In this section complete colorings of $\mathrm{AG}(n, q)$ are presented. These constructions give different bounds on $\psi'(\mathrm{AG}(n,q))$ depending on the parity of n. First, we prove some geometric properties of affine and projective spaces.

Proposition 3.1. *Let $n > 1$ be an integer, Π_1 and Π_2 be subspaces in $\mathrm{PG}(n,q) = \mathrm{AG}(n,q) \cup \mathcal{H}_\infty$. Let d_i denote the dimension of Π_i for $i = 1,2$. Suppose that $\Pi_1 \cap \Pi_2 \cap \mathcal{H}_\infty$ is an m-dimensional subspace and $d_1 + d_2 = n + 1 + m$. Then $\Pi_1 \cap \Pi_2 \cap \mathrm{AG}(n,q)$ is an $(m+1)$-dimensional subspace in $\mathrm{AG}(n,q)$.*

In particular, $\Pi_1 \cap \Pi_2$ is a single point in $\mathrm{AG}(n,q)$ when $\Pi_1 \cap \Pi_2 \cap \mathcal{H}_\infty = \emptyset$ and $d_1 + d_2 = n$.

Proof. Since $\Pi_1 \cap \Pi_2 \cap \mathcal{H}_\infty$ is an m-dimensional subspace, $\dim(\Pi_1 \cap \Pi_2) \le m+1$. On the other hand, the dimension formula yields

$$\dim(\Pi_1 \cap \Pi_2) = \dim \Pi_1 + \dim \Pi_v - \dim\langle \Pi_1, \Pi_2 \rangle \ge d_1 + d_2 - n = m + 1.$$

Thus $\Pi_1 \cap \Pi_2$ is an $(m+1)$-dimensional subspace in $\mathrm{PG}(n,q)$, therefore $\Pi_1 \cap \Pi_2 \cap \mathrm{AG}(n,q)$ is an $(m+1)$-dimensional subspace in $\mathrm{AG}(n,q)$ if $m \ge 0$.

If $m = -1$, then $\Pi_1 \cap \Pi_2 \cap \mathcal{H}_\infty = \emptyset$ and $\dim(\Pi_1 \cap \Pi_2) = 0$. Hence $\Pi_1 \cap \Pi_2$ is a single point in $\mathrm{AG}(n,q)$. □

In the following proposition we present a partition of the points of $\mathrm{PG}(2k,q)$ that we will call a *good partition* in the rest of the paper.

Proposition 3.2. *Let $k \ge 1$ be an integer and $Q \in \mathrm{PG}(2k,q)$ be an arbitrary point. The points of $\mathrm{PG}(2k,q) \setminus \{Q\}$ can be divided into two subsets, say $\mathcal{A}$ and $\mathcal{B}$, and one can assign a subspace $S(P)$ to each point $P \in \mathcal{A} \cup \mathcal{B}$, such that the following holds true:*

- *$P \in S(P)$ for all points;*
- *$|\mathcal{A}| = q^2\left(\frac{q^{2k}-1}{q^2-1}\right)$ and, if $A \in \mathcal{A}$ then $S(A)$ is a k-dimensional subspace;*
- *$|\mathcal{B}| = q\left(\frac{q^{2k}-1}{q^2-1}\right)$ and, if $B \in \mathcal{B}$ then $S(B)$ is a $(k-1)$-dimensional subspace;*
- *$S(A) \cap S(B) = \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

Proof. We prove the assertion by induction on k. If $k = 1$ then let $\{\ell_0, \ell_1, \dots, \ell_q\}$ be the set of lines through Q. Let $\mathcal{A}$ and $\mathcal{B}$ consist of points $\mathrm{PG}(2,q) \setminus \{\ell_0\}$ and $\ell_0 \setminus \{Q\}$, respectively. If $A \in \mathcal{A}$ then let $S(A)$ be the line AQ, if $B \in \mathcal{B}$ then let $S(B)$ be the point B. These sets clearly fulfill the prescribed conditions, so $\mathrm{PG}(2,q)$ admits a good partition.

Now, let us suppose that $\mathrm{PG}(2k,q)$ admits a good partition. In $\mathrm{PG}(2k+2,q)$ take a $2k$-dimensional subspace Π which contains the point Q. Then Π is isomorphic to $\mathrm{PG}(2k,q)$, hence it has a good partition $\{Q\} \cup \mathcal{A}' \cup \mathcal{B}'$ with assigned subspaces $S'(P)$. Let $H_0, \dots, H_q$ be the pencil of hyperplanes in $\mathrm{PG}(2k+2,q)$ with carrier Π. Let $\mathcal{B} = \mathcal{B}' \cup (H_0 \setminus \Pi)$ and $\mathcal{A} = \mathrm{PG}(2k+2,q) \setminus (\mathcal{B} \cup \{Q\})$. Notice that $\mathcal{A}'$ and $\mathcal{B}'$ have the required cardinalities, because

$$\begin{aligned}|\mathcal{A}'| &= \frac{q^{2k+3}-1}{q-1} - (|\mathcal{B}|+1) = (q+1)\frac{q^{2k+3}-1}{q^2-1} - q\left(\frac{q^{2k+2}-1}{q^2-1}\right) - 1 \\ &= q^2\left(\frac{q^{2k+2}-1}{q^2-1}\right), \\ |\mathcal{B}'| &= |\mathcal{B}| + |H_0 \setminus \Pi| = q\left(\frac{q^{2k}-1}{q^2-1}\right) + q^{2k+1} = q\left(\frac{q^{2k+2}-1}{q^2-1}\right).\end{aligned}$$

We assign the subspaces in the following way. If $A \in \mathcal{A}'$ then let $S(A)$ be the $(k+1)$-dimensional subspace $\langle S'(A), P\rangle$ where $P \in \cup_{i=1}^{q} H_i$ is an arbitrary point, whereas, if $A \in (\cup_{i=1}^{q} H_i) \setminus \Pi$ then let $S(A)$ be the $(k+1)$-dimensional subspace $\langle A, S'(P)\rangle$ where $P \in \mathcal{A}'$ is an arbitrary point. In both cases $S(A) \subset \cup_{i=1}^{q} H_i$ for all $A \in \mathcal{A}$. Similarly, if $B \in \mathcal{B}'$ then let $S(B)$ be the k-dimensional subspace $\langle S'(B), P\rangle$ where $P \in H_0$ is an arbitrary point, whereas, if $B \in H_0 \setminus \Pi$ then let $S(B)$ be the k-dimensional subspace $\langle B, S'(P)\rangle$ where $P \in \mathcal{B}'$ is an arbitrary point. Also here, in both cases, $S(B) \subset H_0$ for all $B \in \mathcal{B}$. Moreover, the assigned subspaces satisfy the intersection condition because if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are arbitrary points then

$$S(A) \cap S(B) = (S(A) \cap (\cup_{i=1}^{q} H_i)) \cap (S(B) \cap H_0) = S'(A) \cap S'(B) \cap \Pi = \emptyset.$$

Hence $\mathrm{PG}(2k+2, q)$ also admits a good partition, and the statement is proved. □

The next theorem proves Theorem 1.2 for even dimensional finite affine spaces. Notice that the lower bound depends on the parity of q, but its magnitude is $\frac{\sqrt{v}(v-1)}{2(q-1)}$ in both cases, where $v = q^n$.

Theorem 3.3. *If $k > 1$ then the colorings of the even dimensional affine space, $\mathrm{AG}(2k, q)$, satisfy the inequalities*

$$\psi'(\mathrm{AG}(2k,q)) \geq \begin{cases} \frac{q^k(q^{2k}-1)}{2(q-1)}, & \text{if } q \text{ is odd,} \\ \frac{q^k(q^{2k}-q)}{2(q-1)} + 1, & \text{if } q \text{ is even.} \end{cases}$$

Proof. The hyperplane at infinity in the projective closure of $\mathrm{AG}(2k, q)$, $\mathcal{H}_\infty$, is isomorphic to $\mathrm{PG}(2k-1, q)$, hence it has a $(k-1)$-spread $\mathcal{S} = \{S^1, S^2, \ldots, S^{q^k+1}\}$. The elements of $\mathcal{S}$ are pairwise disjoint $(k-1)$-dimensional subspaces (see [16, Theorem 4.1]). Let $\{P_1^i, P_2^i, \ldots, P_{(q^k-1)/(q-1)}^i\}$ be the set of points of S^i for $i = 1, 2, \ldots, q^k+1$. For a point $P \in \mathcal{H}_\infty$ let $S(P)$ denote the unique element of $\mathcal{S}$ that contains P, and $A(P) = \{\Pi_{P,1}, \Pi_{P,2}, \ldots, \Pi_{P,q^k}\}$ denote the set of the q^k parallel k-dimensional subspaces of $\mathrm{AG}(2k, q)$ whose projective closures intersect $\mathcal{H}_\infty$ in $S(P)$.

We define a pairing on the set of points of $\mathcal{H}_\infty$ which depends on the parity of q. On the one hand, if q is odd then let (P_j^i, P_j^{i+1}) be the pairs for $i = 1, 3, 5, \ldots, q^k$ and $j = 1, 2, \ldots, \frac{q^k-1}{q-1}$. On the other hand, if q is even then $\mathcal{H}_\infty$ has an odd number of points, thus we give the pairing on the set of points $\mathcal{H}_\infty \setminus \{P_1^1\}$: let (P_j^i, P_j^{i+1}) be the pairs for $i = 4, 6, \ldots, q^k$ and $j = 1, 2, \ldots, \frac{q^k-1}{q-1}$, and let (P_j^1, P_j^2), (P_{j+1}^2, P_{j+1}^3), (P_{j+1}^1, P_j^3) and (P_1^2, P_1^3) be the pairs for $i = 1, 2, 3$ and $j = 2, 4, 6, \ldots, \frac{q^k-1}{q-1} - 1$.

Let (U, V) be any pair of points. Then, by defintion, $S(U) \neq S(V)$. Let the color class $C_{U,V,i}$ contain the lines joining either U and a point from $\Pi_{U,i}$, or V and a point from $\Pi_{V,i}$, for $i = 1, 2, \ldots, q^k$. Clearly, (U, V) defines q^k color classes, each one consists of the parallel lines of one subspace in $A(U)$ and the parallel lines of one subspace in $A(V)$. Finally, if q is even, then let the color class C_1 consist of all lines of $\mathrm{AG}(2k, q)$ whose point at infinity is P_1^1.

We divided the points of $\mathcal{H}_\infty$ into $\frac{q^{2k}-1}{2(q-1)}$ pairs if q is odd, and into $\frac{q^{2k}-q}{2(q-1)}$ pairs if q is even. Consequently, the number of color classes is equal to $\frac{q^{2k}-1}{2(q-1)} q^k$ when q is odd, and it is equal to $\frac{q^{2k}-q}{2(q-1)} q^k + 1$ when q is even.

Now, we show that the coloring is complete. The class C_1 obviously intersects any other class. Let $C_{U,V,i}$ and $C_{W,Z,j}$ be two color classes. Then $S(U)$ and $S(V)$ are distinct elements of the spread $\mathcal{S}$ and $S(W)$ is also an element of $\mathcal{S}$. Hence we may assume, without loss of generality, that $S(U) \cap S(W) = \emptyset$. As

$$\dim(S(U) \cup \Pi_{U,i}) = \dim(S(W) \cup \Pi_{W,j}) = k$$

in $\mathrm{PG}(2k, q)$, by Proposition 3.1, we have that $\Pi_{U,i} \cap \Pi_{W,j}$ consists of a single point in $\mathrm{AG}(2k, q)$. Notice that the coloring is not proper, because the same argument shows that $\Pi_{U,i} \cap \Pi_{V,i}$ is also a single point in $\mathrm{AG}(2k, q)$. □

For odd dimensional spaces we have a slightly weaker estimate. In this case, the magnitude of the lower bound is $\frac{1}{\sqrt{q}} \cdot \frac{\sqrt{v}(v-1)}{q-1}$, where $v = q^n$.

Theorem 3.4. *If $k \geq 1$ then the colorings of the odd dimensional affine space, $\mathrm{AG}(2k+1, q)$, satisfy the inequality*

$$q^{k+2}\left(\frac{q^{2k}-1}{q^2-1}\right) + 1 \leq \psi'(\mathrm{AG}(2k+1, q)).$$

Proof. The hyperplane at infinity in the projective closure of $\mathrm{AG}(2k+1, q)$, $\mathcal{H}_\infty$, is isomorphic to $\mathrm{PG}(2k, q)$. Hence, by Proposition 3.2, $\mathcal{H}_\infty$ admits a good partition $\mathcal{H}_\infty = \mathcal{A} \cup \mathcal{B} \cup \{Q\}$ with assigned subspaces $S(U)$. Let $\mathcal{A} = \{P_1, P_2, \ldots, P_t\}$ and $\mathcal{B} = \{R_1, R_2, \ldots, R_s\}$ where $t = q^2\left(\frac{q^{2k}-1}{q^2-1}\right)$ and $s = q\left(\frac{q^{2k}-1}{q^2-1}\right)$.

For a point $P_i \in \mathcal{A}$ let $A(P_i) = \{\Pi_{P_i,1}, \Pi_{P_i,2}, \ldots, \Pi_{P_i,q^k}\}$ denote the set of the q^k parallel $(k+1)$-dimensional subspaces of $\mathrm{AG}(2k+1, q)$ whose projective closures intersect $\mathcal{H}_\infty$ in $S(P_i)$. Similarly, for a point $R_j \in \mathcal{B}$ let $B(R_j) = \{\Pi_{R_j,1}, \Pi_{R_j,2}, \ldots, \Pi_{R_j,q^{k+1}}\}$ denote the set of the q^{k+1} parallel k-dimensional subspaces of $\mathrm{AG}(2k+1, q)$ whose projective closures intersect $\mathcal{H}_\infty$ in $S(R_j)$.

Now, we define the color classes. Let C_1 be the color class that contains all lines of $\mathrm{AG}(2k+1, q)$ whose point at infinity is Q. Let the color class $C_{i,j,m}$ contain the lines joining either $P_{(j-1)q+i}$ and a point from $\Pi_{P_{(j-1)q+i},m}$, or R_j and a point from $\Pi_{R_j,(i-1)q^k+m}$ for $j = 1, 2, \ldots, s$, $i = 1, 2, \ldots, q$ and $m = 1, 2, \ldots, q^k$. Counting the number of color classes of type $C_{i,j,m}$, we obtain $s \cdot q \cdot q^k = q^{k+2}\left(\frac{q^{2k}-1}{q^2-1}\right)$. Each color class consists of the parallel lines of one subspace in $A(P_{(j-1)q+i})$ and the parallel lines of one subspace in $B(R_j)$. Clearly, the total number of color classes is $1 + q^{k+2}\left(\frac{q^{2k}-1}{q^2-1}\right)$. The color class C_1 contains q^{2k} lines and each of the classes of type $C_{i,j,m}$ consists of $q^k + q^{k-1}$ lines.

To prove that the coloring is complete, notice that the class C_1 obviously intersects any other class. Let $C_{i,j,m}$ and $C_{i',j',m'}$ be two color classes other than C_1. Consider the projective closures of those elements of $A(P_{(j-1)q+i})$ and $B(R_{j'})$ whose lines are contained in $C_{i,j,m}$ and in $C_{i',j',m'}$, respectively. One of these subspaces is a $(k+1)$-dimensional, whereas the other one is a k-dimensional subspace in $\mathrm{PG}(2k+1, q)$, and they have no point in common in $\mathcal{H}_\infty$. Thus, by Proposition 3.1, their intersection is a single point in $\mathrm{AG}(2k+1, q)$.

The coloring is not proper, because the same argument shows that $\Pi_{P_{(j-1)q+i},m} \cap \Pi_{R_j,(i-1)q^k+m}$ is also a point in $\mathrm{AG}(2k+1, q)$, thus $C_{i,j,m}$ contains a pair of intersecting lines. □

Now, we are ready to prove our second main theorem.

Proof of Theorem 1.2. If n is even then Theorem 3.3 gives the result at once. If n is odd then $v = q^{2k+1}$, hence $\sqrt{v/q} = q^k$. From the estimate of Theorem 3.4 we get

$$\begin{aligned} q^{k+2}\left(\frac{q^{2k}-1}{q^2-1}\right)+1 &= \frac{q^{3k+2}-q^{k+2}}{q^2-1}+1 \\ &= \frac{(q+1)(q^{3k+1}-q^k)}{q^2-1} - \frac{q^{3k+1}+q^{k+2}-q^{k+1}-q^k}{q^2-1}+1 \\ &= \frac{1}{\sqrt{q}}\frac{\sqrt{v}(v-1)}{q-1} - \frac{q^{3k+1}+q^{k+2}-q^{k+1}-q^k}{q^2-1}+1, \end{aligned}$$

which proves the statement. □

Next, recall that a lower bound for the achromatic index require a proper and complete line-coloring of $\mathrm{AG}(n,q)$. We consider only the even dimensional case.

Theorem 3.5. *Let $k > 1$ and $\epsilon = 0, 1$ or 2, such that $q^k + 1 \equiv \epsilon \pmod 3$. Then the achromatic index of the even dimensional finite affine space $\mathrm{AG}(2k,q)$ satisfies the inequality*

$$\left(\frac{q^k+1-\epsilon}{3}(q^k+2)+\epsilon\right)\frac{q^k-1}{q-1} \le \alpha'(\mathrm{AG}(2k,q)).$$

Proof. The hyperplane at infinity in the projective closure of $\mathrm{AG}(2k,q)$, $\mathcal{H}_\infty$, is isomorphic to $\mathrm{PG}(2k-1,q)$, hence it admits a $(k-1)$-spread $\mathcal{L} = \{\ell_1, \ell_2, \ldots, \ell_{q^k+1}\}$. Let $\mathcal{A}(\ell_i) = \{\Pi_{\ell_i,1}, \Pi_{\ell_i,2}, \ldots, \Pi_{\ell_i,q^k}\}$ denote the set of the q^k parallel k-dimensional subspaces in $\mathrm{AG}(2k,q)$ whose projective closures intersect $\mathcal{H}_\infty$ in ℓ_i. Then, by Proposition 3.1, the intersection $\Pi_{\ell_i,s} \cap \Pi_{\ell_j,t}$ is a single affine point for all $i \neq j$ and $1 \le s,t \le q^k$.

First, to any triple of $(k-1)$-dimensional subspaces, $e, f, g \in \mathcal{L}$, we assign $q^k + 2$ color classes as follows. Take a fourth $(k-1)$-dimensional subspace $d \in \mathcal{L}$, and, for $u = (q^k-1)/(q-1)$, denote the points of the $(k-1)$-dimensional subspaces d, e, f and g by $D_1, D_2, \ldots, D_u$, $E_1, E_2, \ldots, E_u$, $F_1, F_2, \ldots, F_u$ and $G_1, G_2, \ldots, G_u$, respectively. For any triple (D_i, e, g) there is a unique line through D_i which intersects the skew subspaces e and g. We can choose the numbering of the points E_i and G_i such that the line E_iG_i intersects d in D_i for $i = 1, 2, \ldots, u$; the numbering of the points F_i, such that the line D_iF_{i+1} intersects d and g for $i = 1, 2, \ldots, u-1$, and, finally, choose the line D_uF_1 that intersects d and g. Notice that this construction implies that the line D_iF_i does not intersect g for $i = 1, 2, \ldots, u$. Let the points of $\Pi_{d,1}$ denote by $M_1, M_2, \ldots, M_{q^k}$. We can choose the numbering of the elements of $\mathcal{A}(e)$, $\mathcal{A}(f)$ and $\mathcal{A}(g)$ such that $\Pi_{e,i} \cap \Pi_{f,i} \cap \Pi_{g,i} = \{M_i\}$ for $i = 1, 2, \ldots, q^k$.

We define three types of color classes for $i = 1, 2, \ldots, u$ and $j = 1, 2, \ldots, q^k$. Let $B^{i,0}_{e,f,g}$ and $B^{i,1}_{e,f,g}$ be the color classes that contain the lines through M_j whose point at infinity is E_i and F_i, respectively. Let $C^{i,j}_{e,f,g}$ be the color class that contains the lines in $\Pi_{e,i}$ whose point at infinity is E_j, except the line E_jM_i, the lines in $\Pi_{f,i}$ whose point at infinity is F_j, except the line F_jM_i, and the lines in $\Pi_{g,i}$ whose point at infinity is G_j. Hence each of $B^{i,0}_{e,f,g}$ and $B^{i,1}_{e,f,g}$ contains q^k lines and $C^{i,j}_{e,f,g}$ contains $3q^{k-1} - 2$ lines.

Notice that for each $i \in \{1, 2, \ldots, u\}$, the union of the color classes

$$\mathcal{K}^i_{e,f,g} = B^{i,0}_{e,f,g} \cup B^{i,1}_{e,f,g} \cup_{j=1}^{q^k} C^{i,j}_{e,f,g}$$

contains all lines whose point at infinity is E_i, F_i or G_i. Each of the two sets of lines belonging to $B^{i,0}_{e,f,g}$ or $B^{i,1}_{e,f,g}$, naturally defines a $(k+1)$-dimensional subspace of $\mathrm{PG}(2k,q)$, we denote these subspaces by Π_{E_i} and Π_{F_i}, respectively.

For $t = 0, 1, \dots, \lfloor (q^k - 2 - \epsilon)/3 \rfloor$ let $e = \ell_{3t+1}$, $f = \ell_{3t+2}$, $g = \ell_{3t+3}$, $d = \ell_{3t+4}$, define $\ell_{q^k+2-\epsilon}$ as ℓ_1, and make the $q^k + 2$ color classes $B^{i,0}_{e,f,g}$, $B^{i,1}_{e,f,g}$ and $C^{i,j}_{e,f,g}$. Finally, for each point P in the subspace ℓ_{q^k+1} if $\epsilon = 1$, or in ℓ_{q^k} if $\epsilon = 2$, define a new color class D^P which contains all lines whose point at infinity is P.

Clearly, the coloring is proper and it contains, by definition, the required number of color classes. Now, we prove that it is complete. Notice that each color class of type D^P obviously intersects any other color class. In relation to the other cases we have that:

- The color classes $B^{i,j}_{\ell_{3m+1},\ell_{3m+2},\ell_{3m+3}}$ and $B^{i',j'}_{\ell_{3m+1},\ell_{3m+2},\ell_{3m+3}}$ intersect, because both of them contain all points of the k-dimensional subspace $\Pi_{\ell_{3m+4},1}$.
- If $t \neq m$ then the color classes $B^{i,j}_{\ell_{3t+1},\ell_{3t+2},\ell_{3t+3}}$ and $B^{i',j'}_{\ell_{3m+1},\ell_{3m+2},\ell_{3m+3}}$ intersect, because the $(k-1)$-dimensional subspaces ℓ_{3t+4} and ℓ_{3m+4} are skew in $\mathcal{H}_\infty$, hence the 2-dimensional intersection of the $(k+1)$-dimensional subspaces Π_{E_i} or Π_{F_i}, according as $j = 1$ or 2, and $\Pi_{E'_i}$ or $\Pi_{F'_i}$, according as $j' = 1$ or 2, is not a subspace of $\mathcal{H}_\infty$. Thus Proposition 3.1 implies that their intersection contains some affine points.
- The color classes $B^{i,j}_{\ell_{3m+1},\ell_{3m+2},\ell_{3m+3}}$ and $C^{i',j'}_{\ell_{3t+1},\ell_{3t+2},\ell_{3t+3}}$ intersect in both cases $m = t$ and $m \neq t$, because the $(k-1)$-dimensional subspaces ℓ_{3m+4} and ℓ_{3t+3} are skew in $\mathcal{H}_\infty$. Again, Proposition 3.1 implies that the intersection of the k-dimensional subspaces $\Pi_{\ell_{3m+4},1}$ (which is a subspace of either the $(k+1)$-dimensional subspace Π_{E_i} or Π_{F_i}, according as $j = 1$ or 2) and $\Pi_{\ell_{3m+3},i'}$ is an affine point.
- If $t \neq m$ then each pair of color classes $C^{i,j}_{\ell_{3t+1},\ell_{3t+2},\ell_{3t+3}}$ and $C^{i',j'}_{\ell_{3m+1},\ell_{3m+2},\ell_{3m+3}}$, intersects since, as previously, the $(k-1)$-dimensional subspaces ℓ_{3t+3} and ℓ_{3m+3} are skew in $\mathcal{H}_\infty$, thus Proposition 3.1 implies that the projective closures of the k-dimensional subspaces $\Pi_{\ell_{3t+3},i}$ and $\Pi_{\ell_{3m+3},i'}$ intersect each other in $\mathrm{AG}(2k,q)$.
- Finally, we prove that each pair of classes $C^{i,j}_{\ell_{3t+1},\ell_{3t+2},\ell_{3t+3}}$ and $C^{i',j'}_{\ell_{3t+1},\ell_{3t+2},\ell_{3t+3}}$ intersects. It is obvious when $i = i'$. Suppose that $i \neq i'$, let $M_i = \Pi_{\ell_{3t+1},i} \cap \Pi_{\ell_{3t+2},i} \cap \Pi_{\ell_{3t+3},i}$ and $M_{i'} = \Pi_{\ell_{3t+1},i'} \cap \Pi_{\ell_{3t+2},i'} \cap \Pi_{\ell_{3t+3},i'}$. Since the points M_i and $M_{i'}$ are in $\Pi_{\ell_{3t+4},1}$, the line $M_iM_{i'}$ intersects $\mathcal{H}_\infty$ in ℓ_{3t+4}. Take the point $T = M_iM_{i'} \cap \ell_{3t+4}$ and the lines E_jT and F_jT. Clearly, at least one of these lines does not intersect ℓ_{3t+3}, we may assume without loss of generality, that $E_jT \cap \ell_{3t+3} = \emptyset$.

  By Proposition 3.1, there exist affine points $N_i = \Pi_{\ell_{3t+1},i} \cap \Pi_{\ell_{3t+3},i'}$ and $N_{i'} = \Pi_{\ell_{3t+1},i'} \cap \Pi_{\ell_{3t+3},i}$. Suppose that $N_i \in E_{j'}M_{i'}$ and $N_{i'} \in E_jM_i$. Then $\ell_{3t+1} \cap M_iM_{i'} = \emptyset$, hence $\langle \ell_{3t+1}, M_iM_{i'} \rangle$ is a $(k+1)$-dimensional subspace Σ_{k+1}, which intersects $\mathcal{H}_\infty$ in a k-dimensional subspace Σ_k. Obviously, Σ_k also contains the points E_j and $E_{j'}$. Then $\Sigma_k = \langle \ell_{3t+1}, T \rangle$, and $\Sigma_k \cap \ell_{3t+3}$ is a single point, say U. As the lines $N_{i'}M_i$ and $N_iM_{i'}$ are in the k-dimensional subspaces $\Pi_{\ell_{3t+3},i}$ and $\Pi_{\ell_{3t+3},i'}$, respectively, there exist the points $N_{i'}M_i \cap \ell_{3t+3}$ and $N_iM_{i'} \cap \ell_{3t+3}$. Moreover, we have that $N_{i'}M_i \cap \ell_{3t+3} = N_iM_{i'} \cap \ell_{3t+3} = U$. Hence the points $N_i, M_i, N_{i'}$ and $M_{i'}$ are contained in a 2-dimensional subspace Σ_2, and $\Sigma_2 \cap \mathcal{H}_\infty$ contains the points U, E_j, $E_{j'}$ and T. Consequently, $\Sigma_2 \cap \mathcal{H}_\infty$ is the line E_jT and it contains the point U, thus E_jT intersects the subspace ℓ_{3t+3}, contradiction.

Thus $N_i \notin E_{j'}M_{i'}$ or $N_{i'} \notin E_jM_i$. This implies that N_i or $N_{i'}$ is a common point of the color classes $C^{i,j}_{\ell_{3t+1},\ell_{3t+2},\ell_{3t+3}}$ and $C^{i',j'}_{\ell_{3t+1},\ell_{3t+2},\ell_{3t+3}}$.

In consequence, the coloring is complete. □

To conclude this section we prove our third main theorem.

Proof of Theorem 1.3. As $v = q^{2k}$, from Theorem 3.5 we get

$$\left(\frac{q^k+1-\epsilon}{3}(q^k+2)+\epsilon\right)\frac{q^k-1}{q-1} = \frac{q^{3k}+(2-\epsilon)q^{2k}+(2\epsilon-1)q^k-2-\epsilon}{3(q-1)} = \frac{1}{3}\frac{\sqrt{v}(v-1)}{q-1}+\frac{(2-\epsilon)v+2\epsilon\sqrt{v}-2-\epsilon}{3(q-1)},$$

which proves the statement. □

4 Small dimensions

In this section, we improve on our bounds in two and three dimensions. First, we prove the exact values of achromatic and pseudoachromatic indices of finite affine planes. Due to the fact that there exist non-desarguesian affine planes, we use the notation A_q for an arbitrary affine plane of order q. For the axiomatic definition of A_q we refer to [11]. The basic combinatorial properties of A_q are the same as of $\mathrm{AG}(2, q)$.

Theorem 4.1. *Let A_q be any affine plane of order q. Then*

$$\chi'(\mathrm{A}_q) = \alpha'(\mathrm{A}_q) = q + 1.$$

Proof. Let $\mathscr{S}_1, \mathscr{S}_2, \dots, \mathscr{S}_{q+1}$ denote the $q + 1$ parallell classes of lines in A_q. Two lines have a point in common if and only if they belong to distinct parallel classes. Hence, if we define a coloring ϕ with $q + 1$ colors such that a line ℓ gets color i if and only if $\ell \in \mathscr{S}_i$ then ϕ is proper, so $q + 1 \le \chi'(\mathrm{A}_q)$.

Since $\chi'(\mathrm{A}_q) \le \alpha'(\mathrm{A}_q)$, it is enough to prove that $\alpha'(\mathrm{A}_q) \le q + 1$. Suppose to the contrary that ψ is a complete and proper coloring with $m > q + 1$ color classes. As ψ is proper, each color class must be a subset of a parallel class. By the pigeonhole principle, $m > q+1$ implies that there exist at least two color classes that are subsets of the same parallel class. Hence they do not contain intersecting lines, contradicting to the completeness of ψ. Thus $\alpha'(\mathrm{A}_q) \le q + 1$, the theorem is proved. □

Theorem 4.2. *Let A_q be any affine plane of order q. Then*

$$\psi'(\mathrm{A}_q) = \left\lfloor \tfrac{(q+1)^2}{2} \right\rfloor.$$

Proof. First, we prove that $\psi'(\mathrm{A}_q) \le \left\lfloor \frac{(q+1)^2}{2} \right\rfloor$. Suppose to the contrary that φ is a complete coloring of A_q with $\left\lfloor \frac{(q+1)^2}{2} \right\rfloor + 1$ color classes. As A_q has $q^2 + q$ lines, this implies that φ has at most $q^2 + q - \left(\lfloor \frac{(q+1)^2}{2} \rfloor + 1\right)$ color classes of cardinality greater than one. Thus, there are at least

$$\left\lfloor \tfrac{(q+1)^2}{2} \right\rfloor + 1 - \left(q^2 + q - \left(\left\lfloor \tfrac{(q+1)^2}{2} \right\rfloor + 1\right)\right) = \begin{cases} q+2, & \text{if } q \text{ is even}, \\ q+3, & \text{if } q \text{ is odd}, \end{cases}$$

color classes of size one. Hence, again by the pigeonhole principle, there are at least two color classes of size one belonging to the same parallel class. They have empty intersection, so φ is not complete. This contradiction shows that $\psi'(\mathrm{A}_q) \leq \left\lfloor \frac{(q+1)^2}{2} \right\rfloor$.

We go on to give a complete coloring of A_q with $\left\lfloor \frac{(q+1)^2}{2} \right\rfloor$ color classes. Let P be a point and $e_1, e_2, \ldots, e_{q+1}$ be the lines through P. For $i = 1, 2, \ldots, q+1$ let $\mathscr{S}_i$ be the parallel class containing e_i and denote the $q-1$ lines in the set $\mathscr{S}_i \setminus \{e_i\}$ by $\ell_i, \ell_{(q+1)+i}, \ldots, \ell_{(q-2)(q+1)+i}$. Then:

$$\bigcup_{i=1}^{q} (\mathscr{S}_i \setminus \{e_i\}) = \{\ell_1, \ell_2, \ldots, \ell_{q^2-1}\},$$

and ℓ_j and ℓ_{j+1} are non-parallel lines for all $1 \leq j < q^2-1$. For better clarity, we construct $q+1$ color classes with even indices and $\left\lfloor \frac{q^2-1}{2} \right\rfloor$ color classes with odd indices. Let the color class C_{2k} consist of one line, e_k, for $k = 1, 2, \ldots, q+1$. Let the color class C_{2k-1} contain the lines ℓ_{2k-1} and ℓ_{2k} for $k = 1, 2, \ldots, \left\lfloor \frac{q^2-1}{2} \right\rfloor$, finally, if q is even, let the color class C_{q^2-3} contain the line ℓ_{q^2-1}, too.

The coloring is complete, because color classes having even indices intersect at P, and each color class with odd index contains two non-parallel lines whose union intersects all lines of the plane. $\square$

Our last construction gives a lower bound for the achromatic index of $\mathrm{AG}(3,q)$. As $\alpha'(\mathrm{AG}(3,q)) \leq \psi'(\mathrm{AG}(3,q))$, this can be considered as well as a lower estimate on the pseudoachromatic index of $\mathrm{AG}(3,q)$ and this bound is better than the general one proved in Theorem 3.4. We use the cyclic model of $\mathrm{PG}(2,q)$ to make the coloring. The detailed description of this model can be found in [16, Theorem 4.8 and Corollary 4.9]. We collect the most important properties of the cyclic model in the following proposition.

Proposition 4.3. *Let q be a prime power. Then the group $\mathbb{Z}_{q^2+q+1}$ admits a perfect difference set $D = \{d_0, d_1, d_2, \ldots, d_q\}$, that is the q^2+q integers $d_i - d_j$ $(i \neq j)$ are all distinct modulo q^2+q+1. We may assume without loss of generality that $d_0 = 0$ and $d_1 = 1$. The plane $\mathrm{PG}(2,q)$ can be represented in the following way. The points are the elements of $\mathbb{Z}_{q^2+q+1}$, the lines are the subsets*

$$D + j = \{d_i + j : d_i \in D\}$$

for $j = 0, 1, \ldots, q^2+q$, and the incidence is the set theoretical inclusion.

Theorem 4.4. *The achromatic index of $\mathrm{AG}(3,q)$ satisfies the inequality:*

$$\frac{q(q+1)^2}{2} + 1 \leq \alpha'(\mathrm{AG}(3,q)).$$

Proof. The plane at infinity in the projective closure of $\mathrm{AG}(3,q)$, $\mathcal{H}_\infty$, is isomorphic to $\mathrm{PG}(2,q)$, hence it has a cyclic representation (described in Proposition 4.3). Let $v = q^2 + q + 1$, let the points and the lines of $\mathcal{H}_\infty$ be $P_1, P_2, \ldots, P_v$, and $\ell_1, \ell_2, \ldots, \ell_v$, respectively. We can choose the numbering such that for $i = 1, 2, 3, \ldots, v$ the line ℓ_i contains the points P_i, P_{i+1} and P_{i-d} (where $0 \neq d \neq 1$ is a fixed element of the difference set D, and the subscripts are taken modulo v).

Let $\mathcal{A}(P_i) = \{\Pi_{P_i,1}, \Pi_{P_i,2}, \dots, \Pi_{P_i,q}\}$ denote the set of the q parallel planes in $\mathrm{AG}(3,q)$ whose projective closures intersect $\mathcal{H}_\infty$ in ℓ_i, and $\overline{\Pi_{P_i,j}}$ denote the projective closure of $\Pi_{P_i,j}$ for $i = 1, 3, \dots, v$, and $j = 1, 2, \dots, q$. Let W_i be a plane whose projective closure intersects $\mathcal{H}_\infty$ in ℓ_{i-d}. Then the projective closure of each element of $A(P_i) \cup A(P_{i+1})$ intersects W_i in a line whose point at infinity is P_i, so we can choose the numbering of the elements of $A(P_i)$ and $A(P_{i+1})$, such that $\overline{\Pi_{P_i,j}} \cap \overline{\Pi_{P_{i+1},j}} \subset W_i$ for $i = 1, 3, \dots, v-2$, and $j = 1, 2, \dots, q$. Let e^i_j denote the line $\Pi_{P_i,j} \cap \Pi_{P_{i+1},j}$.

We assign $q+1$ color classes to the pair (P_i, P_{i+1}) for $i = 1, 3, \dots, v-2$. Let the color class C^i_0 contain the lines $e^i_1, e^i_2 \dots, e^i_q$. For $j = 1, 2, \dots, q$, let the color class C^i_j contain those lines of $\Pi_{P_i,j}$ whose point at infinity is P_i, except the line e^i_j, and the q parallel lines of $\Pi_{P_{i+1},j}$ whose point at infinity is P_{i+1}. Finally, let the color class C^v contain all lines whose point at infinity is P_v. In this way we constructed

$$(q+1)\frac{v-1}{2} + 1 = \frac{q(q+1)^2}{2} + 1$$

color classes and each line belongs to exactly one of them, because C^i_0 contains q lines, C^i_j contains $2q-1$ lines for each $j = 1, 2, \dots, q$. and C^v contains q^2 lines.

The coloring is proper by construction. The color class C^v obviously intersects any other class. For other pairs of color classes, two major cases are distinguished when we prove the completeness. On the one hand, if $i \neq k$ then we have:

- $C^i_0 \cap C^k_0 \neq \emptyset$, because the planes W_i and W_k intersect each other;
- if $j > 0$ then $C^i_0 \cap C^k_j \neq \emptyset$, because the planes W_i and $\Pi_{P_{k+1},j}$ intersect each other;
- if $m > 0$ and $j > 0$ then $C^i_m \cap C^k_j \neq \emptyset$, because the planes $\Pi_{P_{i+1},m}$ and $\Pi_{P_{k+1},j}$ intersect each other.

On the other hand, color classes having the same superscript also have non-empty intersection:

- $C^i_0 \cap C^i_j \neq \emptyset$, because the planes W_i and $\Pi_{P_{i+1},j}$ intersect each other;
- if $j \neq k$ then the planes $\Pi_{P_i,j}$ and $\Pi_{P_{i+1},k}$ intersect in a line f and $f \neq e^i_j$, hence its points are not removed from $\Pi_{P_i,j}$, so $C^i_j \cap C^i_k \neq \emptyset$.

Hence the coloring is also complete, this proves the theorem. □

References

[1] G. Araujo-Pardo, Gy. Kiss, C. Rubio-Montiel and A. Vázquez-Ávila, On line colorings of finite projective spaces, 2018, `arXiv:1702.06769 [math.CO]`.

[2] G. Araujo-Pardo, J. J. Montellano-Ballesteros, C. Rubio-Montiel and R. Strausz, On the pseudoachromatic index of the complete graph II, *Bol. Soc. Mat. Mex.* **20** (2014), 17–28, doi: 10.1007/s40590-014-0007-9.

[3] G. Araujo-Pardo, J. J. Montellano-Ballesteros, C. Rubio-Montiel and R. Strausz, On the pseudoachromatic index of the complete graph III, *Graphs Combin.* **34** (2018), 277–287, doi: 10.1007/s00373-017-1872-6.

[4] G. Araujo-Pardo, J. J. Montellano-Ballesteros and R. Strausz, On the pseudoachromatic index of the complete graph, *J. Graph Theory* **66** (2011), 89–97, doi:10.1002/jgt.20491.

[5] G. Araujo-Pardo and C. Rubio-Montiel, Pseudoachromatic and connected-pseudoachromatic indices of the complete graph, *Discrete Appl. Math.* **231** (2017), 60–66, doi:10.1016/j.dam.2017.03.019.

[6] G. Araujo-Pardo, C. Rubio-Montiel and A. Vázquez-Ávila, Note on the Erdős-Faber-Lovász conjecture: quasigroups and complete digraphs, *Ars Combin.*, in press.

[7] G. Araujo-Pardo and A. Vázquez-Ávila, A note on Erdős-Faber-Lovász conjecture and edge coloring of complete graphs, *Ars Combin.* **129** (2016), 287–298.

[8] A. Beutelspacher, D. Jungnickel and S. A. Vanstone, On the chromatic index of a finite projective space, *Geom. Dedicata* **32** (1989), 313–318, doi:10.1007/bf00147923.

[9] A. Bouchet, Indice achromatique des graphes multiparti complets et réguliers, *Cahiers Centre Études Rech. Opér.* **20** (1978), 331–340.

[10] C. J. Colbourn and M. J. Colbourn, Greedy colourings of Steiner triple systems, in: A. Barlotti, P. V. Ceccherini and G. Tallini (eds.), *Combinatorics '81*, North-Holland, Amsterdam, volume 18 of *Annals of Discrete Mathematics*, pp. 201–207, 1983, proceedings of the International Conference on Combinatorial Geometries and their Applications held in Rome, June 7 – 12, 1981.

[11] P. Dembowski, *Finite Geometries*, Classics in Mathematics, Springer-Verlag, Berlin, 1997, doi:10.1007/978-3-642-62012-6, reprint of the 1968 original.

[12] P. Erdős, Problems and results in combinatorial analysis and combinatorial analysis, in: C. St. J. A. Nash-Williams and J. Sheehan (eds.), *Proceedings of the Fifth British Combinatorial Conference*, Utilitas Mathematica Publishing, Winnipeg, Manitoba, volume 15 of *Congressus Numerantium*, pp. 169–192, 1976, held at the University of Aberdeen, Aberdeen, July 14 – 18, 1975.

[13] P. Erdős, On the combinatorial problems which I would most like to see solved, *Combinatorica* **1** (1981), 25–42, doi:10.1007/bf02579174.

[14] R. P. Gupta, Bounds on the chromatic and achromatic numbers of complementary graphs, in: W. T. Tutte (ed.), *Recent Progress in Combinatorics*, Academic Press, New York, pp. 229–235, 1969, proceedings of the Third Waterloo Conference on Combinatorics, May 1968.

[15] F. Harary, S. Hedetniemi and G. Prins, An interpolation theorem for graphical homomorphisms, *Portugal. Math.* **26** (1967), 453–462, `http://eudml.org/doc/115047`.

[16] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*, Oxford Mathematical Monographs, The Clarendon Press, New York, 2nd edition, 1998.

[17] R. E. Jamison, On the edge achromatic numbers of complete graphs, *Discrete Math.* **74** (1989), 99–115, doi:10.1016/0012-365x(89)90202-1.

[18] A. Rosa and C. J. Colbourn, Colorings of block designs, in: J. H. Dinitz and D. R. Stinson (eds.), *Contemporary Design Theory*, John Wiley & Sons, New York, Wiley-Interscience Series in Discrete Mathematics and Optimization, pp. 401–430, 1992.

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Pentavalent symmetric graphs of order four times an odd square-free integer*

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Abstract

A graph is said to be symmetric if its automorphism group is transitive on its arcs. Guo et al. in 2011 and Pan et al. in 2013 determined all pentavalent symmetric graphs of order $4pq$. In this paper, we shall generalize this result by determining all connected pentavalent symmetric graphs of order four times an odd square-free integer. It is shown in this paper that, for each such graph Γ, either the full automorphism group $\mathrm{Aut}\,\Gamma$ is isomorphic to $\mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p)\times\mathbb{Z}_2$ or $\mathrm{PGL}(2,p)\times\mathbb{Z}_2$, or Γ is isomorphic to one of 9 graphs.

Keywords: Arc-transitive graph, normal quotient, automorphism group.

Math. Subj. Class.: 05C25, 05E18, 20B25

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1 Introduction

All graphs in this paper are assumed to be finite, simple and undirected. Let Γ be a graph and denote $V\Gamma$ and $A\Gamma$ the vertex set and arc set of Γ, respectively. Let G be a subgroup of the full automorphism group $\mathrm{Aut}\,\Gamma$ of Γ. Then Γ is called *G-vertex-transitive* and *G-arc-transitive* if G is transitive on $V\Gamma$ and $A\Gamma$, respectively. An arc-transitive graph is also called a *symmetric* graph. It is well known that Γ is G-arc-transitive if and only if G is transitive on $V\Gamma$ and the stabilizer $G_\alpha := \{g \in G \mid \alpha^g = \alpha\}$ is transitive on the neighbor set $\Gamma(\alpha)$ of the vertex α of Γ.

The cubic and tetravalent graphs have been studied extensively in the literature. In recent years, attention has moved on to pentavalent symmetric graphs and a series of results have been obtained. For example, all the possibilities of vertex stabilizers of pentavalent symmetric graphs are determined in [7, 20]. Also, for distinct primes p, q and r, the classifications of pentavalent symmetric graphs of order $2pq$ and $2pqr$ are presented in [9, 19], respectively. A classification of 1-regular pentavalent graph (that is, the full automorphism group acts regularly on its arc set) of square-free order is presented in [13]. Recently, pentavalent symmetric graphs of square-free order have been completely classified in [11]. Furthermore, some classifications of pentavalent symmetric graphs of cube-free order also have been obtained in recent years. For example, the classifications of pentavalent symmetric graphs of order $12p$, $4pq$ and $2p^2$ are presented in [8, 16, 5]. More recently, symmetric graphs of any prime valency which admit a soluble arc-transitive group have been classified in [14]. The main purpose of this paper is to extend the results in [8, 16] to four times an odd square-free integer case.

The main result of this paper is the following theorem.

Theorem 1.1. *Let n be an odd square-free integer and let Γ be a pentavalent symmetric graph of order $4n$. If n has at least three prime factors, then one of the following statements holds.*

(1) $\mathrm{Aut}\,\Gamma \cong \mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p) \times \mathbb{Z}_2$ *or* $\mathrm{PGL}(2,p) \times \mathbb{Z}_2$, *where $p \geq 29$ is a prime. Furthermore, the stabilizer $(\mathrm{Aut}\,\Gamma)_\alpha$ and the prime p appear in Table 5 or Table 6.*

(2) The triple $(\Gamma, n, \mathrm{Aut}\,\Gamma)$ lies in the following Table 1.

Remark 1.2 (Remarks on Theorem 1.1)**.**

(a) The graphs in Table 1 are introduced in Example 3.2.

(b) The graphs $\mathcal{C}_{5852}$ and $\mathcal{C}^3_{780}$ in Table 1, and the graphs in part (1) with automorphism group $\mathrm{PSL}(2,p) \times \mathbb{Z}_2$ or $\mathrm{PGL}(2,p) \times \mathbb{Z}_2$ can also be constructed from the bipartite double cover (the definition of bipartite double cover see Section 3) of a pentavalent symmetric graph of square-free order (see [11, Example 4.3 and Example 4.5] and [19, Example 3.9 and Example 3.11] for details on these graphs).

2 Preliminaries

We now give some necessary preliminary results. The first one is a property of the Fitting subgroup, see [18, p. 30, Corollary].

Lemma 2.1. *Let F be the Fitting subgroup of a group G. If G is soluble, then $F \neq 1$ and the centralizer $\mathbf{C}_G(F) \leq F$.*

Table 1: Nine ‘sporadic’ pentavalent symmetric graphs of order four times an odd square-free integer.

Row	Γ	n	$\mathrm{Aut}\,\Gamma$	$(\mathrm{Aut}\,\Gamma)_\alpha$	Transitivity	Bipartite?
1	$\mathcal{C}^1_{17556}$	$3\cdot 7\cdot 11\cdot 19$	J_1	D_{10}	1-transitive	No
2	$\mathcal{C}^2_{17556}$	$3\cdot 7\cdot 11\cdot 19$	J_1	D_{10}	1-transitive	No
3	$\mathcal{C}^3_{17556}$	$3\cdot 7\cdot 11\cdot 19$	J_1	D_{10}	1-transitive	No
4	$\mathcal{C}^4_{17556}$	$3\cdot 7\cdot 11\cdot 19$	J_1	D_{10}	1-transitive	No
5	$\mathcal{C}^5_{17556}$	$3\cdot 7\cdot 11\cdot 19$	J_1	D_{10}	1-transitive	No
6	$\mathcal{C}_{5852}$	$7\cdot 11\cdot 19$	$\mathrm{J}_1\times\mathbb{Z}_2$	A_5	2-transitive	Yes
7	$\mathcal{C}^1_{780}$	$3\cdot 5\cdot 13$	$\mathrm{PSL}(2,25)\times\mathbb{Z}_2$	F_{20}	2-transitive	No
8	$\mathcal{C}^2_{780}$	$3\cdot 5\cdot 13$	$\mathrm{PSL}(2,25)\times\mathbb{Z}_2$	F_{20}	2-transitive	No
9	$\mathcal{C}^3_{780}$	$3\cdot 5\cdot 13$	$\mathrm{PSL}(2,25)\times\mathbb{Z}_2$	F_{20}	2-transitive	Yes

The maximal subgroups of $\mathrm{PSL}(2,p)$ are known, see [4, Section 239].

Lemma 2.2. *Let $T=\mathrm{PSL}(2,p)$, where $p\geq 5$ is a prime. Then a maximal subgroup of T is isomorphic to one of the following groups:*

(1) D_{p-1}, *where* $p\neq 5,7,9,11$*;*

(2) D_{p+1}, *where* $p\neq 7,9$*;*

(3) $\mathbb{Z}_p:\mathbb{Z}_{(p-1)/2}$*;*

(4) A_4, *where* $p=5$ *or* $p\equiv 3,13,27,37 \pmod{40}$*;*

(5) S_4, *where* $p\equiv\pm 1 \pmod 8$

(6) A_5, *where* $p\equiv\pm 1 \pmod 5$.

By [2, Theorem 2], we may easily derive the maximal subgroups of $\mathrm{PGL}(2,p)$.

Lemma 2.3. *Let $T=\mathrm{PGL}(2,p)$ with $p\geq 5$ a prime. Then a maximal subgroup of T is isomorphic to one of the following groups:*

(1) $\mathbb{Z}_p:\mathbb{Z}_{p-1}$*;*

(2) $\mathrm{D}_{2(p+1)}$*;*

(3) $\mathrm{D}_{2(p-1)}$, *where* $p\geq 7$*;*

(4) S_4, *where* $p\equiv\pm 3 \pmod 8$*;*

(5) $\mathrm{PSL}(2,p)$.

From [6, pp. 134–136], we can obtain the following lemma by checking the orders of nonabelian simple groups.

Lemma 2.4. *Let n be an odd square-free integer such that n has at least three prime factors. Let T be a nonabelian simple group of order $2^i\cdot 3^j\cdot 5\cdot n$, where $1\leq i\leq 11$ and $0\leq j\leq 2$. Let p be the largest prime factor of n. Then T is listed in Table 2.*

Table 2: Nonabelian simple groups of order $2^i \cdot 3^j \cdot 5 \cdot n$ with $1 \le i \le 11$ and $0 \le j \le 2$.

T	$\lvert T\rvert$	n
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$3 \cdot 7 \cdot 11$
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$7 \cdot 11 \cdot 23$
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	$7 \cdot 11 \cdot 19$
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$3 \cdot 5 \cdot 7$
$\mathrm{Sz}(32)$	$2^{10} \cdot 5^2 \cdot 31 \cdot 41$	$5 \cdot 31 \cdot 41$
$\mathrm{PSU}(3,4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	$3 \cdot 5 \cdot 13$
$\mathrm{PSp}(4,4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	$3 \cdot 5 \cdot 17$
$\mathrm{PSL}(2,25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	$3 \cdot 5 \cdot 13$
$\mathrm{PSL}(2,2^8)$	$2^8 \cdot 3 \cdot 5 \cdot 17 \cdot 257$	$3 \cdot 17 \cdot 257$
$\mathrm{PSL}(5,2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	$3 \cdot 7 \cdot 31$
$\mathrm{PSL}(2,2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	$3 \cdot 7 \cdot 13$
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$3 \cdot 7 \cdot 11 \cdot 23$
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$3 \cdot 7 \cdot 11 \cdot 23$
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	$3 \cdot 7 \cdot 11 \cdot 19$
$\mathrm{PSL}(2,p)$	$\frac{p(p+1)(p-1)}{2}$ $(p \ge 29)$	

Proof. If T is a sporadic simple group, by [6, p. 135–136], $T = \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{J}_1$ or J_2. If $T = \mathrm{A}_n$ is an alternating group, since 3^4 does not divide $|T|$, we have $n \le 8$, it then easily exclude that $T = \mathrm{A}_5, \mathrm{A}_6, \mathrm{A}_7$ or A_8. Hence no T exists for this case.

Suppose now $T = X(q)$ is a simple group of Lie type, where X is one type of Lie groups, and $q = r^d$ is a prime power. If $r \ge 5$, as $|T|$ has at most three 3-factors, two 5-factors and one p-factor, it easily follows from [6, p. 135] that the only possibility is $T = \mathrm{PSL}(2,p)$ with $p \ge 29$ (note that $\mathrm{PSL}(2,p)$ with $5 \le p \le 23$ does not satisfy the condition of the lemma) or $\mathrm{PSL}(2,25)$, where p is the largest prime factor of n. If $r \le 3$, as 2^{12} and 3^4 do not divide $|T|$, then we have $T = \mathrm{Sz}(32), \mathrm{PSU}(3,4), \mathrm{PSp}(4,4), \mathrm{PSL}(2,2^6)$, $\mathrm{PSL}(2,2^8)$ or $\mathrm{PSL}(5,2)$. □

For a graph $\varGamma$ and a positive integer s, an *s-arc* of $\varGamma$ is a sequence $\alpha_0, \alpha_1, \dots, \alpha_s$ of vertices such that α_{i-1}, α_i are adjacent for $1 \le i \le s$ and $\alpha_{i-1} \ne \alpha_{i+1}$ for $1 \le i \le s-1$. In particular, a 1-arc is just an arc. Then $\varGamma$ is called (G,s)*-arc-transitive* with $G \le \mathrm{Aut}\,\varGamma$ if G is transitive on the set of s-arcs of $\varGamma$. A (G,s)-arc-transitive graph is called (G,s)-*transitive* if it is not $(G,s+1)$-arc-transitive. In particular, a graph $\varGamma$ is simply called *s-transitive* if it is $(\mathrm{Aut}\,\varGamma, s)$-transitive.

Let F_{20} denote the Frobenius group of order 20. The following lemma determines the stabilizers of pentavalent symmetric graphs, refer to [7, 20].

Lemma 2.5. *Let $\varGamma$ be a pentavalent (G,s)-transitive graph, where $G \le \mathrm{Aut}\,\varGamma$ and $s \ge 1$. Let $\alpha \in V\varGamma$. Then one of the following holds.*

(a) If G_α is soluble, then $s \le 3$ and $|G_\alpha| \,\big|\, 80$. Further, the pair (s, G_α) lies in the

following table.

s	G_α
1	$\mathbb{Z}_5,\ \mathrm{D}_{10},\ \mathrm{D}_{20}$
2	$\mathrm{F}_{20},\ \mathrm{F}_{20}\times\mathbb{Z}_2$
3	$\mathrm{F}_{20}\times\mathbb{Z}_4$

(b) *If G_α is insoluble, then $2\le s\le 5$, and $|G_\alpha|\,\big|\,2^9\cdot 3^2\cdot 5$. Further, the pair (s,G_α) lies in the following table.*

s	G_α	$\|G_\alpha\|$
2	$\mathrm{A}_5,\ \mathrm{S}_5$	60, 120
3	$\mathrm{A}_4\times\mathrm{A}_5,\ (\mathrm{A}_4\times\mathrm{A}_5):\mathbb{Z}_2,\ \mathrm{S}_4\times\mathrm{S}_5$	720, 1440, 2880
4	$\mathrm{ASL}(2,4),\ \mathrm{AGL}(2,4),\ \mathrm{A\Sigma L}(2,4),\ \mathrm{A\Gamma L}(2,4)$	960, 1920, 2880, 5760
5	$\mathbb{Z}_2^6:\Gamma\mathrm{L}(2,4)$	23040

A typical method for studying vertex-transitive graphs is taking normal quotients. Let $\varGamma$ be a G-vertex-transitive graph, where $G\le\mathrm{Aut}\,\varGamma$. Suppose that G has a normal subgroup N which is intransitive on $V\varGamma$. Let $V\varGamma_N$ be the set of N-orbits on $V\varGamma$. The *normal quotient graph* $\varGamma_N$ of $\varGamma$ induced by N is defined as the graph with vertex set $V\varGamma_N$, and B is adjacent to C in $\varGamma_N$ if and only if there exist vertices $\beta\in B$ and $\gamma\in C$ such that β is adjacent to γ in $\varGamma$. In particular, if $\mathrm{val}(\varGamma)=\mathrm{val}(\varGamma_N)$, then $\varGamma$ is called a *normal cover* of $\varGamma_N$.

A graph $\varGamma$ is called *G-locally primitive* if, for each $\alpha\in V\varGamma$, the stabilizer G_α acts primitively on $\varGamma(\alpha)$. Obviously, a pentavalent symmetric graph is locally primitive. The following theorem gives a basic method for studying vertex-transitive locally primitive graphs, see [17, Theorem 4.1] and [12, Lemma 2.5].

Theorem 2.6. *Let $\varGamma$ be a G-vertex-transitive locally primitive graph, where $G\le\mathrm{Aut}\,\varGamma$, and let $N\lhd G$ have at least three orbits on $V\varGamma$. Then the following statements hold.*

(i) N is semi-regular on $V\varGamma$, $G/N\le\mathrm{Aut}\,\varGamma_N$, and $\varGamma$ is a normal cover of $\varGamma_N$;

(ii) $G_\alpha\cong(G/N)_\gamma$, where $\alpha\in V\varGamma$ and $\gamma\in V\varGamma_N$;

(iii) $\varGamma$ is (G,s)-transitive if and only if $\varGamma_N$ is $(G/N,s)$-transitive, where $1\le s\le 5$ or $s=7$.

For reduction, we need some information of pentavalent symmetric graphs of order $4pq$, stated in the following lemma, see [8, Theorem 4.1] and [16, Theorem 3.1].

Lemma 2.7. *Let $\varGamma$ be a pentavalent symmetric graph of order $4pq$, where $q>p\ge 3$ are primes. Then the pair $(\mathrm{Aut}\,\varGamma,(\mathrm{Aut}\,\varGamma)_\alpha)$ lies in the following Table 3, where $\alpha\in V\varGamma$.*

Remark 2.8 (Remarks on Lemma 2.7)**.**

(a) Suppose that $\varGamma$ is one of the graphs in Lemma 2.7 and M is an arc-transitive subgroup of $\mathrm{Aut}\,\varGamma$. Then M is insoluble (for convenience, we prove this conclusion in Lemma 4.4 and we remark that Lemma 4.4 is independent where it is used).

(b) By MAGMA [1], the graphs $\mathcal{C}_{66}^{(2)}$ and $\mathcal{C}_{132}^{5}$ in [8, Theorem 4.1] are isomorphic, $\mathrm{Aut}(\mathcal{C}_{132}^{5})\cong\mathrm{PGL}(2,11)\times\mathbb{Z}_2$.

Table 3: Pentavalent symmetric graphs of order $4pq$.

Γ	(p,q)	$\mathrm{Aut}\,\Gamma$	$(\mathrm{Aut}\,\Gamma)_\alpha$
$\mathcal{C}_{60}$	$(3,5)$	$\mathrm{A}_5 \times \mathrm{D}_{10}$	D_{10}
$\mathcal{C}_{132}^1$	$(3,11)$	$\mathrm{PSL}(2,11) \times \mathbb{Z}_2$	D_{10}
$\mathcal{C}_{132}^i,\ 2 \le i \le 4$	$(3,11)$	$\mathrm{PGL}(2,11)$	D_{10}
$\mathcal{C}_{132}^5$	$(3,11)$	$\mathrm{PGL}(2,11) \times \mathbb{Z}_2$	D_{20}
$\mathcal{C}_{574}^{(2)}$	$(7,41)$	$\mathrm{PSL}(2,41) \times \mathbb{Z}_2$	A_5
$\mathcal{C}_{4108}$	$(13,79)$	$\mathrm{PSL}(2,79)$	A_5

The final lemma of this section gives some information about the pentavalent symmetric graphs of square-free order, refer to [19, Theorem 1.1] and [11, Theorem 1.1].

Lemma 2.9. *Let Γ be a pentavalent symmetric graph of order $2n$, where n is an odd square-free integer and has at least three prime factors. Then one of the following statements holds.*

(1) $\mathrm{Aut}\,\Gamma$ *is soluble and* $\mathrm{Aut}\,\Gamma \cong \mathrm{D}_{2n} : \mathbb{Z}_5$.

(2) $\mathrm{Aut}\,\Gamma = \mathrm{PSL}(2,p)$ *or* $\mathrm{PGL}(2,p)$, *where $p \ge 5$ is a prime.*

(3) The triple $(\Gamma, 2n, \mathrm{Aut}\,\Gamma)$ lies in the following Table 4.

Table 4: Two 'sporadic' pentavalent symmetric graphs.

Γ	$2n$	$\mathrm{Aut}\,\Gamma$	$(\mathrm{Aut}\,\Gamma)_\alpha$
$\mathcal{C}_{390}$	390	$\mathrm{PSL}(2,25)$	F_{20}
$\mathcal{C}_{2926}$	2926	J_1	A_5

3 Some examples

In this section, we give some examples of pentavalent symmetric graphs of order $4n$ with n an odd square-free integer.

In order to construct our graphs we first introduce the definition of a coset graph. Let G be a finite group and let H be a core-free subgroup of G. Let $\tau \in G$ and $\tau^2 \in H$. Define the *coset graph* $\mathrm{Cos}(G,H,\tau)$ of G with respect to H as the graph with vertex set $[G:H]$ such that Hx, Hy are adjacent if and only if $yx^{-1} \in H\tau H$. The following lemma about coset graphs is well known and the proof of the lemma follows from the definition of coset graphs.

Lemma 3.1. *Using the notation as above, the coset graph $\Gamma = \mathrm{Cos}(G,H,\tau)$ is G-arc-transitive graph and*

(1) $\mathrm{val}\,\Gamma = |H : H \cap H^\tau|$;

(2) Γ is connected if and only if $\langle H, \tau\rangle = G$.

Conversely, each G-arc-transitive graph Σ is isomorphic to the coset graph $\mathrm{Cos}(G, G_v, \tau)$, *where $\tau \in \mathbf{N}_G(G_{vw})$ is a 2-element such that $\tau^2 \in G_v$, and $v \in V\Sigma$, $w \in \Sigma(v)$.*

We next introduce the definition of the bipartite double cover of a graph. Let Γ be a graph with vertex set $V\Gamma$. The *standard double cover* of Γ is defined as the undirected bipartite graph $\widetilde{\Gamma}$ with biparts V_0 and V_1, where $V_i = \{(v, i) \mid v \in V\Gamma\}$, such that two vertices $(x, 0)$ and $(y, 1)$ are adjacent if and only if x, y are adjacent in Γ. It is easily shown that the standard double cover can be represented as a direct product: $\widetilde{\Gamma} = \Gamma \times \mathrm{K}_2$. Furthermore, $\widetilde{\Gamma}$ is connected if and only if Γ is connected and non-bipartite.

For a given small permutation group X, we may determine all graphs which admit X as an arc-transitive automorphism group by using MAGMA [1]. It is then easy to have the following result.

Example 3.2.

(1) There is a unique pentavalent symmetric graph of order 5852 which admits $\mathrm{J}_1 \times \mathbb{Z}_2$ as an arc-transitive automorphism group; and its full automorphism group is $\mathrm{J}_1 \times \mathbb{Z}_2$. This graph is denoted by $\mathcal{C}_{5832}$ which satisfies the conditions in Row 6 of Table 1.

(2) There are five pentavalent symmetric graphs of order 17556 admitting J_1 as an arc-transitive automorphism group; and their full automorphism group are all isomorphic to J_1. These five graphs are denoted by $\mathcal{C}^i_{17556}$ which satisfy the conditions in Row 1 to Row 5 of Table 1, where $1 \le i \le 5$.

(3) There are three pentavalent symmetric graphs of order 780 which admit $\mathrm{PSL}(2, 25) \times \mathbb{Z}_2$ as an arc-transitive automorphism group; and their full automorphism group are all isomorphic to $\mathrm{PSL}(2, 25) \times \mathbb{Z}_2$. These three graphs are denoted by $\mathcal{C}^j_{780}$ which satisfy the conditions in Row 7 to Row 9 of Table 1, where $1 \le j \le 3$.

Remark 3.3 (Remarks on Example 3.2)**.**

(a) Let Γ be a pentavalent symmetric graph of order $4n$ with n an odd square-free integer and having at least three prime factors. Then the graphs appearing in Example 3.2 are the only sporadic graphs of such Γ. In fact, let $\mathrm{A} = \mathrm{Aut}\,\Gamma$. If A is insoluble and has no nontrivial soluble normal subgroup, then Lemma 4.2 shows that $\mathcal{C}^i_{17556}$ with $1 \le i \le 5$ are the only sporadic graphs. If A is insoluble and has a soluble minimal normal subgroup $N = \mathbb{Z}_2$, then Lemma 4.3 shows that $\mathcal{C}_{5832}$ and $\mathcal{C}^j_{780}$ with $1 \le j \le 3$ are the only sporadic graphs. If A is soluble or has a soluble minimal normal subgroup $N = \mathbb{Z}_r$ with $r > 2$, then Lemma 4.1 and Lemma 4.6 show that no such Γ exists.

(b) Since both $\mathcal{C}_{2926}$ and $\mathcal{C}_{390}$ are non-bipartite, the bipartite double cover of both $\mathcal{C}_{2926}$ and $\mathcal{C}_{390}$ is connected pentavalent symmetric graph of order $4n$. In fact, the graph $\mathcal{C}_{5832}$ is isomorphic to the bipartite double cover of $\mathcal{C}_{2926}$ and the graph $\mathcal{C}^3_{780}$ is isomorphic to the bipartite double cover of $\mathcal{C}_{390}$.

Example 3.4. Let p be a prime such that

$$p \equiv 49, 79, 81, 111 \pmod{160}$$

and let $\mathrm{A} = \mathrm{PSL}(2, p)$. Then by Lemma 2.2, A has a subgroup $H \cong \mathrm{A}_5$. Let $K < H$ with $K \cong \mathrm{A}_4$. Then $\mathbf{N}_\mathrm{A}(K) = K : \langle \tau \rangle \cong \mathrm{S}_4$, where $\tau \in \mathrm{A} - H$ is an involution. Let $\Gamma = \mathrm{Cos}(\mathrm{A}, H, H\tau H)$. Then Γ is a connected pentavalent symmetric graph.

Example 3.5. Let p be a prime such that

$$p \equiv 9, 39, 41, 71 \pmod{80}$$

and let $\mathrm{A} = \mathrm{PGL}(2, p)$. Then by Lemma 2.2 and Lemma 2.3, A has a subgroup $H \cong \mathrm{A}_5$. Let $K < H$ with $K \cong \mathrm{A}_4$. Then $\mathbf{N}_{\mathrm{A}}(K) = K : \langle\tau\rangle \cong \mathrm{S}_4$ is a maximal subgroup of A, where $\tau \in \mathrm{A} - H$ is an involution, and so $\langle H, \tau\rangle = \mathrm{A}$. Let $\Gamma = \mathrm{Cos}(\mathrm{A}, H, H\tau H)$. Then Γ is a connected pentavalent symmetric graph.

Example 3.6. Let p be a prime such that

$$p \equiv 9, 39, 41, 71 \pmod{80}$$

and let $\mathrm{A} = \mathrm{PSL}(2, p) \times \mathbb{Z}_2 = T \times \langle z\rangle$, where $T = \mathrm{PSL}(2, p)$ and $\langle z\rangle = \mathbb{Z}_2$. Then T has a subgroup $H \cong \mathrm{A}_5$. Let $K < H$ with $K \cong \mathrm{A}_4$. Then $\mathbf{N}_{\mathrm{A}}(K) = K : \langle\tau\rangle \times \langle z\rangle \cong \mathrm{S}_4 \times \mathbb{Z}_2$, where $\tau \in T - H$ is an involution. Let $\Gamma = \mathrm{Cos}(\mathrm{A}, H, H\tau z H)$. Then Γ is a connected pentavalent symmetric graph.

Example 3.7. Let p be a prime such that

$$p \equiv 11, 19, 21, 29 \pmod{40}$$

and let $\mathrm{A} = \mathrm{PGL}(2, p) \times \mathbb{Z}_2 = T \times \langle z\rangle$, where $T = \mathrm{PGL}(2, p)$ and $\langle z\rangle = \mathbb{Z}_2$. Then T has a subgroup $H \cong \mathrm{A}_5$. Let $K < H$ with $K \cong \mathrm{A}_4$. Then $\mathbf{N}_{\mathrm{A}}(K) = K : \langle\tau\rangle \times \langle z\rangle \cong \mathrm{S}_4 \times \mathbb{Z}_2$, where $\tau \in T - H$ is an involution. Let $\Gamma = \mathrm{Cos}(\mathrm{A}, H, H\tau z H)$. Then Γ is a connected pentavalent symmetric graph.

4 Proof of Theorem 1.1

Let n be an odd square-free integer and n has at least three prime factors. Let Γ be a pentavalent symmetric graph of order $4n$. Set $\mathrm{A} = \mathrm{Aut}\,\Gamma$. By Lemma 2.5, $|\mathrm{A}_\alpha| \bigm| 2^9 \cdot 3^2 \cdot 5$, and hence $|\mathrm{A}| \bigm| 2^{11} \cdot 3^2 \cdot 5 \cdot n$. Assume that $n = p_1 p_2 \cdots p_s$, where $s \geq 3$ and p_i's are distinct primes.

Lemma 4.1. *The group* A *is insoluble.*

Proof. Suppose to the contrary that A is soluble. Let F be the Fitting subgroup of A. By Lemma 2.1, $F \neq 1$ and $\mathbf{C}_{\mathrm{A}}(F) \leq F$. Further, $F = \mathbf{O}_2(\mathrm{A}) \times \mathbf{O}_{p_1}(\mathrm{A}) \times \mathbf{O}_{p_2}(\mathrm{A}) \times \cdots \times \mathbf{O}_{p_s}(\mathrm{A})$, where $\mathbf{O}_2(\mathrm{A})$, $\mathbf{O}_{p_1}(\mathrm{A})$, $\mathbf{O}_{p_2}(\mathrm{A})$, $\ldots$, $\mathbf{O}_{p_s}(\mathrm{A})$ denote the largest normal 2-, p_1-, p_2-, $\ldots$, p_s-subgroups of A, respectively.

For each $p_i \in \{p_1, p_2, \ldots, p_s\}$, $\mathbf{O}_{p_i}(\mathrm{A})$ has at least three orbits on $V\Gamma$, by Theorem 2.6, $\mathbf{O}_{p_i}(\mathrm{A})$ is semi-regular on $V\Gamma$. Therefore, F is semi-regular on $V\Gamma$ and so $|F|$ divides $|V\Gamma| = 4n$. Since $n = p_1 p_2 \cdots p_s$, we have $\mathbf{O}_{p_i}(\mathrm{A}) \leq \mathbb{Z}_{p_i}$. This argument also proves $\mathbf{O}_2(\mathrm{A}) \leq \mathbb{Z}_4$ or $\mathbb{Z}_2^2$. If $\mathbf{O}_2(\mathrm{A}) = \mathbb{Z}_4$ or $\mathbb{Z}_2^2$, then by Theorem 2.6, the normal quotient graph $\Gamma_{\mathbf{O}_2(\mathrm{A})}$ is a pentavalent symmetric graph of odd order, which is a contradiction. Thus, $\mathbf{O}_2(\mathrm{A}) \leq \mathbb{Z}_2$, $F \cong \mathbb{Z}_m$, where $m \bigm| 2n$. It implies that $\mathbf{C}_{\mathrm{A}}(F) \geq F$, and so $\mathbf{C}_{\mathrm{A}}(F) = F$.

If F has at least three orbits on $V\Gamma$, then, by Theorem 2.6, Γ_F is A/F-arc-transitive. Since $\mathrm{A}/F = \mathrm{A}/\mathbf{C}_{\mathrm{A}}(F) \leq \mathrm{Aut}(F)$ is abelian, we have $(\mathrm{A}/F)_\delta = 1$, where $\delta \in V\Gamma_F$, which is a contradiction.

Thus, F has at most two orbits on $V\Gamma$. If F is transitive on $V\Gamma$, then F is regular on $V\Gamma$, a contradiction with $F \cong \mathbb{Z}_m$, where $m \mid 2n$. Hence F has two orbits on $V\Gamma$ and $F \cong \mathbb{Z}_{2n}$. Let $K = \mathbf{O}_{p_3}(\mathrm{A}) \times \mathbf{O}_{p_4}(\mathrm{A}) \times \cdots \times \mathbf{O}_{p_s}(\mathrm{A})$. Then $K \cong \mathbb{Z}_{p_3p_4\cdots p_s}$. Since $K \trianglelefteq \mathrm{A}$ has $4p_1p_2$ orbits on $V\Gamma$, by Theorem 2.6(i), Γ_K is an A/K-arc-transitive pentavalent graph of order $4p_1p_2$, and hence Γ_K satisfies the conditions in Table 3. Since A/K is soluble, by Remark 2.8, a contradiction occurs. Hence A is insoluble. This completes the proof of the Lemma. □

We now consider the case where A is insoluble and has no nontrivial soluble normal subgroup.

Lemma 4.2. *Assume that* A *is insoluble and has no nontrivial soluble normal subgroup. Then* $\mathrm{Aut}\,\Gamma \cong \mathrm{J}_1$, $\mathrm{PSL}(2,p)$ *or* $\mathrm{PGL}(2,p)$ *with* $p \geq 29$*. Further, if* $\mathrm{Aut}\,\Gamma \cong \mathrm{J}_1$*, then* $\Gamma \cong \mathcal{C}^i_{17556}$ *satisfies the conditions in Row 1 to Row 5 of Table 1 of Theorem 1.1, where* $1 \leq i \leq 5$*. If* $\mathrm{Aut}\,\Gamma \cong \mathrm{PSL}(2,p)$ *or* $\mathrm{PGL}(2,p)$*, then* Γ *satisfies the conditions in Table 5.*

Table 5: $\mathrm{Aut}\,\Gamma$ is almost simple.

$\mathrm{Aut}\,\Gamma$	$(\mathrm{Aut}\,\Gamma)_\alpha$	Γ	Remark
$\mathrm{PSL}(2,p)$	A_5	Example 3.4	$p \equiv 49, 79, 81, 111 \pmod{160}$
$\mathrm{PGL}(2,p)$	A_5	Example 3.5	$p \equiv 9, 39, 41, 71 \pmod{80}$
$\mathrm{PSL}(2,p)$	D_{10}		$p \equiv 9, 39, 41, 71 \pmod{80}$
$\mathrm{PGL}(2,p)$	D_{10}		$p \equiv 11, 19, 21, 29 \pmod{40}$
$\mathrm{PSL}(2,p)$	D_{20}		$p \equiv 49, 79, 81, 111 \pmod{160}$
$\mathrm{PGL}(2,p)$	D_{20}		$p \equiv 9, 39, 41, 71 \pmod{80}$

Proof. Let N be the socle of A. Then N is insoluble and 4 divides $|N|$. If N has more than three orbits on $V\Gamma$, then by Theorem 2.6, Γ_N is a pentavalent symmetric graph of odd order, a contradiction. Hence, N has at most two orbits on $V\Gamma$, so $2n$ divides $|N|$.

Assume that A has at least two minimal normal subgroups N_1 and N_2. Then by a similar argument as above, we have that $2n$ divides both $|N_1|$ and $|N_2|$. Hence $4n^2$ divides $|\mathrm{A}| = 2^{11} \cdot 3^2 \cdot 5 \cdot n$, and so n divides $2^9 \cdot 3^2 \cdot 5$. It implies that $n = 3 \cdot 5$, a contradiction with n having at least three prime factors. So A has a unique minimal normal subgroup and we may write $N = S^d$, where S is a nonabelian simple group and $d \geq 1$.

Since $p_s > 5$, p_s divides $|N|$ and p_s^2 does not divide $|N|$ as $|\mathrm{A}| \mid 2^{11} \cdot 3^2 \cdot 5 \cdot p_1p_2 \cdots p_s$, we conclude that $d = 1$ and $N = S$ is a nonabelian simple group. Hence A is almost simple with socle S.

If $S_\alpha = 1$, then S acts regularly on $V\Gamma$. Hence S is a non-abelian simple group such that $|S| = 4n$. By checking the orders of nonabelian simple groups (see [6, pp. 135–136] for example), we have that $S = \mathrm{PSL}(2,p)$ and so $\mathrm{A} \leq \mathrm{Aut}(S) = \mathrm{PGL}(2,p)$, which is impossible as A is transitive on $A\Gamma$, $|\mathrm{A}| \leq 2|S|$ and $|A\Gamma| = 5|S|$. Hence $S_\alpha \neq 1$. Since Γ is connected and $S \lhd \mathrm{A}$, we have $1 \neq S_\alpha^{\Gamma(\alpha)} \lhd \mathrm{A}_\alpha^{\Gamma(\alpha)}$, it follows that $5 \mid |S_\alpha|$, we thus have $10 \cdot p_1p_2 \cdots p_s$ divides $|S|$.

Thus, $\mathrm{soc}(\mathrm{A}) = S$ is a nonabelian simple group such that $|S| \bigm| 2^{11} \cdot 3^2 \cdot 5 \cdot n$ and $10 \cdot n \bigm| |S|$. Hence the triple $(S, |S|, n)$ lies in Table 2 of Lemma 2.4. We will analyse all the candidates one by one in the following.

Assume $(S, n) = (\mathrm{J}_1, 3 \cdot 7 \cdot 11 \cdot 19)$. Then $|V\Gamma| = 17556$ and $\mathrm{A} \cong \mathrm{J}_1$ as $\mathrm{Out}(\mathrm{J}_1) = 1$. It then follows from Example 3.2 that $\Gamma \cong \mathcal{C}^i_{17556}$ satisfies the conditions in Row 1 to Row 5 of Table 1 of Theorem 1.1, where $1 \leq i \leq 5$.

Assume $(S, n) = (\mathrm{Sz}(32), 5 \cdot 31 \cdot 41)$. Since $\mathrm{Out}(\mathrm{Sz}(32)) \cong \mathbb{Z}_5$ (see Atlas [3] for example), $\mathrm{A} \cong \mathrm{Sz}(32)$ or $\mathrm{Sz}(32).\mathbb{Z}_5$, so $|\mathrm{A}_\alpha| = \frac{|\mathrm{A}|}{4n} = 1280$ or 6400, which is not possible by Lemma 2.5. Similarly, for the case $(S, n) = (\mathrm{PSL}(5, 2), 3 \cdot 7 \cdot 31)$, then $\mathrm{A} \cong \mathrm{PSL}(5, 2)$ or $\mathrm{PSL}(5, 2).\mathbb{Z}_2$ as $\mathrm{Out}(\mathrm{PSL}(5, 2)) \cong \mathbb{Z}_2$. Thus, $|\mathrm{A}_\alpha| = \frac{|\mathrm{A}|}{4n} = 3840$ or 7680, which is impossible by Lemma 2.5. For the case where $(S, n) = (\mathrm{PSL}(2, 2^8), 3 \cdot 17 \cdot 257)$, since $\mathrm{A} \cong \mathrm{PSL}(2, 2^8).O$, where $O \leq \mathrm{Out}(\mathrm{PSL}(2, 2^8)) \cong \mathbb{Z}_8$, we have $|\mathrm{A}_\alpha| = \frac{|\mathrm{A}|}{4n} = 2^k \cdot 5$, where $6 \leq k \leq 9$, which is also impossible by Lemma 2.5. For the case where $(S, n) = (\mathrm{PSU}(3, 4), 3 \cdot 5 \cdot 13)$, since $\mathrm{A} \cong \mathrm{PSU}(3, 4).O$, where $O \leq \mathrm{Out}(\mathrm{PSU}(3, 4)) \cong \mathbb{Z}_4$, we have $|\mathrm{A}_\alpha| = \frac{|\mathrm{A}|}{4n} = 2^k \cdot 5$, where $4 \leq k \leq 6$, which is impossible by Lemma 2.5.

Assume $(S, n) = (\mathrm{PSp}(4, 4), 3 \cdot 5 \cdot 17)$. Since $S \leq \mathrm{A} \leq \mathrm{Aut}(S) \cong \mathrm{PSp}(4, 4).\mathbb{Z}_4$, we have $|\mathrm{A}_\alpha| = \frac{|\mathrm{A}|}{4n} = 960, 1920$ or 3840. If $|\mathrm{A}_\alpha| = 960$ or 1920, then by Lemma 2.5, $\mathrm{A}_\alpha \cong \mathrm{ASL}(2, 4)$ or $\mathrm{A\Sigma L}(2, 4)$. However, by Atlas [3], $\mathrm{PSp}(4, 4)$ has no subgroup isomorphic to $\mathrm{ASL}(2, 4)$ and $\mathrm{PSp}(4, 4).\mathbb{Z}_2$ has no subgroup isomorphic to $\mathrm{A\Sigma L}(2, 4)$. If $|\mathrm{A}_\alpha| = 3840$, then also by Lemma 2.5, a contradiction occurs.

Assume $(S, n) = (\mathrm{PSL}(2, 2^6), 3 \cdot 7 \cdot 13)$. Recall that S has at most two orbits on $V\Gamma$, $|S_\alpha| = \frac{|S|}{4n} = 240$ or $\frac{|S|}{2n} = 480$. However, by Lemma 2.2, $\mathrm{PSL}(2, 2^6)$ has no maximal subgroup with order a multiple of 240, a contradiction occurs. Similarly, for the case $(S, n) = (\mathrm{J}_2, 3 \cdot 5 \cdot 7)$, then $|S_\alpha| = \frac{|S|}{4n} = 2880$ or $\frac{|S|}{2n} = 5760$. By Atlas [3], J_2 has no maximal subgroup with order a multiple of 2880, a contradiction also occurs.

Assume $S \cong \mathrm{M}_{23}$. Then $n = 3 \cdot 7 \cdot 11 \cdot 23$ or $7 \cdot 11 \cdot 23$, and as $\mathrm{Out}(\mathrm{M}_{23}) = 1$, we have $\mathrm{A} = S$ and $|\mathrm{A}_\alpha| = \frac{|\mathrm{M}_{23}|}{4n} = 480$ or 1440. By Lemma 2.5, it is impossible for the case $|\mathrm{A}_\alpha| = 480$. For the latter case, by a direct computation using MAGMA [1], no graph Γ exists. If $(S, n) = (\mathrm{M}_{22}, 7 \cdot 11 \cdot 23)$, as $\mathrm{Out}(\mathrm{M}_{22}) \cong \mathbb{Z}_2$, we have $\mathrm{A} \cong \mathrm{M}_{22}$ or $\mathrm{M}_{22}.\mathbb{Z}_2$, so $|\mathrm{A}_\alpha| = \frac{|\mathrm{A}|}{4n} = 480$ or 960, a computation by MAGMA [1] shows that no graph Γ exists. Similarly, we can exclude the case where $(S, n) = (\mathrm{PSL}(2, 25), 3 \cdot 5 \cdot 13)$ by MAGMA [1].

Assume $(S, n) = (\mathrm{M}_{24}, 3 \cdot 7 \cdot 11 \cdot 23)$ or $(\mathrm{J}_1, 3 \cdot 7 \cdot 11 \cdot 19)$. Since $\mathrm{Out}(\mathrm{M}_{24}) = \mathrm{Out}(\mathrm{J}_1) = 1$, we always have $\mathrm{A} = S$. Hence $|\mathrm{A}_\alpha| = \frac{|\mathrm{A}|}{4n} = 11520$ or 10. A computation by MAGMA [1] also shows that no graph Γ exists.

Finally, assume $S \cong \mathrm{PSL}(2, p)$ with $p \geq 29$ a prime. Then $\mathrm{A} \cong \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$. By Lemma 2.2, Lemma 2.3 and Lemma 2.5, we have $\mathrm{A}_\alpha \cong \mathbb{Z}_5$, D_{10}, D_{20} or A_5. If $\mathrm{A}_\alpha \cong \mathbb{Z}_5$, then Γ is an arc-regular pentavalent graph of order four times an odd square-free integer. However, by [15, Theorem 1.1], no such Γ exists. Hence $\mathrm{A}_\alpha \cong \mathrm{D}_{10}$, D_{20} or A_5. If $\mathrm{A}_\alpha \cong \mathrm{A}_5$, then by Lemma 2.2 and Lemma 2.3, we have $p \equiv \pm 1 \pmod 5$. Since $|\mathrm{A} : \mathrm{A}_\alpha| = 4n$, we have $|\mathrm{A}|$ is divisible by 16, but not by 32. Since $|\mathrm{A}| = |\mathrm{PSL}(2, p)| = \frac{p(p-1)(p+1)}{2}$ or $|\mathrm{PGL}(2, p)| = p(p-1)(p+1)$, we have $p \equiv \pm 15 \pmod{32}$ for $\mathrm{A} \cong \mathrm{PSL}(2, p)$ or $p \equiv \pm 7 \pmod{16}$ for $\mathrm{A} \cong \mathrm{PGL}(2, p)$. Since $p \equiv \pm 1 \pmod 5$, we have $p \equiv 49, 79, 81, 111 \pmod{160}$ for $\mathrm{A} \cong \mathrm{PSL}(2, p)$ or $p \equiv 9, 39, 41, 71 \pmod{80}$ for $\mathrm{A} \cong \mathrm{PGL}(2, p)$. These graphs are constructed in Example 3.4 and Example 3.5. Similarly, if $\mathrm{A}_\alpha \cong \mathrm{D}_{10}$ or D_{20}, then p satisfies the condition in Table 5. This completes the proof of the Lemma. □

We next assume that A has a nontrivial soluble normal subgroup. Let N be a soluble minimal normal subgroup of A. Then there exists a prime $r \mid 4n$ such that $N \cong \mathbb{Z}_r^d$. Further, N has at least three orbits on $V\Gamma$. It follows from Theorem 2.6 that N is semi-regular on $V\Gamma$, and so $|N| = |\mathbb{Z}_r|^d \mid |V\Gamma| = 4n$. If $d \geq 2$, then $(r, d) = (2, 2)$. It follows that Γ_N is an arc-transitive graph of odd order, a contradiction. Hence $d = 1$, $N = \mathbb{Z}_r$. The next lemma consider the case where $r = 2$.

Lemma 4.3. *Assume that* A *is insoluble and has a soluble minimal normal subgroup* $N = \mathbb{Z}_2$. *Then one of the following statements holds:*

(1) $\mathrm{Aut}\,\Gamma \cong \mathrm{PSL}(2,p)\times\mathbb{Z}_2$ *or* $\mathrm{PGL}(2,p)\times\mathbb{Z}_2$, *where* $p \geq 29$ *is a prime. Furthermore,* Γ *satisfies the conditions in Table 6.*

(2) $\mathrm{Aut}\,\Gamma \cong \mathrm{PSL}(2,25)\times\mathbb{Z}_2$ *and* Γ *is isomorphic to* $\mathcal{C}_{780}^i$ *in Table 1, where* $1 \leq i \leq 3$.

(3) $\mathrm{Aut}\,\Gamma \cong \mathrm{J}_1\times\mathbb{Z}_2$ *and* Γ *is isomorphic to* $\mathcal{C}_{5852}$ *in Table 1.*

Table 6: $\mathrm{Aut}\,\Gamma$ has a normal subgroup isomorphic to $\mathbb{Z}_2$.

