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On 2-factors with long cycles in cubic graphs*

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Abstract

Every 2-connected cubic graph G has a 2-factor, and much effort has gone into studying conditions that guarantee G to be Hamiltonian. We show that if G is not Hamiltonian, then G is either the Petersen graph or contains a 2-factor with a cycle of length at least 7. We also give infinite families of, respectively, 2- and 3-connected cubic graphs in which every 2-factor consists of cycles of length at most, respectively, 10 and 16.

Keywords: Cubic graph, 2-factor, long cycle, snark, infinite graph

Math. Subj. Class.: 05C38, 05C45, 05C70

1 Introduction

A theorem of Petersen [6] states that every cubic 2-connected graph G has a perfect matching, and thus a 2-factor. Much effort has gone into studying conditions for when G has a 2-factor consisting of only one cycle, that is G is Hamiltonian. This motivates our definition of $\ell(G)$ as the length of a longest cycle in any 2-factor of G, and any such cycle as a *longest 2-factor cycle*. Obviously G is Hamiltonian precisely when $\ell(G) = |V(G)|$. We also call any 2-factor achieving $\ell(G)$ a *longest* 2-factor of G. (All graphs in this work are simple; that is, there are no loops or multiple edges.)

Petersen's theorem says that $\ell(G)$ is well-defined when G is cubic and 2-connected. A typical proof of Petersen's theorem simply verifies that Tutte's 1-factor condition holds for

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cubic 2-connected graphs. Schönberger [7] (see also [9, Exercise 3.3.17] and [2, Corollary 4.4]) found a strengthening of Petersen's theorem by showing that every edge f in a 2-connected cubic graph G can be extended to a 1-factor, and so G contains a 2-factor *avoiding* f. This also guarantees that G must have a 2-factor *using* any two specified edges e_1, e_2, since we can subdivide e_1, e_2 and connect the new vertices with the new edge f and then consider the 2-factor avoiding f in the new cubic graph.

Schönberger's theorem allows us to study $\ell(G)$ in a slightly more general context. A graph is *near-cubic* if every vertex has degree 3, except for at most one vertex of degree 2. Schönberger's theorem implies that every 2-connected near-cubic graph G has a 2-factor, and thus $\ell(G)$ is defined for this wider class.

Jackson and Yoshimoto [3] prove a theorem that implies that a 3-connected cubic graph G has a 2-factor in which every component has length at least 5. In particular, for a 3-connected cubic graph G, $\ell(G) \geq 5$.

A near-cubic graph is *cyclically k-connected* if the removal of fewer than k edges does not create a graph with cycles in two different components, and we let $\mathcal{C}_k(n)$ denote the class of all near-cubic cyclically k-connected graphs on n vertices. Observe that, in a cubic graph, a minimal size edge-cut is either a matching or the 3 edges incident with a vertex, so that, for $k \leq 3$, the three notions of being cyclically k-connected, being k-edge connected, and being k-connected are all equivalent. Subdividing an edge does not affect cyclic connectivity, so that this notion is a more useful indicator of the structure of near-cubic graphs. We define $L_k(n)$ as the least value of $\ell(G)$, taken over all graphs G in $\mathcal{C}_k(n)$.

The only cubic graphs on at most 6 vertices are the 3-connected graphs K_4, $K_{3,3}$ and the triangular prism $K_3 \square K_2$. All of these are Hamiltonian, as are all other 2-connected cubic graphs on 8 or 10 vertices, with the exception of the Petersen graph $\mathbb{P}$ (see Figure 2.) It is also not hard to see that the near-cubic graphs G^* obtained by subdividing any 2-connected cubic graph G on at most 10 vertices other than $\mathbb{P}$ is also Hamiltonian, and it follows that $L_2(n) = n$ for $4 \leq n \leq 9$, whereas $L_2(10) = 5$ and $L_2(11) = 6$. With some more case analysis it can be shown that $L_2(n) = n - 5$ for $10 \leq n \leq 16$, except $L_2(15) = 9$, and the constructions achieving these values are given at the end of the proof of Theorem 2.1. For general k, observe that any $G \in \mathcal{C}_k(n)$ (with the possible exception of K_4 and $K_{3,3}$) has girth at least k, and thus $\ell(G) \geq k$ and $n \geq 2k$. By considering an arbitrary 2-factor, it is not hard to see that $L_k(n) = n$ for $n = 2k$, except for $\mathbb{P}$ yielding $L_5(10) = 5$.

The obvious question is, what is the behavior of $L_2(n)$, and more generally $L_k(n)$, as n tends to infinity. In Section 2, we show that, for all n, $L_2(n) \leq 11$ and $L_3(n) \leq 16$. In Section 3 we prove that, for all $n \geq 12$, $L_2(n) \geq 7$.

Our consideration of the parameter $L_3(n)$ arose from the question: how does Petersen's Theorem generalize to infinite graphs? Tutte [8] proved his 1-factor theorem for locally finite graphs, so the modern proof of Petersen's Theorem from Tutte's Theorem applies to show that every 2-connected cubic graph has a 2-factor. In the infinite case, we were initially interested in whether there is a 2-factor in which every component is finite or whether there is a 2-factor in which every component is infinite. Thomassen (personal communication) gave us an example of a 2-connected cubic graph with infinitely many ends so that every 2-factor must have an infinite component. The construction showing $L_3(n) \leq 16$ provides an infinite (1-ended) 3-connected cubic graph in which every 2-factor has only cycles of length at most 16. In Section 4, we provide an example of an infinite (1-ended) 3-connected cubic graph in which every 2-factor is a 2-way infinite Hamilton path.

2 Upper bounds

In this section we prove that $L_2(n) \le 11$ and $L_3(n) \le 16$.

For our constructions we first need some simple observations about the Petersen graph $\mathbb{P}$: every 2-factor in $\mathbb{P}$ consists of two 5-cycles, and every 5-cycle is in a unique 2-factor. The edges of $\mathbb{P}$ partition into five sets of three edges, no two of which are together in any path of length 3 in $\mathbb{P}$; any 2-factor uses precisely two edges from each of the five sets and the two edges from the same set are in distinct components of the 2-factor.

The *2-merge* of a graph G (at an edge xy of G) with a graph H (at an edge uv of H) means that we take disjoint copies of G and H and replace the edges xy and uv with the edges xu and yv. If G and H are 2-connected near-cubic graphs, then their 2-merge is also 2-connected and near-cubic, as long as at least one of G and H is cubic.

If G is a cubic graph, then the graph G^* is obtained from G by subdividing one edge of G. If G is a graph and w is a vertex of degree 2 in G, then *suppressing* w results in the graph $G \diamond w$ obtained from $G - w$ by adding in an edge joining the two neighbours of w. If W is a set $\{w_1, w_2, \ldots, w_k\}$ of vertices, all with degree 2 in G, then *suppressing* W results in the graph $(\cdots(G \diamond w_1) \diamond w_2) \diamond \cdots) \diamond w_k$. We will be careful to only suppress vertices in situations when no parallel edges can arise.

For "large" values of n we have the following upper bounds; we conjecture equality holds for n sufficiently large.

Theorem 2.1. *$L_2(n) \le 11$ when n is congruent to 2 or 6 modulo 10, and $L_2(n) \le 10$ otherwise.*

Proof. The result is obviously true for $n \le 10$, and follows for $n = 11$ by the subdivided Petersen graph $\mathbb{P}^*$, and for $n = 12$ by replacing a vertex of $\mathbb{P}$ by a triangle. Thus, we may assume $n > 12$.

Consider $n = 10k$ for some $k \ge 1$. Take disjoint copies $G_1, \ldots G_k$ of $\mathbb{P}$, each with two specified edges u_iv_i and x_iy_i of G_i not in the same cycle of any 2-factor of G_i. We iteratively 2-merge all the G_i, starting with the 2-merge G_1 (at x_1y_1) with G_2 (at u_2v_2), then 2-merging this (at x_2y_2) with G_3 (at u_3v_3), and in general merging G_i at x_iy_i with G_{i+1} at $u_{i+1}v_{i+1}$. The resulting graph $H(k)$ is a cubic 2-connected graph on n vertices.

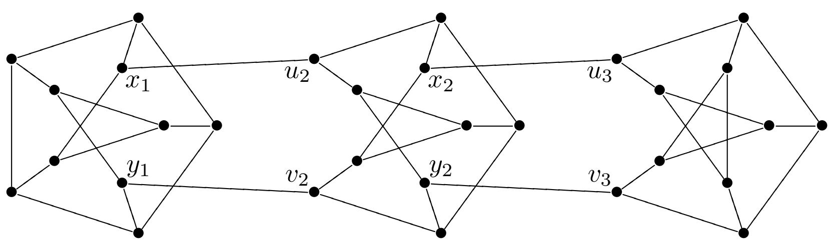

Figure 1: $H(3)$.