$\mathrm{Aut}\,\Gamma$	$(\mathrm{Aut}\,\Gamma)_\alpha$	Γ	Remark
$\mathrm{PSL}(2,p)\times\mathbb{Z}_2$	A_5	Example 3.6	$p \equiv 9, 39, 41, 71 \pmod{80}$
$\mathrm{PGL}(2,p)\times\mathbb{Z}_2$	A_5	Example 3.7	$p \equiv 11, 19, 21, 29 \pmod{40}$
$\mathrm{PSL}(2,p)\times\mathbb{Z}_2$	D_{10}		$p \equiv 11, 19, 21, 29 \pmod{40}$
$\mathrm{PSL}(2,p)\times\mathbb{Z}_2$	D_{20}		$p \equiv 9, 39, 41, 71 \pmod{80}$
$\mathrm{PGL}(2,p)\times\mathbb{Z}_2$	D_{20}		$p \equiv 11, 19, 21, 29 \pmod{40}$

Proof. Since N has more than three orbits on $V\Gamma$, then by Theorem 2.6, Γ_N is an A/N-arc-transitive pentavalent graph of order $\bar{n} = 2n$. It follows that Γ_N is isomorphic to one of the graphs in Lemma 2.9. Since $\mathrm{A}/N \leq \mathrm{Aut}\,\Gamma_N$ and A/N is insoluble, we have that $\mathrm{Aut}\,\Gamma_N$ is insoluble and so $\mathrm{Aut}\,\Gamma_N \cong \mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,25)$ or J_1. Let $\bar{\mathrm{A}} := \mathrm{Aut}\,\bar{\Gamma}$.

Suppose that $\bar{\mathrm{A}} \cong \mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$. Since A/N is insoluble, by Lemma 2.2 and Lemma 2.3, A/N is isomorphic to A_5, $\mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$. If $\mathrm{A}/N \cong \mathrm{A}_5$, then since Γ_N is an A/N-arc-transitive pentavalent graph of order $\bar{n} = 2n$, we have $2n\cdot 5 \mid |\mathrm{A}_5|$. It implies that n divides 6, a contradiction with n having at least three odd prime factors. Thus, A/N is isomorphic to $\mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$. Therefore, $\mathrm{A} \cong N.\mathrm{PSL}(2,p)$ or $N.\mathrm{PGL}(2,p)$, that is, $\mathrm{A} \cong \mathrm{PSL}(2,p)\times\mathbb{Z}_2$, $\mathrm{SL}(2,p)$, $\mathrm{PGL}(2,p)\times\mathbb{Z}_2$ or $\mathrm{SL}(2,p).\mathbb{Z}_2$. Assume first that $\mathrm{A} \cong \mathrm{SL}(2,p)$. Note that $\mathrm{SL}(2,p)$ has a unique central involution. Then by Lemma 2.5, $\mathrm{A}_\alpha \cong \mathbb{Z}_5$. It follows that $|V\Gamma| = |\mathrm{A} : \mathrm{A}_\alpha|$ is divisible by 8 as $|\,\mathrm{SL}(2,p)|$ is divisible by 8, a contradiction. Assume next that $\mathrm{A} \cong \mathrm{SL}(2,p).\mathbb{Z}_2$. Then A contains a normal subgroup H isomorphic to $\mathrm{SL}(2,p)$. Since $8 \mid |H|$, we have $H_\alpha \neq 1$. By Theorem 2.6, H has at most two orbits on $V\Gamma$ and so $\frac{|\mathrm{A}_\alpha|}{|H_\alpha|} \mid 2$. If H is transitive on $V\Gamma$, then H is arc-transitive. A similar argument with the case $\mathrm{A} \cong \mathrm{SL}(2,p)$, a contradiction occurs. Therefore, H has two orbits on $V\Gamma$ and so $H_\alpha = \mathrm{A}_\alpha$. Since H has a unique central involution, by Lemma 2.5, $\mathrm{A}_\alpha \cong \mathbb{Z}_5$, it follows that $|V\Gamma| = |\mathrm{A} : \mathrm{A}_\alpha|$ is divisible by 16, a contradiction. Therefore, $\mathrm{A} \cong \mathrm{PSL}(2,p)\times\mathbb{Z}_2$ or $\mathrm{PGL}(2,p)\times\mathbb{Z}_2$ in this case. By a similar argument as for the case $\mathrm{A} \cong \mathrm{PSL}(2,p)$ (the last paragraph in the proof of Lemma 4.2), we have that Γ satisfies the condition in Table 6. Note that since 16 divides $|\,\mathrm{PGL}(2,p) \times \mathbb{Z}_2|$ and $|\mathrm{A} : \mathrm{A}_\alpha| = 4n$, we have $(\mathrm{A}, \mathrm{A}_\alpha) \neq (\mathrm{PGL}(2,p) \times \mathbb{Z}_2, \mathrm{D}_{10})$.

Suppose that $\bar{\mathrm{A}} \cong \mathrm{PSL}(2,25)$. Since Γ_N is A/N-arc-transitive, we have that $5 \cdot 390 \,\big|\, |\mathrm{A}/N|$. By checking the maximal subgroup of $\mathrm{PSL}(2,25)$ (see Atlas [3] for example), we have that $\mathrm{A}/N = \bar{\mathrm{A}} \cong \mathrm{PSL}(2,25)$. It follows that $\mathrm{A} \cong \mathrm{SL}(2,25)$ or $\mathrm{PSL}(2,25) \times \mathbb{Z}_2$. If $\mathrm{A} \cong \mathrm{PSL}(2,25) \times \mathbb{Z}_2$, then by Example 3.2, $\Gamma \cong \mathcal{C}^i_{780}$ in Table 1, where $1 \leq i \leq 3$. If $\mathrm{A} \cong \mathrm{SL}(2,25)$, then by MAGMA [1], no graph Γ exists.

Suppose that $\bar{\mathrm{A}} \cong \mathrm{J}_1$. Similarly, since Γ_N is A/N-arc-transitive, we have that $5 \cdot 2926 \,\big|\, |\mathrm{A}/N|$. By checking the maximal subgroup of J_1 (see Atlas [3] for example), we have that $\mathrm{A}/N = \bar{\mathrm{A}} \cong \mathrm{J}_1$. Since the Schur multiplier of J_1 is $\mathbb{Z}_1$, $\mathrm{A} \cong N.\mathrm{J}_1 \cong \mathrm{J}_1 \times \mathbb{Z}_2$. By Example 3.2, $\Gamma \cong \mathcal{C}_{5852}$ in Table 1. $\square$

Finally, suppose that $r > 2$. We first prove the following lemma.

Lemma 4.4. *Let Σ be a graph. Assume that Σ is isomorphic to one of the graphs appearing in Lemma 2.7, in Lemma 4.2 or in Lemma 4.3. If M is an arc-transitive subgroup of* $\mathrm{Aut}\,\Sigma$, *then M contains the derived subgroup of* $\mathrm{Aut}\,\Sigma$.

Proof. Let Σ be a graph and isomorphic to one of the graphs appearing in Lemma 2.7, in Lemma 4.2 or in Lemma 4.3. Let M be an arc-transitive subgroup of $B = \mathrm{Aut}\,\Sigma$. Then $B = MB_{\alpha\beta}$, where $(\alpha, \beta) \in A\Sigma$. In particular, $m := |B : M|$ divides $|B_{\alpha\beta}|$. Assume first that Σ is isomorphic to one of the graphs appearing in Lemma 2.7. Then, in the first three rows of Table 3, we have that M has index at most two, and for the fourth row M has index at most four, so in particular, M contains B'. For the last two rows, we have that $m \,\big|\, 12$. Since there is no faithful representation of B in degree m for $2 < m \leq 12$, we have $1 \leq m \leq 2$ and so M also contains B'.

Now assume that Σ is isomorphic to one of the graphs appearing in Lemma 4.2 or in Lemma 4.3. Then B is isomorphic to one of the groups $\mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p) \times \mathbb{Z}_2$, $\mathrm{PGL}(2,p) \times \mathbb{Z}_2$, J_1, $\mathrm{J}_1 \times \mathbb{Z}_2$ or $\mathrm{PSL}(2,25) \times \mathbb{Z}_2$ with $p \geq 29$. If $B \cong \mathrm{J}_1$, then M has index at most two. If $B \cong \mathrm{J}_1 \times \mathbb{Z}_2$, then M has index at most 12. If $B \cong \mathrm{PSL}(2,25) \times \mathbb{Z}_2$, then M has index at most four. For these three cases, by a similar argument as above, we also have M contains B'. If $B \cong \mathrm{PSL}(2,p)$, then since $p \,\big|\, n$ and $20n \,\big|\, |M|$, by Lemma 2.2, $M \leq \mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}}$ or $M = B \cong \mathrm{PSL}(2,p)$. If $M \leq \mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}}$, then $M \cong \mathbb{Z}_p : \mathbb{Z}_l$ for some $l \,\big|\, \frac{p-1}{2}$. Thus, M has a normal subgroup, say $S \cong \mathbb{Z}_p$, which has more than three orbits on $V\Sigma$. It then follows from Theorem 2.6 that the normal quotient graph Σ_S is M/S-arc-transitive, a contradiction occurs as $M/S \cong \mathbb{Z}_l$ is cyclic. Hence, $M \not\leq \mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}}$ and so $M = B' \cong \mathrm{PSL}(2,p)$. If $B \cong \mathrm{PGL}(2,p)$, then since $20n \,\big|\, |M|$, by Lemma 2.3, $\mathrm{M} \leq \mathbb{Z}_p : \mathbb{Z}_{p-1}$, $M \leq \mathrm{PSL}(2,p)$ or $M = B \cong \mathrm{PGL}(2,p)$. With a similar argument, we can conclude that $M \geq B' \cong \mathrm{PSL}(2,p)$. Similarly, we can further show that $M \geq B' \cong \mathrm{PSL}(2,p)$ for the case $B \cong \mathrm{PSL}(2,p) \times \mathbb{Z}_2$ or $\mathrm{PGL}(2,p) \times \mathbb{Z}_2$. $\square$

Now assume that A has a soluble minimal normal subgroup $N = \mathbb{Z}_r$ for $r > 2$.

Lemma 4.5. *Assume that* A *has a soluble minimal normal subgroup $N = \mathbb{Z}_r$ for $r > 2$. Then the normal quotient Γ_N is not isomorphic to any graph appearing in Lemma 2.7, Lemma 4.2 or Lemma 4.3.*

Proof. Suppose to the contrary that Γ_N is isomorphic to one of the graphs appearing in Lemma 2.7, Lemma 4.2 or Lemma 4.3. Let $M/N = (\mathrm{Aut}\,\Gamma_N)'$, and let $\Omega := \{\mathrm{PSL}(2,p), \mathrm{J}_1, \mathrm{PSL}(2,25), \mathrm{A}_5\}$. By checking the graphs appearing in Lemma 2.7, in Lemma 4.2 or in Lemma 4.3, we have that $\mathrm{Aut}\,\Gamma_N$ is isomorphic to one of the groups $\mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p)\times\mathbb{Z}_2$, $\mathrm{PGL}(2,p)\times\mathbb{Z}_2$, J_1, $\mathrm{J}_1\times\mathbb{Z}_2$, $\mathrm{PSL}(2,25)\times\mathbb{Z}_2$ or $\mathrm{A}_5\times\mathrm{D}_{10}$. Thus, M/N is isomorphic to one of the groups in Ω. Since the order of the Schur multiplier of a group in Ω is less than or equal to 2 (see [10, Theorem 7.1.1] for $\mathrm{PSL}(2,p)$ and Atlas [3] for the others) and $r > 2$, we have that $M' \in \Omega$.

By Theorem 2.6, $\mathrm{A}/N \le \mathrm{Aut}\,\Gamma_N$ is transitive on $A\Gamma_N$. It follows from Lemma 4.4 that A/N contains the derived subgroup of $\mathrm{Aut}\,\Gamma_N$, that is, $M/N \le \mathrm{A}/N$. Since $M/N \trianglelefteq \mathrm{Aut}\,\Gamma_N$, we have $M/N \trianglelefteq \mathrm{A}/N$. Therefore, M' char $M \trianglelefteq \mathrm{A}$, it implies that $M' \trianglelefteq \mathrm{A}$. If M' has more than three orbits on $V\Gamma$, then by Theorem 2.6, $\Gamma_{M'}$ is a pentavalent symmetric graph of odd order, a contradiction. Thus, M' has at most two orbits on $V\Gamma$ and so $2n$ divides $|M'|$. Let $\bar{\mathrm{A}} := \mathrm{Aut}\,\Gamma_N$, $\bar{n} := \frac{n}{r}$ and $\bar{M} := M/N$. Then $M' \cong \bar{M}$.

Let ρ be the bijection from the orbits of M' on $V\Gamma$ to the orbits of $\bar{M}$ on $V\Gamma_N$ defined by:

$$\alpha^{M'} \to \delta^{\bar{M}}, \quad \text{where} \quad \alpha \in V\Gamma \quad \text{and} \quad \delta = \alpha^N \in V\Gamma_N.$$

Then we can conclude that, for some $k \in \{2,4\}$, $|M' : (M')_\alpha| = kn$ and $|\bar{M} : \bar{M}_\delta| = \frac{kn}{r}$. It gives $|(M')_\alpha| r = |\bar{M}_\delta|$. Since $|\bar{M}_\delta| \,\big|\, |\bar{\mathrm{A}}_\delta|$ and $|\bar{\mathrm{A}}_\delta| \,\big|\, 2^9\cdot 3^2\cdot 5$, we have $|\bar{M}_\delta| \,\big|\, 2^9\cdot 3^2\cdot 5$ and so $r = 3$ or 5.

Assume first that $r = 5$. Since Γ is connected and $1 \ne M'_\alpha \lhd \mathrm{A}_\alpha$, we have $1 \ne {M'_\alpha}^{\Gamma(\alpha)} \lhd \mathrm{A}_\alpha^{\Gamma(\alpha)}$, it follows that $5 \,\big|\, |M'_\alpha|$. Therefore, $5^2 \,\big|\, |\bar{M}_\delta|$, a contradiction.

Now assume that $r = 3$. Since $\bar{M} \cong M'$ has at most two orbits on $V\Gamma_N$ (if not $(\Gamma_N)_{\bar{M}}$ is a pentavalent symmetric graph of odd order, a contradiction), we have that $|\bar{M} : \bar{M}_\delta| = 2\bar{n}$ or $4\bar{n}$, where $\delta \in V\bar{\Gamma}$. Now $2n$ divides $|\bar{M}|$ and $|\bar{M} : \bar{M}_\delta| = \frac{2n}{r}$ or $\frac{4n}{r}$. It implies that $r = 3$ divides $\bar{M}_\delta$. Therefore $3 \,\big|\, |\bar{\mathrm{A}}_\delta|$. By Lemma 2.5, $\bar{\mathrm{A}}_\delta$ is insoluble, because $|\bar{\mathrm{A}}_\delta|$ does not divide 80, forcing that $\bar{M}_\delta$ is insoluble. Recall that $\bar{M} \cong M' \in \Omega$. If $\bar{M} \cong \mathrm{PSL}(2,p)$, then by Lemma 2.2, $\bar{M}_\delta \cong \mathrm{A}_5$. Hence $M'_\alpha \le (M'N)_\alpha \cong (M'N/N)_\delta = \bar{M}_\delta \cong \mathrm{A}_5$ by Theorem 2.6(ii). Note that $|M'_\alpha| = 20$, it contradicts that A_5 has no subgroup of order 20. If $\bar{M} \cong \mathrm{J}_1$, then $\Gamma_N \cong \mathcal{C}_{5852}$ or $\mathcal{C}^i_{17556}$ in Table 1, where $1 \le i \le 5$. If $\Gamma_N \cong \mathcal{C}^i_{17556}$, then $\bar{\mathrm{A}}_\delta \cong \mathrm{D}_{10}$ is soluble, a contradiction. If $\Gamma_N \cong \mathcal{C}_{5852}$, then $\bar{M}_\delta = \bar{\mathrm{A}}_\delta \cong \mathrm{A}_5$. A similar argument with the case $\bar{M} \cong \mathrm{PSL}(2,p)$ leads to a contradiction. If $\bar{M} \cong \mathrm{A}_5$, then $\Gamma_N \cong \mathcal{C}_{60}$ in Table 3 and $\bar{\mathrm{A}}_\delta \cong \mathrm{D}_{10}$ is soluble, a contradiction. If $\bar{M} \cong \mathrm{PSL}(2,25)$, then $\Gamma_N \cong \mathcal{C}^1_{780}$, $\mathcal{C}^2_{780}$ or $\mathcal{C}^3_{780}$ in Table 1 and $\bar{\mathrm{A}}_\delta \cong \mathrm{F}_{20}$ is soluble, also a contradiction. □

The final lemma completes the proof of Theorem 1.1.

Lemma 4.6. *Assume* A *is insoluble. Then* A *has no soluble minimal normal subgroup isomorphic to* $\mathbb{Z}_r$ *with* $r > 2$.

Proof. Suppose that, on the contrary, A has a soluble minimal normal subgroup $N = \mathbb{Z}_r$ with $r > 2$. We prove the lemma by induction on the order of Γ.

Assume first that $n = pqt$ has three prime factors. (Note that, by Table 3, the conclusion of Lemma 4.6 does not hold for $n = pq$.) Without loss of generality, we may assume that $r = t$. Then Γ_N is a pentavalent symmetric graph of order $4pq$. By Lemma 2.7, Γ_N is isomorphic to one of the graphs in Table 3, which contradicts to Lemma 4.5.

Assume next that n has at least four prime factors. Note that $\mathrm{Aut}\,\Gamma_N$ is insoluble. If $\mathrm{Aut}\,\Gamma_N$ has no nontrivial soluble normal subgroup, then Γ_N is isomorphic to one of the graphs in Lemma 4.2, which contradicts to Lemma 4.5. If $\mathrm{Aut}\,\Gamma_N$ has a soluble minimal normal subgroup $\bar{N}$, then we can also conclude that $\bar{N} \cong \mathbb{Z}_f$ with f a prime. If $f > 2$, then by induction, no such Γ_N exists, a contradiction. If $f = 2$, then Γ_N is isomorphic to one of the graphs appearing in Lemma 4.3, which also contradicts to Lemma 4.5. This completes the proof of the Lemma. □

References

[1] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, *J. Symbolic Comput.* **24** (1997), 235–265, doi:10.1006/jsco.1996.0125.

[2] P. J. Cameron, G. R. Omidi and B. Tayfeh-Rezaie, 3-designs from $\mathrm{PGL}(2, q)$, *Electron. J. Combin.* **13** (2006), #R50, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v13i1r50`.

[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Oxford University Press, Eynsham, 1985.

[4] L. E. Dickson, *Linear Groups: With an Exposition of the Galois Field Theory*, Dover Publications, New York, 1958.

[5] Y.-Q. Feng, J.-X. Zhou and Y.-T. Li, Pentavalent symmetric graphs of order twice a prime power, *Discrete Math.* **339** (2016), 2640–2651, doi:10.1016/j.disc.2016.05.008.

[6] D. Gorenstein, *Finite Simple Groups*, University Series in Mathematics, Plenum Publishing, New York, 1982.

[7] S.-T. Guo and Y.-Q. Feng, A note on pentavalent s-transitive graphs, *Discrete Math.* **312** (2012), 2214–2216, doi:10.1016/j.disc.2012.04.015.

[8] S.-T. Guo, J.-X. Zhou and Y.-Q. Feng, Pentavalent symmetric graphs of order $12p$, *Electron. J. Combin.* **18** (2011), #P233, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v18i1p233`.

[9] X.-H. Hua, Y.-Q. Feng and J. Lee, Pentavalent symmetric graphs of order $2pq$, *Discrete Math.* **311** (2011), 2259–2267, doi:10.1016/j.disc.2011.07.007.

[10] G. Karpilovsky, *The Schur Multiplier*, volume 2 of *London Mathematical Society Monographs New Series*, The Clarendon Press, Oxford, 1987.

[11] C. H. Li, Z. P. Lu and G. Wang, Arc-transitive graphs of square-free order and small valency, *Discrete Math.* **339** (2016), 2907–2918, doi:10.1016/j.disc.2016.06.002.

[12] C. H. Li and J. Pan, Finite 2-arc-transitive abelian Cayley graphs, *European J. Combin.* **29** (2008), 148–158, doi:10.1016/j.ejc.2006.12.001.

[13] Y. Li and Y.-Q. Feng, Pentavalent one-regular graphs of square-free order, *Algebra Colloq.* **17** (2010), 515–524, doi:10.1142/s1005386710000490.

[14] J. Pan, B. Ling and S. Ding, On prime-valent symmetric graphs of square-free order, *Ars Math. Contemp.* **15** (2018), 53–65, doi:10.26493/1855-3974.1161.3b9.

[15] J. Pan and Y. Liu, There exist no arc-regular prime-valent graphs of order four times an odd square-free integer, *Discrete Math.* **313** (2013), 2575–2581, doi:10.1016/j.disc.2013.08.009.

[16] J. Pan, B. Lou and C. Liu, Arc-transitive pentavalent graphs of order $4pq$, *Electron. J. Combin.* **20** (2013), #P36, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v20i1p36`.

[17] C. E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. London Math. Soc.* **47** (1993), 227–239, doi:10.1112/jlms/s2-47.2.227.

[18] M. Suzuki, *Group Theory II*, volume 248 of *Grundlehren der Mathematischen Wissenschaften*, Springer-Verlag, New York, 1986, translated from the Japanese.

[19] D.-W. Yang, Y.-Q. Feng and J.-L. Du, Pentavalent symmetric graphs of order $2pqr$, *Discrete Math.* **339** (2016), 522–532, doi:10.1016/j.disc.2015.09.009.

[20] J.-X. Zhou and Y.-Q. Feng, On symmetric graphs of valency five, *Discrete Math.* **310** (2010), 1725–1732, doi:10.1016/j.disc.2009.11.019.

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The pairing strategies of the 9-in-a-row game

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Abstract

One of the most useful strategies for proving Breaker's win in Maker-Breaker Positional Games is to find a pairing strategy. In some cases there are no pairing strategies at all, in some cases there are unique or almost unique strategies. For the k-in-a-row game, the case $k = 9$ is the smallest (sharp) for which there exists a Breaker winning pairing (paving) strategy. One pairing strategy for this game was given by Hales and Jewett.

In this paper we show that there are other winning pairings for the 9-in-a-row game, all have a very symmetric torus structure. While describing these symmetries we prove that there are only a finite number of non-isomorphic pairings for the game (around 200 thousand), which can be also listed up by a computer program. In addition, we prove that there are no "irregular", non-symmetric pairings. At the end of the paper we also show a pairing strategy for a variant of the 3-dimensional k-in-a-row game.

Keywords: Positional games, k-in-a-row game, pairing strategies, symmetries.

Math. Subj. Class.: 05C65, 05C15

1 Introduction

5-in-a-row is one of the most well known positional games, and its study inspired several deep results in this field. For a very thorough introduction of these, see Beck [2, 3]. In the classical version two players take the squares of a gridpaper (integer lattice), alternately, and the first who achieves five in a row, i.e. five consecutive squares in a vertical, horizontal or diagonal direction, wins the game. John Nash [4] invented the "strategy stealing" argument showing that in these type of games the first player either wins the game or it is a draw. It explains the notion of the so-called *Maker-Breaker* (M-B) version of a game; in these, Maker wins by achieving the original goal, while Breaker wins by preventing Maker

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to doing so. There is a connection between the normal (Maker-Maker) and the M-B versions of a game: If the first player wins the normal game, she wins the M-B one, as well. If Breaker wins the M-B game, then the second player can draw the normal game. However, the reverse statements are not true, see the Tic-Tac-Toe game. In summary, the M-B version is easier for Maker because she does not need to act Breaker's moves.

Allis et al. [1] solved the 5-in-a-row game for the 19×19 and 15×15 boards: the first player wins. However, the case of infinite board is still open (in the normal version). It is natural to ask then what happens in the k-in-a-row game, where the winning condition is to get k consecutive squares in a row. The first result in that direction is due to C. Shannon and H. Pollak [4] who showed that Breaker wins the 9-in-a-row. Later A. Hales and R. Jewett even gave a winning pairing strategy for Breaker. A. Brouwer, under the pseudonym T. G. L. Zetters published in [11] (as a solution to a problem by Guy and Selfridge [6]) that Breaker wins the 8-in-a-row on the infinite board. The cases $k = 6, 7$ are still open, although it is widely believed that those are both draws. (Of course for $k \leq 4$ Maker wins easily.) On the other hand, it was shown that there are no pairing strategies for $k \leq 8$, see [5, 10].

The concept of pairing strategies were useful for other games, the best known are the Harary games. Here Maker's goal is to get a given polyomino on the infinite board; most cases solved by A. Blass, see [4]. The notorious case of "Snaky" is still unsolved, although there are partial results for it. Csernenszky et al. [5] proved a relative existence theorem of pairing strategies for the Snaky: if a pairing is good for the Snaky, it is good for the polyomino P_5 that consists of five consecutive squares vertically or horizontally (but not diagonally). They also managed to give *all* possible pairings for P_5, there are two of those and those are not appropriate for Snaky.

As we mentioned, there is a Hales-Jewett winning pairing strategy for Breaker in the 9-in-a-row, see [3, 4] and also in Figure 1. The easiest way to see that the Hales-Jewett pairing blocks all lines of the board is that this pairing is an extension of a *domino pairing* from the 8×8 *torus* to the whole board. The torus lines consist of not only the rows and columns, but all diagonals, continuing on the opposite side when reaching the border of the 8×8 board.

Since no one has published different pairings for the 9-in-a-row,[1] the highly symmetric structure of the Hales-Jewett pairing, and the other examples of uniqueness or quasi uniqueness of pairings in similar problems; it is natural to think this is the only possible solution.

It turned out that this belief is very far from reality, as we found lots of new ones and will exhibit a few in the following sections. Another belief was that all pairings must be *torical* extensions of their 8×8 section. Somehow surprisingly, this belief is not true either; there are lot of solutions which are connected to the 16×16 torus, but are *not* extensions of the pairings of a 8×8 torus.

In the next sections, we define pairings precisely and give some conditions for their existence and structure. We will show that *all* solutions are either the extension of the pairings of a 8×8 torus (there are $194\,543$ non-isomorphic ones) or some combinations of those resulting in 16×16 toric solutions.

At the end of the paper, we prove a special case of the conjecture of Kruczek and Sundberg [8] about the existence of pairings in higher dimensions.

[1] According to [4], Selfridge also produced a Hales-Jewett pairing, but that pairing is either not different from the known Hales-Jewett pairing or left unpublished.

Since in our paper k-in-a-row type games play an important role, we define $\mathcal{H}_k$, the hypergraph of the k-in-a-row games.

Definition 1.1. The vertices of the **k-in-a-row hypergraph** $\mathcal{H}_k$ are the squares of the infinite (chess)board, i.e., the infinite square grid. The edges of the hypergraph $\mathcal{H}_k$ are the k-element sets of consecutive squares in a row horizontally, vertically or diagonally. We refer to the whole infinite rows as *lines*.

2 Pairing strategies

Given a hypergraph $\mathcal{H} = (V, E)$, where $V = V(\mathcal{H})$ and $E = E(\mathcal{H}) \subseteq \mathcal{P}(\mathcal{H}) = \{S : S \subseteq V\}$ are the set of vertices and edges, respectively. A bijection $\rho\colon X \to Y$, where $X, Y \subset V(\mathcal{H})$ and $X \cap Y = \emptyset$ is a **pairing** on the hypergraph $\mathcal{H}$. An $(x, \rho(x))$ pair **blocks** an $A \in E(\mathcal{H})$ edge, if A contains both elements of the pair. If the pairs of ρ block all edges, we say that ρ is a **good pairing** of $\mathcal{H}$.

Pairings are one way to show that Breaker has a winning strategy in hypergraph games. A good pairing ρ for a hypergraph $\mathcal{H}$ can be turned to a winning strategy for Breaker in the M-B game on $\mathcal{H}$: following ρ on $\mathcal{H}$ in a M-B game, for every $x \in X \cup Y$ element chosen by Maker, Breaker chooses $\rho(x)$ or in case of $x \in Y$ vice versa (if $x \notin X \cup Y$, then Breaker can choose an arbitrary vertex). Hence Breaker can block all edges and wins the game. Since our main topic is the 9-in-a-row game, this will be the first illustration to pairings and pairing strategies.

The following result is due to Hales and Jewett [4, 7]:

Theorem 2.1. *Breaker wins the* 9*-in-a-row M-B game by a pairing strategy, i.e., there exists a good pairing for the* 9*-in-a-row.*

Proof. Figure 1 is an extension of a pairing of an 8×8 torus (framed), where the pairs have a periodicity 8 in every line. Hence, the pairing blocks all 9-in-a-row edges. □

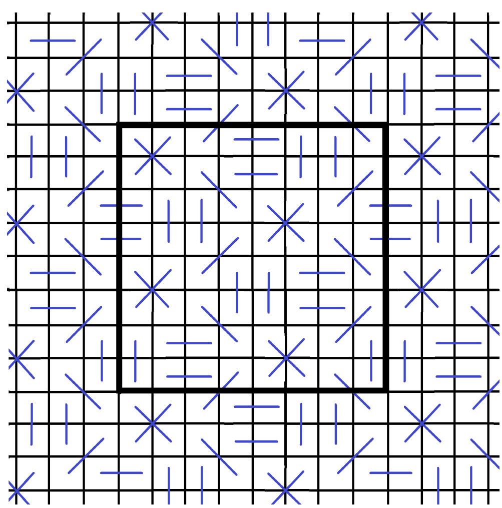

Figure 1: Hales-Jewett pairing blocks the 9-in-a-row.

A pairing is a **domino pairing** or rather a **match(-stick) pairing** on the square grid, if all pairs consist of only neighboring cells (horizontally, vertically or diagonally), called **dominoes**. Note that the pairing in Figure 1 is a domino pairing.

A counting type proposition [5] showed that there is no good pairing strategy for the k-in-a-row hypergraph, if $k < 9$. We will use this proposition, so we formulate the exact statement.

For a hypergraph $\mathcal{H}$ let $d_2(\mathcal{H})$ (briefly d_2) be the greatest number of edges that can be blocked by two vertices of $\mathcal{H}$, i.e., d_2 is the maximal co-degree.

Proposition 2.2 ([5]). *If there is a good pairing ρ for the hypergraph $\mathcal{H} = (V, E)$, then $d_2|X|/2 \geq |\mathcal{G}|$ must hold for all $X \subset V$, where $\mathcal{G} = \{A : A \in E, A \subset X\}$.*

Proof. We will refer to X as a sub-board. The edges of $\mathcal{G}$ can be blocked only by pairs coming from X. There are at most $|X|/2$ such pairs of ρ on the sub-board of size $|X|$. Since a pair blocks maximum d_2 edges, $|X|/2$ pairs block maximum $d_2|X|/2$. So, if there are more edges on the sub-board, there cannot be a good pairing. □

With the help of Proposition 2.2, we can conclude that there is no pairing strategy for $\mathcal{H}_k$ if $k < 9$. In the hypergraph $\mathcal{H}_k$, $d_2 = k - 1$ because a pair blocks at most $k - 1$ edges and this happens if and only if the pair is a domino. If X is an $n \times n$ sub-board for sufficiently large n, then $|\mathcal{G}| = 4n^2 + O(n)$ because four edges start from every square (a vertical, a horizontal and two diagonal, except at the border). By Proposition 2.2 we get $(k-1)n^2/2 \geq 4n^2 + O(n)$; that is, $k \geq 9 + O(1/n)$. One can even compute $O(n)$ exactly: $O(n) = -48n + 128$.

Hales and Jewett gave a pairing for $k = 9$, see [4] or Figure 1. However, there are neither different solutions nor claims of the uniqueness of the Hales-Jewett pairing in the literature. Our main goal is to decide about these questions.

3 Conditions for good pairings of the 9-in-a-row

Consider an $n \times n$ square sub-board of the infinite board. Proposition 2.2 gives $(k - 1)n^2/2 \geq 4n^2 + O(n)$ which implies $k \geq 9 + O(1/n)$. It suggests that one must use the pairs "optimally" to block $\mathcal{H}_9$ that is a pair should block the maximum possible edges of $\mathcal{H}_9$. We make the notion of optimality more precise as follows.

Definition 3.1. A pairing is optimal if:

1. Every pair blocks exactly $k - 1$ edges.
2. There are no overblockings, i.e., every edge is blocked by exactly one pair.
3. There is no empty square, i.e., every square is contained in a pair.

Corollary 3.2. *Let us consider an optimal good pairing for $\mathcal{H}_9$. This pairing is then a domino pairing in which the dominoes are following each other by 8-periodicity in each line and all squares are covered by a pair.*

Proof. The first point of Definition 3.1 implies that the pairing is a domino pairing, while the second gives the 8-periodicity since otherwise it would cause either overblocking or resulting in an unblocked edge. The lack of empty squares is just the repetition of the third condition. □

Definition 3.3. We call a square of a pairing **anomaly** where the 8-periodicity is violated, a non-domino type pair or an empty square appears in the pairing.

Of course the Hales-Jewett pairing is anomaly-free.

Remark 3.4. Because of the $O(n)$, there might be anomalies even in a good pairing of $\mathcal{H}_9$.[2] However, in Section 6 we will show that a good pairing of $\mathcal{H}_9$ is always anomaly-free.

The first step towards this is the following lemma:

Lemma 3.5. *For every good pairing of $\mathcal{H}_9$ there is an arbitrarily big, anomaly-free square sub-board.*

Proof. Let us take any $n \times n$ sub-board X and cut it up to smaller $\sqrt{n}/100 \times \sqrt{n}/100$ sub-boards. According to Proposition 2.2, there are at most $48n - 128$ anomalies in X. Hence, most of the $10000n$ sub-squares of X must be anomaly-free. □

From now on we describe the structure of anomaly-free pairings of $\mathcal{H}_9$. Let us divide a good pairing of $\mathcal{H}_9$ into 8×8 sub-boards and designate one that we call *Central square*, shortly C. We keep only the (domino) pairs touching C and examine where pairs should be on the neighboring 8×8 sub-boards of C. In order to talk about these sub-boards we call the 8×8 sub-boards Eastern (E), North-Eastern (NE) etc., while for the individual squares of the sub-boards the usual algebraic chess notations ($A1$ to $H8$) are used, see Figure 2.

Lemma 3.6. *Suppose we have an anomaly-free good pairing of $\mathcal{H}_9$ and we have nine 8×8 squares, $(C, E, NE, \ldots)$ as above. The horizontal and vertical dominoes touching the Central square C appear on the same places in all eight neighboring sub-boards of C. The diagonal dominoes also must appear on the sub-boards NE, NW, SW, SE in the same places. However, while the diagonal pattern of C may extend to the other four sub-boards, namely the E, S, W, N, it cannot be guaranteed. That is: the whole plane is the periodic copies either of the 8×8 sub-board C or the 16×16 square consisting of the sub-boards C, S, SE, E.*

Proof. It suffices to check the following five steps. We designate a general square in a 8×8 sub-board by $Xi \in \{A1, \ldots, H8\}$ according to the chess notation. If a domino d covers the same pair of squares e.g., in the C and E square, we say that *d extends to E from C*.

1. Because of the 8-periodicity of the domino pairs on horizontal (vertical) lines, the pairs of C extend uniquely to the same places of W and E (N and S respectively). The slope $+1$ diagonal dominoes extend similarly to SW and NE, while the slope -1 diagonal dominoes to SE and NW.
2. To see the horizontal (vertical) extension of dominoes to N and S (W and E respectively) we need a little case study. We have already seen that the vertical dominoes of C extend to north and south. Suppose for example that there is a vertical domino v at the Xi square of C. If the Xi square of W is covered by a slope $+1$ (or -1)

[2] A pairing with anomalies might be called "quasi-crystal" referring to the highly symmetric, crystal like appearance of known anomaly-free examples such as the Hales-Jewett or the ones shown in [5].

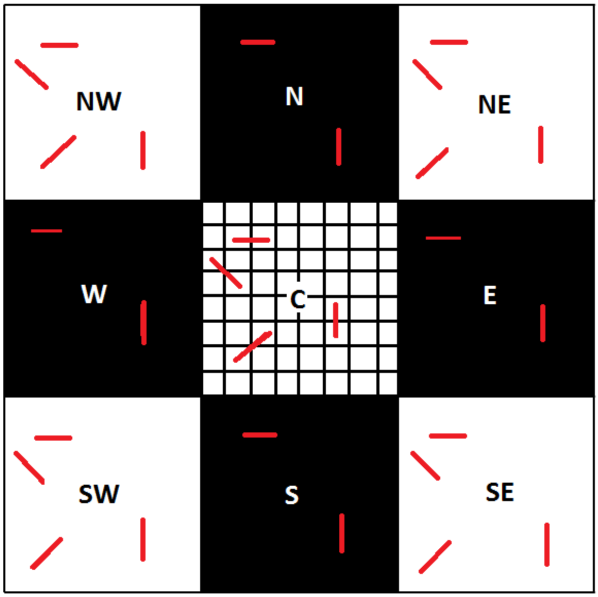

Figure 2: The extension of a pairing of C.

diagonal domino, then the 8-periodicity implies that the Xi square of N (or S) is also covered by a diagonal domino. This is a contradiction because we know from the previous point, that the Xi square of N is covered by a copy of the vertical domino v. The same is true for the sub-board E. If the Xi square of W (or E) is covered by a horizontal domino, then C should contain the copy of that horizontal domino at Xi by 8-periodicity, which is also a contradiction. We get that the vertical domino v in C extends to W and E, moreover, by 8-periodicity v extends to SW, NW, NE, SE, too. *So, we have seen that the vertical dominoes of C extend to all its eight neighboring sub-boards.* The same is true for the horizontal dominoes of C.

3. Let us check the diagonal dominoes. At the first and second step all slope $+1$ diagonal dominoes of C extend to SW and NE. Since there are no empty squares or overblockings, the remaining squares in SW and NE can be covered only by -1 slope diagonal dominoes. The same is true for $+1$ slope diagonal dominoes in SE and NW. *That is so far, all dominoes of C extend to the SW, SE, NE, NW, furthermore, the vertical and horizontal dominoes of C extend to S, E, N and W.*

4. We can see that the diagonal dominoes of C do not necessarily extend to the sub-boards S, E, N, W (colored by black in Figure 2). However, by 8-periodicity the diagonal pairs of E extend to S, N and W, that is the black sub-boards S, E, N, W have the exactly same structure of pairs. □

Remark 3.7. The diagonal dominoes of C *may* extend to the sub-boards S, E, N, W, and then all 8×8 sub-boards of the infinite board are the exact copy of C. However, it is possible that there are two different diagonal structures on the whole board, one in the

C, NW, NE, SE and SW types 8×8 sub-boards (colored by white in Figure 2) and a different diagonal structure in the sub-boards S, E, N and W (black ones). We will see a few examples in the next section.

Definition 3.8. A pairing of the infinite board (or of an anomaly-free sub-board) is **k-toric** if it is an extension of a $k \times k$ torus, but not for a smaller value.

Now we can summarize the previous lemmas and remarks in one central theorem:

Theorem 3.9. *Suppose we have an anomaly-free good pairing of $\mathcal{H}_9$. Then that pairing is either 8-toric or 16-toric.*

Proof. Lemma 3.6 and Remark 3.7 gives the proof of the Theorem. □

Observation 3.10. There are 8-toric good pairings of $\mathcal{H}_9$ that are not isomorphic to the Hales-Jewett pairing.

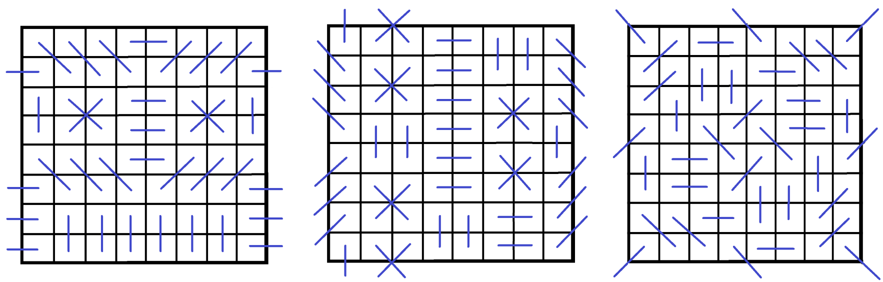

Figure 3: Some other pairings for 9-in-a-row.

Proof. The extensions of the 8×8 pairings in Figure 3 to the infinite board result in three different 8-toric pairings. Note that the pairings on the left have reflectional symmetry, while the pairing on the right has rotational symmetry. □

It is somehow surprising that there exist also some 16-toric pairings of $\mathcal{H}_9$. To understand their structure we refine the argument of the proof of Lemma 3.9 in the next section.

4 Diagonal alternating cycles

The 8-toric and 16-toric good pairings of $\mathcal{H}_9$ can be considered as special perfect matchings of graphs based on $\mathcal{H}_9$. The vertex sets are the basic tori, and each vertex is connected to the eight neighbors of the square it represents. A domino of a pairing is an edge, and the whole pairing is not only a perfect matching but has the additional property that it contains exactly one edge (domino) from each torus line.

It is well known that the union of two perfect matchings on the same vertex set consist of parallel edges and alternating cycles. So if we take the (graph theoretic) union of two good pairings (e.g. of C and W) which have the same horizontal and vertical edges, then the non-trivial alternating cycles contain only diagonal edges. Identifying the vertices in the case of non-isomorphic G_C and G_W the system of diagonal alternating cycles gives the possible ways to get the 16-toric good pairings.

We arrive to the following simple corollary.

Corollary 4.1. *If there exists a* 16*-toric good pairing for* $\mathcal{H}_9$*, then we can derive two* 8*-toric good pairings from it (in case of non-isomorphic* G_C *and* G_W*) differing only in some diagonal cycles.*

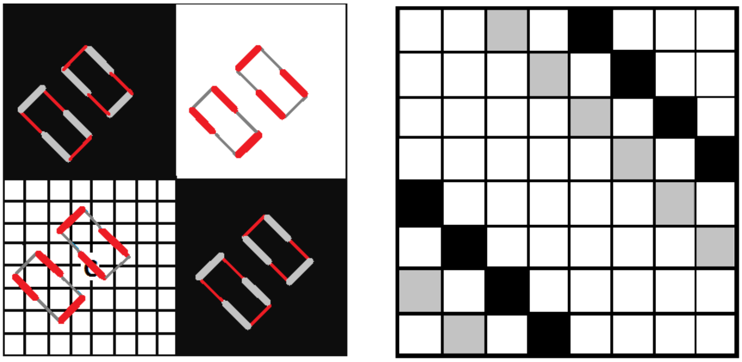

Figure 4: Diagonal alternating cycles give 16-toric pairing (left) and some -1 slope diagonal torus lines (right).

Theorem 4.2. *An* 8*-toric solution* C *gives a* 16*-toric solution if and only if another* 8*-toric solution* W *exists, differing in some diagonal dominoes, such that their union gives a system of diagonal alternating cycles. There are only two possible systems of diagonal alternating cycles which are shown in Figure 5; the left and middle ones.*

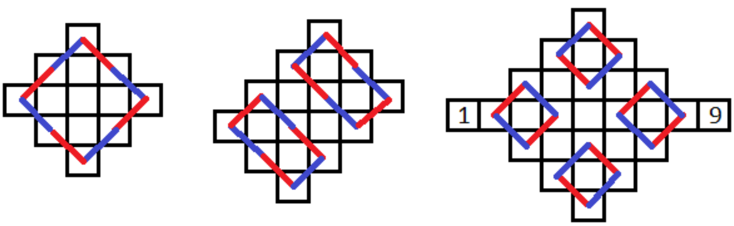

Figure 5: The diagonal alternating cycles.

Proof. Since there is exactly one domino in each torus line of an arbitrarily chosen 8×8 square sub-board of a 8-toric solution, then the alternating cycles coming from the diagonal dominoes of the union of C and W must meet the torus lines either in zero or two dominoes. (If they meet only in one, then there will be an unblocked torus line in C or W. Meeting more than two times would mean overblocking.)

An easy case study gives that only the systems of diagonal alternating cycles of Figure 5 may come into consideration. However, the third one would make a horizontal line (namely the 1-9) impossible to be blocked by a domino. □

Remark 4.3. There are only two different systems of alternating cycles, but it is possible that there is more than one such system in one 16-toric pairing. In that case we can deduce more than two (four or eight) 8-toric pairings from that 16-toric one. An example is shown in the right of Figure 6.

Observation 4.4. There exist good pairings for $\mathcal{H}_9$ containing the first or the second type of (the systems of) diagonal alternating cycles.

Proof. In Figure 6, one can see examples of the statement. Taking bold (thin) pairs of the alternating cycle for C (W) we get a 16-toric pairing. Naturally this 16-toric pairing is *not* 8-toric. □

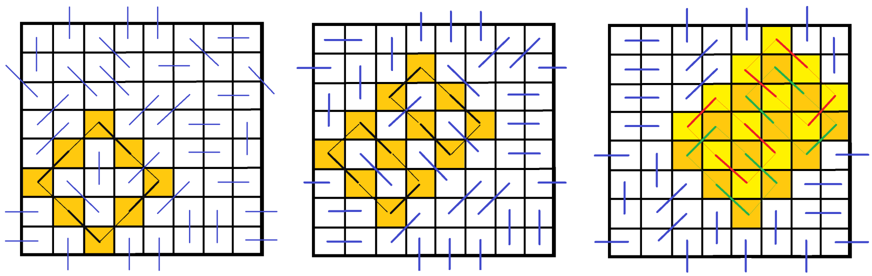

Figure 6: Examples of the alternating circles.

5 Pairings of the 8 torus

We have seen that pairings on the anomaly-free sub-boards are either 8-toric or 16-toric. Since the 16-toric solutions can be reduced to 8-toric ones or conversely, they can be constructed from 8-toric solutions we examine only the later ones in detail.

Definition 5.1. The 8×8 Maker-Breaker torus game is played on the 64 squares of the discrete torus, where there are 32 winning sets; the eight rows and columns and the diagonal torus lines of slope ± 1 (see the right side of Figure 4). We will call $\mathcal{T}_8$ the hypergraph of that game.

Observation 5.2. An arbitrary good 8-toric pairing for $\mathcal{H}_9$ provides a good pairing for $\mathcal{T}_8$.

Remark 5.3. The reverse is not true, since $\mathcal{T}_8$ has good pairings which are not domino types. However, considering only domino pairings we can always extend a good pairing of $\mathcal{T}_8$ into a good 8-toric pairing of $\mathcal{H}_9$.

To find all good domino pairings for the 8×8 torus is a finite task, which is not hard using a computer. However, one has to check the torus symmetries to list the non-isomorphic pairings, which gives the difficulty of the problem. The number of non-isomorphic domino type good pairings is 194 543, which turns out to be a prime. The pairings themselves can be downloaded at the page [9].

6 There is no quasi crystal pairing for the infinite board

We have an open problem left: are there any pairings for $\mathcal{H}_9$ with anomalies? Note that on a $n \times n$ sub-board there can be $O(n)$ anomalies which might result in infinitely many (and possibly untraceable) solutions. Fortunately, this is not the case as we will see.

Lemma 6.1. *A given anomaly-free pairing of a large enough square sub-board can be extended uniquely to the whole plane.*

Proof. We have seen that all anomaly-free pairings of a square sub-board is an extension of a domino pairing of either a 8×8 or a 16×16 torus. Continuing the extension to the whole plane gives a good pairing. □

Lemma 6.2. *Let us assume that a pairing of the whole plane is an extension of an anomaly-free half-plane R. Then the whole pairing is anomaly-free.*

Proof. To prove by contradiction, assume that we have an extension AL containing anomalies. Let AF be the anomaly-free extension of the half-plane pairing that exists by Lemma 6.1. Obviously AL is not equal to AF.

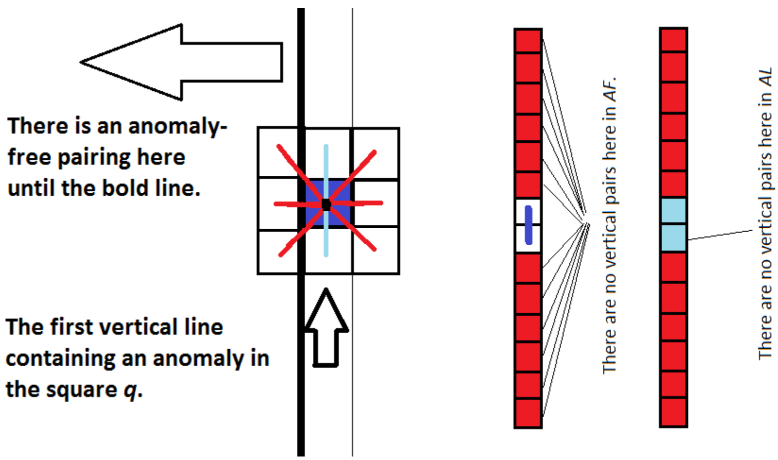

Figure 7: There are no quasicrystals.

Let us take a square with an anomaly which is one of the closest to R, and denote it by q. As it is pictured in Figure 7 we may assume that the border line of the half-plane R is vertical and the pairing AL is anomaly-free left to the square q. Let $AF(q)$ be the domino covering the square q in AF. If $AF(q)$ is placed horizontally and q is the right half of it, then AL does not contain the domino $AF(q)$ at the square q which leaves a 9-in-a-row edge unblocked by AL. A similar argument to diagonally placed dominoes shows that $AF(q)$ can be nothing but a vertical domino. Let us take the six squares above and below $AF(q)$. Because of 8-periodicity, there are no other squares containing a vertical pair in AF covering these 12 squares, but there must be a half of a vertical pair on those places (e.g. in s) in AL, because of the blocking condition. The domino $AF(s)$ is either horizontal or diagonal, and since $AF(s)$ is not in AL, it results in an unblocked horizontal or diagonal edge in AL. □

To answer the main question at the beginning of this section, we will need the ideas of the previous lemma.

Theorem 6.3. *An anomaly-free pairing of a big enough square sub-board extends uniquely and anomaly-free to the whole board.*

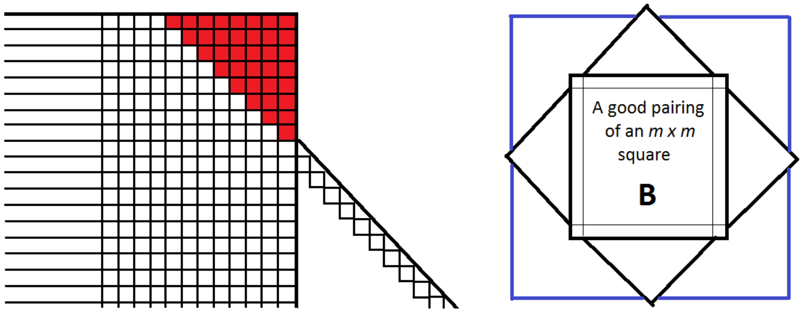

Figure 8: The extension of an anomaly-free pairing.

Proof. Fix a good pairing for $\mathcal{H}_9$ and take an $m \times m$ sub-board B that is anomaly-free; this exists by Lemma 3.5. The pairing on B extends anomaly-free to a large part of the right side of B, like in Lemma 6.2. The extension surely contains the right-angled triangle whose hypotenuse length is $m - 16$, and touches the right side of B, see Figure 8. The argument of Lemma 6.2 does not work next to the top and the bottom of B, since there are no diagonal dominoes there in B which were used before.

Doing the same trick to extend the pairing on the other sides of B, that results in a bigger (the size is about $(\sqrt{2}m - 16) \times (\sqrt{2}m - 16))$ rotated square. Repeating this procedure, we can see that the anomaly-free pairing of B is forced to extend to the whole plane. □

7 A pairing strategy in 3D

Kruczek and Sundberg [8] conjectured upper bounds matching with the lower bound of Proposition 2.2 for k-in-a-row type games in d dimension.

Conjecture 7.1 ([8])**.** *In the Maker-Breaker game on $\mathbb{Z}^d$ where there is a finite set $S \subset \mathbb{Z}^d$ of winning line direction-vectors, Breaker has a pairing strategy that allows him to win if the length of each winning line is at least $2|S| + 1$, i.e., Breaker has a winning pairing-strategy for the game k-in-a-row if $k \geq 2|S| + 1$.*

The special case of the *plane* gives back that Breaker has winning pairing strategy in the k-in-a-row if and only if $k \geq 9$. The higher dimensional versions are mainly open. One possible form of the M-B game in 3-dimension is when the winning directions are given by 13 vectors: $\{(0,0,1),\ (0,1,0),\ (1,0,0),\ (0,1,1),\ (1,0,1),\ (1,1,0),\ (0,1,-1),\ (1,0,-1),\ (1,-1,0),\ (1,1,1),\ (1,1,-1),\ (1,-1,1),\ (-1,1,1)\}$. Here Proposition 2.2 implies that k should be at least 27 to have a good pairing. According to Conjecture 7.1, we may expect good pairings for $k = 27$.

We have examined a related problem in 3-dimension. If the directions of winning lines are given by three vectors: $\{(0,0,1),\ (0,1,0),\ (1,0,0)\}$, then one expects pairing strategies if $k \geq 7$. (In other words, this is a Harary-type game [4] in 3-dimension, where the winning polyomino is the P_7, i.e. the seven connected consecutive cubes in a row.)

In fact, a computer search confirms this expectation, see Figure 9. This is a domino pairing of 3-dimensional torus type, we give the pairing on the $6 \times 6 \times 6$ torus in layers. The horizontal and vertical pairs of the same layer are obvious, while the pairs between the layers are denoted by points and circles.

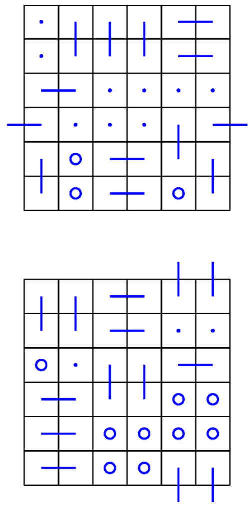

Figure 9: A good pairing of the 3D 7-in-a-row.

8 Conclusion

We have shown some new pairings for $\mathcal{H}_9$. We have proved that a good pairing for $\mathcal{H}_9$ is either 8-toric or 16-toric. There are 194 543 pairings which are 8-toric, and it is possible to construct 16-toric good pairings of $\mathcal{H}_9$ from some of those. There are no good pairings on the infinite board containing anomalies.

References

[1] L. V. Allis, H. J. van den Herik and M. P. H. Huntjens, Go-Moku solved by new search techniques, *Comput. Intell.* **12** (1996), 7–23, doi:10.1111/j.1467-8640.1996.tb00250.x.

[2] J. Beck, Van der Waerden and Ramsey type games, *Combinatorica* **1** (1981), 103–116, doi:10.1007/bf02579267.

[3] J. Beck, *Combinatorial Games: Tic-Tac-Toe Theory*, volume 114 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 2008, `http://www.cambridge.org/9780521461009`.

[4] E. R. Berlekamp, J. H. Conway and R. K. Guy, *Winning Ways for Your Mathematical Plays, Volume 3*, A K Peters/CRC Press, Natick, Massachusetts, 2nd edition, 2003.

[5] A. Csernenszky, R. R. Martin and A. Pluhár, On the complexity of chooser-picker positional games, *Integers* **12** (2012), 427–444, doi:10.1515/integ.2011.113.

[6] R. K. Guy and J. L. Selfridge, Problems and solutions: Problems dedicated to Emory P. Starke: S10, *Amer. Math. Monthly* **86** (1979), 306–306, doi:10.2307/2320758.

[7] A. W. Hales and R. I. Jewett, Regularity and positional games, *Trans. Amer. Math. Soc.* **106** (1963), 222–229, doi:10.2307/1993764.

[8] K. Kruczek and E. Sundberg, A pairing strategy for tic-tac-toe on the integer lattice with numerous directions, *Electron. J. Combin.* **15** (2008), #N42, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v15i1n42`.

[9] G. Makay, 9-in-a-row game, http://www.math.u-szeged.hu/~makay/amoba/, accessed on 6 December 2016.

[10] P. Mukkamala and D. Pálvölgyi, Asymptotically optimal pairing strategy for tic-tac-toe with numerous directions, *Electron. J. Combin.* **17** (2010), #N33, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v17i1n33.

[11] T. G. L. Zetters, Problems and solutions: Solutions of problems dedicated to Emory P. Starke: S10, *Amer. Math. Monthly* **87** (1980), 575–576, doi:10.2307/2321433.

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Weight choosability of oriented hypergraphs

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Abstract

The 1-2-3 conjecture states that every simple graph (with no isolated edges) has an edge weigthing by numbers $1, 2, 3$ such that the resulting weighted vertex degrees form a proper coloring of the graph. We study a similar problem for oriented hypergraphs. We prove that every oriented hypergraph has an edge weighting satisfying a similar condition, even if the weights are to be chosen from arbitrary lists of size two. The proof is based on the Combinatorial Nullstellensatz and a theorem of Schur for permanents of positive semi-definite matrices. We derive several consequences of the main result for uniform hypergraphs. We also point on possible applications of our results to problems of 1-2-3 type for non-oriented hypergraphs.

Keywords: Oriented hypergraphs, 1-2-3 *conjecture, combinatorial nullstellensatz, list weighting.*

Math. Subj. Class.: 05C15, 05C65, 05C78

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1 Introduction

The famous 1-2-3 conjecture, posed by Karoński, Łuczak, and Thomason [8], states that every simple graph (with no isolated edges) has an edge weighting with integers $1, 2, 3$ such that no two adjacent vertices get the same weighted degrees. This innocently looking problem remains open for more than ten years, despite serious attacks based on various techniques, including the celebrated Combinatorial Nullstellensatz of Alon [1] (see [2, 10, 12]). The best result up to now [6] confirms the conjecture when the set of allowable weights includes also 4 and 5. There were also many variants considered involving lists, orientations, and most recently hypergraphs (see [2, 3, 5, 7]). It is known, for instance, that any oriented graph has a weighting with numbers 1 and 2 only, such that the resulting weighted degrees are different for every pair of adjacent vertices [2, 9].

In this paper we consider the list version of the 1-2-3 conjecture for oriented hypergraphs. A *hypergraph* H on the set of vertices V is a family E of non-empty subsets of V, called the edges of H. A hypergraph H is k*-uniform* if each of its edges has size exactly k. So, a 2-uniform hypergraph is just a simple graph. Let I_H denote the *incidence graph* of a hypergraph H, that is, a bipartite graph with color classes V and E, whose edges are of the form ve, with $v \in V$, $e \in E$, whenever $v \in e$.

Now, by an *orientation* of a hypergraph H we mean any function $\mu \colon E(I_H) \to \mathbb{C}$ assigning non-zero complex "signs" to the edges of the graph I_H. The *cumulated degree* of a vertex $v \in V$ is defined by

$$D_v = \sum_{e \ni v} \mu(ev).$$

If the range of the mapping μ is confined to the set $\{-1, +1\}$, or more generally, to the set of complex roots of unity, then we get a natural generalization of traditional orientation of a simple graph. The orientation μ is conveniently represented by the *oriented incidence matrix* $X = [\mu_{ev}]$ of dimension $|E| \times |V|$, where $\mu_{ev} = \mu(ev)$ if $ev \in E(I_H)$, and $\mu_{ev} = 0$, otherwise. By an *oriented hypergraph* we mean a hypergraph H together with some fixed orientation μ.

Suppose now that each edge $e \in E$ of an oriented hypergraph (H, μ) is assigned a complex weight w_e. Then the resulting *weighted degree* of a vertex $v \in V$ is computed as

$$W_v = \sum_{e \ni v} \mu_{ev} w_e.$$

For each $e \in E$ we now define

$$W_e = \sum_{v \in e} \mu_{ev}^* W_v,$$

where x^* denotes the complex conjugate of x. Observe that in case of a usual oriented graph, W_e is exactly the difference between weighted degrees of both ends of e. We say that the weighting w is *virtuous* if for each edge $e \in E$ we have

$$W_e \neq 0.$$

Suppose that a list of complex numbers L_e is assigned to each $e \in E$. We say that an oriented hypergraph H is t*-weight choosable* if for any lists satisfying $|L_e| \geq t$ we are able to choose weights $w_e \in L_e$ so that w is a virtuous weighting of H. Our main theorem extends the results of [2] and [9] in the following way.

Theorem 1.1. *Every oriented hypergraph is* 2*-weight choosable.*

In the next section we will give an algebraic proof of this theorem based on the Combinatorial Nullstellensatz. It is inspired by the approach applied in [9]. In the last section we will speculate on possible consequences of this result for non-oriented hypergraphs, in particular, to the recent intriguing conjecture of Karoński, Kalkowski, and Pfender [7].

2 Proof of the main result

First recall the celebrated Combinatorial Nullstellensatz of Alon [1].

Theorem 2.1 (Alon)**.** *Let K be an arbitrary field, and let $F = F(x_1, x_2, \ldots, x_n)$ be a polynomial in $K[x_1, x_2, \ldots, x_n]$. Suppose that the total degree of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ is nonzero. If $S_1, S_2, \ldots, S_n$ are subsets of K, with $|S_i| > t_i$, then there are $s_1 \in S_1$, $s_2 \in S_2$, ..., $s_n \in S_n$ so that*

$$F(s_1, s_2, \ldots, s_n) \neq 0.$$

Let $A = (a_{ij})_{n\times n}$ be a square matrix with complex entries. Recall that the *permanent* of A is defined by:

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where S_n denotes the group of permutations of the set $\{1, 2, \ldots, n\}$. This is seemingly almost the same as the determinant of A, only the signs of permutations σ are ignored. The following classic result of Schur [11] gives a relation between permanent and determinant of a Hermitian matrix.

Theorem 2.2 (Schur)**.** *If A is a positive semi-definite Hermitian matrix, then* $\operatorname{per}(A) \geq \det(A)$*, with equality if and only if A is diagonal or A has a zero row.*

Proof of Theorem 1.1. Let H be a hypergraph with n vertices, m edges, and fixed orientation μ. Assume that a list L_e with two complex numbers has been assigned to each edge $e \in E$ of H, as well as a complex variable x_e. Let us define

$$P_v = \sum_{e \ni v} \mu_{ev} x_e$$

for each vertex $v \in V$ of H, and

$$P_e = \sum_{v \in e} \mu_{ev}^* P_v$$

for each $e \in E$. Finally, let us define the complex multivariate polynomial

$$P_H = \prod_{e \in E} P_e.$$

We see that H is 2-weight choosable if

$$P_H(w_{e_1}, w_{e_2}, \ldots, w_{e_m}) \neq 0$$

for some choice of weights $w_e \in L_e$, with $e \in E$.

Each monomial in P_H has total degree m. We are going to show that the coefficient of

$$\prod_{e \in E} x_e$$

is nonzero. This will finish the proof by Theorem 2.1.

Let us expand P_H. We have:

$$P_e = \sum_{v \in e} \mu^*_{ev} P_v = \sum_{v \in e} \mu^*_{ev} \sum_{f \ni v} \mu_{fv} x_f = \sum_{v \in e} \mu^*_{ev} \sum_{f \in E} \mu_{fv} x_f.$$

The sums in the last expression are independent, thus we can write

$$P_e = \sum_{v \in e} \sum_{f \in E} \mu^*_{ev} \mu_{fv} x_f = \sum_{f \in E} \sum_{v \in e} \mu^*_{ev} \mu_{fv} x_f.$$

Let

$$m_{ef} = \sum_{v \in e} \mu^*_{ev} \mu_{fv}.$$

Let M be the $m \times m$ matrix consisting of the m_{ef}. From the above definition it follows that $M = XX^*$, where X is the oriented incidence matrix of H, and X^* is its conjugate transpose.

We have

$$P_e = \sum_{f \in E} m_{ef} x_f,$$

and

$$P_H = \prod_{e \in E} P_e = \prod_{e \in E} \sum_{f \in E} m_{ef} x_f.$$

It follows that the coefficient of

$$\prod_{e \in E} x_e$$

is equal to $\operatorname{per}(M)$. So, it is enough to prove that $\operatorname{per}(M) \neq 0$ (see the permanent lemma in [1]). To get this we will apply Schur's theorem. First notice that for any complex vector z we have

$$zMz^* = zXX^*z^* = zX(zX)^* = |zX|^2.$$

As the last number is real and non-negative, M is positive semi-definite. Notice also that by assumption H has no empty edges and $\mu_{ev} \neq 0$ for all $v \in e$. Therefore all entries on the main diagonal of M are strictly positive real numbers. In particular, M has no zero row. Hence, if M is not diagonal, then by Schur's theorem we get

$$\operatorname{per}(M) > \det(M) \geq 0.$$

If M is diagonal, then

$$\operatorname{per}(M) = \det(M) = \prod_{e \in E} m_{ee} > 0.$$

This completes the proof. □

3 Some applications for uniform hypergraphs

We give two applications of Theorem 1.1 extending some results from [2, 5] and [9]. For simplicity we confine ourselves to uniform hypergraphs with specific orientations defined as follows. Let $k \geq 2$ be a fixed positive integer and let U_k denote the multiplicative group of k-th complex roots of unity. If ε is a primitive root in U_k, then we may write $U_k = \{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{k-1}\}$.

Let H be a k-uniform hypergraph. Consider a *canonical* orientation of H given by the mapping $\mu\colon E(I_H) \to U_k$ such that $\mu(eu) \neq \mu(ev)$ for every edge $e \in E$ and any two vertices $u, v \in e$. Notice that for $k = 2$ we get the traditional orientation of a simple graph. Recall that a coloring of the vertices of a hypergraph is *proper* if no edge is monochromatic.

Corollary 3.1. *Every canonically oriented k-uniform hypergraph H has an edge weighting with numbers* $1, 2$ *such that weighted vertex degrees give a proper coloring of H.*

Proof. Let H be a given k-uniform hypergraph and let μ be any canonical orientation of H. Assign the lists $L_e = \{1, 2\}$ to all edges of H. Then by Theorem 1.1 there exists a virtuous edge weighting w such that $w_e \in \{1, 2\}$ for every $e \in E$. We claim that this weighting satisfies the assertion of the corollary. Suppose on the contrary that this is not the case, and that there is some edge $e = \{v_1, v_2, \ldots, v_k\}$ such that all weighted degrees W_{v_i} are equal:

$$W_{v_1} = \cdots = W_{v_k} = W.$$

This implies that

$$W_e = \sum_{i=1}^{k} \mu(ev_i)^* W_{v_i} = W \sum_{i=1}^{k} \mu(ev_i)^* = 0,$$

since

$$\sum_{i=1}^{k} \mu(ev_i)^* = 1 + \varepsilon + \cdots + \varepsilon^{k-1} = 0.$$

This contradicts the virtue of the weighting w. □

Corollary 3.2. *Every k-uniform hypergraph H has a canonical orientation such that cumulated vertex degrees give a proper coloring of H.*

Proof. Let H be a given k-uniform hypergraph and let μ be any of its canonical orientations. Assign the list $L_e = \{1, \varepsilon\}$ to every edge $e \in E$. Then by Theorem 1.1 there exists a virtuous edge weighting w such that $w_e \in \{1, \varepsilon\}$ for every $e \in E$. Consider now a new orientation μ' defined by $\mu'(ev) = \mu(ev)w_e$ for every edge ev of the incidence graph I_H. We claim that this orientation satisfies the assertion of the corollary. Suppose on the contrary that this is not the case, and that there is some edge $e = \{v_1, v_2, \ldots, v_k\}$ with all cumulated degrees D_{v_i} equal in orientation μ':

$$D_{v_1} = \cdots = D_{v_k} = D.$$

This implies that also weighted degrees W_{v_i} are equal to D, since

$$W_{v_i} = \sum_{e \ni v_i} \mu(ev_i) w_e = \sum_{e \ni v_i} \mu'(ev_i) = D_{v_i} = D.$$

In consequence, we get

$$W_e = \sum_{i=1}^{k} \mu^*(ev_i) W_{v_i} = D \sum_{i=1}^{k} \mu(ev_i)^* = 0,$$

which contradicts the fact that w is virtuous. □

4 Discussion

We conclude the paper with pointing on some possible applications of our results to non-oriented problems of "1-2-3" type. Actually, the original 1-2-3 conjecture can be stated in a setting similar to virtuous weightings of oriented hypergraphs. To see this consider a bipartite graph B on bipartition classes X and Y with some signing $\mu\colon E(B) \to \{-1, +1\}$. Let $w\colon X \to \mathbb{Z}$ be a weighting of one part of B. Then for every vertex $y \in Y$ we may define the induced weight of y by

$$W_y = \sum_{x \in N(y)} w(x)\mu(xy),$$

where $N(y)$ denote the set of neighbors of y. A weighting w is called *half-virtuous* if $W_y \neq 0$ for every $y \in Y$. A natural problem is to find for a given signed graph B, the least integer k such that B has a half-virtuous weighting with $w(x) \in \{1, 2, \dots, k\}$ for each $x \in X$. It is not hard to see that 1-2-3-conjecture is equivalent to the statement that certain signed bipartite graphs arising from simple graphs have half-virtuous weightings with weights in the range $\{1, 2, 3\}$. Consider a simple graph G and the related bipartite graph B_G on bipartition classes $X = Y = E(G)$, with $xy \in E(B_G)$ whenever x and y are incident edges of G. An appropriate signing μ is defined so that for a fixed $y \in Y$, the edges $xy, x'y \in E(B_G)$ have the same sign if and only if x and x' are incident in G to the same end of y. A half-virtuous weighting of B_G is then equivalent to a weighting of the edges of G with different sums over edges incident to opposite ends of any given edge.

It is also possible that our results could be useful in a recently introduced version of the 1-2-3 conjecture for (non-oriented) hypergraphs [7]. Let H be a k-uniform hypergraph and let w denote a weighting of its edges. For every vertex v define its weighted degree by

$$W_v = \sum_{e \ni v} w_e.$$

Recall that a *proper* coloring of H is a coloring of its vertices such that no edge is monochromatic. A hypergraph H is *r-weight colorable* if there is a weighting $w\colon E(H) \to \{1, 2, \dots, r\}$ such that the weighted degrees W_v form a proper coloring of H. The following conjecture is stated in [4] (see also [7]).

Conjecture 4.1. *Every k-uniform hypergraph ($k \geq 2$) with no isolated edges is 3-weight colorable.*

The conjecture holds for random uniform hypergraphs in a stronger sense that even non-weighted degrees give a proper coloring, as proved recently in [4]. Another strong support for validity of the conjecture is given in [7], where it is proved that every non-trivial hypergraph (not only uniform) is $(2, 3)$-weight colorable. This means that a proper coloring of a hypergraph is obtained by using weights $\{1, 2, 3\}$ for the edges, and weights $\{1, 2\}$ for the vertices, with weighted vertex degrees computed by a formula:

$$W_v = w_v + \sum_{e \ni v} w_e.$$

This statement in restriction to simple graphs was formerly proved by Kalkowski in his PhD thesis. Then the result was extended to the list version by Wong and Zhu [12] who applied the Combinatorial Nullstellensatz. This encourages us to conclude the paper with the following general conjecture.

Conjecture 4.2. *Every k-uniform hypergraph ($k \geq 2$) with no isolated edges is 3-weight choosable.*

References

[1] N. Alon, Combinatorial Nullstellensatz, *Combin. Probab. Comput.* **8** (1999), 7–29, doi:10.1017/s0963548398003411.

[2] T. Bartnicki, J. Grytczuk and S. Niwczyk, Weight choosability of graphs, *J. Graph Theory* **60** (2009), 242–256, doi:10.1002/jgt.20354.

[3] O. Baudon, J. Bensmail and É. Sopena, An oriented version of the 1-2-3 conjecture, *Discuss. Math. Graph Theory* **35** (2015), 141–156, doi:10.7151/dmgt.1791.

[4] P. Bennett, A. Dudek, A. Frieze and L. Helenius, Weak and strong versions of the 1-2-3 conjecture for uniform hypergraphs, *Electron. J. Combin.* **23** (2016), #P2.46, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v23i2p46`.

[5] M. Borowiecki, J. Grytczuk and M. Pilśniak, Coloring chip configurations on graphs and digraphs, *Inform. Process. Lett.* **112** (2012), 1–4, doi:10.1016/j.ipl.2011.09.011.

[6] M. Kalkowski, M. Karoński and F. Pfender, Vertex-coloring edge-weightings: towards the 1-2-3-conjecture, *J. Comb. Theory Ser. B* **100** (2010), 347–349, doi:10.1016/j.jctb.2009.06.002.

[7] M. Kalkowski, M. Karoński and F. Pfender, The 1-2-3-conjecture for hypergraphs, *J. Graph Theory* **85** (2017), 706–715, doi:10.1002/jgt.22100.

[8] M. Karoński, T. Łuczak and A. Thomason, Edge weights and vertex colours, *J. Comb. Theory Ser. B* **91** (2004), 151–157, doi:10.1016/j.jctb.2003.12.001.

[9] M. Khatirinejad, R. Naserasr, M. Newman, B. Seamone and B. Stevens, Digraphs are 2-weight choosable, *Electron. J. Combin.* **18** (2011), #P21, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v18i1p21`.

[10] J. Przybyło and M. Woźniak, Total weight choosability of graphs, *Electron. J. Combin.* **18** (2011), #P112, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v18i1p112`.

[11] I. Schur, Über endliche Gruppen und Hermitesche Formen, *Math. Z.* **1** (1918), 184–207, doi:10.1007/bf01203611.

[12] T.-L. Wong and X. Zhu, Every graph is $(2, 3)$-choosable, *Combinatorica* **36** (2016), 121–127, doi:10.1007/s00493-014-3057-8.

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The Doyen-Wilson theorem for 3-sun systems*

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Abstract

A solution to the existence problem of G-designs with given subdesigns is known when G is a triangle with $p = 0, 1$, or 2 disjoint pendent edges: for $p = 0$, it is due to Doyen and Wilson, the first to pose such a problem for Steiner triple systems; for $p = 1$ and $p = 2$, the corresponding designs are kite systems and bull designs, respectively. Here, a complete solution to the problem is given in the remaining case where G is a 3-sun, i.e. a graph on six vertices consisting of a triangle with three pendent edges which form a 1-factor.

Keywords: 3*-sun systems, embedding, difference set.*

Math. Subj. Class.: 05B05, 05B30

1 Introduction

If G is a graph, then let $V(G)$ and $E(G)$ be the vertex-set and edge-set of G, respectively. The graph K_n denotes the complete graph on n vertices. The graph $K_m \setminus K_n$ has vertex-set $V(K_m)$ containing a distinguished subset H of size n; the edge-set of $K_m \setminus K_n$ is $E(K_m)$ but with the $\binom{n}{2}$ edges between the n distinguished vertices of H removed. This graph is sometimes referred to as a complete graph of order m with a *hole* of size n.

Let G and Γ be finite graphs. A *G-design* of Γ is a pair $(X, \mathcal{B})$ where $X = V(\Gamma)$ and $\mathcal{B}$ is a collection of isomorphic copies of G (*blocks*), whose edges partition $E(\Gamma)$. If $\Gamma = K_n$, then we refer to such a design as a *G-design of order n*.

A G-design $(X_1, \mathcal{B}_1)$ of order n is said to be *embedded* in a G-design $(X_2, \mathcal{B}_2)$ of order m provided $X_1 \subseteq X_2$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2$ (we also say that $(X_1, \mathcal{B}_1)$ is a *subdesign* (or *subsystem*) of $(X_2, \mathcal{B}_2)$ or $(X_2, \mathcal{B}_2)$ contains $(X_1, \mathcal{B}_1)$ as subdesign). Let $N(G)$ denote the set of integers n such that there exists a G-design of order n. A natural question to ask is: given $n, m \in N(G)$, with $m > n$, and a G-design $(X, \mathcal{B})$ of order n, does exists a G-design of order m containing $(X, \mathcal{B})$ as subdesign? Doyen and Wilson were the first to

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pose this problem for $G = K_3$ (Steiner triple systems) and in 1973 they showed that given $n, m \in N(K_3) = \{v : v \equiv 1, 3 \pmod 6\}$, then *any Steiner triple system of order n can be embedded in a Steiner triple system of order m if and only if $m \geq 2n+1$ or $m = n$* (see [3]). Over the years, any such problem has come to be called a "Doyen-Wilson problem" and any solution a "Doyen-Wilson type theorem". The work along these lines is extensive ([1, 4, 5, 6, 7, 8, 9, 10, 13]) and the interested reader is referred to [2] for a history of this problem.

In particular, taking into consideration the case where G is a triangle with $p = 0, 1, 2$, or 3 mutually disjoint pendent edges, a solution to the Doyen-Wilson problem is known when $p = 0$ (Steiner triple systems, [3]), $p = 1$ (kite systems, [9, 10]) and $p = 2$ (bull designs, [4]). Here, we deal with the remaining case ($p = 3$) where G is a 3-*sun*, i.e. a graph on six vertices consisting of a triangle with three pendent edges which form a 1-factor, by giving a complete solution to the Doyen-Wilson problem for G-designs where G is a 3-sun (3-*sun systems*).

2 Notation and basic lemmas

The 3-sun consisting of the triangle (a, b, c) and the three mutually disjoint pendent edges $\{a, d\}, \{b, e\}, \{c, f\}$ is denoted by $(a, b, c; d, e, f)$. A 3-sun system of order n (briefly, 3SS(n)) exists if and only if $n \equiv 0, 1, 4, 9 \pmod{12}$ and if $(X, \mathcal{S})$ is a 3SS(n), then $|\mathcal{S}| = \frac{n(n-1)}{12}$ (see [14]).

Let $n, m \equiv 0, 1, 4, 9 \pmod{12}$, with $m = u + n$, $u \geq 0$. The Doyen-Wilson problem for 3-sun systems is equivalent to the existence problem of decompositions of $K_{u+n} \setminus K_n$ into 3-suns.

Let r and s be integers with $r < s$, define $[r, s] = \{r, r+1, \ldots, s\}$ and $[s, r] = \emptyset$. Let $Z_u = [0, u-1]$ and $H = \{\infty_1, \infty_2, \ldots, \infty_t\}$, $H \cap Z_u = \emptyset$. If $S = (a, b, c; d, e, f)$ is a 3-sun whose vertices belong to $Z_u \cup H$ and $i \in Z_u$, let $S + i = (a+i, b+i, c+i; d+i, e+i, f+i)$, where the sums are modulo u and $\infty + i = \infty$, for every $\infty \in H$. The set $(S) = \{S + i : i \in Z_u\}$ is called the *orbit of S under Z_u* and S is a *base block* of (S).

To solve the Doyen-Wilson problem for 3-sun systems we use the *difference method* (see [11, 12]). For every pair of distinct elements $i, j \in Z_u$, define $|i - j|_u = \min\{|i - j|, u - |i - j|\}$ and set $D_u = \{|i - j|_u : i, j \in Z_u\} = \{1, 2, \ldots, \lfloor \frac{u}{2} \rfloor\}$. The elements of D_u are called *differences* of Z_u. For any $d \in D_u$, $d \neq \frac{u}{2}$, we can form a single 2-factor $\{\{i, d+i\} : i \in Z_u\}$, while if u is even and $d = \frac{u}{2}$, then we can form a 1-factor $\{\{i, i + \frac{u}{2}\} : 0 \leq i \leq \frac{u}{2} - 1\}$. It is also worth remarking that 2-factors obtained from distinct differences are disjoint from each other and from the 1-factor.

If $D \subseteq D_u$, denote by $\langle Z_u \cup H, D \rangle$ the graph with vertex-set $V = Z_u \cup H$ and the edge-set $E = \{\{i, j\} : |i - j|_u = d,\ d \in D\} \cup \{\{\infty, i\} : \infty \in H,\ i \in Z_u\}$. The graph $\langle Z_u \cup H, D_u \rangle$ is the complete graph $K_{u+t} \setminus K_t$ based on $Z_u \cup H$ and having H as a hole. The elements of H are called *infinity points*.

Let X be a set of size $n \equiv 0, 1, 4, 9 \pmod{12}$. The aim of the paper is to decompose the graph $\langle Z_u \cup X, D_u \rangle$ into 3-suns. To obtain our main result the $\langle Z_u \cup X, D_u \rangle$ will be regarded as a union of suitable edge-disjoint subgraphs of type $\langle Z_u \cup H, D \rangle$ (where $H \subseteq X$ may be empty, while $D \subseteq D_u$ is always non empty) and then each subgraph will be decomposed into 3-suns by using the lemmas given in this section. From here on suppose $u \equiv 0, 1, 3, 4, 5, 7, 8, 9, 11 \pmod{12}$.

Lemmas 2.1 – 2.4 give decompositions of subgraphs of type $\langle Z_u \cup H, D \rangle$ where D

contains particular differences, more precisely, $D = \{2\}$, $D = \{2, 4\}$ or $D = \{1, \frac{u}{3}\}$.

Lemma 2.1. *Let* $u \equiv 0 \pmod 4$, $u \geq 8$. *Then the graph* $\langle Z_u \cup \{\infty_1, \infty_2\}, \{2\}\rangle$ *can be decomposed into* 3*-suns.*

Proof. Consider the 3-suns

$$(\infty_1, 2+4i, 4i; 3+4i, 4+4i, \infty_2),$$
$$(\infty_2, 3+4i, 1+4i; 2+4i, 5+4i, \infty_1),$$

for $i = 0, 1, \ldots, \frac{u}{4} - 1$. □

Lemma 2.2. *Let* $u \equiv 0 \pmod{12}$. *Then the graph* $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2\}\rangle$ *can be decomposed into* 3*-suns.*

Proof. Consider the 3-suns

$$(\infty_1, 12i, 2+12i; 7+12i, , \infty_3, \infty_4),$$
$$(\infty_1, 4+12i, 6+12i; 9+12i, \infty_3, \infty_4),$$
$$(\infty_1, 8+12i, 10+12i; 11+12i, \infty_3, \infty_4),$$
$$(\infty_2, 2+12i, 4+12i; 1+12i, \infty_3, \infty_4),$$
$$(\infty_2, 6+12i, 8+12i; 7+12i, \infty_3, \infty_4),$$
$$(\infty_2, 10+12i, 12+12i; 11+12i, \infty_3, \infty_4),$$
$$(\infty_3, 1+12i, 3+12i; 9+12i, \infty_1, \infty_2),$$
$$(\infty_3, 5+12i, 7+12i; 11+12i, \infty_1, 9+12i),$$
$$(\infty_4, 3+12i, 5+12i; 1+12i, \infty_1, \infty_2),$$
$$(\infty_4, 9+12i, 11+12i; 7+12i, \infty_2, 13+12i),$$

for $i = 0, 1, \ldots, \frac{u}{12} - 1$. □

Lemma 2.3. *The graph* $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\}\rangle$, $u \geq 7$, $u \neq 8$, *can be decomposed into* 3*-suns.*

Proof. Let $u = 4k + r$, with $r = 0, 1, 3$, and consider the 3-suns

$$(\infty_1, 4+4i, 6+4i; 5+4i, 8+4i, \infty_4),$$
$$(\infty_2, 5+4i, 7+4i; 6+4i, 9+4i, \infty_1),$$
$$(\infty_3, 6+4i, 8+4i; 7+4i, 10+4i, \infty_2),$$
$$(\infty_4, 7+4i, 9+4i; 8+4i, 11+4i, \infty_3),$$

for $i = 0, 1, \ldots, k-3$, $k \geq 3$, plus the following blocks as the case may be.

If $r = 0$,

$$(\infty_1, 0, 2; 1, 4, \infty_4),$$
$$(\infty_2, 1, 3; 2, 5, \infty_1),$$
$$(\infty_3, 2, 4; 3, 6, \infty_2),$$
$$(\infty_4, 3, 5; 4, 7, \infty_3),$$
$$(\infty_1, 4k-4, 4k-2; 4k-3, 0, \infty_4),$$

$$(\infty_2, 4k-3, 4k-1; 4k-2, 1, \infty_1),$$
$$(\infty_3, 4k-2, 0; 4k-1, 2, \infty_2),$$
$$(\infty_4, 4k-1, 1; 0, 3, \infty_3).$$

If $r = 1$,

$$(\infty_1, 0, 2; 1, 4, \infty_2),$$
$$(\infty_2, 1, 3; 0, 5, \infty_1),$$
$$(\infty_3, 2, 4; 3, 6, \infty_2),$$
$$(\infty_4, 3, 5; 4, 7, \infty_3),$$
$$(\infty_1, 4k-4, 4k-2; 4k-3, 4k, \infty_2),$$
$$(\infty_2, 4k-3, 4k-1; 4k, 0, \infty_1),$$
$$(\infty_3, 4k-2, 4k; 4k-1, 1, \infty_1),$$
$$(\infty_4, 4k-1, 0; 4k-2, 2, \infty_3),$$
$$(\infty_4, 4k, 1; 2, 3, \infty_3).$$

If $r = 3$,

$$(\infty_1, 0, 2; 1, 4, \infty_4),$$
$$(\infty_2, 1, 3; 2, 5, \infty_1),$$
$$(\infty_3, 2, 4; 3, 6, \infty_2),$$
$$(\infty_4, 3, 5; 4, 7, \infty_3),$$
$$(\infty_1, 4k-4, 4k-2; 4k-3, 4k, \infty_4),$$
$$(\infty_2, 4k-3, 4k-1; 4k-2, 4k+1, \infty_1),$$
$$(\infty_3, 4k-2, 4k; 4k-1, 4k+2, \infty_2),$$
$$(\infty_4, 4k-1, 4k+1; 4k, 0, \infty_3),$$
$$(\infty_1, 4k, 4k+2; 4k+1, 1, \infty_4),$$
$$(\infty_2, 4k+1, 0; 4k+2, 2, \infty_4),$$
$$(\infty_3, 4k+2, 1; 0, 3, \infty_4).$$

With regard to the difference 4 in Z_7, note that $|4|_7 = 3$ and the seven distinct blocks obtained for $k = 1$ and $r = 3$ gives a decomposition of $\langle Z_7 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 3\}\rangle$ into 3-suns. □

Lemma 2.4. *Let* $u \equiv 0 \pmod 3$*,* $u \geq 12$*. Then the graph* $\langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_8\}, \{1, \frac{u}{3}\}\rangle$ *can be decomposed into* 3*-suns.*

Proof. If $u \equiv 0 \pmod 6$ consider the 3-suns:

$$(\infty_1, 2i, \tfrac{u}{3} + 2i; 2\tfrac{u}{3} + 2i, \infty_5, \infty_6),\ i = 0, 1, \ldots, \tfrac{u}{6} - 1,$$
$$(\infty_1, 1 + 2i, \tfrac{u}{3} + 1 + 2i; 2\tfrac{u}{3} + 1 + 2i, \infty_6, \infty_5),\ i = 0, 1, \ldots, \tfrac{u}{6} - 1,$$
$$(\infty_2, 2\tfrac{u}{3} + 2i, \tfrac{u}{3} + 2i; 2 + 2i, 2i, \infty_5),\ i = 0, 1, \ldots, \tfrac{u}{6} - 2,$$
$$(\infty_2, 2\tfrac{u}{3} + 1 + 2i, \tfrac{u}{3} + 1 + 2i; 3 + 2i, 1 + 2i, \infty_6),\ i = 0, 1, \ldots, \tfrac{u}{6} - 2,$$
$$(\infty_2, u - 2, 2\tfrac{u}{3} - 2; 0, \tfrac{u}{3} - 2, \infty_5),$$

$(\infty_2, u-1, 2\frac{u}{3}-1; 1, \frac{u}{3}-1, \infty_6)$,
$(\infty_3, 2i, 1+2i; 2\frac{u}{3}+2i, \infty_7, \infty_8),\ i = 0, 1, \ldots, \frac{u}{6}-1$,
$(\infty_3, \frac{u}{3}+2i, \frac{u}{3}+1+2i; 2\frac{u}{3}+1+2i, \infty_7, \infty_8),\ i = 0, 1, \ldots, \frac{u}{6}-1$,
$(\infty_4, 1+2i, 2+2i; 2\frac{u}{3}+2+2i, \infty_7, \infty_8),\ i = 0, 1, \ldots, \frac{u}{6}-1$,
$(\infty_4, \frac{u}{3}+1+2i, \frac{u}{3}+2+2i; 2\frac{u}{3}+1+2i, \infty_7, \infty_8),\ i = 0, 1, \ldots, \frac{u}{6}-1$,
$(\infty_5, 2\frac{u}{3}+2i, 2\frac{u}{3}+1+2i; 1+2i, \infty_7, \infty_8),\ i = 0, 1, \ldots, \frac{u}{6}-1$,
$(\infty_6, 2\frac{u}{3}+3+2i, 2\frac{u}{3}+4+2i; 2+2i, \infty_7, \infty_8),\ i = 0, 1, \ldots, \frac{u}{6}-2$,
$(\infty_6, 2\frac{u}{3}+1, 2\frac{u}{3}+2; 2\frac{u}{3}, \infty_7, \infty_8)$.