Claim 1. *For $k \ge 2$, $\ell(H(k)) = 10$.*

Proof. It is easy to see that any 2-factor of $H(k)$ arises from a 2-factor of the G_i's by doing the specified edge replacements where necessary, and thus the length of each cycle in a 2-factor of $H(k)$ is a multiple of 5. Taking a 2-factor in each G_i containing both of the specified edges, we obtain a 2-factor of $H(k)$ consisting of two 5-cycles (one each in G_1 and G_k) and $k-1$ 10-cycles, so that $\ell(H(k)) \geq 10$.

To see that equality holds, observe that any 2-factor F of $H(k)$ induces a 2-factor F_i in each G_i. Each F_i has two cycles of length 5 and none uses both edges that are switched to connect G_i with G_{i-1} and G_{i+1}. Thus, each component of F can intersect at most two consecutive G_i and, therefore, has length at most 10. △

To complete the proof of Theorem 2.1, let $n = 10k + r$, with $1 \leq r \leq 9$.

Case $r = 6$**.** In this case 2-merge $H(k)$ at u_1v_1 with $K_{3,3}$, which at the most can lengthen the 5-cycle in $H(k)$ through u_1v_1 by six vertices to an 11-cycle, and hence $L(10k+6) \leq 11$.

Case $r = 2$**.** Here we 2-merge $H(k-1)$ with a $K_{3,3}$ each at u_1v_1 and $x_{k-1}y_{k-1}$ to obtain $L(10(k-1)+12) \leq 11$, since here $k \geq 2$.

The following construction is useful in many of the remaining cases. The $(2k-1)$-*replacement in* G *at the edge* w_1z_1, is obtained by subdividing the edge w_1z_1 $2k-1$ times, thereby creating the path $(w_1, w_2, \ldots, w_k, v, z_k, z_{k-1}, \ldots, z_1)$, and then adding all the edges w_iz_i, $i = 2, 3, \ldots, k$. It is easy to see that, if $k \geq 2$, for any $(2k-1)$-replacement G' in G and any 2-factor F of G, there is a 2-factor F' that naturally extends F; in particular, $\ell(G') \geq \ell(G)$. In the case $k = 1$, the $(2k-1)$-replacement is simply subdividing w_1z_1.

If r is not 2 or 6, then there exist $s \in \{0, 1, 3, 4, 5\}$ and $t \in \{0, 4\}$ such that $r = s + t$. Start with $H(k)$. If $t = 4$, then 2-merge $H(k)$ at x_ky_k with a K_4 to obtain a graph G' on $10k + t$ vertices. If $k \geq 2$, then $\ell(G') = 10$, while for $k = 1$ $\ell(G') = 9$. If $s \in \{1, 3, 5\}$, s-replace u_1v_1, while if $s = 4$ we 2-merge G' at u_1v_1 with a K_4. None of these operations increases ℓ beyond 10. □

To obtain a construction for the 3-connected case, let w, z be non-adjacent vertices in $\mathbb{P}$, let c be their common neighbor, and let the remaining neighbors of w be u, v and the remaining neighbors of z be x, y, where ux and vy are edges of $\mathbb{P}$. Observe that every 5-cycle in $\mathbb{P}$ that contains w and z also contains c and exactly one of ux and vy. Moreover, $\mathbb{P}-z$ has exactly two 2-factors: a 9-cycle through wc and wu and a 9-cycle through wc and wv.

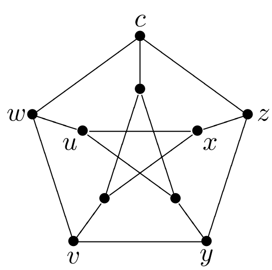

Figure 2: Petersen graph $\mathbb{P}$.

A *3-merge of a cubic graph* G *(at a vertex* w *in* G*) with a near-cubic graph* H *(at a degree 3 vertex* z *of* H*)* is obtained from disjoint copies of $G - w$ and $H - z$ by adding a matching u_1x_1, u_2x_2, u_3x_3 from the neighborhood $\{u_1, u_2, u_3\}$ of w to the neighborhood $\{x_1, x_2, x_3\}$ of z. The resulting near-cubic graph is cyclically 3-connected if and only if both G and H are.