If $u \equiv 3 \pmod 6$ consider the 3-suns:

$(\infty_1, 2i, \frac{u}{3}+2i; 2\frac{u}{3}+2i, \infty_5, \infty_6),\ i = 0, 1, \ldots, \frac{u-3}{6}$,
$(\infty_1, 1+2i, \frac{u}{3}+1+2i; 2\frac{u}{3}+1+2i, \infty_6, \infty_5),\ i = 0, 1, \ldots, \frac{u-9}{6}$,
$(\infty_2, 2\frac{u}{3}+2i, \frac{u}{3}+2i; 2+2i, 2i, \infty_5),\ i = 0, 1, \ldots, \frac{u-9}{6}$,
$(\infty_2, u-1, 2\frac{u}{3}-1; 0, \frac{u}{3}-1, \infty_5)$,
$(\infty_2, 2\frac{u}{3}+1+2i, \frac{u}{3}+1+2i; 3+2i, 1+2i, \infty_6),\ i = 0, 1, \ldots, \frac{u-15}{6}$,
$(\infty_2, u-2, 2\frac{u}{3}-2; 1, \frac{u}{3}-2, \infty_6)$,
$(\infty_3, 2i, 1+2i; 2\frac{u}{3}+2i, \infty_7, \infty_8),\ i = 2, 3, \ldots, \frac{u-3}{6}$,
$(\infty_3, 0, 1; 2\frac{u}{3}, \infty_6, \infty_8)$,
$(\infty_3, 2, 3; 2\frac{u}{3}+2, \infty_6, \infty_8)$,
$(\infty_3, \frac{u}{3}+1+2i, \frac{u}{3}+2+2i; 2\frac{u}{3}+1+2i, \infty_7, \infty_8),\ i = 0, 1, \ldots, \frac{u-9}{6}$,
$(\infty_4, 1+2i, 2+2i; 2\frac{u}{3}+2+2i, \infty_7, \infty_8),\ i = 0, 1, \ldots, \frac{u-9}{6}$,
$(\infty_4, \frac{u}{3}+2i, \frac{u}{3}+1+2i; 2\frac{u}{3}+1+2i, \infty_7, \infty_8),\ i = 0, 1, \ldots, \frac{u-3}{6}$,
$(\infty_5, 2\frac{u}{3}+2i, 2\frac{u}{3}+1+2i; 1+2i, \infty_7, \infty_8),\ i = 0, 1, \ldots, \frac{u-9}{6}$,
$(\infty_6, 2\frac{u}{3}+1+2i, 2\frac{u}{3}+2+2i; 4+2i, \infty_7, \infty_8),\ i = 0, 1, \ldots, \frac{u-15}{6}$,
$(\infty_6, u-2, u-1; 2\frac{u}{3}, \infty_7, \infty_8)$,
$(\infty_7, u-1, 0; 2, \infty_5, \infty_8)$. □

Lemmas 2.5 – 2.9 allow to decompose $\langle Z_u \cup H, D\rangle$ where u is even and D contains the difference $\frac{u}{2}$.

Lemma 2.5. *Let u be even, $u \geq 8$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \frac{u}{2}\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the 3-suns

$(\infty_1, 2i, 1+2i; \frac{u}{2}+2+2i, \frac{u}{2}+2i, \infty_3),\ i = 0, 1, \ldots, \frac{u}{4}-2$,
$(\infty_1, \frac{u}{2}-2, \frac{u}{2}-1; \frac{u}{2}, u-2, \infty_3)$,
$(\infty_2, 1+2i, \frac{u}{2}+1+2i; 2i, 2+2i, \infty_1),\ i = 0, 1, \ldots, \frac{u}{4}-1$,
$(\infty_3, \frac{u}{2}+1+2i, \frac{u}{2}+2i; 2i, \frac{u}{2}+2+2i, \infty_2),\ i = 0, 1, \ldots, \frac{u}{4}-1$. □

Lemma 2.6. *Let $u \equiv 0 \pmod{12}$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \frac{u}{2}\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the 3-suns

$$
\begin{aligned}
&(\infty_1, 6i, \tfrac{u}{2}+6i; 4+6i, \infty_3, \infty_2),\\
&(\infty_1, 1+6i, \tfrac{u}{2}+1+6i; 5+6i, \infty_4, \infty_2),\\
&(\infty_1, 2+6i, \tfrac{u}{2}+2+6i; \tfrac{u}{2}+3+6i, \infty_4, \infty_3),\\
&(\infty_2, 1+6i, 6i; \tfrac{u}{2}+3+6i, \infty_3, \infty_4),\\
&(\infty_2, 2+6i, 3+6i; \tfrac{u}{2}+4+6i, 1+6i, \infty_4),\\
&(\infty_2, 5+6i, 4+6i; \tfrac{u}{2}+5+6i, 6+6i, 3+6i),\\
&(\infty_3, 3+6i, \tfrac{u}{2}+3+6i; 2+6i, \infty_1, \tfrac{u}{2}+2+6i),\\
&(\infty_3, 4+6i, \tfrac{u}{2}+4+6i; \tfrac{u}{2}+6i, \infty_4, \infty_1),\\
&(\infty_3, 5+6i, \tfrac{u}{2}+5+6i; \tfrac{u}{2}+1+6i, \infty_4, \infty_1),\\
&(\infty_4, \tfrac{u}{2}+1+6i, \tfrac{u}{2}+2+6i; \tfrac{u}{2}+3+6i, \tfrac{u}{2}+6i, \infty_2),\\
&(\infty_4, \tfrac{u}{2}+4+6i, \tfrac{u}{2}+5+6i; \tfrac{u}{2}+6i, \tfrac{u}{2}+3+6i, \tfrac{u}{2}+6+6i),
\end{aligned}
$$

for $i = 0, 1, \ldots, \frac{u}{12} - 1$. □

Lemma 2.7. *Let u be even, $u \geq 8$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \frac{u}{2}\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the 3-suns

$$
\begin{aligned}
&(\infty_1, 2i, 1+2i; \tfrac{u}{2}+2+2i, \tfrac{u}{2}+2i, \infty_3),\ i = 0, 1, \ldots, \tfrac{u}{4}-2,\\
&(\infty_1, \tfrac{u}{2}-2, \tfrac{u}{2}-1; \tfrac{u}{2}, u-2, \infty_3),\\
&(\infty_2, 1+2i, \tfrac{u}{2}+1+2i; 2i, \infty_6, \infty_1),\ i = 0, 1, \ldots, \tfrac{u}{4}-1,\\
&(\infty_3, \tfrac{u}{2}+1+2i, \tfrac{u}{2}+2i; 2i, \infty_6, \infty_2),\ i = 0, 1, \ldots, \tfrac{u}{4}-1,\\
&(\infty_4, 1+2i, 2+2i; \tfrac{u}{2}+2+2i, \infty_5, \infty_6),\ i = 0, 1, \ldots, \tfrac{u}{4}-1,\\
&(\infty_5, \tfrac{u}{2}+1+2i, \tfrac{u}{2}+2+2i; 2+2i, \infty_4, \infty_6),\ i = 0, 1, \ldots, \tfrac{u}{4}-1.
\end{aligned}
$$
□

Lemma 2.8. *Let $u \equiv 0 \pmod{12}$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_7\}, \{1, \frac{u}{2}\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the 3-suns

$$
\begin{aligned}
&(\infty_1, 6i, \tfrac{u}{2}+6i; 4+6i, \infty_7, \infty_2),\\
&(\infty_1, 1+6i, \tfrac{u}{2}+1+6i; \tfrac{u}{2}+3+6i, \infty_7, \infty_4),\\
&(\infty_1, 2+6i, \tfrac{u}{2}+2+6i; \tfrac{u}{2}+5+6i, \infty_5, \infty_2),\\
&(\infty_2, 3+6i, \tfrac{u}{2}+3+6i; 6i, \infty_1, \infty_4),\\
&(\infty_2, 4+6i, \tfrac{u}{2}+4+6i; 2+6i, \infty_7, \infty_1),\\
&(\infty_2, 5+6i, \tfrac{u}{2}+5+6i; \tfrac{u}{2}+1+6i, \infty_1, \infty_7),\\
&(\infty_3, 6i, 1+6i; \tfrac{u}{2}+6i, \infty_5, \infty_6),\\
&(\infty_3, 2+6i, 3+6i; \tfrac{u}{2}+2+6i, \infty_7, \infty_6),\\
&(\infty_3, 4+6i, 5+6i; \tfrac{u}{2}+5+6i, \infty_5, \infty_6),\\
&(\infty_4, 1+6i,\ 2+6i; \tfrac{u}{2}+6+6i, \infty_2, \infty_6),\\
&(\infty_4, 3+6i, 4+6i; \tfrac{u}{2}+4+6i, \infty_7, \infty_6),
\end{aligned}
$$

$$(\infty_4, 5+6i, 6+6i; \tfrac{u}{2}+5+6i, \infty_7, \infty_6),$$
$$(\infty_5, \tfrac{u}{2}+6i, \tfrac{u}{2}+1+6i; 1+6i, \infty_7, \infty_3),$$
$$(\infty_5, \tfrac{u}{2}+2+6i, \tfrac{u}{2}+3+6i; 3+6i, \infty_7, \infty_3),$$
$$(\infty_5, \tfrac{u}{2}+4+6i, \tfrac{u}{2}+5+6i; 5+6i, \infty_7, \tfrac{u}{2}+6+6i),$$
$$(\infty_6, \tfrac{u}{2}+1+6i, \tfrac{u}{2}+2+6i; \tfrac{u}{2}+5+6i, \infty_7, \infty_4),$$
$$(\infty_6, \tfrac{u}{2}+3+6i, \tfrac{u}{2}+4+6i; \tfrac{u}{2}+6+6i, \infty_7, \infty_3),$$

for $i = 0, 1, \ldots, \frac{u}{12} - 1$. □

Lemma 2.9. *Let* $u \equiv 0 \pmod{12}$. *Then the graph* $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 2, \frac{u}{2}\}\rangle$ *can be decomposed into* 3*-suns.*

Proof. Consider the 3-suns

$$(\infty_1, 6i, 1+6i; \tfrac{u}{2}+1+6i, \tfrac{u}{2}+6i, 3+6i),$$
$$(\infty_1, 2+6i, 3+6i; \tfrac{u}{2}+5+6i, \tfrac{u}{2}+2+6i, 5+6i),$$
$$(\infty_1, 4+6i, 5+6i; \tfrac{u}{2}+2+6i, \tfrac{u}{2}+4+6i, 7+6i),$$
$$(\infty_1, \tfrac{u}{2}+3+6i, \tfrac{u}{2}+4+6i; \tfrac{u}{2}+6i, \tfrac{u}{2}+2+6i, \infty_2),$$
$$(\infty_2, 1+6i, \tfrac{u}{2}+1+6i; \tfrac{u}{2}+3+6i, 2+6i, \tfrac{u}{2}+2+6i),$$
$$(\infty_2, 3+6i, 4+6i; 2+6i, \tfrac{u}{2}+3+6i, 6+6i),$$
$$(\infty_2, 5+6i, \tfrac{u}{2}+5+6i; \tfrac{u}{2}+2+6i, 6+6i, \tfrac{u}{2}+6+6i),$$
$$(\infty_3, 2+6i, 6i; 1+6i, 4+6i, \infty_2),$$
$$(\infty_3, \tfrac{u}{2}+2+6i, \tfrac{u}{2}+6i; 4+6i, \tfrac{u}{2}+4+6i, \infty_2),$$
$$(\infty_3, \tfrac{u}{2}+1+6i, \tfrac{u}{2}+3+6i; 3+6i, \tfrac{u}{2}+6i, \tfrac{u}{2}+5+6i),$$
$$(\infty_3, \tfrac{u}{2}+5+6i, \tfrac{u}{2}+4+6i; 5+6i, \tfrac{u}{2}+7+6i, \tfrac{u}{2}+6+6i),$$

for $i = 0, 1, \ldots, \frac{u}{12} - 1$. □

The following lemma "combines" one infinity point with one difference $d \neq \frac{u}{2}, \frac{u}{3}$ such that $\frac{u}{\gcd(u,d)} \equiv 0 \pmod 3$ (therefore, $u \equiv 0 \pmod 3$).

Lemma 2.10. *Let* $u \equiv 0 \pmod 3$ *and* $d \in D_u \setminus \{\frac{u}{2}, \frac{u}{3}\}$ *such that* $p = \frac{u}{\gcd(u,d)} \equiv 0 \pmod 3$. *Then the graph* $\langle Z_u \cup \{\infty\}, \{d\}\rangle$ *can be decomposed into* 3*-suns.*

Proof. The subgraph $\langle Z_u, \{d\}\rangle$ can be decomposed into $\frac{u}{p}$ cycles of length $p = 3q$, $q \geq 2$.

If $q > 2$, let $(x_1, x_2, \ldots, x_{3q})$ be a such cycle and consider the 3-suns

$$(\infty, x_{2+3i}, x_{3+3i}; x_{7+3i}, x_{1+3i}, x_{4+3i}),$$

for $i = 0, 1, \ldots, q-1$ (where the sum is modulo $3q$).

If $q = 2$, let $(x_1^{(j)}, x_2^{(j)}, x_3^{(j)}, x_4^{(j)}, x_5^{(j)}, x_6^{(j)})$, $j = 0, 1, \ldots, \frac{u}{6} - 1$, be the 6-cycles decomposing $\langle Z_u, \{d\}\rangle$ and consider the 3-suns

$$(\infty, x_2^{(j)}, x_3^{(j)}; x_1^{(j+1)}, x_1^{(j)}, x_4^{(j)}),$$
$$(\infty, x_5^{(j)}, x_6^{(j)}; x_4^{(j+1)}, x_4^{(j)}, x_1^{(j)}),$$

for $j = 0, 1, \ldots, \frac{u}{6} - 1$ (where the sums are modulo $\frac{u}{6}$). □

Subsequent Lemmas 2.11 – 2.14 allow to decompose $\langle Z_u \cup H, D\rangle$, where $|H| = 1, 2, 3, 5$, $|D| = 6 - |H|$ and $\frac{u}{2} \notin D$; here, u and D are any with the unique condition that if D contains at least three differences d_1, d_2, d_3, then $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$.

Lemma 2.11. *Let $d_1, d_2, d_3, d_4, d_5 \in D_u \setminus \{\frac{u}{2}\}$ such that $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$. Then the graph $\langle Z_u \cup \{\infty\}, \{d_1, d_2, d_3, d_4, d_5\}\rangle$ can be decomposed into 3-suns.*

Proof. If $d_3 = d_2 - d_1$, consider the orbit of

$$(d_1, d_2, 0; \infty, d_2 + d_5, d_4)$$

(or $(d_1, d_2, 0; \infty, d_2 + d_5, -d_4)$, if $d_2 + d_5 = d_4$) under Z_u. If $d_1 + d_2 + d_3 = u$, consider the orbit of

$$(-d_1, d_2, 0; \infty, d_2 + d_5, d_4)$$

(or $(-d_1, d_2, 0; \infty, d_2 + d_5, -d_4)$, if $d_2 + d_5 = d_4$) under Z_u. □

Lemma 2.12. *Let $d_1, d_2, d_3, d_4 \in D_u \setminus \{\frac{u}{2}\}$ such that $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2\}, \{d_1, d_2, d_3, d_4\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the orbit of $(d_1, d_2, 0; \infty_1, \infty_2, d_4)$ or $(-d_1, d_2, 0; \infty_1, \infty_2, d_4)$ under Z_u when, respectively, $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$. □

Lemma 2.13. *Let $d_1, d_2, d_3 \in D_u \setminus \{\frac{u}{2}\}$ such that $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{d_1, d_2, d_3\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the orbit of $(d_1, d_2, 0; \infty_1, \infty_2, \infty_3)$ or $(-d_1, d_2, 0; \infty_1, \infty_2, \infty_3)$ under Z_u when, respectively, $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$. □

Lemma 2.14. *Let $d \in D_u \setminus \{\frac{u}{2}\}$, the graph $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}, \{d\}\rangle$ can be decomposed into 3-suns.*

Proof. The subgraph $\langle Z_u, \{d\}\rangle$ is regular of degree 2 and so can be decomposed into l-cycles, $l \geq 3$. Let $(x_1, x_2, \dots, x_l)$ be a such cycle. Put $l = 3q + r$, with $r = 0, 1, 2$, and consider the 3-suns with the sums modulo l

$$\begin{aligned}&(\infty_1, x_{1+3i}, x_{2+3i}; x_{3+3i}, \infty_4, \infty_5),\\&(\infty_2, x_{2+3i}, x_{3+3i}; x_{4+3i}, \infty_4, \infty_5),\\&(\infty_3, x_{3+3i}, x_{4+3i}; x_{5+3i}, \infty_4, \infty_5),\end{aligned}$$

for $i = 0, 1, \dots, q-2$, $q \geq 2$, plus the following blocks as the case may be.

If $r = 0$,

$$\begin{aligned}&(\infty_1, x_{3q-2}, x_{3q-1}; x_{3q}, \infty_4, \infty_5),\\&(\infty_2, x_{3q-1}, x_{3q}; x_1, \infty_4, \infty_5),\\&(\infty_3, x_{3q}, x_1; x_2, \infty_4, \infty_5).\end{aligned}$$

If $r = 1$,

$$\begin{aligned}&(\infty_1, x_{3q-2}, x_{3q-1}; x_{3q+1}, \infty_4, \infty_5),\\&(\infty_2, x_{3q-1}, x_{3q}; x_1, \infty_4, \infty_1),\\&(\infty_3, x_{3q}, x_{3q+1}; x_2, \infty_4, \infty_2),\\&(\infty_5, x_{3q+1}, x_1; x_{3q}, \infty_4, \infty_3).\end{aligned}$$

If $r = 2$,

$$(\infty_1, x_{3q-2}, x_{3q-1}; x_{3q+2}, \infty_4, \infty_5),$$
$$(\infty_2, x_{3q-1}, x_{3q}; x_1, \infty_4, \infty_5),$$
$$(\infty_3, x_{3q}, x_{3q+1}; x_2, \infty_1, \infty_2),$$
$$(\infty_4, x_{3q+1}, x_{3q+2}; x_{3q}, \infty_1, \infty_3),$$
$$(\infty_5, x_{3q+2}, x_1; x_{3q+1}, \infty_2, \infty_3). \qquad \square$$

Finally, after settling the infinity points by using the above lemmas, if u is large we need to decompose the subgraph $\langle Z_u, L\rangle$, where L is the set of the differences unused (*difference leave*). Since by applying Lemmas 2.1 – 2.13 it could be necessary to use the differences 1, 2 or 4, while Lemma 2.14 does not impose any restriction, it is possible to combine infinity points and differences in such a way that the difference leave L is the set of the "small" differences, where 1, 2 or 4 could possibly be avoided.

Lemma 2.15. *Let $\alpha \in \{0, 4, 8\}$ and u, s be positive integers such that $u > 12s + \alpha$. Then there exists a decomposition of $\langle Z_u, L\rangle$ into 3-suns, where:*

i) $\alpha = 0$ *and* $L = [1, 6s]$*;*

ii) $\alpha = 4$ *and* $L = [3, 6s + 2]$*;*

iii) $\alpha = 8$ *and* $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$.

Proof.

i) Consider the orbits (S_j) under Z_u, where $S_j = (5s + 1 + j,\ 5s - j,\ 0; 3s,\ s,\ u - 2 - 2j)$, $j = 0, 1, \ldots, s - 1$.

ii) Consider the orbits in i), where (S_0) is replaced with the orbit of $(6s+1,\ 4s,\ 0; s,\ 9s,\ 6s + 2)$.

iii) Consider the orbits in i), where the orbits (S_0) and (S_1) are replaced with the orbits of $(6s + 1,\ 4s,\ 0; s,\ 9s,\ 6s + 4)$ and $(5s + 2,\ 5s - 1,\ 0; 3s,\ s,\ 6s + 2)$. $\square$

3 The main result

Let $(X, \mathcal{S})$ be a $3\mathrm{SS}(n)$ and $m \equiv 0, 1, 4, 9 \pmod{12}$.

Lemma 3.1. *If $(X, \mathcal{S})$ is embedded in a 3-sun system of order $m > n$, then $m \geq \frac{7}{5}n + 1$.*

Proof. Suppose $(X, \mathcal{S})$ is embedded in $(X', \mathcal{S}')$, with $|X'| = m = n + u$ (u positive integer). Let c_i be the number of 3-suns of $\mathcal{S}'$ each of which contains exactly i edges in $X' \setminus X$. Then $\sum_{i=1}^{6} i \times c_i = \binom{u}{2}$ and $\sum_{i=1}^{5}(6 - i)c_i = u \times n$, from which it follows $6c_2 + 12c_3 + 18c_4 + 24c_5 + 30c_6 = \frac{u(5u-2n-5)}{2}$ and so $u \geq \frac{2}{5}n + 1$ and $m \geq \frac{7}{5}n + 1$. $\square$

By previous Lemma:

1. if $n = 60k + 5r$, $r = 0, 5, 8, 9$, then $m \geq 84k + 7r + 1$;
2. if $n = 60k + 5r + 1$, $r = 0, 3, 4, 7$, then $m \geq 84k + 7r + 3$;
3. if $n = 60k + 5r + 2$, $r = 2, 7, 10, 11$, then $m \geq 84k + 7r + 4$;
4. if $n = 60k + 5r + 3$, $r = 2, 5, 6, 9$, then $m \geq 84k + 7r + 6$;

5. if $n = 60k + 5r + 4$, $r = 0, 1, 4, 9$, then $m \geq 84k + 7r + 7$.

In order to prove that the necessary conditions for embedding a 3-sun system $(X, \mathcal{S})$ of order n in a 3-sun system of order $m = n + u$, $u > 0$ are also sufficient, the graph $\langle Z_u \cup X, D_u \rangle$ will be expressed as a union of edge-disjoint subgraphs $\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup X, D \rangle \cup \langle Z_u, L \rangle$, where $L = D_u \setminus D$ is the difference leave, and $\langle Z_u \cup X, D \rangle$ (if necessary, expressed itself as a union of subgraphs) will be decomposed by using Lemmas 2.1 – 2.14, while if $L \neq \emptyset$, $\langle Z_u, L \rangle$ will be decomposed by Lemma 2.15. To obtain our main result we will distinguish the five cases 1. – 5. listed before by giving a general proof for any $k \geq 0$ with the exception of a few cases for $k = 0$, which will be indicated by a star $\star$ and solved in Appendix. Finally, note that:

a) $u \equiv 0, 1, 4$, or $9 \pmod{12}$, if $n \equiv 0 \pmod{12}$;

b) $u \equiv 0, 3, 8$, or $11 \pmod{12}$, if $n \equiv 1 \pmod{12}$;

c) $u \equiv 0, 5, 8$, or $9 \pmod{12}$, $n \equiv 4 \pmod{12}$;

d) $u \equiv 0, 3, 4$, or $7 \pmod{12}$, if $n \equiv 9 \pmod{12}$.

Proposition 3.2. *For any $n = 60k + 5r$, $r = 0, 5, 8, 9$, there exists a decomposition of $K_{n+u} \setminus K_n$ into 3-suns for every admissible $u \geq 24k + 2r + 1$.*

Proof. Let $X = \{\infty_1, \infty_2, \ldots, \infty_{60k+5r}\}$, $r = 0, 5, 8, 9$, and $u = 24k + 2r + 1 + h$, with $h \geq 0$. Set $h = 12s + l$, $0 \leq l \leq 11$ (l depends on r), and distinguish the following cases.

Case 1: $r = 0, 5, 8, 9$ and $l = 0$ (odd u).

Write $\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup X, D \rangle \cup \langle Z_u, L \rangle$, where $D = [6s + 1, 12k + r + 6s]$, $|D| = 12k + r$, and $L = [1, 6s]$, and apply Lemmas 2.14 and 2.15.

Case 2: $r = 0, 9$ and $l = 8$ (odd u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{2, 6s+3, 6s+5\} \rangle \cup \\ \langle Z_u \cup \{\infty_4\}, \{1\} \rangle \cup \langle Z_u \cup \{\infty_5\}, \{6s+4\} \rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D' \rangle \cup \langle Z_u, L \rangle,\end{aligned}$$

where $D' = [6s + 6, 12k + r + 6s + 4]$, $|D'| = 12k + r - 1$, and $L = [3, 6s + 2]$, and apply Lemmas 2.13, 2.10, 2.14 and 2.15.

Case 3: $r = 5, 8$ and $l = 4$ (odd u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\} \rangle \cup \langle Z_u \cup \{\infty_5\}, \{1\} \rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D' \rangle \cup \langle Z_u, L \rangle,\end{aligned}$$

where $D' = [6s + 3, 12k + r + 6s + 2] \setminus \{6s + 4\}$, $|D'| = 12k + r - 1$, and $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$, and apply Lemmas 2.3, 2.10, 2.14 and 2.15.

Case 4: $r = 0, 8$ and $l = 3$ (even u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\} \rangle \cup \langle Z_u \cup \{\infty_4, \infty_5\}, \{2\} \rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D' \rangle \cup \langle Z_u, L \rangle,\end{aligned}$$

where $D' = [6s+3, 12k+r+6s+1]$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.5, 2.1, 2.14 and 2.15.

Case 5: $r = 0$ and $l = 11$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 2, \tfrac{u}{2}\}\rangle \cup\\ &\langle Z_u \cup \{\infty_4, \infty_5\}, \{4, 6s+3, 6s+5, 6s+7\}\rangle \cup\\ &\langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+6, 12k+6s+5] \setminus \{6s+7\}$, $|D'| = 12k-1$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.9, 2.12, 2.14 and 2.15.

Case 6: $r = 5$ and $l = 1$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup\\ &\langle Z_u \cup \{\infty_7, \infty_8, \infty_9, \infty_{10}\}, \{2\}\rangle \cup\\ &\langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_{10}\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+3, 12k+6s+5]$, $|D'| = 12k+3$, and $L = [3, 6s+2]$, and apply Lemmas 2.7, 2.2, 2.14 and 2.15.

Case 7: $r = 5, 9$ and $l = 9$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup\\ &\langle Z_u \cup \{\infty_4, \infty_5\}, \{2, 6s+3, 6s+4, 6s+5\}\rangle \cup\\ &\langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+6, 12k+r+6s+4]$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.5, 2.12, 2.14 and 2.15.

Case 8: $r = 8$ and $l = 7$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 2, \tfrac{u}{2}\}\rangle \cup\\ &\langle Z_u \cup \{\infty_4\}, \{4\}\rangle \cup \langle Z_u \cup \{\infty_5\}, \{6s+5\}\rangle \cup\\ &\langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+3, 12k+6s+11] \setminus \{6s+4, 6s+5\}$, $|D'| = 12k+7$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.9, 2.10, 2.14 and 2.15.

Case 9: $r = 9$ and $l = 5$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_4\}, \{2\}\rangle \cup\\ &\langle Z_u \cup \{\infty_5\}, \{4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+3, 12k+6s+11] \setminus \{6s+4\}$, $|D'| = 12k+8$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.5, 2.10, 2.14 and 2.15. □

Proposition 3.3. *For any $n = 60k + 5r + 1$, $r = 0, 3, 4, 7$, there exists a decomposition of $K_{n+u} \setminus K_n$ into 3-suns for every admissible $u \geq 24k + 2r + 2$.*

Proof. Let $X = \{\infty_1, \infty_2, \ldots, \infty_{60k+5r+1}\}$, $r = 0, 3, 4, 7$, and $u = 24k + 2r + 2 + h$, with $h \geq 0$. Set $h = 12s + l$, $0 \leq l \leq 11$, and distinguish the following cases.

Case 1: $r = 0, 3$ and $l = 1$ (odd u).

Write $\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty\}, \{6s + 2\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty\}), D'\rangle \cup \langle Z_u, L\rangle$, where $D' = [6s + 1, 12k + r + 6s + 1] \setminus \{6s + 2\}$, $|D'| = 12k + r$, and $L = [1, 6s]$, and apply Lemmas 2.10, 2.14 and 2.15.

Case 2: $r = 0, 3, 4, 7$ and $l = 9$ (odd u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 6s + 3, 6s + 4\}\rangle \cup \\ \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s + 5, 6s + 7\}\rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+6, 12k+r+6s+5] \setminus \{6s+7\}$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.13, 2.14 and 2.15.

Case 3: $r = 4^\star, 7$ and $l = 5$ (odd u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\}\rangle \cup \langle Z_u \cup \{\infty_5\}, \{1\}\rangle \cup \\ \langle Z_u \cup \{\infty_6\}, \{6s + 8\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s + 3, 12k + r + 6s + 3] \setminus \{6s + 4, 6s + 8\}$, $|D'| = 12k + r - 1$, and $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$, and apply Lemmas 2.3, 2.10, 2.14 and 2.15.

Case 4: $r = 0, 4$ and $l = 6$ (even u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \\ \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s + 3, 6s + 5\}\rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+4, 12k+r+6s+3] \setminus \{6s+5\}$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.5, 2.13, 2.14 and 2.15.

Case 5: $r = 0$ and $l = 10$ (even u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \\ \langle Z_u \cup \{\infty_7, \infty_8, \infty_9, \infty_{10}\}, \{2\}\rangle \cup \\ \langle Z_u \cup \{\infty_{11}\}, \{4, 6s + 3, 6s + 5, 6s + 6, 6s + 7\}\rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_{11}\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s + 8, 12k + 6s + 5]$, $|D'| = 12k - 2$, and $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$, and apply Lemmas 2.7, 2.2, 2.11, 2.14 and 2.15.

Case 6: $r = 3, 7$ and $l = 0$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = \{2\} \cup [6s+3, 12k+r+6s]$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.7, 2.14 and 2.15.

Case 7: $r = 3$ and $l = 4$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_5\}, \{2\}\rangle \cup \langle Z_u \cup \{\infty_6\}, \{6s+5\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+3, 12k+6s+5] \setminus \{6s+5\}$, $|D'| = 12k+2$, and $L = [3, 6s+2]$, and apply Lemmas 2.6, 2.10, 2.14 and 2.15.

Case 8: $r = 4$ and $l = 2$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_5, \infty_6\}, \{2\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+3, 12k+6s+5]$, $|D'| = 12k+3$, and $L = [3, 6s+2]$, and apply Lemmas 2.6, 2.1, 2.14 and 2.15.

Case 9: $r = 7$ and $l = 8$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 2, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{4, 6s+3, 6s+7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+5, 12k+6s+11] \setminus \{6s+7\}$, $|D'| = 12k+6$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.9, 2.13, 2.14 and 2.15. □

Proposition 3.4. *For any $n = 60k+5r+2$, $r = 2, 7, 10, 11$, there exists a decomposition of $K_{n+u} \setminus K_n$ into 3-suns for every admissible $u \geq 24k+2r+2$.*

Proof. Let $X = \{\infty_1, \infty_2, \ldots, \infty_{60k+5r+2}\}$, $r = 2, 7, 10, 11$, and $u = 24k+2r+2+h$, with $h \geq 0$. Set $h = 12s+l$, $0 \leq l \leq 11$, and distinguish the following cases.

Case 1: $r = 2, 11$ and $l = 3$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1\}, \{6s+2\}\rangle \cup \langle Z_u \cup \{\infty_2\}, \{6s+4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+1, 12k+r+6s+2] \setminus \{6s+2, 6s+4\}$, $|D'| = 12k+r$, and $L = [1, 6s]$, and apply Lemmas 2.10, 2.14 and 2.15.

Case 2: $r = 2, 7, 10, 11$ and $l = 7$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2\}, \{1, 2, 6s+3, 6s+4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+5, 12k+r+6s+4]$, $|D'| = 12k+r$, and $L = [3, 6s+2]$, and apply Lemmas 2.12, 2.14 and 2.15.

Case 3: $r = 7, 10$ and $l = 11$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 6s+3, 6s+4\}\rangle \cup \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s+5, 6s+7\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{6s+8\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+6, 12k+r+6s+6] \setminus \{6s+7, 6s+8\}$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.13, 2.10, 2.14 and 2.15.

Case 4: $r = 2$ and $l = 6$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_5, \infty_6, \infty_7\}, \{2, 6s+3, 6s+5\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+4, 12k+6s+5] \setminus \{6s+5\}$, $|D'| = 12k+1$, and $L = [3, 6s+2]$, and apply Lemmas 2.6, 2.13, 2.14 and 2.15.

Case 5: $r = 2, 10$ and $l = 10$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \dots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{2, 6s+3, 6s+4, 6s+5, 6s+6\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+7, 12k+r+6s+5]$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.7, 2.11, 2.14 and 2.15.

Case 6: $r = 7, 11$ and $l = 4$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_4, \infty_5, \infty_6, \infty_7\}, \{2, 4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s + 3, 12k + r + 6s + 2] \setminus \{6s + 4\}$, $|D'| = 12k + r - 1$, and $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$, and apply Lemmas 2.5, 2.3, 2.14 and 2.15.

Case 7: $r = 7$ and $l = 8$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s + 3, 6s + 5\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{6s + 7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+4, 12k+6s+11] \setminus \{6s+5, 6s+7\}$, $|D'| = 12k+6$, and $L = [3, 6s+2]$, and apply Lemmas 2.5, 2.13, 2.10, 2.14 and 2.15.

Case 8: $r = 10$ and $l = 2$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{2\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s + 3, 12k + 6s + 11]$, $|D'| = 12k + 9$, and $L = [3, 6s + 2]$, and apply Lemmas 2.7, 2.10, 2.14 and 2.15.

Case 9: $r = 11$ and $l = 0$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_7\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = \{2\} \cup [6s + 3, 12k + 6s + 11]$, $|D'| = 12k + 10$, and $L = [3, 6s + 2]$, and apply Lemmas 2.8, 2.14 and 2.15. □

Proposition 3.5. *For any $n = 60k + 5r + 3$, $r = 2, 5, 6, 9$, there exists a decomposition of $K_{n+u} \setminus K_n$ into 3-suns for every admissible $u \geq 24k + 2r + 3$.*

Proof. Let $X = \{\infty_1, \infty_2, \ldots, \infty_{60k+5r+3}\}$, $r = 2, 5, 6, 9$, and $u = 24k + 2r + 3 + h$, with $h \geq 0$. Set $h = 12s + l$, $0 \leq l \leq 11$, and distinguish the following cases.

Case 1: $r = 2, 5, 6, 9$ and $l = 4$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 6s + 3, 6s + 4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = \{2\} \cup [6s + 5, 12k + r + 6s + 3]$, $|D'| = 12k + r$, and $L = [3, 6s + 2]$, and apply Lemmas 2.13, 2.14 and 2.15.

Case 2: $r = 2, 5$ and $l = 8$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2\}, \{1, 6s + 3, 6s + 4, 6s + 5\}\rangle \cup \langle Z_u \cup \{\infty_3\}, \{2\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s + 6, 12k + r + 6s + 5]$, $|D'| = 12k + r$, and $L = [3, 6s + 2]$, and apply Lemmas 2.12, 2.10, 2.14 and 2.15.

Case 3: $r = 6, 9$ and $l = 0$ (odd u).

If $s = 0$, then write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_8\}, \{1, \tfrac{u}{3}\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_8\}), D'\rangle,$$

where $D' = [2, 12k + r + 1] \setminus \{\frac{u}{3}\}$, $|D'| = 12k + r - 1$, and apply Lemmas 2.4 and 2.14.

If $s > 0$, then write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 5s, 5s + 1\}\rangle \cup \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s + 1, 6s + 3\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{6s + 2\}\rangle \cup \langle Z_u \cup \{\infty_8\}, \{6s + 4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_8\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = \{2s + 1, 4s\} \cup [6s + 5, 12k + r + 6s + 1]$, $|D'| = 12k + r - 1$, and $L = [3, 6s] \setminus \{2s + 1, 4s, 5s, 5s + 1\}$, and apply Lemmas 2.13, 2.10 and 2.14 to decompose the first five subgraphs, while to decompose the last one apply Lemma 2.15 i) and delete the orbit (S_0).

Case 4: $r = 2, 6$ and $l = 1$ (even u).

Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = \{2\} \cup [6s + 3, 12k + r + 6s + 1]$, $|D'| = 12k + r$, and $L = [3, 6s + 2]$, and apply Lemmas 2.5, 2.14 and 2.15.

Case 5: $r = 2^\star$ and $l = 5$ (even u).

Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_7, \infty_8, \infty_9, \infty_{10}\}, \{2\}\rangle \cup \langle Z_u \cup \{\infty_{11}, \infty_{12}, \infty_{13}\}, \{4, 6s + 3, 6s + 7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_{13}\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+5, 12k+6s+5] \setminus \{6s+7\}$, $|D'| = 12k$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.7, 2.2, 2.13, 2.14 and 2.15.

Case 6: $r = 5, 9$ and $l = 7$ (even u).

Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_7, \infty_8\}, \{2, 6s + 3, 6s + 4, 6s + 5\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_8\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+6, 12k+r+6s+4]$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.7, 2.12, 2.14 and 2.15.

Case 7: $r = 5$ and $l = 11$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_5, \infty_6\}, \{2, 6s+3, 6s+5, 6s+6\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{4\}\rangle \cup \langle Z_u \cup \{\infty_8\}, \{6s+7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_8\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+8, 12k+6s+11]$, $|D'| = 12k+4$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.6, 2.12, 2.10, 2.14 and 2.15.

Case 8: $r = 6$ and $l = 9$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s+3, 6s+5\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{4\}\rangle \cup \langle Z_u \cup \{\infty_8\}, \{6s+7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_8\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+6, 12k+6s+11] \setminus \{6s+7\}$, $|D'| = 12k+5$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.5, 2.13, 2.10, 2.14 and 2.15.

Case 9: $r = 9$ and $l = 3$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 2, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+3, 12k+6s+11]$, $|D'| = 12k+9$, and $L = [3, 6s+2]$, and apply Lemmas 2.9, 2.14 and 2.15. □

Proposition 3.6. *For any $n = 60k+5r+4$, $r = 0, 1, 4, 9$, there exists a decomposition of $K_{n+u} \setminus K_n$ into 3-suns for every admissible $u \geq 24k+2r+3$.*

Proof. Let $X = \{\infty_1, \infty_2, \ldots, \infty_{60k+5r+4}\}$, $r = 0, 1, 4, 9$, and $u = 24k+2r+3+h$, with $h \geq 0$. Set $h = 12s+l$, $0 \leq l \leq 11$, and distinguish the following cases.

Case 1: $r = 0, 1^\star, 4, 9$ and $l = 2$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = \{1, 6s+3\} \cup [6s+5, 12k+r+6s+2]$, $|D'| = 12k+r$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.3, 2.14 and 2.15.

Case 2: $r = 0, 9$ and $l = 6$ (odd u).
Write

$$\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 6s+3, 6s+4\}\rangle \cup \langle Z_u \cup \{\infty_4\}, \{2\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+5, 12k+r+6s+4]$, $|D'| = 12k+r$, and $L = [3, 6s+2]$, and apply Lemmas 2.13, 2.10, 2.14 and 2.15.

Case 3: $r = 1, 4$ and $l = 10$ (odd u).
Write

$$\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2\}, \{1, 6s+3, 6s+5, 6s+6\}\rangle \cup \langle Z_u \cup \{\infty_3\}, \{2\}\rangle \cup \langle Z_u \cup \{\infty_4\}, \{6s+4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+7, 12k+r+6s+6]$, $|D'| = 12k+r$, and $L = [3, 6s+2]$, and apply Lemmas 2.12, 2.10, 2.14 and 2.15.

Case 4: $r = 0, 4$ and $l = 5$ (even u).
Write

$$\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_7, \infty_8, \infty_9\}, \{2, 6s+3, 6s+5\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_9\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+4, 12k+r+6s+3] \setminus \{6s+5\}$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.7, 2.13, 2.14 and 2.15.

Case 5: $r = 0$ and $l = 9$ (even u).
Write

$$\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_5, \infty_6, \infty_7\}, \{2, 6s+3, 6s+5\}\rangle \cup \langle Z_u \cup \{\infty_8\}, \{4\}\rangle \cup \langle Z_u \cup \{\infty_9\}, \{6s+7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_9\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+6, 12k+6s+5] \setminus \{6s+7\}$, $|D'| = 12k-1$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.6 , 2.13, 2.10, 2.14 and 2.15.

Case 6: $r = 1$ and $l = 7$ (even u).
Write

$$\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_7\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_8, \infty_9\}, \{2, 4, 6s+3, 6s+5\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_9\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+6, 12k+6s+5]$, $|D'| = 12k$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.8, 2.12, 2.14 and 2.15.

Case 7: $r = 1, 9$ and $l = 11$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \\ \langle Z_u \cup \{\infty_4\}, \{2, 4, 6s+3, 6s+5, 6s+6\}\rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+7, 12k+r+6s+6]$, $|D'| = 12k+r$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.5, 2.11, 2.14 and 2.15.

Case 8: $r = 4$ and $l = 1$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\}\rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = \{2\} \cup [6s+3, 12k+6s+5]$, $|D'| = 12k+4$, and $L = [3, 6s+2]$, and apply Lemmas 2.6, 2.14 and 2.15.

Case 9: $r = 9$ and $l = 3$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_4\}, \{2\}\rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+3, 12k+6s+11]$, $|D'| = 12k+9$, and $L = [3, 6s+2]$, and apply Lemmas 2.5, 2.10, 2.14 and 2.15. □

Combining Lemma 3.1 and Propositions 3.2 – 3.6 gives our main theorem.

Theorem 3.7. *Any* 3SS(n) *can be embedded in a* 3SS(m) *if and only if*

$$m \geq \tfrac{7}{5}n + 1 \quad \text{or} \quad m = n.$$

References

[1] D. E. Bryant and C. A. Rodger, The Doyen-Wilson theorem extended to 5-cycles, *J. Comb. Theory Ser. A* **68** (1994), 218–225, doi:10.1016/0097-3165(94)90101-5.

[2] D. E. Bryant and C. A. Rodger, On the Doyen-Wilson theorem for m-cycle systems, *J. Combin. Des.* **2** (1994), 253–271, doi:10.1002/jcd.3180020405.

[3] J. Doyen and R. M. Wilson, Embeddings of Steiner triple systems, *Discrete Math.* **5** (1973), 229–239, doi:10.1016/0012-365x(73)90139-8.

[4] C.-M. Fu, Y.-L. Lin, S.-W. Lo, Y.-F. Hsu and W.-C. Huang, The Doyen-Wilson theorem for bull designs, *Discrete Math.* **313** (2013), 498–507, doi:10.1016/j.disc.2012.11.023.

[5] H.-L. Fu and C. C. Lindner, The Doyen-Wilson theorem for maximum packings of K_n with 4-cycles, *Discrete Math.* **183** (1998), 103–117, doi:10.1016/s0012-365x(97)00080-0.

[6] H.-L. Fu, C. C. Lindner and C. A. Rodger, Two Doyen-Wilson theorems for maximum packings with triples, *Discrete Math.* **178** (1998), 63–71, doi:10.1016/s0012-365x(97)81817-1.

[7] H. Gao and J. Wang, Doyen-Wilson theorem for perfect hexagon triple systems, *Discrete Math.* **311** (2011), 1006–1014, doi:10.1016/j.disc.2011.03.004.

[8] W.-C. Huang and W.-C. Yang, The Doyen-Wilson theorem for extended directed triple systems, *Ars Combin.* **84** (2007), 77–83.

[9] G. Lo Faro and A. Tripodi, The Doyen-Wilson theorem for kite systems, *Discrete Math.* **306** (2006), 2695–2701, doi:10.1016/j.disc.2006.03.074.

[10] G. Lo Faro and A. Tripodi, Embeddings of λ-fold kite systems, $\lambda \geq 2$, *Australas. J. Combin.* **36** (2006), 143–150, `https://ajc.maths.uq.edu.au/pdf/36/ajc_v36_p143.pdf`.

[11] R. Peltesohn, Eine Lösung der beiden Heffterschen Differenzenprobleme, *Compositio Math.* **6** (1939), 251–257, `http://www.numdam.org/item?id=CM_1939__6__251_0`.

[12] G. Stern and H. Lenz, Steiner triple systems with given subspaces; another proof of the Doyen-Wilson-theorem, *Boll. Un. Mat. Ital. A Serie 5* **17** (1980), 109–114.

[13] J. Wang, Perfect dexagon triple systems with given subsystems, *Discrete Math.* **309** (2009), 2930–2933, doi:10.1016/j.disc.2008.07.004.

[14] J.-X. Yin and B.-S. Gong, Existence of G-designs with $|V(G)| = 6$, in: W. D. Wallis, H. Shen, W. Wei and L. Zhu (eds.), *Combinatorial Designs and Applications*, Marcel Dekker, New York, volume 126 of *Lecture Notes in Pure and Applied Mathematics*, 1990 pp. 201–218.

Appendix

- $n = 21$, $u = 12s + 15$

  Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\}\rangle \cup \\ \langle Z_u \cup \{\infty_5\}, \{1\}\rangle \cup \langle Z_u \cup \{\infty_6\}, \{6s + 7\}\rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), \{6s + 3, 6s + 5, 6s + 6\}\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

  where $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$, and apply Lemmas 2.3, 2.10, 2.14 and 2.15.

- $n = 13$, $u = 12s + 12$

  Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, 6s + 6\}\rangle \cup \\ \langle Z_u \cup \{\infty_7, \infty_8, \infty_9, \infty_{10}\}, \{2\}\rangle \cup \\ \langle Z_u \cup \{\infty_{11}, \infty_{12}, \infty_{13}\}, \{4, 6s + 3, 6s + 5\}\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

  where $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$, and apply Lemmas 2.7, 2.2, 2.13 and 2.15.

- $n = 9$, $u = 12s + 7$

  Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\}\rangle \cup \\ \langle Z_u \cup \{\infty_5, \infty_6, \infty_7, \infty_8, \infty_9\}, \{1\}\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

  where $L = [3, 6s + 3] \setminus \{4\}$, and apply Lemmas 2.3, 2.14 and decompose $\langle Z_u, L\rangle$ as in Lemma 2.15 iii), taking in account that $|6s + 4|_{12s+7} = 6s + 3$.

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The conductivity of superimposed key-graphs with a common one-dimensional adjacency nullspace

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Abstract

Two connected labelled graphs H_1 and H_2 of nullity one, with identical one-vertex deleted subgraphs $H_1 - z_1$ and $H_2 - z_2$ and having a common eigenvector in the nullspace of their 0-1 adjacency matrix, can be overlaid to produce the superimposition Z. The graph Z is $H_1 + z_2$ and also $H_2 + z_1$ whereas $Z + e$ is obtained from Z by adding the edge $\{z_1, z_2\}$. We show that the nullity of Z cannot take all the values allowed by interlacing. We propose to classify graphs with two chosen vertices according to the type of the vertices occurring by using a 3-type-code. Out of the 27 values it can take, only 9 are hypothetically possible for Z, 8 of which are known to exist. Moreover, the SSP molecular model predicts conduction or insulation at the Fermi level of energy for 11 possible types of devices consisting of a molecule and two prescribed connecting atoms over a small bias voltage. All 11 molecular device types are realizable for general molecules, but the structure of Z and of $Z + e$ restricts the number to just 5.

Keywords: Nullity, core vertices, key-graphs, superimposition, circuit.

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1 Introduction

The graphs we consider are simple, that is they are undirected with no multiple edges or loops. The 0-1 *adjacency matrix* $\mathbf{G} = (a_{ij})$ of a labelled graph G on n vertices is a $n \times n$ matrix such that $a_{ij} = 1$ if there is an edge between the vertices i and j, and $a_{ij} = 0$ otherwise. The degree of a vertex v is the number of non-zero entries in the vth row (or column) of $\mathbf{G}$. A graph is *singular* if $\mathbf{G}$ has zero as an eigenvalue, and *non-singular* otherwise. The multiplicity of zero in the spectrum of $\mathbf{G}$ is the *nullity* $\eta = \eta(G)$ of the graph G. A *kernel eigenvector* $\mathbf{x}$ of G is a nonzero vector that satisfies $\mathbf{Gx} = \mathbf{0}$. The *nullspace* $\ker(\mathbf{G})$ of $\mathbf{G}$ is generated by a basis of η linearly independent kernel eigenvectors. Thus, a graph G is singular if and only if $\dim(\ker(\mathbf{G})) \geq 1$.

A *core vertex* (CV) of G corresponds to a nonzero entry in some kernel eigenvector. The set of CVs is an invariant of G, that is, it is independent of the basis chosen for $\ker(\mathbf{G})$ [14, 15, 16]. A vertex which is not a CV is a *core-forbidden vertex* (CFV), recently referred to as a *Fiedler vertex* [1, 10]. Proposition 1.1 characterizes a CV in a singular graph, and Corollary 1.2 is its direct consequence for graphs of nullity one.

Proposition 1.1 ([18])**.** *Let $G + u$ be a graph obtained from G by adding a vertex u. Then $\eta(G + u) = \eta(G) + 1$ if and only if u is a CV of $G + u$.*

Corollary 1.2 ([17])**.** *Let v be a CV of a graph G of nullity one. Then the graph $G - v$ is non-singular.*

We make use of a result on the nullity of graphs derived from Cauchy's Interlacing Theorem for real symmetric matrices.

Theorem 1.3 ([11, p. 119])**.** *Let v be any vertex of a graph G on $n \geq 2$ vertices. Then*

$$\eta(G) - 1 \leq \eta(G - v) \leq \eta(G) + 1.$$

Theorem 1.3 permits the nullity of a graph to change by at most one on the deletion or addition of a vertex. Thus, a vertex u in a graph G can be one of three *types*, depending on the difference of the nullity of $G - u$ from the nullity of G. Following the terminology used in [4], a vertex u is a CV, a *middle core-forbidden vertex* (CFV_{mid}) or an *upper core-forbidden vertex* (CFV_{upp}) if the nullity of the graph $G - u$ obtained from G upon deleting the vertex u is $\eta(G) - 1$, $\eta(G)$, or $\eta(G) + 1$, respectively. It follows from Proposition 1.1 that CFVs are vertices corresponding to a zero entry in each kernel eigenvector in the nullspace of G. For the eigenvalue zero, CFVs were renamed *F-vertices*. For the specific case of CFV_{upp}, they were renamed and *P-vertices* [2]. Whether electricity flows through a molecule or not is mainly determined by the types of the two vertices (atoms of the molecule) chosen as terminals with a bias voltage across them [5].

From the definitions of the possible types of vertices in a graph, the following result is immediate.

Lemma 1.4. *Let H_1 and H_2 be two graphs such that $\eta(H_1) = \eta(H_2)$. Then $\eta(H_1 - z_1) = \eta(H_2 - z_2)$ if and only if z_1 and z_2 are of the same type in H_1 and in H_2, respectively.*

1.1 Superimpositions

To explore the structure of singular graphs, basic subgraphs of nullity one that are found in singular graphs are constructed in [14, 15, 16]. In Proposition 4.3 of [16], it is proved that a singular graph of nullity η has η induced subgraphs of nullity one having the least possible number of vertices (called *singular configurations*).

The kernel eigenvectors are key to determining the substructures that make a singular graph. Focus is placed on singular graphs of nullity one; otherwise distinct singular configurations which are induced subgraphs in a singular graph of nullity more than one may be masked by others belonging to linearly independent kernel eigenvectors. The vertices of a singular configuration corresponding to the nonzero entries in a kernel eigenvector $\mathbf{x}$ are the CVs of G and the remaining vertices, if any, are $\mathrm{CFV_{upp}}$ [16]. By Proposition 1.1, deleting a CV reduces the nullity, whereas deleting an $\mathrm{CFV_{upp}}$ increases the nullity. The question then arises: what are the conditions that need to be satisfied by a graph H of nullity one so that, for some vertex v, the graph $H + v$ retains nullity one and has the same nonzero entries of a kernel eigenvector as H? The investigations in this paper stem from the quest to answer this question.

First we fix some notation. Two labelled graphs G_1 and G_2 are *identical* if they are isomorphic to a labelled graph G and have the same labelling as G; we write $G_1 \equiv G_2 \equiv G$. We consider pairs of graphs of nullity one which have a common kernel eigenvector for some labelling of their vertices, and such that each of the two graphs have a vertex which, when deleted, yields two identical graphs. One such example is illustrated in Figure 1, where $\mathbf{x} = (1, -1, 1, -1, 0, 0, 0)^t$ is a kernel eigenvector of both H_1 and H_2 with the labelling shown, such that when the vertex labelled 7 is deleted from each, the resulting graphs are identical.

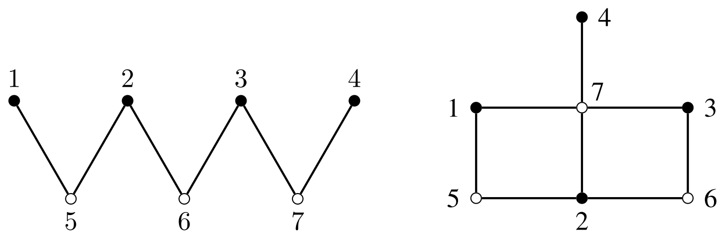


Figure 1: A pair of non-isomorphic graphs $G + 7$ of nullity one having a common kernel eigenvector $\mathbf{x} = (1, -1, 1, -1, 0, 0, 0)^t$.

If two graphs H_1 and H_2 are not isomorphic, but the deletion of a vertex z_1 of H_1 yields a graph identical to that obtained by deleting a vertex z_2 from H_2, then the difference in the dimensions of the nullspaces of H_1 and H_2 is bounded as given in the following theorem.

Theorem 1.5. *Let H_1 and H_2 be two graphs having vertices z_1 and z_2, respectively, such that $G \equiv H_1 - z_1 \equiv H_2 - z_2$. Then $|\eta(H_1) - \eta(H_2)| = 2$ when one of the vertices is a CV and the other is an $\mathrm{CFV_{upp}}$, and $|\eta(H_1) - \eta(H_2)| \leq 1$, otherwise.*

Proof. By Theorem 1.3, $\eta(G)-1 \leq \eta(H_1) \leq \eta(G)+1$ and $\eta(G)-1 \leq \eta(H_2) \leq \eta(G)+1$. Thus, $|\eta(H_1)-\eta(H_2)| \leq 2$. Equality holds only when, without loss of generality, $\eta(H_1) = \eta(G) - 1$ and $\eta(H_2) = \eta(G) + 1$, in which case z_1 is an $\mathrm{CFV_{upp}}$ in H_1 and z_2 is a CV in H_2. □

In the sequel, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two connected labelled graphs of order $n \geq 3$ whose 0-1 adjacency matrix has nullity one with a common kernel eigenvector $\mathbf{x}$ such that, for some vertex z_1 in $\mathcal{H}_1$ and some vertex z_2 in $\mathcal{H}_2$, $\mathcal{H}_1 - z_1 \equiv \mathcal{H}_2 - z_2 \equiv G$. The graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ are termed *key-graphs*. It follows immediately that the label of z_1 in $\mathcal{H}_1$ is the same as that of z_2 in $\mathcal{H}_2$. We choose the labels of the vertices z_1 and z_2 to be the last in the two graphs.

The *superimposition* of the key-graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ is the graph Z obtained from G by adding both vertices z_1 and z_2 adjacent to the same neighbours as those of z_1 in $\mathcal{H}_1$ and z_2 in $\mathcal{H}_2$. The graph $Z + e$ is obtained from Z by adding the edge $e = z_1z_2$.

In the next section, we look at some examples so that the concept of superimpositions and its possible effects on the type of vertices of a graph becomes clearer.

1.2 Motivation

A conjugated hydrocarbon molecule has a π-system where each carbon atom contributes a delocalized electron in the neutral molecule. The Hückel/Tight-Binding model simplifies Schrödinger's equation to $\mathbf{A}\mathbf{x} = E\mathbf{x}$ where $\mathbf{A}$ is the adjacency matrix of the carbon skeleton of the molecule, $\mathbf{x}$ represents a molecular orbital and E is the orbital energy. Since carbon has a valency of four, *chemical graphs* for π-systems have at most three sigma bonds per atom (edges meeting at any vertex). In this article we extend our study to any graph where the vertex degree (or valency) can be larger than three.

In chemistry, the role of the electrons in the molecule is crucial in determining the physical and chemical properties of the molecule. The discrete energy levels that an electron may occupy within a molecule are the solutions to Schrödinger's time-independent equation in quantum mechanics. The wave function as a solution of Schrödinger's equation predicts the electron probability density, which in Hückel theory is a sum of orbital densities.

The Hamiltonian for the n-atomic molecular system turns out to be a linear function of the 0-1 $n \times n$ adjacency matrix $\mathbf{G}$ of the labelled molecular graph G, whose eigenvalues give the energy E of the electron orbitals. The non-zero entries of $\mathbf{G}$ correspond to the sigma bonds between pairs of atoms.

In this article we investigate the change in nullity when forming Z and $Z + e$. The conduction of electricity through a molecular graph with a bias electrical potential across two vertices $\bar{L}$ and $\bar{R}$ depends on the nullities of $\mathbf{G}$ and of three of its induced subgraphs obtained by deleting $\bar{L}$ and $\bar{R}$ separately and jointly [4, 13]. Note that the deletion of a CFV typically preserves the chemical nature of the graph (unless it is a cut vertex), but addition typically does not. In Section 5, conductivity of molecular devices is discussed for examples of molecular graphs of the form Z and $Z + e$.

In Figure 2, the four vertices $1, 2, 3$ and 4 are CVs in both H and Z, but the vertex 5 is a CFV_{upp} in H, whereas each of the vertices 5 and 6 is a CFV_{mid} in Z. Each vertex of $Z + e$ is a CV. Both H and Z have nullity one and, since H has a kernel eigenvector $(1, 1, -1, -1, 0)^t$ and Z has a kernel eigenvector $(1, 1, -1, -1, 0, 0)^t$, there is a kernel eigenvector of $H - 5$ with the same nonzero entries as for H and Z. It is interesting to note that Z is obtained by superimposing two isomorphic copies $\mathcal{H}_1$ and $\mathcal{H}_2$ of the singular configuration H, but Z itself is not a singular configuration.

It is thus natural to ask whether $\mathcal{H}_1$ and $\mathcal{H}_2$ need to be isomorphic (as in the example discussed above) to retain nullity one in Z obtained from $\mathcal{H}_1$ by adding the vertex z_2. Also, is this a condition that $\mathcal{H}_1$ and $\mathcal{H}_2$ must satisfy so that the nonzero entries of a kernel eigenvector are preserved in Z?

The graph Z shown in Figure 3 is obtained by superimposing the two graphs of Fig-

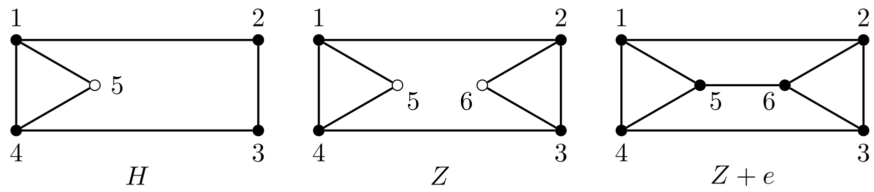


Figure 2: Two copies of H, of nullity one, are induced subgraphs of the superimposition Z, also of nullity one. The graph $Z + e$ is of nullity two. For all three graphs, there is a vector in the respective nullspace with the same nonzero part $1, 1, -1, -1$ associated with the first four labelled vertices.

ure 1. Adding the edge e between $z_1 = 7$ and $z_2 = 8$ produces $Z+e$. The nullity of Z is two whereas that of $Z + e$ is one. This example shows that $\mathcal{H}_1$ and $\mathcal{H}_2$ need not be isomorphic for the nullity to be one in $Z+e$. A kernel eigenvector of $Z+e$ is $(1, -1, 1, -1, 0, 0, 0, 0)^t$, and thus the nonzero entries of a kernel eigenvector of $\mathcal{H}_1$ and $\mathcal{H}_2$ are also preserved, even though $\mathcal{H}_1$ and $\mathcal{H}_2$ are not isomorphic.

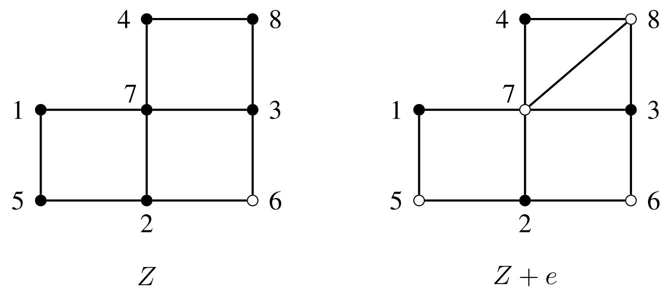


Figure 3: The graph $Z + e$ with nullity one, having a kernel eigenvector $(1, -1, 1, -1, 0, 0, 0, 0)^t$, is obtained from the superimposition Z (which has nullity two) of the graphs in Figure 1.

Observe that the nullities of Z and of $Z + e$ are different in the two examples discussed above. The vertices z_1 in $\mathcal{H}_1$ and z_2 in $\mathcal{H}_2$ in both examples are CFV_{upp}. They become CFV_{mid} in Z in the example of Figure 2 and also in $Z + e$ in the example of Figure 3. However, z_1 and z_2 become CV in $Z + e$ in the example of Figure 2 and also in Z in the example of Figure 3. The following results follow immediately from the definitions of CFV_{mid} and CV.

Proposition 1.6. *The vertices z_1 and z_2 are CFV_{mid} in the superimposition Z (respectively in $Z + e$) if and only if $\eta(Z) = 1$ (respectively $\eta(Z + e) = 1$).*

Proposition 1.7. *The vertices z_1 and z_2 are CV in the superimposition Z (respectively in $Z + e$) if and only if $\eta(Z) = 2$ (respectively $\eta(Z + e) = 2$).*

As we shall show, graphs satisfying Propositions 1.6 and 1.7 exist. However, is it possible that both z_1 and z_2 be CFV_{upp} in Z or in $Z + e$? Do the types of the vertices z_1

and z_2 determine the type in Z or $Z+e$? We shall investigate all possible combinations of the vertex type of z_1 in H_1 and z_2 in H_2.

By Lemma 1.4, the type of the vertex z_1 in $\mathcal{H}_1$ and of the vertex z_2 in $\mathcal{H}_2$ is the same. Moreover, as we shall see in Lemmas 2.2 and 3.1, vertices z_1 and z_2 are of the same type in Z and of the same type in $Z+e$ (the type in the latter graph $Z+e$ possibly different from that in the former Z). We thus propose a 3-*type-code*[1] where a type is denoted by:

1. C if it corresponds to a core vertex;
2. M if it corresponds to a middle core-forbidden vertex; and
3. U if it corresponds to an upper core-forbidden vertex.

The code consists of an ordered string of three types and, thus, it has three available positions, namely y_1, y_2 and y_3. Each of the positions y_1, y_2 and y_3 is filled with the symbol C, M or U, depending on the type of the vertices z_1 and z_2 in the key-graphs, in Z and in $Z+e$, in that order. The 3-type-code presents 27 classes of graphs. Algebraic considerations show that only 9 may exist.

The case when the two vertices z_1 and z_2 are both CFVs in the respective key-graphs is discussed in Section 2, yielding 8 possible classes of graphs. In Section 3, the case when they are both CVs produces just one class of graphs. For the graphs $\{Z\}$ and $\{Z+e\}$, what factors determine that the nullity of a graph remains unchanged on deleting a vertex? When does the type of a pair of adjacent vertices remain unchanged after deleting the edge between them? These questions are answered in Section 4. Chemical implications for the conductivity of a molecule which has a graph that is a superimposition are discussed in Section 5.

2 Core-forbidden vertices in the key-graphs

In this section, the vertices z_1 and z_2 are CFVs in the key-graphs $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Thus, the last entry of the common kernel eigenvector $\mathbf{x}$ of $\mathcal{H}_1$ and of $\mathcal{H}_2$ (which corresponds to z_1 and z_2) is zero. We write $\mathbf{x} = \left(\frac{\mathbf{v}}{0}\right)$ where $\mathbf{v} \neq \mathbf{0}$. Letting $\mathbf{z_1}$ and $\mathbf{z_2}$ denote the characteristic vectors representing the adjacencies of z_1 and z_2 to the vertices of G, we obtain

$$\mathbf{H_1 x} = \mathbf{H_1}\left(\frac{\mathbf{v}}{0}\right) = \left(\begin{array}{c|c} \mathbf{G} & \mathbf{z_1} \\ \hline \mathbf{z_1}^t & 0 \end{array}\right)\left(\frac{\mathbf{v}}{0}\right) = \left(\frac{\mathbf{Gv}}{\mathbf{z_1}^t\mathbf{v}}\right) = \mathbf{0} \tag{2.1}$$

and

$$\mathbf{H_2 x} = \mathbf{H_2}\left(\frac{\mathbf{v}}{0}\right) = \left(\begin{array}{c|c} \mathbf{G} & \mathbf{z_2} \\ \hline \mathbf{z_2}^t & 0 \end{array}\right)\left(\frac{\mathbf{v}}{0}\right) = \left(\frac{\mathbf{Gv}}{\mathbf{z_2}^t\mathbf{v}}\right) = \mathbf{0}, \tag{2.2}$$

for some $\mathbf{v} \neq \mathbf{0}$.

The following lemma explores the nullspaces of Z and of $Z+e$. On adding a vertex to the key graph $\mathcal{H}_1$, of nullity one, the graph Z or $Z+e$ produced is never non-singular.

Lemma 2.1. *If z_1 and z_2 are CFVs in $\mathcal{H}_1$ and in $\mathcal{H}_2$, respectively, then*

(i) $(\mathbf{x}, 0)^t = (\mathbf{v}, 0, 0)^t$ *is a kernel eigenvector of both Z and $Z+e$;*

[1]Different three letter acronyms are proposed in [6, 7] to classify classes of molecular graphs as conductors or insulators with respect to the graph-theoretical distance between two connecting vertices of the graph across which there is a small bias voltage.

(ii) $1 \leq \eta(Z) \leq 2$ *and* $1 \leq \eta(Z+e) \leq 2$.

Proof. Let $\mathbf{Z}$ be the adjacency matrix of the graph Z and let $\mathbf{W}$ be the adjacency matrix of $Z+e$, where z_1 and z_2 are respectively the n^{th} and $(n+1)^{\text{th}}$ labelled vertices of Z and of $Z+e$.

(i) Since

$$\mathbf{Z}\begin{pmatrix}\mathbf{v}\\ \hline 0\\ 0\end{pmatrix} = \left(\begin{array}{c|cc}\mathbf{G} & \mathbf{z_1} & \mathbf{z_2}\\ \hline \mathbf{z_1}^t & 0 & 0\\ \mathbf{z_2}^t & 0 & 0\end{array}\right)\begin{pmatrix}\mathbf{v}\\ \hline 0\\ 0\end{pmatrix} = \begin{pmatrix}\mathbf{Gv}\\ \hline \mathbf{z_1}^t\mathbf{v}\\ \mathbf{z_2}^t\mathbf{v}\end{pmatrix}$$

and

$$\mathbf{W}\begin{pmatrix}\mathbf{v}\\ \hline 0\\ 0\end{pmatrix} = \left(\begin{array}{c|cc}\mathbf{G} & \mathbf{z_1} & \mathbf{z_2}\\ \hline \mathbf{z_1}^t & 0 & 1\\ \mathbf{z_2}^t & 1 & 0\end{array}\right)\begin{pmatrix}\mathbf{v}\\ \hline 0\\ 0\end{pmatrix} = \begin{pmatrix}\mathbf{Gv}\\ \hline \mathbf{z_1}^t\mathbf{v}\\ \mathbf{z_2}^t\mathbf{v}\end{pmatrix},$$

then by (2.1) and (2.2), $(\mathbf{v},0,0)^t$ is a kernel eigenvector of $\mathbf{Z}$ and of $\mathbf{W}$.

(ii) By Theorem 1.3, $0 \leq \eta(Z) \leq 2$ and $0 \leq \eta(Z+e) \leq 2$. From (i) above, $\eta(Z) \geq 1$ and $\eta(Z+e) \geq 1$. Thus, $1 \leq \eta(Z) \leq 2$ and $1 \leq \eta(Z+e) \leq 2$. □

Next we show that the vertices z_1 and z_2 must be of the same type in each of the graphs Z and in $Z+e$, and that they cannot be CFV_{upp}.

Lemma 2.2. *Let z_1 and z_2 be CFVs in $\mathcal{H}_1$ and in $\mathcal{H}_2$, respectively. Then in each of the graphs Z and $Z+e$, the two vertices z_1 and z_2 are either both CFV_{mid} or both CV.*

Proof. Suppose first that z_1 and z_2 are not of the same type in Z. Then, deleting z_1 from Z yields the graph $\mathcal{H}_2$ which has a different nullity from the graph $\mathcal{H}_1$ obtained on deleting z_2 from Z, a contradiction since $\eta(\mathcal{H}_1) = \eta(\mathcal{H}_2) = 1$. A similar argument yields that the type of vertices z_1 and z_2 in $Z+e$ must be the same. From Lemma 2.1, $1 \leq \eta(Z) \leq 2$ and $1 \leq \eta(Z+e) \leq 2$, and thus by Propositions 1.6 and 1.7, z_1 and z_2 are either both CFV_{mid} or both CV. □

Remark 2.3. From Lemma 2.2 it follows that when z_1 and z_2 are CFVs in the key-graphs, each of the two positions y_2 and y_3 in the 3-type-code can be filled in two ways, namely C and M. Therefore, there are only eight possible different classes of the 3-type-code graphs having the first position y_1 filled with either M or U.

In the case when both z_1 and z_2 are CFVs in the key-graphs, we have the following necessary and sufficient condition.

Theorem 2.4. *Let z_1 and z_2 be CFVs in $\mathcal{H}_1$ and in $\mathcal{H}_2$, respectively. In the graph Z or $Z+e$, z_1 and z_2 are CV if and only if they correspond to nonzero entries in exactly one kernel eigenvector of the basis of the nullspace of the graph Z or $Z+e$.*

Proof. By Proposition 1.7, the dimension of the nullspace of Z and of $Z+e$ is two. From Lemma 2.1, $(\mathbf{x},0)^t = (\mathbf{v},0,0)^t$ is a kernel eigenvector of Z and of $Z+e$. Since z_1 and z_2 are CV in Z, they correspond to nonzero entries in a kernel eigenvector $(\mathbf{y_1},\alpha_1,\beta_1)^t$ of Z, for $\alpha_1 \neq 0$ and $\beta_1 \neq 0$. A similar argument holds for $Z+e$. □

We note that although there are 18 possible 3-type-code classes of graphs when the first entry of the code is not C, Lemma 2.2 restricts the number of possible classes to just 8.

3 Core vertices in key-graphs

In this section we show that for the case when z_1 and z_2 are both CV in the key-graphs $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, only one 3-type-code class may occur. This case is completely different from the case discussed in Section 2 in that, as we prove in Proposition 3.2 and Theorem 3.5, the nullity of each of the graphs Z and $Z + e$ can take only one value and it is not the same value in the two graphs.

Recall that $\mathcal{H}_1$ and $\mathcal{H}_2$ have a common kernel eigenvector generating their nullspace. Since z_1 and z_2 are CVs, the last entry of a common kernel eigenvector $\mathbf{x}$ of $\mathcal{H}_1$ and of $\mathcal{H}_2$ is nonzero, that is $\mathbf{x} = \begin{pmatrix} \mathbf{v} \\ \hline \alpha \end{pmatrix}$ for $\mathbf{v} \neq \mathbf{0}$ and $\alpha \neq 0$. Thus, letting $\mathbf{z_1}$ and $\mathbf{z_2}$ denote the characteristic vectors representing the adjacencies of z_1 and z_2 to the vertices of G, we obtain

$$\mathbf{H_1} \begin{pmatrix} \mathbf{v} \\ \hline \alpha \end{pmatrix} = \left(\begin{array}{c|c} \mathbf{G} & \mathbf{z_1} \\ \hline \mathbf{z_1}^t & 0 \end{array}\right) \begin{pmatrix} \mathbf{v} \\ \hline \alpha \end{pmatrix} = \begin{pmatrix} \mathbf{G}\mathbf{v} + \alpha \mathbf{z_1} \\ \hline \mathbf{z_1}^t \mathbf{v} \end{pmatrix} = \mathbf{0} \tag{3.1}$$

and

$$\mathbf{H_2} \begin{pmatrix} \mathbf{v} \\ \hline \alpha \end{pmatrix} = \left(\begin{array}{c|c} \mathbf{G} & \mathbf{z_2} \\ \hline \mathbf{z_2}^t & 0 \end{array}\right) \begin{pmatrix} \mathbf{v} \\ \hline \alpha \end{pmatrix} = \begin{pmatrix} \mathbf{G}\mathbf{v} + \alpha \mathbf{z_2} \\ \hline \mathbf{z_2}^t \mathbf{v} \end{pmatrix} = \mathbf{0}, \tag{3.2}$$

for some $\mathbf{v} \neq \mathbf{0}$ and $\alpha \neq 0$.

An argument similar to that used in the proof of Lemma 2.2 yields the following result.

Lemma 3.1. *Let z_1 and z_2 be CV in $\mathcal{H}_1$ and in $\mathcal{H}_2$, respectively. Then in each of the graphs Z and $Z + e$, the two vertices z_1 and z_2 are of the same type.*

The unique value that the dimension of the nullspace of Z can take is given next.

Proposition 3.2. *If z_1 and z_2 are CV in $\mathcal{H}_1$ and in $\mathcal{H}_2$, respectively, then $\eta(Z) = 2$.*

Proof. Let $\mathbf{Z}$ be the adjacency matrix of the graph Z, where $\mathbf{z_1}$ and $\mathbf{z_2}$ are the n^{th} and $(n+1)^{\text{th}}$ columns corresponding to the characteristic vectors of z_1 and z_2, respectively. Since

$$\mathbf{Z} \begin{pmatrix} \mathbf{v} \\ \hline \alpha \\ 0 \end{pmatrix} = \left(\begin{array}{c|cc} \mathbf{G} & \mathbf{z_1} & \mathbf{z_2} \\ \hline \mathbf{z_1}^t & 0 & 0 \\ \mathbf{z_2}^t & 0 & 0 \end{array}\right) \begin{pmatrix} \mathbf{v} \\ \hline \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{G}\mathbf{v} + \alpha \mathbf{z_1} \\ \hline \mathbf{z_1}^t \mathbf{v} \\ \mathbf{z_2}^t \mathbf{v} \end{pmatrix}$$

and

$$\mathbf{Z} \begin{pmatrix} \mathbf{v} \\ \hline 0 \\ \alpha \end{pmatrix} = \left(\begin{array}{c|cc} \mathbf{G} & \mathbf{z_1} & \mathbf{z_2} \\ \hline \mathbf{z_1}^t & 0 & 0 \\ \mathbf{z_2}^t & 0 & 0 \end{array}\right) \begin{pmatrix} \mathbf{v} \\ \hline 0 \\ \alpha \end{pmatrix} = \begin{pmatrix} \mathbf{G}\mathbf{v} + \alpha \mathbf{z_2} \\ \hline \mathbf{z_1}^t \mathbf{v} \\ \mathbf{z_2}^t \mathbf{v} \end{pmatrix},$$

then by (3.1) and (3.2), $(\mathbf{v}, \alpha, 0)^t$ and $(\mathbf{v}, 0, \alpha)^t$ are two linearly independent kernel eigenvectors of $\mathbf{Z}$ and hence $\eta(Z) \geq 2$. By Theorem 1.3, $0 \leq \eta(Z) \leq 2$. Thus $\eta(Z) = 2$, and $(\mathbf{v}, \alpha, 0)^t$ and $(\mathbf{v}, 0, \alpha)^t$ span $\ker(\mathbf{Z})$. □

A consequence which has important implications on the construction of Z and, eventually, of $Z + e$, is the following.

Corollary 3.3. *If z_1 and z_2 are CV in $\mathcal{H}_1$ and in $\mathcal{H}_2$, respectively, then z_1 and z_2 are duplicates in Z and $\mathcal{H}_1 \equiv \mathcal{H}_2$.*

Proof. Since $(\mathbf{v}, \alpha, 0)^t$ and $(\mathbf{v}, 0, \alpha)^t$ are kernel eigenvectors of Z, then $(\mathbf{0}, \alpha, -\alpha)^t$ is also a kernel eigenvector of Z. Thus $\mathbf{z_1} = \mathbf{z_2}$. Hence, z_1 and z_2 are duplicate vertices in Z, implying that $\mathcal{H}_1$ and $\mathcal{H}_2$ are equivalent graphs. □

The dimension of the nullspace of $Z + e$ turns out to be different from that of Z. The result is stated in Theorem 3.5 and the proof follows from Corollary 3.3 and the following lemma. We remark that the graph H in the following lemma plays the role of each of the key-graphs $\mathcal{H}_1$ and $\mathcal{H}_2$, and hence equations (3.1) and (3.2) still hold for H.

Lemma 3.4. *Let z_1 be a CV in a graph H of nullity one and let Z be obtained from H by duplicating the vertex z_1 to obtain a new vertex z_2. Then $\eta(Z + e) = 0$, where e is the edge $z_1 z_2$.*

Proof. Let $\mathbf{W}$ be the adjacency matrix of $Z + e$, where $\mathbf{z_1}$ and $\mathbf{z_2}$ are the n^{th} and $(n+1)^{\text{th}}$ columns corresponding to the characteristic vectors of z_1 and z_2, respectively. Let $\mathbf{x} = \left(\frac{\mathbf{v}}{\alpha}\right)$, where $\mathbf{v} \neq \mathbf{0}$ and $\alpha \neq 0$, be a kernel eigenvector of H and let $G = H - z_1$. From (3.1) and (3.2), it follows that $\mathbf{W}(\mathbf{v}, \alpha, 0)^t = (\mathbf{0}, 0, \alpha)^t$ and $\mathbf{W}(\mathbf{v}, 0, \alpha)^t = (\mathbf{0}, \alpha, 0)^t$, and thus neither $(\mathbf{v}, \alpha, 0)^t$ nor $(\mathbf{v}, 0, \alpha)^t$ are kernel eigenvectors of $Z + e$.

We claim that $Z + e$ does not have any kernel eigenvectors. For, suppose $(\mathbf{u}, \beta, \delta)^t$ is a kernel eigenvector of $Z + e$. Since z_1 and z_2 are duplicates in Z, and hence co-duplicates in $Z + e$, then

$$\mathbf{W}\begin{pmatrix} \mathbf{u} \\ \hline \beta \\ \delta \end{pmatrix} = \left(\begin{array}{c|cc} \mathbf{G} & \mathbf{z_1} & \mathbf{z_1} \\ \hline \mathbf{z_1}^t & 0 & 1 \\ \mathbf{z_1}^t & 1 & 0 \end{array}\right)\begin{pmatrix} \mathbf{u} \\ \hline \beta \\ \delta \end{pmatrix} = \begin{pmatrix} \mathbf{G}\mathbf{u} + (\beta+\delta)\mathbf{z_1} \\ \hline \mathbf{z_1}^t\mathbf{u} + \delta \\ \mathbf{z_1}^t\mathbf{u} + \beta \end{pmatrix} = \mathbf{0},$$

implying that $\beta = \delta$. Thus, a kernel eigenvector of $Z + e$ must be of the form $(\mathbf{u}, \beta, \beta)^t$. If $\beta = 0$, then $\mathbf{G}\mathbf{u} = \mathbf{0}$ and hence $(\mathbf{u}, 0)^t$ is another kernel eigenvector of H which is linearly independent of $\mathbf{x} = (\mathbf{v}, \alpha)^t$, a contradiction since $\eta(H) = 1$. Thus $\beta \neq 0$ and we can choose $\beta = \alpha$ such that an eigenvector of $Z + e$ is $(\mathbf{w}, \alpha, \alpha)^t$. Thus,

$$\mathbf{W}\begin{pmatrix} \mathbf{w} \\ \hline \alpha \\ \alpha \end{pmatrix} = \left(\begin{array}{c|cc} \mathbf{G} & \mathbf{z_1} & \mathbf{z_1} \\ \hline \mathbf{z_1}^t & 0 & 1 \\ \mathbf{z_1}^t & 1 & 0 \end{array}\right)\begin{pmatrix} \mathbf{w} \\ \hline \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} \mathbf{G}\mathbf{w} + 2\alpha\mathbf{z_1} \\ \hline \mathbf{z_1}^t\mathbf{w} + \alpha \\ \mathbf{z_1}^t\mathbf{w} + \alpha \end{pmatrix} = \mathbf{0}.$$

But from (3.1), $\mathbf{G}\mathbf{v} + \alpha\mathbf{z_1} = \mathbf{0}$ and thus $\mathbf{G}(\mathbf{w} - 2\mathbf{v}) = \mathbf{0}$. Hence

- either $\mathbf{w} - 2\mathbf{v} = \mathbf{0}$, in which case $\mathbf{v} = \frac{1}{2}\mathbf{w}$. From (3.1), $\mathbf{z_1}^t\mathbf{v} = 0$, implying that $\mathbf{z_1}^t\mathbf{w} = 0$ and hence $\alpha = 0$, a contradiction;

- or $\mathbf{w} - 2\mathbf{v}$ is a kernel eigenvector of G, in which case $\eta(G) \geq 1$, a contradiction since z_1 is a CV in H and $\eta(G) = \eta(H - z_1) = 0$.

Hence, $\eta(Z + e) = 0$. □

Lemma 3.4 is now applied to the particular case when $Z + e$ is obtained from the superimposition of $\mathcal{H}_1$ and $\mathcal{H}_2$ with core vertices z_1 and z_2, respectively.

Theorem 3.5. *If z_1 and z_2 are CV in $\mathcal{H}_1$ and in $\mathcal{H}_2$, respectively, then $\eta(Z + e) = 0$ and z_1 and z_2 are both CFV$_{upp}$ in $Z + e$.*

Proof. By Corollary 3.3, z_1 and z_2 are duplicate vertices. The first part of the result follows by applying Lemma 3.4. Also, since $\eta(Z+e) = 0$, then z_1 and z_2 cannot be CV in $Z+e$. Noting that $\eta(Z+e-z_1) = \eta(\mathcal{H}_2) = 1$ and $\eta(Z+e-z_2) = \eta(\mathcal{H}_1) = 1$, we get that z_1 and z_2 are both CFV_{upp} in $Z+e$. □

Remark 3.6. Proposition 1.7, Proposition 3.2 and Theorem 3.5 imply that when z_1 and z_2 are CV in the key-graphs, each of the two positions y_2 and y_3 in the 3-type-code can be filled in only one way, namely C in the position y_2 and U in the position y_3. Therefore, there is only one possible class of the 3-type-code graphs having C in its first position y_1, namely CCU. Moreover the two key graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ are induced subgraphs in both Z and $Z+e$.

4 Three-type-code

Were it not for the restrictions of Lemmas 1.4, 2.2 and 3.1, the type of vertices would allow 81 classes of graphs for Z and another 81 for $Z+e$. These Lemmas allow only 27 potential classes and by eigenvector techniques, even these are further restricted to just nine with a specific 3-type-code. In Figure 4, three molecular graphs $\{Z+e\}$ that are not chemical are presented for the each type of vertex z_1 in $\mathcal{H}_1$.

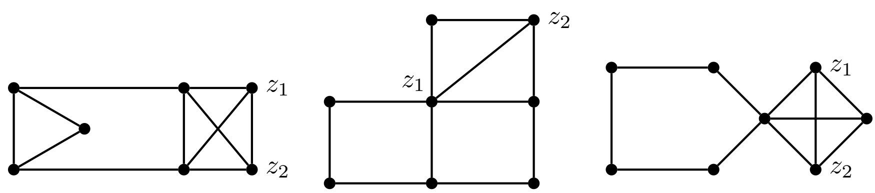


Figure 4: Graphs $Z+e$, having type M, U and C respectively, for the vertex z_1 in $\mathcal{H}_1$.

Except for the case UCC (that is, when $\{z_1, z_2\}$ are CFV_{upp} in H_1 and in H_2 and CV in Z and in $Z+e$), examples for all the remaining eight possible 3-type-codes graphs are known to exist (see Table 1). It is worth noting that among the eight graphs $\{Z+e\}$ and the associated graphs $\{Z\}$ drawn in Table 1, six are chemical graphs. The occurrence, or otherwise, of the UCC class remains open. Table 1 illustrates the different types of z_1 and z_2 in $\{H_1, H_2\}$, in Z and in $Z+e$, the associated code, the corresponding nullities of Z and of $Z+e$, and an example of a possible graph $Z+e$ (when existence is known).

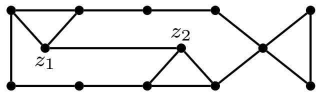
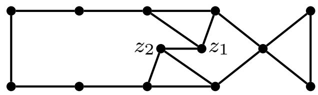
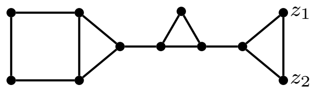
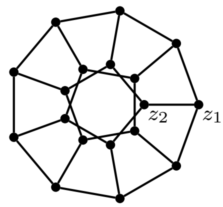
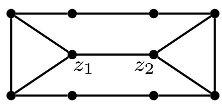
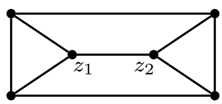
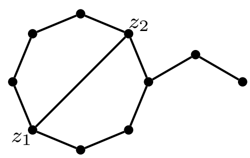
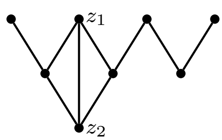
Observe that although the interlacing theorem allows three values for the nullity of Z, this value can never be zero.

At this stage, we can provide answers to the questions we posed at the end of Section 1.2 for the subclasses of graphs $\{Z\}$ and $\{Z+e\}$.

(i) On deleting the vertex z_1 or z_2 from Z, the nullity remains unchanged only for MMM, MMC, UMM and UMC out of the nine possibilities for the 3-type-code with z_1 and z_2 CFV_{mid} in Z. Similarly, on deleting the vertex z_1 or z_2 from $Z+e$, the nullity remains unchanged for MMM, MCM, UMM and UCM.

(ii) On deleting the edge $e = z_1z_2$ in $Z+e$, the type of the vertices z_1 and z_2 remains unchanged when the 3-type-code is one of MMM, MCC, UMM and, possibly, UCC.

Table 1: All possible cases of superimpositions $\{Z\}$ and the derived class $\{Z{+}e\}$ of graphs.

Type of z_1 and of z_2 in			Code	$\eta(Z)$	$\eta(Z+e)$	Example of $Z+e$
$\mathcal{H}_1$ & $\mathcal{H}_2$	Z	$Z+e$				
$\mathrm{CFV_{mid}}$	$\mathrm{CFV_{mid}}$	$\mathrm{CFV_{mid}}$	MMM	1	1	
	$\mathrm{CFV_{mid}}$	CV	MMC	1	2	
	CV	$\mathrm{CFV_{mid}}$	MCM	2	1	
	CV	CV	MCC	2	2	
$\mathrm{CFV_{upp}}$	$\mathrm{CFV_{mid}}$	$\mathrm{CFV_{mid}}$	UMM	1	1	
	$\mathrm{CFV_{mid}}$	CV	UMC	1	2	
	CV	$\mathrm{CFV_{mid}}$	UCM	2	1	
	CV	CV	UCC	2	2	(not known)
CV	CV	$\mathrm{CFV_{upp}}$	CCU	2	0	

We observe that no 3-type-code has U in both positions y_2 and y_3. This can be explained since if z_1 and z_2 are CFV$_{\text{upp}}$ in Z and $\eta(\mathcal{H}_1) = \eta(\mathcal{H}_2) = 1$, then $\eta(Z) = 0$, which never occurs by Lemma 2.1 and Proposition 3.2. Another point worth noting is that, in the case where z_1 and z_2 are CFV in the key-graphs, the results do not depend on whether they are upper or middle. Thus, the type of the core-forbidden vertices z_1 and z_2 in $\mathcal{H}_1$ and $\mathcal{H}_2$ is not a factor that determines their type in Z and in $Z + e$. Could it be that distinguishing between the types U and M for z_1 and z_2 in $\mathcal{H}_1$ and $\mathcal{H}_2$ would determine the existence or otherwise of UCC?

5 Electrical conductivity

A model device consists of the molecule with a pair of semi-infinite wires attached to it, so that a voltage can be applied across the molecule. The molecule, the wires and the contact atoms are represented by an augmented molecular graph with vertices for atoms and with edges for the sigma bonds. Left and right wires are represented by two special source and sink vertices L and R outside the molecule, which are then in contact with the molecule through single (usually distinct) vertices (contact atoms) labelled $\bar{L}$ and $\bar{R}$. Coulomb and resonance integrals are assigned to the wires and molecule-wire contacts. This model gives a Hückel/Tight-Binding model for ballistic currents which is the simplest version of the SSP (Source-and-Sink Potential) model for ballistic conduction through simple molecular electronic devices [8, 9, 12]. The approximations lead to a non-Hermitian set of linear equations of order $n + 2$, with an implicit dependence of the SSP matrix entries on the eigenvalue. Linear algebraic techniques are used to describe the solutions of the larger problem in terms of characteristic polynomials derived from the original $n \times n$ adjacency matrix.

Conduction or insulation of the unsaturated molecular device can then be predicted. The criteria for conduction at zero energy (the Fermi or non-bonding level) depend on the changes in nullity when the contact vertices $\bar{L}$ and $\bar{R}$ are deleted from the molecular graph on n vertices, separately and together [3, 5, 19].

5.1 Transmission

A molecular device G can be considered as a graph on n vertices with two prescribed vertices $\bar{L}$ and $\bar{R}$ connected by wires to two vertices L and R outside the molecule. The transmission at energy E, from the sink R, of ballistic electrons entering at the source L, can be expressed in terms of the characteristic polynomials $s(E)$, $u(E)$, $t(E)$ and $v(E)$ of G, $G - \bar{L}$, $G - \bar{R}$ and $G - \bar{L} - \bar{R}$, respectively, as functions of E. To determine whether a device conducts or bars conduction at the Fermi level of energy ($E = 0$), it suffices to consider the possible nullity signatures as an ordered quadruple $(g_s = \eta(G), g_u = \eta(G - \bar{L}), g_t = \eta(G - \bar{R}), g_v = \eta(G - \bar{L} - \bar{R}))$. Cauchy's inequalities for the eigenvalues of real symmetric matrices and of their principal submatrices lead to the interlacing theorem for graphs. As a consequence, the change in the nullity on deleting a vertex can be at most one. Hence relative to g_s, each of g_u and g_t can take 3 values whereas g_v can take 5. Thus the quadruple signature can in principle take 45 values with respect to g_s.

However interlacing, device symmetry, and the Jacobi-Sylvester theorem (that is $u(E)t(E) - s(E)v(E)$ is a perfect square $j^2_{\bar{L}\bar{R}}$, where $j_{\bar{L}\bar{R}}$ is the $\bar{L}\bar{R}^{\text{th}}$ entry of the adjugate of $EI - A$) restrict the number from 45 to just 11 [5, 19]. Table 2 gives the signatures of all possible classes of π-conjugated devices and their conducting/insulating properties at the

Fermi level of energy.

Table 2: The conductivity of all devices (G, L, R), their *variety* [19] and *case* [5].

$\text{Signature}(g_s, g_t, g_u, g_v)$	Nullity of G	Variety	Case	Transmission
Two CVs		1		
$(g_s, g_s - 1, g_s - 1, g_s - 2)$	$\eta_G \geq 2$	1(i)	11	Insulator
$(g_s, g_s - 1, g_s - 1, g_s)$	$\eta_G \geq 1$	1(ii)	9	Conductor
$(g_s, g_s - 1, g_s - 1, g_s - 1)$	$\eta_G \geq 1$	1(iii)	10	Conductor
CV and CFV		2		
$(g_s, g_s + 1, g_s - 1, g_s)$	$\eta_G \geq 1$	2a	5	Insulator
$(g_s, g_s, g_s - 1, g_s - 1)$	$\eta_G \geq 1$	2b	8	Insulator
Two CFVs		3		
$(g_s, g_s + 1, g_s + 1, g_v)$		3a		
$(g_s, g_s + 1, g_s + 1, g_s)$		3a(i)	2	Conductor
$(g_s, g_s + 1, g_s + 1, g_s + 2)$		3a(ii)	1	Insulator
$(g_s, g_s + 1, g_s, g_v)$		3b		
$(g_s, g_s + 1, g_s, g_s + 1)$		3b(i)	3	Insulator
$(g_s, g_s + 1, g_s, g_s)$		3b(ii)	4	Conductor
(g_s, g_s, g_s, g_v)		3c		
$(g_s, g_s, g_s, g_s + 1)$		3c(i)	6	Conductor
(g_s, g_s, g_s, g_s)		3c(ii)	7	
(g_s, g_s, g_s, g_s) & $j_a(0) \neq 0$		3c(iiA)	7i	Conductor
(g_s, g_s, g_s, g_s) & $j_a(0) = 0$		3c(iiB)	7ii	Insulator

5.2 A superimposition device

The superimposition Z and the derived graph $Z + e$ have a structure that restricts the number of device classes to which they can belong. Their signature can be determined from Table 1.

All the 11 device classes are realizable by molecular graphs. Table 3 shows that, of these, the superimpositions Z may be of only 5 cases and the derived graphs $Z + e$ may also be of 5 cases. Both Z and $Z + e$ can be of case 7 which assumes conductivity or insulating properties according to the vanishing or otherwise of $j_{z_1,z_2}(0)$. From Table 2, the derived graph $Z + e$ may be an insulator only for case 7 when both z_1 and z_2 are middle core-forbidden vertices in their respective key-graphs $\mathcal{H}_1$ and $\mathcal{H}_2$. Apart from case 7, a superimposition Z is an insulator only when both z_1 and z_2 are core vertices in their respective key-graphs.

Table 3: All possible cases of superimpositions $\{Z\}$ and the derived class $\{Z\}$ of graphs.

Type of z_1 and of z_2 in			Code	Signature(Z)		Signature$(Z+e)$	
$\mathcal{H}_1$ & $\mathcal{H}_2$	Z	$Z+e$		Variety	Case	Variety	Case
$\mathrm{CFV}_{\mathrm{mid}}$	$\mathrm{CFV}_{\mathrm{mid}}$	$\mathrm{CFV}_{\mathrm{mid}}$	MMM	(g_s, g_s, g_s, g_s)	$3c/7$	(g_s, g_s, g_s, g_s)	$3c/7$
	$\mathrm{CFV}_{\mathrm{mid}}$	CV	MMC	(g_s, g_s, g_s, g_s)	$3c/7$	$(g_s, g_s-1, g_s-1, g_s-1)$	$1iii/10$
	CV	$\mathrm{CFV}_{\mathrm{mid}}$	MCM	$(g_s, g_s-1, g_s-1, g_s-1)$	$1iii/10$	(g_s, g_s, g_s, g_s)	$3c/7$
	CV	CV	MCC	$(g_s, g_s-1, g_s-1, g_s-1)$	$1iii/10$	$(g_s, g_s-1, g_s-1, g_s-1)$	$1iii/10$
$\mathrm{CFV}_{\mathrm{upp}}$	$\mathrm{CFV}_{\mathrm{mid}}$	$\mathrm{CFV}_{\mathrm{mid}}$	UMM	(g_s, g_s, g_s, g_s+1)	$3ci/6$	(g_s, g_s, g_s, g_s+1)	$3ci/6$
	$\mathrm{CFV}_{\mathrm{mid}}$	CV	UMC	(g_s, g_s, g_s, g_s+1)	$3ci/6$	(g_s, g_s-1, g_s-1, g_s)	$1ii/9$
	CV	$\mathrm{CFV}_{\mathrm{mid}}$	UCM	(g_s, g_s-1, g_s-1, g_s)	$1ii/9$	(g_s, g_s, g_s, g_s+1)	$3ci/6$
	CV	CV	UCC	(g_s, g_s-1, g_s-1, g_s)	$1ii/9$	(g_s, g_s-1, g_s-1, g_s)	$1ii/9$
CV	CV	$\mathrm{CFV}_{\mathrm{upp}}$	CCU	$(g_s, g_s-1, g_s-1, g_s-2)$	$1i/11$	(g_s, g_s+1, g_s+1, g_s)	$3ai/2$

References

[1] D. A. Ali, J. B. Gauci, I. Sciriha and K. R. Sharaf, Coalescing Fiedler and core vertices, *Czechoslovak Math. J.* **66** (2016), 971–985, doi:10.1007/s10587-016-0304-8.

[2] M. Anđelić, C. M. da Fonseca and R. Mamede, On the number of P-vertices of some graphs, *Linear Algebra Appl.* **434** (2011), 514–525, doi:10.1016/j.laa.2010.09.017.

[3] P. W. Fowler, B. T. Pickup, T. Z. Todorova, M. Borg and I. Sciriha, Omni-conducting and omni-insulating molecules, *J. Chem. Phys.* **140** (2014), 054115, doi:10.1063/1.4863559.

[4] P. W. Fowler, B. T. Pickup, T. Z. Todorova, R. De Los Reyes and I. Sciriha, Omni-conducting fullerenes, *Chem. Phys. Lett.* **568–569** (2013), 33–35, doi:10.1016/j.cplett.2013.03.022.

[5] P. W. Fowler, B. T. Pickup, T. Z. Todorova and W. Myrvold, A selection rule for molecular conduction, *J. Chem. Phys.* **131** (2009), 044104, doi:10.1063/1.3182849.

[6] P. W. Fowler, I. Sciriha, M. Borg and B. T. Pickup, Molecular graphs and molecular conduction: the d-omniconductors, to appear.

[7] P. W. Fowler, I. Sciriha, M. Borg, V. E. Seville and B. T. Pickup, Near omni-conductors and insulators: Alternant hydrocarbons in the SSP model of ballistic conduction, *J. Chem. Phys.* **147** (2017), 164115, doi:10.1063/1.4995544.

[8] A. Goker, F. Goyer and M. Ernzerhof, Bond dissociation and correlation effects in molecular electronic devices, *J. Chem. Phys.* **129** (2008), 194901, doi:10.1063/1.3013815.

[9] F. Goyer and M. Ernzerhof, Correlation effects in molecular conductors, *J. Chem. Phys.* **134** (2011), 174101, doi:10.1063/1.3581096.

[10] I.-J. Kim and B. L. Shader, On Fiedler- and Parter-vertices of acyclic matrices, *Linear Algebra Appl.* **428** (2008), 2601–2613, doi:10.1016/j.laa.2007.12.022.

[11] M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston, 1964.

[12] B. T. Pickup and P. W. Fowler, An analytical model for steady-state currents in conjugated systems, *Chem. Phys. Lett.* **459** (2008), 198–202, doi:10.1016/j.cplett.2008.05.062.

[13] B. T. Pickup, P. W. Fowler, M. Borg and I. Sciriha, A new approach to the method of source-sink potentials for molecular conduction, *J. Chem. Phys.* **143** (2015), 194105, doi:10.1063/1.4935716.

[14] I. Sciriha, On the construction of graphs of nullity one, *Discrete Math.* **181** (1998), 193–211, doi:10.1016/s0012-365x(97)00036-8.

[15] I. Sciriha, On the rank of graphs, in: Y. Alavi, D. R. Lick and A. Schwenk (eds.), *Combinatorics, Graph Theory, and Algorithms, Vol. II*, New Issues Press, Kalamazoo, Michigan, 1999 pp. 769–778, proceedings of the 8th Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms, and Applications, dedicated to the memory of Paul Erdős, held at Western Michigan University, Kalamazoo, Michigan, June 3 – 7, 1996.

[16] I. Sciriha, A characterization of singular graphs, *Electron. J. Linear Algebra* **16** (2007), 451–462, doi:10.13001/1081-3810.1215.

[17] I. Sciriha, Coalesced and embedded nut graphs in singular graphs, *Ars Math. Contemp.* **1** (2008), 20–31, doi:10.26493/1855-3974.20.7cc.

[18] I. Sciriha, Maximal core size in singular graphs, *Ars Math. Contemp.* **2** (2009), 217–229, doi:10.26493/1855-3974.115.891.

[19] I. Sciriha, M. Debono, M. Borg, P. W. Fowler and B. T. Pickup, Interlacing-extremal graphs, *Ars Math. Contemp.* **6** (2013), 261–278, doi:10.26493/1855-3974.275.574.

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Regular polygonal systems

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Abstract

Let $M = M(\Omega)$ be any triangle-free tiling of a planar polygonal region Ω with regular polygons. We prove that its *face vector* $f(M) = (f_3, f_4, f_5, \dots)$, its *symmetry group* $S(M)$ and the tiling M itself are uniquely determined by its *boundary angles code* $c_a(M) = c_a(\Omega) = (t_1, \dots, t_r)$, a cyclical sequence of numbers t_i describing the shape of Ω.

Keywords: Regular polygonal system, boundary code, face vector, symmetry group, reconstructibility from the boundary.

Math. Subj. Class.: 05B40, 05B45

1 Introduction

Systems of regular (planar or spherical) polygons joined edge to edge arise in various contexts (in tilings, in polyhedral maps, in nature, in chemistry, in art). A rich theory of such systems may be developed. Researchers usually focus on some particular class of such systems (defined by some conditions), try to determine all its elements and explore various questions related to their combinatorial description, parameters, enumeration, characterization, classification, coding, etc. To unify the investigations of such objects and to emphasize their common characteristics we propose a general concept of a *regular polygonal system* and make some first few steps towards a general theory of such systems.

Here we give definitions, examples and remarks; the general reconstruction problem is presented in Section 2; results are gathered in Section 3.

A *polygonal system* M is a (finite or infinite) incidence structure $M = (V, E, F)$ whose elements are called vertices, edges and faces: faces are *abstract polygons* – cyclical sequences of vertices $(v_1, \dots, v_n)$, and edges are pairs of vertices $\{v_i, v_{i+1}\}$. Two faces are

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incident if they share an edge, two edges are incident if they share a vertex. If M consists only of n-gons it is called a n-*system*; if the number of these n-gons is m, it is a n_m-*system*. If all these faces are congruent polygons X, it is a *monohedral system* denoted X_m. If M consists only of n_1-gons, n_2-gons, ..., n_k-gons, it is called a $(n_1, \dots, n_k)$-*system*. The *face vector* (or just the f-vector) of the polygonal system M is the sequence $f(M) = (f_3, f_4, f_5, \dots)$ where $f(i) = f_i(M)$ are the numbers of its faces with i edges. We use also the notation $f(M) = (3_{f(3)}, 4_{f(4)}, 5_{f(5)}, \dots)$.

A planar or spherical polygon P is called *regular* if there is a cyclic group $G = \langle R \mid R^n = I \rangle$ of rotations acting transitively on the vertices and edges of P, and if its boundary ∂P is simply connected (thus we exclude star polygons, as in Kepler solids). A *regular polygonal system* $M = M(\Omega)$ consists of regular polygons joined edge to edge, covering a polygonal planar or spherical region Ω. The *symmetry group* of $M = M(\Omega)$ is the group of the rotations and reflections of Euclidean space E^2 or E^3 preserving the incidences in M.

A *code* $c(X)$ of a given mathematical object X is any (not necessarily discrete) structure from which X can be completely reconstructed (up to isomorphism). A *boundary code* of a given m-dimensional object X is any code $c(\partial X)$ of the shape of its $(m-1)$-dimensional boundary ∂X. The *boundary angles code* of the planar or spherical regular polygonal system $M = M(\Omega)$ covering (tiling) a polygonal region Ω is the cyclical sequence $c_a(M) = c_a(\Omega) = (t_1, t_2, \dots, t_r)$ of the angles $t_i = 180° - \alpha_i \in (-180°, 180°)$, where α_i are the interior angles between adjacent edges $A_{i-1}A_i$ and A_iA_{i+1} of Ω. The *boundary faces-edges code* $c_{f,e}(M)$ is the cyclical sequence of symbols $f(i)_{e(i)}$, where $f(i)$ is the number of edges of i-th boundary face and $e(i)$ is the number of the boundary edges of this face (see examples in Appendix A). A few examples and remarks will help us better understand these definitions and the motivation for them.

In chemistry, the mathematical model of benzenoid molecules are called *benzenoid systems* or *polyhexes* (composed of regular hexagons). Similar systems are *polydiamonds* (composed of equilateral triangles) and *polyominoes* (composed of squares) (see Figure 1).

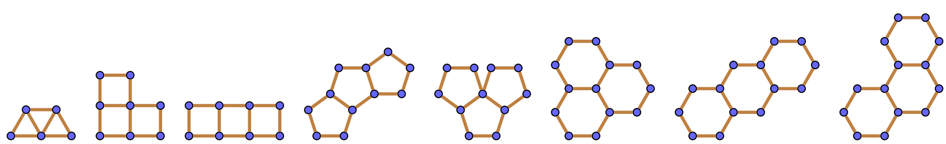


Figure 1: Planar regular n_3-systems for $n = 3, 4, 5, 6$.

The motivation for the boundary angles code c_a comes from the "turtle geometry" [1]. The definition of boundary faces-edges code $c_{f,e}$ is motivated by the boundary codes of various n-systems presented below.

Several boundary codes exist for the planar regular *hexagonal systems* B. One of them is the *boundary edges code* [6]. This code $c_e(B) = (k_1, k_2, \dots, k_r)$ is a cyclical sequence counting the numbers k_i of boundary edges in boundary hexagons; we travel around the boundary in the clockwise direction, starting at any hexagon, and having the interior of B always at our right (see [2, 7]).

The code for the regular planar *triangular systems* (or 3-systems) T may be defined exactly in the same way – as the boundary edges code $c_e(T)$ counting the number of boundary edges in boundary triangles of T. Thus, for the 3-system T in Figure 1 its code is $c_e(T) = (2, 1, 2)$.

The case of the planar *square systems* (or 4-systems) S is trickier. To get the right numbers in $c_e(S)$ we must count also the "zeros" of boundary edges in boundary vertices adjacent to no boundary edge. We can use also a simple *boundary vertices-faces code* $c_{v,f}(S)$, a cyclical sequence counting for each boundary vertex how many squares (1, 2 or 3) are incident with that vertex. For the first 4-system S in Figure 1 we have $c_e(S) = (2,3,0,3)$ and $c_{v,f}(S) = (1,2,1,1,3,1,1,2)$.

There is also a code for the regular *pentagonal systems* P (described in [3]) that actually counts the numbers of vertices incident only with boundary edges. For the second 5-system from Figure 1 this code is $(3,3,2)$.

One can easily imagine polygonal systems composed of regular polygons with different numbers of edges, and also non-planar generalizations of these concepts.

Example 1.1. The face vector of the planar regular $(3,4,5)$-system M in Figure 2 is $f(M) = (3_3, 4_1, 5_2) = T_3 S_1 P_2$, since there are three triangles T ($f_3 = 3$), one square S ($f_4 = 1$) and two pentagons P ($f_5 = 2$). Its symmetry group $S(M)$ is generated by one reflection (over the vertical axis).

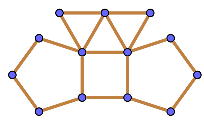


Figure 2: A polygonal system M with the symmetry group $S(M) = \mathbb{Z}_2$.

Example 1.2. The boundary faces-edges code of the planar system M in Figure 3 is $c_{f,e}(M) = (5_4, 4_0, 3_2, 4_2)$. The boundary angles codes $c_a(M)$ of the spherical systems with the same $c_{f,e}(M)$ depend on the size of the spherical triangle T. Increasing the size of T the interior angles α_i increase as well, hence the angles $t_i = 180° - \alpha_i$ decrease. If T is big enough, the angle between the pentagon and the triangle vanishes and we get a region with $c_{f,e}(M) = (5_3, 3_1, 4_2)$.

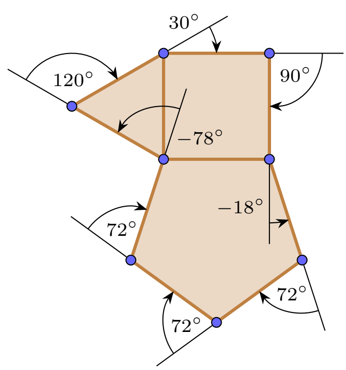


Figure 3: A planar regular polygonal $(3_1, 4_1, 5_1)$-system M with the boundary angles code $c_a(M) = (120°, 30°, 90°, -18°, 72°, 72°, 72°, -78°)$.

The maps of Platonic solids are examples of regular 3-systems (tetrahedron 3_4, octahedron 3_8, icosahedron 3_{20}), 4-systems (cube 4_6) and 5-systems (dodecahedron 5_{12}). The polygonal system of the square pyramid is (B, C, D, E), (A, B, C), (A, C, D), (A, D, E), (A, E, B), thus it is a $(3, 4)$-system. The system (A, B, C, D), (A, B, D, C) represents the Möbius band tiled with two quadrilaterals. In a polygonal system more than two faces may share the same edge, as in the "3-page book" (A, B, C, D), (A, B, D, E), (A, B, F, G), which is a 3-system, too.

2 The reconstruction problem

The first motivation for this research was the realization that it is possible to generalize the method that worked so well for benzenoid systems B: "*Encode the boundary of a system as a cyclical sequence of numbers, and then obtain various information about B from its boundary code $b(B)$*" to other planar polycyclic molecules. Thus $b(B)$ was used to find various relations between the parameters of B, to determine its symmetry scheme, to calculate its face count, etc. (as in [5, 7, 8]). With this motivation in mind we have in the Introduction already defined two boundary codes $c_{e,f}(M)$ and $c_a(M)$. Now we can ask: *Is it possible to reconstruct the face vector $f(M)$, the symmetry group $S(M)$ or even the whole planar regular polygonal system $M = M(\Omega)$ from its boundary code $c_{f,e}(M)$ or $c_a(M)$?* Obviously, some regular polygonal systems M are reconstructible from the chosen boundary code of the region Ω covered by M. How to characterize such systems? This question may be stated in a more general form: *Given some class $\mathcal{C}$ of objects X whose boundary is determined by some boundary code $c(\partial X)$ characterize those X from $\mathcal{C}$ that can be reconstructed from its boundary code, in other words, find X for which $c(\partial X)$ is also the code $c(X)$ of X.*

The code is not necessarily a cyclical sequence of numbers. The boundary of any simply connected *polycube* P (a solid composed of cubes of unit length joined face to face) may be coded with a graph $G(P)$ whose vertices are boundary square faces of P and whose edges are labeled with angles ($0, \pi/2, -\pi/2$) between adjacent boundary faces. The number of cubes in P, the symmetry group $S(P)$ and the polycube P itself are obviously determined by its boundary. However, their actual reconstruction from the graph $G(P)$ is complicated.

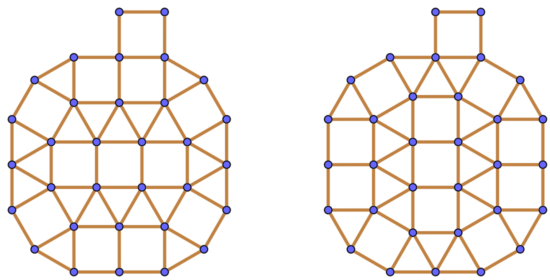
The code is not always a discrete structure. In analysis, a differentiable function f is reconstructible from its derivative f' by the Newton-Leibniz formula up to an additive constant C. Likewise, a harmonic function $f\colon \Omega \to \mathbb{R}$ is determined by its values on the boundary of Ω.

Not all the codes of the same object contain the same amount of information. For example, two similar planar triangles have different codes (lengths a, b, c and a', b', c'), but the same boundary code (consisting of angles α, β, γ). In some cases the chosen boundary code $c(\partial X)$ contains some additional information about the structure of X that cannot be deduced from ∂X alone. Thus it may happen that it is possible to reconstruct X from the code of ∂X, although it is not possible to reconstruct X only from ∂X.

Example 2.1. Let Ω be a planar polygonal region composed of a regular 12-gon and a square sharing one side. There are two tilings M_1 and M_2 of Ω with 12 squares S and 20 equilateral triangles T, having the same boundary angles code $(90°, 90°, 60°, 30°, 30°, 0°, 30°, 30°, 30°, 0°, 30°, 30°, 30°, 0°, 30°, 30°, 30°, -90°)$, but different boundary faces-edges codes (see Figure 4):

$$c_{e,f}(M_1) = (4_3, 4_0, 3_1, 4_1, 3_1, 3_0, 3_1, 4_1, 3_1, 4_1, 4_1, 3_1, 4_1, 3_1, 3_0, 3_1, 4_1, 3_1, 4_1, 4_0),$$
$$c_{e,f}(M_2) = (4_3, 3_0, 4_1, 3_1, 4_1, 4_1, 3_1, 4_1, 3_1, 3_0, 3_1, 4_1, 3_1, 4_1, 4_1, 3_1, 4_1, 3_1, 3_0, 3_0).$$

Hence, it is not possible to reconstruct M from the shape of the boundary $\partial\Omega$, encoded by $c_a(\Omega)$. However, M is reconstructible from $c_{f,e}(M)$, since there are only two possible tilings M_1 and M_2 of Ω. In this sense, $c_{f,e}(M)$ is a stronger code (containing more information), and $c_a(M) = c_a(\Omega)$ is weaker (containing less information).



(a) M_1 (b) M_2

Figure 4: Two different planar $(3, 4)$-systems M_1 and M_2 with the same boundary and with the same face vector $(3_{20}, 4_{12})$.

Remark 2.2. Two closed planar polygonal regions Ω and Ω' may have the same boundary angles code $c_a = (t_1, \ldots, t_r)$, but if the ratio of the lengths of the corresponding edges A_iA_{i+1} and $A'_iA'_{i+1}$ are not all the same, then the shapes of these regions are different. Note that the lengths of the edges of a region Ω tiled by regular polygons joined edge to edge are the same. This is the reason why the boundary angles code $c_a(\Omega)$ suffices to describe the shape of the boundary of Ω, tiled by regular polygons.[1]

3 Results

In this Section we show: if M is a regular triangle-free tiling of a planar polygonal region Ω, then we can use $c_a(M)$ to find the face vector $f(M)$ (Theorems 3.1 and 3.8), to determine the symmetry group $S(M)$ (Theorem 3.12), or even to completely reconstruct $M = M(\Omega)$ (Theorem 3.5).

Theorem 3.1. *Let $M = M(\Omega)$ be a planar regular $(3, 4)$-system covering the polygonal region Ω. Then the boundary angles code $c_a(M) = c_a(\Omega)$ determines the face vector $f(M)$ of M.*

Proof. The area of Ω can be calculated from the boundary angles code $c_a(\Omega) = (t_1, \ldots, t_n)$ as follows: fix the coordinates of two adjacent vertices $A_1 = (0, 0)$ and $A_2 = (1, 0)$ of Ω, use vectors to find the coordinates of other vertices A_i, triangulate Ω and sum the areas

[1] However, as we see in Figure 4, the tiling $M(\Omega)$ may not be uniquely determined by the shape of its boundary, although it is tiled by regular polygons.

of all these triangles. The area of the regular n-gon P_n with side 1 is $\mathrm{Area}(P_n) = n \cdot \cot(360°/n)$. These numbers are incommensurable at least for $n = 3, 4$ since $\mathrm{Area}(P_3) = \sqrt{3}/4$, $\mathrm{Area}(P_4) = 1$. Now it is easy to see that the integer solutions of the equation $x_1(\sqrt{3})/4 + y_1 = x_2(\sqrt{3})/4 + y_2$ are possible only if $x_1 = x_2, y_1 = y_2$. Thus $f(M)$ is determined by $c_a(M)$. Solving the equation $x_1(\sqrt{3})/4 + y_1 = \mathrm{Area}(M)$ is easy, since the calculated expression for $\mathrm{Area}(M)$ must appear in this form, from which we just read x_1 and y_1. □

Theorem 3.2. *The sum of the angles in the boundary angles code $c_a(\Omega) = (t_1, t_2, \ldots, t_r)$ of a planar polygonal region Ω is $t_1 + t_2 + \cdots + t_r = 360°$.*

Proof. This follows from the formula $\sum_{i=1}^{n} \alpha_i = (n-2) \cdot 180°$ for the sum of the interior angles of a n-gon: $\sum_{i=1}^{n} t_i = \sum_{i=1}^{n}(180° - \alpha_i)$. Another proof in the context of the "turtle geometry" is given in [1, p. 175]. □

Regular (planar or spherical) n-gons are usually denoted by the symbol $\{n\}$ or just n. The *vertex type* of the interior vertex of the regular polygonal system M is defined as the cyclical sequence $(a.b.c.\ldots)$ of the faces $a, b, c, \ldots$ surrounding it. The planar vertex types in this notation are listed in [4]. It is easy to check the following very useful observation ([4, p. 60]).

Theorem 3.3. *If the planar regular n_1-gon, …, n_r-gon surround a vertex without gaps and overlaps, then $3 \leq r \leq 6$ and $(n_1 - 2)/n_1 + \cdots + (n_r - 2)/n_r = 2$, hence there are 21 types of vertices surrounded by regular polygons in a plane without gaps or overlaps:*

3.3.3.3.3.3,	3.3.3.6,	3.3.3.4.4,	3.3.4.3.4,	3.3.4.12,	3.4.3.12,	3.3.6.6,
3.6.3.6,	3.4.4.6,	3.4.6.4,	3.7.42,	3.9.18,	3.8.24,	3.10.15,
3.12.12,	4.4.4.4,	4.5.20,	4.6.12,	4.8.8,	5.5.10,	6.6.6.

Theorem 3.4. *The possible (interior or boundary) vertex types in spherical regular polygonal systems are (if we exclude spherical 2-gons):*

- 5 *triangles:* 3.3.3.3.3*;*
- 4 *triangles:* 3.3.3.3.4, 3.3.3.3.5*;*
- 3 *triangles:* 3.3.3, 3.3.3.4, 3.3.3.5*;*
- 2 *triangles:* 3.3, 3.3.m, 3.m.3 *(*$m \geq 4$*)*, 3.3.4.5, 3.4.3.5, 3.3.5.5*;*
- 1 *triangle:* 3, 3.m *(*$m \geq 4$*)*, 3.4.n, 3.n.4 *(*$n \geq 4$*)*, 3.5.n, 3.n.5 *(*$n \geq 5$*);*
- 0 *triangles:* m, 4.m, 4.4.m, 4.m.4 *(*$m \geq 4$*)*, 4.5.n, 4.n.5 *(*$19 \geq n \geq 4$*)*, 5.5.5.

Proof. We just use the fact that for the interior angle α_n of the spherical n-gon it holds that $180° > \alpha_n > 180° - 360°/n$, and check all the possible cases. A similar treatment of faces is given in Appendix B. □

Theorem 3.5. *Let $M = M(\Omega)$ be a planar regular polygonal system covering the polygonal region Ω. If M contains no triangles then it is reconstructible from its boundary angles code $c_a(M)$. Likewise, M is reconstructible from $c_a(M)$ also in the case M is without squares and hexagons.*

Proof. The theorem is certainly true if M contains only one face. Now suppose it is true for the systems with m or less faces and let $M = M(\Omega)$ be a system with $m + 1$ faces. Since the sum of the r inner angles α_i of Ω is $\sum_{i=1}^{r} \alpha_i = (r - 2) \cdot 180°$, at least one α_i must be smaller than $180°$. The vertex A_i of such $\alpha_i < 180°$ is incident with at most three regular polygons: with two or three triangles, with one triangle and one square or pentagon or with two squares, or with a single n-gon with the inner angle $(n - 2) \cdot 180°/n$.

Hence, if M is triangle-free, then the angle $\alpha_i < 180°$ is incident with only one n-gon P_n whose interior angle is $(n - 2) \cdot 180°/n = \alpha_i$. Removing this P_n from M we get a smaller system, whose tiling is unique, by the induction hypothesis. Hence the position of each polygon in M is uniquely determined.

Similarly, if there are no squares and hexagons in M, then every interior angle $\alpha_i < 180°$ is incident either with a single polygon P_n (and then we proceed as before) or with a triangle and a pentagon. But there is no planar interior vertex type containing faces 3 and 5 (see Theorem 3.3). Therefore both the vertices of the edge AC shared by a boundary triangle ABC and a pentagon are boundary vertices (see Figure 5). If we interchange the

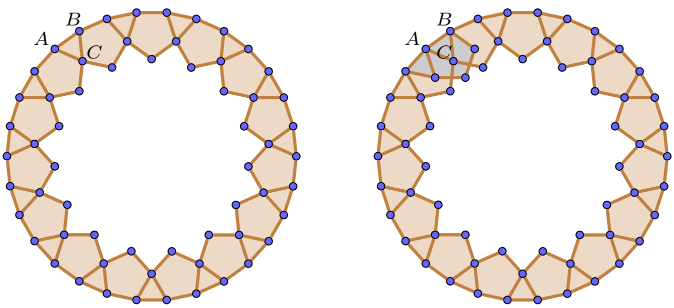


Figure 5: Left: A $(3, 5)$-system with two boundary components and with the cyclical symmetry group C_{15}. Right: An illustration of the fact that such system cannot have two different tilings.

positions of two adjacent boundary faces 3 and 5 along the adjacent boundary edges then the interchanged face 5 covers a neighbourhood of C (as in Figure 5 right). So there is only one possible tiling of M in the neighbourhood of all boundary points incident with 3.5 or 5.3. Removing this triangle and pentagon we get a smaller region for which the theorem is true by the induction hypothesis. □

Remark 3.6. It may happen that the boundary of $M - P_n$ is no longer a simple closed curve (see Figure 6); it may have crossings and it may have more than one component. However, by repeating the process of removing the polygons corresponding to interior angles $\alpha_i < 180°$ from the system and calculating the boundary angles code of the smaller system we may find the exact locations of each face in the system algorithmically. For example, removing a boundary square P_4 from M changes the boundary code from $c_a(M) = (t_1, t_2, \ldots, t_{i-1}, t_i = \alpha_4 = 90°, t_{i+1}, \ldots, t_r)$ into $c_a(M - P_4) = (t_1, t_2, t_{i-1} + \alpha_4, \alpha_4 - 180°, \alpha_4 + t_{i+1}, \ldots, t_r)$, as in Figure 6. For $n \geq 5$ we get a more complicated formula for $c_a(M - P_n)$, dependent on how many successive angles $t_i, t_{i+1}, \ldots t_{i+k}$ in $c_a(M)$ are equal to $(n - 2) \cdot 180°/n$.

Just as in the planar case, there are polygonal regions on the sphere admitting more

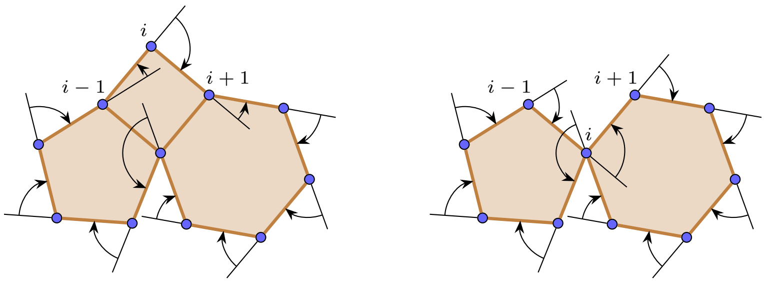


Figure 6: Removing of a boundary face may produce a system that is no more face-connected.

than one regular tiling, too. A spherical pentagon, covered by five regular triangles sharing a vertex of the spherical icosahedron is an example of such a region.

Theorem 3.7. *Let M be a spherical regular polygonal system M without triangles and squares. If the boundary of M has only one component, then M is reconstructible from its boundary faces-edges code $c_{f,e}(M)$.*

Proof. By Theorem 3.4 M has no interior points. Hence there can be no cycle composed of adjacent faces in M, and $c_{f,e}(M)$ completely describes the system M. □

Theorem 3.8. *The face vector $f(M) = (f_3, f_4, f_5, \dots)$ and consequently also the number of faces $f = f_3 + f_4 + f_5 + \cdots$ of any regular planar polygonal system $M = M(\Omega)$ without hexagons or triangles and with all sides of length 1 is uniquely determined by the area of Ω, and this area is uniquely determined by its boundary angles code $c_a(\Omega) = c_a(M)$.*

Proof. If $M = M(\Omega)$ has at most two faces, then its face vector is obviously determined by its boundary code, except in the case of a hexagon, which may be decomposed into six triangles. Suppose the theorem is true for any system with n faces. Take a system M with $n+2$ faces. In every boundary vertex with a positive angle t_i we can repeat the procedure of taking out the polygons of M as in the proof of Theorem 3.5. In the boundary vertices with the interior angles filled with 3.4, 4.3, 3.5, 5.3, we take away from M both combinations and get two smaller systems M^* and M^{**} composed of n faces which must have the same area, hence they have the same face vector by the induction hypothesis. Therefore M has the unique face vector, too. □

Theorem 3.9. *The symmetry group $S(M)$ of a planar or spherical regular polygonal system $M = M(\Omega)$ is a subgroup of $S(\Omega) = S(\partial\Omega)$.*

Proof. Any rotation or reflection preserving M preserves the region Ω, tiled by the polygons of M. The symmetry group of the boundary of Ω is obviously isomorphic to the symmetry group of Ω. □

A symmetry of the boundary of Ω does not necessarily induce a symmetry of $M = M(\Omega)$. For example, if we glue together the two 12-gons from Figure 4 (without the added top square) along the vertical edge, we get a region with two reflection symmetries (over the vertical axis and over the horizontal axis), yet its tiling has only one reflection symmetry (over the horizontal axis).

Theorem 3.10. *If there is only one regular polygonal system $M = M(\Omega)$ covering the region Ω then the groups $S(M)$ and $S(\Omega)$ are isomorphic.*

Proof. If the tiling M of Ω with regular polygons is unique, then every symmetry of Ω automatically induces an automorphism of M. □

Lemma 3.11. *Let Ω be a planar polygonal system with given lengths of its edges $l_i = A_iA_{i+1}$ and with the boundary angles code $c(\Omega) = (t_1, \dots, t_r)$.*

- *(i) If $t_{i+k} = t_k$ and if $l_{i+k} = l_k$ for some $k \geq 2$, then Ω has a rotational symmetry for the angle $360°/(r/k)$.*
- *(ii) If $c_a(\Omega) = (t_1, \dots, t_r) = (t_r, \dots, t_1)$ and if the cyclical sequence $c_l = (l_1, l_2, \dots, l_r)$ of the lengths l_i of the boundary edges A_iA_{i+1} has a reflection symmetry, then Ω has a reflection symmetry.*

Proof. (i) This is obviously true for $k = 1$, for in that case we have $c_a(\Omega) = (t, t, \dots, t)$, hence all t_i are the same, and all l_i are the same, hence Ω is a regular polygon P_r invariant for the rotation for the angle $360°/r$.

In the case $k = 2$ we have $c_a(\Omega) = (t, s, t, s, \dots, t, s)$. Removing from Ω the congruent triangles $\triangle A_1A_2A_3, \triangle A_3A_4A_5, \dots, \triangle A_{2n-1}A_{2n}A_1$ we get a region with $r/k = r/2$ boundary edges where all t_i are the same and all l_i the same (case $k = 1$) and for which we know (i) is true; hence Ω has the rotation for the angle $360°/(r/2)$, too.

Similarly, for the $k \geq 3$ we remove $r/k = n$ congruent triangles from Ω to get a smaller region with $r - \frac{r}{k} = nk - n = n(k-1)$ boundary angles and with a period $k - 1$, hence by the induction hypothesis having a rotation R_n where $n = \frac{r}{k} = \frac{n(k-1)}{k-1}$. Hence Ω has the rotation for the angle $360°/(r/k)$, too.

(ii) Reflection symmetry of the sequence $c_l(\Omega)$ may have three forms[2]:

(a) $c_l = \quad (y, z, \dots, b, a, a, b, \dots, y, z)$,

(b) $c_l = (x, y, z, \dots, b, a, a, b, \dots, y, z)$,

(c) $c_l = (x, y, z, \dots, b, a, a, b, \dots, y, z, w)$,

and the same holds for the cyclical sequences of the lengths l_i.

Obviously (ii) is true if Ω has $3, 4, 5$ or 6 sides, since the only possible cases of $c_l(\Omega)$ are: (x, y, y), (y, z, z, y), (x, y, y, z), (x, y, z, z, y), (x, y, z, z, y, w) and (x, y, z, z, y, x). Now we can use a simple induction argument to see that if Ω has more sides than 6, then we can cut it in two pieces with the boundary angles code of the types (c) or (b), and each of these two pieces has the same reflection symmetry (whose axis is the symmetral of the same line XY), hence Ω has the same reflection symmetry (see Figure 7 middle). This is clear for regions of the type (a); the regions of the types (b) and (c) are obtained from a region of the type (a) by glueing one or two triangles to it. □

Theorem 3.12. *Let $M = M(\Omega)$ be a regular planar polygonal system without triangles covering the region Ω. Then the symmetry group $S(M)$ of M and the center of the rotation R_n are determined by its boundary angles code $c_a(M)$ as follows:*

- *(i) if $c_a(M)$ has a reflection symmetry $c_a(M) = (t_1, \dots, t_r) = (t_r, \dots, t_1)$ then M has a reflection symmetry;*

[2] The letters $x, y, z, a, b, \dots$ are used here to denote lengths l_i of boundary edges of Ω.

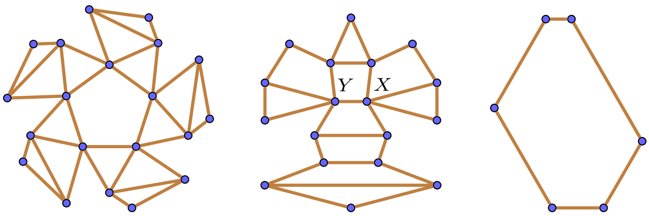


Figure 7: Left: a region with rotational symmetry; middle: a region with reflection symmetry; right: a region whose all boundary angles are identical (equal to $120°$, as in a regular hexagon), yet it has no symmetry.

(ii) *if* $c_a(M) = (t_1, \ldots, t_r)$ *is a periodical cyclical sequence:* $t_{i+k} \equiv t_i$ *for some* $k \geqslant 2$ *then* M *has a rotational symmetry for the angle* $360°/(r/k)$*;*

(iii) *if* $f \equiv 1 \pmod n$ *then the center of the rotation* R_n *is in a face;*

(iv) *if* $f \equiv 0 \pmod n$ *and* $n > 2$ *then the center of the rotation is in the vertex (if* $n > 2$*);*

(v) *if* $f \equiv 0 \pmod n$ *and* $n = 2$ *then the center of the rotation is in an edge.*

Proof. By Theorem 3.5, $M = M(\Omega)$ is uniquely determined by its boundary angles code $c_a(M) = c_a(\Omega)$. By Theorem 3.9, the group of symmetries of M is isomorphic to the group of symmetries of the boundary of Ω. Now (i) and (ii) follow directly from Lemma 3.11. Indeed, if the boundary angles code $c_a(M) = c_a(\Omega)$ of length r remains the same after the shift $k \pmod m$, then $r = kn$ is divisible by k and Ω has a rotational symmetry R_n (see Lemma 3.11).

Likewise, the reflection symmetry of $c_a(\Omega)$ implies the reflection symmetry of Ω, since all the edges of Ω are of the same size (since they are tiled by the regular polygons of M joined edge to edge) and we can apply Lemma 3.11. Now we use again $S(\Omega) = S(M)$, implied by Theorem 3.5.

The center of the rotational symmetry R_n of any planar regular polygonal system can be in a vertex (this is possible only in the cases when $n = 3, 4, 6$), in an edge center (this is possible only if $n = 2$) or in a center of a m-gon, where n divides m. If there are no triangles in the system, and if n is odd, then the center of the rotation R_n is in a face center.

If the rotational symmetry R_n has the center in a m-gon, then $f_m \equiv 1 \pmod n$ while the number of all other i-gons f_i is divisible by n, hence $f \equiv 1 \pmod n$. If the center of the rotation is in a vertex or in an edge center, then $f_i \equiv 0 \pmod n$ for every i, hence $f \equiv 0 \pmod n$. By Theorem 3.8 the face vector of M without triangles may be obtained from the boundary angles code $c_a(M)$ so the numbers f_i are known, hence we can easily distinguish between the two possible cases with rotational symmetry: $f \equiv 1 \pmod n$ and $f \equiv 0 \pmod n$.

If $f \equiv 0 \pmod n$ and $n > 2$ then the center of the rotation must be in a vertex. If $f \equiv 0 \pmod n$ and $n = 2$ then the center of the rotation must be in an edge. □

Remark 3.13. So the information hidden in the boundary angles code suffices to find out whether the center of rotation is located in a face, in a vertex or in the middle of an edge.[3]

[3] Another possible approach to this question is to calculate the center of rotation directly via orbit barycenters, and compare its coordinates with the coordinates of all the vertices, face centers and edge centers of the tiling M.

4 Summary and open questions

We proposed a concept of a regular polygonal system, a mathematical model of chemical (planar or non-planar) molecules composed of chains of atoms forming regular n-gonal cycles of equal or various length ($n = 3, 4, 5, 6, \dots$).

We proved that the structure (and hence the symmetry group) of any regular planar polygonal system M without triangles or without squares and hexagons is reconstructible from its boundary angles code $c_a(M)$.

The proof of Theorem 3.1 implicitly uses a simple incommensurability argument. This type of argument can be generalized as in the following definition and lemma:

Definition 4.1. The quantities $q_1, q_2, \dots, q_n$ are called *incommensurable*, if they satisfy the following condition: if two linear combinations of these quantities with integer coefficients are the same, i.e. $\sum_{i=1}^{n} a_i q_i = \sum_{i=1}^{n} b_i q_i$, then their corresponding coefficients must be the same, i.e. $a_i = b_i$.

From this definition immediately follows:

Lemma 4.2. *If any quantity q can be expressed as an integer linear combination*

$$q = \sum_{i=1}^{n} a_i q_i$$

of incommensurable quantities, then the coefficients a_i are uniquely determined by q.

Conjecture 4.3. *The areas of all planar regular polygons with the same side length (except the triangle or hexagon) are incommensurable quantities.*

Conjecture 4.4. *The volumes of all Platonic and Archimedean solids with the same side are incommensurable quantities.*

If one could prove Conjecture 4.3, this would automatically prove also Theorem 3.8. However, the converse is not true. If one could prove Conjecture 4.4 this would prove another conjecture:

Conjecture 4.5. *If we glue together copies of Platonic and Archimedean solids (with unit sides) face to face then we can find the number of each of them just by knowing the volume of such composed solid.*

References

[1] H. Abelson and A. diSessa, *Turtle Geometry: The Computer as a Medium for Exploring Mathematics*, MIT Press Series in Artificial Intelligence, MIT Press, Cambridge, Massachusetts, 1981.

[2] A. T. Balaban, Chemical graphs–VII: Proposed nomenclature of branched cata-condensed benzenoid polycyclic hydrocarbons, *Tetrahedron* **25** (1969), 2949–2956, doi:10.1016/s0040-4020(01)82827-1.

[3] M. Deza, P. W. Fowler and V. P. Grishukhin, Allowed boundary sequences for fused polycyclic patches and related algorithmic problems, *J. Chem. Inf. Comput. Sci.* **41** (2001), 300–308, doi:10.1021/ci000060o.

[4] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman and Company, New York, 1987.

[5] I. Gutman and S. J. Cyvin, *Introduction to the Theory of Benzenoid Hydrocarbons*, Springer-Verlag, Berlin, 1989, doi:10.1007/978-3-642-87143-6.

[6] P. Hansen, C. Lebatteux and M. Zheng, The boundary-edges code for polyhexes, *J. Mol. Struct. THEOCHEM* **363** (1996), 237–247, doi:10.1016/0166-1280(95)04139-7.

[7] J. Kovič, How to obtain the number of hexagons in a benzenoid system from its boundary edges code, *MATCH Commun. Math. Comput. Chem.* **72** (2014), 27–38, http://match.pmf.kg.ac.rs/electronic_versions/Match72/n1/match72n1_27-38.pdf.

[8] J. Kovič, T. Pisanski, A. T. Balaban and P. W. Fowler, On symmetries of benzenoid systems, *MATCH Commun. Math. Comput. Chem.* **72** (2014), 3–26, http://match.pmf.kg.ac.rs/electronic_versions/Match72/n1/match72n1_3-26.pdf.

[9] N. J. A. Sloane (ed.), The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org.

Appendix A Planar regular $(3, 4)$-systems

These systems may be classified by two parameters: the numbers f_3 and f_4 of triangular and square faces, as in Figure 8.

As we see, the number of these systems grows very quickly with the total number of faces $f = f_3 + f_4$. There are 16 such systems with 1, 2 or 3 faces.

In Figure 9 they are arranged by the increasing total number of faces. We give also their boundary faces-edges codes $c_{f,e}$ and boundary angles codes c_a. Observe that in all these cases all the faces are boundary faces.

Since for the planar regular polygonal $(3, 4)$-systems all their boundary angles are multiples of $30°$, we give the boundary angles codes also in the form of multiples of $30°$.

The corresponding boundary faces-edges code and boundary angles codes of these $(3, 4)$-systems, ordered as in Figure 9, are:

$c_{f,e}(M)$	$c_a(M)$		
(3_3)	$(120, 120, 120)°$	$=$	$30°(4, 4, 4)$
(4_4)	$(90, 90, 90, 90)°$	$=$	$30°(3, 3, 3, 3)$
$(3_2, 3_2)$	$(60, 120, 60, 120)°$	$=$	$30°(2, 4, 2, 4)$
$(4_3, 3_2)$	$(120, 30, 90, 90, 30)°$	$=$	$30°(4, 1, 3, 3, 1)$
$(4_3, 4_3)$	$(90, 0, 90, 90, 0, 90)°$	$=$	$30°(3, 0, 3, 3, 0, 3)$
$(3_2, 3_1, 3_2, 3_0)$	$(60, 60, 120, 0, 120)°$	$=$	$30°(2, 2, 4, 0, 4)$
$(3_2, 3_0, 4_3, 3_1)$	$(-30, 90, 90, 30, 60, 120)°$	$=$	$30°(-1, 3, 3, 1, 2, 4)$
$(3_2, 3_1, 4_3, 3_0)$	$(90, 90, -30, 120, 60, 30)°$	$=$	$30°(3, 3, -1, 4, 2, 1)$
$(3_2, 4_1, 3_2, 4_1)$	$(120, 30, 30, 120, 30, 30)°$	$=$	$30°(4, 1, 1, 4, 1, 1)$
$(3_2, 4_0, 3_2, 4_2)$	$(120, -30, 120, 30, 90, 30)°$	$=$	$30°(4, -1, 4, 1, 3, 1)$
$(4_3, 4_0, 3_2, 4_2)$	$(90, -60, 120, 30, 90, 0, 90)°$	$=$	$30°(3, -2, 4, 1, 3, 0, 3)$
$(4_2, 3_2, 4_0, 4_3)$	$(30, 120, -60, 90, 90, 0, 90)°$	$=$	$30°(1, 4, -2, 3, 3, 0, 3)$
$(4_3, 4_1, 3_2, 4_1)$	$(90, 0, 30, 120, 30, 0, 90)°$	$=$	$30°(3, 0, 1, 4, 1, 0, 3)$
$(4_3, 3_0, 4_3, 3_1)$	$(90, 90, -60, 90, 90, 30, 30)°$	$=$	$30°(3, 3, -2, 3, 3, 1, 1)$
$(4_3, 4_1, 4_3, 4_1)$	$(90, 90, 0, 0, 90, 90, 0, 0)°$	$=$	$30°(3, 3, 0, 0, 3, 3, 0, 0)$
$(4_3, 4_0, 4_3, 4_2)$	$(90, 90, -90, 90, 90, 0, 90, 0)°$	$=$	$30°(3, 3, -3, 3, 3, 0, 3, 0)$

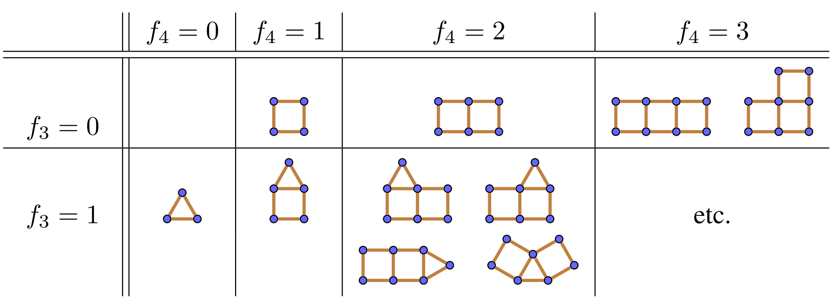


Figure 8: Regular planar $(3, 4)$-systems classified by two parameters.

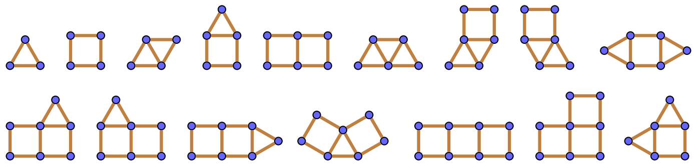


Figure 9: Planar regular polygonal $(3,4)$-systems with $f = f_3 + f_4 \leq 3$ faces.

Appendix B Types of faces

The idea to classify the types of hexagons in a benzenoid system with respect to their contacts to adjacent faces in a system [2, 7] may be generalized to any n-gonal faces in any polygonal system as follows:

Definition B.1. The type of the n-gonal face f in a polygonal system M is the cyclical binary sequence $(b_1, \dots, b_n)$ where $b_i = 0$ if the i-th edge of f is incident to at least one other face of M and $b_i = 1$ if it is a boundary edge of f. The number of types of n-gons having k entries 1 in the binary code is denoted $T(n,k)$. The number of possible types of a n-gon is denoted $T(n)$.

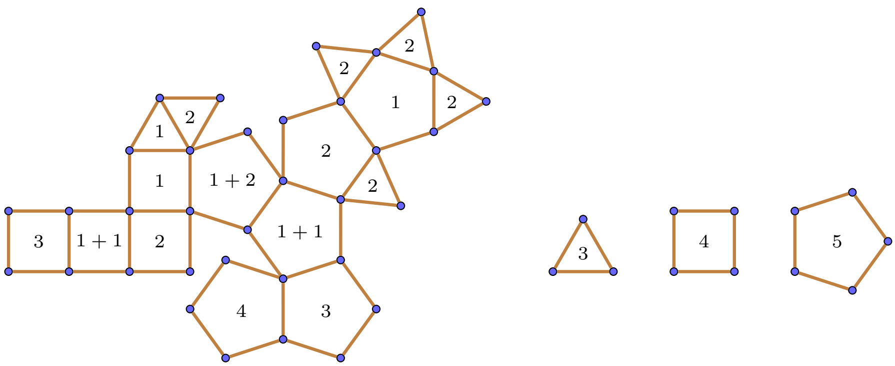


Figure 10: Boundary face types T_1, T_2, T_3, $S_1, S_2, S_{1+1}, S_3, S_4$, P_1, P_2, P_{1+1}, P_3, P_{1+2}, P_4, P_5.

Theorem B.2. *The following relations hold:*

(i) $T(n)$ *equals the number of binary cyclical sequences of length* n.

(ii) $T(3) = 4, T(4) = 6, T(5) = 8, T(6) = 13.$

Proof. (i) This is obvious, and trivially implies also relations $T(n,k) = T(n, n-k)$ and $T(n) = T(n,0) + T(n,1) + \cdots + T(n,n)$. To each type t of a n-gon exists the opposite type t^* with the entries $b_i^* = 1 - b_i$, hence $t^{**} = t$.

For (ii) see Figure 10. Summing the adjacent entries 1 and ignoring the entries 0 we can, at least for the triangles T, squares S and pentagons P, denote their possible boundary types like this:

$$\begin{array}{lll} T_0 = (0,0,0), & T_1 = (1,0,0), & T_2 = (1,1,0) = T_1^*, \\ T_3 = (1,1,1) = T_0^*, & S_0 = (0,0,0,0), & S_1 = (1,0,0,0), \\ S_2 = (1,1,0,0) = S_2^*, & S_{1+1} = (1,0,1,0) = S_{1+1}^*, & S_3 = (1,1,1,0) = S_1^*, \\ S_4 = (1,1,1,1), & P_0 = (0,0,0,0,0), & P_1 = (1,0,0,0,0), \\ P_2 = (1,1,0,0,0), & P_{1+1} = (1,0,1,0,0), & P_{1+2} = (1,0,1,1,0) = P_{1+2}^*, \\ P_3 = (1,1,1,0,0) = P_2^*, & P_4 = (1,1,1,1,0) = P_1^*, & P_5 = (1,1,1,1,1). \end{array}$$

The inner faces are

$$T_0 = (0,0,0), \qquad S_0 = (0,0,0,0), \qquad P_0 = (0,0,0,0,0).$$

The 13 types of boundary hexagons H with $1, 2, 3, 4$ or 5 boundary edges, e.g.

$$H_{1+1+1} = (1,0,1,0,1,0) \quad \text{or} \quad H_2 = (1,1,0,0,0,0)$$

are given in [2] and more precisely classified in [7]. □

Remark B.3. The sequence $T(n)$ (although defined only for $n \geqslant 3$) corresponds to the sequence A000029 in OEIS [9].

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$\mathscr{F}$-WORM colorings of some 2-trees: partition vectors

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Abstract

Suppose $\mathscr{F} = \{F_1, \ldots, F_t\}$ is a collection of distinct subgraphs of a graph $G = (V, E)$. An $\mathscr{F}$-*WORM* coloring of G is the coloring of its vertices such that no copy of each subgraph $F_i \in \mathscr{F}$ is monochrome or rainbow. This generalizes the notion of F-WORM coloring that was introduced recently by W. Goddard, K. Wash, and H. Xu. A (restricted) partition vector $(\zeta_\alpha, \ldots, \zeta_\beta)$ is a sequence whose terms ζ_r are the number of $\mathscr{F}$-WORM colorings using exactly r colors, with $\alpha \leq r \leq \beta$. The partition vectors of complete graphs and those of some 2-trees are discussed. We show that, although 2-trees admit the same partition vector in classic proper vertex colorings which forbid monochrome K_2, their partition vectors in K_3-WORM colorings are different.

Keywords: 2*-tree, maximal outerplanar, partition, Stirling numbers.*

Math. Subj. Class.: 05C15, 05C10

1 Preliminaries

A *partition* σ of a set S is a set of nonempty subsets or *blocks* of S such that each element of S is in exactly one of the subsets of S. The number of blocks of σ is its *rank* and a partition of rank r is simply called an r-*partition*. For instance, the Stirling number of the second kind, $\left\{ {n \atop r} \right\}$ counts the number r-partitions of the set $[n] = \{1, 2, \ldots, n\}$.

Consider the mapping $c\colon S \to [x]$ being an x-coloring of the elements of S. A subset $A \subseteq S$ is said to be *monochrome* if all of its elements share the same color and A is *rainbow* if all of its elements have different colors. As such, a coloring $c(S)$ is a partition of the set

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S since all of the elements of S are assigned a color; elements that share the same color belong to the same block (monochrome subsets), and different blocks are used for those with distinct colors (rainbow subsets).

Let $G = (V, E)$ denote a simple graph and suppose $\mathscr{F} = \{F_1, \ldots, F_t\}$ is a collection of some distinct subgraphs $F_i \subseteq G$, $1 \leq i \leq t$. An *$\mathscr{F}$-WORM* coloring of G is the coloring of the vertices of G such that no copy of each subgraph F_i is monochrome or rainbow. When $\mathscr{F}$ has only one member, say F, we write F-WORM coloring; this special case was first introduced by W. Goddard, K. Wash and H. Xu, and independently studied by Cs. Bujtás and Zs. Tuza [5, 7, 8, 12, 13]. We note that this coloring requirement makes sense only if each $F_i \in \mathscr{F}$ is of order three or more. However, for a generalization purpose if some F_j is of order 2, we allow only rainbow copies of F_j in order to meet the classic proper (vertex) coloring requirement. Suppose $\mathscr{H} = (\mathcal{V}, \mathcal{E})$ is a hypergraph. If $|e| = s$ for each hyperedge $e \in \mathcal{E}$, then $\mathscr{H}$ is said to be *s-uniform.* Given any vertex coloring of $\mathscr{H}$, if no $e \in \mathcal{E}$ is monochrome, $\mathscr{H}$ is called a *$\mathcal{D}$-hypergraph* or simply a hypergraph. When no $e \in \mathcal{E}$ is rainbow, $\mathscr{H}$ is called a *cohypergraph.* In the event no $e \in \mathcal{E}$ is monochrome or rainbow, then $\mathscr{H}$ is called a *bihypergraph.* Moreover, if G is a hypergraph and each subgraph $F_i = E_r$, the null graph on r-vertices, then an $\mathscr{F}$-WORM coloring of G is a proper (vertex) coloring of an r-uniform bihypergraph; Cs. Bujtás and Zs. Tuza [7, 8] also noted this strong relation between $\mathscr{F}$-WORM coloring and mixed hypergraph colorings, a theory that was first introduced by the second author [25, 26]. Thus, the notion of $\mathscr{F}$-WORM colorings generalizes several well known coloring constraints. Given an $\mathscr{F}$-WORM coloring, the sequence $(\zeta_\alpha, \ldots, \zeta_\beta)$ whose terms, ζ_r, are the number of r-partitions is called a *(restricted) partition vector*, with $\alpha \leq r \leq \beta$. In general, partition vectors have some added benefits in the study of log-concave and unimodal sequences which often arise in algebra, combinatorics, computer science, even in probability and statistics (see for e.g., [2, 4, 11]). A sequence of non-negative terms $(a_0, \ldots, a_n)$ is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for $i = 1, \ldots, n-1$. Such sequence is also said to be *unimodal* if it has no gap (i.e., there is no i with $a_{i-1} \neq 0$, $a_i = 0$ and $a_{i+1} \neq 0$) and there is an index $0 \leq j \leq n$ such that $a_0 \leq \ldots \leq a_j \geq \ldots \geq a_n$. Further, partition vectors are closely related to colorings; each ζ_r gives the number of $\mathscr{F}$-WORM colorings using exactly r colors, in which case α and β are the *lower* and *upper chromatic numbers*, respectively. In [7], it is shown that it is NP-hard to determine α and it is NP-complete to decide whether or not a graph G admits a K_3-WORM coloring using $k \geq 2$ colors. Moreover, the integer set $S = \{x : \alpha \leq x \leq \beta\}$ commonly known as *feasible set*, has been the subject of numerous research publications (see e.g., [6, 15, 18, 27]). We note that the term chromatic spectrum has also been used for feasible set in some of the aforementioned literatures. Further, we call the rank-generating function

$$\sigma(G|_{\mathscr{F}}; x) = \sum_{k=\alpha}^{\beta} \zeta_k x^k$$

the *restricted partition polynomial* of G subject to an $\mathscr{F}$-WORM coloring. Note that, when x^i is replaced by the falling factorial power $x^{\underline{i}} = x(x-1)(x-2)\cdots(x-i+1)$, the polynomial

$$\sigma(G|_{\mathscr{F}}; x) = \sum_{k=\alpha}^{\beta} \zeta_k x^{\underline{k}}$$

counts all $\mathscr{F}$-WORM colorings using at most x colors. Some variants of restricted partition polynomials have been well studied. For instance when $G = E_n$, $\sigma(G|_\emptyset; x)$ is the

Bell polynomial which is a widely studied tool in combinatorial analysis [9, 24]. Also, $\sigma(G|_{K_2}; x)$ has been recently called Stirling polynomial [11] although it was first introduced by Korfhage as σ-polynomial [17]. In particular, when written in the falling factorial power of x, $\sigma(G|_{K_2}; x) = \chi(G; x)$ is the well known chromatic polynomial [3, 22]. Thus, a restricted partition polynomial extends both the chromatic polynomial and the Stirling polynomial of graphs. In this paper, in Section 2, we determine the partition vectors of some mixed hypergraphs. Later, in Section 3, we investigate K_3-WORM colorings of some 2-trees. We find that, while 2-trees admit the same partition vector given any (classic) proper vertex coloring, it is not true for their K_3-WORM colorings. To support this argument, we present two non-isomorphic members of 2-trees which have different partition vectors. In Section 4, we conclude this paper with $\mathscr{F}$-WORM colorings when $\mathscr{F}$ includes a family of 2 or more graphs such as Path, Star or Cycle.

2 Coloring K_n with forbidden monochrome or rainbow subgraphs

We begin by establishing a connection between K_s-WORM coloring of a complete graph K_n and mixed hypergraph colorings.

Theorem 2.1. *The partition vector in a K_s-WORM coloring of K_n is $(\zeta_{\lfloor \frac{n}{s-1} \rfloor}, \ldots, \zeta_{s-1})$, where $\zeta_r = \left\{ {n \atop r} \right\}$ for all $3 \le s < n$.*

Proof. A partition of the vertices of K_n into r blocks, i.e., $\left\{ {n \atop r} \right\}$, guarantees that no subset of $V(K_n)$ with size $r \le s-1$ is rainbow. Moreover, to forbid a monochrome K_s, it suffices to ensure that no subset contains s or more elements, given each r-partition. This implies that $n \le r(s-1)$. Hence, $\lfloor \frac{n}{s-1} \rfloor \le r \le s-1$, giving the result. □

An s-uniform hypergraph whose hyperedges are all the subsets of size $s \ge 3$ of its vertex set is called an *s-uniform complete hypergraph*.

Corollary 2.2. *The partition vector of an s-uniform complete bihypergraph is $(\zeta_{\lfloor \frac{n}{s-1} \rfloor}, \ldots, \zeta_{s-1})$, where $\zeta_r = \left\{ {n \atop r} \right\}$ for all $s \ge 3$.*

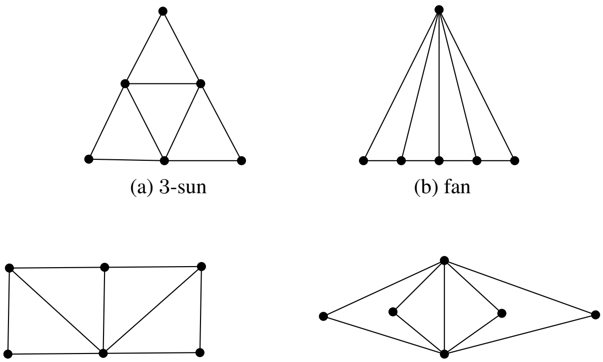
Removing either restriction on r gives each of the next result.

Corollary 2.3. *The partition vector of an s-uniform complete hypergraph is $(\zeta_{\lfloor \frac{n}{s-1} \rfloor}, \ldots, \zeta_s)$, where $\zeta_r = \left\{ {n \atop r} \right\}$ for all $s \ge 2$.*

Corollary 2.4. *The partition vector of an s-uniform complete cohypergraph is $(\zeta_1, \ldots, \zeta_{s-1})$, where $\zeta_r = \left\{ {n \atop r} \right\}$ for all $s \ge 3$.*

3 Partition vectors of some 2-trees

As a generalization of a tree, a *k-tree* on n vertices (with $1 \le k \le n$) is a graph which arises from a K_k by adding $n-k \ge 1$ new vertices, each joined to a K_k in the old graph; this process generates several non-isomorphic k-trees, $k \ge 1$. Figure 1 depicts four non-isomorphic 2-trees on 6 vertices. K-trees are chordal graphs which are known to admit at least one simplicial elimination ordering ([10]). Recall, a graph is *chordal* if it does not contain an induced cycle of length 4 or more. The characterization of families of graphs by forbidden subgraphs is an old tradition in graph theory and k-trees, despite



(a) 3-sun (b) fan

(c) snake (d) $\theta(1, 2, 2, 2, 2)$

Figure 1: Some non-isomorphic 2-trees.

being ubiquitous, have yet to be fully classified even in the case when $k = 1$. Adding some additional restrictions on the coloring of certain subgraphs besides K_2 and E_n may help in the analysis of the structure of the graphs that contain them. To help support this claim, we begin with the partition vectors in the coloring of 2-trees when monochrome K_2 are forbidden. These vectors do not characterize any member of k-trees, since non-isomorphic k-trees do share the same partition vector as shown, later, in Corollary 3.2.

Proposition 3.1. *The equality*

$$x^{\underline{k}}(x-k)^{n-k} = \sum_{t=k+1}^{n} \left\{ {n-k \atop t-k} \right\} x^{\underline{t}}$$

holds for all $1 \leq k \leq n$.

Proof. Since $x^n = \sum_{t=1}^{n} \left\{ {n \atop t} \right\} x^{\underline{t}}$, this implies that $(x-k)^{n-k} = \sum_{t=1}^{n-k} \left\{ {n-k \atop t} \right\} (x-k)^{\underline{t}}$ and

$$\begin{aligned} x^{\underline{k}}(x-k)^{n-k} &= \sum_{t=1}^{n-k} \left\{ {n-k \atop t} \right\} x(x-1)\cdots(x-k-1)(x-k)^{\underline{t}} \\ &= \sum_{t=1}^{n-k} \left\{ {n-k \atop t} \right\} x^{\underline{t+k}} \\ &= \sum_{t=k+1}^{n} \left\{ {n-k \atop t-k} \right\} x^{\underline{t}}, \end{aligned}$$

giving the result. □

Corollary 3.2. *The partition vector of any k-tree on $n-k$ simplicial vertices such that no K_2 is monochrome is* $(\zeta_{k+1}, \ldots, \zeta_n)$, *where* $\zeta_r = \left\{ {n-k \atop r-k} \right\}$.

Proof. It is easy to see that the left side of the equality of the formula in Proposition 3.1 is that of the chromatic polynomial of any k-tree, $k \geq 1$. The result follows from the right side of that equality. □

We note the quantity $\left\{ {n-k \atop r-k} \right\} = \left\{ {n \atop r} \right\}^{(k)}$ is known for counting the number of k-nonconsecutive r-partitions of n elements (see e.g., [16]); a partition of the set $\{1, \ldots, n\}$ is said to be $k-$*nonconsecutive* whenever x and y are in the same block, $|x - y| \geq k$.

Recall that a graph is called *outerplanar* if it can be embedded in the plane in such a way that every vertex lies on the outer cycle. A *maximal outerplanar (MOP)* graph is an outerplanar graph with a maximum number of edges [21]. Graphs such as 3-sun, fan and snake are some well known MOPs; these graphs are depicted in Figures 1(a), 1(b) and 1(c), respectively. Laskar and Mulder [19, 20] characterized MOPs as the intersection of any two of the following graphs: chordal, path-neighborhood, and triangle graphs $T(G)$ which are trees. Recall, a *path-neighborhood graph* is a graph in which every vertex neighborhood induces a path and the *triangle graph* $T(G)$ of G has the triangles of G as its vertices, and two vertices of $T(G)$ are adjacent whenever their corresponding triangles in G share an edge [1]. Simply put, MOPs are members of 2-trees. Here, by considering K_3-WORM colorings of 2-trees, we have found that their partition vectors are uniquely determined and the process reveals that MOPs are 2-trees with the characteristic that every edge is shared by at most two triangles.

Theorem 3.3. *Suppose G is a 2-tree such that its triangle graph is a path. Then the number of colorings of its vertices such that no triangle is monochrome or rainbow is*

$$P(G) = \sum_{1 \leq j \leq \lfloor \frac{n}{2} \rfloor} \phi(n-1, j) x(x-1)^j,$$

where

$$\phi(n-1, j) = \begin{cases} a_{n-1,j} + a_{n-1, \lfloor \frac{n-1}{2} \rfloor + j} & \text{if } j < \frac{n}{2} \\ a_{n-1, \frac{n}{2}} & \text{otherwise} \end{cases}$$

and the values $a_{i,j}$'s satisfying,

(i)

$$a_{1,1} = 1, a_{i,1} = 2 \text{ for each } 2 \leq i \leq n-1$$

and, for each $k = 1, \ldots, \lfloor \frac{i}{2} \rfloor$,

(ii)

$$a_{2k,j} = \begin{cases} a_{2k-1,j} + a_{2k-1,j+k-1} & \text{if } 2 \leq j \leq k \\ a_{2k-1,j-k} & \text{if } k+1 \leq j \leq i \end{cases}$$

(iii)

$$a_{2k+1,j} = \begin{cases} a_{2k,j} + a_{2k,j+k-1} & \text{if } 2 \leq j \leq k \\ 1 & \text{if } j = k+1 \\ a_{2k,j-k-1} & \text{if } k+2 \leq j \leq i. \end{cases}$$

Proof. Suppose $G = (V, E)$ is a 2-tree on $n \geq 3$ whose triangle graph is a path. Then, there exists a simplicial elimination ordering $\pi = \{u_1, \ldots, u_n\}$, such that u_i is adjacent to the edge with endpoints (u_{i-1}, u_{i-2}). Let $G_1 := u_1$, $G_2 := u_1 u_2$, and $G_i := G_{i-1} \cup \{u_i\}$ where u_i is adjacent to the pair (u_{i-1}, u_{i-2}) in G_i for all $i \geq 3$. Suppose c is any coloring of G and denote by $P(G)$ the restricted number of colorings of G. For $n = 3$, we count the colorings when $u_1 u_2$ is rainbow and when $u_1 u_2$ is monochrome, separately. If we denote

$A_1 = x(x-1)$ and $B_1 = x$ then $P(G_3) = A_1+(x-1)B_1+A_1$. Set $A_2 := A_1+(x-1)B_1$ and $B_2 := A_1$ and clearly A_2 and B_2 count the number of colorings where $c(u_3) \neq c(u_2)$ and $c(u_3) = c(u_2)$, respectively. For all $n \geq 3$, at each iteration, we separate the terms that count $c(u_i) \neq c(u_{i-1})$ from those that count $c(u_i) = c(u_{i-1})$, giving the recursion $P(G_n) = A_{n-1} + B_{n-1}$, where $A_{n-1} := A_{n-2} + (x-1)B_{n-2}$ and $A_{n-1} := A_{n-2}$. Now use A_1 and B_1 as basis for the previous recursion and record at each iterative step the coefficients $a_{i,j}$'s of each expression $(x-1)^k$, for $1 \leq i, j \leq n-1$. By letting $a_{1,1} = 1$, it is easy to verify that the coefficients $a_{i,j}$'s satisfy the conditions (i) – (iii). For instance, when $n = 3$, $a_{2,1} = 2$, $a_{2,2} = a_{1,1} = 1$. Now, define an $(n-1) \times (n-1)$ matrix A whose entries are the coefficients $a_{i,j}$'s of $P(G_{i+1})$ with $P(G_2) = x(x-1)$. It follows that $P := xA \cdot Q$, where

$$P = \begin{bmatrix} P(G_2) \\ P(G_3) \\ \vdots \\ P(G_n) \end{bmatrix}, \qquad A = \begin{bmatrix} a_{1,1} & & & \\ a_{2,1} & a_{2,2} & & \\ \vdots & \vdots & \ddots & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{bmatrix},$$

and

$$Q = \left[Q^1 \mid Q^2\right]^T,$$

with

$$Q^1 = \left[(x-1)^1 \quad \cdots \quad (x-1)^{\lceil \frac{n-1}{2} \rceil}\right] \text{ and}$$
$$Q^2 = \left[(x-1)^1 \quad \cdots \quad (x-1)^{\lfloor \frac{n-1}{2} \rfloor}\right].$$

Thus,

$$P(G) = P(G_n) = \\ x\Big(\sum_{1 \leq k \leq \lceil \frac{n-1}{2} \rceil} a_{n-1,k}(x-1)^k + \sum_{1+\lceil \frac{n-1}{2} \rceil \leq k \leq n-1} a_{n-1,k}(x-1)^{k-\lceil \frac{n+1}{2} \rceil} \Big) = \\ \sum_{1 \leq j \leq \lfloor \frac{n}{2} \rfloor} \phi(n-1,j) x (x-1)^j, \quad (3.1)$$

where

$$\phi(n-1,j) = \begin{cases} a_{n-1,j} + a_{n-1,\lfloor \frac{n-1}{2} \rfloor + j} & \text{if } j < \frac{n}{2} \\ a_{n-1,\frac{n}{2}} & \text{otherwise,} \end{cases}$$

giving the result. □

Remark 3.4. The argument in Theorem 3.3 requires that, at each iteration, no newly added vertex is joined to a previously used edge of a triangle. Also, this argument obviously applies to the case when one endpoint (but not both) of an edge is reused during the iteration process as in the case of a fan, for example. Moreover, because the recursive process is independent of the choice of the edge in the old graph, the result includes all such 2-trees which have the characteristic that each edge in the graph is shared by at most two triangles; this is a unique characteristic of MOPs, giving the next result.

Corollary 3.5. *If G is a MOP then its partition vector given any K_3-WORM coloring is $(\zeta_2, \ldots, \zeta_{\lfloor \frac{n}{2} \rfloor+1})$, where*

$$\zeta_r = \sum_{j=r-1}^{\lfloor \frac{n}{2} \rfloor} \phi(n-1, j) \left\{ \begin{matrix} j \\ r-1 \end{matrix} \right\},$$

with

$$\phi(n-1, j) = \begin{cases} a_{n-1,j} + a_{n-1,\lfloor \frac{n-1}{2} \rfloor + j} & \text{if } j < \frac{n}{2} \\ a_{n-1,\frac{n}{2}} & \text{otherwise,} \end{cases}$$

and the values $a_{i,j}$'s satisfying,

(i)

$$a_{1,1} = 1, a_{i,1} = 2 \text{ for each } 2 \le i \le n-1$$

and, for each $k = 1, \ldots, \lfloor \frac{i}{2} \rfloor$,

(ii)

$$a_{2k,j} = \begin{cases} a_{2k-1,j} + a_{2k-1,j+k-1} & if\ 2 \le j \le k \\ a_{2k-1,j-k} & if\ k+1 \le j \le i \end{cases}$$

(iii)

$$a_{2k+1,j} = \begin{cases} a_{2k,j} + a_{2k,j+k-1} & if\ 2 \le j \le k \\ 1 & if\ j = k+1 \\ a_{2k,j-k-1} & if\ k+2 \le j \le i. \end{cases}$$

Proof. Apply Proposition 3.1 (when $k = 1$) to the factors of the parameter ϕ in (3.1) and combine like expressions, to obtain the restricted partition polynomial

$$\sigma(G|_{K_3}) = \sum_{r=2}^{\lfloor \frac{n}{2} \rfloor+1} \Big(\sum_{j=r-1}^{\lfloor \frac{n}{2} \rfloor} \phi(n-1, j) \left\{ \begin{matrix} j \\ r-1 \end{matrix} \right\} x^{\underline{r}} \Big),$$

giving the result. □

Observations.

1. When $r = \lfloor \frac{n}{2} \rfloor + 1$, the upper partition number $\zeta_{\lfloor \frac{n}{2} \rfloor+1} = \phi(n-1, \lfloor \frac{n}{2} \rfloor)$, where

   $$\phi(n-1, \lfloor \tfrac{n}{2} \rfloor) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 + \frac{n-1}{2} & \text{otherwise.} \end{cases}$$

   These values indicate that MOPs with even number of vertices admit a unique K_3-WORM coloring.

2. Also, it is worth noting that when $r = 2$, we have the lower partition number $\zeta_2 = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \phi(n, j) = \sum_j a_{i,j}$. Further, if we let $b_{i-1} = \sum_j a_{i,j}$ for all $i \ge 2$, the sequence $\{b_n\}$ satisfies the shifted Fibonacci recurrence given by $b_1 = 3$, $b_2 = 5$ and $b_n = b_{n-1} + b_{n-2}$, for $n \ge 3$.

3. If each triangle of G is replaced by a hyperedge (of size 3), the previous result also gives the partition vector of several nonlinear 3-uniform acyclic bihypergraphs, which include the complete 3-uniform interval bihypergraphs [26]; 3-uniform bihypergraphs often appear in communication models for cyber security [14].

Obviously, there are other members of 2-trees who have 3 or more triangles sharing the same edge as subgraphs. Here, we present the other extremal case of 2-trees when all triangles share a single edge, say u_1u_2. This 2-tree, often denoted by $\theta(1, 2, \ldots, 2)$, is a member of the well known n-bridge graphs. See Figure 1(d) for an example of a 5-bridge. Note that $\theta(1, 2, \ldots, 2)$ is a maximal planar graph but not a MOP, for all $n \geq 5$.

Corollary 3.6. *Suppose $G = \theta(1, 2, \ldots, 2)$, an $(n-1)$-bridge graph on $n \geq 3$ vertices. The partition vector of a K_3-WORM coloring of G is $(\zeta_2, \ldots, \zeta_{n-1})$ where*

$$\zeta_r = \begin{cases} 2^{n-2} + 1 & \text{if } r = 2 \\ \left\{ \begin{matrix} n-2 \\ r-1 \end{matrix} \right\} & \text{otherwise.} \end{cases}$$

Proof. Count the number of colorings when the shared edge u_1u_2 is monochrome and when it is rainbow, giving $x(x-1)^{n-2} + 2^{n-2}x(x-1)$ colorings. Now apply Proposition 3.1 (when $k = 1$) to the terms of the expression to obtain the result. □

We leave it to the reader to verify that the previous values in the partition vector when $G = \theta(1, 2, \ldots, 2)$ are different from those of MOPs, for all $n \geq 5$.

4 Conclusion

We've shown that while 2-trees admit the same partition vector given any proper vertex coloring, it is not the case with their K_3-WORM colorings. We hope these results indicate the importance of WORM colorings in general in the analysis of the structures of some well-known graphs which could not be classified with the usual proper vertex colorings. For a potential future research, we introduce some generalizations of $\mathscr{F}$-WORM colorings when $\mathscr{F}$ includes multiple graphs such as Path, Star or Cycle. In the next results, C_n, $K_{1,n-1}$, and P_n^* denote an n-cycle, an n-star, and an n-path that includes a fixed vertex (apex) u_1, respectively.

Corollary 4.1. *Suppose G is a fan on $n \geq 4$ vertices. If G has a K_3-WORM coloring then G admits an $\mathscr{F}$-WORM coloring with $\mathscr{F} = \{P_s^*, K_{1,t}, C_r, \theta(1, n_1, n_2)\}$ where $s \geq 4$, $\lfloor \frac{n-1}{2} \rfloor \leq t \leq n-1$, $r \geq 3$, and $2 \leq n_1 \leq n_2$ such that $n_1 + n_2 \leq n$.*

Proof. Suppose G is a fan on $n \geq 4$ vertices which we can construct as follow: start with a triangle, say (u_1, u_2, u_3), and iteratively add $n-3$ new vertices such that each additional vertex u_i is adjacent to the pair (u_1, u_{i-1}), for $i = 4, \ldots, n$. Assume G admits a K_3-WORM coloring.

(i) Observe that for $s \geq 4$, every path $P_s^* \subseteq G$ contains the subgraph $u_1u_iu_{i+1}$ for some i ($2 \leq i \leq n-2$). If some 3-path (that includes u_1) is monochrome/rainbow then the triangle (u_1, u_i, u_{i+1}) is monochrome/rainbow, violating the K_3-WORM coloring assumption. Hence G admits a P_s^*-WORM coloring for all $s \geq 4$.

(ii) By letting the vertices of $K_{1,t} \subseteq G$ be all the vertices of G, it follows that $t \leq n-1$. Now, consider the coloring such $c(u_1) = c(u_{2k})$ and $c(u_1) \neq c(u_{2k+1})$ for $k = 1, \ldots, \lceil \frac{n-1}{2} \rceil$. Clearly, such coloring does not violate our assumption of K_3-WORM coloring of G. Hence, the lower bound of t is satisfied by letting the vertices of $K_{1,t}$ be $\{u_1, u_2\} \cup \{u_{2k+1} : k = 1, \ldots, \lceil \frac{n-1}{2} \rceil\}$, which guarantees a $K_{1,t}$-WORM coloring for all $t \geq \lceil \frac{n-1}{2} \rceil$.

(iii) For $r \geq 4$, since every cycle $C_r \subseteq G$ includes the apex u_1, there exists an $s \leq r$ such that $P_s^* \subseteq C_r$, with $4 \leq s \leq r \leq n$. From (i), G admits a C_r-WORM coloring. The case when $r = 3$ is trivial.

(iv) Likewise, since $\theta(1, n_1, n_2)$ contains $C_{1+q} \subseteq G$ with $q \in \{n_1, n_2\}$, the result follows from (iii) that, for all $2 \leq n_1 \leq n_2$ such that $n_1 + n_2 \leq n$, G admits a $\theta(1, n_1, n_2)$-WORM coloring. □

Note that the converse of the statement in Corollary 4.1 is not true. Using a similar argument as in the previous proof establishes the next result; recall, a *snake* (see Figure 1(c)) is a 3-sun-free maximal outerplanar graph with at least four vertices.

Corollary 4.2. *Suppose G is a snake on $n \geq 4$ vertices. If G has a K_3-WORM coloring then G admits an $\mathscr{F}$-WORM coloring, where $\mathscr{F} = \{C_r, \theta(1,2,2)\}$ with $3 \leq r \leq n$.*

References

[1] R. Balakrishnan, Triangle graphs, in: R. Balakrishnan, H. M. Mulder and A. Vijayakumar (eds.), *Graph Connections*, Allied Publishers Limited, New Delhi, 1999 p. 44, proceedings of the conference held at Cochin University of Science and Technology, Cochin, January 28 – 31, 1998.

[2] E. Bertin and R. Theodorescu, On the unimodality of discrete probability measures, *Math Z.* **201** (1989), 131–137, doi:10.1007/bf01162000.

[3] G. D. Birkhoff and D. C. Lewis, Chromatic polynomials, *Trans. Amer. Math. Soc.* **60** (1946), 355–451, doi:10.1090/s0002-9947-1946-0018401-4.

[4] F. Brenti, Expansions of chromatic polynomials and log-concavity, *Trans. Amer. Math. Soc.* **332** (1992), 729–756, doi:10.1090/S0002-9947-1992-1069745-7.

[5] Cs. Bujtás, E. Sampathkumar, Zs. Tuza, M. S. Subramanya and C. Dominic, 3-consecutive C-colorings of graphs, *Discuss. Math. Graph Theory* **30** (2010), 393–405, doi:10.7151/dmgt.1502.

[6] Cs. Bujtás and Zs. Tuza, Uniform mixed hypergraphs: the possible numbers of colors, *Graphs Combin.* **24** (2008), 1–12, doi:10.1007/s00373-007-0765-5.

[7] Cs. Bujtás and Zs. Tuza, K_3-WORM colorings of graphs: lower chromatic number and gaps in the chromatic spectrum, *Discuss. Math. Graph Theory* **36** (2016), 759–772, doi:10.7151/dmgt.1891.

[8] Cs. Bujtás, Zs. Tuza and V. Voloshin, Hypergraph colouring, in: L. W. Beineke and R. J. Wilson (eds.), *Topics in Chromatic Graph Theory*, Cambridge University Press, Cambridge, volume 156 of *Encyclopedia of Mathematics and its Applications*, pp. 230–254, 2015, doi:10.1017/cbo9781139519793.014.