Theorem 2.2. $L_3(n) \leq 16$.

Proof. The result is obviously true for $n \leq 16$.

We start by considering the case $n = 8k + 2$, $k \geq 2$. Take disjoint copies $G_1, \dots G_k$ of $\mathbb{P}$, each with the specified vertices $u_i, v_i, w_i, c_i, x_i, y_i, z_i$ as above. We obtain the graph $J(k)$ by iteratively 3-merging the G_i, beginning with G_1 (at z_1) and G_2 (at w_2), and then 3-merging this (at z_2) with G_3 (at w_3), and so on. The specific 3-merges we use include the edges in $M_i = \{c_iu_{i+1}, x_iv_{i+1}, y_ic_{i+1}\}$ (see Figure 3.)

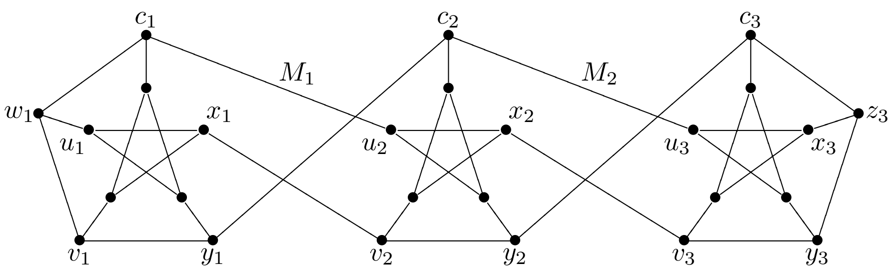

Figure 3: $J(3)$.

Claim 1. *Suppose $k \geq 3$ and F is a 2-factor of $J(k)$. If, for some i with $1 < i < k$, C is a cycle in F meeting both M_{i-1} and M_i, then $y_{i-1}c_i$ and c_iu_{i+1} are both edges of C, and $V(C) \cap V(G_i)$ is $\{c_i, u_i, x_i\}$ or $\{c_i, v_i, y_i\}$.*

Proof. Because F is a 2-factor, two edges incident with c_i are in F. Thus, either $y_{i-1}c_i$ or c_iu_{i+1} (or both) is in F. Since C is the only cycle in F that meets $M_{i-1} \cup M_i$ it follows that c_i is in C. Obtain a 2-factor F_i in G_i from F by deleting all of $G_1, \dots, G_{i-1}$, $G_{i+1}, \dots, G_k$, and adding back w_i and z_i, together with the edges corresponding to those in $F \cap M_{i-1}$ as incident with w_i, and likewise with z_i. The cycle C now corresponds to the 5-cycle C_i in F_i containing c_i.

As C_i contains edges incident with both w_i and z_i, the remark preceding Theorem 2.2 shows that C_i contains exactly one of u_ix_i and v_iy_i. Also, both w_ic_i and c_iz_i are in C_i and, therefore, $y_{i-1}c_i$ and c_iu_{i+1} are in C and $V(C) \cap V(G_i)$ is either $\{c_i, u_i, x_i\}$ or $\{c_i, v_i, y_i\}$, as required. △

Claim 2. *If C is a cycle of a 2-factor F of $J(k)$ that is contained in at most two G_i, then C has length at most 16.*

Proof. If C is contained in a single G_i, then obviously C has length at most 9 (which could occur for G_1 or G_k). If C meets both G_i and G_{i+1}, then C has length at most 16: this is obvious if $1 < i < k - 1$, while in the remaining case, C can have at most 4 vertices in either G_1 or G_k and, therefore, has length at most 12. △

Claim 3. *For $k \geq 4$, $\ell(J(k)) = 16$.*

Proof. It is easy to see that $\ell(J(k)) \geq 16$: pick the 9-cycle in $G_1 - z_1$, a cycle of length 16 using only vertices from G_2 and G_3, and any 2-factor to cover the remainder. (If k is even, then this can be done with two 9-cycles and $(k-2)/2$ 16-cycles; if k is odd, then this can be done with one 9-cycle, $(k-3)/2$ 16-cycles, one 12-cycle, and one 5-cycle.)

For the reverse inequality, let C be a cycle of a 2-factor F of $J(k)$. By Claim 2, if C is contained in at most two consecutive G_i, then C has length at most 16. Thus, we may assume that C meets all of G_{i-1}, G_i, and G_{i+1}. From Claim 1, we see that c_i, y_{i-1}, and u_{i+1} are in C, and that C meets G_i in exactly 3 vertices.