[9] C. B. Collins, The role of Bell polynomials in integration, *J. Comput. Appl. Math.* **131** (2001), 195–222, doi:10.1016/s0377-0427(00)00274-0.

[10] G. A. Dirac, On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg* **25** (1961), 71–76, doi:10.1007/bf02992776.

[11] D. Galvin and D. T. Thanh, Stirling numbers of forests and cycles, *Electron. J. Combin.* **20** (2013), #P73, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v20i1p73`.

[12] W. Goddard, K. Wash and H. Xu, WORM colorings forbidding cycles or cliques, *Congr. Numer.* **219** (2014), 161–173.

[13] W. Goddard, K. Wash and H. Xu, WORM colorings, *Discuss. Math. Graph Theory* **35** (2015), 571–584, doi:10.7151/dmgt.1814.

[14] A. Jaffe, T. Moscibroda and S. Sen, On the price of equivocation in byzantine agreement, in: D. Kowalski and A. Panconesi (eds.), *PODC'12, ACM Symposium on Principles of Distributed Computing*, ACM, 2012 pp. 309–318, doi:10.1145/2332432.2332491, proceedings of the 2012 ACM Symposium on Principles of Distributed Computing held in Funchal, Madeira, Portugal, July 16 – 18, 2012.

[15] T. Jiang, D. Mubayi, Zs. Tuza, V. Voloshin and D. B. West, The chromatic spectrum of mixed hypergraphs, *Graphs Combin.* **18** (2002), 309–318, doi:10.1007/s003730200023.

[16] Zs. Kereskényi-Balogh and G. Nyul, Stirling numbers of the second kind and Bell numbers for graphs, *Australas. J. Combin.* **58** (2014), 264–274, `https://ajc.maths.uq.edu.au/pdf/58/ajc_v58_p264.pdf`.

[17] R. R. Korfhage, σ-polynomials and graph coloring, *J. Comb. Theory Ser. B* **24** (1978), 137–153, doi:10.1016/0095-8956(78)90015-1.

[18] D. Král̆, On feasible sets of mixed hypergraphs, *Electron. J. Combin.* **11** (2004), #R19, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v11i1r19`.

[19] R. C. Laskar and H. M. Mulder, Path-neighborhood graphs, *Discuss. Math. Graph Theory* **33** (2013), 731–745, doi:10.7151/dmgt.1700.

[20] R. C. Laskar, H. M. Mulder and B. Novick, Maximal outerplanar graphs as chordal graphs, path-neighborhood graphs, and triangle graphs, *Australas. J. Combin.* **52** (2012), 185–195, `https://ajc.maths.uq.edu.au/pdf/52/ajc_v52_p185.pdf`.

[21] S. L. Mitchell, *Algorithms on Trees and Maximal Outerplanar Graphs: Design, Complexity Analysis and Data Structures Study*, Ph.D. thesis, The University of Virginia, Charlottesville, Virginia, 1977, `https://search.proquest.com/docview/302868872`.

[22] A. Mohr and T. D. Porter, Applications of chromatic polynomials involving Stirling numbers, *J. Combin. Math. Combin. Comput.* **70** (2009), 57–64.

[23] N. J. Pullman, Clique covering of graphs IV: Algorithms, *SIAM J. Comput.* **13** (1984), 57–75, doi:10.1137/0213005.

[24] J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley Publications in Mathematical Statistics, John Wiley & Sons, New York, 1958, `http://www.jstor.org/stable/j.ctt7zv5hs`.

[25] V. I. Voloshin, On the upper chromatic number of a hypergraph, *Australas. J. Combin.* **11** (1995), 25–45, `https://ajc.maths.uq.edu.au/pdf/11/ocr-ajc-v11-p25.pdf`.

[26] V. I. Voloshin, *Coloring Mixed Hypergraphs: Theory, Algorithms and Applications*, volume 17 of *Fields Institute Monographs*, American Mathematical Society, Providence, RI, 2002, doi:10.1090/fim/017.

[27] P. Zhao, K. Diao and K. Wang, The chromatic spectrum of 3-uniform bi-hypergraphs, *Discrete Math.* **311** (2011), 2650–2656, doi:10.1016/j.disc.2011.08.007.

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Relating the total domination number and the annihilation number of cactus graphs and block graphs*

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Abstract

The total domination number $\gamma_t(G)$ of a graph G is the order of a smallest set $D \subseteq V(G)$ such that each vertex of G is adjacent to some vertex in D. The annihilation number $a(G)$ of G is the largest integer k such that there exist k different vertices in G with degree sum of at most $|E(G)|$. It is conjectured that $\gamma_t(G) \le a(G) + 1$ holds for every nontrivial connected graph G. The conjecture was proved for graphs with minimum degree at least 3, and remains unresolved for graphs with minimum degree 1 or 2. In this paper we establish the conjecture for cactus graphs and block graphs.

Keywords: Total domination number, annihilation number, cactus graph, block graph.

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1 Introduction

All graphs considered in this paper are nontrivial, finite, simple and undirected. By a nontrivial graph we mean a graph on at least two vertices. If $G = (V, E)$ is a graph, then $V = V(G)$ is the set of vertices of order $n(G) = |V|$, and $E = E(G)$ is the set of edges of size $m(G) = |E|$. The *degree* of a vertex $v \in V$ in graph G will be denoted by $d_G(v)$. A vertex v of degree 1 is a *leaf*, while its only neighbor is called a *support vertex*. If u has at least two neighbors which are leaves, then u is referred to as a *strong support vertex*. The minimum and maximum degree among the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For $v \in V(G)$, the set of its neighbors is denoted by $N_G(v)$ and called the *open neighborhood of* v. We use a similar notation for a set $A \subseteq V(G)$, it is defined as $N_G(A) = \bigcup_{v \in A} N_G(v)$. If G is clear from the context, we simply write $d(v)$, $N(v)$ and $N(A)$ instead of $d_G(v)$, $N_G(v)$ and $N_G(A)$, respectively.

For a graph G, a set $D \subseteq V(G)$ is a *total dominating set* if every $v \in V(G)$ has at least one neighbor in D; i.e., if $N(D) = V(G)$. If G does not contain isolated vertices, such a set D always exists, and the minimum cardinality of a total dominating set, denoted by $\gamma_t(G)$, is the *total domination number* of G. A survey on total domination can be found in [8], and more recently, the topic was thoroughly covered in the book [9]. If C_n is a cycle of length n, its total domination number can be obtained as follows:

$$\gamma_t(C_n) = \begin{cases} \frac{n}{2} + 1, & \text{if } n \equiv 2 \pmod 4; \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

For a set $B \subseteq V$ we denote by $G - B$ the graph which is obtained from G by deleting the vertices in B and all edges incident with them. Moreover, if $v_1v_2 \in E$ and $v_1v_2 \notin E$ with $v_1, v_2 \in V$, we use the notations $G - v_1v_2$ and $G + v_1v_2$ for the graphs $(V, E \setminus \{v_1v_2\})$ and $(V, E \cup \{v_1v_2\})$, respectively. Let G_1 and G_2 be two vertex-disjoint graphs and let $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. The *identification of vertices* v_1 and v_2 results in a graph G with $V(G) = (V(G_1) \cup V(G_2) \cup \{v\}) \setminus \{v_1, v_2\}$ such that $N_G(v) = N_{G_1}(v_1) \cup N_{G_2}(v_2)$. Moreover, for any vertex $u \neq v$, the open neighborhood of u remains the same.

The *subdivided star* $S(K_{1,\ell})$ is the graph on $2\ell + 1$ vertices which is constructed from the star $K_{1,\ell}$ by subdividing each edge exactly once (left-hand side of Figure 1). The *paw* is the graph P obtained from K_4 by deleting two neighboring edges (right-hand side of Figure 1). A connected graph is called *cactus graph* if its cycles are pairwise edge-disjoint. Moreover, G is a *block graph* if each 2-connected component of G is a clique.

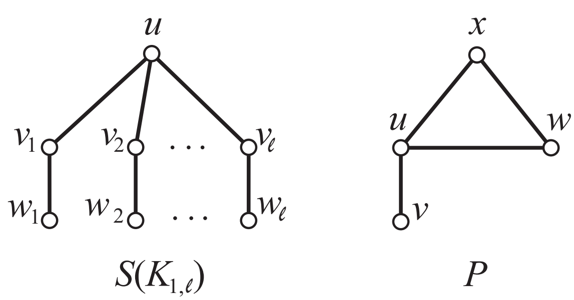

Figure 1: The subdivided star $S(K_{1,\ell})$, $\ell \geq 2$, and the paw graph P.

For a subset $S \subseteq V(G)$ we define

$$\sum(S,G) = \sum_{v \in S} d_G(v).$$

Let $v_1, v_2, \dots, v_n$ be an ordering of the vertices of G such that $d(v_1) \le d(v_2) \le \cdots \le d(v_n)$. The *annihilation number* $a(G)$ is the largest integer k such that $\sum_{i=1}^{k} d(v_i) \le m(G)$. Equivalently, $a = a(G)$ is the only integer satisfying both

$$\sum_{i=1}^{a} d(v_i) \le m(G) \quad \text{and} \quad \sum_{i=1}^{a+1} d(v_i) \ge m(G) + 1.$$

It is clear by definition that every independent set[1] A satisfies $\sum_{v \in A} d(v) \le m(G)$ and consequently, the annihilation number is an upper bound on the independence number [11]. The annihilation number was first introduced by Pepper in [12]. The 'annihilation process', which is referred to in this original definition, is very similar to the 'Havel-Hakimi process' (see [7] and [11] for exact descriptions).

In general, a set S of vertices is called an *annihilation set* if $\sum_{v \in S} d(v) \le m(G)$; and S is an *optimal annihilation set*, if

$$|S| = a(G) \quad \text{and} \quad \max\{d(v) \mid v \in S\} \le \min\{d(u) \mid u \in V(G) \setminus S\}.$$

In particular, if G is a connected graph on at least 3 vertices, any optimal annihilation set of G contains all leaves.

Assuming that S is an optimal annihilation set, we introduce the following notations. First, denote by $d^*(G)$ (or simply by d^*) the minimum vertex degree over the set $V(G) \setminus S$. Note that $d^*(G) = d(v_{a(G)+1})$, and consequently, the value of $d^*(G)$ is independent from the choice of the optimal annihilation set S.

The following conjecture can be found in a slightly different form in Graffiti.pc [4], and was later reformulated in [5].

Conjecture 1.1 ([4, 5]). *If G is a connected nontrivial graph, then*

$$\gamma_t(G) \le a(G) + 1. \tag{1.1}$$

By definition, every graph satisfies $a(G) \ge \lfloor \frac{n(G)}{2} \rfloor$. Hence, the formulas given for $\gamma_t(C_n)$ above show that each cycle C_n satisfies the conjecture. Further, if $\delta(G) \ge 3$, it was observed that the total domination number is at most $\lfloor \frac{n(G)}{2} \rfloor$ [1, 3, 13, 14]. Hence, if $\delta(G) \ge 3$, then $\gamma_t(G) \le a(G)$ clearly holds, even if G is disconnected. Therefore, it is interesting to study this conjecture for graphs with small minimum degree, i.e. $\delta(G) \in \{1, 2\}$. So far, Conjecture 1.1 has been proved for only one further important graph class. The following result was established by Desormeaux, Haynes, and Henning in 2013.

Theorem 1.2 ([5]). *If T is a nontrivial tree, then $\gamma_t(T) \le a(T) + 1$, and the bound is sharp.*

[1] A set $A \subseteq V(G)$ is called an independent set if it induces an edgeless subgraph in G. The largest cardinality of such a vertex set is the independence number of G and denoted by $\alpha(G)$.

A similar result was proved by Desormeaux, Henning, Rall, and Yeo [6] for the 2-domination number of trees. Very recently, a different proof was given for the same statement by Lyle and Patterson [10]. Namely, their result can be obtained if we replace the total domination number with the 2-domination number in Theorem 1.2.

In this paper we prove Conjecture 1.1 over two further graph classes, namely for cactus graphs and block graphs. These are two natural generalizations of trees and also, for a cactus graph G we have $\delta(G) \leq 2$ and there exist block graphs with small minimum degree. Remark that cactus and block graphs are well-studied classes with several applications, see for instance [2]. Our main results are the following ones.

Theorem 1.3. *If G is a nontrivial cactus graph, then $\gamma_t(G) \leq a(G) + 1$.*

Theorem 1.4. *If G is a nontrivial block graph, then $\gamma_t(G) \leq a(G) + 1$.*

To formulate and to prove our results we will use the following *function f* defined for every finite graph G as

$$f(G) = n(G) + 3m(G) + n_1(G),$$

where $n_1(G)$ denotes the number of leaves in G. Remark that f is strictly monotone in the sense that if G' is a proper subgraph of G, then $f(G') < f(G)$. Indeed, $n(G') + m(G') < n(G) + m(G)$ clearly holds, and $2m(G') + n_1(G') \leq 2m(G) + n_1(G)$ is true because the deletion of an edge may result in at most two new leaves. Also note that we have $f(G) \geq 7$ for any nontrivial, finite and connected graph G.

The paper is organized as follows. In Section 2, we establish several lemmas which will be referred to in later proofs. In Section 3 and 4 we prove Theorem 1.3 and 1.4, respectively. In the last section we discuss the sharpness of our main theorems and arise some related problems.

2 Preliminary results

Here we present some preliminary results on how we can obtain a smaller graph G' from G (mainly, by deleting some edges and/or vertices from G) such that $\gamma_t(G') \leq a(G') + 1$ implies $\gamma_t(G) \leq a(G) + 1$. First we consider changes related to vertices of small degree.

Lemma 2.1. *Assume that G is a connected graph on at least three vertices and it fulfills at least one of the following properties:*

(i) $d^*(G) \leq 2$*;*

(ii) *G has a strong support vertex;*

(iii) *G contains an induced path $vu_1u_2u_3w$ such that $d(u_1) = d(u_2) = d(u_3) = 2$;*

(iv) *G contains a path $u_1u_2u_3v$ such that u_1 is a leaf and $d(u_2) = d(u_3) = 2$;*

(v) *G contains two adjacent support vertices.*

Then, there exists a nontrivial connected graph G' with $f(G') < f(G)$ such that $\gamma_t(G') \leq a(G') + 1$ implies $\gamma_t(G) \leq a(G) + 1$. Moreover, if G is a cactus graph, then G' can be chosen to be a cactus graph as well; and if G is a block graph, in cases (ii) – (v), G' can be chosen to be a block graph.

Proof. Since trees and cycles satisfy Conjecture 1.1, we may suppose throughout that G is neither a tree nor a cycle.

(i) First assume that $d^*(G) \leq 2$. Since G is neither a tree nor a cycle, there exists a vertex $v \in V(G)$ with $d(v) \geq 3$ which is incident to a cycle. Let $e = vu$ be an edge from that cycle. Clearly, $G' = G - e$ is connected, $f(G') < f(G)$ and $m(G') = m(G) - 1$. The deletion of an edge does not decrease the total domination number. This establishes $\gamma_t(G) \leq \gamma_t(G')$. Consider now an optimal annihilation set S' of G'. By definition, it satisfies $\sum(S', G') \leq m(G') = m(G) - 1$. If $u, v \notin S'$ then $\sum(S', G) = \sum(S', G') \leq m(G) - 1$; if S' contains exactly one of u and v, then $\sum(S', G) = \sum(S', G') + 1 \leq m(G)$. In either case $a(G) \geq |S'| = a(G')$ follows. In the third case $u, v \in S'$ and $\sum(S', G) = \sum(S', G') + 2 \leq m(G) + 1$. Let $V_{1,2}$ denote the set of vertices which have degree 1 or 2 in G. Our assumption $d^*(G) \leq 2$ implies $\sum(V_{1,2}, G) \geq m(G) + 1$. Since $d(v) \geq 3$, we have $\sum(V_{1,2} \cup \{v\}, G) \geq m(G) + 4$. Therefore, $(V_{1,2} \cup \{v\}) \nsubseteq S'$ implies that we have a vertex $v^* \in V_{1,2}$ which is not contained in S'. If v is replaced with v^* in S', we obtain a set S with $\sum(S, G) \leq \sum(S', G) - 1 \leq m(G)$. This proves $a(G) \geq |S| = a(G')$. If G' satisfies (1.1), we may conclude that the same is true for G:

$$\gamma_t(G) \leq \gamma_t(G') \leq a(G') + 1 \leq a(G) + 1.$$

In the sequel of the proof we will assume that $d^*(G) \geq 3$.

(ii) Assume that a vertex $v \in V(G)$ has two neighbors u_1 and u_2 which are leaves in G. Since v remains a support vertex in $G' = G - \{u_1\}$, it is contained in every total dominating set of G'. This implies $\gamma_t(G') = \gamma_t(G)$. On the other hand, every optimal annihilation set of G contains u_1 and hence $a(G') \leq a(G)$. Then, $f(G') < f(G)$, and $\gamma_t(G') \leq a(G') + 1$ implies $\gamma_t(G) \leq a(G) + 1$.

(iii) If $vu_1u_2u_3w$ is an induced path in G and $d(u_1) = d(u_2) = d(u_3) = 2$, consider the graph $G' = G - \{u_1, u_2, u_3\} + vw$. Observe that $n(G') = n(G) - 3$, $m(G') = m(G) - 3$, $n_1(G') = n_1(G)$ and hence, $f(G') = f(G) - 12$. Let D' be an optimal total dominating set of G' and define D as follows:

$$D = \begin{cases} D' \cup \{u_1, u_3\}, & \text{if } v, w \in D'; \\ D' \cup \{u_2, u_3\}, & \text{if } w \notin D'; \\ D' \cup \{u_1, u_2\}, & \text{if } w \in D' \text{ and } v \notin D'. \end{cases}$$

In either case, D is a total dominating set in G. Hence, $\gamma_t(G) \leq \gamma_t(G') + 2$. Consider next an optimal annihilation set S' of G'. Since $d_G(v) = d_{G'}(v)$ and $d_G(w) = d_{G'}(w)$, $\sum(S', G) = \sum(S', G') \leq m(G') = m(G) - 3$. Our assumption $d^*(G) \geq 3$ implies that every vertex x with degree $d(x) \leq 2$ is contained in every optimal annihilation set of G. Hence, either $S' \cup \{u_1, u_2, u_3\}$ is a subset of an optimal annihilation set of G and $a(G) \geq a(G') + 3$, or there is a vertex $v^* \in S'$ with $d(v^*) \geq 3$. In the latter case, consider $S = (S' \setminus \{v^*\}) \cup \{u_1, u_2, u_3\}$, and observe that $\sum(S, G) \leq \sum(S', G) - 3 + 3 \cdot 2 \leq m(G)$. Therefore, $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. If G' satisfies inequality (1.1), we have

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1$$

and that proves the statement for property *(iii)* .

(iv) Let $u_1u_2u_3v$ be a path in G such that $d(u_1) = 1$ and $d(u_2) = d(u_3) = 2$. Since G is connected and not a path, $G' = G - \{u_1, u_2, u_3\}$ is nontrivial, and we have $f(G') < f(G)$.

If D' is an optimal total dominating set of G', then $D = D' \cup \{u_2, u_3\}$ totally dominates all vertices in G. Thus, $\gamma_t(G) \leq |D| \leq \gamma_t(G') + 2$. Next, we choose an optimal annihilation set S' in G' and consider three cases concerning v and S'.

- If $d(v) = 2$, then G contains three consecutive degree-2 vertices and, as we have already proved it in *(iii)*, there exists a graph G' with the required property.
- If $v \notin S'$, then $\sum(S', G) = \sum(S', G')$, and $\sum(S', G) \leq m(G') = m(G) - 3$. Hence, $S = S' \cup \{u_1, u_2\}$ satisfies $\sum(S, G) = \sum(S', G) + 3 \leq m(G)$, and $a(G) \geq a(G') + 2$. This, together with the assumption $\gamma_t(G') \leq a(G') + 1$, establishes inequality (1.1) for G.
- In the last case we assume that both $v \in S'$ and $d(v) \geq 3$ hold. Then, $\sum(S', G) = \sum(S', G') + 1 \leq m(G') + 1 = m(G) - 2$. We define $S = (S' \setminus \{v\}) \cup \{u_1, u_2, u_3\}$ and observe that $\sum(S, G) = \sum(S', G) - d(v) + 5 \leq m(G)$. Hence, S is an annihilation set in G and we may conclude $a(G) \geq |S| \geq a(G') + 2$. The statement of the lemma is proved by the following chain of inequalities: $\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1$.

(v) Let u and v be two leaves in G with support vertices u' and v' respectively such that $uu', vv', u'v' \in E(G)$. Since G is not a path, at least one of these two support vertices, say u', is of degree of at least 3. Then, we define $G' = G - uu' + uv$ and observe that $f(G') = f(G) - 1$. Let D' be an optimal total dominating set of G'. Since v is a support vertex in G', $v \in D'$ must hold. Moreover, since $N_{G'}(u) \subseteq N_{G'}(v')$, we can choose D' such that u does not belong to it. Then, $D = (D' \setminus \{v\}) \cup \{u'\}$ is a total dominating set in G. Hence, $\gamma_t(G) \leq |D| = |D'| = \gamma_t(G')$. By construction, every vertex has the same degree in G as in G' with the two exceptions v and u', for which $d_G(u') = d_{G'}(u') + 1$ and $d_G(v) = d_{G'}(v) - 1$. Hence, any optimal annihilation set S' of G' satisfies one of the following cases.

- If $u', v \in S'$ or $u', v \notin S'$, then $\sum(S', G) = \sum(S', G') \leq m(G') = m(G)$. Therefore, $a(G) \geq |S'| = a(G')$.
- If $v \in S'$ and $u' \notin S'$, then $\sum(S', G) = \sum(S', G') - 1 \leq m(G') - 1 = m(G) - 1$. Therefore, $a(G) \geq |S'| = a(G')$.
- If $u' \in S'$ and $v \notin S'$, then $\sum(S', G) = \sum(S', G') + 1 \leq m(G') + 1 = m(G) + 1$. We define $S = (S' \setminus \{u'\}) \cup \{v\}$. By our assumption, $d_G(u') \geq 3$ and so, we have $\sum(S, G) = \sum(S', G) - d_G(u') + 1 \leq m(G) + 1 - d_G(u') + 1 \leq m(G) - 1$. This implies $a(G) \geq a(G')$.

We have seen that for all possible cases $a(G') \leq a(G)$ and $\gamma_t(G) \leq \gamma_t(G')$. Together with the condition that G' satisfies (1.1), these imply $\gamma_t(G) \leq \gamma_t(G') \leq a(G') + 1 \leq a(G) + 1$.

At the end of the proof we remark that all the above transformations result in a cactus graph G', if G was of the same type. Further, with the only exception of *(i)*, the obtained graphs stay block graphs if G is a block graph. □

Lemma 2.2.

(i) *For an integer $\ell \geq 3$, let $Q \cong K_\ell$ be a complete subgraph of the connected graph G such that Q contains exactly one vertex, say x, of degree larger than $\ell - 1$. Assume further that $G' = G - (V(Q) \setminus \{x\})$ satisfies $\gamma_t(G') \leq a(G') + 1$. Then, $\gamma_t(G) \leq a(G) + 1$ follows.*

(ii) Let C be a cycle in a connected graph G such that C contains exactly one vertex which is of degree larger than 2. Then, there exists a nontrivial connected graph G' with $f(G') < f(G)$ such that $\gamma_t(G') \le a(G') + 1$ implies $\gamma_t(G) \le a(G) + 1$. Moreover, if G is a cactus graph, then G' can be chosen to be a cactus graph as well.

Proof. (i) We suppose $d(x) \ge \ell \ge 3$ and $V(Q) = \{v_1, v_2, \dots, v_{\ell-1}, x\}$. By definition, $m(G') = m(G) - \binom{\ell}{2}$. For any total dominating set D' of G', $D' \cup \{x\}$ is a total dominating set of G. Hence, $\gamma_t(G) \le \gamma_t(G') + 1$. Now, let S' be an optimal annihilation set in G'.

- If $x \in S'$, we define $S = (S' \setminus \{x\}) \cup \{v_1, \dots, v_{\lfloor \frac{\ell}{3} \rfloor + 1}\}$. Since $d_{G'}(x) \ge 1$, we $\sum(S, G) \le \sum(S', G') - 1 + \left(\lfloor \frac{\ell}{3} \rfloor + 1\right)(\ell - 1)$.
- If $x \notin S'$, let $S = S' \cup \{v_1, \dots, v_{\lfloor \frac{\ell}{3} \rfloor}\}$. Then, since $\ell \ge 3$, we have

$$\sum(S, G) \le \sum(S', G') + \left\lfloor \frac{\ell}{3} \right\rfloor (\ell - 1) \le \sum(S', G') - 1 + \left(\left\lfloor \frac{\ell}{3} \right\rfloor + 1 \right) (\ell - 1).$$

Observe that in either case and for every $\ell \ge 3$, the relation $|S| \ge |S'| + 1$ holds. Moreover, as $\sum(S', G') \le m(G')$, we may estimate $\sum(S, G)$ as follows:

$$\begin{aligned} \sum(S, G) &\le m(G') - 1 + \left(\left\lfloor \frac{\ell}{3} \right\rfloor + 1 \right) (\ell - 1) \\ &= m(G) - \binom{\ell}{2} - 1 + \left(\left\lfloor \frac{\ell}{3} \right\rfloor + 1 \right) (\ell - 1) \\ &= m(G) - \left[(\ell - 1) \left(\frac{\ell}{2} - \left\lfloor \frac{\ell}{3} \right\rfloor - 1 \right) + 1 \right] \le m(G). \end{aligned}$$

Here, the last inequality can be directly checked for $\ell = 3$, 4 and 5. If $\ell \ge 6$, this clearly follows from $\frac{\ell}{2} - \frac{\ell}{3} - 1 \ge 0$. We conclude that S is an annihilation set in G and therefore, $a(G) \ge |S| \ge |S'| + 1 = a(G') + 1$. Together with the condition given in *(i)* for G',

$$\gamma_t(G) \le \gamma_t(G') + 1 \le a(G') + 2 \le a(G) + 1$$

follows. This finishes the proof of *(i)*.

(ii) Since $C_3 = K_3$, it suffices to prove *(ii)* for cycles $C \cong C_\ell$ of length $\ell \ge 4$. If $d^*(G) \le 2$ or $\ell \ge 6$, Lemma 2.1*(i)* and 2.1*(iii)* establish the statement. Henceforth, we will suppose that $d^*(G) \ge 3$ and $\ell = 4$ or 5. Let $xv_1 \dots v_{\ell-1}x$ be the cycle C such that $d(x) \ge 3$.

First, assume that $\ell + d_G(x) \ge 8$; i.e., at least one of $\ell = 5$ and $d_G(x) \ge 4$ holds. Let $G' = G - (V(C) \setminus \{x\})$ and let D' be an optimal total dominating set of G'. Observe that $D = D' \cup \{v_2, v_3\}$ is a total dominating set in G and consequently, $\gamma_t(G) \le \gamma_t(G') + 2$. Now, fix an optimal annihilation set S' in G' and consider the following two subcases.

- If $x \notin S'$, we have $\sum(S', G) = \sum(S', G') \le m(G') = m(G) - \ell$. Then, we define $S = S' \cup \{v_1, v_2\}$ and observe that $\sum(S, G) = \sum(S', G) + 2 \cdot 2 \le m(G) - \ell + 4 \le m(G)$. This proves $a(G) \ge |S| = a(G') + 2$.
- If $x \in S'$, we have $\sum(S', G) = \sum(S', G') + 2 \le m(G') + 2 = m(G) - \ell + 2$. In this case, consider $S = (S' \setminus \{x\}) \cup \{v_1, v_2, v_3\}$. For this set,

$$\sum(S, G) \le \sum(S', G) - d_G(x) + 3 \cdot 2 \le m(G) - \ell - d_G(x) + 8 \le m(G)$$

holds under the present assumption $\ell + d_G(x) \geq 8$. Therefore, we have $a(G) \geq |S| = a(G') + 2$.

In either subcase, if G' satisfies (1.1), we may conclude that

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1.$$

In the other case, $C \cong C_4$ and $d_G(x) = 3$. Here, we define $G' = G - V(C)$. Since $d_G(x) = 3$, G' is connected. If G' consists of only one vertex, $\gamma_t(G) = 2 < a(G) + 1$ can be proved directly. Hence, we may assume that G' is nontrivial. Let D' be an optimal total dominating set in G' and observe that, also in this case, $D = D' \cup \{v_2, v_3\}$ is a total dominating set in G. Hence, $\gamma_t(G) \leq \gamma_t(G') + 2$. On the other hand, let S' be an optimal annihilation set in G'. Since there is at most one edge between S' and $V(C)$, $\sum(S', G) \leq \sum(S', G') + 1 \leq m(G') + 1 = m(G) - 4$. Moreover, for $S = S' \cup \{v_1, v_2\}$, we obtain $\sum(S, G) = \sum(S', G) + 4 \leq m(G)$, from which $a(G) \geq a(G') + 2$ follows. Thus, if G' satisfies (1.1), the desired inequality $\gamma_t(G) \leq a(G) + 1$ holds again. □

The analogue of the following proof was given by Desormeaux et al. [5] inside the proof of Theorem 1.2. There, both H and T were restricted to be a tree. Here, we restate and prove the lemma in a more general form, where H can be an arbitrary connected graph.

Lemma 2.3. *Let H be a nontrivial connected graph and T be a tree such that $V(H) \cap V(T) = \emptyset$. Suppose that $w \in V(H)$, $u \in V(T)$, and v is a leaf in T such that $d(u, v) \geq 3$. If G is obtained from H and T by identifying w and u, there exists a connected graph G' with $f(G') < f(G)$ such that $\gamma_t(G') \leq a(G') + 1$ implies $\gamma_t(G) \leq a(G) + 1$.*

Proof. First note that the statement follows from Lemma 2.1*(i)* if $d^*(G) \leq 2$. Hence, we may suppose that $d^*(G) \geq 3$. Assume that T is rooted in u and choose a leaf $v_1 \in V(T)$ which is of maximum distance from u. Let v_2 be the parent of v_1, and v_3 be the parent of v_2. By assumption, $d(u, v_1) \geq 3$ and hence, $v_i \neq u$ $(i = 1, 2, 3)$.

We will consider graphs G' obtained from G by removing a set of vertices from $V(T)$ in such a way that G' will stay connected. Throughout, S' will denote an optimal annihilation set in G'.

If v_2 is a strong support vertex, Lemma 2.1*(ii)* implies the statement. So, we may suppose that v_1 is the only leaf of the support vertex v_2. Since v_1 is of maximum distance from u, $d(v_2) = 2$ also follows. Remark that the same is true for any other leaf and its support vertex, if the leaf is of maximum distance from u. Suppose that $d(v_3) \geq 3$ and let $G' = G - \{v_1, v_2\}$. So $m(G') = m(G) - 2$. If v_3 is a support vertex in G', then v_3 belongs to a minimum total dominating set D' of G'. If v_3 is not a support vertex, then every child of v_3 is a support vertex of degree 2. If a leaf-neighbor of a child of v_3 belongs to D', then we can simply replace it in D' with the vertex v_3. In either case, we may assume that $v_3 \in D'$. Thus the set $D = D' \cup \{v_2\}$ is a total dominating set of G, and so $\gamma_t(G) \leq |D| = |D'| + 1 = \gamma_t(G') + 1$. Independently of whether vertex v_3 lies in S' or not we have $\sum(S', G) \leq \sum(S', G') + 1 \leq m(G') + 1 = m(G) - 1$. Consider $S = S' \cup \{v_1\}$. Then $\sum(S, G) = \sum(S', G) + d(v_1) \leq m(G)$, implying that $a(G) \geq |S| = |S'| + 1 = a(G') + 1$. By assumption, we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 1 \leq a(G') + 2 \leq a(G) + 1.$$

So, we may suppose that $d(v_3) = 2$. Now we have three consecutive vertices v_1, v_2, v_3 with degrees $d(v_1) = 1$ and $d(v_2) = d(v_3) = 2$. Thus, by Lemma 2.1*(iv)*, there exists a graph G' with $f(G') < f(G)$ which satisfies the statement. $\square$

The following lemmas will be needed to cover two specific cases in the proofs of Theorems 1.3 and 1.4. Therefore, we give the proof for both cases here.

Lemma 2.4. *Let H and $F \cong S(K_{1,\ell})$ be two vertex-disjoint graphs with $n(H) \geq 3$ and $\ell \geq 2$. Assume that w is a vertex of H such that $H - \{w\}$ is connected and u is the central vertex of the subdivided star F. If G is the graph obtained from H and F by identifying w and u, and $\gamma_t(G - V(F)) \leq a(G - V(F)) + 1$, then $\gamma_t(G) \leq a(G) + 1$.*

Proof. Suppose the subgraph F of G is rooted in u. We denote with $v_1, \ldots, v_\ell$ the children of u, and with $w_1, \ldots, w_\ell$ the leaves. By our assumption, $G' = G - V(F) = G - \{u, v_1, \ldots, v_\ell, w_1, \ldots, w_\ell\}$ is a nontrivial connected graph, and $m(G') = m(G) - d_G(u) - \ell$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, v_1, \ldots, v_\ell\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + \ell + 1 = \gamma_t(G') + \ell + 1$. Now, consider an optimal annihilation set S' in G'. Independently of whether the vertices in $N_{G'}(u)$ are inside S' or not, we have $\sum(S', G) \leq \sum(S', G') + d_G(u) - \ell \leq m(G') + d_G(u) - \ell = m(G) - 2\ell$. Let $S = S' \cup \{v_1, w_1, \ldots, w_\ell\}$. Then $\sum(S, G) = \sum(S', G) + 2 + \ell \leq m(G) - 2\ell + \ell + 2 \leq m(G)$, since $\ell \geq 2$. Then, we have $a(G) \geq |S| = |S'| + \ell + 1 = a(G') + \ell + 1$. By assumption, we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + \ell + 1 \leq a(G') + \ell + 2 \leq a(G) + 1. \qquad \square$$

Lemma 2.5. *Let H and P be two vertex-disjoint graphs, where P is the paw graph and H is a nontrivial connected graph. Moreover, let z be a vertex of H and let x be a vertex of P with $d_P(x) = 2$. Assume that G is the graph obtained from H and P by identifying z and x. Then, there exists a connected graph G' with $f(G') < f(G)$ such that $\gamma_t(G') \leq a(G') + 1$ implies $\gamma_t(G) \leq a(G) + 1$.*

Proof. If $H \cong K_2$, then G is a graph of order 5 satisfying $\gamma_t(G) = 2$ and $a(G) = 3$. Thus, (1.1) holds for G. From now on, we assume that $n(H) \geq 3$. We denote the neighbors of x in P with u and w, and let v be the leaf neighbor of u. Two subcases will be considered depending on the degree $d(x)$ of x in G.

First suppose that $d(x) = 3$. Denote the third neighbor of x outside P with y. Since H had at least three vertices, y is not a leaf, and hence $G' = G - V(P) = G - \{x, u, v, w\}$ is not a trivial graph. Also, $m(G') = m(G) - 5$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, w\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 2 = \gamma_t(G') + 2$. If S' is an optimal annihilation set of G', we have $\sum(S', G) \leq \sum(S', G') + 1 \leq m(G') + 1 = m(G) - 4$. Let $S = S' \cup \{u, v\}$. Then $\sum(S, G) = \sum(S', G) + d(u) + d(v) \leq m(G) - 4 + 3 + 1 = m(G)$, and we have $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. Then, $\gamma_t(G') \leq a(G') + 1$ implies

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1. \tag{2.1}$$

Now, suppose $d(x) \geq 4$. In this case let $G' = G - \{u, v, w\}$, and so $m(G') = m(G) - 4$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, w\}$ is a total dominating set of graph G, and hence $\gamma_t(G) \leq |D| = |D'| + 2 = \gamma_t(G') + 2$. Now, let S' be an optimal annihilation set in G'. If $x \notin S'$, then $\sum(S', G) = \sum(S', G')$. In this case, let

$S = S' \cup \{u, v\}$. Then $\sum(S, G) = \sum(S', G) + d(u) + d(v) \leq m(G) - 4 + 3 + 1 = m(G)$, and we have $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. If $\gamma_t(G') \leq a(G') + 1$, the chain (2.1) of inequalities verifies the statement.

But, if $x \in S'$, then $\sum(S', G) = \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 4 + 2 = m(G) - 2$. In this case, let $S = (S' \backslash \{x\}) \cup \{u, v, w\}$. Since $d(x) \geq 4$ we have $\sum(S, G) = \sum(S', G) - d(x) + 3 + 1 + 2 \leq m(G) - 2 - 4 + 6 = m(G)$, implying that $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. By assumption we have that $\gamma_t(G') \leq a(G') + 1$. Therefore, we get again (2.1) which proves the lemma. □

3 Cactus graphs

Recall that a cactus graph is a connected graph such that any two of its cycles are pairwise edge-disjoint. If the cactus graph does not contain any cycles, then it is a tree. Let C^1 and C^2 be two cycles in the cactus graph. We define

$$d\left(C^1, C^2\right) = \min\{d(u, v) \mid u \in V\left(C^1\right), v \in V\left(C^2\right)\},$$

where $d(u, v)$ denotes the distance between vertices u and v. Let $x_1 \in V\left(C^1\right)$ and $x_2 \in V\left(C^2\right)$ be two vertices such that $d(x_1, x_2) = d\left(C^1, C^2\right)$. Then we call x_1 and x_2 *exit-vertices* of cycles C^1 and C^2, respectively. A cycle is said to be an *outer cycle* if it has at most one exit-vertex. If a cactus graph is not a tree, then by the definition of a cactus graph it must contain at least one outer cycle. Note that a cactus graph, which is neither a tree nor a cycle, does not contain exit-vertices if and only if it is unicyclic. In this case, we will take an arbitrary vertex of the unique cycle whose degree is at least 3 for the role of the exit-vertex x. In the right-hand side graph of Figure 2, we have three possibilities for the choice of that vertex x (either x_1 or x_2 or x_3). In both cases, whether a cactus graph has one or more cycles, vertex x will always have degree $d(x) \geq 3$.

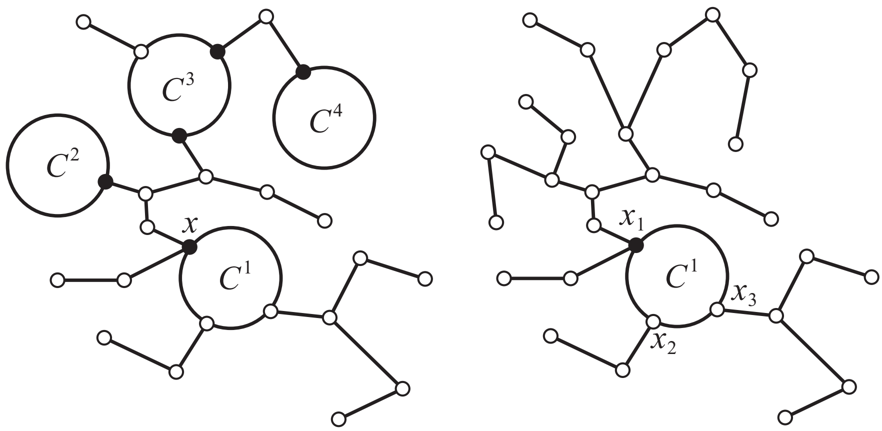

Figure 2: Two examples of cactus graphs. The first one has three outer cycles (C^1, C^2, C^4), its exit-vertices are filled with black. The second cactus graph is unicyclic with one outer cycle, and has no exit-vertices.

In this section we prove Conjecture 1.1 for cactus graphs. Recall the corresponding statement.

Theorem 1.3. *If G is a nontrivial cactus graph, then $\gamma_t(G) \le a(G) + 1$.*

Proof. We proceed by induction on the value of function $f(G) \ge 7$. For $f(G) = 7$ we have $G \cong K_2$, and $\gamma_t(K_2) = 2 = a(K_2) + 1$. For the inductive hypothesis, let $f(G) \ge 8$ and assume that for every nontrivial cactus graph G' with $f(G') < f(G)$ we have $\gamma_t(G') \le a(G') + 1$. If G is a tree, then by Theorem 1.2 the result follows. Also, if G is a cycle, the statement is true. Thus, we may suppose that G contains at least one cycle as a proper subgraph. We denote with C_k, $k \ge 3$, an outer cycle of G.

Through most part of the proof, we will consider cactus graphs G' formed from G by removing a set of vertices in such a way that graph G' will still be a connected cactus graph and consequently $f(G') < f(G)$ will hold. Throughout, S' will denote an optimal annihilation set in G'. We consider two cases.

Case 1: All vertices from $V(C_k)\backslash\{x\}$ have degree 2.

Lemma 2.2*(ii)* and our inductive hypothesis together imply that $\gamma_t(G) \le a(G) + 1$.

Case 2: There exists a vertex from $V(C_k)\backslash\{x\}$ that has degree at least 3.

Since $V(C_k)\backslash\{x\}$ contains some vertices of degree at least 3, and C_k is an outer cycle, there are trees attached to those vertices. Suppose, we root all trees in the vertices $V(C_k)\backslash\{x\}$ to which these trees are attached. Amongst those trees we consider the tree T with the largest height $h(T) = \max\{d(u,v) \mid u = V(C_k) \cap V(T), v \in V(T)\}$. Denote this maximum height with $h \ge 1$ and let u be the vertex of $V(C_k)\backslash\{x\}$ to which tree T is attached. We consider three subcases.

Case 2.1: $h \ge 3$.

Since $h \ge 3$, there exists a leaf $v \in V(T)$ such that $d(u,v) = h \ge 3$. By Lemma 2.3 and our inductive hypothesis, graph G satisfies (1.1).

Case 2.2: $h = 2$.

We only need to consider the four cases shown in Figure 3. All other cases for $h = 2$ can be proved with the help of Lemma 2.1*(ii)* and 2.1*(v)*.

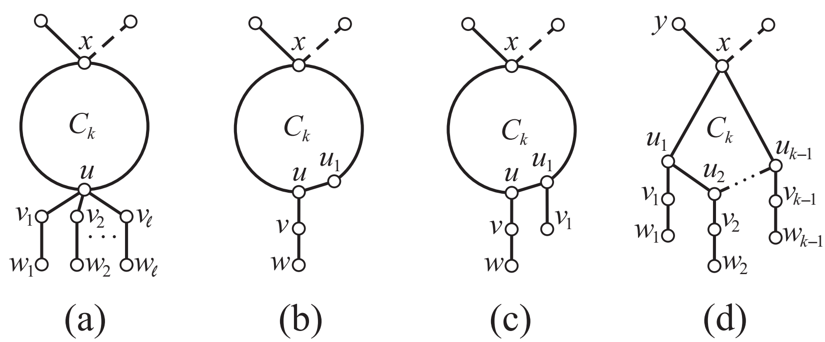

Figure 3: Cases for $h = 2$.

We first start with the case in Figure 3(a). In this case, we have a subdivided star $K_{1,\ell}$, $\ell \ge 2$, attached to the outer cycle, and hence, by Lemma 2.4 and our inductive hypothesis for $G' = G - V(S(K_{1,\ell}))$, graph G satisfies (1.1).

Next, we consider the case in Figure 3(b). Vertex u has only one path of length 2 attached to it, i.e. $d(u) = 3$. We suppose that u has a neighbor u_1 in $V(C_k)\backslash\{x\}$ with degree $d(u_1) = 2$. We denote with v the only child of u, and with w the only child of v. Let $G' = G - \{u, u_1, v, w\}$, and so $m(G') = m(G) - 5$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, v\}$ is a total dominating set of graph G, and hence $\gamma_t(G) \leq |D| = |D'| + 2 = \gamma_t(G') + 2$. Independently of whether the neighbors of u and u_1 in G' are inside S' or not, we have $\sum(S', G) \leq \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 3$. Let $S = S' \cup \{v, w\}$. Then $\sum(S, G) = \sum(S', G) + d(v) + d(w) \leq m(G) - 3 + 2 + 1 = m(G)$, and we have $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1.$$

We proceed with the case in Figure 3(c). Vertex u has again only one path of length 2 attached to it, i.e. $d(u) = 3$. We suppose that u has a neighbor u_1 in $V(C_k)\backslash\{x\}$ with degree $d(u_1) = 3$, and a path of length 1 attached to it. Denote its child with v_1. We also denote with v the only child of u, and with w the only child of v. Let $G' = G - \{u, v, w, u_1, v_1\}$, and so $m(G') = m(G) - 6$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, v, u_1\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 3 = \gamma_t(G') + 3$. Independently of whether the neighbors of u and u_1 in G' are inside S' or not, we have $\sum(S', G) \leq \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 4$. Let $S = S' \cup \{v, w, v_1\}$. Then $\sum(S, G) = \sum(S', G) + d(v) + d(w) + d(v_1) \leq m(G) - 4 + 2 + 1 + 1 = m(G)$, and we have $a(G) \geq |S| = |S'| + 3 = a(G') + 3$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 3 \leq a(G') + 4 \leq a(G) + 1.$$

The last case to consider is the one shown in Figure 3(d). Denote with $u_1, \ldots, u_{k-1}$ all vertices of $V(C)\backslash\{x\}$. Each of those vertices must have one path of length 2 attached to it, i.e. $d(u_i) = 3$ for every $i \in \{1, \ldots, k-1\}$, since otherwise this case would be covered by one of the previous three cases. Clearly, vertices u_1 and u_{k-1} are neighbors of x. Denote for every $i \in \{1, \ldots, k-1\}$ with v_i the only child of u_i, and with w_i the only child of v_i. We consider two subcases.

First, suppose that $d(x) = 3$. Denote the third neighbor of x outside C_k with y. If vertex y was a leaf, then we could exchange vertex x with one of u_i's, and use the proof for the case in Figure 3(c). Hence, we may assume that y is not a leaf and graph $G' = G - \{x, u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1}, w_1, \ldots, w_{k-1}\}$ is not a trivial cactus graph. Also, $m(G') = m(G) - 3k + 1$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1}\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 2k - 2 = \gamma_t(G') + 2k - 2$. Independently of whether y is inside S' or not we have $\sum(S', G) \leq \sum(S', G') + 1 \leq m(G') + 1 = m(G) - 3k + 2$. Let $S = S' \cup \{v_1, \ldots, v_{k-1}, w_1, \ldots, w_{k-1}\}$. Then $\sum(S, G) = \sum(S', G) + 2(k-1) + (k-1) \leq m(G) - 3k + 2 + (3k - 3) = m(G) - 1$, and we have $a(G) \geq |S| = |S'| + 2k - 2 = a(G') + 2k - 2$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2k - 2 \leq a(G') + 2k - 1 \leq a(G) + 1.$$

Now, suppose that $d(x) \geq 4$. Let $G' = G - \{u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1}, w_1, \ldots, w_{k-1}\}$, and so $m(G') = m(G) - 3k + 2$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1}\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 2k - 2 = \gamma_t(G') + 2k - 2$. If $x \notin S'$, then $\sum(S', G) = \sum(S', G')$.

In this case, let $S = S' \cup \{v_1, \ldots, v_{k-1}, w_1, \ldots, w_{k-1}\}$. Then $\sum(S, G) = \sum(S', G) + 2(k-1)+(k-1) \leq m(G)-1$, and we have $a(G) \geq |S| = |S'|+2k-2 = a(G')+2k-2$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2k - 2 \leq a(G') + 2k - 1 \leq a(G) + 1.$$

If $x \in S'$, then $\sum(S', G) = \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 3k + 2 + 2 = m(G) - 3k + 4$. In this case, let $S = (S' \backslash \{x\}) \cup \{u_1, v_1, \ldots, v_{k-1}, w_1, \ldots, w_{k-1}\}$. Since $d(x) \geq 4$ we have $\sum(S, G) = \sum(S', G) - d(x) + d(u_1) + 2(k-1) + (k-1) \leq m(G) - 3k + 4 - 4 + 3 + 3(k-1) = m(G)$, implying that $a(G) \geq |S| = |S'|+2k-2 = a(G')+2k-2$. By our inductive hypothesis, we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2k - 2 \leq a(G') + 2k - 1 \leq a(G) + 1.$$

Case 2.3: $h = 1$.

It suffices to consider only those cases shown in Figure 4. Note that all other cactus graphs with $h = 1$ would involve two leaves at distance of at most 3, and hence these cases can be reduced to the direct application of Lemma 2.1*(ii)* and 2.1*(v)*.

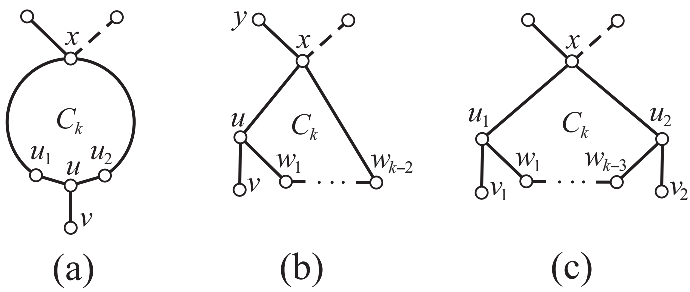

Figure 4: Cases for $h = 1$.

First, consider Figure 4(a). Here, we assume that vertex u has degree $d(u) = 3$, and its neighbors in $V(C_k)\backslash\{x\}$, namely u_1 and u_2, are of degree 2. Denote the child of u with v. In this case we want u_1 and u_2 to be different from the exit-vertex x. Let $G' = G - \{u, v, u_1, u_2\}$, and so $m(G') = m(G) - 5$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, u_i\}$ with $i = 1$ or $i = 2$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'|+2 = \gamma_t(G')+2$. Independently of whether the neighbors of u_1 and u_2 in G' are inside S' or not, we have $\sum(S', G) \leq \sum(S', G')+2 \leq m(G')+2 = m(G)-3$. Let $S = S' \cup \{u_1, v\}$. Then $\sum(S, G) = \sum(S', G) + d(u_1) + d(v) \leq m(G) - 3 + 2 + 1 = m(G)$, and we have $a(G) \geq |S| = |S'|+2 = a(G')+2$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1.$$

We proceed with the case in Figure 4(b). Denote with u the vertex of $V(C_k)\backslash\{x\}$ with one path of length 1 attached to it, i.e. $d(u) = 3$, and let v be its only child. One of the neighbors of u must clearly be vertex x because otherwise we would have the case in Figure 4(a). Suppose that all other vertices in $V(C_k)\backslash\{x\}$, denote them with $w_1, \ldots, w_{k-2}$, have degree 2.

First suppose that $k = 3$. In this case x, u, v and w_1 induce the paw graph. Then, by Lemma 2.5 and our inductive hypothesis, graph G satisfies (1.1).

Suppose that $k \geq 4$. Let $G' = G - \{u, v, w_1, w_2\}$, and so $m(G') = m(G) - 5$. Remark that G' remains a cactus graph. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, w_1\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 2 = \gamma_t(G') + 2$. Independently of whether x and w_3 is inside S' or not we have $\sum(S', G) \leq \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 3$. Let $S = S' \cup \{v, w_1\}$. Then $\sum(S, G) = \sum(S', G) + d(v) + d(w_1) \leq m(G) - 3 + 1 + 2 = m(G)$, and we have $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1.$$

We finish with the case in Figure 4(c). Denote with u_1 and u_2 two vertices in $V(C_k)\backslash\{x\}$ each with one path of length 1 attached to it, i.e. $d(u_1) = d(u_2) = 3$, and let v_1 and v_2 be the only child of u_1 and u_2, respectively. The exit-vertex x must be the neighbor of both u_1 and u_2 because otherwise we would have the case in Figure 4(a). We denote all vertices in $V(C_k)\backslash\{x\}$ between vertex u_1 and u_2 with $w_1, \ldots, w_{k-3}$. Those vertices have all degree 2.

If $k = 3$, the statement follows immediately from the hypothesis and Lemma 2.1*(v)*, since in this case the support vertices of v_1 and v_2 are adjacent. Thus, we first suppose that $k = 4$. Let $G' = G - \{u_1, v_1, u_2, v_2, w_1\}$, and so $m(G') = m(G) - 6$. If D' is a minimum total dominating set of G', then $D = D' \cup \{x, u_1, u_2\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 3 = \gamma_t(G') + 3$. Independently of whether $x \in S'$ or $x \notin S'$, we have $\sum(S', G) \leq \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 4$. Let $S = S' \cup \{v_1, v_2, w_1\}$. Then $\sum(S, G) = \sum(S', G) + d(v_1) + d(v_2) + d(w_1) \leq m(G) - 4 + 1 + 1 + 2 = m(G)$, and we have $a(G) \geq |S| = |S'| + 3 = a(G') + 3$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 3 \leq a(G') + 4 \leq a(G) + 1.$$

Now, suppose that $k = 5$. We make a similar cut than the one for $k = 4$. Let $G' = G - \{u_1, v_1, u_2, v_2, w_1, w_2\}$, and so $m(G') = m(G) - 7$. If D' is a minimum total dominating set of G', then $D = D' \cup \{x, u_1, u_2\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 3 = \gamma_t(G') + 3$. For any optimal annihilation set S' of G', we have $\sum(S', G) \leq \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 5$. Let $S = S' \cup \{v_1, v_2, w_1\}$. Then $\sum(S, G) = \sum(S', G) + d(v_1) + d(v_2) + d(w_1) \leq m(G) - 1$, and $a(G) \geq |S| = |S'| + 3 = a(G') + 3$ follows. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 3 \leq a(G') + 4 \leq a(G) + 1.$$

For the last case, let $k \geq 6$. We have three consecutive vertices w_1, w_2, w_3 with degree $d(w_1) = d(w_2) = d(w_3) = 2$. Furthermore, vertices u_1 and w_4 (or u_1 and u_2, if $k = 6$) are not adjacent. Thus, by Lemma 2.1*(iii)* and our inductive hypothesis, graph G satisfies (1.1).

These cover all possible cases which can occur in a cactus graph which is neither a tree nor a cycle. Hence, Conjecture 1.1 is true for the family of cactus graphs. □

4 Block graphs

Recall that a block graph is a connected graph in which every 2-connected component (block) is a clique. Block graphs have minimum degree at least 3 if its building blocks are complete graphs K_k, $k \geq 4$. Thus, Conjecture 1.1 obviously holds for them. On the other hand, block graphs also contain blocks K_2 and K_3, and therefore, it clearly makes sense to study Conjecture 1.1 on block graphs.

We proceed with a similar definition than the one for cactus graphs. If all cliques in a block graph are K_2, then it is a tree. For every $k \geq 3$ we will call complete graph K_k a *complex clique*. Let K^1 and K^2 be two complex cliques in the block graph. We define

$$d\left(K^1, K^2\right) = \min\{d(u,v) \mid u \in V\left(K^1\right), v \in V\left(K^2\right)\},$$

where $d(u,v)$ denotes the distance between vertices u and v. Let $x_1 \in V\left(K^1\right)$ and $x_2 \in V\left(K^2\right)$ be two vertices such that $d(x_1,x_2) = d(K^1,K^2)$. Then we call x_1 and x_2 exit-vertices of complex cliques K^1 and K^2, respectively. Notice that a complex clique might not have any exit-vertices if it is the only complex clique in the block graph. A complex clique will be called an *outer complex clique* if it has at most one exit-vertex. If a block graph is not a tree, then by the definition of a block graph it must contain at least one outer complex clique.

Now, we are ready to present a proof of Theorem 1.4. Recall its statement.

Theorem 1.4. *If G is a nontrivial block graph, then $\gamma_t(G) \leq a(G) + 1$.*

Proof. We proceed by induction on the value of function $f(G)$. For $f(G) = 7$ we have $G \cong K_2$, and $\gamma_t(K_2) = 2 = a(K_2) + 1$. For the inductive hypothesis, let $f(G) \geq 8$ and assume that for every nontrivial block graph G' with $f(G') < f(G)$ we have $\gamma_t(G') \leq a(G') + 1$. If G does not contain complex cliques, then it is a tree, and by Theorem 1.2 the result follows. Also, if G is a complete graph, i.e. $G \cong K_\ell$, $\ell \geq 2$, we have $\gamma_t(K_\ell) = 2 \leq a(K_\ell) + 1$. Thus, we may suppose that G is neither a tree nor a complete graph, but contains at least one complex clique as a proper subgraph. We denote with K_k an outer complex clique of G. Similarly as in the proof for cactus graphs, all outer complex cliques in the figures will be drawn with an exit-vertex x even though a unique complex clique in a block graph does not have one. In the latter case, we denote with x an arbitrary vertex of clique K_k whose degree is at least k. In both cases, whether a block graph has one or more complex cliques, vertex x will have degree $d(x) \geq k$.

Through most part of the proof, we will consider block graphs G' formed from G by removing a set of vertices in such a way that graph G' will still be a connected block graph and consequently $f(G') < f(G)$ will hold. Throughout, S' will denote an optimal annihilation set in G'. We consider two cases.

Case 1: All vertices from $V(K_k)\backslash\{x\}$ have degree $k-1$.

Let $u_1, \ldots, u_{k-1}$ be vertices from $V(K_k)\backslash\{x\}$ with degree $k-1$. By Lemma 2.2*(i)*, and inductive hypothesis for $G' = G - \{u_1, \ldots, u_{k-1}\}$, graph G satisfies (1.1).

Case 2: There exists a vertex from $V(K_k)\backslash\{x\}$ that has degree at least k.

Since $V(K_k)\backslash\{x\}$ contains vertices of degree at least k, and K_k is an outer complex clique, there are trees attached to those vertices. Suppose, we root all trees in the vertices $V(K_k)\backslash\{x\}$ to which these trees are attached. Amongst those trees we consider the tree

T with the largest height h. Let u be the vertex of $V(K_k)\backslash\{x\}$ to which this tree T is attached. We split the problem into three subcases.

Case 2.1: $h \geq 3$.

Since $h \geq 3$, there exists a leaf $v \in V(T)$ such that $d(u,v) = d \geq 3$. By Lemma 2.3 and our inductive hypothesis, graph G satisfies (1.1).

Case 2.2: $h = 2$.

We only need to consider cases shown in Figure 5. All other cases for $h = 2$ can be proved directly with the help of Lemma 2.1*(ii)* and 2.1*(v)*.

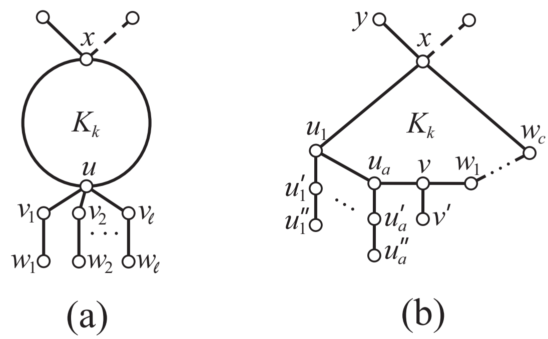

Figure 5: Cases for $h = 2$.

We start with the case in Figure 5(a) and suppose that there exists a subdivided star $S(K_{1,\ell})$, $\ell \geq 2$, attached to the outer complex clique. By Lemma 2.4 and our inductive hypothesis for $G' = G - V(S(K_{1,\ell}))$, graph G satisfies (1.1).

In the case shown in Figure 5(b), there are vertices in $V(K_k)\backslash\{x\}$ such that a path of length 2 is attached to them. We denote such vertices with $u_1, \ldots, u_a$. Since $h = 2$, we must have at least one such vertex. Thus, $a \in \{1, \ldots, k-1\}$. For each $i \in \{1, \ldots, a\}$ we denote with u'_i the child of u_i, and with u''_i the child of u'_i. Also, we may suppose that at most one vertex in $V(K_k)\backslash\{x\}$ has a path of length 1 attached to it. If we had more such vertices, then we would have two adjacent support vertices and we could prove the statement by referring to Lemma 2.1*(v)*. Hence, denote this vertex with v and let b denote the Boolean value whether it exists in $V(K_k)\backslash\{x\}$ or not, i.e. $b \in \{0,1\}$. We denote the child of v with v'. There may also be some vertices in $V(K_k)\backslash\{x\}$ without a path attached to them. Denote them with $w_1, \ldots, w_c$, $c \in \{0, \ldots, k-2\}$. Clearly, we have $a + b + c = k - 1$. Let $G' = G - \{u_1, \ldots u_a, u'_1, \ldots u'_a, u''_1, \ldots, u''_a, v, v', w_1, \ldots, w_c\}$, and so $m(G') = m(G) - (\frac{k(k-1)}{2} + 2a + b)$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u_1, \ldots, u_a, u'_1, \ldots, u'_a, v\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 2a + b = \gamma_t(G') + 2a + b$. Independently of whether x is inside S' or not we have

$$\begin{aligned}\sum(S', G) &\leq \sum(S', G') + (k-1) \\ &= m(G') + (k-1) = m(G) - \left(\frac{k(k-1)}{2} + 2a + b\right) + k - 1.\end{aligned}$$

Let $S = S' \cup \{u'_1, \ldots, u'_a, u''_1, \ldots u''_a, v'\}$ and observe that

$$\begin{aligned}\sum(S,G) &= \sum(S',G) + d(u'_1) + \cdots + d(u'_a) + d(u''_1) + \cdots + d(u''_a) + d(v') \\ &\leq m(G) - \left(\frac{k(k-1)}{2} + 2a + b\right) + k - 1 + 3a + b.\end{aligned}$$

First, suppose that $1 \leq a \leq k-2$ holds. Then,

$$\begin{aligned}m(G) - \left(\frac{k(k-1)}{2} + 2a + b\right) + k - 1 + 3a + b &= m(G) - \frac{1}{2}k^2 + \frac{3}{2}k + a - 1 \\ &\leq m(G) - \frac{1}{2}k^2 + \frac{5}{2}k - 3 = m(G) - \frac{1}{2}(k-2)(k-3) \leq m(G).\end{aligned}$$

Similarly, under the conditions $a = k - 1 \geq 3$, the following relations hold:

$$m(G) - \left(\frac{k(k-1)}{2} + 2a + b\right) + k - 1 + 3a + b \leq m(G) - \frac{1}{2}k^2 + \frac{5}{2}k - 2 \leq m(G).$$

In both cases we get $\sum(S,G) \leq m(G)$, which implies $a(G) \geq |S| = |S'| + 2a + b = a(G') + 2a + b$. By our inductive hypothesis, G' satisfies Conjecture 1.1. Consequently,

$$\gamma_t(G) \leq \gamma_t(G') + 2a + b \leq a(G') + 2a + b + 1 \leq a(G) + 1.$$

What remains is to establish the statement for $k = 3$ and $a = k - 1 = 2$. We consider two subcases. First, suppose that $d(x) = 3$. Denote the third neighbor of x outside K_3 with y. If vertex y was a leaf, then we could exchange vertex x either with u_1 or u_2, and apply the proof for the case $a = 1 = k - 2$. Hence, we may assume that y is not a leaf, and therefore, graph $G' = G - \{x, u_1, u'_1, u''_1, u_2, u'_2, u''_2\}$ is not a trivial block graph. Also, $m(G') = m(G) - 8$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u_1, u'_1, u_2, u'_2\}$ is a total dominating set of graph G, and hence $\gamma_t(G) \leq |D| = |D'| + 4 = \gamma_t(G') + 4$. Independently of whether y is inside S' or not we have $\sum(S', G) \leq \sum(S', G') + 1 \leq m(G') + 1 = m(G) - 8 + 1 = m(G) - 7$. Let $S = S' \cup \{u'_1, u''_1, u'_2, u''_2\}$. Then $\sum(S,G) = \sum(S',G) + d(u'_1) + d(u''_\ell) + d(u'_2) + d(u''_2) \leq m(G) - 7 + 2 + 1 + 2 + 1 = m(G) - 1$, and we have $a(G) \geq |S| = |S'| + 4 = a(G') + 4$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 4 \leq a(G') + 5 \leq a(G) + 1.$$

Now, suppose that $d(x) \geq 4$. Let $G' = G - \{u_1, u'_1, u''_1, u_2, u'_2, u''_2\}$, and so $m(G') = m(G) - 7$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u_1, u'_1, u_2, u'_2\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 4 = \gamma_t(G) + 4$. If $x \notin S'$, then $\sum(S', G) = \sum(S', G')$. In this case, let $S = S' \cup \{u'_1, u''_1, u'_2, u''_2\}$. Then $\sum(S,G) = \sum(S',G) + d(u'_1) + d(u''_1) + d(u'_2) + d(u''_2) \leq m(G) - 7 + 2 + 1 + 2 + 1 = m(G) - 1$, and we have $a(G) \geq |S| = |S'| + 4 = a(G') + 4$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 4 \leq a(G') + 5 \leq a(G) + 1.$$

If $x \in S'$, then $\sum(S', G) = \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 7 + 2 = m(G) - 5$. In this case, let $S = (S' \backslash \{x\}) \cup \{u_1, u'_1, u''_1, u'_2, u''_2\}$. Since $d(x) \geq 4$ we have $\sum(S,G) =$

$\sum(S', G) - d(x) + d(u_1) + d(u_1') + d(u_1'') + d(u_2') + d(u_2'') \leq m(G) - 5 - 4 + 3 + 2 + 1 + 2 + 1 = m(G)$, implying that $a(G) \geq |S| = |S'| + 4 = a(G') + 4$. Applying again our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 4 \leq a(G') + 5 \leq a(G) + 1.$$

Case 2.3: $h = 1$.

We need to consider only one case which is shown in Figure 6. As we have already seen in Case 2.2, all other cases for $h = 1$ can be proved with the help of Lemma 2.1*(ii)* and 2.1*(v)*.

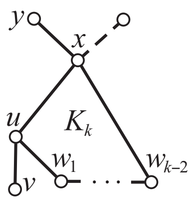

Figure 6: The case for $h = 1$.

We may also suppose that there is at most one vertex in $V(K_k)\backslash\{x\}$ which has a path of length 1 attached to it. If we had more such vertices, then we would have adjacent support vertices and we could prove this case with Lemma 2.1*(v)*. Denote this vertex with u and its child with v. There are also vertices in $V(K_k)\backslash\{x\}$ without a path attached to them. Denote them with $w_1, \ldots, w_{k-2}$. Let $G' = G - \{u, v, w_1, \ldots, w_{k-2}\}$, and so $m(G') = m(G) - (\frac{k(k-1)}{2} + 1)$. If D' is a minimum total dominating set of G', then $D = D' \cup \{x, u\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 2 = \gamma_t(G') + 2$. Independently of whether x is inside S' or not we have $\sum(S', G) \leq \sum(S', G') + k - 1 \leq m(G') + k - 1 = m(G) - (\frac{k(k-1)}{2} + 1) + k - 1$. Let $S = S' \cup \{v, w_1\}$. Then,

$$\begin{aligned}\sum(S, G) &= \sum(S', G) + d(v) + d(w_1) \\ &\leq m(G) - \left(\frac{k(k-1)}{2} + 1\right) + k - 1 + 1 + k - 1.\end{aligned}$$

For $k \geq 4$, this gives

$$\sum(S, G) \leq m(G) - \frac{1}{2}k^2 + \frac{5}{2}k - 2 \leq m(G).$$

Hence, $a(G) \geq |S| = |S'| + 2 = a(G') + 2$ and applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1.$$

We end the proof with $k = 3$. In this case, x, u, v and w_1 induce the paw graph and, by Lemma 2.5 and our inductive hypothesis, graph G satisfies (1.1).

We have considered all possible cases which can occur in a block graph which is neither a tree nor a complete graph. Hence, Conjecture 1.1 is true over the family of block graphs. □

5 Concluding remarks

To show that our main results, namely Theorem 1.3 and 1.4, are sharp, we remark that trees are included in both classes. Therefore, we may refer to the family of trees characterized in [5] which satisfy Conjecture 1.1 with equality.

We may also observe that even cycles C_n, where $n \equiv 2 \pmod 4$, have $\gamma_t(C_n) = \frac{n}{2}+1$ and $a(C_n) = \frac{n}{2}$. Also, there are other cactus graphs which are neither trees nor cycles, but satisfy $\gamma_t(G) = a(G) + 1$. Take two vertex-disjoint cycles C_6, and connect any vertex from the first cycle and any vertex from the second cycle with a path of length 3. We get a cactus graph G on $n = 14$ vertices and $m = 15$ edges. It is easy to see that $\gamma_t(G) = 8$ and $a(G) = 7$. Thus, Theorem 1.3 holds with equality for the graph G constructed this way. One can also use other cycles C_n with $n \equiv 2 \pmod 4$ and connect them with different paths to obtain other extremal examples. Hence, the following characterization problem remains open.

Problem 5.1. Characterize cactus graphs G which satisfy $\gamma_t(G) = a(G) + 1$.

For block graphs, already the following question might be interesting.

Problem 5.2. Does there exist a block graph G which is neither a tree nor a K_3 but satisfies $\gamma_t(G) = a(G) + 1$?

References

[1] D. Archdeacon, J. Ellis-Monaghan, D. Fisher, D. Froncek, P. C. B. Lam, S. Seager, B. Wei and R. Yuster, Some remarks on domination, *J. Graph Theory* **46** (2004), 207–210, doi:10.1002/jgt.20000.

[2] B. Brimkov and I. V. Hicks, Memory efficient algorithms for cactus graphs and block graphs, *Discrete Appl. Math.* **216** (2017), 393–407, doi:10.1016/j.dam.2015.10.032.

[3] V. Chvátal and C. McDiarmid, Small transversals in hypergraphs, *Combinatorica* **12** (1992), 19–26, doi:10.1007/bf01191201.

[4] E. DeLaViña, Written on the Wall II, `http://cms.dt.uh.edu/faculty/delavinae/research/wowII/`.

[5] W. J. Desormeaux, T. W. Haynes and M. A. Henning, Relating the annihilation number and the total domination number of a tree, *Discrete Appl. Math.* **161** (2013), 349–354, doi:10.1016/j.dam.2012.09.006.

[6] W. J. Desormeaux, M. A. Henning, D. F. Rall and A. Yeo, Relating the annihilation number and the 2-domination number of a tree, *Discrete Math.* **319** (2014), 15–23, doi:10.1016/j.disc.2013.11.020.

[7] J. R. Griggs and D. J. Kleitman, Independence and the Havel-Hakimi residue, *Discrete Math.* **127** (1994), 209–212, doi:10.1016/0012-365x(92)00479-b.

[8] M. A. Henning, A survey of selected recent results on total domination in graphs, *Discrete Math.* **309** (2009), 32–63, doi:10.1016/j.disc.2007.12.044.

[9] M. A. Henning and A. Yeo, *Total Domination in Graphs*, Springer Monographs in Mathematics, Springer, New York, 2013, doi:10.1007/978-1-4614-6525-6.

[10] J. Lyle and S. Patterson, A note on the annihilation number and 2-domination number of a tree, *J. Comb. Optim.* **33** (2017), 968–976, doi:10.1007/s10878-016-0019-7.

[11] R. Pepper, On the annihilation number of a graph, in: V. Zafiris, M. Benavides, K. Gao, S. Hashemi, K. Jegdic, G. A. Kouzaev, P. Simeonov, L. Vladareanu and C. Vobach (eds.), *AMATH'09 Proceedings of the 15th American Conference on Applied Mathematics*, World Scientific and Engineering Academy and Society (WSEAS), Stevens Point, Wisconsin, USA, 2009 pp. 217–220, held in Houston, USA, April 30 – May 02, 2009.

[12] R. D. Pepper, *Binding Independence*, Ph.D. thesis, University of Houston, Houston, Texas, 2004, https://search.proquest.com/docview/305196562.

[13] S. Thomassé and A. Yeo, Total domination of graphs and small transversals of hypergraphs, *Combinatorica* **27** (2007), 473–487, doi:10.1007/s00493-007-2020-3.

[14] Zs. Tuza, Covering all cliques of a graph, *Discrete Math.* **86** (1990), 117–126, doi:10.1016/0012-365x(90)90354-k.

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The validity of Tutte's 3-flow conjecture for some Cayley graphs*

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Abstract

Tutte's 3-flow conjecture claims that every bridgeless graph with no 3-edge-cut admits a nowhere-zero 3-flow. In this paper we verify the validity of Tutte's 3-flow conjecture on Cayley graphs of certain classes of finite groups. In particular, we show that every Cayley graph of valency at least 4 on a generalized dicyclic group has a nowhere-zero 3-flow. We also show that if G is a solvable group with a cyclic Sylow 2-subgroup and the connection sequence S with $|S| \geq 4$ contains a central generator element, then the corresponding Cayley graph $\mathrm{Cay}(G, S)$ admits a nowhere-zero 3-flow.

Keywords: Nowhere-zero flow, Cayley graph, Tutte's 3*-flow conjecture, connection sequence, solvable group, nilpotent group.*

Math. Subj. Class.: 05C25, 05C21, 20D10

1 Introduction

Let D be an orientation of a graph Γ and let k be a positive integer. A *k-flow* on a graph Γ is a pair (D, f) where f is an integer valued function

$$f\colon E(\Gamma) \to \mathbb{Z}$$

such that $|f(e)| < k$ for every $e \in E(\Gamma)$, and for every $v \in V(\Gamma)$,

$$\sum_{e \in E(v)^+} f(e) = \sum_{e \in E(v)^-} f(e),$$

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where $E(v)^+$ and $E(v)^-$ are the all edges with tails at v and heads at v, respectively. *A nowhere-zero k-flow* (abbreviated a k-NZF) is a pair (D, f) such that for every $e \in E(\Gamma)$, $f(e) \neq 0$.

The following conjecture is due to Tutte and is known as Tutte's 3-flow conjecture:

Conjecture 1.1 (Tutte's 3-flow conjecture [8, 9])**.** *Every bridgeless graph with no* 3*-edge-cut has a* 3*-NZF.*

Although Tutte's 3-flow conjecture has been studied by many authors, it is still widely open.

Let G be a finite group with identity 1 and $S = (s_1, s_2, \ldots, s_n)$ be a sequence of elements of $G \setminus \{1\}$ such that the mapping $s_i \to s_i^{-1}$ permutes the entries of S. We call S a *connection sequence* (note that all entries of S are distinct unless stated otherwise). A *Cayley graph*, denoted by $\mathrm{Cay}(G, S)$, is a graph whose vertex set is G with adjacency defined by

$$g \sim h \quad \text{if and only if} \quad g^{-1}h \in S,$$

for every $g, h \in G$. We see at once that if S generates G, then $\mathrm{Cay}(G, S)$ is connected.

Alspach et al. [1] conjectured that every Cayley graph of valency at least 3 has a nowhere-zero 4-flow. They also showed their conjecture to be true for solvable groups. Their result was significantly strengthened and extended by Nedela and Škoviera to a much wider class of groups [5].

By combining the fact that a k-valent Cayley graph is k-edge-connected graph with the fact that every 4-edge-connected graph has a 4-NZF [2], we deduce that every Cayley graph of valency at least 4 has a 4-NZF. Thus the question about the existence of a nowhere-zero 4-flow is interesting only for cubic Cayley graphs. Since 4-regular graphs admit a nowhere-zero 2-flow, the important question about flows on Cayley graphs of valency greater than 3 is whether every Cayley graph of valency at least 5 has a nowhere-zero 3-flow. In other words, it is interesting to verify whether Tutte's 3-flow conjecture holds on such Cayley graphs.

In [6], it has been proved that every abelian Cayley graph of valency k, where $k \geq 4$, admits a 3-NZF. Nánásiová and Škoviera [4] improved the above result to Cayley graphs on a group G whose Sylow 2-subgroup is the direct factor of G, and as a consequence, they showed that every Cayley graph of valency at least 4 on a nilpotent group has a 3-NZF. Recently, Yang and Li [11] showed the same fact for a Cayley graph on a dihedral group, and L. Li and X. Li [3] verified Tutte's 3-flow conjecture for Cayley graphs on generalized dihedral groups and generalized quaternion groups.

In this paper, we investigate Tutte's 3-flow conjecture for Cayley graphs on a solvable group with a suitable normal subgroup (Theorems 3.1 and 3.2 and Remark 3.5) and as a consequence of these theorems, we show that every Cayley graph of valency at least 4 on a generalized dicyclic group satisfies Tutte's 3-flow conjecture. By using Theorem 3.6 we can obtain the results of [3] and [11] by a different method.

In [4], the authors showed that a Cayley graph of valency at least 4 with the connection sequence containing a central involution admits a 3-NZF. In Theorem 3.6, we extend this result to the case when Sylow 2-subgroups of G are cyclic and the connection sequence of G contains a central generator element. As a consequence of this theorem, we show that if a Cayley graph of valency at least 4 on a solvable group G, with a cyclic Sylow 2-subgroup, admits a 3-NZF, then every Cayley graph of valency at least 4 on the direct product of G and a nilpotent group admits a 3-NZF.

2 Notation and preliminaries

The terminology and notation used in this paper are standard both in group theory and graph theory, see for instance [7, 10].

An element g of G is called an *involution* if g has order 2. Let $Z(G)$ be the center of a group G. We say that an element x of G is *central* if $x \in Z(G)$. The group generated by a sequence S is denoted by $\langle S \rangle$ and the element $x \in G$ is named a *generator element* of G in S if $\langle S \setminus \{x\} \rangle \neq \langle S \rangle$. For integers $m, n \geq 2$, a cycle of length n and a path of length $m-1$ are denoted by C_n and P_m, respectively. For an integer $m \geq 3$ and for $n \in \mathbb{Z}_m$, the Cayley graph $\mathrm{Cay}(\mathbb{Z}_m, \{-1, 1, -n, n\})$ will be denoted by $C(m, n)$. Let N be a subgroup of G and x belongs to a left transversal set of N in G. The image of $\mathrm{Cay}(N, S)$ under left translation by x is denoted by $x\,\mathrm{Cay}(N, S)$. The *Cartesian product* $H_1 \,\square\, H_2$ of graphs H_1 and H_2 is a graph such that $V(H_1) \times V(H_2)$ is its vertex set and any two vertices (u, u') and (v, v') are adjacent in $H_1 \,\square\, H_2$ if and only if either $u = v$ and $u'v' \in E(H_2)$ or $u' = v'$ and $uv \in E(H_1)$. Set $L = P_n \,\square\, K_2$, where $V(P_n) = \{1, 2, \ldots, n\}$ and $V(K_2) = \{1, 2\}$. The *Möbius ladder* ML_n is a graph obtained by adding the edges $(12)(n1)$ and $(11)(n2)$ to L. Also, by adding the edges $(11)(n1)$ and $(12)(n2)$ to L, we obtain a graph is called the *circular ladder* CL_n. In fact $CL_n \cong C_n \,\square\, K_2$. Any graph isomorphic to either CL_n or ML_n for some n will be referred to as a *closed ladder*. It is easy to check that the circular ladder is bipartite if and only if n is even while the Möbius ladder is bipartite if and only if n is odd.

Lemma 2.1 ([4, Theorems 3.3 and 4.3]). *Let* $\mathrm{Cay}(G, S)$ *be a Cayley graph of valency* k*, where* $k \geq 4$*. If* S *contains a central involution, then* $\mathrm{Cay}(G, S)$ *has a* 3*-NZF. In particular, if* G *is nilpotent, then* $\mathrm{Cay}(G, S)$ *has a* 3*-NZF.*

Lemma 2.2 ([4, Proposition 4.1]). *Let* G *be a group,* H *be a normal subgroup of* G *and let* S *be a connection sequence with no intersection with* H*. If* $\mathrm{Cay}(G/H, S/H)$ *has a* 3*-NZF, then so does* $\mathrm{Cay}(G, S)$*.*

Note that in Lemma 2.2, according to the paragraph before Proposition 4.1 in [4], for distinct elements $s, t \in S$, we regard sH and tH as distinct elements of S/H. So, $\mathrm{Cay}(G/H, S/H)$ may have parallel edges even when $\mathrm{Cay}(G, S)$ is simple and $|S/H| = |S|$.

Lemma 2.3 ([6, Theorem 1.1]). *Every abelian Cayley graph of valency* k*, where* $k \geq 4$*, admits a* 3*-NZF.*

Lemma 2.4 ([6, Proposition 2.5]). *Let* $m, n \geq 3$ *be integers. Then the graph* $C_n \,\square\, C_m \,\square\, K_2$ *admits a* 3*-NZF.*

Lemma 2.5 ([6, Proposition 2.6]). *Let* $m, n \geq 3$ *be two integers such that* $m > n \geq 1$ *and* $m \geq 3$*. Then the graph* $C(m, n) \,\square\, K_2$ *admits a* 3*-NZF.*

Lemma 2.6 ([6, Corollary 2.2]). *A regular bipartite graph of valency at least* 2 *admits a* 3*-NZF.*

Lemma 2.7 ([10, page 308]). *A cubic graph has a* 3*-NZF if and only if it is bipartite.*

Lemma 2.8. *Let* G *be a group and* N *be a subgroup of* G *of index* 2*. Then* $\mathrm{Cay}(G, S \setminus (S \cap N))$ *is bipartite.*

Proof. Since the index of N in G is 2, there exists $d \in G \setminus N$ such that $G = N \cup dN$. So, we can consider the vertices of $\mathrm{Cay}(G, S)$ as two partitions N and dN. Since for every $m, n \in N$, m and n are adjacent, and dm and dn are adjacent if and only if $m^{-1}n \in S \cap N$, we obtain that $\mathrm{Cay}(G, S \setminus S \cap N)$ is a bipartite graph with partite sets N and dN. □

Lemma 2.9 ([10, page 308]). *A graph has a 2-NZF if and only if it is an even graph.*

Remark 2.10. According to the above lemma, for discussion about a nowhere-zero 3-flow in a Cayley graph with a connection sequence S, it is enough to investigate the case when $|S|$ is odd.

Remark 2.11. Let G be a group and N be a subgroup of G. Let $T = \{x_1, \dots, x_t\}$, where $t \in \mathbb{N}$, be a left transversal set of N in G. If S is a connection sequence of N such that $\mathrm{Cay}(N, S)$ is connected, then

$$\{x_i \,\mathrm{Cay}(N, S) : 1 \leq i \leq t\}$$

is the set of connected components of $\mathrm{Cay}(G, S)$. For every x_i where $i \in \{1, \dots, t\}$, $\mathrm{Cay}(N, S)$ and $x_i \,\mathrm{Cay}(N, S)$ are isomorphic, because for every $m, n \in N$,

$$\begin{aligned} x_i m \sim x_i n \quad (\text{in } x_i \,\mathrm{Cay}(N, S \cap N)) \qquad &\text{if and only if} \\ (x_i m)^{-1}(x_i n) \in S \cap N \qquad \text{if and only if} \qquad & m^{-1}n \in S \cap N \\ &\text{if and only if} \qquad m \sim n \quad (\text{in } \mathrm{Cay}(N, S \cap N)). \end{aligned}$$

Thus if $\mathrm{Cay}(N, S)$ has a 3-NZF, then $\mathrm{Cay}(G, S)$ has a 3-NZF. Hence for finding a 3-NZF in $\mathrm{Cay}(G, S)$, we reduce to find a 3-NZF in $\mathrm{Cay}(N, S)$.

3 Main results

In this section we show the validity of Tutte's 3-flow conjecture for a solvable group with a suitable normal subgroup. As examples, we show the same result for Cayley graphs on generalized dicyclic groups, generalized dihedral groups and quaternion groups. We also prove that every Cayley graph $\mathrm{Cay}(G, S)$ on a solvable group G with a cyclic Sylow 2-subgroup such that the connection sequence S contains a central generator element, admits a 3-NZF.

Theorem 3.1. *Let G be a solvable group, N be a subgroup of G of index 2 and let S be a connection sequence of G such that $|S| \geq 5$ is odd and $S \cap Z(N) \neq \emptyset$. If*

(1) $\mathrm{Cay}(N, S \cap N)$ admits a 3-NZF and

(2) for every $d \in S \setminus N$, $\quad d^{-1}(S \cap N)d = S \cap N$,

then $\mathrm{Cay}(G, S)$ has a 3-NZF.

Proof. Without loss of generality, we can assume that there exists an element $d \in S \setminus N$, because otherwise $S \subset N$ and by Condition (1), we could conclude that $\mathrm{Cay}(G, S)$ has a 3-NZF. Thus, there is $d \in S \setminus N$. Note that $|S|$ is odd.

We continue the proof in the following two cases:

Case 1. If $|S \cap N|$ is odd, then since $|S \setminus (S \cap N)| = |S| \setminus |S \cap N|$ is even, Lemma 2.9 shows that $\mathrm{Cay}(G, S \setminus (S \cap N))$ admits a 3-NZF. Also by Condition (1), $\mathrm{Cay}(N, S \cap N)$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S) = \mathrm{Cay}(G, S \setminus (S \cap N)) \cup \mathrm{Cay}(G, S \cap N)$.

Case 2. If $|S \cap N|$ is even, then the proof will be divided into two subcases:

Subcase 1. Assume that $|S\backslash(S\cap N)| \geq 2$. By Lemma 2.8, $\mathrm{Cay}(G, S\backslash(S\cap N))$ is bipartite. So Lemma 2.6 shows that $\mathrm{Cay}(G, S \setminus (S \cap N))$ admits a 3-NZF. Since $\mathrm{Cay}(G, S \cap N)$ admits a 3-NZF, we deduce that $\mathrm{Cay}(G, S)$ has a 3-NZF.

Subcase 2. Assume that $|S \setminus (S \cap N)| = 1$. Thus $\{S \setminus (S \cap N)\} = \{d\}$, so $O(d) = 2$ and it is not hard to check that G is the semidirect product of N and $\langle d\rangle$. We want to show that $\mathrm{Cay}(N, S \cap N) \,\Box\, \mathrm{Cay}(\langle d\rangle, \{d\}) \cong \mathrm{Cay}(G, S)$. For this purpose, we define $\phi\colon\ \mathrm{Cay}(N, S \cap N) \,\Box\, \mathrm{Cay}(\langle d\rangle, \{d\}) \to \mathrm{Cay}(G, S)$ such that $\phi(m, x) = mx$ for every $m \in N$ and $x \in \langle d\rangle$. Since G is the semidirect product of N and $\langle d\rangle$, it is obvious that ϕ is a bijective function. Now we will show that ϕ is homomorphism. For every $m, n \in N$ and $x, y \in \langle d\rangle$, we have:

$$(m, x) \sim (n, y) \quad (\text{in } \mathrm{Cay}(N, S \cap N) \,\Box\, \mathrm{Cay}(\langle d\rangle, \{d\})) \quad \text{if and only if} \quad m = n,\ x \sim y \quad \text{or} \quad n \sim m,\ x = y.$$

We should check the following cases:

(1) If $m = n$, $x = 1$ and $y = d$, then $(\phi(m, x))^{-1}\phi(n, y) = m^{-1}nd = d \in S$. Thus $\phi(m, x) \sim \phi(n, y)$ in $\mathrm{Cay}(G, S)$.

(2) If $m = n$, $x = d$ and $y = 1$, then $(\phi(m, x))^{-1}\phi(n, y) = d^{-1}m^{-1}n = d \in S$. Thus $\phi(m, x) \sim \phi(n, y)$ in $\mathrm{Cay}(G, S)$.

(3) If $m \sim n$ and $x = y = 1$, then $m^{-1}n \in S \cap N$. Thus $(\phi(m, x))^{-1}\phi(n, y) = (mx)^{-1}(ny) = m^{-1}n \in N \cap S$. So $\phi(m, x) \sim \phi(n, y)$ in $\mathrm{Cay}(G, S)$.

(4) If $m \sim n$ and $x = y = d$, then $m^{-1}n \in S \cap N$. Thus $(\phi(m, x))^{-1}\phi(n, y) = d^{-1}(m^{-1}n)d \in d^{-1}(S \cap N)d = N \cap S \subset S$. So $\phi(m, x) \sim \phi(n, y)$ in $\mathrm{Cay}(G, S)$.

Now, let $t_1 \sim t_2$ in $\mathrm{Cay}(G, S)$. Since G is the semidirect product of N and $\langle d\rangle$, there exist $m, n \in N$ and $x, y \in \langle d\rangle$ such that $t_1 = mx$ and $t_2 = ny$. We continue the proof in the following cases:

(i) If $x = 1$ and $y = d$, then $m^{-1}nd = t_1^{-1}t_2 \in S\backslash(S\cap N) = \{d\}$. Therefore, $m^{-1}n = 1$ and so $m = n$. From this, we have $\phi^{-1}(t_1) = (m, x) \sim (n, y) = \phi^{-1}(t_2)$.

(ii) If $x = d$ and $y = 1$, the above reason shows that $\phi^{-1}(t_1) = (m, x) \sim (n, y) = \phi^{-1}(t_2)$.

(iii) If $x = y = 1$, then $m^{-1}n = t_1^{-1}t_2 \in S \cap N$. Therefore $m \sim n$ in $\mathrm{Cay}(N, S \cap N)$ and hence $\phi^{-1}(t_1) = (m, x) \sim (n, y) = \phi^{-1}(t_2)$.

(iv) If $x = y = d$, then $d^{-1}m^{-1}nd = t_1^{-1}t_2 \in d(S \cap N)d^{-1} = S \cap N$. Therefore $m^{-1}n \in d(S \cap N)d^{-1} = S \cap N$ and hence, $\phi^{-1}(t_1) = (m, x) \sim (n, y) = \phi^{-1}(t_2)$.

These show that $\mathrm{Cay}(N, S\cap N) \,\Box\, \mathrm{Cay}(\langle d\rangle, \{d\}) \cong \mathrm{Cay}(G, S)$. Now, suppose that the theorem is false, and let G be the smallest group satisfying the hypothesis and $\mathrm{Cay}(G, S)$ does not admit a 3-NZF. Note that $|S| \geq 5$. We examine the following possibilities:

Subcase 2.1. If there is $y \in S \cap Z(N)$ of order $n > 2$ such that $d^{-1}yd \notin \{y, y^{-1}\}$, then since $Z(N)$ is normal in G, the assumption guarantees the existence of an element $z \in S \cap Z(N)$ such that $d^{-1}yd = z$. Since $O(d) = 2$, we see that $d^{-1}zd = y$.

Thus $\langle y, y^{-1}, z, z^{-1}\rangle \trianglelefteq \langle y, y^{-1}, z, z^{-1}, d\rangle$. If $G \neq \langle y, y^{-1}, z, z^{-1}, d\rangle$, then by our assumption on G, $\mathrm{Cay}(\langle y, y^{-1}, z, z^{-1}, d\rangle, \{y, y^{-1}, z, z^{-1}, d\})$ admits a 3-NZF. Thus since $|S \setminus \{y, y^{-1}, z, z^{-1}, d\}|$ is even, we get that $\mathrm{Cay}(G, S)$ admits a 3-NZF. This is a contradiction. Therefore, we can assume that $G = \langle y, y^{-1}, z, z^{-1}, d\rangle$, $N = \langle y, y^{-1}, z, z^{-1}\rangle$, $S = \{y, y^{-1}, z, z^{-1}, d\}$ and $S \cap N = \{y, y^{-1}, z, z^{-1}\}$. Let K be a minimal normal subgroup of G such that $K \leq Z(N)$. If $K \cap S = \emptyset$, then $N/K \trianglelefteq G/K$ with $[G/K : N/K] = 2$ and $Z(N/K) \cap S/K \neq \emptyset$. Note that $|S/K| = 5$ and $|(S \cap N)/K| = 4$. So $\mathrm{Cay}(N/K, (S \cap N)/K)$ admits a 3-NZF. Also $|G/K| < |G|$. Thus our assumption on G leads us to see that $\mathrm{Cay}(G/K, S/K)$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S)$ by Lemma 2.2. This is a contradiction. Thus $K \cap S \neq \emptyset$. Without loss of generality, we can suppose that $y \in K$, so $d^{-1}yd = z \in K$. Therefore, $K = N$. This forces N to be cyclic or elementary abelian. Thus either $N = \langle y\rangle$ or $N = \langle S \cap N\rangle = \langle y\rangle \times \langle z\rangle$ and hence, either $z = y^i$ and

$$\mathrm{Cay}(N, N \cap S) = \mathrm{Cay}(\langle y\rangle, \{y, y^{-1}, y^i, y^{-i}\}) \cong C(n, i) \quad \text{or}$$
$$\mathrm{Cay}(N, N \cap S) = \mathrm{Cay}(\langle y\rangle, \{y, y^{-1}\}) \,\square\, \mathrm{Cay}(\langle z\rangle, \{z, z^{-1}\}) \cong C_n \,\square\, C_n.$$

Note that $\mathrm{Cay}(G, S) = \mathrm{Cay}(N, S \cap N) \,\square\, K_2$. Thus $\mathrm{Cay}(G, S)$ is isomorphic to either $C(n, i) \,\square\, K_2$ or $(C_n \,\square\, C_n) \,\square\, K_2$. So Lemmas 2.5 and 2.4 guarantee that $\mathrm{Cay}(G, S)$ admits a 3-NZF. This is a contradiction.

Subcase 2.2. If $S \cap Z(N)$ contains an involution y such that $d^{-1}yd \neq y$, then there exists an element $z \in S \cap Z(N)$ such that $d^{-1}yd = z$. Therefore, $\langle y, z\rangle$ is an elementary abelian 2-group of order 4. So $\mathrm{Cay}(\langle y, z, d\rangle, \{y, z, d\})$ is the circular ladder CL_4 (see Figure 1) which is bipartite and hence, it admits a 3-NZF. Also, $\mathrm{Cay}(G, S \setminus \{y, z, d\})$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S)$. This is a contradiction.

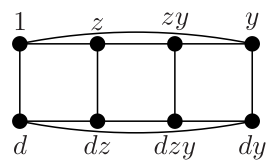

Figure 1: The circular ladder CL_4.

Subcase 2.3. Suppose that for every $y \in Z(N) \cap S$, $d^{-1}yd \in \{y, y^{-1}\}$. Applying the above argument shows that there exists an element $y \in Z(N) \cap S$ such that $\langle y\rangle$ is a minimal normal subgroup of G. If the order of y is 2, then y is a central involution and hence, $\mathrm{Cay}(G, S)$ admits a 3-NZF. This is a contradiction. Thus the order of y is an odd prime number. Now if $N \cap S$ contains an element z such that $O(z) \geq 3$ and $d^{-1}zd \in \{z, z^{-1}\}$, then applying the same argument as that of used in Subcase 2.1 leads us to get a contradiction. Now suppose that there exists an element $z \in (S \cap N) \setminus \{y, y^{-1}\}$ such that $O(z) \geq 3$ and $d^{-1}zd \notin \{z, z^{-1}\}$. So our assumption on G allows us to assume that $S = \{y, y^{-1}, z, z^{-1}, d^{-1}zd, d^{-1}z^{-1}d, d\}$. Let K be a normal subgroup of G containing y such that $K \leq N$ and it is maximal with the property $K \cap (S \setminus \{y, y^{-1}\}) = \emptyset$. If M/K is a minimal normal subgroup of G/K such that $M/K \leq N/K$, then our assumption on K shows that $M \cap (S \setminus \{y, y^{-1}\}) \neq \emptyset$. Without loss of generality, we can assume that $z \in M$. Since M is normal in G, we deduce that $d^{-1}zd \in M$ and hence, $S - \{d\} \subseteq M$. Thus $M = N$. Set $S_1 = \{z, z^{-1}, d^{-1}zd, d^{-1}z^{-1}d, d\}$. Moreover $M/K = N/K$ is abelian and normal in G/K of index 2 such that $S_1/K \setminus (S_1/K \cap M/K) = \{dK\}$ and $dK(S_1/K \cap M/K)dK = (S_1/K \cap M/K)$. By our assumption on G, $\mathrm{Cay}(G/K, S_1/K)$ admits a 3-NZF. But $S_1 \cap K = \emptyset$, so Lemma 2.2 shows that $\mathrm{Cay}(G, S_1)$ admits a 3-NZF. In addition, since $|S \setminus S_1| = 2$, $\mathrm{Cay}(G, S \setminus S_1)$ admits a 3-NZF and hence, $\mathrm{Cay}(G, S)$ admits a 3-NZF. This is a contradiction. Finally, let $N \cap S$ contain an element z of order 2. Since $|S \cap N|$ is even, our assumption on G allows us to assume that there exists an involution $w \in (S \cap N) \setminus \{z\}$ such that $G = \langle y, y^{-1}, z, w, d\rangle$. Since z, w are distinct involutions, we have that either $\langle z, w\rangle$ is an elementary abelian 2-group of order 4 or a dihedral group. We can see at once that $\mathrm{Cay}(\langle w, z, d\rangle, \{w, z, d\})$ is a circular ladder CL_k, for some even number k, which is bipartite. Therefore, $\mathrm{Cay}(\langle w, z, d\rangle, \{w, z, d\})$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S)$. This is a contradiction.

This shows that $\mathrm{Cay}(G, S)$ admits a 3-NZF, as desired. □

Theorem 3.2. *Let G be a group, N be an abelian subgroup of G of index* 2 *and let S be a connection sequence of G such that $|S| \geq 4$. If there exists $d \in S \setminus (S \cap N)$ such that $d^{-1}(S \cap N)d = S \cap N$, then $\mathrm{Cay}(G, S)$ admits a 3-NZF.*

Proof. First, assume that $|S \cap N| \geq 4$. By Lemma 2.3, $\mathrm{Cay}(N, S \cap N)$ has a 3-NZF. Since $|G/N| = 2$, we can assume that $G/N = \langle dN\rangle$, and hence for every $y \in S \setminus (S \cap N)$, $yN \in \langle dN\rangle$. Thus there exists $t \in N$ such that $y = td$ and

$$\text{for every } s \in S \cap N \text{ and } y \in S \setminus (S \cap N), \quad y^{-1}sy \in S \cap N. \tag{3.1}$$

So the Conditions (1) and (2) of Theorem 3.1 are fulfilled and hence $\mathrm{Cay}(G, S)$ admits a 3-NZF. Now, we assume that $|S \cap N| \leq 3$. The proof falls naturally into several parts. If $|S \cap N| = 0$, then by Lemma 2.8, $\mathrm{Cay}(G, S)$ is bipartite, and hence Lemma 2.6 shows that $\mathrm{Cay}(G, S)$ admits a 3-NZF. Moreover, if $|S \cap N| = 2$, then Lemma 2.9 forces $\mathrm{Cay}(N, S \cap N)$ to admit a 3-NZF. Also by (3.1), for every $s \in S \cap N$, $y^{-1}sy = d^{-1}sd \in S \cap N$. So Theorem 3.1 completes the proof. Therefore, $|S \cap N| \in \{1, 3\}$. We consider these possibilities in the following cases:

Case 1. Assume that $|S \cap N| = 1$. So $S \cap N = \{x\}$. Clearly, $O(x) = 2$ and $d^{-1}xd = x^{-1} = x$. Also, for every $y \in S \setminus (S \cap N)$, we have $yN \in \langle dN\rangle$ and hence, $y = md$ for some $m \in N$. Therefore, we can see $y^{-1}xy = x$. Thus $x \in Z(\langle S\rangle)$ is of order 2. Hence by Lemma 2.1, we have $\mathrm{Cay}(\langle S\rangle, S)$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S)$.

Case 2. Assume that $|S \cap N| = 3$. We continue the proof in two subcases:

Subcase 1. Let $S \cap N = \{x, y, y^{-1}\}$, where $O(x) = 2$ and $O(y) \geq 3$. Since $d^{-1}xd \in S \cap N$ and $O(d^{-1}xd) = 2$, the same argument as that of used in Case 1 completes the proof.

Subcase 2. Let $S \cap N = \{x, y, z\}$, where $O(x) = O(y) = O(z) = 2$. First, assume that none of the elements in $S \cap N$ generates by the other ones. Since x, y, z are of order 2 and N is abelian, we have

$$\langle N \cap S\rangle = \{x^i y^j z^l \mid 1 \leq i, j, k \leq 2\} = \langle x\rangle \times \langle y\rangle \times \langle z\rangle \leq N.$$

It is easy to check that $\mathrm{Cay}(\langle N \cap S\rangle, S \cap N)$ is bipartite (similar to Figure 1) and hence by Lemma 2.6, $\mathrm{Cay}(N, S \cap N)$ admits a 3-NZF. The rest of the proof runs as the case when $|S \cap N| \geq 4$.

Otherwise, without loss of generality, assume that $S \cap N = \{x, y, xy\}$. Set $S_1 = \{d, d^{-1}\}$. Note that $|S|$ is odd. Thus $|S \setminus ((S \cap N) \cup S_1)| = 0$ or $2k$ where $k \in \mathbb{N}$. Set $S_2 = S \setminus ((S \cap N) \cup S_1)$ and $H = \langle (S \cap N) \cup S_1 \rangle$. In fact,

$$\mathrm{Cay}(G, S_2) \cup \mathrm{Cay}(G, (S \cap N) \cup S_1) = \mathrm{Cay}(G, S)$$

and $\mathrm{Cay}(G, S_2)$ admits a 3-NZF. So it is sufficient to find a 3-NZF in $\mathrm{Cay}(G, (S \cap N) \cup S_1)$.

We know that $d^{-1}xd \in S \cap N$. If $d^{-1}xd = x$, then since N is abelian, we have $x \in Z(H)$ and its order is 2, so the proof is complete by Lemma 2.1. Now, assume that $d^{-1}xd = y$. Since $N \neq dN \in G/N$ and $|G/N| = 2$, we have $O(dN) = 2$, and hence $d^2 \in N$. It follows that $x = d^2xd^{-2} = dyd^{-1}$. Therefore,

$$d^{-1}xyd = d^{-1}xdd^{-1}yd = yx = xy.$$

Thus $xy \in Z(H)$ and $O(xy) = 2$. Lemma 2.1 shows that $\mathrm{Cay}(H, (S \cap N) \cup S_1)$ admits a 3-NZF, and so does $\mathrm{Cay}(G, (S \cap N) \cup S_1)$, as desired. The same reasoning can be applied to the case $d^{-1}xd = xy$. □

In the following we show that Theorem 3.2 guarantees the existence of a 3-NZF in a Cayley graph on a generalized dicyclic group.

Example 3.3. Let H be an abelian group, having a specific element $y \in H$ of order 2. A group G is called a *generalized dicyclic group*, $\mathrm{Dic}(H, y)$, if it is generated by H and an additional element x. Moreover, we have $[G : H] = 2$, $x^2 = y$ and $x^{-1}ax = a^{-1}$ for every $a \in H$. It is easy to see that every Cayley graph of valency at least 4 on $\mathrm{Dic}(H, y)$ has a 3-NZF by applying Theorem 3.2.

Note that in [3, 11], as the main theorems, it is showed that the graphs mentioned in Example 3.4 admit nowhere-zero 3-flows.

Example 3.4.

(1) Let H be an abelian group. The *generalized dihedral group* D_H is a group of order $2|H|$ generated by H and an element p where $p \notin H$, $p^2 = 1$ and $p^{-1}hp = h^{-1}$ for all $h \in H$. We see at once that every Cayley graph of valency at least 4 on D_H satisfies the conditions of Theorem 3.2, and hence it admits a 3-NZF. In particular, $G = \langle x, a \mid a^n = x^2 = 1, x^{-1}ax = a^{-1} \rangle$ is a special case of D_H, where $H = \langle a \rangle$, $p = x$ and it is called a *dihedral group* and denoted by D_{2n}.

(2) Let $G = \langle z, a \mid a^n = z^2, a^n = 1, z^{-1}az = a^{-1} \rangle$ which is called a *generalized quaternion group*, denoted by Q_{4n}. Note that G is a special case of a generalized dicyclic group where $\langle a \rangle$ and z play the roles of H and x, respectively. Thus every Cayley graph of valency at least 4 on Q_{4n} admits a 3-NZF.

Remark 3.5. Let G be a group, N be a normal subgroup of G of an odd index at least 3 and S be a connection sequence of G such that $|S| \geq 4$. Assume that $T = \{x_1, \ldots, x_{2k+1}\}$ is a left transversal set of N in G and $\mathrm{Cay}(N, S \cap N))$ has a 3-NZF. Note that by Remark 2.11,

$$\mathrm{Cay}(G, S) = \left(\bigcup_{i=1}^{2k+1} x_i \, \mathrm{Cay}(N, S \cap N) \right) \cup \mathrm{Cay}(G, S \setminus (S \cap N)).$$

By the assumption, for every $i \in \{1, \dots, 2k+1\}$, $x_i \operatorname{Cay}(N, S \cap N)$ admits a 3-NZF. For finding a 3-NZF in $\operatorname{Cay}(G, S)$, it is enough to find a 3-NZF in $\operatorname{Cay}(G, S \setminus (S \cap N))$. If $|S \setminus (S \cap N)|$ is odd, then there exists $y \in S \setminus (S \cap N)$ such that $O(y) = 2$ and hence $yN \in G/N$ and $O(yN) = 2$. So we have $2 \mid |G/N|$. This is impossible. Thus $|S \setminus (S \cap N)|$ is even and hence $\operatorname{Cay}(G, S \setminus (S \cap N))$ admits a 3-NZF by Lemma 2.9. Therefore if $\operatorname{Cay}(N, S \cap N)$ has a 3-NZF, then so does $\operatorname{Cay}(G, S)$.

Theorem 3.6. *Let G be a solvable group with a cyclic Sylow 2-subgroup and let S be a connection sequence of G with $|S| \geq 4$. If there exists an element $x \in Z(G) \cap S$ such that x is a generator element of G in S, then $\operatorname{Cay}(G, S)$ admits a 3-NZF.*

Proof. Suppose that G is the smallest counterexample satisfies the above conditions, but $\operatorname{Cay}(G, S)$ does not admit a 3-NZF. Without loss of generality, we can assume that $|S| = 5$ and $x \in Z(G) \cap S$. Thus $O(x) \geq 3$ by Lemma 2.1. If there exists $u \in Z(G)$ such that $\langle u \rangle \cap S = \emptyset$, then $|S/\langle u \rangle| = |S|$, $x\langle u \rangle \in Z(G/\langle u \rangle) \cap S/\langle u \rangle$ and $|G/\langle u \rangle| < |G|$. If $x\langle u \rangle$ is a generator element of $G/\langle u \rangle$ in $S/\langle u \rangle$, then by our assumption, $\operatorname{Cay}(G/\langle u \rangle, S/\langle u \rangle)$ admits a 3-NZF. Lemma 2.2 forces $\operatorname{Cay}(G, S)$ to admit a 3-NZF, a contradiction. Thus $x\langle u \rangle$ is not a generator element. Therefore, there exist an element $t \in \langle S \setminus \{x, x^{-1}\} \rangle$ and $i \in \mathbb{N}$ such that $xu^i = t$ and hence $t \in Z(G)$. If there exists $t_1 \in \langle t \rangle \cap S$, then as stated above, we can see that $O(t_1) \geq 3$. Thus $Z(G) \cap S = \{x, x^{-1}, t_1, t_1^{-1}\}$. Therefore $|G/Z(G)| \in \{1, 2\}$ and hence, $G/Z(G)$ is cyclic. So G is an abelian group. This forces $\operatorname{Cay}(G, S)$ to admit a 3-NZF, a contradiction. Thus $\langle t \rangle \cap S = \emptyset$. Moreover, we can see at once that $x\langle t \rangle$ is a generator element of $G/\langle t \rangle$ in $S/\langle t \rangle$, $|S/\langle t \rangle| = |S|$ and $|G/\langle t \rangle| < |G|$. Therefore, our assumption forces $\operatorname{Cay}(G/\langle t \rangle, S/\langle t \rangle)$ to admit a 3-NZF, and so does $\operatorname{Cay}(G, S)$ by Lemma 2.2. This is a contradiction. So for every $u \in Z(G)$, we have $\langle u \rangle \cap S \neq \emptyset$. We continue the proof in two cases:

Case 1. Suppose that $|Z(G)|$ is even. So there exists $w \in Z(G)$ of order 2. By our assumption, $\langle w \rangle \cap S \neq \emptyset$, and hence S contains a central involution. Lemma 2.1 shows that $\operatorname{Cay}(\langle S \rangle, S)$ admits a 3-NZF, and so does $\operatorname{Cay}(G, S)$. This is a contradiction.

Case 2. Let $|Z(G)|$ be odd. Since $|S| = 5$, S contains an involution y. We continue the proof in three subcases:

Subcase 1. Suppose that $|S \cap Z(G)|$ is odd, so $Z(G)$ contains an involution. This is a contradiction, because $|Z(G)|$ is odd.

Subcase 2. Suppose that $|Z(G) \cap S| = 2$. So we have $Z(G) \cap S = \{x, x^{-1}\}$, where $O(x)$ is an odd prime number p. Therefore, $\langle x \rangle$ is a cyclic subgroup of order p. By the assumption, $x \notin \langle S \setminus \{x, x^{-1}\} \rangle$ and hence, we deduce that $G = \langle x \rangle \times M$, where $M = \langle S \setminus \{x, x^{-1}\} \rangle$ is a maximal subgroup of G. Let N be a minimal normal subgroup of G such that $N \leq M$. So N is an elementary abelian q-group, where q is a prime number. If $N \cap S = \emptyset$, then $x\langle N \rangle \in Z(G/N) \cap S/N$ is a generator element of G/N in S/N, $|G/N| < |G|$ and $|S/N| = 5$. Thus by our assumption on G, $\operatorname{Cay}(G/N, S/N)$ admits a 3-NZF, and so does $\operatorname{Cay}(G, S)$. This contradicts our assumption. If $N \cap S \neq \emptyset$, then the proof falls naturally into several parts:

- **(a)** If $y \in N \cap S$ such that $O(y) = 2$, then $2 \mid |N|$. Since N is elementary abelian, we get that N is an elementary abelian 2-group. Thus $|N| = 2$ by the assumption. Therefore $y \in N \leq Z(G)$, and hence $|Z(G)|$ is even. This is a contradiction.

(b) If $N \cap S = \{z, z^{-1}\}$, where $O(z) \geq 3$, then $S \setminus \{x, x^{-1}\} = \{z, z^{-1}, y\}$. Since N is an elementary abelian q-group where q is a prime number, we get $O(z) = q \neq 2$. So $y \notin N$. If $yz = zy$, then G is an abelian group and hence, Lemma 2.3 forces $\mathrm{Cay}(G, S)$ to admit a 3-NZF, a contradiction. If $yz \neq zy$ and $O(yz) = 2$, then we have $yzy = z^{-1}$. Thus $L = \langle x, x^{-1}, z, z^{-1} \rangle \lhd G = \langle x, x^{-1}, z, z^{-1}, y \rangle$. Therefore, $[G : L] = 2$ and $L \lhd G$. We thus get that $\mathrm{Cay}(G, S)$ admits a 3-NZF by Theorem 3.1. This is a contradiction. Now, suppose that $yz \neq zy$ and $O(yz) \geq 3$. Since $O(z) = q$, $z \in N$ and $|M/N| = |\langle yN \rangle| = 2$, we have $|M| = 2q^t$, where $t \in \mathbb{N}$. If $O(yz) = q^n$, where $n \leq t$, then $yz \in N$. So $y \in N$, a contradiction. Suppose that $O(yz) = 2q^n$ where $n \leq t$. Since $\gcd(2, q^n) = 1$, there exist $k, s \in \mathbb{Z}$ such that $2s + kq^n = 1$. So, $O((yz)^{2s}) = q^n$ and $O((yz)^{kq^n}) = 2$. Thus we have $(yz)^{2s} \in N$. Since $z \in N$ and N is abelian, we can see that $(yz)^{2s}y = y(yz)^{2s}$. Therefore $(yz)^{2s} \in Z(M) \leq Z(G)$. Thus $\langle (yz)^{2s} \rangle$ is a normal subgroup of G and $\langle (yz)^{2s} \rangle \leq N$. So $z \in N = \langle (yz)^{2s} \rangle \leq Z(M) \leq Z(G)$ and hence $yz = zy$. This is a contradiction with the above statements.

Subcase 3. Suppose that $|S \cap Z(G)| = 4$. Since $|S| = 5$, we can see $|S| \setminus |S \cap Z(G)| = 1$. It follows that $[\langle S \rangle : \langle S \cap Z(G) \rangle] = 2$. So $\langle S \rangle / (\langle S \cap Z(G) \rangle)$ is a cyclic group. On the other hand, $\langle S \cap Z(G) \rangle \leq Z(\langle S \rangle)$. Therefore $\langle S \rangle$ is abelian, and hence Lemma 2.3 yields that $\mathrm{Cay}(\langle S \rangle, S)$ admits a 3-NZF, and so does $\mathrm{Cay}(G, S)$, a contradiction. □

Corollary 3.7. *Let G be a solvable group such that the Sylow 2-subgroups of G are cyclic and every Cayley graph of valency at least 4 on G admits a 3-NZF. If H is a nilpotent group, then every Cayley graph of valency at least 4 on $G \times H$ admits a 3-NZF.*

Proof. Suppose that H is the smallest nilpotent group such that $\mathrm{Cay}(G \times H, S)$ does not admit a 3-NZF. Note that by the assumption on G, we have $H \neq 1$. If there exists $1 \neq t \in Z(H)$ such that $\langle t \rangle \cap S = \emptyset$, then since $\langle t \rangle \lhd G \times H$, our assumption on H shows that $\mathrm{Cay}((G \times H)/\langle t \rangle, S/\langle t \rangle)$ admits a 3-NZF. So Lemma 2.2 forces $\mathrm{Cay}(G \times H, S)$ to admit a 3-NZF. This is a contradiction. Thus for every $t \in Z(H)$, $\langle t \rangle \cap S \neq \emptyset$. If $|H|$ is even, then S contains a central involution and hence, Lemma 2.1 shows that $\mathrm{Cay}(G \times H, S)$ admits a 3-NZF, a contradiction. Thus $|H|$ is odd. Let the order of $t \in Z(H) \cap S$ be odd. If $|H \cap S|$ is odd, then $2 \mid |H|$. This is a contradiction. If $|H \cap S| = 2$, then $H \cap S = \{x, x^{-1}\}$ and hence, $Z(H) \cap S = \{x, x^{-1}\}$ and $O(x)$ is a prime number. Since G is solvable, we can assume that K is a normal subgroup of $G \times H$ such that $K \leq G$ and K is maximal with the property that $S \cap K = \emptyset$. If $G = K$, then $(G \times H)/G$ is nilpotent and $|S/G| = |S|$, and hence, $\mathrm{Cay}((G \times H)/G, S/G)$ admits a 3-NZF, and so does $\mathrm{Cay}(G \times H, S)$. This is a contradiction. Thus $G \neq K$ and for a minimal normal subgroup M/K of $(G \times H)/K$ such that $M/K \leq G/K$, we have $M \cap S \neq \emptyset$. So one of the following possibilities occurs:

(I) Suppose that $M \cap S$ contains an involution z. Then $2 \mid |M/K|$. Since M/K is elementary abelian and the Sylow 2-subgroups of G are cyclic, we have $M/K = \langle zK \rangle$ and hence $\langle zK \rangle \leq Z((G \times H)/K)$. Therefore, Lemma 2.1 shows that $\mathrm{Cay}((G \times H)/K, S/K)$ admits a 3-NZF, and so does $\mathrm{Cay}(G \times H, S)$ by Lemma 2.2, a contradiction.

(II) If $M \cap S$ does not contain any involution, then $|M \cap S|$ is an even number. Since $|S|$ is odd, we get that $S \setminus (M \cap S)$ contains an involution z. But $|H|$ is odd, so $z \in G$. Let $S_1 = (M \cap S) \cup \{z, x, x^{-1}\}$. We have $\langle S_1 \rangle = \langle M \cap S, z \rangle \times \langle x \rangle$ and $|S_1| \geq 5$ is an odd number. Thus Theorem 3.6 shows that $\mathrm{Cay}(\langle S_1 \rangle, S_1)$ admits a 3-NZF, so does $\mathrm{Cay}(G \times H, S_1)$. Since $|S \setminus S_1|$ is even, $\mathrm{Cay}(G \times H, S)$ admits a 3-NZF, a contradiction.

If $|H \cap S| \geq 4$, then there exists an element $x \in S$ such that $O(x) = 2$. Since $|H|$ is odd, we have $x \notin H \cap S$ and the Sylow 2-subgroups of $G \times H$ are the Sylow 2-subgroups of G and hence, $x \in G$. Therefore $x \in C_{G \times H}(H \cap S)$, the centralizer of $H \cap S$ in $G \times H$, and hence $x \in Z(\langle H \cap S \rangle \times \langle x \rangle)$. So Lemma 2.1 forces $\mathrm{Cay}(\langle H \cap S \rangle \times \langle x \rangle, (H \cap S) \cup \{x\})$ to admit a 3-NZF, so does $\mathrm{Cay}(G \times H, (H \cap S) \cup \{x\})$. But $|S \setminus ((H \cap S) \cup \{x\})|$ is even, So $\mathrm{Cay}(G \times H, S)$ admits a 3-NZF, a contradiction. □

Corollary 3.8. *If L is a nilpotent group, then for every generalized dihedral group D_H, the Cayley graph of valency at least 4 on $D_H \times L$ admits a 3-NZF.*

Proof. Let D_H be the smallest generalized dihedral group such that the Cayley graph of valency at least 4 on $D_H \times L$ does not admit a 3-NZF. If $|H|$ is odd, then the Sylow 2-subgroups of D_H are cyclic, and hence Corollary 3.7 shows that $\mathrm{Cay}(D_H \times L, S)$ admits a 3-NZF, a contradiction. If $|H|$ is even, then H contains a central involution t. If $t \in S$, then Lemma 2.1 shows that the Cayley graph of valency at least 4 on $D_H \times L$ admits a 3-NZF, a contradiction. If $t \notin S$, then by our assumption, $\mathrm{Cay}((D_H \times L)/\langle t \rangle, S/\langle t \rangle)$ admits a 3-NZF. It follows that $\mathrm{Cay}(D_H \times L, S)$ admits a 3-NZF by Lemma 2.2. This is impossible. These contradictions show that every Cayley graph of valency at least 4 on $D_H \times L$ admits a 3-NZF. □

References

[1] B. Alspach, Y. Liu and C. Zhang, Nowhere-zero 4-flows and Cayley graphs on solvable groups, *SIAM J. Discrete Math.* **9** (1996), 151–154, doi:10.1137/s0895480193258017.

[2] F. Jaeger, Flows and generalized coloring theorems in graphs, *J. Comb. Theory Ser. B* **26** (1979), 205–216, doi:10.1016/0095-8956(79)90057-1.

[3] L. Li and X. Li, Nowhere-zero 3-flows in Cayley graphs on generalized dihedral group and generalized quaternion group, *Front. Math. China* **10** (2015), 293–302, doi:0.1007/s11464-014-0378-2.

[4] M. Nánásiová and M. Škoviera, Nowhere-zero 3-flows in Cayley graphs and Sylow 2-subgroups, *J. Algebraic Combin.* **30** (2009), 103–111, doi:10.1007/s10801-008-0153-0.

[5] R. Nedela and M. Škoviera, Cayley snarks and almost simple groups, *Combinatorica* **21** (2001), 583–590, doi:10.1007/s004930100014.

[6] M. Potočnik, P. Škoviera and R. Škrekovski, Nowhere-zero 3-flows in abelian Cayley graphs, *Discrete Math.* **297** (2005), 119–127, doi:10.1016/j.disc.2005.04.013.

[7] J. S. Rose, *A Course on Group Theory*, Cambridge University Press, Cambridge, 1978.

[8] W. T. Tutte, A contribution to the theory of chromatic polynomials, *Canad. J. Math.* **6** (1954), 80–91, doi:10.4153/cjm-1954-010-9.

[9] W. T. Tutte, On the algebraic theory of graph colorings, *J. Comb. Theory* **1** (1966), 15–50, doi:10.1016/s0021-9800(66)80004-2.

[10] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, NJ, 1996.

[11] F. Yang and X. Li, Nowhere-zero 3-flows in dihedral Cayley graphs, *Inform. Process. Lett.* **111** (2011), 416–419, doi:10.1016/j.ipl.2011.01.017.

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Edge-transitive bi-p-metacirculants of valency p

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Abstract

Let p be an odd prime. A graph is called a bi-p-metacirculant on a metacyclic p-group H if admits a metacyclic p-group H of automorphisms acting semiregularly on its vertices with two orbits. A bi-p-metacirculant on a group H is said to be abelian or non-abelian according to whether or not H is abelian.

By the results of Malnič et al. in 2004 and Feng et al. in 2006, we see that up to isomorphism, the Gray graph is the only cubic edge-transitive non-abelian bi-p-metacirculant on a group of order p^3. This motivates us to consider the classification of cubic edge-transitive bi-p-metacirculants. Previously, we have proved that a cubic edge-transitive non-abelian bi-p-metacirculant exists if and only if $p = 3$. In this paper, we give a classification of connected edge-transitive non-abelian bi-p-metacirculants of valency p, and consequently, we complete the classification of connected cubic edge-transitive non-abelian bi-p-metacirculants.

Keywords: Bi-p-metacirculant, edge-transitive, inner-abelian p-group.

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1 Introduction

Given a group H, let $\mathcal{R}$, $\mathcal{L}$ and S be three subsets of H such that $\mathcal{R}^{-1} = \mathcal{R}$, $\mathcal{L}^{-1} = \mathcal{L}$ and $\mathcal{R} \cup \mathcal{L}$ does not contain the identity element of H. The *bi-Cayley graph* over H with respect to the triple $(\mathcal{R}, \mathcal{L}, S)$, denoted by $\mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$, is the graph having vertex set the union $H_0 \cup H_1$ of two copies of H, and edges of the form $\{h_0, (xh)_0\}$, $\{h_1, (yh)_1\}$ and $\{h_0, (zh)_1\}$ with $x \in \mathcal{R}, y \in \mathcal{L}, z \in S$ and $h_0 \in H_0$, $h_1 \in H_1$ representing a given $h \in H$. It is easy to see that a graph is a bi-Cayley graph over a group H if and only if it admits H as a semiregular automorphism group with two orbits.

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Let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$. For $g \in H$, define a permutation $R(g)$ on the vertices of Γ by the rule

$$h_i^{R(g)} = (hg)_i, \quad \forall i \in \mathbb{Z}_2,\ h \in H.$$

Then $R(H) = \{R(g) \mid g \in H\}$ is a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ which is isomorphic to H and has H_0 and H_1 as its two orbits. When $R(H)$ is normal in $\mathrm{Aut}(\Gamma)$, the bi-Cayley graph $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ is said to be *normal* (see [24]). When $N_{\mathrm{Aut}(\Gamma)}(R(H))$ is transitive on the edge set of Γ, we say that Γ is *normal edge-transitive* (see [7]).

Bi-Cayley graphs are useful in constructing edge-transitive graphs (see [7, 24]). However, it is difficult in general to decide whether a bi-Cayley graph is edge-transitive. So it is natural to investigate the edge-transitive bi-Cayley graphs over some given groups. Note that metacylic groups are widely used in constructing graphs with some kinds of symmetry, see, for example, [1, 11, 12, 13, 14, 18]. (A group G is called *metacyclic* if it contains a cyclic normal subgroup N such that G/N is cyclic.) In this paper, we shall focus on the bi-Cayley graphs over a metacyclic p-group with p an odd prime. For convenience, a bi-Cayley graph over a (resp. non-abelian or abelian) metacyclic p-group is simply called a (resp. *non-abelian* or *abelian*) *bi-p-metacirculant*.

Note that the Gray graph [6], the smallest cubic semisymmetric graph, is a non-abeian bi-3-metacirculant of order $2 \cdot 3^3$. Malnič et al. in [8, 17] gave a classification of cubic edge-transitive graphs of order $2p^3$ for each prime p. Actually, it is easy to prove that every cubic edge-transitive graphs of order $2p^3$ is a bi-Cayley graph over a group of order p^3. Rather than describe the classification in detail, we would simply like to point out one striking feature: except the Gray graph, there do not exist other cubic edge-transitive non-abelian bi-p-metacirculants of order $2 \cdot p^3$ for every odd prime p. This seems to suggest that cubic edge-transitive non-abelian bi-p-metacirculants are rare. Motivated by this, we are going to consider the following problem:

Problem 1.1. Classify cubic edge-transitive non-abelian bi-p-metacirculants for every odd prime p.

In [19], we gave a partial answer to this problem. We first proved that a cubic edge-transitive non-abelian bi-p-metacirculant exists if and only if $p = 3$, and then we gave a classification of cubic edge-transitive bi-Cayley graphs over an inner-abelian metacyclic p-group for each odd prime p. (A non-abelian group is called an *inner-abelian group* if all of its proper subgroups are abelian.) In view of this, to solve Problem 1.1, it suffices to classify cubic edge-transitive non-abelian bi-3-metacirculants. Naturally, the following problem arises.

Problem 1.2. Classify edge-transitive non-abelian bi-p-metacirculants of valency p for every odd prime p.

The following is the main result of this paper which gives a solution of Problem 1.2.

Theorem 1.3. *Let p be an odd prime, and let Γ be a connected edge-transitive non-abelian bi-p-metacirculants of valency p. Then $p = 3$ and Γ is isomorphic to one of the following graphs:*

(i)

$$\Gamma_r = \mathrm{BiCay}(\mathcal{G}_r, \emptyset, \emptyset, \{1, a, a^{-1}b\}),$$
$$\mathcal{G}_r = \left\langle a, b \mid a^{3^{r+1}} = b^{3^r} = 1, b^{-1}ab = a^{1+3^r} \right\rangle,$$

(ii)

$$\Sigma_r = \mathrm{BiCay}(\mathcal{H}_r, \emptyset, \emptyset, \{1, b, b^{-1}a\}),$$
$$\mathcal{H}_r = \left\langle a, b \mid a^{3^{r+1}} = b^{3^{r+1}} = 1, b^{-1}ab = a^{1+3^r} \right\rangle,$$

where r is a positive integer.

Remark 1.4. The graphs Γ_r and Σ_r are actually those graphs what we have found in [19]. By [19], Γ_r is semisymmetric while Σ_r is symmetric. To the best of our knowledge, the graphs Γ_r form the first known infinite family of cubic semisymmetric graphs of order twice a power of 3.

From the above theorem and [19, Theorem 1], we may immediately obtain the following result which gives a solution of Problem 1.1.

Corollary 1.5. *Let p be an odd prime. A connected cubic non-abelian bi-p-metacirculant is edge-transitive if and only if it is isomorphic to one the graphs given in Theorem 1.3.*

Remark 1.6. The classification of cubic edge-transitive bi-Cayley graphs on abelian groups has been given in [10, 23]. So our result actually completes the classification of all cubic edge-transitive bi-p-metacirculants for each odd prime p.

2 Preliminaries

2.1 Definitions and notation

Throughout this paper, groups are assumed to be finite, and graphs are assumed to be finite, connected, simple and undirected. For the group-theoretic and the graph-theoretic terminology not defined here we refer the reader to [4, 21].

Let G be a permutation group on a set Ω and take $\alpha \in \Omega$. The stabilizer G_α of α in G is the subgroup of G fixing the point α. The group G is said to be *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular.

For a positive integer n, denote by $\mathbb{Z}_n$ the cyclic group of order n and by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n. For a finite group G, the full automorphism group and the derived subgroup of G will be denoted by $\mathrm{Aut}(G)$ and G', respectively. Denote by $\exp(G)$ the exponent of G. For any $x \in G$, denote by $o(x)$ the order of x. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. A non-abelian group is called an *inner-abelian group* if all of its proper subgroups are abelian.

For a graph Γ, we denote by $V(\Gamma)$ the set of all vertices of Γ, by $E(\Gamma)$ the set of all edges of Γ, by $A(\Gamma)$ the set of all arcs of Γ, and by $\mathrm{Aut}(\Gamma)$ the full automorphism group of Γ. For $u, v \in V(\Gamma)$, denote by $\{u, v\}$ the edge incident to u and v in Γ. If a subgroup G of $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, $E(\Gamma)$ or $A(\Gamma)$, we say that Γ is *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive*, respectively. In the special case when $G = \mathrm{Aut}(\Gamma)$ we say that Γ is *vertex-transitive*, *edge-transitive* or *arc-transitive*, respectively. An arc-transitive graph is also called a *symmetric graph*. A graph Γ is said to be *semisymmetric* if Γ is regular and is edge- but not vertex-transitive.

2.2 Quotient graph

Let Γ be a connected graph with an edge-transitive group G of automorphisms and let N be a normal subgroup of G. The *quotient graph* Γ_N of Γ relative to N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two orbits adjacent if there exists an edge in Γ between the vertices lying in those two orbits. Below we introduce two propositions of which the first is a result of [15, Theorem 9].

Proposition 2.1. *Let p be an odd prime and Γ be a graph of valency p, and let $G \leq \mathrm{Aut}(\Gamma)$ be arc-transitive on Γ. Then G is an s-arc-regular subgroup of $\mathrm{Aut}(\Gamma)$ for some integer s. If $N \trianglelefteq G$ has more than two orbits in $V(\Gamma)$, then N is semiregular on $V(\Gamma)$, Γ_N is a symmetric graph of valency p with G/N as an s-arc-regular subgroup of automorphisms.*

In view of [16, Lemma 3.2], we have the following proposition.

Proposition 2.2. *Let p be an odd prime and Γ be a graph of valency p, and let $G \leq \mathrm{Aut}(\Gamma)$ be transitive on $E(\Gamma)$ but intransitive on $V(\Gamma)$. Then Γ is a bipartite graph with two partition sets, say V_0 and V_1. If $N \trianglelefteq G$ is intransitive on each of V_0 and V_1, then N is semiregular on $V(\Gamma)$, Γ_N is a graph of valency p with G/N as an edge- but not vertex-transitive group of automorphisms.*

2.3 Bi-Cayley graphs

Proposition 2.3 ([23, Lemma 3.1])**.** *Let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a connected bi-Cayley graph over a group H. Then the following hold:*

(1) H is generated by $\mathcal{R} \cup \mathcal{L} \cup S$.

(2) Up to graph isomorphism, S can be chosen to contain the identity of H.

(3) For any automorphism α of H, $\mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S) \cong \mathrm{BiCay}(H, \mathcal{R}^\alpha, \mathcal{L}^\alpha, S^\alpha)$.

(4) $\mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S) \cong \mathrm{BiCay}(H, \mathcal{L}, \mathcal{R}, S^{-1})$.

Let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a bi-Cayley graph over a group H. Recall that for each $g \in H$, $R(g)$ is a permutation on $V(\Gamma)$ defined by the rule

$$h_i^{R(g)} = (hg)_i, \quad \forall i \in \mathbb{Z}_2,\ h, g \in H,$$

and $R(H) = \{R(g) \mid g \in H\} \leq \mathrm{Aut}(\Gamma)$. For an automorphism α of H and $x, y, g \in H$, define two permutations on $V(\Gamma) = H_0 \cup H_1$ as following:

$$\delta_{\alpha,x,y} \colon h_0 \mapsto (xh^\alpha)_1,\ h_1 \mapsto (yh^\alpha)_0,\ \forall h \in H,$$
$$\sigma_{\alpha,g} \colon h_0 \mapsto (h^\alpha)_0,\ h_1 \mapsto (gh^\alpha)_1,\ \forall h \in H.$$

Set

$$I = \{\delta_{\alpha,x,y} \mid \alpha \in \mathrm{Aut}(H) \text{ s.t. } \mathcal{R}^\alpha = x^{-1}\mathcal{L}x, \mathcal{L}^\alpha = y^{-1}\mathcal{R}y, S^\alpha = y^{-1}S^{-1}x\},$$
$$F = \{\sigma_{\alpha,g} \mid \alpha \in \mathrm{Aut}(H) \text{ s.t. } \mathcal{R}^\alpha = \mathcal{R}, \mathcal{L}^\alpha = g^{-1}\mathcal{L}g, S^\alpha = g^{-1}S\}.$$

Proposition 2.4 ([24, Theorem 3.4])**.** *Let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a connected bi-Cayley graph over the group H. Then $N_{\mathrm{Aut}(\Gamma)}(R(H)) = R(H) \rtimes F$ if $I = \emptyset$ and $N_{\mathrm{Aut}(\Gamma)}(R(H)) = R(H)\langle F, \delta_{\alpha,x,y}\rangle$ if $I \neq \emptyset$ and $\delta_{\alpha,x,y} \in I$. Furthermore, for any $\delta_{\alpha,x,y} \in I$, we have the following:*

(1) $\langle R(H), \delta_{\alpha,x,y}\rangle$ *acts transitively on* $V(\Gamma)$*;*

(2) if α *has order* 2 *and* $x = y = 1$*, then* Γ *is isomorphic to the Cayley graph* $\mathrm{Cay}(\bar{H}, \mathcal{R} \cup \alpha S)$*, where* $\bar{H} = H \rtimes \langle\alpha\rangle$*.*

3 Some basic properties of metacyclic p-groups

In this section, we will give some properties of metacyclic p-groups.

Proposition 3.1. *Any metacyclic* p*-group* G *(*p *an odd prime) has the following presentation:*

$$G = \left\langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, a^b = a^{1+p^r} \right\rangle,$$

where r, s, t, u *are non-negative integers with* $u \le r$*. Different values of the parameters* r, s, t, u *with the above conditions give non-isomorphic metacyclic* p*-groups. Furthermore, the following hold:*

(1) If $|G'| = p^n$*, then for any* $m \ge n$*, we have*

$$(xy)^{p^m} = x^{p^m} y^{p^m}, \quad \forall x, y \in G.$$

(2) For any positive integer k *and for any* $x, y \in G$*,*

$$x^{p^k} = y^{p^k} \iff (x^{-1}y)^{p^k} = 1 \iff (xy^{-1})^{p^k} = 1.$$

Proof. By [22, Theorem 2.1], it suffices to prove the items (1) and (2). Since G' is cyclic, (1) follows from [9, Chapter 3, §10, Theorem 10.2 (c) and Theorem 10.8 (g)]. Item (2) follows from [9, Chapter 3, §10, Theorem 10.2 (c) and Theorem 10.6 (a)]. □

Lemma 3.2. *Let* p *be an odd prime, and let* H *be a metacyclic* p*-group generated by* a, b *with the following defining relations:*

$$a^{p^m} = b^{p^n} = 1, \qquad b^{-1}ab = a^{1+p^r},$$

where m, n, r *are positive integers such that* $r < m \le n + r$*. Then the following hold:*

(1) For any $i \in \mathbb{Z}_{p^m}$*,* $j \in \mathbb{Z}_{p^n}$*, we have*

$$a^i b^j = b^j a^{i(1+p^r)^j}.$$

(2) For any positive integer k *and for any* $i \in \mathbb{Z}_{p^m}$*,* $j \in \mathbb{Z}_{p^n}$*, we have*

$$(b^j a^i)^k = b^{kj} a^{i \sum_{s=0}^{k-1} (1+p^r)^{sj}}.$$

(3) For any positive integers t, k *and any element* x *of* H*, if* $x^{p^{2t}} = 1$*, then*

$$x^{(1+p^t)^k} = x^{1+k \cdot p^t}.$$

(4) The subgroup of H *of order* p *is one of the following groups:*

$$\left\langle a^{p^{m-1}} \right\rangle, \quad \left\langle b^{p^{n-1}} a^{i' p^{m-1}} \right\rangle \ (i' \in \mathbb{Z}_p).$$

Proof. From [19, Lemma 14 (1)–(2)], we have the items (1)–(2).

For (3), the result is clearly true if $k = 1$. In what follows, assume $k \geq 2$. Since $x^{p^{2t}} = 1$, we have $x^{p^{kt}} = 1$. Then

$$\begin{aligned} x^{(1+p^t)^k} &= x^{[C_k^0 \cdot 1^k \cdot (p^t)^0 + C_k^1 \cdot 1^{k-1} \cdot (p^t)^1 + C_k^2 \cdot 1^{k-2} \cdot (p^t)^2 + \cdots + C_k^k \cdot 1^0 \cdot (p^t)^k]} \\ &= x^{C_k^0 \cdot (p^t)^0} \cdot x^{C_k^1 \cdot (p^t)^1} \cdot x^{C_k^2 \cdot (p^t)^2} \cdots x^{C_k^k \cdot (p^t)^k} \\ &= x \cdot (x^{p^t})^{C_k^1} \cdot (x^{p^{2t}})^{C_k^2} \cdots (x^{p^{kt}})^{C_k^k} \\ &= x \cdot x^{k \cdot p^t} \\ &= x^{1+k \cdot p^t}, \end{aligned}$$

and so (3) holds. (Here for any integers $N \geq l \geq 0$, we denote by C_N^l the binomial coefficient, that is, $C_N^l = \frac{N!}{l!(N-l)!}$.)

For (4), let $\Omega_1(H) = \langle x \in H \mid o(x) = p \rangle$. Since H is a metacyclic p-group, by [2, Exercise 85], we have $\Omega_1(H) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. It implies that H has $p + 1$ subgroups of order p. Furthermore, the subgroup of H of order p is one of the following groups:

$$\left\langle a^{p^{m-1}} \right\rangle, \quad \left\langle b^{p^{n-1}} a^{i' p^{m-1}} \right\rangle \; (i' \in \mathbb{Z}_p),$$

as required. □

4 Inner-abelian bi-p-metacirculants of valency p

In this section, we focus on edge-transitive bi-Cayley graphs over inner-abelian metacyclic p-groups of valency p. For convenience, a bi-Cayley graph over an inner-abelian metacyclic p-group is simply called an *inner-abelian bi-p-metacirculant.*

In [19, Theorem 2], we gave a classification of cubic edge-transitive inner-abelian bi-p-metacirculants.

Proposition 4.1 ([19, Theorem 2])**.** *Let Γ be a connected cubic edge-transitive bi-Cayley graph over an inner-abelian metacyclic 3-group H. Then $H \cong \mathcal{G}_r$ or $\mathcal{H}_r$, and $\Gamma \cong \Gamma_r$ or Σ_r, where the groups $\mathcal{G}_r$, $\mathcal{H}_r$, and the graphs Γ_r, Σ_r are defined as in Theorem 1.3. In particular, $H/H' \cong \mathbb{Z}_{3^r} \times \mathbb{Z}_{3^r}$ or $\mathbb{Z}_{3^r} \times \mathbb{Z}_{3^{r+1}}$.*

In this section, we shall prove the following theorem.

Theorem 4.2. *Let H be an inner-abelian metacyclic p-group with p an odd prime, and let Γ be a connected edge-transitive bi-Cayley graph over H of valency p. Then $p = 3$, and Γ is isomorphic to one of the graphs given in Theorem 1.3.*

4.1 Two technical lemmas

Lemma 4.3. *Let p be an odd prime and let Γ be a connected edge-transitive graph of valency p. If $G \leq \mathrm{Aut}(\Gamma)$ is transitive on the edges of Γ, then for each $v \in V(\Gamma)$, $|G_v| = pm$ with $(m, p) = 1$.*

Proof. Since G is transitive on the edges of Γ, for each $v \in V(\Gamma)$, the order of a vertex stabilizer G_v must be divisible by p. Suppose, by way of contradiction, that $|G_v|$ is divisible by p^2. Let G_v^* be the subgroup of G_v fixing the neighborhood $\Gamma(v)$ of v in Γ pointwise.

Then $G_v/G_v^* \lesssim S_p$, forcing that $p \mid |G_v^*|$. Then G_v^* contains an element α of order p. Note that each orbit of $\langle\alpha\rangle$ has length either 1 or p. Since $\langle\alpha\rangle$ fixes v and each vertex in $\Gamma(v)$, the connectedness of Γ implies that each orbit of $\langle\alpha\rangle$ has length 1, and so $\alpha = 1$, a contradiction. □

Lemma 4.4. *Let H be a p-group with p an odd prime, and let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a connected edge-transitive bi-Cayley graph of valency p. Then*

(1) Γ is normal edge-transitive, $\mathcal{R} = \mathcal{L} = \emptyset$, and $S = \{1, h, hh^\alpha, \ldots, hh^\alpha \cdots h^{\alpha^{p-2}}\}$ for some $1 \neq h \in H$ and $\alpha \in \mathrm{Aut}(H)$ satisfying $hh^\alpha h^{\alpha^2} \cdots h^{\alpha^{p-1}} = 1$ and $o(\alpha) \mid p$;

(2) if H has a characteristic subgroup K such that H/K is isomorphic to $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, then $|m - n| \leq 1$.

Proof. Let $A = \mathrm{Aut}(\Gamma)$, and let P be a sylow p-subgroup of A such that $R(H) \leq P$. Since Γ is edge-transitive, Lemma 4.3 gives that $|A| = |R(H)| \cdot p \cdot m$, where $(p, m) = 1$. It follows that $|P| = p|R(H)|$, and hence $P \leq N_A(R(H))$. Furthermore, for any $e \in E(\Gamma)$, we have $|A : A_e| = |E(\Gamma)| = p|R(H)|$, and so $|A_e| = m$. It follows that $P_e = P \cap A_e = 1$, and hence $|P : P_e| = |P| = p|R(H)| = |E(\Gamma)|$. Thus, P is transitive on the edges of Γ. Thus, Γ is normal edge-transitive.

Let $N = N_A(R(H))$. Then N is transitive on the edges of Γ. Since $R(H) \trianglelefteq N$, the two orbits H_0, H_1 of $R(H)$ do not contain any edge of Γ, and so $\mathcal{R} = \mathcal{L} = \emptyset$. By Proposition 2.3, we may assume that $1 \in S$. Since N is transitive on the edges of Γ and Γ has valency p, N_{1_0} has an element $\sigma_{\alpha,h}$ of order p for some $\alpha \in \mathrm{Aut}(H)$ and $1 \neq h \in H$. Furthermore, $\sigma_{\alpha,h}$ cyclically permutes the elements in $\Gamma(1_0)$. So we have $\Gamma(1_0) = \{1_1, h_1, (hh^\alpha)_1, \ldots, (hh^\alpha \cdots h^{\alpha^{p-2}})_1\}$ and $hh^\alpha h^{\alpha^2} \cdots h^{\alpha^{p-1}} = 1$. This implies that

$$S = \{1, h, hh^\alpha, \ldots, hh^\alpha \cdots h^{\alpha^{p-2}}\},$$

and $h^{\alpha^p} = h$. Since Γ is connected, one has $H = \langle S \rangle = \langle h^{\alpha^i} \mid 0 \leq i \leq p-1 \rangle$. As $h^{\alpha^p} = h$, α^p is a trivial automorphism of H. Consequently, we have $o(\alpha) = 1$ or p and (1) is proved.

For (2), without loss of generality, assume that $H/K \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ with $m > n$, where K is a characteristic subgroup of H. Let $T = \langle R(x) \in R(H) \mid x^{p^n} \in K \rangle$. Then T is characteristic in $R(H)$ and $R(H)/T \cong \mathbb{Z}_{p^{m-n}}$. Propositions 2.1 and 2.2 implies that the quotient graph Γ_T of Γ relative to T is a graph of valency p with N/T as an edge-transitive group of automorphisms. Clearly, $R(H)/T$ is semiregular on $V(\Gamma_T)$ with two orbits and $R(H)/T \trianglelefteq N/T$, so Γ_T is a normal edge-transitive bi-Cayley graph over $R(H)/T \cong \mathbb{Z}_{p^{m-n}}$ of valency p.

So to complete the proof, it suffices to show that if $H \cong \mathbb{Z}_{p^m}$ then $m \leq 1$. Suppose to the contrary that $H \cong \mathbb{Z}_{p^m}$ with $m \geq 2$. Since $H = \langle h^{\alpha^i} \mid 0 \leq i \leq p-1 \rangle$, we have $H = \langle h \rangle$. Let $h^\alpha = h^\lambda$ for some $\lambda \in \mathbb{Z}_{p^m}^*$. Then

$$1 = hh^\alpha h^{\alpha^2} \cdots h^{\alpha^{p-1}} = h^{1+\lambda+\lambda^2+\cdots+\lambda^{p-1}},$$

and then

$$1 + \lambda + \lambda^2 + \cdots + \lambda^{p-1} \equiv 0 \pmod{p^m}.$$

It follows that $\lambda^p \equiv 1 \pmod{p^m}$, and hence $\lambda \equiv 1 \pmod{p}$. Let $\lambda = kp+1$ for some integer k. Since $m \geq 2$, we have

$$1 + (kp+1) + (kp+1)^2 + \cdots + (kp+1)^{p-1} \equiv 0 \pmod{p^2}.$$

It follows that

$$1 + (kp+1) + (2kp+1) + \cdots + ((p-1)kp+1) \equiv 0 \pmod{p^2},$$

and hence

$$p + \frac{1}{2}p(p-1)kp \equiv 0 \pmod{p^2}.$$

A contradiction occurs. □

4.2 Proof of Thorem 4.2

Throughout this subsection, we shall always let H be an inner-abelian metacyclic p-group with p an odd prime, and Γ be a connected edge-transitive bi-Cayley graph over H of valency p.

In view of Lemma 4.4(1) and since H is inner abelian, we may make the following assumption throughout this subsection.

Assumption 4.5. $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$, *where* $S = \{1, h, hh^\alpha, \ldots, hh^\alpha \cdots h^{\alpha^{p-2}}\}$ *for some* $1 \neq h \in H$ *and* $\alpha \in \mathrm{Aut}(H)$ *satisfying* $hh^\alpha h^{\alpha^2} \cdots h^{\alpha^{p-1}} = 1$ *and* $o(\alpha) = p$.

Proof of Theorem 4.2. Suppose to the contrary that $p > 3$. Since H is an inner-abelian metacyclic p-group, by elementary group theory (see also [20] or [3, Lemma 65.2]), we may assume that

$$H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^s} = 1, b^{-1}ab = a^{p^t+1} \right\rangle,$$

where $t \geq 1, s \geq 1$. Note that $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^s}$. By Lemma 4.4, we have $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^t}$, $\mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{t+1}}$ or $\mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{t-1}}$.

If $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{t-1}}$, then $s = t-1$ and

$$H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^{t-1}} = 1, b^{-1}ab = a^{p^t+1} \right\rangle.$$

Let $T = \langle R(x) \mid x \in H, x^{p^{t-1}} = 1 \rangle$. Then T is characteristic in $R(H)$ and $R(H)/T$ is isomorphic to $\mathbb{Z}_{p^2}$. However, by the proof of Lemma 4.4, this is impossible.

If $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^t}$, then $s = t$ and

$$H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^t} = 1, b^{-1}ab = a^{p^t+1} \right\rangle,$$

where $t \geq 1$. We shall show that this is impossible in Lemma 4.6.

If $H/H' = \langle aH' \rangle \times \langle bH' \rangle \cong \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{t+1}}$, then $s = t+1$ and

$$H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^{t+1}} = 1, b^{-1}ab = a^{p^t+1} \right\rangle,$$

where $t \geq 1$. We shall show that this is impossible in Lemma 4.7. □

Lemma 4.6. *If* $H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^t} = 1, b^{-1}ab = a^{p^t+1} \right\rangle$ $(t > 0)$*, then* $p = 3$.

Proof. Suppose to the contrary that $p > 3$. We first define the following four maps. Let

$$\gamma\colon a \mapsto a^{1+p}, b \mapsto b, \qquad \delta\colon a \mapsto a, b \mapsto b^{1+p},$$
$$\sigma\colon a \mapsto a, b \mapsto ba^p, \qquad \tau\colon a \mapsto ba, b \mapsto b.$$

Let $x_1 = a^{1+p}$, $x_2 = x_3 = a$, $x_4 = ba$, $y_1 = y_4 = b$, $y_2 = b^{1+p}$ and $y_3 = ba^p$. Since H is an inner-abelian metacyclic-p group, by Proposition 3.1 and a direct computation, we have $o(x_{i_1}) = o(a) = p^{t+1}$, $o(y_{i_1}) = o(b) = p^t$ and it is direct to check that x_{i_1} and y_{i_1} have the same relations as do a and b, where $i_1 \in \{1, 2, 3, 4\}$. Moreover, for any $i_1 \in \{1, 2, 3, 4\}$, we have $\langle x_{i_1}, y_{i_1} \rangle = H$. It follows that each of the above four maps induces an automorphism of H.

Set $P = \langle \sigma, \gamma, \delta, \tau \rangle$. By a direct computation, we have $o(\gamma) = p^t, o(\delta) = p^{t-1}$ and $o(\sigma) = o(\tau) = p^t$. Furthermore, $\gamma\delta = \delta\gamma$, $\gamma^{-1}\sigma\gamma = \sigma^{p+1}$ and $\delta^{-1}\sigma\delta = \sigma^{\ell}$ with $\ell(p+1) \equiv 1 \pmod{p^t}$. As both γ and δ fixes the subgroup $\langle b \rangle$ while σ does not, one has

$$\langle \sigma, \gamma, \delta \rangle = \langle \sigma \rangle \rtimes (\langle \gamma \rangle \times \langle \delta \rangle) \cong \mathbb{Z}_{p^t} \rtimes (\mathbb{Z}_{p^t} \times \mathbb{Z}_{p^{t-1}}).$$

Observing that $\langle \sigma, \gamma, \delta \rangle$ fixes the subgroup $\langle a \rangle$ setwise but τ does not, it follows that $\langle \sigma, \gamma, \delta \rangle \cap \langle \tau \rangle = 1$, and hence $|P| \geq p^{4t-1}$. In view of [13, Theorem 2.8], $\mathrm{Aut}(H)$ has a normal Sylow p-subgroup of order p^{4t-1}. It follows that $P = \langle \sigma, \gamma, \delta, \tau \rangle$ is the unique Sylow p-subgroup of $\mathrm{Aut}(H)$. In particular, we have $P = \langle \gamma \rangle \langle \delta \rangle \langle \sigma \rangle \langle \tau \rangle$.

Recall that $S = \{1, h, hh^{\alpha}, \ldots, hh^{\alpha} \cdots h^{\alpha^{p-2}}\}$. Assume that $h = b^u a^v$ for some $u \in \mathbb{Z}_{p^t}$ and $v \in \mathbb{Z}_{p^{t+1}}$. Since $H = \langle S \rangle$, we have $o(h) = \exp(H)$. It follows that $(v, p) = 1$. Then the map $\varphi_1\colon a \mapsto a^v, b \mapsto b$ induces an automorphism of H. Let $\varphi = (\tau^u \varphi_1)^{-1}$. Then $\varphi \in \mathrm{Aut}(H)$ and $h^{\varphi} = a$. By Proposition 2.4(3), we have that $\Gamma \cong \Gamma' = \mathrm{BiCay}(H, \emptyset, \emptyset, S^{\varphi})$. Let $\beta = \varphi^{-1}\alpha\varphi$. Then $\sigma_{\beta,a} \in \mathrm{Aut}(\Gamma')$ cyclically permutates the elements in $\Gamma'(1_0)$. It follows that

$$S^{\varphi} = \{1, a, aa^{\beta}, aa^{\beta}a^{\beta^2}, \ldots, aa^{\beta}a^{\beta^2} \cdots a^{\beta^{p-2}}\},$$

and $aa^{\beta}a^{\beta^2} \cdots a^{\beta^{p-1}} = 1$. Clearly, $o(\beta) = o(\alpha) = p$, so $\beta \in P$. We assume that $\beta = \gamma^i \delta^j \sigma^k \tau^l$ for some $i, k, l \in \mathbb{Z}_{p^t}$ and $j \in \mathbb{Z}_{p^{t-1}}$.

By Lemma 3.2(2) – (3) and Proposition 3.1(1), we have

$$\beta\colon \begin{cases} a \mapsto (b^l a)^{(1+p)^i} = b^{(1+p)^i l} a^{(1+p)^i} \\ b \mapsto (b \cdot (b^l a)^{pk})^{(1+p)^j} = b^{(1+p)^j (1+pkl)} a^{(1+p)^j pk} \end{cases} \tag{4.1}$$

Let $\mho_1(H) = \{x^p \mid x \in H\}$. Then $\mho_1(H) \leq Z(H)$ and

$$\beta\colon \begin{cases} a \mapsto b^l a \cdot w \\ b \mapsto b \cdot w' \end{cases} \tag{4.2}$$

for some w, $w' \in \mho_1(H)$. Since Γ' is connected, by Proposition 2.3, we have $H = \langle S^{\varphi} \rangle$. By Proposition 3.1(1), it follows that $(l, p) = 1$.

We shall finish the proof by the following steps.

Step 1: $t > 1$.

Suppose to the contrary that $t = 1$. Then $H = \langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p}\rangle$. We shall first show that for any $r \geq 1$,

$$a^{\beta^r} = b^{rl}a^{1+\frac{1}{2}r(r-1)klp+irp} \tag{4.3}$$

By Equation (4.1) we have

$$\beta\colon \begin{cases} a \mapsto b^l a^{1+ip} \\ b \mapsto ba^{kp} \end{cases}$$

So Equation (4.3) holds when $r = 1$. Now assume that $r > 1$ and

$$a^{\beta^{r-1}} = b^{(r-1)l}a^{1+\frac{1}{2}(r-1)(r-2)klp+i(r-1)p}.$$

By a direct computation, we have

$$\begin{aligned} a^{\beta^r} &= (b^{(r-1)l}a^{1+\frac{1}{2}(r-1)(r-2)klp+i(r-1)p})^\beta \\ &= (ba^{kp})^{(r-1)l}(b^l a^{1+ip})^{1+\frac{1}{2}(r-1)(r-2)klp+i(r-1)p} \\ &= b^{(r-1)l}a^{(r-1)lkp}b^l a^{1+\frac{1}{2}[(r-1)^2-(r-1)]klp+irp} \\ &= b^{(r-1)l+l}a^{1+[\frac{1}{2}(r-1)^2-\frac{1}{2}(r-1)+(r-1)]klp+irp} \\ &= b^{rl}a^{1+\frac{1}{2}r(r-1)klp+irp} \end{aligned}$$

By induction, we have Equation (4.3).

Now we show that for any $r \geq 1$,

$$a \cdot a^\beta \cdots a^{\beta^r} = b^{\frac{1}{2}r(r+1)l}a^{(r+1)+[\frac{1}{6}r(r+1)(2r+1)l+\frac{1}{2}r(r+1)i+\frac{1}{6}(r-1)r(r+1)kl]p}. \tag{4.4}$$

By Equation (4.3) and Lemma 3.2(1)&(3), we have

$$a \cdot a^\beta = a \cdot b^l a^{1+ip} = b^l a^{(1+p)^l}a^{1+ip} = b^l a^{1+lp}a^{1+ip} = b^l a^{2+(l+i)p}.$$

So Equation 4.4 holds when $r = 1$. Now assume that $r > 1$ and

$$a \cdot a^\beta \cdots a^{\beta^{r-1}} = b^{\frac{1}{2}(r-1)rl}a^{r+[\frac{1}{6}(r-1)r(2r-1)l+\frac{1}{2}(r-1)ri+\frac{1}{6}(r-2)(r-1)rkl]p}.$$

By a direct computation, we have

$$\begin{aligned} &aa^\beta a^{\beta^2}\cdots a^{\beta^r} \\ &= b^{\frac{1}{2}(r-1)rl}a^{r+[\frac{1}{6}(r-1)r(2r-1)l+\frac{1}{2}(r-1)ri+\frac{1}{6}(r-2)(r-1)rkl]p} \cdot b^{rl}a^{1+\frac{1}{2}r(r-1)klp+irp} \\ &= b^{\frac{1}{2}r(r+1)l}a^{\{r+[\frac{1}{6}(r-1)r(2r-1)l+\frac{1}{2}(r-1)ri+\frac{1}{6}(r-2)(r-1)rkl]p\}\cdot(1+rlp)+1+\frac{1}{2}r(r-1)klp+irp} \\ &= b^{\frac{1}{2}r(r+1)l}a^{r(1+rlp)+[\frac{1}{6}(r-1)r(2r-1)l+\frac{1}{2}(r-1)ri+\frac{1}{6}(r-2)(r-1)rkl]p+1+\frac{1}{2}r(r-1)klp+irp} \\ &= b^{\frac{1}{2}r(r+1)l}a^{(r+1)+[\frac{1}{6}(r-1)r(2r-1)+r^2]lp+[\frac{1}{2}(r-1)r+r]ip+[\frac{1}{6}(r-2)(r-1)+\frac{1}{2}r(r-1)]rklp} \\ &= b^{\frac{1}{2}r(r+1)l}a^{(r+1)+[\frac{1}{6}r(r+1)(2r+1)l+\frac{1}{2}r(r+1)i+\frac{1}{6}(r-1)r(r+1)kl]p}. \end{aligned}$$

By induction, we have Equation (4.4).

Since p is a prime and $p > 3$, by Equation (4.4), we have

$$aa^{\beta}a^{\beta^2}\cdots a^{\beta^{p-1}} = b^{\frac{1}{2}(p-1)pl}a^{p+[\frac{1}{6}(p-1)p(2p-1)l+\frac{1}{2}(p-1)pi+\frac{1}{6}(p-2)(p-1)pkl]p} = a^p \neq 1,$$

a contradiction.

Step 2: A final contradiction

Let $\mho_2(H) = \{x^{p^2} \mid x \in H\}$. Then $\mho_2(H) \leq Z(H)$. By Equation (4.1), we have

$$a^{\beta} = b^{(1+ip)l}a^{1+ip} \cdot \varpi,$$
$$b^{\beta} = b^{1+jp+pkl}a^{pk} \cdot \varpi',$$

for some $\varpi, \varpi' \in \mho_2(H)$. Let $m \equiv il \pmod p$, $n \equiv i \pmod p$, $f \equiv j + kl \pmod p$ for some $m, n, f \in \mathbb{Z}_p$. Then

$$\beta\colon \begin{cases} a \mapsto b^{mp+l}a^{np+1} \cdot \varpi_1 \\ b \mapsto b^{fp+1}a^{kp} \cdot \varpi_1' \end{cases} \tag{4.5}$$

for some $\varpi_1, \varpi_1' \in \mho_2(H)$.

We shall first prove the following claim.

Claim. *For any $r \geq 2$, $a^{\beta^r} = b^{c_rp+2l}a^{d_rp}\varpi_r$ for some $c_r, d_r \in \mathbb{Z}_p$ and $\varpi_r \in \mho_2(H)$.*

Since $t > 1$, for any positive integer i_0, by Lemma 3.2(1)&(3), we have

$$ab^{i_0} = b^{i_0}a^{(1+p^t)^{i_0}} = b^{i_0}a^{1+i_0p^t} = b^{i_0}a \cdot \varpi_0, \tag{4.6}$$

for some $\varpi_0 \in \mho_2(H)$. Then by Equations (4.5) and (4.6), we have

$$\begin{aligned} a^{\beta^2} &= (b^{fp+1}a^{kp} \cdot \varpi_1')^{mp+l}(b^{mp+l}a^{np+1} \cdot \varpi_1)^{np+1} \cdot \varpi_1^{\beta} \\ &= b^{(2m+fl+nl)p+2l}a^{(2n+kl)p} \cdot \varpi_2, \end{aligned}$$

for some $\varpi_2 \in \mho_2(H)$. Take $c_2, d_2 \in \mathbb{Z}_p$ such that $2m + fl + nl \equiv c_2 \pmod p$ and $2n + kl \equiv d_2 \pmod p$. If $r = 2$, then Claim is clearly true. Now assume that $r > 2$ and Claim holds for any positive integer less than r. Then

$$a^{\beta^{r-1}} = b^{c_{r-1}p+2l}a^{d_{r-1}p} \cdot \varpi_{r-1},$$

for some $c_{r-1}, d_{r-1} \in \mathbb{Z}_p$ and $\varpi_{r-1} \in \mho_2(H)$, and then

$$\begin{aligned} a^{\beta^r} &= (b^{fp+1}a^{kp} \cdot \varpi_1')^{c_{r-1}p+2l}(b^{mp+l}a^{np+1} \cdot \varpi_1)^{d_{r-1}p} \cdot \varpi_{r-1}^{\beta} \\ &= b^{(c_{r-1}+2fl+ld_{r-1})p+2l}a^{(2kl+d_{r-1})p} \cdot \varpi_r, \end{aligned}$$

for some $\varpi_r \in \mho_2(H)$. Take $c_r, d_r \in \mathbb{Z}_p$ such that $c_{r-1} + 2fl + ld_{r-1} \equiv c_r \pmod p$ and $2kl + d_{r-1} \equiv d_r \pmod p$. By induction, we complete the proof of Claim.

Now by our Claim, we have

$$a^{\beta^p} = b^{c_pp+2l}a^{d_pp} \cdot \varpi_p = a,$$

for some $c_p, d_p \in \mathbb{Z}_p$ and $\varpi_p \in \mho_2(H)$. It follows that $c_pp + 2l \equiv 0 \pmod{p^2}$, a contradiction. This completes the proof of our lemma. □

Lemma 4.7. *If* $H = \left\langle a, b \mid a^{p^{t+1}} = b^{p^{t+1}} = 1, b^{-1}ab = a^{p^t+1} \right\rangle$ $(t > 0)$*, then* $p = 3$.

Proof. Suppose to the contrary that $p > 3$. We first define the following four maps. Let

$$\gamma\colon a \mapsto a^{1+p}, b \mapsto b, \qquad \delta\colon a \mapsto a, b \mapsto b^{1+p},$$
$$\sigma\colon a \mapsto b^p a, b \mapsto b, \qquad \tau\colon a \mapsto a, b \mapsto ba.$$

Let $x_1 = a^{1+p}$, $x_2 = x_4 = a$, $x_3 = b^p a$, $y_1 = y_3 = b$, $y_2 = b^{1+p}$ and $y_4 = ba$. Since H is an inner-abelian metacyclic-p group, by Proposition 3.1 and a direct computation, we have $o(x_{i_1}) = o(a) = p^{t+1}$, $o(y_{i_1}) = o(b) = p^t$ and it is direct to check that x_{i_1} and y_{i_1} have the same relations as do a and b, where $i_1 \in \{1, 2, 3, 4\}$. Moreover, for any $i_1 \in \{1, 2, 3, 4\}$, we have $\langle x_{i_1}, y_{i_1} \rangle = H$. It follows that each of the above four maps induces an automorphism of H.

Set $P = \langle \sigma, \gamma, \delta, \tau \rangle$. By a direct computation, we have $o(\gamma) = o(\delta) = p^t$, $o(\sigma) = p^t$ and $o(\tau) = p^{t+1}$. Moreover, we have $\gamma\delta = \delta\gamma$, $\delta^{-1}\sigma\delta = \sigma^{p+1}$ and $\gamma^{-1}\sigma\gamma = \sigma^\ell$ with $\ell(p + 1) \equiv 1 \pmod{p^{t+1}}$. As both γ and δ fixes the subgroup $\langle a \rangle$ while σ does not, one has

$$\langle \sigma, \gamma, \delta \rangle = \langle \sigma \rangle \rtimes (\langle \gamma \rangle \times \langle \delta \rangle) \cong \mathbb{Z}_{p^t} \rtimes (\mathbb{Z}_{p^t} \times \mathbb{Z}_{p^t}).$$

Observing that $\langle \sigma, \gamma, \delta \rangle$ fixes the subgroup $\langle b \rangle$ setwise but τ does not, it follows that $\langle \sigma, \gamma, \delta \rangle \cap \langle \tau \rangle = 1$, and hence $|P| \geq p^{4t+1}$. In view of [13, Theorem 2.8], $\mathrm{Aut}(H)$ has a normal Sylow p-subgroup of order p^{4t+1}. It follows that $P = \langle \sigma, \gamma, \delta, \tau \rangle$ is the unique Sylow p-subgroup of $\mathrm{Aut}(H)$. In particular, we have $P = \langle \gamma \rangle \langle \delta \rangle \langle \sigma \rangle \langle \tau \rangle$.

Recall that $S = \{1, h, hh^\alpha, \ldots, hh^\alpha \cdots h^{\alpha^{p-2}}\}$ and $o(\alpha) = p$. Assume that $h = b^u a^v$ for some $u \in \mathbb{Z}_{p^{t+1}}$ and $v \in \mathbb{Z}_{p^{t+1}}$. Since $H = \langle S \rangle$, we obtain that $o(h) = \exp(H)$. It follows that $(u, p) = 1$. Then there exists $u' \in \mathbb{Z}^*_{p^{t+1}}$ such that $u \equiv u'v \pmod{p^{t+1}}$. Let $\varphi = \sigma^{u'}(\delta^u)^{-1}(\tau^v)^{-1}$. Then $\varphi \in \mathrm{Aut}(H)$ and $h^\varphi = b$. By Proposition 2.4(3), we have $\Gamma \cong \mathrm{BiCay}(H, \emptyset, \emptyset, S^\varphi)$. Let $\Gamma' = \mathrm{BiCay}(H, \emptyset, \emptyset, S^\varphi)$ and $\beta = \varphi^{-1}\alpha\varphi$. Then $\sigma_{\beta,b} \in \mathrm{Aut}(\Gamma')$ cyclically permutates the elements in $\Gamma'(1_0)$. It follows that $bb^\beta b^{\beta^2} \cdots b^{\beta^{p-1}} = 1$ and

$$S^\varphi = \{1, b, bb^\beta, bb^\beta b^{\beta^2}, \ldots, bb^\beta b^{\beta^2} \cdots b^{\beta^{p-2}}\}.$$

Since $o(\beta) = o(\alpha) = p$, we have $\beta \in P$. Assume that $\beta = \gamma^i\delta^j\sigma^k\tau^l$ for some $i, j, k \in \mathbb{Z}_{p^t}$ and $l \in \mathbb{Z}_{p^{t+1}}$. Then by Lemma 3.2(2) – (3) and Proposition 3.1(1), we have

$$\beta\colon \begin{cases} a \mapsto (ba^l)^{(1+p)^i kp} a^{(1+p)^i} = b^{(1+p)^i kp} a^{(1+p)^i(1+klp)} \\ b \mapsto (ba^l)^{(1+p)^j} = b^{(1+p)^j} a^{(1+p)^j l} \end{cases} \tag{4.7}$$

and then

$$\beta\colon \begin{cases} a \mapsto a \cdot w \\ b \mapsto ba^l \cdot w' \end{cases} \tag{4.8}$$

for some $w, w' \in \mho_1(H)$. Since $\Gamma' \cong \Gamma$ is connected, we derive from Proposition 2.3 that $H = \langle S^\varphi \rangle$. By Proposition 3.1(1), it follows that $(l, p) = 1$. We shall finish the proof by the following steps.

Step 1: $t > 1$.

Suppose to the contrary that $t = 1$. Then $H = \langle a, b \mid a^{p^2} = b^{p^2} = 1, b^{-1}ab = a^{1+p}\rangle$. We shall first show that for any $r \geq 1$,

$$b^{\beta^r} = b^{1+(rj+\frac{1}{2}r(r-1)kl)p}a^{rl+\frac{1}{2}r(r+1)jlp+\frac{1}{2}r(r-1)(i+kl)lp+\frac{1}{6}r(r-1)(r-2)kl^2p}. \tag{4.9}$$

By Equation (4.7), we have

$$\beta\colon \begin{cases} a \mapsto b^{kp}a^{1+(i+kl)p} \\ b \mapsto b^{1+jp}a^{l+jlp}. \end{cases}$$

Thus Equation (4.9) holds when $r = 1$. Now assume that $r > 1$ and

$$b^{\beta^{r-1}} = b^{1+((r-1)j+\frac{1}{2}(r-1)(r-2)kl)p} \cdot a^{(r-1)l+\frac{1}{2}(r-1)rjlp+\frac{1}{2}(r-1)(r-2)(i+kl)lp+\frac{1}{6}(r-1)(r-2)(r-3)kl^2p}.$$

By a direct computation, we have

$$\begin{aligned} b^{\beta^r} &= (b^{1+jp}a^{l+jlp})^{1+((r-1)j+\frac{1}{2}(r-1)(r-2)kl)p} \\ &\quad\cdot (b^{kp}a^{1+(i+kl)p})^{(r-1)l+\frac{1}{2}(r-1)rjlp+\frac{1}{2}(r-1)(r-2)(i+kl)lp+\frac{1}{6}(r-1)(r-2)(r-3)kl^2p} \\ &= b^{1+(rj+\frac{1}{2}(r-1)(r-2)kl+(r-1)kl)p} \cdot a^{l+jlp+[(r-1)lj+\frac{1}{2}(r-1)(r-2)kl^2]p} \\ &\quad\cdot a^{(r-1)l(1+(i+kl)p)+\frac{1}{2}(r-1)rjlp+\frac{1}{2}(r-1)(r-2)(i+kl)lp+\frac{1}{6}(r-1)(r-2)(r-3)kl^2p} \\ &= b^{1+rjp+[\frac{1}{2}(r-1)(r-2)+(r-1)]klp} \\ &\quad\cdot a^{[l+(r-1)l]+[1+(r-1)+\frac{1}{2}(r-1)r]jlp+\frac{1}{2}r(r-1)(i+kl)lp+[\frac{1}{2}+\frac{1}{6}(r-3)](r-1)(r-2)kl^2p} \\ &= b^{1+(rj+\frac{1}{2}r(r-1)kl)p}a^{rl+\frac{1}{2}r(r+1)jlp+\frac{1}{2}r(r-1)(i+kl)lp+\frac{1}{6}r(r-1)(r-2)kl^2p}. \end{aligned}$$

By induction, we have Equation (4.9). Then by Equation (4.9), we have

$$b^{\beta^p} = b^{1+(pj+\frac{1}{2}p(p-1)kl)p}a^{pl+\frac{1}{2}p(p+1)jlp+\frac{1}{2}p(p-1)(i+kl)lp+\frac{1}{6}p(p-1)(p-2)kl^2p} = ba^{pl} \neq b,$$

a contradiction.

Step 2: A final contradiction.

Let $\mho_2(H) = \{x^{p^2} \mid x \in H\}$. Then $\mho_2(H) \leq Z(H)$. By Equation (4.7), we have

$$\begin{aligned} a^\beta &= b^{kp}a^{(i+kl)p+1} \cdot \varpi' \\ b^\beta &= b^{jp+1}a^{jlp+l} \cdot \varpi \end{aligned}$$

for some $\varpi, \varpi' \in \mho_2(H)$. Let $f \equiv i + kl \pmod p$, $n \equiv j \pmod p$, $m \equiv jl \pmod p$ for some $m, n, f \in \mathbb{Z}_p$. Then

$$\beta\colon \begin{cases} a \mapsto b^{kp}a^{fp+1} \cdot \varpi_1' \\ b \mapsto b^{np+1}a^{mp+l} \cdot \varpi_1 \end{cases} \tag{4.10}$$

for some $\varpi_1, \varpi_1' \in \mho_2(H)$.

We shall first prove the following claim.

Claim. *For any* $r \geq 1$, $b^{\beta^r} = b^{rnp+\frac{r(r-1)}{2}klp+1}a^{rmp+\frac{r(r-1)}{2}(n+f)lp+\frac{r(r-1)(r-2)}{6}kl^2p+rl} \cdot \varpi_r$ *with* $\varpi_r \in \mho_2(H)$.

If $r = 1$, then by Equation (4.10), Claim is clearly true. Now assume that $r > 1$ and Claim holds for any positive integer less than r. Then

$$b^{\beta^{r-1}} = b^{(r-1)np+\frac{(r-1)(r-2)}{2}klp+1} \cdot a^{(r-1)mp+\frac{(r-1)(r-2)}{2}(n+f)lp+\frac{(r-1)(r-2)(r-3)}{6}kl^2p+(r-1)l} \cdot \varpi_{r-1},$$

for some $\varpi_{r-1} \in \mho_2(H)$. Since $t > 1$, for any positive integer i_0, by Lemma 3.2(1)&(3), we have

$$ab^{i_0} = b^{i_0}a^{(1+p^t)^{i_0}} = b^{i_0}a^{1+i_0p^t} = b^{i_0}a \cdot \varpi_0, \tag{4.11}$$

for some $\varpi_0 \in \mho_2(H)$. Then by Equations (4.10) and (4.11), we have

$$\begin{aligned} b^{\beta^r} &= (b^{np+1}a^{mp+l} \cdot \varpi_1)^{(r-1)np+\frac{(r-1)(r-2)}{2}klp+1} \\ &\quad \cdot (b^{kp}a^{fp+1} \cdot \varpi_1')^{(r-1)mp+\frac{(r-1)(r-2)}{2}(n+f)lp+\frac{(r-1)(r-2)(r-3)}{6}kl^2p+(r-1)l} \cdot \varpi_{r-1}^{\beta} \\ &= b^{(r-1)np+\frac{(r-1)(r-2)}{2}klp+np+1+k(r-1)lp} \cdot \varpi_r \cdot a^{(r-1)nlp+\frac{(r-1)(r-2)}{2}kl^2p} \\ &\quad \cdot a^{mp+l+(r-1)mp+\frac{(r-1)(r-2)}{2}(n+f)lp+\frac{(r-1)(r-2)(r-3)}{6}kl^2p+(r-1)l+(r-1)flp} \\ &= b^{rnp+\frac{r(r-1)}{2}klp+1} \cdot a^{rmp+\frac{r(r-1)}{2}(n+f)lp+\frac{r(r-1)(r-2)}{6}kl^2p+rl} \cdot \varpi_r. \end{aligned}$$

for some $\varpi_r \in \mho_2(H)$. By induction, we complete the proof of Claim.

Now by our Claim and $o(\beta) = p$, we have

$$b^{\beta^p} = b^{np^2+\frac{(p-1)}{2}klp^2+1} \cdot a^{mp^2+\frac{(p-1)}{2}(n+f)lp^2+\frac{(p-1)(p-2)}{6}kl^2p^2+pl} \cdot \varpi_p = b$$

for some $\varpi_p \in \mho_2(H)$. It follows that $pl \equiv 0 \pmod{p^2}$, a contradiction. This completes the proof of our lemma. □

5 Proof of Theorem 1.3

We first prove a lemma.

Lemma 5.1. *Let p be an odd prime, and let H be a metacyclic p-group. If Γ is a connected edge-transitive bi-Cayley graph over H of valency p, then H is either abelian or inner-abelian.*

Proof. We may assume that H is non-abelian. By Proposition 3.1, the group H has the following presentation:

$$H = \left\langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, a^b = a^{1+p^r} \right\rangle,$$

where r, s, t, u are non-negative integers with $u \leq r$ and $r \geq 1$.

Let $\Gamma = \mathrm{BiCay}(H, \mathcal{R}, \mathcal{L}, S)$ be a connected edge-transitive bi-p-Cayley graph over H of valency p. Let $A = \mathrm{Aut}(\Gamma)$, and let P be a Sylow p-subgroup of A such that

$R(H) \leq P$. From the proof of Lemma 4.4(1), we see that P is transitive on the edges of Γ. Since $H' = \langle a^{p^r} \rangle \cong \mathbb{Z}_{p^{s+u}}$, we have

$$H/H' = \left\langle \overline{a}, \overline{b} \mid \overline{a}^{p^r} = \overline{b}^{p^{r+s+t}} = 1, \overline{a}^{\overline{b}} = \overline{a} \right\rangle \cong \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^{r+s+t}},$$

where $\overline{a} = aH'$ and $\overline{b} = bH'$. By Lemma 4.4(2), we have $s + t = 0$ or 1, and so $(s, t) = (0, 0), (1, 0)$ or $(0, 1)$.

Let $n = 2r+2s+u+t$. We use induction on n. If $n = 1$ or 2, then H is clearly abelian, as desired. Assume $n \geq 3$. Let N be a minimal normal subgroup of P and $N \leq R(H)$. Since H is metacyclic, we have $N \cong \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. Suppose that $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Note that $R(H)' \cong \mathbb{Z}_{p^{s+u}}$. Let Q be the subgroup of $R(H)'$ of order p. Since Q is characteristic in $R(H)'$ and $R(H)'$ is characteristic in $R(H)$, $R(H) \trianglelefteq P$ gives that $Q \trianglelefteq P$. By Lemma 3.2(4), each subgroup of $R(H)$ of order p is contained in N. It follows that $Q < N$, contrary to the minimality of N. Thus $N \cong \mathbb{Z}_p$.