Suppose first that neither c_{i-1} nor c_{i+1} is in C. Then Claim 1 implies C meets neither M_{i-2} or M_{i+2}. In this case (or if $i = 2$ or $i = k-1$), C meets both G_{i-1} and G_{i+1} in precisely four vertices, as the cycles containing c_{i-1} and c_{i+1} are different from C. As there are three vertices of G_i in C, C has exactly 11 vertices.

Suppose C contains c_{i-1}. We claim c_{i+1} is not in C. To see this, observe that Claim 2 implies $c_{i-1}u_i$ and $y_{i-1}c_i$ are in C, as is u_iv_i. This implies that y_i is not in C, so Claim 2 implies c_{i+1} is not in C.

It follows that C meets G_{i+1} in four vertices. Likewise, C meets G_{i-2} (assuming $i \geq 3$) in four vertices, for a total length of 14. △

Suppose $n = 8k + 2 + r$, for $k \geq 2$ and $0 \leq r \leq 7$. We exhibit a graph $J_r(k)$ so that $\ell(J_r(k)) \leq 16$. We start with $J_0(k) = J(k)$. Observe that the same argument as above shows that if a cycle in a 2-factor of $J(k)$ contains exactly one of w_1 or z_k, then its length is at most 12, whereas if it contains both, then it has length at most 8 (and $k \leq 3$).

We get $J_1(k)$ by subdividing an edge of $J(k)$ incident with w_1. When $2 \leq r \leq 4$, $J_r(k)$ is a 3-merge of $J(k)$ at w_1 with any graph in $\mathcal{C}_3(r+2)$.

When $5 \leq r \leq 7$, $J_r(k)$ is the 3-merge of $J_{r-4}(k)$ at z_k with $K_{3,3}$. The only remaining case is $n = 17$, in which case we take a 3-merge of $\mathbb{P}$ with any graph in $\mathcal{C}_3(9)$ to see that $L_3(17) \leq 12$. □

3 Lower bound

The previous section shows that $L_2(n) \leq 11$. In this section, we show that $L_2(n) \geq 7$ for $n \geq 12$. More precisely, we prove the following.

Theorem 3.1. *If G is a 2-connected near-cubic graph, then either $\ell(G) \geq 7$ or $\ell(G) = |V(G)|$ or G is $\mathbb{P}$ or G is $\mathbb{P}^*$.*

The proof will be by induction. We start with some useful terminology and basic lemmas that will serve as reductions in the proof of Theorem 3.1.

We always let t be the vertex of degree 2 in G, if it exists. Given an edge e in a near-cubic graph G, let $\ell(G, e)$ denote the length of a longest cycle through e that is contained in a 2-factor of G. The first lemma yields the base case for our induction, and will help in many of the reductions. It can be verified by checking all small cases.

Lemma 3.2. *Let $n \leq 10$ and let G be a 2-connected near-cubic graph on n vertices.*

1. *$\ell(G) = n$, except if $G = \mathbb{P}$.*
2. *If G is cyclically 3-connected, then, for each edge e of G, $\ell(G, e) = \ell(G)$.*

Observe that the smallest near-cubic graph with cyclic connectivity 2 is the Hamiltonian graph G on 7 vertices obtained from a 3-replacement of K_4, but that the edge w_2z_2 is in no Hamilton cycle of G. Recall (see the paragraph following the case $r = 2$ in the proof of Theorem 2.1) that any $(2k-1)$-replacement of an edge in a cubic graph still has a 2-factor.

Lemma 3.3. *If G is the $(2k-1)$-replacement of the edge e in a 2-connected cubic graph G', then $\ell(G) \geq \ell(G', e) + (2k-1)$ when $k \geq 1$. If $k \geq 2$, then also $\ell(G) \geq \ell(G')$.*

Proof. A 2-factor in G' that contains a cycle C through e can be extended to a 2-factor of G by including the new vertices in C, immediately yielding the first bound. If a longest 2-factor of G' does not use the edge e, then, when $k \geq 2$, it can still be extended to a 2-factor of G by including a new cycle consisting of all the replacement vertices. □

Given disjoint induced subgraphs H, K of a graph G so that $V(G) = V(H) \cup V(K)$, we use $[H, K]$ to denote the set of edges having one end in each of H and K, that is, the (edge) cut that separates H from K. Moreover, $[H, K]$ is a *k-cut*, provided that its size $|[H, K]|$ is k.