Consider the quotient graph Γ_N of Γ corresponding to the orbits of N. Clearly, N is intransitive on both H_0 and H_1, the two orbits of $R(H)$ on $V(\Gamma)$, and by Propositions 2.1 and 2.2, N is semiregular and Γ_N is a graph of valency p with P/N as an edge-transitive group of automorphisms. Clearly, Γ_N is a bi-Cayley graph over the group $R(H)/N$ of order $2 \cdot p^{n_1}$ with $n_1 < n$. By induction, we have $R(H)/N$ is either abelian or inner-abelian. If $R(H)/N$ is abelian, then $R(H)' \leq N \cong \mathbb{Z}_p$. It follows that $R(H)' = 1$ or $R(H)' \cong \mathbb{Z}_p$, implying that $H \cong R(H)$ is abelian or inner-abelian, as required.

In what follows, we always assume that $R(H)/N$ is inner-abelian, and for any element $h \in H$, we use $\overline{h}$ to denote hN.

By Theorem 4.2, we have $p = 3$. Recall that $(s, t) = (0, 0), (1, 0)$ or $(0, 1)$.

Case 1: $(s, t) = (0, 0)$.

In this case, we have

$$H = \left\langle a, b \mid a^{3^{r+u}} = 1, b^{3^r} = a^{3^r}, a^b = a^{1+3^r} \right\rangle.$$

Let $x = a$ and $y = ba^{-1}$. Since $b^{3^r} = a^{3^r}$, by Proposition 3.1(2), we conclude that $y^{3^r} = (ba^{-1})^{3^r} = 1$ and

$$x^y = a^{ba^{-1}} = (a^b)^{a^{-1}} = (a^{1+3^r})^{a^{-1}} = a^{1+3^r} = x^{1+3^r}.$$

Then

$$R(H) \cong H = \left\langle x, y \mid x^{3^{r+u}} = y^{3^r} = 1, x^y = x^{1+3^r} \right\rangle.$$

Recall that $N \cong \mathbb{Z}_3$ and $N \leq R(H)$. By Lemma 3.2(4), N is one of the following four groups: $\langle x^{3^{r+u-1}} \rangle$, $\langle y^{3^{r-1}} \rangle$, $\langle y^{3^{r-1}} x^{3^{r+u-1}} \rangle$, $\langle y^{3^{r-1}} x^{2 \cdot 3^{r+u-1}} \rangle$.

First suppose that $N \neq \langle x^{3^{r+u-1}} \rangle$. Then $\overline{x}$ has order 3^{r+u}. We shall show that H/N has the following presentation:

$$H/N = \left\langle \overline{x}, \overline{h} \mid \overline{x}^{3^{r+u}} = \overline{h}^{3^{r-1}} = \overline{1}, \overline{x}^{\overline{h}} = \overline{x}^{1+3^r} \right\rangle.$$

Actually, if $N = \langle y^{3^{r-1}} \rangle$, then we may take $h = y$. If $N = \langle y^{3^{r-1}} x^{3^{r+u-1}} \rangle$, then take $h = yx^{3^u}$, and then by Lemma 3.2(2) – (3), we have

$$\begin{aligned}(yx^{3^u})^{3^{r-1}} &= y^{3^{r-1}} x^{3^u[1+(1+3^r)+(1+3^r)^2+\cdots+(1+3^r)^{3^{r-1}-1}]}\\ &= y^{3^{r-1}} x^{3^u[1+(1+3^r)+(1+2\cdot 3^r)+\cdots+(1+(3^{r-1}-1)\cdot 3^r)]}\\ &= y^{3^{r-1}} x^{3^u \cdot 3^{r-1}}\\ &= y^{3^{r-1}} x^{3^{u+r-1}} \in N.\end{aligned}$$

If $N = \langle y^{3^{r-1}} x^{2\cdot 3^{r+u-1}} \rangle$, then take $h = yx^{2\cdot 3^u}$, and then by Lemma 3.2(2) – (3), we have

$$\begin{aligned}(yx^{2\cdot 3^u})^{3^{r-1}} &= y^{3^{r-1}} x^{2\cdot 3^u[1+(1+3^r)+(1+3^r)^2+\cdots+(1+3^r)^{3^{r-1}-1}]}\\ &= y^{3^{r-1}} x^{2\cdot 3^u[1+(1+3^r)+(1+2\cdot 3^r)+\cdots+(1+(3^{r-1}-1)\cdot 3^r)]}\\ &= y^{3^{r-1}} x^{2\cdot 3^u \cdot 3^{r-1}}\\ &= y^{3^{r-1}} x^{2\cdot 3^{u+r-1}} \in N.\end{aligned}$$

Clearly, in each case, we have $\overline{x}^{\overline{h}} = \overline{x}^{1+3^r}$. So H/N always has the above presentation. Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 1$. However, by Proposition 4.1, there is no cubic edge-transitive bi-Cayley graph over $R(H)/N$, a contradiction.

Suppose now that $N = \langle x^{3^{r+u-1}} \rangle$. Then

$$H/N = \left\langle \overline{x}, \overline{y} \mid \overline{x}^{3^{r+u-1}} = \overline{y}^{3^r} = \overline{1}, \overline{x}^{\overline{y}} = \overline{x}^{1+3^r} \right\rangle,$$

Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 2$. Then

$$H = \left\langle x, y \mid x^{3^{r+2}} = y^{3^r} = 1, x^y = x^{1+3^r} \right\rangle,$$

where $r \geq 1$.

If $r = 1$, then by MAGMA [5], there is no cubic edge-transitive bi-Cayley graph over H, a contradiction. If $r \geq 2$, then by Lemma 4.4(1), we have $\mathcal{R} = \mathcal{L} = \emptyset$. Assume that $S = \{1, g, h\}$. Since Γ is connected, by Proposition 2.3(1), we have $H = \langle S \rangle = \langle g, h \rangle$. It follows that $o(g) = o(h) = \exp(H) = 3^{r+2}$, and so $H' = \langle x^{3^r} \rangle = \langle g^{3^r} \rangle = \langle h^{3^r} \rangle$. Moreover, by Lemma 4.4(1), there exists $\alpha \in \mathrm{Aut}(H)$ such that $g^\alpha = g^{-1}h$, $h^\alpha = g^{-1}$ and $o(\alpha) \mid 3$. Suppose that α is trivial. Then $h = g^{-1}$, and then $H = \langle g \rangle$, a contradiction. Thus, α has order 3. Assume that $(g^{3^r})^\alpha = g^{\lambda\cdot 3^r}$ for some $\lambda \in \mathbb{Z}_9^*$. Then $(h^{3^r})^\alpha = h^{\lambda\cdot 3^r}$. Since $g^\alpha = g^{-1}h$ and $h^\alpha = g^{-1}$, we have $g^{\lambda\cdot 3^r} = g^{-3^r}h^{3^r}$ and $h^{\lambda\cdot 3^r} = g^{-3^r}$. Then

$$g^{\lambda^2\cdot 3^r} = (g^{\lambda\cdot 3^r})^\lambda = (g^{-3^r}h^{3^r})^\lambda = g^{-\lambda\cdot 3^r}h^{\lambda\cdot 3^r} = g^{-\lambda\cdot 3^r}g^{-3^r} = g^{(-\lambda-1)\cdot 3^r}.$$

It follows that $g^{(\lambda^2+\lambda+1)\cdot 3^r} = 1$, and so $9 \mid \lambda^2 + \lambda + 1$, a contradiction.

Case 2: $(s, t) = (1, 0)$.

In this case, we have

$$H = \left\langle a, b \mid a^{3^{r+u+1}} = 1, b^{3^{r+1}} = a^{3^{r+1}}, a^b = a^{1+3^r} \right\rangle.$$

Let $x = a$ and $y = ba^{-1}$. Since $b^{3^{r+1}} = a^{3^{r+1}}$, by Proposition 3.1(2), we obtain that $y^{3^{r+1}} = (ba^{-1})^{3^{r+1}} = 1$ and

$$x^y = a^{ba^{-1}} = (a^b)^{a^{-1}} = (a^{1+3^r})^{a^{-1}} = a^{1+3^r} = x^{1+3^r}.$$

Then

$$R(H) \cong H = \left\langle x, y \mid x^{3^{r+u+1}} = y^{3^{r+1}} = 1, x^y = x^{1+3^r} \right\rangle,$$

Recall that $N \cong \mathbb{Z}_3$ and $N \leq R(H)$. By Lemma 3.2(4), N is one of the following four groups: $\langle x^{3^{r+u}} \rangle$, $\langle y^{3^r} \rangle$, $\langle y^{3^r} x^{3^{r+u}} \rangle$, $\langle y^{3^r} x^{2\cdot 3^{r+u}} \rangle$.

Suppose first that $N \neq \langle x^{3^{r+u}} \rangle$. Then $\overline{x}$ has order 3^{r+u+1}. We shall show that H/N has the following presentation:

$$H/N = \left\langle \overline{x}, \overline{h} \mid \overline{x}^{3^{r+u+1}} = \overline{h}^{3^r} = \overline{1}, \overline{x}^{\overline{h}} = \overline{x}^{1+3^r} \right\rangle.$$

Actually, if $N = \langle y^{3^r} \rangle$, then we may take $h = y$. If $N = \langle y^{3^r} x^{3^{r+u}} \rangle$, then take $h = yx^{3^u}$, and then by Lemma 3.2(2) – (3), we have

$$\begin{aligned}
(yx^{3^u})^{3^r} &= y^{3^r} x^{3^u[1+(1+3^r)+(1+3^r)^2+\cdots+(1+3^r)^{3^r-1}]} \\
&= y^{3^r} x^{3^u[1+(1+3^r)+(1+2\cdot 3^r)+\cdots+(1+(3^r-1)\cdot 3^r)]} \\
&= y^{3^r} x^{3^u[3^r+\frac{3^r\cdot(3^r-1)}{2}\cdot 3^r]} \\
&= y^{3^r} x^{3^{u+r}} \in N.
\end{aligned}$$

If $N = \langle y^{3^r} x^{2\cdot 3^{r+u}} \rangle$, then take $h = yx^{2\cdot 3^u}$, and then by Lemma 3.2(2) – (3), we have

$$\begin{aligned}
(yx^{2\cdot 3^u})^{3^r} &= y^{3^r} x^{2\cdot 3^u[1+(1+3^r)+(1+3^r)^2+\cdots+(1+3^r)^{3^r-1}]} \\
&= y^{3^r} x^{2\cdot 3^u[1+(1+3^r)+(1+2\cdot 3^r)+\cdots+(1+(3^r-1)\cdot 3^r)]} \\
&= y^{3^r} x^{2\cdot 3^u[3^r+\frac{3^r\cdot(3^r-1)}{2}\cdot 3^r]} \\
&= y^{3^r} x^{2\cdot 3^{u+r}} \in N.
\end{aligned}$$

Clearly, in each case, we have $\overline{x}^{\overline{h}} = \overline{x}^{1+3^r}$. So H/N always has the above presentation. Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 0$. Then

$$H = \left\langle x, y \mid x^{3^{r+1}} = y^{3^{r+1}} = 1, x^y = x^{1+3^r} \right\rangle,$$

where $r \geq 1$. By [20] or [3, Lemma 65.2], H is inner-abelian, as required.

Suppose now $N = \langle x^{3^{r+u}} \rangle$. Then

$$R(H)/N = \left\langle \overline{x}, \overline{y} \mid \overline{x}^{3^{r+u}} = \overline{y}^{3^{r+1}} = \overline{1}, \overline{x}^{\overline{y}} = \overline{x}^{1+3^r} \right\rangle.$$

Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 1$. Then

$$H = \left\langle x, y \mid x^{3^{r+2}} = y^{3^{r+1}} = 1, x^y = x^{1+3^r} \right\rangle,$$

where $r \geq 1$.

If $r = 1$, then by MAGMA [5], there is no cubic edge-transitive bi-Cayley graph over H, a contradiction. If $r \geq 2$, then by Lemma 4.4(1), we have $\mathcal{R} = \mathcal{L} = \emptyset$. Assume that $S = \{1, g, h\}$. Since Γ is connected, by Proposition 2.3(1), we have $H = \langle S \rangle = \langle g, h \rangle$. It follows that $o(g) = o(h) = \exp(H) = 3^{r+2}$. By Lemma 4.4(1), there exists $\alpha \in \mathrm{Aut}(H)$ such that $g^\alpha = g^{-1}h$, $h^\alpha = g^{-1}$ and $o(\alpha) \mid 3$. Suppose that α is trivial. Then $h = g^{-1}$, and then $H = \langle g \rangle$, a contradiction. Thus, α has order 3. Note that

$$\Omega_r(H) = \left\langle z^{3^r} \mid z \in H \right\rangle = \left\langle x^{3^r} \right\rangle \times \left\langle y^{3^r} \right\rangle \cong \mathbb{Z}_9 \times \mathbb{Z}_3$$

and $g^{3^r}, h^{3^r} \in \Omega_r(H)$.

If $\langle g^{3^r} \rangle = \langle h^{3^r} \rangle$, then we may assume that $(g^{3^r})^\alpha = g^{\lambda \cdot 3^r}$ for some $\lambda \in \mathbb{Z}_9^*$. Then $(h^{3^r})^\alpha = h^{\lambda \cdot 3^r}$. Since $g^\alpha = g^{-1}h$ and $h^\alpha = g^{-1}$, we have $g^{\lambda \cdot 3^r} = g^{-3^r}h^{3^r}$ and $h^{\lambda \cdot 3^r} = g^{-3^r}$. Then

$$g^{\lambda^2 \cdot 3^r} = (g^{\lambda \cdot 3^r})^\lambda = (g^{-3^r}h^{3^r})^\lambda = g^{-\lambda \cdot 3^r}h^{\lambda \cdot 3^r} = g^{-\lambda \cdot 3^r}g^{-3^r} = g^{(-\lambda-1)\cdot 3^r}.$$

It follows that $g^{(\lambda^2+\lambda+1)\cdot 3^r} = 1$, and so $9 \mid \lambda^2 + \lambda + 1$, a contradiction.

Suppose $\langle g^{3^r} \rangle \neq \langle h^{3^r} \rangle$. Then $\Omega_r(H) = \langle g^{3^r}, h^{3^r} \rangle$ and $H' = \langle x^{3^r} \rangle \cong \mathbb{Z}_9$. Assume that $x^{3^r} = g^{i \cdot 3^r}h^{j \cdot 3^r}$ for some $i, j \in \mathbb{Z}_9$. Then either $(i, 3) = 1$ or $(j, 3) = 1$. Since $H' = \langle x^{3^r} \rangle$, we have $\langle x^{3^r} \rangle^\alpha = \langle x^{3^r} \rangle$. So $(g^{i \cdot 3^r}h^{j \cdot 3^r})^\alpha = (g^{i \cdot 3^r}h^{j \cdot 3^r})^k$ for some $k \in \mathbb{Z}_9$. Then

$$g^{ik \cdot 3^r}h^{jk \cdot 3^r} = (g^{i \cdot 3^r}h^{j \cdot 3^r})^\alpha = (g^\alpha)^{i \cdot 3^r}(h^\alpha)^{j \cdot 3^r} = g^{-i \cdot 3^r}h^{i \cdot 3^r}g^{-j \cdot 3^r} = g^{-(i+j)\cdot 3^r}h^{i \cdot 3^r}.$$

It follows that $-(i + j) \equiv ik \pmod 9$ and $i \equiv jk \pmod 9$. Then $-(jk + j) \equiv jk^2 \pmod 9$, and so $j(1+k+k^2) \equiv 0 \pmod 9$, forcing that $3 \mid j$. Furthermore, since $i \equiv jk \pmod 9$, we have $3 \mid i$, a contradiction.

Case 3: $(s, t) = (0, 1)$.

In this case, we have

$$H = \left\langle a, b \mid a^{3^{r+u}} = 1, b^{3^{r+1}} = a^{3^r}, a^b = a^{1+3^r} \right\rangle.$$

Let $x = b, y = b^3a^{-1}$. Since $a^b = a^{1+3^r}$, we have $b^{-1}aba^{-1} = a^{3^r}$, and then

$$aba^{-1} = ba^{3^r} = bb^{3^{r+1}} = b^{1+3^{r+1}}.$$

Since $b^{3^{r+1}} = a^{3^r}$, by Proposition 3.1(2), we have

$$x^{3^{r+u+1}} = b^{3^{r+u+1}} = a^{3^{r+u}} = 1, \qquad y^{3^r} = (b^3a^{-1})^{3^r} = 1,$$
$$x^y = b^{b^3a^{-1}} = (b)^{a^{-1}} = aba^{-1} = b^{1+3^{r+1}} = x^{1+3^{r+1}}.$$

Then

$$R(H) \cong H = \left\langle x, y \mid x^{3^{r+u+1}} = y^{3^r} = 1, x^y = x^{1+3^{r+1}} \right\rangle.$$

Recall that $N \cong \mathbb{Z}_3$ and $N \leq R(H)$. By Lemma 3.2(4), N is one of the following four groups: $\langle x^{3^{r+u}} \rangle$, $\langle y^{3^{r-1}} \rangle$, $\langle y^{3^{r-1}}x^{3^{r+u}} \rangle$, $\langle y^{3^{r-1}}x^{2 \cdot 3^{r+u}} \rangle$.

Suppose first that $N \neq \langle x^{3^{r+u}} \rangle$. Then $\overline{x}$ has order 3^{r+u+1}. We shall show that H/N has the following presentation:

$$H/N = \left\langle \overline{x}, \overline{h} \mid \overline{x}^{3^{r+u+1}} = \overline{h}^{3^{r-1}} = \overline{1}, \overline{x}^{\overline{h}} = \overline{x}^{1+3^{r+1}} \right\rangle.$$

Actually, if $N = \langle y^{3^{r-1}} \rangle$, then we may take $h = y$. If $N = \langle y^{3^{r-1}} x^{3^{r+u}} \rangle$, then take $h = yx^{3^{u+1}}$, and then by Lemma 3.2(2) – (3), we have

$$\begin{aligned}(yx^{3^{u+1}})^{3^{r-1}} &= y^{3^{r-1}} x^{3^{u+1}[1+(1+3^{r+1})+(1+3^{r+1})^2+\cdots+(1+3^{r+1})^{3^{r-1}-1}]} \\ &= y^{3^{r-1}} x^{3^{u+1}[1+(1+3^{r+1})+(1+2\cdot 3^{r+1})+\cdots+(1+(3^{r-1}-1)\cdot 3^{r+1})]} \\ &= y^{3^{r-1}} x^{3^{u+1}[3^{r-1}+\frac{3^{r-1}\cdot(3^{r-1}-1)}{2}\cdot 3^{r+1}]} \\ &= y^{3^{r-1}} x^{3^{u+r}} \in N.\end{aligned}$$

If $N = \langle y^{3^{r-1}} x^{2\cdot 3^{r+u}} \rangle$, then take $h = yx^{2\cdot 3^{u+1}}$, and then by Lemma 3.2(2) – (3), we have

$$\begin{aligned}(yx^{2\cdot 3^{u+1}})^{3^{r-1}} &= y^{3^{r-1}} x^{2\cdot 3^{u+1}[1+(1+3^{r+1})+(1+3^{r+1})^2+\cdots+(1+3^{r+1})^{3^{r-1}-1}]} \\ &= y^{3^{r-1}} x^{2\cdot 3^{u+1}[1+(1+3^{r+1})+(1+2\cdot 3^{r+1})+\cdots+(1+(3^{r-1}-1)\cdot 3^{r+1})]} \\ &= y^{3^{r-1}} x^{2\cdot 3^{u+1}[3^{r-1}+\frac{3^{r-1}\cdot(3^{r-1}-1)}{2}\cdot 3^{r+1}]} \\ &= y^{3^{r-1}} x^{2\cdot 3^{u+r}} \in N.\end{aligned}$$

Clearly, in each case, we have $\overline{x}^{\overline{h}} = \overline{x}^{1+3^r}$. So H/N always has the above presentation. Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 1$. However, by Proposition 4.1, there is no cubic edge-transitive bi-Cayley graph over $R(H)/N$, a contradiction.

Suppose now that $N = \langle x^{3^{r+u}} \rangle$. Then

$$R(H)/N = \left\langle \overline{x}, \overline{y} \mid \overline{x}^{3^{r+u}} = \overline{y}^{3^r} = \overline{1}, \overline{x}^{\overline{y}} = \overline{x}^{1+3^{r+1}} \right\rangle.$$

Since $R(H)/N$ is inner-abelian, by [20] or [3, Lemma 65.2], we have $u = 2$. However, by Proposition 4.1, there is no cubic edge-transitive bi-Cayley graph over $R(H)/N$, a contradiction. □

Now we are ready to finish the proof of Theorem 1.3.

Proof of Theorem 1.3. By Lemma 5.1, if H is non-abelian, then H is inner-abelian. By Theorem 4.2, we have $p = 3$, and then by Proposition 4.1, Γ is isomorphic to either Γ_r or Σ_r, as desired. □

References

[1] B. Alspach and T. D. Parsons, A construction for vertex-transitive graphs, *Canad. J. Math.* **34** (1982), 307–318, doi:10.4153/cjm-1982-020-8.

[2] Y. Berkovich, *Groups of Prime Power Order, Volume 1*, volume 46 of *De Gruyter Expositions in Mathematics*, De Gruyter, Kammergericht, Berlin, 2008, doi:10.1515/9783110208221.

[3] Y. Berkovich and Z. Janko, *Groups of Prime Power Order, Volume 2*, volume 47 of *De Gruyter Expositions in Mathematics*, De Gruyter, Kammergericht, Berlin, 2008, doi:10.1515/9783110208238.

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier Publishing Co., New York, 1976.

[5] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, *J. Symbolic Comput.* **24** (1997), 235–265, doi:10.1006/jsco.1996.0125.

[6] I. Z. Bouwer, An edge but not vertex transitive cubic graph, *Canad. Math. Bull.* **11** (1968), 533–535, doi:10.4153/cmb-1968-063-0.

[7] M. Conder, J.-X. Zhou, Y.-Q. Feng and M.-M. Zhang, Edge-transitive bi-Cayley graphs, 2016, `arXiv:1606.04625 [math.CO]`.

[8] Y.-Q. Feng, J. H. Kwak and M.-Y. Xu, Cubic s-regular graphs of order $2p^3$, *J. Graph Theory* **52** (2006), 341–352, doi:10.1002/jgt.20169.

[9] B. Huppert, *Endliche Gruppen I*, volume 134 of *Die Grundlehren der Mathematischen Wissenschaften*, Springer-Verlag, Berlin, 1967.

[10] H. Koike and I. Kovács, Arc-transitive cubic abelian bi-Cayley graphs and BCI-graphs, *Filomat* **30** (2016), 321–331, doi:10.2298/fil1602321k.

[11] C. H. Li, J. Pan, S. J. Song and D. Wang, A characterization of a family of edge-transitive metacirculant graphs, *J. Comb. Theory Ser. B* **107** (2014), 12–25, doi:10.1016/j.jctb.2014.02.002.

[12] C. H. Li and H.-S. Sim, Automorphisms of Cayley graphs of metacyclic groups of prime-power order, *J. Aust. Math. Soc.* **71** (2001), 223–231, doi:10.1017/s144678870000286x.

[13] C. H. Li and H.-S. Sim, On half-transitive metacirculant graphs of prime-power order, *J. Comb. Theory Ser. B* **81** (2001), 45–57, doi:10.1006/jctb.2000.1992.

[14] C. H. Li, S. J. Song and D. J. Wang, A characterization of metacirculants, *J. Comb. Theory Ser. A* **120** (2013), 39–48, doi:10.1016/j.jcta.2012.06.010.

[15] P. Lorimer, Vertex-transitive graphs: symmetric graphs of prime valency, *J. Graph Theory* **8** (1984), 55–68, doi:10.1002/jgt.3190080107.

[16] Z. Lu, C. Wang and M. Xu, On semisymmetric cubic graphs of order $6p^2$, *Sci. China Ser. A* **47** (2004), 1–17, doi:10.1360/02ys0241.

[17] A. Malnič, D. Marušič and C. Wang, Cubic edge-transitive graphs of order $2p^3$, *Discrete Math.* **274** (2004), 187–198, doi:10.1016/s0012-365x(03)00088-8.

[18] D. Marušič and P. Šparl, On quartic half-arc-transitive metacirculants, *J. Algebraic Combin.* **28** (2008), 365–395, doi:10.1007/s10801-007-0107-y.

[19] Y.-L. Qin and J.-X. Zhou, Cubic edge-transitive bi-p-metacirculant, *Electron. J. Combin.* **25** (2018), #P3.28, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v25i3p28`.

[20] L. Rédei, Das "schiefe Produkt" in der Gruppentheorie mit Anwendung auf die endlichen nichtkommutativen Gruppen mit lauter kommutativen echten Untergruppen und die Ordnungszahlen, zu denen nur kommutative Gruppen gehören, *Comment. Math. Helv.* **20** (1947), 225–264, doi:10.1007/bf02568131.

[21] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964, translated from the German by R. Bercov.

[22] M. Xu and Q. Zhang, A classification of metacyclic 2-groups, *Algebra Colloq.* **13** (2006), 25–34, doi:10.1142/s1005386706000058.

[23] J.-X. Zhou and Y.-Q. Feng, Cubic bi-Cayley graphs over abelian groups, *European J. Combin.* **36** (2014), 679–693, doi:10.1016/j.ejc.2013.10.005.

[24] J.-X. Zhou and Y.-Q. Feng, The automorphisms of bi-Cayley graphs, *J. Comb. Theory Ser. B* **116** (2016), 504–532, doi:10.1016/j.jctb.2015.10.004.

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Comparing the expected number of random elements from the symmetric and the alternating groups needed to generate a transitive subgroup

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Abstract

Given a transitive permutation group of degree n, we denote by $e_{\mathcal{T}}(G)$ the expected number of elements of G which have to be drawn at random, with replacement, before a set of generators of a transitive subgroup of G is found. We compare $e_{\mathcal{T}}(\mathrm{Sym}(n))$ and $e_{\mathcal{T}}(\mathrm{Alt}(n))$.

Keywords: Transitive groups, generation, expectation.

Math. Subj. Class.: 20B30, 20P05

1 Introduction

Let $n \in \mathbb{N}$ and suppose that we are in the following situation. There are two boxes, one is blue and one is red. The balls in the blue box correspond to the elements of $\mathrm{Sym}(n)$, the balls in the red box correspond to the elements of $\mathrm{Alt}(n)$. We choose one of the boxes, and then we extract balls from the chosen box, with replacement, until a transitive permutation group of degree n is generated. In order to minimize the number of extractions, is it better to choose the red box or the blue one? We are going to prove that the answer depends on the parity of n. If n is even the best choice is the blue box, if n is odd the red one.

In order to formulate and discuss this problem in an appropriate way, we need to introduce some definitions. Let G be a transitive permutation group of degree n and $x = (x_m)_{m\in\mathbb{N}}$ be a sequence of independent, uniformly distributed G-valued random variables. We may define a random variable τ_G by setting

$$\tau_G = \min\{t \geq 1 \mid \langle x_1, \ldots, x_t\rangle \text{ is a transitive subgroup of } \mathrm{Sym}(n)\} \in [1, +\infty].$$

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We denote with $e_{\mathcal{T}}(G) = \sum_{t \geq 1} tP(\tau_G = t)$ the expectation of the random variable τ_G. Thus $e_{\mathcal{T}}(G)$ is the expected number of elements of G which have to be drawn at random, with replacement, before a set of generators of a transitive subgroup of G is found.

The case when $G = \mathrm{Sym}(n)$ has been studied in [2, Section 5]. Denote by Π_n the set of partitions of n, i.e. nondecreasing sequences of natural numbers whose sum is n. Given $\omega = (n_1, \dots, n_k) \in \Pi_n$ with

$$n_1 = \cdots = n_{k_1} > n_{k_1+1} = \cdots = n_{k_1+k_2} > \cdots > n_{k_1+\cdots+k_{r-1}+1} = \cdots = n_{k_1+\cdots+k_r}$$

define

$$\begin{aligned} \mu(\omega) &= (-1)^{k-1}(k-1)!, \\ \iota(\omega) &= \frac{n!}{n_1! n_2! \cdots n_k!}, \\ \nu(\omega) &= k_1! k_2! \cdots k_r!. \end{aligned}$$

It turns out (see [2, Theorem 9]) that for every $n \geq 2$,

$$e_{\mathcal{T}}(\mathrm{Sym}(n)) = -\sum_{\omega \in \Pi_n^*} \frac{\mu(\omega)\iota(\omega)^2}{\nu(\omega)(\iota(\omega) - 1)},$$

where Π_n^* is the set of partitions of n into at least two subsets. The aim of this paper is to consider the case $G = \mathrm{Alt}(n)$. Our main result is the following.

Theorem 1.1. *For every natural number $n \geq 3$*

$$e_{\mathcal{T}}(\mathrm{Sym}(n)) - e_{\mathcal{T}}(\mathrm{Alt}(n)) = \frac{(-1)^{n+1} n!(n-1)!}{(n! - 1)(n! - 2)}.$$

So the difference $e_{\mathcal{T}}(\mathrm{Sym}(n)) - e_{\mathcal{T}}(\mathrm{Alt}(n))$ tends to zero when n tends to infinity, but it is positive if n is odd and negative otherwise. To explain this behaviour notice that, if $G \leq \mathrm{Sym}(n)$, then $P(\tau_G = 1)$ concides with the probability $P_{\mathcal{T}}(G, 1)$ that one randomly chosen element g in G generates a transitive subgroup of $\mathrm{Sym}(n)$, i.e. that g is an n-cycle: in particular $P_{\mathcal{T}}(\mathrm{Sym}(n), 1) = 1/n$, $P_{\mathcal{T}}(\mathrm{Alt}(n), 1) = 2/n$ if n is odd and $P_{\mathcal{T}}(\mathrm{Alt}(n), 1) = 0$ if n is even.

In [2, Section 5], it is proved that $\lim_{n \to \infty} e_{\mathcal{T}}(\mathrm{Sym}(n)) = 2$ and

$$2 = e_{\mathcal{T}}(\mathrm{Sym}(2)) \leq e_{\mathcal{T}}(\mathrm{Sym}(n)) \leq e_{\mathcal{T}}(\mathrm{Sym}(4)) = \frac{7982}{3795} \sim 2.1033.$$

A similar result can be obtained in the alternating case.

Theorem 1.2. *Assume $n \geq 3$.*

1. *If n is odd, then $\frac{3}{2} = e_{\mathcal{T}}(\mathrm{Alt}(3)) \leq e_{\mathcal{T}}(\mathrm{Alt}(n)) < 2$.*
2. *If n is even, then $2 < e_{\mathcal{T}}(\mathrm{Alt}(n)) \leq e_{\mathcal{T}}(\mathrm{Alt}(4)) = \frac{394}{165} \sim 2.3879$.*

Moreover $\lim_{n \to \infty} e_{\mathcal{T}}(\mathrm{Alt}(n)) = 2$.

2 Proof of Theorem 1.1

Let $\Lambda = (X, \leq)$ be a finite poset. Recall that the Möbius function μ_Λ on the poset Λ is the unique function $\mu_\Lambda \colon X \times X \to \mathbb{Z}$, satisfying $\mu(x, y) = 0$ unless $x \leq y$ and the recursion formula

$$\sum_{x \leq y \leq z} \mu_\Lambda(y, z) = \begin{cases} 1 & \text{if } x = z, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a transitive subgroup of $\mathrm{Sym}(n)$. We denote with $P_{\mathcal{T}}(G, t)$ the probability that t randomly chosen elements of G generate a transitive subgroup of $\mathrm{Sym}(n)$. Notice that $\tau_G > t$ if and only if $\langle x_1, \dots, x_t \rangle$ is not a transitive subgroup of G, so we have

$$P(\tau_G > t) = 1 - P_{\mathcal{T}}(G, t).$$

We get that

$$\begin{aligned} e_{\mathcal{T}}(G) &= \sum_{t \geq 1} t P(\tau_G = t) = \sum_{t \geq 1} \left(\sum_{m \geq t} P(\tau_G = m) \right) \\ &= \sum_{t \geq 1} P(\tau_G \geq t) = \sum_{t \geq 0} P(\tau_G > t) = \sum_{t \geq 0} (1 - P_{\mathcal{T}}(G, t)). \end{aligned} \tag{2.1}$$

Consider the poset $\mathcal{X}_G$ of the intransitive subgroups of G, let $\mathcal{I}_G$ be the set of subgroups of G than can be obtained as intersection of maximal elements of the the poset $\mathcal{X}_G$, and let $\mathcal{J}_G = \mathcal{I}_G \cup \{G\}$. From [1, Section 2] we have that

$$P_{\mathcal{T}}(G, t) = \sum_{H \in \mathcal{L}_{\mathcal{T}}(G)} \frac{\mu_{\mathcal{T},G}(H, G)}{|G : H|^t} = \sum_{H \in \mathcal{J}_G} \frac{\mu_{\mathcal{T},G}(H, G)}{|G : H|^t},$$

where $\mu_{\mathcal{T},G}$ denotes the Möbius function on the lattice $\mathcal{L}_{\mathcal{T}}(G) = \mathcal{X}_G \cup \{G\}$. So in order to compute the function $P_{\mathcal{T}}(G, t)$ we need information about the subgroups in $\mathcal{J}_G$. Let $\mathcal{P}_n$ be the poset of partitions of $\{1, \dots, n\}$, ordered by refinement. The maximum $\hat{1}$ of $\mathcal{P}_n$ is $\{\{1, \dots, n\}\}$ (the partition into only one part), while the minimum $\hat{0}$ is $\{\{1\}, \{2\}, \dots, \{n\}\}$ (the partition into n parts of size 1). The orbit lattice of G is defined as

$$\mathcal{P}_n(G) = \{\sigma \in \mathcal{P}_n \mid \text{the orbits of some } H \leq G \text{ are the parts of } \sigma\}.$$

If $\sigma = \{\Omega_1, \dots, \Omega_k\} \in \mathcal{P}_n$, then we define

$$G(\sigma) = (\mathrm{Sym}(\Omega_1) \times \cdots \times \mathrm{Sym}(\Omega_k)) \cap G.$$

If $\sigma \in \mathcal{P}_n(G)$, then $G(\sigma)$ is the maximal element in the lattice of those subgroups of G whose orbits are precisely the parts of σ. Notice that $H \in \mathcal{J}_G$ if and only if there exists $\sigma \in \mathcal{P}_n(G)$ with $H = G(\sigma)$; moreover $\mu_{\mathcal{T},G}(G(\sigma), G) = \mu_{\mathcal{P}_n(G)}(\sigma, \hat{1})$ so

$$P_{\mathcal{T}}(G, t) = \sum_{\sigma \in \mathcal{P}_n(G)} \frac{\mu_{\mathcal{P}_n(G)}(\sigma, \hat{1})}{|G : G(\sigma)|^t}. \tag{2.2}$$

We want now to use (2.2) in order to compute $P_{\mathcal{T}}(\mathrm{Sym}(n), t) - P_{\mathcal{T}}(\mathrm{Alt}(n), t)$. Let $\mathcal{P}_{2,n}$ be the subset of $\mathcal{P}_n$ consisting of the partitions of $\{1, \dots, n\}$ into $n-1$ parts (one of size 2, the others of size 1) and let $\mathcal{P}^*_{2,n} = \mathcal{P}_{2,n} \cup \{\hat{0}\}$. The following two lemmas are immediate but crucial in our computation.

Lemma 2.1. *$\mathcal{P}_n(\mathrm{Sym}(n)) = \mathcal{P}_n$ and $\mathcal{P}_n(\mathrm{Alt}(n)) = \mathcal{P}_n \setminus \mathcal{P}_{2,n}$.*

Lemma 2.2. *If $\sigma \in \mathcal{P}_n \setminus \mathcal{P}^*_{2,n}$, then*

1. $\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\sigma, \hat{1}) = \mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\sigma, \hat{1}) = \mu_{\mathcal{P}_n}(\sigma, \hat{1})$;
2. $|\,\mathrm{Sym}(n) : \mathrm{Sym}(n)(\sigma)| = |\,\mathrm{Alt}(n) : \mathrm{Alt}(n)(\sigma)|$.

Lemma 2.3. *We have*

1. $\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$;
2. $\mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)! + \frac{(-1)^{n-2}n!}{2}$.

Proof. We use the following known result (see for example [3, p. 128]):

$$\mu_{\mathcal{P}_n}(\{\Omega_1, \ldots, \Omega_k\}, \hat{1}) = (-1)^{k-1}(k-1)!. \tag{2.3}$$

This immediately implies $\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\hat{0}, \hat{1}) = \mu_{\mathcal{P}_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$. Moreover

$$\begin{aligned}
\mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\hat{0}, \hat{1}) &= -\sum_{\sigma \in \mathcal{P}_n(\mathrm{Alt}(n)) \setminus \{\hat{0}\}} \mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\sigma, \hat{1}) \\
&= -\sum_{\sigma \in \mathcal{P}_n \setminus \mathcal{P}^*_{2,n}} \mu_{\mathcal{P}_n}(\sigma, \hat{1}) \\
&= -\sum_{\sigma \in \mathcal{P}_n \setminus \{\hat{0}\}} \mu_{\mathcal{P}_n}(\sigma, \hat{1}) + \sum_{\sigma \in \mathcal{P}_{2,n}} \mu_{\mathcal{P}_n}(\sigma, \hat{1}) \\
&= \mu_{\mathcal{P}_n}(\hat{0}, \hat{1}) + \sum_{\sigma \in \mathcal{P}_{2,n}} \mu_{\mathcal{P}_n}(\sigma, \hat{1}) \\
&= (-1)^{n-1}(n-1)! + \binom{n}{n-2}(-1)^{n-2}(n-2)! \\
&= (-1)^{n-1}(n-1)! + \frac{(-1)^{n-2}n!}{2}.
\end{aligned}$$

□

Theorem 2.4. *For every natural number $n \geq 2$*

$$P_{\mathcal{T}}(\mathrm{Sym}(n), t) - P_{\mathcal{T}}(\mathrm{Alt}(n), t) = \frac{(-1)^{n+1}(n-1)!(2^t - 1)}{(n!)^t}.$$

Proof. For every $t \in \mathbb{N}$ (and using (2.3) and Lemma 2.3) let

$$\begin{aligned}
\eta_1(n, t) &= \sum_{\sigma \in \mathcal{P}_{2,n}} \frac{\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\sigma, \hat{1})}{|\,\mathrm{Sym}(n) : \mathrm{Sym}(n)(\sigma)|^t} \\
&= \binom{n}{2} \frac{(-1)^{n-2}(n-2)!2^t}{(n!)^t} = \frac{(-1)^{n-2}(n!)2^t}{2(n!)^t}, \\
\eta_2(n, t) &= \frac{\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\hat{0}, \hat{1})}{|\,\mathrm{Sym}(n) : \mathrm{Sym}(n)(\hat{0})|^t} = \frac{(-1)^{n-1}(n-1)!}{(n!)^t}, \\
\eta_3(n, t) &= \frac{\mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\hat{0}, \hat{1})}{|\,\mathrm{Alt}(n) : \mathrm{Alt}(n)(\hat{0})|^t} = \left((-1)^{n-1}(n-1)! + \frac{(-1)^{n-2}n!}{2}\right)\left(\frac{2}{n!}\right)^t.
\end{aligned}$$

From (2.2), Lemma 2.1 and Lemma 2.2, we deduce that

$$\begin{aligned}
P_{\mathcal{T}}(\mathrm{Sym}(n),t) &= \sum_{\sigma\in\mathcal{P}_n(\mathrm{Sym}(n))} \frac{\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\sigma,\hat{1})}{|\,\mathrm{Sym}(n):\mathrm{Sym}(n)(\sigma)|^t} \\
&= \sum_{\sigma\in\mathcal{P}_n(\mathrm{Alt}(n))} \frac{\mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\sigma,\hat{1})}{|\,\mathrm{Alt}(n):\mathrm{Alt}(n)(\sigma)|^t} \\
&\qquad + \sum_{\sigma\in\mathcal{P}_{2,n}(\mathrm{Sym}(n))} \frac{\mu_{\mathcal{P}_n}(\sigma,\hat{1})}{|\,\mathrm{Sym}(n):\mathrm{Sym}(n)(\sigma)|^t} \\
&\qquad + \frac{\mu_{\mathcal{P}_n(\mathrm{Sym}(n))}(\hat{0},\hat{1})}{|\,\mathrm{Sym}(n):\mathrm{Sym}(n)(\hat{0})|^t} - \frac{\mu_{\mathcal{P}_n(\mathrm{Alt}(n))}(\hat{0},\hat{1})}{|\,\mathrm{Alt}(n):\mathrm{Alt}(n)(\hat{0})|^t} \\
&= P_{\mathcal{T}}(\mathrm{Alt}(n),t) + \eta_1(n,t) + \eta_2(n,t) - \eta_3(n,t) \\
&= P_{\mathcal{T}}(\mathrm{Alt}(n),t) + \frac{(-1)^n(n-1)!(2^t-1)}{(n!)^t}. \qquad \square
\end{aligned}$$

Proof of Theorem 1.1. Using equation (2.1) we obtain that

$$\begin{aligned}
e_{\mathcal{T}}(\mathrm{Sym}(n)) - e_{\mathcal{T}}(\mathrm{Alt}(n)) &= \sum_{t\geq 0} \left(P_{\mathcal{T}}(\mathrm{Alt}(n),t) - P_{\mathcal{T}}(\mathrm{Sym}(n),t)\right) \\
&= \sum_{t\geq 0} \frac{(-1)^{n+1}(n-1)!(2^t-1)}{(n!)^t} \\
&= (-1)^{n+1}(n-1)!\left(\sum_{t\geq 0}\left(\frac{2}{n!}\right)^t - \sum_{t\geq 0}\left(\frac{1}{n!}\right)^t\right) \\
&= (-1)^{n+1}(n-1)!\left(\frac{n!}{n!-2} - \frac{n!}{n!-1}\right) \\
&= \frac{(-1)^{n+1}n!(n-1)!}{(n!-1)(n!-2)}. \qquad \square
\end{aligned}$$

3 Examples

In this section we want to verify Theorem 1.1 in the particular case when $n \in \{3,4\}$ using some direct, elementary arguments to compute $e_{\mathcal{T}}(\mathrm{Sym}(n))$ and $e_{\mathcal{T}}(\mathrm{Alt}(n))$.

First assume $n = 3$. Notice that $\tau_{\mathrm{Alt}(3)}$ is a geometric random variable with parameter $\frac{2}{3}$, so $e_{\mathcal{T}}(\mathrm{Alt}(3)) = \frac{3}{2}$. To generate a transitive subgroup of $\mathrm{Sym}(3)$ first of all we have to search for a nontrivial element of $\mathrm{Sym}(3)$. The numbers of trials needed to obtain a nontrivial element x of $\mathrm{Sym}(3)$ is a geometric random variable of parameter $\frac{5}{6}$: its expectation is equal to $E_0 = \frac{6}{5}$. If this element has order 3, we have already obtained a transitive subgroup. However, with probability $p_1 = \frac{3}{5}$, the nontrivial element x is a transposition: in this case in order to generate a transitive subgroup we need to find an element $y \notin \langle x\rangle$ and the number of trials needed to find $y \notin \langle x\rangle$ is a geometric random variable with parameter $\frac{2}{3}$ and expectation $E_1 = \frac{3}{2}$. Definitely

$$e_{\mathcal{T}}(\mathrm{Sym}(3)) = E_0 + p_1E_1 = \frac{6}{5} + \frac{3}{5}\cdot\frac{3}{2} = \frac{21}{10}.$$

In particular

$$e_{\mathcal{T}}(\mathrm{Sym}(3)) - e_{\mathcal{T}}(\mathrm{Alt}(3)) = \frac{21}{10} - \frac{3}{2} = \frac{3}{5},$$

according with Theorem 1.1.

Now assume $n = 4$. The transitive subgroups of $\mathrm{Alt}(4)$ are the noncyclic subgroups. Thus the subgroup $\langle x_1, \ldots, x_t \rangle$ of $\mathrm{Alt}(4)$ is transitive if and only if there exist $1 \leq i < j \leq t$ such that $x_i \neq 1$ and $x_j \notin \langle x_i \rangle$. The numbers of trials needed to obtain a nontrivial element x of $\mathrm{Alt}(4)$ is a geometric random variable of parameter $\frac{11}{12}$ and expectation $E_0 = \frac{12}{11}$. With probability $p_1 = \frac{3}{11}$ the nontrivial element x has order 2: in this case the number of trials needed to find an element $y \notin \langle x \rangle$ is a geometric random variable of parameter $\frac{10}{12}$ and expectation $E_1 = \frac{12}{10}$. On the other hand, with probability $p_2 = \frac{8}{11}$ the nontrivial element x has order 3: in this second case the number of trials needed to find an element $y \notin \langle x \rangle$ is a geometric random variable of parameter $\frac{9}{12}$ and expectation $E_2 = \frac{12}{9}$. Thus

$$e_{\mathcal{T}}(\mathrm{Alt}(4)) = E_0 + p_1 E_1 + p_2 E_2 = \frac{394}{165}.$$

The case of $\mathrm{Sym}(4)$ is more complicated. To generate a transitive subgroup of $\mathrm{Sym}(4)$ first of all we have to search for a nontrivial element of $\mathrm{Sym}(4)$. The numbers of trials needed to obtain a nontrivial element x of $\mathrm{Sym}(4)$ is a geometric random variable of parameter $\frac{23}{24}$: its expectation is equal to $E_0 = \frac{24}{23}$. If x is a 4-cycle, then we have already generated a transitive subgroup. With probability $p_1 = \frac{3}{23}$, x is a product of two disjoint transposition: in this case to generate a transitive subgroup it is sufficient to find an element $y \notin \langle x \rangle$ and the number of trials needed to find such an element is a geometric random variable of parameter $\frac{20}{24}$ and expectation $E_1 = \frac{24}{20}$. With probability $p_2 = \frac{8}{23}$, x is a 3-cycle: to generate a transitive subgroup we need an elements y which does not normalizes $\langle x \rangle$: the number of trials needed to find such an element is a geometric random variable of parameter $\frac{18}{24}$ and expectation $E_2 = \frac{24}{18}$. Finally, with probability $p_3 = \frac{6}{23}$, x is a transposition. To find an element $y \notin \langle x \rangle$ we need $E_3 = \frac{24}{22}$ trials. If y is a 4-cycle or a 3-cycle with $|\operatorname{supp}(y) \cap \operatorname{supp}(x)| = 1$ or a product of two disjoint transpositions $(a,b)\,(c,d)$ with $x \notin \{(a,b),(c,d)\}$, then we have already generated a transitive subgroup. With probability $q_1 = \frac{2}{22}$, $\langle x, y \rangle$ is an intransitive subgroup of order 4: to generate a transitive subgroup we need an elements $z \notin \langle x, y \rangle$. The number of trials needed to find such an element is a geometric random variable of parameter $\frac{20}{24}$ and expectation $E_1^* = \frac{24}{20}$. With probability $q_2 = \frac{8}{22}$, $\langle x, y \rangle \cong \mathrm{Sym}(3)$ and to generate a transitive subgroup we need other $E_2^* = \frac{24}{18}$ trials. Definitely

$$\begin{aligned} e_{\mathcal{T}}(\mathrm{Sym}(4)) &= E_0 + p_1 E_1 + p_2 E_2 + p_3(E_3 + q_1 E_1^* + q_2 E_2^*) \\ &= \frac{24}{23} + \frac{3}{23} \cdot \frac{24}{20} + \frac{8}{23} \cdot \frac{24}{18} + \frac{6}{23} \left(\frac{24}{22} + \frac{2}{22} \cdot \frac{24}{20} + \frac{8}{22} \cdot \frac{24}{18} \right) = \frac{7982}{3795}. \end{aligned}$$

In particular

$$e_{\mathcal{T}}(\mathrm{Sym}(4)) - e_{\mathcal{T}}(\mathrm{Alt}(4)) = \frac{7982}{3795} - \frac{394}{165} = -\frac{72}{253},$$

according with Theorem 1.1.

4 Proof of Theorem 1.2

Lemma 4.1. *Let $\epsilon = 0$ if n is even, $\epsilon = 1$ if n is odd. Then*

$$e_{\mathcal{T}}(\mathrm{Alt}(n)) \leq 2 - \frac{2\epsilon}{n} + \frac{1}{n-1} + \frac{2}{n(n-1)-2} + \frac{3n}{n(n-1)(n-2)-6}.$$

Proof. Since an element of $\mathrm{Alt}(n)$ generates a transitive subgroup if and only if it is a cycle of length n, we have that $P_{\mathcal{T}}(\mathrm{Alt}(n), 1) = 2\epsilon/n$. Let now $t \geq 2$ and let $x_1, \dots, x_t \in \mathrm{Alt}(n)$ and $Y = \langle x_1, \dots, x_t \rangle \leq \mathrm{Alt}(n)$. If Y is contained in an intransitive maximal subgroup, then Y is contained in a subgroup conjugate to $\mathrm{Sym}(k) \times \mathrm{Sym}(n-k)$ for some $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. Let $k \in \{1, \dots, n-1\}$. The probability that Y is contained in a subgroup conjugate to $\mathrm{Sym}(k) \times \mathrm{Sym}(n-k)$ is bounded by $\binom{n}{k}^{1-t}$. So

$$1 - P_{\mathcal{T}}(\mathrm{Alt}(n), t) \leq \sum_{1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor} \binom{n}{k}^{1-t}.$$

Notice that

$$\sum_{3 \leq k \leq \lfloor \frac{n-1}{2} \rfloor} \binom{n}{k}^{1-t} \leq \frac{n}{2} \binom{n}{3}^{1-t}.$$

Hence

$$\begin{aligned}
e_{\mathcal{T}}(\mathrm{Alt}(n)) &= \sum_{t \geq 0} (1 - P_{\mathcal{T}}(\mathrm{Alt}(n), t)) \\
&= (1 - P_{\mathcal{T}}(\mathrm{Alt}(n), 0)) + (1 - P_{\mathcal{T}}(\mathrm{Alt}(n), 1)) + \sum_{t \geq 2} (1 - P_{\mathcal{T}}(\mathrm{Alt}(n), t)) \\
&\leq 2 - \frac{2\epsilon}{n} + \sum_{t \geq 2} \left(n^{1-t} + \binom{n}{2}^{1-t} + \frac{n}{2} \binom{n}{3}^{1-t} \right) \\
&= 2 - \frac{2\epsilon}{n} + \frac{1}{n-1} + \frac{1}{\binom{n}{2} - 1} + \frac{n}{2} \frac{1}{\binom{n}{3} - 1} \\
&= 2 - \frac{2\epsilon}{n} + \frac{1}{n-1} + \frac{2}{n(n-1)-2} + \frac{3n}{n(n-1)(n-2)-6}.
\end{aligned}$$

□

Proof of Theorem 1.2. Let

$$f(n) = \frac{(-1)^{n+1} n!(n-1)!}{(n!-1)(n!-2)}.$$

In [2, Section 5] it has been proved that $\lim_{n \to \infty} e_{\mathcal{T}}(\mathrm{Sym}(n)) = 2$. This implies

$$\lim_{n \to \infty} e_{\mathcal{T}}(\mathrm{Alt}(n)) = \lim_{n \to \infty} (e_{\mathcal{T}}(\mathrm{Sym}(n)) - f(n)) = \lim_{n \to \infty} e_{\mathcal{T}}(\mathrm{Sym}(n)) - \lim_{n \to \infty} f(n) = 2.$$

Moreover, again by [2, Section 5], if $n \geq 2$, then

$$2 \leq e_{\mathcal{T}}(\mathrm{Sym}(n)) \leq e_{\mathcal{T}}(\mathrm{Sym}(4)) \sim 2.1033. \tag{4.1}$$

The values of $e_{\mathcal{T}}(\mathrm{Alt}(n))$ and $e_{\mathcal{T}}(\mathrm{Sym}(n))$ when $n \in \{3, 4\}$ have been discussed in the previous section. So we may assume $n \geq 5$. Notice that $|f(n)|$ is a decreasing function and that $f(n) < 0$ if n is even, $f(n) > 0$ otherwise.

Assume that n is even:

$$e_{\mathcal{T}}(\mathrm{Alt}(n)) = e_{\mathcal{T}}(\mathrm{Sym}(n)) - f(n) \geq 2 - f(n) > 2,$$
$$e_{\mathcal{T}}(\mathrm{Alt}(n)) = e_{\mathcal{T}}(\mathrm{Sym}(n)) - f(n) \leq e_{\mathcal{T}}(\mathrm{Sym}(4)) - f(4) = e_{\mathcal{T}}(\mathrm{Alt}(4)).$$

Assume that n is odd: it follows immediately from Lemma 4.1, that $e_{\mathcal{T}}(\mathrm{Alt}(n)) < 2$ if $n \geq 9$. Moreover

$$e_{\mathcal{T}}(\mathrm{Alt}(5)) = \frac{2205085}{1170324} \sim 1.8842,$$
$$e_{\mathcal{T}}(\mathrm{Alt}(7)) = \frac{1493015628619946854486}{779316363245447358045} \sim 1.9158.$$

Finally

$$e_{\mathcal{T}}(\mathrm{Alt}(n)) = e_{\mathcal{T}}(\mathrm{Sym}(n)) - f(n) \geq 2 - f(5) \geq 2 - \frac{1440}{7021} > \frac{3}{2} = e_{\mathcal{T}}(\mathrm{Alt}(3)). \quad \square$$

References

[1] E. Detomi and A. Lucchini, Some generalizations of the probabilistic zeta function, in: T. Hawkes, P. Longobardi and M. Maj (eds.), *Ischia Group Theory 2006*, World Scientific Publishing, Hackensack, NJ, 2007 pp. 56–72, doi:10.1142/9789812708670_0007, proceedings of the conference in honor of Akbar Rhemtulla held in Ischia, March 29 – April 1, 2006.

[2] A. Lucchini, The expected number of random elements to generate a finite group, *Monatsh. Math.* **181** (2016), 123–142, doi:10.1007/s00605-015-0789-5.

[3] R. P. Stanley, *Enumerative Combinatorics, Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 1997, doi:10.1017/cbo9780511805967.

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On the domination number and the total domination number of Fibonacci cubes

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Abstract

Fibonacci cubes are special subgraphs of the hypercube graphs. Their domination numbers and total domination numbers are obtained for some small dimensions by integer linear programming. For larger dimensions upper and lower bounds on these numbers are given. In this paper, we present the up-down degree polynomials for Fibonacci cubes containing the degree information of all vertices in more detail. Using these polynomials we define optimization problems whose solutions give better lower bounds on the domination numbers and total domination numbers of Fibonacci cubes. Furthermore, we present better upper bounds on these numbers.

Keywords: Fibonacci cubes, domination number, total domination number, integer linear programming.

Math. Subj. Class.: 05C69, 68R10, 11B39

1 Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. $D \subseteq V(G)$ is a *dominating set* of G if every vertex in $V(G)$ either belongs to D or is adjacent to some vertex in D. The *domination number* $\gamma(G)$ is defined as the minimum cardinality of a dominating set of the graph G. Similarly, $D \subseteq V(G)$ is a *total dominating set* if every vertex in $V(G)$ is adjacent to some vertex in D and the *total domination number* $\gamma_t(G)$ is defined as the minimum cardinality of a total dominating set of G. Note that the total domination number is defined for isolate-free graphs and it is not defined for the graphs that contain isolated vertices. The domination number of the Fibonacci cubes Γ_n is first given

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in [14] and [2]. These results are extended in [8] by using integer linear programming for some cases. Total domination number of Γ_n is considered in [1], in which an upper bound and a lower bound on $\gamma_t(\Gamma_n)$ are obtained. The exact values of $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$ are also considered by using integer programming in [1]. The upper bound on $\gamma_t(\Gamma_n)$ given in [1] is improved in [15]. We summarize these results in Section 2. The aim of this work is to improve some of the results given in [1] and [15].

The hypercube Q_n of dimension $n \geq 1$ is the graph with vertex set $V(Q_n) = \{0,1\}^n$, in which two vertices are adjacent if they differ in one coordinate. For convenience $Q_0 = K_1$. All the vertices of Q_n are labeled by the binary strings of length n. The Fibonacci cubes Γ_n are special subgraphs of Q_n and they were introduced by Hsu [7] as a model of interconnection networks. In literature, many interesting properties of the Fibonacci cubes have been investigated, see survey [9] for details. In recent years results on disjoint hypercubes in Γ_n are presented in [5, 13, 16] and the cube enumerator polynomial of Γ_n is considered in [10, 11, 17] and many combinatorial results are given. The domination-type invariants of Γ_n are considered in [1, 2, 8, 14, 15] and some numerical results and bounds are presented.

It is known that Fibonacci strings of length n are the binary strings of length n that contain no consecutive ones. For this reason we can write

$$V(\Gamma_n) = \{b_1 b_2 \cdots b_n \mid b_i \in \{0,1\},\ 1 \leq i \leq n, \text{ and } b_i \cdot b_{i+1} = 0 \text{ for } 1 \leq i < n\} \text{ and}$$
$$E(\Gamma_n) = \{(u,v) \mid u,v \in V(\Gamma_n),\ d_H(u,v) = 1\},$$

where $d_H(u,v)$ denotes the Hamming distance between u and v, that is, the number of different coordinates in u and v. The number of vertices of the Fibonacci cubes Γ_n is F_{n+2}, where F_n are the Fibonacci numbers defined as $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $n \geq 2$ we will use the following formulation for the fundamental decomposition of Γ_n (see, [9]):

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}. \tag{1.1}$$

Here note that $0\Gamma_{n-1}$ is the subgraph of Γ_n induced by the vertices that start with 0 and $10\Gamma_{n-2}$ is the subgraph of Γ_n induced by the vertices that start with 10. Furthermore, $0\Gamma_{n-1}$ has a subgraph isomorphic to $00\Gamma_{n-2}$, and there is a matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$ (see Figure 1).

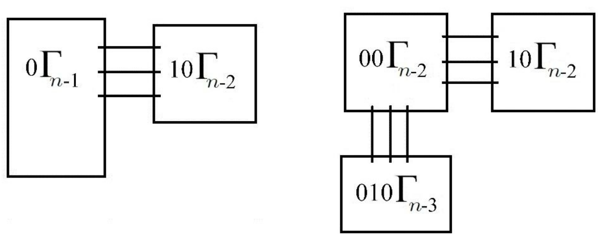

Figure 1: Fundamental decompositions of the Fibonacci cube Γ_n, $n \geq 3$.

In this paper, we present upper bounds on $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$. Furthermore, we introduce the up-down degree polynomials for Γ_n containing the degree information of all vertices $V(\Gamma_n)$ in more detail. Using these polynomials we define optimization problems whose solutions give lower bound on $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$.

2 Known results and new upper bounds on $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$

In this section, first we summarize some known results on the domination number and the total domination number of Fibonacci cubes and then we present new uppper bounds for these numbers. We start with Figure 2 and Figure 3 showing a dominating set and a total dominating set for small dimensional Γ_n's.

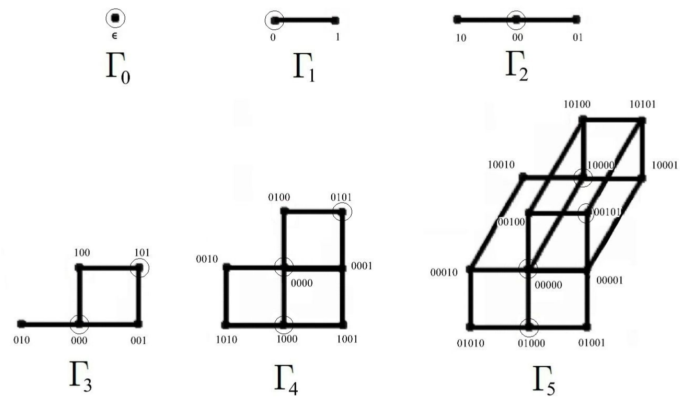

Figure 2: $\Gamma_0, \ldots, \Gamma_5$ and their dominating sets.

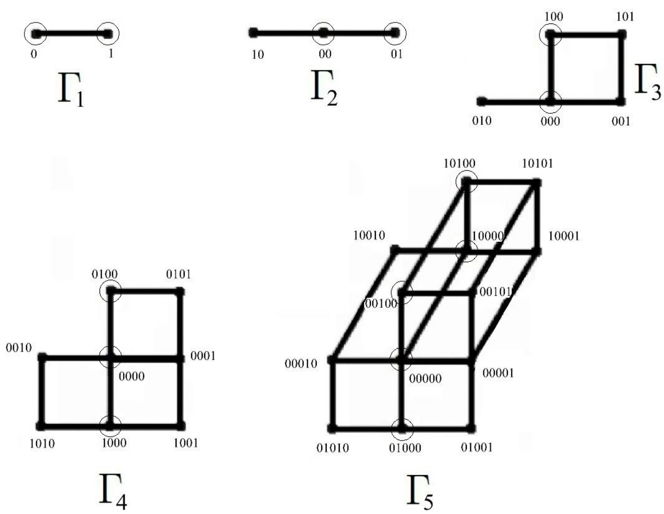

Figure 3: $\Gamma_1, \ldots, \Gamma_5$ and their total dominating sets.

We collect the known values of $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$ in Table 1. The values of $\gamma(\Gamma_n)$ for $n \leq 8$ are obtained in [14]. The other values of $\gamma(\Gamma_n)$ are obtained by integer programming. For $n = 9$ and $n = 10$ they are obtained in [8] and for $n = 11$ and $n = 12$ they are obtained in [1]. Similarly, all the values of $\gamma_t(\Gamma_n)$ given in Table 1 are obtained by computer using integer programming in [1].

Table 1: Known values of $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert V(\Gamma_n)\rvert$	2	3	5	8	13	21	34	55	89	144	233	377	610
$\gamma(\Gamma_n)$	1	1	2	3	4	5	8	12	17	25	39	54 – 61	
$\gamma_t(\Gamma_n)$	2	2	2	3	5	7	10	13	20	30	44	65	97 – 101

Now we describe the integer linear programming used in [8] and [1]. Suppose each vertex $v \in V(\Gamma_n)$ is associated with a binary variable x_v. Let $N(v)$ be the set of vertices adjacent to v and $N[v] = N(v) \cup \{v\}$. The problems of determining $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$ can be expressed as a problem of minimizing the objective function

$$\sum_{v \in V(\Gamma_n)} x_v \tag{2.1}$$

subject to the condition that for every $v \in V(\Gamma_n)$ we have

$$\sum_{a \in N[v]} x_a \geq 1 \quad \text{(for domination number)},$$
$$\sum_{a \in N(v)} x_a \geq 1 \quad \text{(for total domination number)}.$$

The value of the objective function is then $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$ respectively. Note that this problem has F_{n+2} variables and F_{n+2} constraints. In [1], it is stated that $\gamma(\Gamma_{12})$ and $\gamma_t(\Gamma_{13})$ were not computed in real time using the above optimization problem. They got the estimates

$$54 \leq \gamma(\Gamma_{12}) \leq 61 \quad \text{and} \quad 97 \leq \gamma_t(\Gamma_{13}) \leq 101.$$

Here, the main difficulty is the order of Γ_n which equals to the number of variables and the number of constraints.

By using the degree information of the vertices in Γ_n the following lower bound on $\gamma(\Gamma_n)$ is presented in [14].

Theorem 2.1 ([14]). *If $n \geq 9$, then*

$$\gamma(\Gamma_n) \geq \left\lceil \frac{F_{n+2} - 2}{n - 2} \right\rceil.$$

By using a similar technique the following lower bound on $\gamma_t(\Gamma_n)$ is obtained in [1].

Theorem 2.2 ([1]). *If $n \geq 9$, then*

$$\gamma_t(\Gamma_n) \geq \left\lceil \frac{F_{n+2} - 11}{n - 3} \right\rceil - 1.$$

In Section 3 we propose an optimization problem having less number of variables and constraints to estimate lower bounds on $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$. Our results improve the lower bounds given in Theorem 2.1 and Theorem 2.2 and we present some numerical values in Table 2 and Table 3.

By using the exact values in Table 1 and the fundamental decomposition (1.1) of Γ_n, the following upper bound on $\gamma_t(\Gamma_n)$ is obtained in [1].

Theorem 2.3 ([1])**.** *If* $n \geq 11$*, then* $\gamma_t(\Gamma_n) \leq 21F_{n-8} + 2F_{n-10}$.

In [1], using the computer result $\gamma_t(\Gamma_{13}) \leq 101$ the upper bound in Theorem 2.3 improved to

$$\gamma_t(\Gamma_n) \leq 601F_{n-1} - 371F_n, \quad n \geq 12.$$

These two upper bounds further improved in [15] by using the values of $\gamma(\Gamma_n)$ and the fundamental decomposition (1.1) of Γ_n more than once.

Theorem 2.4 ([15])**.** *If* $n \geq 15$*, then*

$$\begin{aligned}\gamma(\Gamma_n) \leq \gamma_t(\Gamma_n) &\leq 3\gamma(\Gamma_{n-3}) + 2\gamma(\Gamma_{n-4}) \\ &\leq 116F_n - 187F_{n-1} = 21F_{n-8} - (2F_{n-10} + F_{n-12}) .\end{aligned}$$

Furthermore, $\gamma_t(\Gamma_{14}) \leq 166$.

We implemented the same integer linear programming problem (2.1) using CPLEX in NEOS Server [3, 4, 6] for $n = 13$ and obtain the estimates (takes approximately 2 hours)

$$78 \leq \gamma(\Gamma_{13}) \leq 93.$$

Using this result with $\gamma(\Gamma_{12}) \leq 61$ we obtain the following bound on the domination number of Γ_n.

Theorem 2.5. *If* $n \geq 12$*, then* $\gamma(\Gamma_n) \leq 21F_{n-8} - (2F_{n-10} + 8F_{n-12})$.

Proof. The proof mimics the proof of [1, Theorem 2.1]. By the fundamental decomposition (1.1) of Γ_n we have $\gamma(\Gamma_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-2})$. We know that $\gamma(\Gamma_{12}) \leq 61$ and $\gamma(\Gamma_{13}) \leq 93$. For $n \geq 12$ define the sequence (b_n) with $b_n = b_{n-1} + b_{n-2}$ where $b_{12} = 61$ and $b_{13} = 93$. Then by induction we have $b_n = 21F_{n-8} - 2F_{n-10} - 8F_{n-12}$ for any $n \geq 12$. We complete the proof since $\gamma(\Gamma_n) \leq b_n$ for $n \geq 12$. □

Combining the results in Theorem 2.5 and Theorem 2.4 we get the following result which improves Theorem 2.3 and Theorem 2.4.

Theorem 2.6. *If* $n \geq 16$*, then*

$$\gamma_t(\Gamma_n) \leq 21F_{n-8} - (2F_{n-10} + 8F_{n-12}) .$$

3 Up-down degree enumerator polynomial

In this section we present the up-down degree enumerator polynomial for Γ_n. It contains the degree information of all vertices $V(\Gamma_n)$ in more detail. Using this polynomial we write optimization problems whose solutions are lower bounds on $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$.

For each fixed $v \in V(\Gamma_n)$ we write a monomial $x^u y^d$ where $d = w(v)$ is the Hamming weight of v and u is $\deg(v) - d$ (that is, $\deg(v) = u + d$). Recall that, by the definition of Γ_n, $(v, v') \in E(\Gamma_n)$ if and only if $d_H(v, v') = 1$. Therefore, the number of neighbors of v

whose weight is one more than the weight of v (say up neighbors of v, $w(v') = w(v) + 1$) is u and the number of neighbors of v whose weight is one less than the weight of v (say down neighbors of v, $w(v') = w(v) - 1$) is d. For this reason we call the polynomial

$$P_n(x, y) = \sum_{v \in V(\Gamma_n)} x^{\deg(v) - w(v)} y^{w(v)} = \sum_{v \in V(\Gamma_n)} x^u y^d$$

as the up-down degree enumerator polynomial of Γ_n. By using the fundamental decomposition (1.1) of Γ_n we obtain the following recursive relation which will be useful to calculate $P_n(x, y)$.

Theorem 3.1. *Let $P_n(x, y)$ be the up-down degree enumerator polynomial of Γ_n. Then for $n \geq 3$ we have*

$$P_n(x, y) = xP_{n-1}(x, y) + yP_{n-2}(x, y) + yP_{n-3}(x, y) - xyP_{n-3}(x, y),$$

where

$$P_0(x, y) = 1, \quad P_1(x, y) = x + y \quad \textit{and} \quad P_2(x, y) = x^2 + 2y.$$

Proof. P_0, P_1 and P_2 are clear from Figure 2. Assume that $n \geq 3$. We know that the up-down degree enumerator polynomials of Γ_{n-1}, Γ_{n-2} and Γ_{n-3} are $P_{n-1}(x, y)$, $P_{n-2}(x, y)$ and $P_{n-3}(x, y)$ respectively. By (1.1) we have

$$\begin{aligned} \Gamma_n &= 0\Gamma_{n-1} + 10\Gamma_{n-2} && (3.1) \\ &= (00\Gamma_{n-2} + 010\Gamma_{n-3}) + 10\Gamma_{n-2} && (3.2) \end{aligned}$$

and there is a matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$ (see also Figure 1). From this decomposition we have the following three different cases:

1. Assume that $v \in 10\Gamma_{n-2}$. These vertices are the ones in Γ_{n-2} whose weights $d = w(v)$ increase by one in Γ_n. Furthermore, their degrees increase by one due to the matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$, which means that $u = \deg(v) - w(v)$ remains the same in Γ_n. Therefore, these vertices contribute $yP_{n-2}(x, y)$ to $P_n(x, y)$.
2. Assume that $v \in 010\Gamma_{n-3}$. These vertices are the ones in Γ_{n-3} whose weights $d = w(v)$ increase by one in Γ_n and their degrees increase by one due to the matching between $010\Gamma_{n-3}$ and $000\Gamma_{n-3} \subset 00\Gamma_{n-2}$, which means that $u = \deg(v) - w(v)$ remains the same in Γ_n. Therefore, these vertices contribute $yP_{n-3}(x, y)$ to $P_n(x, y)$.
3. Assume that $v \in 00\Gamma_{n-2}$. These vertices are the ones in $0\Gamma_{n-1}$ that are not in $010\Gamma_{n-3}$. In Γ_{n-1} the up-down degree enumerator polynomial of these vertices becomes $P_{n-1}(x, y) - yP_{n-3}(x, y)$. The weights $d = w(v)$ of all such vertices remain the same in Γ_n but their degrees increase by one due to the matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$, that is, $u = \deg(v) - w(v)$ increase by one in Γ_n. Therefore, these vertices contribute $x(P_{n-1}(x, y) - yP_{n-3}(x, y))$ to $P_n(x, y)$.

By adding all of the above contributions we get the desired result. □

Now we describe an optimization problem using the up-down degree enumerator polynomial $P_n(x, y)$. Let D_T be a total dominating set of Γ_n. Then by the definition of Fibonacci cubes for every vertex $v \in V(\Gamma_n)$ with weight $w(v)$ then there must exist a vertex

$v_D \in N(v) \cap D_T$ with $w(v_D) = w(v) \mp 1$. Furthermore, assume that for any fixed vertex $v_D \in D_T$ its corresponding monomial be $x^u y^d$ in the $P_n(x,y)$. This means that v_D dominates u distinct vertices $v \in V(\Gamma_n)$ with weight $w(v) = w(v_D) + 1$ and d distinct vertices $v \in V(\Gamma_n)$ with weight $w(v) = w(v_D) - 1$. Note that for all $v_D \in D_T$ some of the dominated vertices may coincide. Now assume that

$$P_n(x,y) = \sum_{v \in V(\Gamma_n)} x^u y^d = \sum c_d^u x^u y^d. \tag{3.3}$$

For each pair (u,d) in $P_n(x,y)$ we associate an integer variable z_d^u which counts the number of vertices in D_T with weight d and degree $u+d$, that is, the number of vertices in D_T having d down neighbors and u up neighbors. Clearly, we have the bounds $0 \le z_d^u \le c_d^u$. Our aim is to minimize $|D_T|$, that is, our objective function is to minimize

$$\sum_{u,d} z_d^u.$$

Then by the above observation to dominate all the vertices having a fixed weight d such that $1 \le d \le \lceil \frac{n}{2} \rceil - 1$ we must have

$$r_d\colon \; \sum_u \left(u \cdot z_{d-1}^u + (d+1) \cdot z_{d+1}^u \right) \ge \sum_u c_d^u$$

since any vertex with weight $d-1$ having u up neighbors can dominate u distinct vertices with weight d and any vertex with weight $d+1$ (all have $d+1$ down neighbors) can dominate $d+1$ distinct vertices with weight d. By the same argument, for $d = 0$ we must have

$$r_0\colon \; \sum_u z_1^u \ge \sum_u c_0^u = 1$$

and for $d = \lceil \frac{n}{2} \rceil$ we must have

$$r_{\lceil \frac{n}{2} \rceil}\colon \; \sum_u u \cdot z_{\lceil \frac{n}{2} \rceil - 1}^u \ge \sum_u c_{\lceil \frac{n}{2} \rceil}^u = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \frac{n}{2} + 1 & \text{if } n \text{ is even.} \end{cases}$$

Now subject to these constraints $r_0, \ldots, r_{\lceil \frac{n}{2} \rceil}$ the value of the objective function will be a lower bound on $\gamma_t(\Gamma_n)$. Similarly, to find a lower bound on $\gamma(\Gamma_n)$ we need to modify all of the constraints r_d, $0 \le d \le \lceil \frac{n}{2} \rceil$. By the definition of the dominating set, for each fixed d we need to add all of the variables z_d^u to the left side of the constraint r_d.

Remark 3.2. We remark that using [12, Theorem 4.6] we can easily obtain the coefficients c_d^u in (3.3). By the definition of the up-down degree enumerator polynomial we know that c_d^u is the number of vertices in Γ_n whose number of up neighbors is u and weight is d. That is, c_d^u equals to the number of vertices of Γ_n having degree $u+d$ and weight d. Therefore, [12, Theorem 4.6] gives

$$c_d^u = \binom{d+1}{n-2d-u+1} \binom{n-2d}{u}.$$

Remark 3.3. We know that the number of vertices of Γ_n with weight d is equal to the right hand side of the above constraints r_d. By definition of Γ_n this number is equal to the number of Fibonacci strings of length n and weight d. Therefore we have

$$\sum_u c_d^u = \binom{n-d+1}{d}.$$

Remark 3.4. To find the number of variables z_d^u we need to find the number of monomials in $P_n(x,y)$. Assume that n is even. Then by the structure of the vertices in Fibonacci cubes (it can also be seen from the structure of Fibonacci strings) $n-3d \le u \le n-2d$. Therefore the number of variables z_d^u becomes

$$\sum_{d=0}^{\lfloor \frac{n}{3} \rfloor}(d+1) + \sum_{d=\lfloor \frac{n}{3} \rfloor+1}^{\frac{n}{2}} (n-2d+1)$$

which is equal to

$$s^2 - 2sr + \frac{3r(r+1)}{2} + 1$$

where $r = \lfloor \frac{n}{3} \rfloor$ and $s = n/2$. Similarly, if n is odd we obtain that the number of variables z_d^u is

$$s^2 - s(2r+1) + \frac{r(3r+5)}{2} + 2$$

where $r = \lfloor \frac{n}{3} \rfloor$ and $s = \lceil \frac{n}{2} \rceil$.

Now we illustrate our optimization problem for $n = 14$. We have the following polynomial by Theorem 3.1.

$$\begin{aligned} P_{14}(x,y) = {} & 8y^7 + \\ & 7y^6x^2 + 42y^6x + 35y^6 + \\ & 6y^5x^4 + 60y^5x^3 + 120y^5x^2 + 60y^5x + 6y^5 + \\ & 5y^4x^6 + 60y^4x^5 + 150y^4x^4 + 100y^4x^3 + 15y^4x^2 + \\ & 4y^3x^8 + 48y^3x^7 + 112y^3x^6 + 56y^3x^5 + \\ & 3y^2x^{10} + 30y^2x^9 + 45y^2x^8 + \\ & 2yx^{12} + 12yx^{11} + \\ & x^{14} \end{aligned}$$

and this polynomial corresponds to the following optimization problem:

Objective function:

$$\begin{aligned} min\colon\ & z_0^{14} + z_1^{12} + z_1^{11} + z_2^{10} + z_2^9 + z_2^8 + z_3^8 + z_3^7 + z_3^6 + z_3^5 + \\ & z_4^6 + z_4^5 + z_4^4 + z_4^3 + z_4^2 + z_5^4 + z_5^3 + z_5^2 + z_5^1 + z_5^0 + z_6^2 + z_6^1 + z_6^0 + z_7^0 \end{aligned}$$

Constraints for $\gamma_t(\Gamma_{14})$*:*

$$\begin{aligned}
r_7 &: & 2z_6^2 + z_6^1 &\geq 8\\
r_6 &: & 4z_5^4 + 3z_5^3 + 2z_5^2 + z_5^1 + 7z_7^0 &\geq 84\\
r_5 &: & 6z_4^6 + 5z_4^5 + 4z_4^4 + 3z_4^3 + 2z_4^2 + 6z_6^2 + 6z_6^1 + 6z_6^0 &\geq 252\\
r_4 &: & 8z_3^8 + 7z_3^7 + 6z_3^6 + 5z_3^5 + 5z_5^4 + 5z_5^3 + 5z_5^2 + 5z_5^1 + 5z_5^0 &\geq 330\\
r_3 &: & 10z_2^{10} + 9z_2^9 + 8z_2^8 + 4z_4^6 + 4z_4^5 + 4z_4^4 + 4z_4^3 + 4z_4^2 &\geq 220\\
r_2 &: & 12z_1^{12} + 11z_1^{11} + 3z_3^8 + 3z_3^7 + 3z_3^6 + 3z_3^5 &\geq 78\\
r_1 &: & 14z_0^{14} + 2z_2^{10} + 2z_2^9 + 2z_2^8 &\geq 14\\
r_0 &: & z_1^{12} + z_1^{11} &\geq 1
\end{aligned}$$

Constraints for $\gamma(\Gamma_{14})$*:*

$$\begin{aligned}
r_7 &: & 2z_6^2 + z_6^1 + z_7^0 &\geq 8\\
r_6 &: & 4z_5^4 + 3z_5^3 + 2z_5^2 + z_5^1 + z_6^2 + z_6^1 + z_6^0 + 7z_7^0 &\geq 84\\
r_5 &: & 6z_4^6 + 5z_4^5 + 4z_4^4 + 3z_4^3 + 2z_4^2 + z_5^4 + z_5^3 + &\\
& & z_5^2 + z_5^1 + z_5^0 + 6z_6^2 + 6z_6^1 + 6z_6^0 &\geq 252\\
r_4 &: & 8z_3^8 + 7z_3^7 + 6z_3^6 + 5z_3^5 + z_4^6 + z_4^5 + z_4^4 + z_4^3 + &\\
& & z_4^2 + 5z_5^4 + 5z_5^3 + 5z_5^2 + 5z_5^1 + 5z_5^0 &\geq 330\\
r_3 &: & 10z_2^{10} + 9z_2^9 + 8z_2^8 + z_3^8 + z_3^7 + z_3^6 + &\\
& & z_3^5 + 4z_4^6 + 4z_4^5 + 4z_4^4 + 4z_4^3 + 4z_4^2 &\geq 220\\
r_2 &: & 12z_1^{12} + 11z_1^{11} + z_2^{10} + z_2^9 + z_2^8 + 3z_3^8 + 3z_3^7 + 3z_3^6 + 3z_3^5 &\geq 78\\
r_1 &: & 14z_0^{14} + z_1^{12} + z_1^{11} + 2z_2^{10} + 2z_2^9 + 2z_2^8 &\geq 14\\
r_0 &: & z_0^{14} + z_1^{12} + z_1^{11} &\geq 1
\end{aligned}$$

Bounds:

$z_0^{14} \leq 1$	$z_1^{12} \leq 2$	$z_1^{11} \leq 12$	$z_2^{10} \leq 3$	$z_2^9 \leq 30$
$z_2^8 \leq 45$	$z_3^8 \leq 4$	$z_3^7 \leq 48$	$z_3^6 \leq 112$	$z_3^5 \leq 56$
$z_4^6 \leq 5$	$z_4^5 \leq 60$	$z_4^4 \leq 150$	$z_4^3 \leq 100$	$z_4^2 \leq 15$
$z_5^4 \leq 6$	$z_5^3 \leq 60$	$z_5^2 \leq 120$	$z_5^1 \leq 60$	$z_5^0 \leq 6$
$z_6^2 \leq 7$	$z_6^1 \leq 42$	$z_6^0 \leq 35$	$z_7^0 \leq 8$	

The value of the objective function gives a lower bound on $\gamma(\Gamma_{14})$ and $\gamma_t(\Gamma_{14})$ respectively. Note that the above problem have only 24 variables and 8 constraints (instead of having 987 variables and 987 constraints, see Section 2). In general using the up-down degree enumerator polynomial $P_n(x,y)$ of Γ_n in Theorem 3.1 we can write an optimization problem having less number of variables z_d^u (see Remark 3.4) and $\lceil \frac{n}{2} \rceil + 1$ constraints r_d. The solutions of these problem give lower bounds on $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$. One can easily see that the number of variables and the number of constraints are very smaller than the ones in the optimization problem described in Section 2.

For illustration we implemented the above integer linear programming problem using CPLEX in NEOS Server [3, 4, 6] for $13 < n \leq 26$ and immediately obtain the lower bounds presented in Table 2 and Table 3. Note that for $n = 26$, the number of variables in our optimization problem is 70 by Remark 3.4 and it is $F_{28} = 317811$ for the general optimization problem (2.1). In addition, the upper bounds in these tables are obtained by Theorem 2.5 for $n \geq 14$ and Theorem 2.6 for $n \geq 16$. Note that the first bounds in both tables are obtained in [1] and the upper bounds on $\gamma_t(\Gamma_{14})$ and $\gamma_t(\Gamma_{15})$ comes from Theorem 2.4.

Table 2: Current best bounds on $\gamma(\Gamma_n)$, $12 \leq n \leq 26$.

n	$\gamma(\Gamma_n)$	n	$\gamma(\Gamma_n)$	n	$\gamma(\Gamma_n)$
12	54 – 61	17	344 – 648	22	3060 – 7189
13	78 – 93	18	528 – 1049	23	4748 – 11632
14	98 – 154	19	819 – 1697	24	7381 – 18821
15	148 – 247	20	1270 – 2746	25	11472 – 30453
16	224 – 401	21	1970 – 4443	26	17912 – 49274

Table 3: Current best bounds on $\gamma_t(\Gamma_n)$, $13 \leq n \leq 26$.

n	$\gamma(\Gamma_n)$	n	$\gamma(\Gamma_n)$	n	$\gamma(\Gamma_n)$
13	97 – 101	18	578 – 1049	23	5075 – 11632
14	110 – 166	19	890 – 1697	24	7865 – 18821
15	164 – 261	20	1374 – 2746	25	12191 – 30453
16	246 – 401	21	2121 – 4443	26	19033 – 49274
17	376 – 648	22	3281 – 7189		

Remark 3.5. For $n = 12$, Theorem 2.1 gives $\gamma(\Gamma_{12}) \geq 38$ and Theorem 2.2 gives $\gamma_t(\Gamma_{12}) \geq 40$. The values of the objective function in our optimization problems having 19 variables and 7 constraints give lower bounds $\gamma(\Gamma_{12}) \geq 44$ and $\gamma_t(\Gamma_{12}) \geq 50$.

For the case $n = 13$, Theorem 2.1 gives $\gamma(\Gamma_{13}) \geq 56$ and Theorem 2.2 gives $\gamma_t(\Gamma_{13}) \geq 59$. The values of the objective function in our optimization problems having 22 variables and 8 constraints give lower bounds $\gamma(\Gamma_{13}) \geq 65$ and $\gamma_t(\Gamma_{13}) \geq 75$.

References

[1] J. Azarija, S. Klavžar, Y. Rho and S. Sim, On domination-type invariants of Fibonacci cubes and hypercubes, *Ars Math. Contemp.* **14** (2018), 387–395, doi:10.26493/1855-3974.1172.bae.

[2] A. Castro, S. Klavžar, M. Mollard and Y. Rho, On the domination number and the 2-packing number of Fibonacci cubes and Lucas cubes, *Comput. Math. Appl.* **61** (2011), 2655–2660, doi:10.1016/j.camwa.2011.03.012.

[3] J. Czyzyk, M. P. Mesnier and J. J. Moré, The NEOS server, *IEEE Comput. Sci. Eng.* **5** (1998), 68–75, doi:10.1109/99.714603.

[4] E. D. Dolan, *NEOS Server 4.0 Administrative Guide*, Technical Memorandum ANL/MCS-TM-250, Mathematics and Computer Science Division, Argonne National Laboratory, May 2001.

[5] S. Gravier, M. Mollard, S. Špacapan and S. S. Zemljič, On disjoint hypercubes in Fibonacci cubes, *Discrete Appl. Math.* **190/191** (2015), 50–55, doi:10.1016/j.dam.2015.03.016.

[6] W. Gropp and J. J. Moré, Optimization environments and the NEOS server, in: M. D. Buhmann and A. Iserles (eds.), *Approximation Theory and Optimization*, Cambridge University Press, Cambridge, pp. 167–182, 1997, selected papers from the Conference on Numerical Mathematics, in honor of M. J. D. Powell on the occasion of his 60th birthday, held in Cambridge, July 27 – 30, 1996.

[7] W.-J. Hsu, Fibonacci cubes—a new interconnection topology, *IEEE Trans. Parallel Distrib. Syst.* **4** (1993), 3–12, doi:10.1109/71.205649.

[8] A. Ilić and M. Milošević, The parameters of Fibonacci and Lucas cubes, *Ars Math. Contemp.* **12** (2017), 25–29, doi:10.26493/1855-3974.915.f48.

[9] S. Klavžar, Structure of Fibonacci cubes: a survey, *J. Comb. Optim.* **25** (2013), 505–522, doi:10.1007/s10878-011-9433-z.

[10] S. Klavžar and M. Mollard, Cube polynomial of Fibonacci and Lucas cubes, *Acta Appl. Math.* **117** (2012), 93–105, doi:10.1007/s10440-011-9652-4.

[11] S. Klavžar and M. Mollard, Daisy cubes and distance cube polynomial, *European J. Combin.* (2018), doi:10.1016/j.ejc.2018.02.019.

[12] S. Klavžar, M. Mollard and M. Petkovšek, The degree sequence of Fibonacci and Lucas cubes, *Discrete Math.* **311** (2011), 1310–1322, doi:10.1016/j.disc.2011.03.019.

[13] M. Mollard, Non covered vertices in Fibonacci cubes by a maximum set of disjoint hypercubes, *Discrete Appl. Math.* **219** (2017), 219–221, doi:10.1016/j.dam.2016.10.029.

[14] D. A. Pike and Y. Zou, The domination number of Fibonacci cubes, *J. Combin. Math. Combin. Comput.* **80** (2012), 433–444.

[15] E. Saygı, Upper bounds on the domination and total domination number of fibonacci cubes, *SDU J. Nat. Appl. Sci.* **21** (2017), 782–785, doi:10.19113/sdufbed.05851.

[16] E. Saygı and Ö. Eğecioğlu, Counting disjoint hypercubes in Fibonacci cubes, *Discrete Appl. Math.* **215** (2016), 231–237, doi:10.1016/j.dam.2016.07.004.

[17] E. Saygı and Ö. Eğecioğlu, q-cube enumerator polynomial of Fibonacci cubes, *Discrete Appl. Math.* **226** (2017), 127–137, doi:10.1016/j.dam.2017.04.026.

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Pappus's Theorem in Grassmannian $Gr(3, \mathbb{C}^n)$

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Abstract

In this paper we study intersections of quadrics, components of the hypersurface in the Grassmannian $Gr(3, \mathbb{C}^n)$ introduced by S. Sawada, S. Settepanella and S. Yamagata in 2017. This lead to an alternative statement and proof of Pappus's Theorem retrieving Pappus's and Hesse configurations of lines as special points in the complex projective Grassmannian. This new connection is obtained through a third purely combinatorial object, the intersection lattice of Discriminantal arrangement.

Keywords: Discriminantal arrangements, intersection lattice, Grassmannian, Pappus's Theorem.

Math. Subj. Class.: 52C35, 05B35, 14M15

1 Introduction

Pappus's hexagon Theorem, proved by Pappus of Alexandria in the fourth century A.D., began a long development in algebraic geometry.

> *In its changing expressions one can see reflected the changing concerns of the field, from synthetic geometry to projective plane curves to Riemann surfaces to the modern development of schemes and duality.*
>
> (D. Eisenbud, M. Green and J. Harris [4])

There are several knowns proofs of Pappus's Theorem including its generalizations such as Cayley Bacharach Theorem (see Chapter 1 of [9] for a collection of proofs of Pappus's Theorem and [4] for proofs and conjectures in higher dimension).

In this paper, by mean of recent results in [6] and [10], we connect Pappus's hexagon configuration to intersections of well defined quadrics in the Grassmannian providing a new statement and proof of Pappus's Theorem as an original result on dependency conditions for defining polynomials of those quadrics. This result enlightens a new connection

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between special configurations of points (lines) in the projective plane and hypersurfaces in the projective Grassmannian $Gr(3, \mathbb{C}^n)$. This connection is made through a third combinatorial object, the intersection lattice of the *Discriminantal arrangement*. Introduced by Manin and Schechtman in 1989, it is an arrangement of hyperplanes generalizing classical braid arrangement (cf. [7, p. 209]). Fixed a generic arrangement $\mathcal{A} = \{H_1^0, \dots, H_n^0\}$ in $\mathbb{C}^k$, the Discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$, $n, k \in \mathbb{N}$ for $k \geq 2$ ($k = 1$ corresponds to Braid arrangement), consists of parallel translates $H_1^{t_1}, \dots, H_n^{t_n}$, $(t_1, \dots, t_n) \in \mathbb{C}^n$, of $\mathcal{A}$ which fail to form a generic arrangement in $\mathbb{C}^k$. The combinatorics of $\mathcal{B}(n, k, \mathcal{A})$ is known in the case of *very generic arrangements*, i.e. $\mathcal{A}$ belongs to an open Zariski set $\mathcal{Z}$ in the space of generic arrangements H_i^0, $i = 1, \dots, n$ (see [7], [1] and [2]), but still almost unknown for $\mathcal{A} \notin \mathcal{Z}$. In 2016, Libgober and Settepanella (cf. [6]) gave a sufficient geometric condition for an arrangement $\mathcal{A}$ not to be very generic, i.e. $\mathcal{A} \notin \mathcal{Z}$. In particular in the case $k = 3$, their result shows that multiplicity 3 codimension 2 intersections of hyperplanes in $\mathcal{B}(n, 3, \mathcal{A})$ appears if and only if collinearity conditions for points at infinity of lines, intersections of certain planes in $\mathcal{A}$, are satisfied (Theorem 3.8 in [6]). More recently (see [10]) authors applied this result to show that points in a specific degree 2 hypersurface in the Grassmannian $Gr(3, \mathbb{C}^n)$ correspond to generic arrangements of n hyperplanes in $\mathbb{C}^3$ with associated discriminantal arrangement having intersections of multiplicity 3 in codimension 2 (Theorem 5.4 in [10]). In this paper we look at Pappus's configuration (see Figure 1) as a generic arrangement of 6 lines in $\mathbb{P}^2$ which intersection points satisfy certain collinearity conditions (see Figure 2). This allows us to apply results on [6] and [10] to restate and re-prove Pappus's Theorem.

More in details, let $\mathcal{A}$ be a generic arrangement in $\mathbb{C}^3$ and $\mathcal{A}_\infty$ the arrangement of lines in $H_\infty \simeq \mathbb{P}^2$ directions at infinity of planes in $\mathcal{A}$. The space of generic arrangements of n lines in $(\mathbb{P}^2)^n$ is Zariski open set U in the space of all arrangements of n lines in $(\mathbb{P}^2)^n$. On the other hand in $Gr(3, \mathbb{C}^n)$ there is open set U' consisting of 3-spaces intersecting each coordinate hyperplane transversally (i.e. having dimension of intersection equal 2). One has also one set $\tilde{U}$ in $\mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^n)$ consisting of embeddings with image transversal to coordinate hyperplanes and $\tilde{U}/\,\mathrm{GL}(3) = U'$ and $\tilde{U}/(\mathbb{C}^*)^n = U$. Hence generic arrangements in $\mathbb{C}^3$ can be regarded as points in $Gr(3, \mathbb{C}^n)$. Let $\{s_1 < \cdots < s_6\} \subset \{1, \dots, n\}$ be a set of indices of a generic arrangement $\mathcal{A} = \{H_1^0, \dots, H_n^0\}$ in $\mathbb{C}^3$, α_i the normal vectors of H_i^0's and $\beta_{ijl} = \det(\alpha_i, \alpha_j, \alpha_l)$. For any permutation $\sigma \in \mathbf{S_6}$ denote by $[\sigma] = \{\{i_1, i_2\}, \{i_3, i_4\}, \{i_5, i_6\}\}$, $i_j = s_{\sigma(j)}$, and by Q_σ the quadric in $Gr(3, \mathbb{C}^n)$ of equation $\beta_{i_1 i_3 i_4}\beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4}\beta_{i_1 i_5 i_6} = 0$. The following theorem, equivalent to the Pappus's hexagon Theorem, holds.

Theorem 5.3 (Pappus's Theorem)**.** *For any disjoint classes $[\sigma_1]$ and $[\sigma_2]$, there exists a unique class $[\sigma_3]$ disjoint from $[\sigma_1]$ and $[\sigma_2]$ such that $\{\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}\}$ is a Pappus configuration, i.e.*

$$\mathrm{Q}_{\sigma_{i_1}} \cap \mathrm{Q}_{\sigma_{i_2}} = \bigcap_{i=1}^{3} \mathrm{Q}_{\sigma_i}$$

for any $\{i_1, i_2\} \subset [3]$.

In the rest of the paper, we retrieve the Hesse configuration of lines studying intersections of six quadrics of the form Q_σ for opportunely chosen $[\sigma]$. This lead to a better understanding of differences in the combinatorics of Discriminantal arrangement in the complex and real case. Indeed it turns out that this difference is connected with existence of the Hesse arrangement (see [8]) in $\mathbb{P}^2(\mathbb{C})$, but not in $\mathbb{P}^2(\mathbb{R})$.

From above results it seems very likely that a deeper understanding of combinatorics of Discriminantal arrangements arising from non very generic arrangements of hyperplanes in $\mathbb{C}^k$ (i.e. $\mathcal{A} \notin \mathcal{Z}$), could lead to new connections between higher dimensional special configurations of hyperplanes (points) in the projective space and Grassmannian. Vice versa, known results in algebraic geometry could help in understanding the combinatorics of Discriminantal arrangements in the non very generic case. Moreover we conjecture that regularity in the geometry of Discriminantal arrangement could lead to results on hyperplanes arrangements with high multiplicity intersections, e.g., in the case $k=3$, line arrangements in $\mathbb{P}^2$ with high number of triple points (see Remark 6.6). This will be object of further studies.

The content of the paper is the following. In Section 2 we recall definition of Discriminantal arrangement from [7], basic notions on Grassmannian, and definitions and results from [10]. In Section 3 we provide an example of the case of 6 hyperplanes in $\mathbb{C}^3$. In Section 4 we define and study Pappus hypersurface. Section 5 contains Pappus's theorem in $Gr(3,\mathbb{C}^n)$ and its proof. In the last section we study intersections of higher numbers of quadrics and Hesse configuration.

2 Preliminaries

2.1 Discriminantal arrangement

Let $H_i^0, i=1,\dots,n$ be a generic arrangement in $\mathbb{C}^k$, $k<n$ i.e. a collection of hyperplanes such that $\operatorname{codim}\bigcap_{i\in K,|K|=p} H_i^0 = p$. Space of parallel translates $(H_1^0,\dots,H_n^0)$ (or simply when dependence on H_i^0 is clear or not essential) is the space of n-tuples $H_1,\dots,H_n$ such that either $H_i \cap H_i^0 = \emptyset$ or $H_i = H_i^0$ for any $i=1,\dots,n$. One can identify with n-dimensional affine space $\mathbb{C}^n$ in such a way that $(H_1^0,\dots,H_n^0)$ corresponds to the origin. In particular, an ordering of hyperplanes in $\mathcal{A}$ determines the coordinate system in (see [6]).

We will use the compactification of $\mathbb{C}^k$ viewing it as $\mathbb{P}^k(\mathbb{C})\setminus H_\infty$ endowed with collection of hyperplanes $\bar{H}_i^0$ which are projecive closures of affine hyperplanes H_i^0. Condition of genericity is equivalent to $\bigcup_i \bar{H}_i^0$ being a normal crossing divisor in $\mathbb{P}^k(\mathbb{C})$.

Given a generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$ formed by hyperplanes H_i, $i=1,\dots,n$ *the trace at infinity*, denoted by $\mathcal{A}_\infty$, is the arrangement formed by hyperplanes $H_{\infty,i} = \bar{H}_i^0 \cap H_\infty$ in the space $H_\infty \simeq \mathbb{P}^{k-1}(\mathbb{C})$. The trace $\mathcal{A}_\infty$ of an arrangement $\mathcal{A}$ determines the space of parallel translates $\mathbb{S}$ (as a subspace in the space of n-tuples of hyperplanes in $\mathbb{P}^k$).

Fixed a generic arrangement $\mathcal{A}$, consider the closed subset of $\mathbb{S}$ formed by those collections which fail to form a generic arrangement. This subset of $\mathbb{S}$ is a union of hyperplanes $D_L \subset \mathbb{S}$ (see [7]). Each hyperplane D_L corresponds to a subset $L=\{i_1,\dots,i_{k+1}\} \subset [n] := \{1,\dots,n\}$ and it consists of n-tuples of translates of hyperplanes $H_1^0,\dots,H_n^0$ in which translates of $H_{i_1}^0,\dots,H_{i_{k+1}}^0$ fail to form a general position arrangement. The arrangement $\mathcal{B}(n,k,\mathcal{A})$ of hyperplanes D_L is called *Discriminantal arrangement* and has been introduced by Manin and Schechtman in [7]. Notice that $\mathcal{B}(n,k,\mathcal{A})$ depends on the trace at infinity $\mathcal{A}_\infty$ hence it is sometimes more properly denoted by $\mathcal{B}(n,k,\mathcal{A}_\infty)$.

2.2 Good $3s$-partitions

Given $s\geq 2$ and $n\geq 3s$, *a good* $3s$*-partition* (see [10]) is a set $\mathbb{T}=\{L_1,L_2,L_3\}$, with L_i subsets of $[n]$ such that $|L_i|=2s$, $|L_i\cap L_j| = s$ $(i\neq j)$, $L_1\cap L_2\cap L_3=\emptyset$ (in particular $|\bigcup L_i| = 3s$), i.e. $L_1=\{i_1,\dots,i_{2s}\}, L_2=\{i_1,\dots,i_s,i_{2s+1},\dots,i_{3s}\}, L_3 =$

$\{i_{s+1}, \dots, i_{3s}\}$.

Notice that given a generic arrangement $\mathcal{A}$ in $\mathbb{C}^{2s-1}$, subsets L_i define hyperplanes D_{L_i} in the Discriminantal arrangement $\mathcal{B}(n, 2s-1, \mathcal{A}_\infty)$. In this paper we are mainly interested in the case $s = 2$ corresponding to generic arrangements in $\mathbb{C}^3$.

2.3 Matrices $A(\mathcal{A}_\infty)$ and $A_\mathbb{T}(\mathcal{A}_\infty)$

Let $\alpha_i = (a_{i1}, \dots, a_{ik})$ be the normal vectors of hyperplanes H_i, $1 \le i \le n$, in the generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$. Normal here is intended with respect to the usual dot product

$$(a_1, \dots, a_k) \cdot (v_1, \dots, v_k) = \sum_i a_i v_i.$$

Then the normal vectors to hyperplanes D_L, $L = \{s_1 < \cdots < s_{k+1}\} \subset [n]$ in $\mathbb{S} \simeq \mathbb{C}^n$ are nonzero vectors of the form

$$\alpha_L = \sum_{i=1}^{k+1} (-1)^i \det(\alpha_{s_1}, \dots, \hat{\alpha_{s_i}}, \dots, \alpha_{s_{k+1}}) e_{s_i}, \tag{2.1}$$

where $\{e_j\}_{1 \le j \le n}$ is the standard basis of $\mathbb{C}^n$ (cf. [2]).

Let $\mathcal{P}_{k+1}([n]) = \{L \subset [n] \mid |L| = k+1\}$ be the set of cardinality $k+1$ subsets of $[n]$. Following [10] we denote by

$$A(\mathcal{A}_\infty) = (\alpha_L)_{L \in \mathcal{P}_{k+1}([n])}$$

the matrix having in each row the entries of vectors α_L normal to hyperplanes D_L and by $A_\mathbb{T}(\mathcal{A}_\infty)$ the submatrix of $A(\mathcal{A}_\infty)$ with rows α_L, $L \in \mathbb{T}$, $\mathbb{T} \subset \mathcal{P}_{k+1}([n])$. In this paper we are mainly interested in the matrix $A_\mathbb{T}(\mathcal{A}_\infty)$ in the case of $\mathbb{T}$ good 6-partition.

2.4 Grassmannian $Gr(k, \mathbb{C}^n)$

Let $Gr(k, \mathbb{C}^n)$ be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ and

$$\begin{aligned} \gamma\colon\ Gr(k, \mathbb{C}^n) &\to \mathbb{P}(\bigwedge^k \mathbb{C}^n) \\ \langle v_1, \dots, v_k \rangle &\mapsto [v_1 \wedge \cdots \wedge v_k], \end{aligned}$$

the Plücker embedding. Then $[x] \in \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ is in $\gamma(Gr(k, \mathbb{C}^n))$ if and only if the map

$$\begin{aligned} \varphi_x\colon \mathbb{C}^n &\to \bigwedge^{k+1} \mathbb{C}^n \\ v &\mapsto x \wedge v \end{aligned}$$

has kernel of dimension k, i.e. $\ker \varphi_x = \langle v_1, \dots, v_k \rangle$. If $e_1, \dots, e_n$ is a basis of $\mathbb{C}^n$ then $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$, $I = \{i_1, \dots, i_k\} \subset [n]$, $i_1 < \cdots < i_k$, is a basis for $\bigwedge^k \mathbb{C}^n$ and $x \in \bigwedge^k \mathbb{C}^n$ can be written uniquely as

$$x = \sum_{\substack{I \subseteq [n] \\ |I| = k}} \beta_I e_I = \sum_{1 \le i_1 < \cdots < i_k \le n} \beta_{i_1 \dots i_k} (e_{i_1} \wedge \cdots \wedge e_{i_k})$$

where homogeneous coordinates β_I are the Plücker coordinates on $\mathbb{P}(\bigwedge^k \mathbb{C}^n) \simeq \mathbb{P}^{\binom{n}{k}-1}(\mathbb{C})$ associated to the ordered basis $e_1,\dots,e_n$ of $\mathbb{C}^n$. With this choice of basis for $\mathbb{C}^n$ the matrix M_x associated to φ_x is a $\binom{n}{k+1}\times n$ matrix with rows indexed by subsets $I=\{i_1,\dots,i_k\}\subset[n]$ and entries

$$b_{ij}=\begin{cases}(-1)^l\beta_{I\setminus\{j\}} & \text{if } j=i_l\in I,\\ 0 & \text{otherwise.}\end{cases}$$

Plücker relations, i.e. conditions for $\dim(\ker\varphi_x)=k$, are vanishing conditions of all $(n-k+1)\times(n-k+1)$ minors of M_x. It is well known (see for instance [5]) that Plücker relations are degree 2 relations and they can also be written as

$$\sum_{l=0}^{k}(-1)^l\beta_{p_1\dots p_{k-1}q_l}\beta_{q_0\dots\hat{q_l}\dots q_k}=0 \tag{2.2}$$

for any $2k$-tuple $(p_1,\dots,p_{k-1},q_0,\dots,q_k)$.

Remark 2.1. Notice that vectors α_L in the equation (2.1) normal to hyperplanes D_L correspond to rows indexed by L in the Plücker matrix M_x, that is

$$A(\mathcal{A}_\infty)=M_x,$$

up to permutation of rows. Notice that, in particular, $\det(\alpha_{s_1},\dots,\hat{\alpha_{s_i}},\dots,\alpha_{s_{k+1}})$ is the Plücker coordinate β_I, $I=\{s_1,s_2,\dots,s_{k+1}\}\setminus\{s_i\}$.

2.5 Relation between intersections of lines in $\mathcal{A}_\infty$ and quadrics in $Gr(3,\mathbb{C}^n)$

Let $\mathcal{A}=\{H_1^0,\dots,H_n^0\}$ be a generic arrangement in $\mathbb{C}^3$. If there exist $L_1,L_2,L_3\subset[n]$ subsets of indices of cardinality 4, such that codimension of $D_{L_1}\cap D_{L_2}\cap D_{L_3}$ is 2 then $\mathcal{A}$ is *non very generic arrangement* (see [2]).

Let $\mathbb{T}=\{L_1,L_2,L_3\}$ be a good 6-partition of indices $\{s_1,\dots,s_6\}\subset[n]$. In [6], authors proved that the codimension of $D_{L_1}\cap D_{L_2}\cap D_{L_3}$ is 2 if and only if points

$$\bigcap\nolimits_{t\in L_1\cap L_2}H_{\infty,t},\quad \bigcap\nolimits_{t\in L_1\cap L_3}H_{\infty,t}\quad\text{and}\quad\bigcap\nolimits_{t\in L_2\cap L_3}H_{\infty,t}$$

are collinear in H_∞ ([6, Lemma 3.1]).

Since α_{L_i} is vector normal to D_{L_i}, the codimension of $D_{L_1}\cap D_{L_2}\cap D_{L_3}$ is 2 if and only if $\operatorname{rank}A_{\mathbb{T}}(\mathcal{A}_\infty)=2$, i.e. all 3×3 minors of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ vanish. In [10] authors proved the following Lemma.

Lemma 2.2 ([10, Lemma 5.3])**.** *Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^3$ and*

$$\sigma.\mathbb{T}=\{\{i_1,i_2,i_3,i_4\},\ \{i_1,i_2,i_5,i_6\},\ \{i_3,i_4,i_5,i_6\}\}$$

a good 6-partition of indices $s_1<\dots<s_6\in[n]$ such that $i_j=s_{\sigma(j)}$, σ permutation in $\mathbf{S}_6$. Then $\operatorname{rank}A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty)=2$ if and only if $\mathcal{A}$ is a point in the quadric of Grassmannian $Gr(3,\mathbb{C}^n)$ of equation

$$\beta_{i_1i_3i_4}\beta_{i_2i_5i_6}-\beta_{i_2i_3i_4}\beta_{i_1i_5i_6}=0. \tag{2.3}$$

As consequence of above results, we obtain correspondence between points

$$x = \sum_{\substack{I \subset [n] \\ |I|=3}} \beta_I e_I, \ \beta_I \neq 0,$$

in the quadric of equation (2.3) and generic arrangements of n hyperplanes $\mathcal{A}$ in $\mathbb{C}^3$ such that $H_{\infty,i_1} \cap H_{\infty,i_2}$, $H_{\infty,i_3} \cap H_{\infty,i_4}$ and $H_{\infty,i_5} \cap H_{\infty,i_6}$ are collinear in H_∞. Notice that condition $\beta_I \neq 0$ is direct consequence of $\mathcal{A}$ being generic arrangement.

3 Motivating example of Pappus's Theorem for quadrics in $Gr(3, \mathbb{C}^n)$

In classical projective geometry the following theorem is known as Pappus's theorem or Pappus's hexagon theorem.

Theorem 3.1 (*Pappus*)**.** *On a projective plane, consider two lines l_1 and l_2, and a couple of triple points A, B, C and A', B', C' which are on l_1 and l_2 respectively. Let X, Y, Z be points of $AB' \cap A'B$, $AC' \cap A'C$ and $BC' \cap B'C$ respectively. Then there exists a line l_3 passing through the three points X, Y, Z (see Figure 1).*

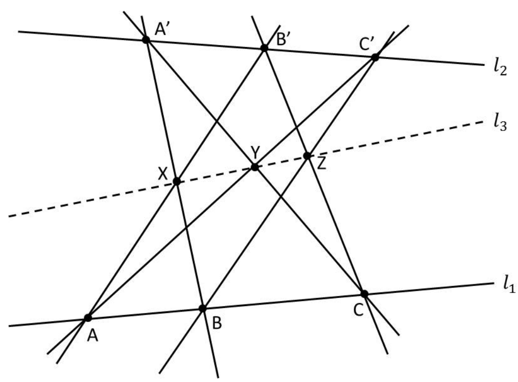

Figure 1: Original Pappus's Theorem.

This theorem was originally stated by Pappus of Alexandria around 290–350 A.D.

In this section, we restate this classical theorem in terms of quadrics in the Grassmannian. Indeed the six lines $AB', A'B, BC', B'C, AC', A'C \in \mathbb{P}^2(\mathbb{C})$ correspond to lines in the trace at infinity $\mathcal{A}_\infty$ of a generic arrangement $\mathcal{A}$ in $\mathbb{C}^3$ and lines l_1, l_2 and l_3 correspond to collinearity conditions for intersection points of lines in $\mathcal{A}_\infty$.

Consider a generic arrangement $\mathcal{A} = \{H_1, \ldots, H_6\}$ of 6 hyperplanes in $\mathbb{C}^3$, $\mathcal{A}_\infty$ its trace at infinity and $\mathbb{T} = \{L_1, L_2, L_3\}$ the good 6-partition defined by $L_1 = \{1, 2, 3, 4\}$, $L_2 = \{1, 2, 5, 6\}$, $L_3 = \{3, 4, 5, 6\}$. By Lemma 2.2 we get that the triple points

$$\bigcap_{i \in L_1 \cap L_2} \bar{H}_i \cap H_\infty, \quad \bigcap_{i \in L_1 \cap L_3} \bar{H}_i \cap H_\infty, \quad \bigcap_{i \in L_2 \cap L_3} \bar{H}_i \cap H_\infty$$

are collinear if and only if $\mathcal{A}$ is a point of the quadric

$$\mathrm{Q}_1\colon \beta_{134}\beta_{256} - \beta_{234}\beta_{156} = 0$$

in $Gr(3, \mathbb{C}^6)$.

Analogously if

$$\mathbb{T}' = \{L'_1, L'_2, L'_3\},\ L'_1 = \{4, 6, 2, 5\},\ L'_2 = \{4, 6, 1, 3\},\ L'_3 = \{2, 5, 1, 3\}$$

and

$$\mathbb{T}'' = \{L''_1, L''_2, L''_3\},\ L''_1 = \{2, 4, 1, 6\},\ L''_2 = \{2, 4, 3, 5\},\ L''_3 = \{1, 6, 3, 5\}$$

are different good 6-partitions then triple points

$$\bigcap\nolimits_{i\in L'_1\cap L'_2} \bar{H}_i \cap H_\infty, \quad \bigcap\nolimits_{i\in L'_1\cap L'_3} \bar{H}_i \cap H_\infty, \quad \bigcap\nolimits_{i\in L'_2\cap L'_3} \bar{H}_i^{t_i} \cap H_\infty$$

and

$$\bigcap\nolimits_{i\in L''_1\cap L''_2} \bar{H}_i \cap H_\infty, \quad \bigcap\nolimits_{i\in L''_1\cap L''_3} \bar{H}_i \cap H_\infty, \quad \bigcap\nolimits_{i\in L''_2\cap L''_3} \bar{H}_i \cap H_\infty$$

are collinear if and only if $\mathcal{A}$ is, respectively, a point of quadrics

$$\mathrm{Q}_2\colon \beta_{425}\beta_{613} - \beta_{625}\beta_{413} = 0 \quad \text{and}$$
$$\mathrm{Q}_3\colon \beta_{216}\beta_{435} - \beta_{416}\beta_{235} = 0.$$

With above remarks and notations we can restate Pappus's Theorem as follows (see Figure 2).

Theorem 3.2 (Pappus's Theorem)**.** *Let $\mathcal{A} = \{H_1, \ldots, H_6\}$ be a generic arrangement of hyperplanes in $\mathbb{C}^3$. If $\mathcal{A}$ is a point of two of three quadrics $\mathrm{Q}_1, \mathrm{Q}_2$ and Q_3 in the Grassmannian $Gr(3, \mathbb{C}^6)$, then $\mathcal{A}$ is also a point of the third. In other words*

$$\mathrm{Q}_{i_1} \cap \mathrm{Q}_{i_2} = \bigcap_{i=1}^{3} \mathrm{Q}_i, \quad \{i_1, i_2\} \subset [3].$$

We develop this argument in the following sections providing in Theorem 5.3 a general statement on quadrics in the Grassmannian which implies Pappus hexagon Theorem in the projective plane.

4 Pappus Variety

In this section, we consider a generic arrangement $\{H_1, \ldots, H_n\}$ in $\mathbb{C}^3$ ($n \geq 6$). Let's introduce basic notations that we will use in the rest of the paper.

Notation. *Let $\{s_1, \ldots, s_6\}$ be a subset of indices $\{1, \ldots, n\}$ and $\mathbb{T} = \{L_1, L_2, L_3\}$ be the good 6-partition given by*

$$L_1 = \{s_1, s_2, s_3, s_4\},\ L_2 = \{s_1, s_2, s_5, s_6\} \text{ and } L_3 = \{s_3, s_4, s_5, s_6\}.$$

Then for any permutation $\sigma \in \boldsymbol{S}_6$ we denote by $\sigma.\mathbb{T} = \{\sigma.L_1, \sigma.L_2, \sigma.L_3\}$ the good 6-partition given by subsets

$$\sigma.L_1 = \{i_1, i_2, i_3, i_4\},\ \sigma.L_2 = \{i_1, i_2, i_5, i_6\} \text{ and } \sigma.L_3 = \{i_3, i_4, i_5, i_6\}$$

with $i_j = s_{\sigma(j)}$. Accordingly, we denote by Q_σ the quadric in $Gr(3, \mathbb{C}^n)$ of equation

$$\mathrm{Q}_\sigma\colon \beta_{i_1 i_3 i_4}\beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4}\beta_{i_1 i_5 i_6} = 0.$$

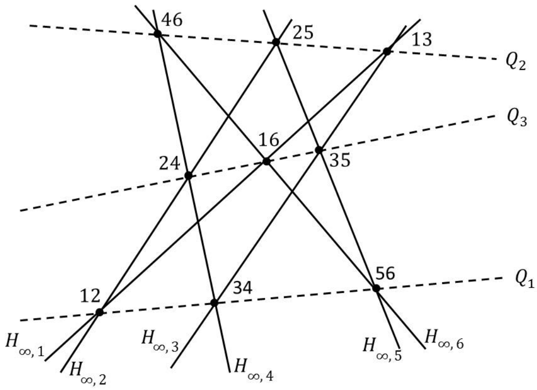

Figure 2: Trace at infinity of $\mathcal{A} \in \bigcap_{i=1}^{3} \mathrm{Q}_i$. In the figure ij denotes $H_{\infty,i} \cap H_{\infty,j}$.

The following lemma holds.

Lemma 4.1. *Let $\sigma, \sigma' \in \boldsymbol{S}_6$ be distinct permutations, then $\mathrm{Q}_\sigma = \mathrm{Q}_{\sigma'}$ if and only if there exists $\tau \in \boldsymbol{S}_3$ such that $\sigma.L_i \cap \sigma.L_j = \sigma'.L_{\tau(i)} \cap \sigma'.L_{\tau(j)}$ $(1 \leq i < j \leq 3)$.*

Proof. By definition of good 6-partition we have that

$$L_1 = (L_1 \cap L_2) \cup (L_1 \cap L_3),$$
$$L_2 = (L_2 \cap L_1) \cup (L_2 \cap L_3),$$
$$L_3 = (L_3 \cap L_1) \cup (L_3 \cap L_2).$$

Then there exists $\tau \in \mathbf{S}_3$ such that σ and σ' satisfy $\sigma.L_i \cap \sigma.L_j = \sigma'.L_{\tau(i)} \cap \sigma'.L_{\tau(j)}$ $(1 \leq i < j \leq 3)$ if and only if $\sigma.L_l = \sigma'.L_{\tau(l)}$ for $l = 1, 2, 3$, that is $A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty)$ is obtained by permuting rows of $A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty)$. It follows that $\operatorname{rank} A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty) = 2$ if and only if $\operatorname{rank} A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty) = 2$ and hence by Lemma 2.2 this is equivalent to $\mathrm{Q}_\sigma \cap N_{s_1,\dots,s_6} = \mathrm{Q}_{\sigma'} \cap N_{s_1,\dots,s_6}$, where

$$N_{s_1,\dots,s_6} = \{x = \sum_{\substack{I \subseteq [n] \\ |I|=3}} \beta_I e_I \mid \beta_I \neq 0 \text{ for any } I \subset \{s_1, \dots, s_6\}\}.$$

Since $N_{s_1,\dots,s_6}$ is dense open set in $\gamma(Gr(3, \mathbb{C}^n))$, $\mathrm{Q}_\sigma \cap N_{s_1,\dots,s_6} = \mathrm{Q}_{\sigma'} \cap N_{s_1,\dots,s_6}$ if and only if $\mathrm{Q}_\sigma = \mathrm{Q}_{\sigma'}$. Vice versa if $\mathrm{Q}_\sigma \cap N_{s_1,\dots,s_6} = \mathrm{Q}_{\sigma'} \cap N_{s_1,\dots,s_6}$, then any generic arrangement $\mathcal{A}$ corresponding to a point in $\mathrm{Q}_\sigma \cap N_{s_1,\dots,s_6}$ corresponds to a point in $\mathrm{Q}_{\sigma'} \cap N_{s_1,\dots,s_6}$, that is $\operatorname{rank} A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty) = 2$ if and only if $\operatorname{rank} A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty) = 2$. It follows that $A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty)$ and $A_{\sigma'.\mathbb{T}}(\mathcal{A}_\infty)$ are submatrices of $A(\mathcal{A}_\infty)$ defined by the same three rows, i.e. $\sigma.L_l = \sigma'.L_{\tau(l)}$ for $l = 1, 2, 3$. □

Definition 4.2. For any 6 fixed indices $T = \{s_1, \dots, s_6\} \subset [n]$ the Pappus Variety is the hypersurface in $Gr(3, \mathbb{C}^n)$ given by

$$\mathcal{P}_T = \bigcup_{\sigma \in \mathbf{S}_6} \mathrm{Q}_\sigma.$$

Notice that all the content of this section and the following section is based on the choice of six indices $\{s_1 < \cdots < s_6\} \subset [n]$. This is related to result in Theorem 3.8 in [6] and, consequently, Lemma 5.3 in [10] (Lemma 2.2 in this paper). Indeed Theorem 3.8 in [6] states that in order to study special configurations of n lines in $\mathbb{P}^2$, that is non very generic arrangements of n lines in $\mathbb{P}^2$, it is sufficient to study subsets of six lines out of n. On the other hand since Pappus Variety can be defined inside $Gr(3, \mathbb{C}^n)$, we decided to keep the discussion more general picking six indices $\{s_1 < \cdots < s_6\} \subset [n]$ instead of simply study the case $Gr(3, \mathbb{C}^6)$ (see also Remark 6.7).

For $\sigma, \sigma' \in \mathbf{S}_6$ we define the equivalence relation $\sigma.\mathbb{T} \sim \sigma'.\mathbb{T}$ corresponding to $\mathrm{Q}_\sigma = \mathrm{Q}_{\sigma'}$ as following:

$$\sigma.\mathbb{T} \sim \sigma'.\mathbb{T} \Leftrightarrow \exists\, \tau \in \mathbf{S}_3 \text{ such that } \sigma.L_i \cap \sigma.L_j = \sigma'.L_{\tau(i)} \cap \sigma'.L_{\tau(j)} \ (1 \le i < j \le 3).$$

We denote by $[\sigma]$ the equivalence class containing $\sigma.\mathbb{T}$ and by Q_σ the corresponding quadric (notice that σ in the notation Q_σ can be any representative of $[\sigma]$). By Lemma 4.1 $[\sigma]$ only depends on couples $L_i \cap L_j$ hence for each class $[\sigma]$ we can choice a representative

$$\tilde{\sigma}.\mathbb{T}_0 = \big\{\{j_1, j_2, j_3, j_4\},\ \{j_1, j_2, j_5, j_6\},\ \{j_3, j_4, j_5, j_6\}\big\}$$

such that $j_1 < j_2$, $j_3 < j_4$, $j_5 < j_6$ and $j_1 < j_3 < j_5$ and we can equivalently define

$$[\sigma] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\}.$$

Since the number of choices of $[\sigma]$ is $\frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!} = 15$, Pappus Variety is composed by 15 quadrics. Finally remark that

$$[\sigma] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\} \text{ and}$$
$$[\sigma'] = \big\{\{j'_1, j'_2\},\ \{j'_3, j'_4\},\ \{j'_5, j'_6\}\big\}$$

are *disjoint*, i.e. $[\sigma] \cap [\sigma'] = \emptyset$, if and only if $\{j_{2l-1}, j_{2l}\} \neq \{j'_{2l'-1}, j'_{2l'}\}$ for any $1 \le l, l' \le 3$.

Definition 4.3 (Pappus configuration)**.** Let $[\sigma_1], [\sigma_2]$ and $[\sigma_3]$ be disjoint classes, a Pappus configuration is a set $\{\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}\}$ of quadrics in $Gr(3, \mathbb{C}^n)$ such that

$$\mathrm{Q}_{\sigma_{i_1}} \cap \mathrm{Q}_{\sigma_{i_2}} = \bigcap_{i=1}^{3} \mathrm{Q}_{\sigma_i}$$

for any $\{i_1, i_2\} \subset [3]$.

Quadrics $\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}$ are said to be in Pappus configuration if $\{\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}\}$ is a Pappus configuration.

Remark 4.4. Fixed a class of good 6-partition $[\sigma] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\}$, we shall count the number of disjoint classes.

First let's count the number of classes $[\sigma'] = \big\{\{j'_1, j'_2\},\ \{j'_3, j'_4\},\ \{j'_5, j'_6\}\big\}$ not disjoint and distinct from $[\sigma]$. Since $[\sigma]$ and $[\sigma']$ are distinct, only one couple $\{j'_l, j'_{l+1}\}$ is contained in $[\sigma]$. Without lost of generality we can assume $\{j_l, j_{l+1}\} = \{j'_1, j'_2\}$ (l is either $1, 3$ or 5) then pairs $\{j'_3, j'_4\}$ and $\{j'_5, j'_6\}$ are not in the same set, i.e. we have two possibilities:

$$\{j'_3, j'_5\} \text{ and } \{j'_4, j'_6\} \in [\sigma],$$

or

$$\{j'_3, j'_6\} \text{ and } \{j'_4, j'_5\} \in [\sigma].$$

Hence there are $2 \cdot 3 + 1 = 7$ not disjoint classes from $[\sigma]$ and, since the number of all classes is 15, we get that any fixed $[\sigma]$ admits exactly $15 - 7 = 8$ disjoint classes.

5 Pappus's Theorem

In this section we restate Pappus's Theorem for quadrics in $Gr(3, \mathbb{C}^n)$ by using notation introduced in the previous section. For a fixed class $[\sigma] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\}$ let's denote by $G_{[\sigma]}$ the free group generated by permutations of elements in each subset of $[\sigma]$, that is

$$G_{[\sigma]} = \big\langle (j_{2l-1}\, j_{2l}) \in \mathbf{S}_6 \mid l = 1, 2, 3 \big\rangle,$$

and, for any class, $[\sigma']$ let's define the set

$$\operatorname{orbit}_{G_{[\sigma]}}([\sigma']) = \big\{\tau[\sigma'] \mid \tau \in G_{[\sigma]}\big\}$$

where τ acts naturally as permutation of entries of each set in $[\sigma']$.

Remark 5.1. The action of $G_{[\sigma]}$ on class $[\sigma']$ disjoint from $[\sigma]$ is faithful. Indeed let $\tau, \tau' \in G_{[\sigma]}$ be such that $\tau[\sigma'] = \tau'[\sigma']$ then $\tau^{-1}\tau'[\sigma'] = [\sigma']$, i.e. $\tau^{-1}\tau' \in G_{[\sigma']}$. Thus we get $\tau^{-1}\tau' \in G_{[\sigma]} \cap G_{[\sigma']}$. Since $[\sigma]$ and $[\sigma']$ are disjoint, $G_{[\sigma]} \cap G_{[\sigma']} = \{e\}$, i.e., $\tau = \tau'$. Remark that $|\operatorname{orbit}_{G_{[\sigma]}}([\sigma'])| = |G_{[\sigma]}| = 8$ and $\tau[\sigma] = [\sigma]$ for any $\tau \in G_{[\sigma]}$.

Lemma 5.2. *Let $[\sigma]$ and $[\sigma']$ be disjoint classes, then*

$$\operatorname{orbit}_{G_{[\sigma]}}([\sigma']) = \big\{[\sigma''] \mid [\sigma] \cap [\sigma''] = \emptyset\big\}.$$

Proof. First we prove that $\operatorname{orbit}_{G_{[\sigma]}}([\sigma']) \subset \big\{[\sigma''] \mid [\sigma] \cap [\sigma''] = \emptyset\big\}$. Let

$$[\sigma] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\} \text{ and}$$
$$[\sigma'] = \big\{\{j'_1, j'_2\},\ \{j'_3, j'_4\},\ \{j'_5, j'_6\}\big\}$$

be disjoint, then $|\{j_{2l-1}, j_{2l}\} \cap \{j'_{2m-1}, j'_{2m}\}| \leq 1$. Since $\tau \in G_{[\sigma]}$ permutes only j_{2l-1} and j_{2l} then $\tau[\sigma'] \cap [\sigma] = \emptyset$, that is $\tau[\sigma']$ is disjoint from $[\sigma]$, i.e. $\tau[\sigma'] \in \big\{[\sigma''] \mid [\sigma] \cap [\sigma''] = \emptyset\big\}$. Since $G_{[\sigma]}$ is faithful, $|\operatorname{orbit}_{G_{[\sigma]}}([\sigma'])| = 8$ and, by calculations in the Remark 4.4, $|\big\{[\sigma''] \mid [\sigma] \cap [\sigma''] = \emptyset\big\}| = 8$, it follows that $\operatorname{orbit}_{G_{[\sigma]}}([\sigma']) = \big\{[\sigma''] \mid [\sigma] \cap [\sigma''] = \emptyset\big\}$. □

The following theorem holds.

Theorem 5.3 (Pappus's Theorem)**.** *For any disjoint classes $[\sigma]$ and $[\sigma']$, there exists a unique class $[\sigma'']$ disjoint from $[\sigma]$ and $[\sigma']$ such that $\{\mathrm{Q}_\sigma, \mathrm{Q}_{\sigma'}, \mathrm{Q}_{\sigma''}\}$ is a Pappus configuration.*

Remark 5.4. Let $[\sigma_1]$ and $[\sigma_2]$, $[\sigma_i] = \{\{j_{1,i}, j_{2,i}\},\ \{j_{3,i}, j_{4,i}\},\ \{j_{5,i}, j_{6,i}\}\}$, $i = 1, 2$ be classes of indices in $\{1, \dots, 6\}$. Recall the following facts (see Section 2 and Lemma 2.2):

i) If

$$x = \sum_{\substack{I \subseteq [6] \\ |I|=3}} \beta_I e_I,\ \beta_I \neq 0$$

is a point in Q_{σ_i} then any arrangement $\mathcal{A} \in \mathbb{C}^3$ such that $A(\mathcal{A}_\infty) = M_x$ is an arrangement of 6 planes in general position in $\mathbb{C}^3$ with lines in $\mathcal{A}_\infty$ such that points

$$H_{\infty,j_{1,i}} \cap H_{\infty,j_{2,i}}, \quad H_{\infty,j_{3,i}} \cap H_{\infty,j_{4,i}} \quad \text{and} \quad H_{\infty,j_{5,i}} \cap H_{\infty,j_{6,i}}$$

are collinear.

ii) Vice versa if $\mathcal{A}$ is an arrangement of 6 lines in general position in $\mathbb{C}^3$ with the intersection points

$$H_{\infty,j_{1,i}} \cap H_{\infty,j_{2,i}}, \quad H_{\infty,j_{3,i}} \cap H_{\infty,j_{4,i}} \quad \text{and} \quad H_{\infty,j_{5,i}} \cap H_{\infty,j_{6,i}}$$

collinear, then any point

$$x = \sum_{\substack{I \subseteq [6] \\ |I|=3}} \beta_I e_I$$

such that $M_x = A(\mathcal{A}_\infty)$ verifies $\beta_I \neq 0$ and $x \in \mathrm{Q}_{\sigma_i}$.

From ii) it follows that if $\mathcal{A}_\infty$ is an arrangement of 6 lines in general position in $\mathbb{P}^2$ such that

$$H_{\infty,j_{1,i}} \cap H_{\infty,j_{2,i}}, \quad H_{\infty,j_{3,i}} \cap H_{\infty,j_{4,i}} \quad \text{and} \quad H_{\infty,j_{5,i}} \cap H_{\infty,j_{6,i}}$$

are collinear for $i = 1, 2$, then any point

$$x = \sum_{\substack{I \subseteq [6] \\ |I|=3}} \beta_I e_I$$

such that $M_x = A(\mathcal{A}_\infty)$ belongs to $\mathrm{Q}_{\sigma_1} \cap \mathrm{Q}_{\sigma_2}$. Moreover $[\sigma_1]$ and $[\sigma_2]$ are disjoint classes. By Theorem 5.3 there exists a third class

$$[\sigma_3] = \{\{j_{1,3}, j_{2,3}\}, \{j_{3,3}, j_{4,3}\}, \{j_{5,3}, j_{6,3}\}$$

such that $\{\mathrm{Q}_{\sigma_1}, \mathrm{Q}_{\sigma_2}, \mathrm{Q}_{\sigma_3}\}$ is a Pappus configuration. Then $x \in \bigcap_{1 \leq i \leq 3} \mathrm{Q}_{\sigma_i}$ which implies, by i) that also

$$H_{\infty,j_{1,3}} \cap H_{\infty,j_{2,3}}, \quad H_{\infty,j_{3,3}} \cap H_{\infty,j_{4,3}} \quad \text{and} \quad H_{\infty,j_{5,3}} \cap H_{\infty,j_{6,3}}$$

have to be collinear. That is Theorem 5.3 implies Pappus hexagon Theorem in the plane (see Figure 2).

Notice that Theorem 5.3 is slightly more general than Pappus hexagon Theorem since it also applies to the case in which some $\beta_I = 0$.

Proof of Theorem 5.3. Following example in Section 3, for any class

$$[\omega_1] = \big\{\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}\big\}$$

let's consider disjoint classes

$$[\omega_2] = \big\{\{j_1, j_3\}, \{j_2, j_5\}, \{j_4, j_6\}\big\} \text{ and}$$
$$[\omega_3] = \big\{\{j_1, j_6\}, \{j_2, j_4\}, \{j_3, j_5\}\big\}.$$

The corresponding quadrics have equations:

$$\begin{aligned}
\mathrm{Q}_{\omega_1} &: \beta_{j_1j_3j_4}\beta_{j_2j_5j_6} - \beta_{j_2j_3j_4}\beta_{j_1j_5j_6} = 0,\\
\mathrm{Q}_{\omega_2} &: \beta_{j_4j_2j_5}\beta_{j_6j_1j_3} - \beta_{j_6j_2j_5}\beta_{j_4j_1j_3} = 0,\\
\mathrm{Q}_{\omega_3} &: \beta_{j_5j_1j_6}\beta_{j_3j_2j_4} - \beta_{j_3j_1j_6}\beta_{j_5j_2j_4} = 0.
\end{aligned}$$

By definition of β_{ijk}, equations of Q_{ω_2} and Q_{ω_3} can equivalently be written as

$$\begin{aligned}
\mathrm{Q}_{\omega_2} &: \beta_{j_2j_4j_5}\beta_{j_1j_3j_6} + \beta_{j_2j_5j_6}\beta_{j_1j_3j_4} = 0,\\
\mathrm{Q}_{\omega_3} &: \beta_{j_1j_5j_6}\beta_{j_2j_3j_4} + \beta_{j_1j_3j_6}\beta_{j_2j_4j_5} = 0.
\end{aligned}$$

If we denote left side of defining equations of Q_{ω_i} by P_{ω_i} then

$$P_{\omega_2} - P_{\omega_1} = P_{\omega_3},$$

that is zeros of any two polynomials $P_{\omega_{i_1}}, P_{\omega_{i_2}}$ are zeros of $P_{\omega_{i_3}}$, $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. We get

$$\mathrm{Q}_{\omega_{i_1}} \cap \mathrm{Q}_{\omega_{i_2}} = \bigcap_{i=1}^{3} \mathrm{Q}_{\omega_i}$$

for any $\{i_1, i_2\} \subset [3]$, i.e. $\mathrm{Q}_{\omega_1}, \mathrm{Q}_{\omega_2}$ and Q_{ω_3} are in Pappus configuration.

By Lemma 5.2, since $[\omega_1] \cap [\omega_2] = \emptyset$, the set of disjoint classes from $[\omega_1]$ is given by

$$\big\{[\sigma_0] \mid [\omega_1] \cap [\sigma_0] = \emptyset\big\} = \big\{\tau_0[\omega_2] \mid \tau_0 \in G_{[\omega_1]}\big\}.$$

Then if $[\sigma']$ is disjoint from $[\omega_1]$, there exists a unique element $\tau \in G_{[\omega_1]}$ such that $[\sigma'] = \tau[\omega_2]$. That is, for a generic class $[\omega_1]$, any disjoint couple $([\omega_1], [\sigma'])$ is of the form $([\omega_1], \tau[\omega_2]) = (\tau[\omega_1], \tau[\omega_2])$ and we have

$$\begin{aligned}
\mathrm{Q}_{\omega_1} = \mathrm{Q}_{\tau\omega_1} &: \beta_{\tau(j_1)\tau(j_3)\tau(j_4)}\beta_{\tau(j_2)\tau(j_5)\tau(j_6)} - \beta_{\tau(j_2)\tau(j_3)\tau(j_4)}\beta_{\tau(j_1)\tau(j_5)\tau(j_6)} = 0,\\
\mathrm{Q}_{\sigma'} = \mathrm{Q}_{\tau\omega_2} &: \beta_{\tau(j_4)\tau(j_2)\tau(j_5)}\beta_{\tau(j_6)\tau(j_1)\tau(j_3)} - \beta_{\tau(j_6)\tau(j_2)\tau(j_5)}\beta_{\tau(j_4)\tau(j_1)\tau(j_3)} = 0.
\end{aligned}$$

By antisymmetric property of indices of β_{ijk}, if we denote by P_{ω_1} and $P_{\sigma'}$ the left side of above equations, i.e.

$$\begin{aligned}
P_{\omega_1} &= \beta_{\tau(j_1)\tau(j_3)\tau(j_4)}\beta_{\tau(j_2)\tau(j_5)\tau(j_6)} - \beta_{\tau(j_2)\tau(j_3)\tau(j_4)}\beta_{\tau(j_1)\tau(j_5)\tau(j_6)},\\
P_{\sigma'} &= \beta_{\tau(j_4)\tau(j_2)\tau(j_5)}\beta_{\tau(j_6)\tau(j_1)\tau(j_3)} - \beta_{\tau(j_6)\tau(j_2)\tau(j_5)}\beta_{\tau(j_4)\tau(j_1)\tau(j_3)}
\end{aligned}$$

then

$$P_{\sigma''} := P_{\sigma'} - P_{\omega_1} = \beta_{\tau(j_5)\tau(j_1)\tau(j_6)}\beta_{\tau(j_3)\tau(j_2)\tau(j_4)} - \beta_{\tau(j_3)\tau(j_1)\tau(j_6)}\beta_{\tau(j_5)\tau(j_2)\tau(j_4)}$$

is the defining polynomial of $\mathrm{Q}_{\tau\omega_3}$. That is $[\sigma'']$ is uniquely determined by disjoint couple $([\omega_1], [\sigma'])$. □

From proof of Theorem 5.3 we get that for any class

$$[\omega_1] = \big\{\{j_1, j_2\},\ \{j_3, j_4\},\ \{j_5, j_6\}\big\}$$

if we denote

$$[\omega_2] = \big\{\{j_1, j_3\},\ \{j_2, j_5\},\ \{j_4, j_6\}\big\} \text{ and}$$
$$[\omega_3] = \big\{\{j_1, j_6\},\ \{j_2, j_4\},\ \{j_3, j_5\}\big\},$$

then all Pappus configurations are of the form $\{\mathrm{Q}_{\tau\omega_1}, \mathrm{Q}_{\tau\omega_2}, \mathrm{Q}_{\tau\omega_3}\}$, $\tau \in G_{[\omega_1]}$ and the following Corollary holds.

Notice that the proof of Theorem 5.3 only uses equations of quadrics Q_σ and hence provides alternative proof to Pappus hexagon Theorem. In particular it is also alternative to classical proof based on Grassmann-Plücker relations. Indeed the latter proof uses the fact that points in Pappus configurations verify the Grassmann-Plücker relations while, in our cases, quadrics Q_σ are proper quadrics in the Grassmannian, i.e. equations of quadrics Q_σ are not Grassmann-Plücker relations.

Corollary 5.5. *The number of Pappus configurations* $\{\mathrm{Q}_\sigma, \mathrm{Q}_{\sigma'}, \mathrm{Q}_{\sigma''}\}$ *in* $Gr(3,\mathbb{C}^6)$ *is* 20.

Proof. By Remark 4.4 the number of $[\sigma]$ is 15 and by Lemma 5.2 each fixed class $[\sigma]$ admits 8 disjoint classes. By Theorem 5.3 if $[\sigma]$ and $[\sigma']$ are fixed, $[\sigma'']$ is uniquely determined, thus the number of the sets $\big\{[\sigma], [\sigma'], [\sigma'']\big\}$ is $15 \times 8/3! = 20$. □

Corollary 5.5 establishes that for any given 6 lines in $\mathbb{P}^2$ there are 20 possible combinations of their intersections that give rise to a Pappus's configuration like the one in Figure 2.

6 Intersections of quadrics

In this section we study intersections of quadrics in $Gr(3,\mathbb{C}^n)$. In particular we are interested in the intersection of sets

$$\mathrm{Q}_\sigma^\circ = \mathrm{Q}_\sigma \cap \big\{x = \sum_{\substack{I\subset[n]\\ |I|=3}} \beta_I e_I \mid \beta_I \neq 0 \text{ for any } I \subset \{s_1, \dots, s_6\}\big\}$$

of points in quadrics Q_σ that correspond to arrangements of lines in $\mathbb{P}^2(\mathbb{C})$ with subarrangement $\{H_{s_1}, \dots, H_{s_6}\}$ generic. The following lemma holds.

Lemma 6.1. *If* $[\sigma_1], [\sigma_2], [\sigma_3]$ *are distinct and pairwise not disjoint classes then*

$$\mathrm{Q}_{\sigma_1}^\circ \cap \mathrm{Q}_{\sigma_2}^\circ \cap \mathrm{Q}_{\sigma_3}^\circ = \emptyset.$$

Proof. If $[\sigma_1], [\sigma_2], [\sigma_3]$ are not disjoint then either

(1) $|[\sigma_1] \cap [\sigma_2] \cap [\sigma_3]| = 1$ or

(2) $|[\sigma_{i_1}] \cap [\sigma_{i_2}]| = 1\, (1 \leq i_1 < i_2 \leq 3)$ and $[\sigma_1] \cap [\sigma_2] \cap [\sigma_3] = \emptyset$.

(1) Assume $[\sigma_1] \cap [\sigma_2] \cap [\sigma_3] = \{i_1, i_2\}$. Let $[\sigma_1] = \big\{\{i_1, i_2\}, \{i_3, i_4\}, \{i_5, i_6\}\big\}$, $[\sigma_2] = \big\{\{i_1, i_2\}, \{i_3, i_5\}, \{i_4, i_6\}\big\}$, and $[\sigma_3] = \big\{\{i_1, i_2\}, \{i_3, i_6\}, \{i_4, i_5\}\big\}$ then we obtain the following quadrics

$$\mathrm{Q}_{\sigma_1} : \beta_{i_1 i_3 i_4}\beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4}\beta_{i_1 i_5 i_6} = 0,$$
$$\mathrm{Q}_{\sigma_2} : \beta_{i_1 i_3 i_5}\beta_{i_2 i_4 i_6} - \beta_{i_2 i_3 i_5}\beta_{i_1 i_4 i_6} = 0,$$
$$\mathrm{Q}_{\sigma_3} : \beta_{i_1 i_3 i_6}\beta_{i_2 i_4 i_5} - \beta_{i_2 i_3 i_6}\beta_{i_1 i_4 i_5} = 0.$$

Any point $x \in \mathrm{Q}^\circ_{\sigma_1} \cap \mathrm{Q}^\circ_{\sigma_2}$ belongs to $Gr(3, \mathbb{C}^n)$, that is x satisfies Plücker relations in (2.2). In particular $x \in Pl_1 \cap Pl_2$ where Pl_1 and Pl_2 are the quadrics:

$$Pl_1 \colon \beta_{i_1 i_3 i_2}\beta_{i_4 i_5 i_6} - \beta_{i_1 i_3 i_4}\beta_{i_2 i_5 i_6} + \beta_{i_1 i_3 i_5}\beta_{i_2 i_4 i_6} - \beta_{i_1 i_3 i_6}\beta_{i_2 i_4 i_5} = 0,$$
$$Pl_2 \colon \beta_{i_2 i_3 i_1}\beta_{i_4 i_5 i_6} - \beta_{i_2 i_3 i_4}\beta_{i_1 i_5 i_6} + \beta_{i_2 i_3 i_5}\beta_{i_1 i_4 i_6} - \beta_{i_2 i_3 i_6}\beta_{i_1 i_4 i_5} = 0.$$

Notice that Pl_1 and Pl_2 can be obtained from equations in (2.2) considering the 6-tuples $(p_1, p_2, q_0, q_1, q_2, q_3) = (i_1, i_3, i_2, i_4, i_5, i_6)$ and $(i_2, i_3, i_1, i_4, i_5, i_6)$ respectively. We get

$$\mathrm{Q}_{\sigma_2} - \mathrm{Q}_{\sigma_1} - Pl_1 + Pl_2 \colon \beta_{i_1 i_3 i_6}\beta_{i_2 i_4 i_5} - \beta_{i_2 i_3 i_6}\beta_{i_1 i_4 i_5} + 2(\beta_{i_1 i_2 i_3}\beta_{i_4 i_5 i_6}) = 0.$$

Since $\beta_{i_1 i_2 i_3} \neq 0$ and $\beta_{i_4 i_5 i_6} \neq 0$ then $\beta_{i_1 i_2 i_3}\beta_{i_4 i_5 i_6} \neq 0$ and hence

$$\beta_{i_1 i_3 i_6}\beta_{i_2 i_4 i_5} - \beta_{i_2 i_3 i_6}\beta_{i_1 i_4 i_5} \neq 0,$$

that is $x \notin \mathrm{Q}^\circ_{\sigma_3}$.

(2) Assume $[\sigma_1] \cap [\sigma_2] = \{i_1, i_2\}$, $[\sigma_1] \cap [\sigma_3] = \{i_3, i_4\}$ and $[\sigma_2] \cap [\sigma_3] = \{i_5, i_6\}$ and name $P_1 = \{i_1, i_2\}$, $P_2 = \{i_3, i_4\}$, $P_3 = \{i_5, i_6\}$. To any point $x \in \mathrm{Q}^\circ_{\sigma_1} \cap \mathrm{Q}^\circ_{\sigma_2} \cap \mathrm{Q}^\circ_{\sigma_3}$ corresponds the existence of an arrangement with a generic sub-arrangement indexed by $\{i_1, \ldots, i_6\}$ which trace at infinity $\{H_{\infty,i_1}, \ldots, H_{\infty,i_6}\}$ satisfyies collinearity conditions as in Figure 3. That is there exist couples $P_4 \in [\sigma_1], P_5 \in [\sigma_2]$ and $P_6 \in [\sigma_3]$ that correspond, respectively, to intersection points p_4, p_5 and p_6 of lines in $\{H_{\infty,i_1}, \ldots, H_{\infty,i_6}\}$ (see Figure 3).

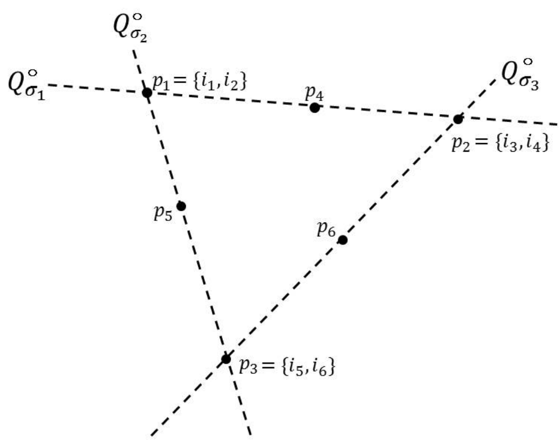

Figure 3: Case (2) trace at infinity of $\mathcal{A} \in \bigcap_{i=1}^{3} \mathrm{Q}^\circ_{\sigma_i}$, $\{i, j\}$ corresponds to $H_{\infty,i} \cap H_{\infty,j}$.

By definition of P_1, P_2 and P_3 we have

$$P_3 = \{i_5, i_6\} \in \big(\{i_1, \ldots, i_6\} \setminus P_1\big) \cap \big(\{i_1, \ldots, i_6\} \setminus P_2\big).$$

On the other hand, if P_4 is different from P_1 and P_2 in $\mathrm{Q}^\circ_{\sigma_1}$ then $P_4 = \big(\{i_1, \ldots, i_6\} \setminus P_1\big) \cap \big(\{i_1, \ldots, i_6\} \setminus P_2\big)$. Thus we get $P_3 = P_4$ and, similarly, $P_5 = P_2$ and $P_6 = P_1$, that is $\mathrm{Q}^\circ_{\sigma_1} = \mathrm{Q}^\circ_{\sigma_2} = \mathrm{Q}^\circ_{\sigma_3}$ which contradict hypothesis. □

Lemma 6.2. *For any three pairwise disjoint classes* $[\sigma_1],[\sigma_2],[\sigma_3]$, *either* $\{\mathrm{Q}_{\sigma_1},\mathrm{Q}_{\sigma_2},\mathrm{Q}_{\sigma_3}\}$ *is a Pappus configuration or*

$$\bigcap_{i=1}^{3}\mathrm{Q}^{\circ}_{\sigma_i}=\emptyset.$$

Proof. By Pappus's Theorem, for any two disjoint classes $[\sigma_i],[\sigma_j]$, there exists $[\sigma_{ij}]$ such that $\{\mathrm{Q}_{\sigma_i},\mathrm{Q}_{\sigma_j},\mathrm{Q}_{\sigma_{ij}}\}$ is Pappus configuration. If $[\sigma_{ij}]=[\sigma_k]$ for some $k\in[3]$, then $\{\mathrm{Q}_{\sigma_1},\mathrm{Q}_{\sigma_2},\mathrm{Q}_{\sigma_3}\}$ is a Pappus configuration. Thus assume all $[\sigma_{ij}]\neq[\sigma_k]$ for any $k=1,2,3$. Moreover $[\sigma_{12}],[\sigma_{13}],[\sigma_{23}]$ are distinct since if $[\sigma_{ij}]=[\sigma_{ik}]$ then $[\sigma_j]=[\sigma_k]$.

If $[\sigma_{12}]\cap[\sigma_{13}]\neq\emptyset$, $[\sigma_{12}]\cap[\sigma_{23}]\neq\emptyset$ and $[\sigma_{13}]\cap[\sigma_{23}]\neq\emptyset$, then

$$\bigcap_{1\le l_1<l_2\le 3}\mathrm{Q}^{\circ}_{\sigma_{l_1l_2}}=\emptyset$$

by Lemma 6.1 and

$$\bigcap_{i=1}^{3}\mathrm{Q}^{\circ}_{\sigma_i}=(\bigcap_{i=1}^{3}\mathrm{Q}^{\circ}_{\sigma_i})\cap(\bigcap_{1\le l_1<l_2\le 3}\mathrm{Q}^{\circ}_{\sigma_{l_1l_2}})=\emptyset.$$

Otherwise assume $[\sigma_{12}]\cap[\sigma_{13}]=\emptyset$, we get a new Pappus configuration. Since the number of disjoint classes is finite, iterating the process, we will eventually get 3 classes $[\sigma_{l_1}]$, $[\sigma_{l_2}]$, $[\sigma_{l_3}]$ pairwise not disjoint and

$$\bigcap_{i=1}^{3}\mathrm{Q}^{\circ}_{\sigma_i}=(\bigcap_{i=1}^{3}\mathrm{Q}^{\circ}_{\sigma_i})\cap\mathrm{Q}^{\circ}_{\sigma_{l_1}}\cap\mathrm{Q}^{\circ}_{\sigma_{l_2}}\cap\mathrm{Q}^{\circ}_{\sigma_{l_3}}=\emptyset.$$ □

Lemma 6.3. *If* $[\sigma_1],[\sigma_2],[\sigma_3]$ *are distinct classes such that* $[\sigma_1]\cap[\sigma_2]\neq\emptyset$ *and* $[\sigma_i]\cap[\sigma_3]=\emptyset$ *for* $i=1,2$, *then*

$$\bigcap_{i=1}^{3}\mathrm{Q}^{\circ}_{\sigma_i}=\emptyset.$$

Proof. Since $[\sigma_1],[\sigma_3]$ and $[\sigma_2],[\sigma_3]$ are disjoint, there exist $[\sigma_4]$ and $[\sigma_5]$ such that $\{\mathrm{Q}_{\sigma_1},\mathrm{Q}_{\sigma_3},\mathrm{Q}_{\sigma_4}\}$ and $\{\mathrm{Q}_{\sigma_2},\mathrm{Q}_{\sigma_3},\mathrm{Q}_{\sigma_5}\}$ are Pappus configurations and

$$[\sigma_1]\cap[\sigma_5]\neq\emptyset,\quad[\sigma_2]\cap[\sigma_4]\neq\emptyset,\quad[\sigma_4]\cap[\sigma_5]\neq\emptyset.$$

Indeed if one of them is empty, we obtain 3 disjoint classes not in Pappus configuration and by Lemma 6.2, it follows

$$\bigcap_{i=1}^{3}\mathrm{Q}^{\circ}_{\sigma_i}=\bigcap_{i=1}^{5}\mathrm{Q}^{\circ}_{\sigma_i}=\emptyset.$$

Since $[\sigma_1]\cap[\sigma_2]\neq\emptyset$, we can assume $\{i_1,i_2\}=[\sigma_1]\cap[\sigma_2]$ and we can set

$$\begin{aligned}[\sigma_1]&=\big\{\{i_1,i_2\},\ \{i_3,i_4\},\ \{i_5,i_6\}\big\},\\ [\sigma_2]&=\big\{\{i_1,i_2\},\ \{i'_3,i'_4\},\ \{i'_5,i'_6\}\big\},\\ [\sigma_3]&=\big\{\{j_1,j_2\},\ \{j_3,j_4\},\ \{j_5,j_6\}\big\}.\end{aligned}$$

To any point

$$x \in \bigcap_{i=1}^{3} \mathrm{Q}^{\circ}_{\sigma_i} \neq \emptyset$$

corresponds an arrangement $\mathcal{A}$ with generic subarrangement $\{H_{i_1}, \dots, H_{i_6}\}$ with trace at infinity $\{H_{\infty,i_1}, \dots, H_{\infty,i_6}\}$ intersecting as in Figures 4 and 5 (up to rename). It follows

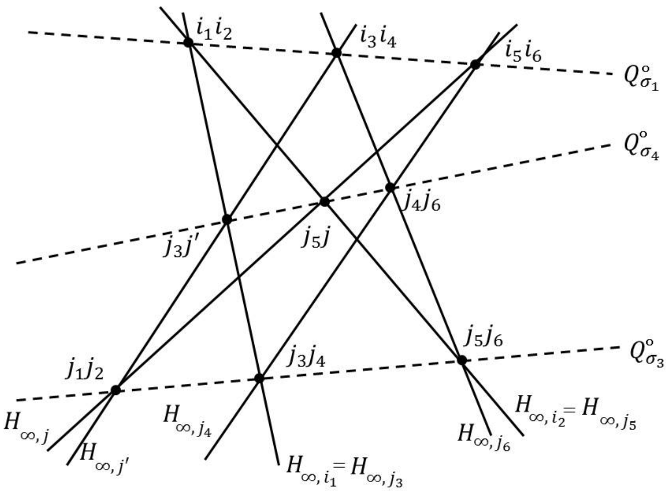

Figure 4: Each j, j' is j_1 or j_2.

that $\{j_4, j_6\} \in [\sigma_4]$ and since $\{j_3, j_5\} = \{i_1, i_2\} \in [\sigma_1]$ and $[\sigma_1] \cap [\sigma_4] = \emptyset$ (see Figure 4), there are two possibilities:

$$[\sigma_4] = \big\{\{j_4, j_6\},\ \{j_1, j_3\},\ \{j_2, j_5\}\big\}$$

or

$$[\sigma_4] = \big\{\{j_4, j_6\},\ \{j_1, j_5\},\ \{j_2, j_3\}\big\}.$$

Analogously (see Figure 5) class $[\sigma_5]$ is of the form

$$[\sigma_5] = \big\{\{j_4, j_6\}, \{j_1, j_3\}, \{j_2, j_5\}\big\}$$

or

$$[\sigma_5] = \big\{\{j_4, j_6\}, \{j_1, j_5\}, \{j_2, j_3\}\big\}.$$

Since $[\sigma_1] \cap [\sigma_5] \neq \emptyset$ and $[\sigma_5] \not\ni \{j_3, j_5\} = \{i_1, i_2\}$, we deduce that $\{j_4, j_6\} = \{i_3, i_4\}$ or $\{i_5, i_6\}$, which is not possible by $[\sigma_1] \cap [\sigma_4] = \emptyset$. Hence

$$\bigcap_{i=1}^{3} \mathrm{Q}^{\circ}_{\sigma_i} = \emptyset. \qquad \square$$

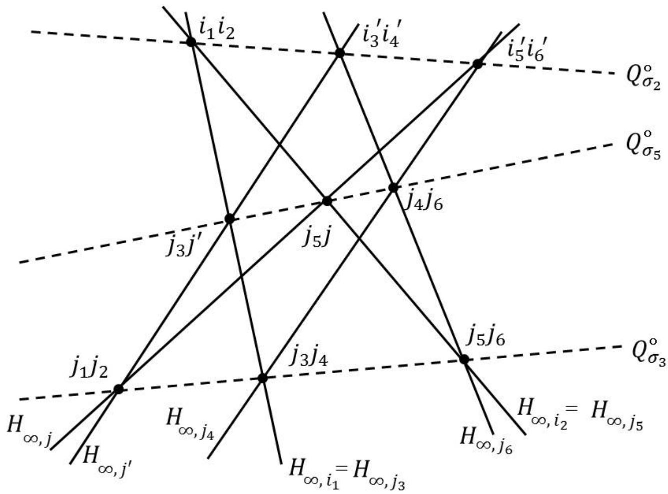

Figure 5: Each j, j' is j_1 or j_2.

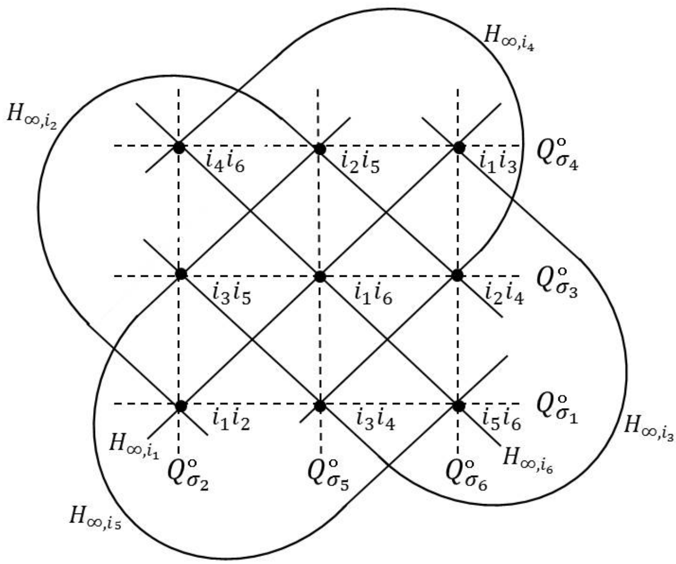

Figure 6: Hesse arrangement with $H_{\infty i_1}, \ldots, H_{\infty i_6}$ and $\bigcap_{i=1}^{6} Q^\circ_{\sigma_i} \neq \emptyset$.

Notice that the Hesse arrangement in $\mathbb{P}^2(\mathbb{C})$ (see Figure 6) can be regarded as a generic arrangement of 6 lines which intersection points satisfy 6 collinearity conditions.

Definition 6.4 (Hesse configuration)**.** Let $[\sigma_i]$, $1 \leq i \leq 6$ be distinct classes, we call Hesse configuration a set $\{\mathrm{Q}_{\sigma_1}, \ldots, \mathrm{Q}_{\sigma_6}\}$ of quadrics in $Gr(3, \mathbb{C}^n)$ such that there exist disjoint sets $I, J \subset [6]$, $|I| = |J| = 3$ such that $\{\mathrm{Q}_{\sigma_i}\}_{i \in I}, \{\mathrm{Q}_{\sigma_j}\}_{j \in J}$ are Pappus configurations and $[\sigma_i] \cap [\sigma_j] \neq \emptyset$ for any $i \in I$, $j \in J$.

With above notations, the following classification Theorem holds.

Theorem 6.5. *For any choice of indices $\{s_1, \ldots, s_6\} \subset [n]$ sets Q°_σ, $\sigma \in \boldsymbol{S}_6$, in the Grassmannian $Gr(3, \mathbb{C}^n)$ intersect as follows.*

(1) For any disjoint classes $[\sigma_1]$ and $[\sigma_2]$, there exist $[\sigma_3], \ldots, [\sigma_6]$ such that $\{\mathrm{Q}_{\sigma_1}, \ldots, \mathrm{Q}_{\sigma_6}\}$ is an Hesse configuration for $I = \{1, 2, 3\}$, $J = \{4, 5, 6\}$ and

$$\bigcap_{i=1}^{2} \mathrm{Q}^\circ_{\sigma_i} = \bigcap_{i=1}^{3} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{4} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{6} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \emptyset.$$

(2) For any not disjoint classes $[\sigma_1]$ and $[\sigma_2]$, there exist $[\sigma_3], \ldots, [\sigma_6]$ such that $\{\mathrm{Q}_{\sigma_1}, \ldots, \mathrm{Q}_{\sigma_6}\}$ is an Hesse configuration for $I = \{1, 3, 4\}$, $J = \{2, 5, 6\}$ and

$$\bigcap_{i=1}^{2} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{3} \mathrm{Q}^\circ_{\sigma_i} = \bigcap_{i=1}^{4} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \bigcap_{i=1}^{6} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \emptyset.$$

All other intersections are empty.

Remark 6.6. Notice that, since Hesse configuration only exists in the complex case, in $Gr(3, \mathbb{C}^n)$ we can find 6 quadrics $\{\mathrm{Q}_{\sigma_1}, \ldots, \mathrm{Q}_{\sigma_6}\}$ such that

$$\bigcap_{i=1}^{6} \mathrm{Q}^\circ_{\sigma_i} \supsetneq \emptyset,$$

while in $Gr(3, \mathbb{R}^n)$,

$$\bigcap_{\substack{j \in J \subset [6] \\ |J| > 4}} \mathrm{Q}^\circ_{\sigma_j} = \emptyset.$$

It follows that in the real case, for any choice of indices $\{s_1, \ldots, s_6\} \subset [n]$, we have at most 4 collinearity conditions (see Figure 7) corresponding to 15 hyperplanes in the Discriminantal arrangement with 4 multiplicity 3 intersections in codimension 2 (see Figure 8). While in the complex case Hesse configuration (see Figure 6) gives rise to a Discriminantal arrangement containing 15 hyperplanes intersecting in 6 multiplicity 3 spaces in codimension 2.

This remark allows a better understanding of differences in the combinatorics of Discriminantal arrangement in the real and complex cases. Indeed the existence of a discriminantal arrangement of 15 hyperplanes intersecting in 6 multiplicity 3 spaces in codimension 2 in $\mathbb{C}$ but not in $\mathbb{R}$ implies that there exist combinatorics of Discriminantal arrangements that cannot be realised in any field. This is especially interesting since in the case known until now, i.e. in the case of very generic arrangements $\mathcal{A}$, the combinatorics of Discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ is independent from the field (see [1]).

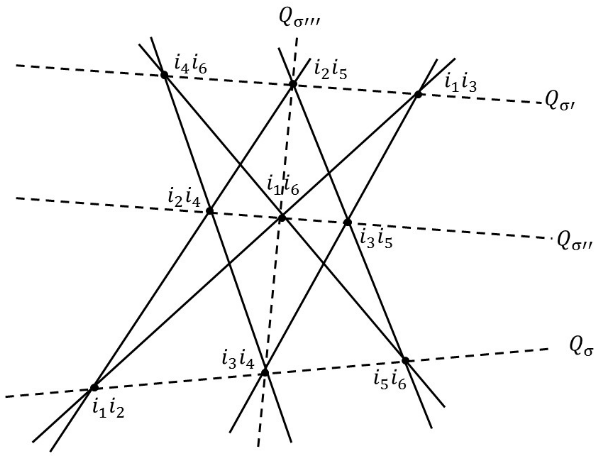

Figure 7: Generic arrangement $\mathcal{A}$ in $\mathbb{R}^3$ containing 6 lines satisfying 4 collinearity conditions.

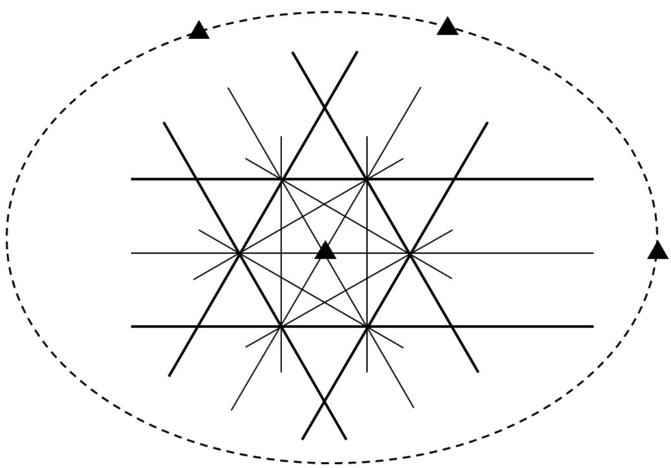

Figure 8: Codimension 2 intersections of 15 hyperplanes in $\mathcal{B}(n, 3, \mathcal{A}_\infty)$ indexed in $\{s_1, \dots, s_6\} \subset [n]$ with 4 multiplicity 3 points ▲ corresponding to intersections $\bigcap_{i=1}^3 D_{\sigma.L_i}, \bigcap_{i=1}^3 D_{\sigma'.L_i}, \bigcap_{i=1}^3 D_{\sigma''.L_i}$ and $\bigcap_{i=1}^3 D_{\sigma'''.L_i}$, $\sigma, \sigma', \sigma'', \sigma'''$ as in Figure 7.

Remark 6.7. Finally Theorem 6.5 implies that the maximum number of intersections of multiplicity 3 in codimension 2 in the complex case is strictly higher than the one in the real case. This agrees with results on maximum number of triple points in an arrangement of lines in $\mathbb{P}^2$ (see [3] for a discussion on line arrangements with maximal number of triple points over arbitrary fields). Those observations suggest that special configurations of lines in the projective plane intersecting in a big number of triple points could be understood by studying Discriminantal arrangements with maximum number of multiplicity 3 intersections in codimension 2. Indeed each multiplicity 3 intersection in codimension 2 of $\mathcal{B}(n, 3, \mathcal{A}_\infty)$ corresponds to a collinearity condition for lines in $\mathcal{A}_\infty$ which is equivalent to the possibility to add a line that gives rise to "higher" number of triple points. It seems hence interesting to study exact number of intersections of type (1) and (2) in Theorem 6.5 in the Grassmannian $Gr(3, \mathbb{C}^n)$. This will be object of further studies.

References

[1] C. A. Athanasiadis, The largest intersection lattice of a discriminantal arrangement, *Beiträge Algebra Geom.* **40** (1999), 283–289, `https://www.emis.de/journals/BAG/vol.40/no.2/1.html`.

[2] M. M. Bayer and K. A. Brandt, Discriminantal arrangements, fiber polytopes and formality, *J. Algebraic Combin.* **6** (1997), 229–246, doi:10.1023/a:1008601810383.

[3] M. Dumnicki, Ł. Farnik, A. Główka, M. Lampa-Baczyńska, G. Malara, T. Szemberg, J. Szpond and H. Tutaj-Gasińska, Line arrangements with the maximal number of triple points, *Geom. Dedicata* **180** (2016), 69–83, doi:10.1007/s10711-015-0091-7.

[4] D. Eisenbud, M. Green and J. Harris, Cayley-Bacharach theorems and conjectures, *Bull. Amer. Math. Soc. (N. S.)* **33** (1996), 295–324, doi:10.1090/s0273-0979-96-00666-0.

[5] J. Harris, *Algebraic Geometry: A First Course*, volume 133 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1992, doi:10.1007/978-1-4757-2189-8.

[6] A. Libgober and S. Settepanella, Strata of discriminantal arrangements, *J. Singul.* **18** (2018), 440–454, doi:10.5427/jsing.2018.18w.

[7] Yu. I. Manin and V. V. Schechtman, Arrangements of hyperplanes, higher braid groups and higher Bruhat orders, in: J. Coates, R. Greenberg, B. Mazur and I. Satake (eds.), *Algebraic Number Theory*, Academic Press, Boston, MA, volume 17 of *Advanced Studies in Pure Mathematics*, pp. 289–308, 1989, in honor of K. Iwasawa on the occasion of his 70th birthday on September 11, 1987.

[8] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften*, Springer-Verlag, Berlin, 1992, doi:10.1007/978-3-662-02772-1.

[9] J. Richter-Gebert, *Perspectives on Projective Geometry: A Guided Tour Through Real and Complex Geometry*, Springer, Heidelberg, 2011, doi:10.1007/978-3-642-17286-1.

[10] S. Sawada, S. Settepanella and S. Yamagata, Discriminantal arrangement, 3×3 minors of Plücker matrix and hypersurfaces in Grassmannian $Gr(3, n)$, *C. R. Acad. Sci. Paris Ser. I* **355** (2017), 1111–1120, doi:10.1016/j.crma.2017.10.011.

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- Vida Dujmović (University of Ottawa, Canada)
- Miguel Angel Pizaña (Universidad Autónoma Metropolitana-Iztapalapa, Mexico)
- Jeroen Schillewaert (University of Auckland, New Zealand)
- Klara Stokes (Maynooth University, Ireland)

Students will have opportunity to present their results in 15-minutes talks, and a 3-member committee will decide the winner of the Best Students Talk award. The winner will get a certificate and free participation at the 10th PhD Summer School in Discrete Mathematics.

Students can also apply for financial support, which includes half board accommodation and the exemption from conference fee payment. Students that would like to apply for financial support need to indicate it in the registration form and send their motivation letter and short CV (in English). Deadline for the financial support application is February 28, 2019.

Scientific Committee: Ademir Hujdurović, Klavdija Kutnar, Aleksander Malnič, Dragan Marušič, Štefko Miklavič, Primož Šparl.

Organizing Committee: Boštjan Frelih, Ademir Hujdurović, Boštjan Kuzman, Rok Požar.

Further information: https://conferences.famnit.upr.si/e/rogla2019/

Organized by:

- University of Primorska, IAM and University of Primorska, FAMNIT, and
- Slovenian Discrete and Applied Mathematics Society

in collaboration with:

- Centre for Discrete Mathematics, UL PeF.

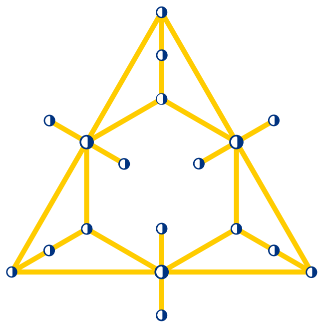
Sponsors:

- EMS – European Mathematical Society,
- ARRS – Slovenian Research Agency,
- MIZS – Ministry of Education, Science and Sport.

9th Slovenian International Conference on Graph Theory (Bled '19)

Bled, Slovenia, June 23 – 29, 2019

https://conferences.matheo.si/e/bled19



Bled '19 is the 9th edition of the quadrennial Slovenian Graph Theory Conference, which has become one of the largest – and for several areas of graph theory, the premier – graph theory conference series, and is attended by leading researchers in graph theory, as well as many postdocs and talented Ph.D. students. In addition to the keynote speakers named below, the conference will consist of minisymposia from specific fields, ranging across algebraic, algorithmic, geometric, topological, and other aspects of graph theory, a general session and poster session. A special session will be organized to celebrate the 70th birthday of our friend, colleague, and one of the founders of Slovenian Graph Theory, Tomaž Pisanski.

List of Keynote Speakers:

- Noga Alon (Tel Aviv University, Israel)
- Marco Buratti (University of Perugia, Italy)
- Gareth Jones (University of Southampton, UK)
- Gábor Korchmáros (University of Basilicata, Italy)
- Daniel Kráľ (Warwick University, UK and Masaryk University, Czech Republic)
- Daniela Kühn (University of Birmingham, UK)
- Sergei Lando (National Research University Higher School of Economics, Russia)
- János Pach (EPFL, Lausanne and Rényi Institute, Hungary)
- Cheryl E. Praeger (University of Western Australia, Australia)
- Zsolt Tuza (University of Pannonia, Hungary)
- Xuding Zhu (Zhejiang Normal University, China)

Scientific Committee: Sandi Klavžar, Dragan Marušič, Bojan Mohar (chair), Tomaž Pisanski.

Further information: https://conferences.matheo.si/e/bled19

The 31st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2019)

Ljubljana, Slovenia, July 1 – July 5, 2019

http://fpsac2019.fmf.uni-lj.si/

The International Conference on Formal Power Series and Algebraic Combinatorics is a major annual combinatorial conference that is organized in a different city every year. The 31st edition of FPSAC will take place in Ljubljana, Slovenia. Topics include all aspects of combinatorics and their relations with other parts of mathematics, physics, computer science, and biology. The conference will include invited lectures, contributed presentations, poster sessions, and software demonstrations. There will be no parallel sessions.

List of Keynote Speakers:

- Andrej Bauer (University of Ljubljana & IMFM, Slovenia)
- Alin Bostan (Inria, France)
- Sandra Di Rocco (KTH, Royal Institute of Technology, Sweden)
- Eric Katz (The Ohio State University, United States)
- Caroline Klivans (Brown University, United States)
- Nataša Pržulj (University College London, United Kingdom)
- Vic Reiner (University of Minnesota, United States)
- Stephan Wagner (Stellenbosch University, South Africa)
- Chuánmíng Zōng (Tianjin University, People's Republic of China)

Program Committee Chairs: Sara Billey, Marko Petkovšek, Günter Ziegler.

Organizing Committee Chair: Matjaž Konvalinka.

Further information: http://fpsac2019.fmf.uni-lj.si/

Organized by:

- UL FMF – University of Ljubljana, Faculty of Mathematics and Physics

in collaboration with:

- UM FNM – University of Maribor, Faculty of Natural Sciences and Mathematics,
- UP FAMNIT – University of Primorska, Faculty of Mathematics, Natural Sciences and Information Technologies,
- IMFM – Institute of Mathematics, Physics and Mechanics, Ljubljana, and
- SDAMS – Slovenian Discrete and Applied Mathematics Society,

and is sponsored by a variety of organizations and companies, including the National Science Foundation.

ARS MATHEMATICA CONTEMPORANEA

Maps ∩ Configurations ∩ Polytopes ∩ Molecules ⊆ Graphs: The mathematics of Tomaž Pisanski on the occasion of his 70th birthday

Ljubljana, Slovenia, May 23 – 25, 2019

https://conferences.matheo.si/e/mcpm

Tomaž Pisanski

It is our great pleasure to announce this conference in Discrete Mathematics, dedicated to our dear colleague Tomaž Pisanski in celebration of his mathematics and the occasion of his 70th birthday.

Tomaž Pisanski, known as Tomo to his friends, works in several areas of discrete and computational mathematics. Combinatorial configurations, abstract polytopes, maps on surfaces and chemical graph theory are just a few areas of his broad research interests. Tomo is the author or coauthor of over 160 original scientific papers. Together with Brigitte Servatius he authored the book *Configurations from a Graphical Viewpoint*, which was published in 2013 by Birkhäuser. Tomo's scientific work has been cited over 3400 times according to Google Scholar. In 2008, together with Dragan Marušič, he cofounded *Ars Mathematica Contemporanea*, the first international mathematical journal to be published in Slovenia. Many mathematicians refer to Tomo as "the father of Slovenian discrete mathematics". The Mathematics Genealogy Project lists 16 of his PhD students and 79 academic descendants.

The invited speakers listed below are world-class mathematicians who work in areas of mathematics that are close to Tomo's research interests. In addition to invited talks, participants will have the opportunity to deliver short, 15-minute presentations.

Venue: The conference will take place at UL FMF in Ljubljana, Slovenia.

Invited Speakers:

- Vladimir Batagelj (University of Ljubljana and University of Primorska, Slovenia)
- Gunnar Brinkmann (Ghent University, Belgium)
- Patrick W. Fowler (The University of Sheffield, United Kingdom)
- Gábor Gévay (University of Szeged, Hungary)
- Wilfried Imrich (Montanuniversität Leoben, Austria)
- Asia Ivić Weiss (York University, Canada)
- Sandi Klavžar (University of Ljubljana and University of Maribor, Slovenia)
- Dimitri Leemans (Université Libre de Bruxelles, Belgium)
- Dragan Marušič (University of Primorska, Slovenia)
- Alexander D. Mednykh (Sobolev Institute of Mathematics and Novosibirsk State University, Russian Federation)

- Bojan Mohar (Simon Fraser University, Canada, and IMFM, Slovenia)
- Daniel Pellicer Covarrubias (National Autonomous University of Mexico, Mexico)
- Egon Schulte (Northeastern University, United States)
- Brigitte Servatius (Worcester Polytechnic Institute, United States)
- Martin Škoviera (Comenius University in Bratislava, Slovakia)
- Thomas W. Tucker (Colgate University, United States)
- Steve Wilson (Northern Arizona University, United States)

Scientific Committee: Klavdija Kutnar, Primož Potočnik, Arjana Žitnik.

Organising Committee: Nino Bašić, Marko Boben, Boris Horvat, Jurij Kovič, Alen Orbanić, Arjana Žitnik.

Further information: https://conferences.matheo.si/e/mcpm

Organised by:
- UL FMF – University of Ljubljana, Faculty of Mathematics and Physics

in collaboration with:
- UP FAMNIT – University of Primorska, Faculty of Mathematics, Natural Sciences and Information Technologies
- UP IAM – University of Primorska, Andrej Marušič Institute
- SDAMS – Slovenian Discrete and Applied Mathematics Society
- IMFM – Institute of Mathematics, Physics, and Mechanics
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