A k-cut $[H, K]$ in a 2-connected near-cubic graph is *cyclic* if both H and K contain a cycle; otherwise it is *non-cyclic*. It is easy to see that if $[H, K]$ is a non-cyclic 2- or 3-cut, with H not containing a cycle, then H is either a single vertex of degree $|[H, K]|$ or H consists of two adjacent vertices, one of which is the degree 2 vertex t of G. In general the edges in a cyclic cut of minimum size in a near-cubic graph G form a matching.

In order to deal with 2-cuts, we need to understand their structure in more detail. Let $(u_0, u_1, \ldots, u_k)$ and $(v_0, v_1, \ldots, v_k)$ be disjoint paths in G so that, for each i with $0 < i < k$, u_i and v_i are adjacent in G. Then the union of the two paths and the edges u_iv_i, $i = 1, 2, \ldots, k-1$ is a (u_0, u_k, v_0, v_k)*-ladder in G*. The two paths are the *rails* of the ladder and the edges u_iv_i are its *rungs*.

Let $[H', K']$ be a 2-cut in G such that H' contains the degree 2 vertex t, if it exists. If t is in a triangle, then our earlier remarks (see the paragraph following the case $r = 2$ in the proof of Theorem 2.1) show G is a $(2k-1)$-replacement in some cubic graph, and $k \geq 2$. If G is cubic, then [9, Lemma 7.3.3.] shows there are induced subgraphs H and K of H' and K', respectively, with distinct vertices u_H, v_H of H', not adjacent in H, and u_K, v_K of K', not adjacent in K, and there is a (u_H, u_K, v_H, v_K)-ladder L in G so that $G = H \cup L \cup K$. Moreover H and K have at least four vertices each. In the remaining case that t exists and is not in a triangle, the preceding sentence applies to the graph obtained by suppressing t, and we may restore t in H, by possibly shortening the ladder if t is in L. The decomposition $G = H \cup L \cup K$ of G is a *cyclic 2-cut ladder decomposition*.

The graphs $H_{(2)}$ and $K_{(2)}$ are obtained from H and K, respectively by adding the edges $e_H = u_Hv_H$ and $e_K = u_Kv_K$, respectively. Observe that $H_{(2)}$ and $K_{(2)}$ are near-cubic and 2-connected.

Lemma 3.4. *Suppose G is a 2-connected near-cubic graph that has a cyclic 2-cut ladder decomposition $H \cup L \cup K$. Specifically, the vertex t, if it exists, is not in a triangle, but is in H. Let the rails of L have length k.*

1. $\ell(G) \geq \ell(H_{(2)}, e_H) + \ell(K_{(2)}, e_K) + 2k - 2$.
2. $\ell(G) \geq \ell(H_{(2)})$.

Proof. Suppose that F is a 2-factor of $H_{(2)}$ containing e_H in the cycle C and that F' is a 2-factor of $K_{(2)}$ containing e_K in the cycle C'. Obtain a 2-factor of G from $(F \setminus \{e_H\}) \cup (F' \setminus \{e_K\})$ by adding the rails of L. The cycle of this 2-factor of G that contains the ladder rails has length $|V(C)| + |V(C')| + 2k - 2$.

For the second part, let F be a longest 2-factor of $H_{(2)}$. If some cycle C of F contains e_H, then consider any 2-factor of $K_{(2)}$ that contains e_K and find a 2-factor for G as above. Observe that the longest cycle in $H_{(2)}$ either remains unchanged or is extended, so we are done.

Finally, suppose that e_H is not in F. If $k = 1$, then let F' be a 2-factor of $K_{(2)}$ not using e_K, and now $F \cup F'$ is a 2-factor of G. If $k > 1$, then let F' be a 2-factor of $K_{(2)}$ using e_K in some cycle C, and convert F' into a 2-factor F'' of $G - V(H)$ by including the ladder vertices in C. Now $F \cup F''$ is a 2-factor of G. In either case, the 2-factor of G contains F and the second part follows. □

The next lemma helps us to deal with cyclic 3-cuts $[H, K]$.

Lemma 3.5. *Suppose G is a 3-merge of a 2-connected near-cubic graph $H_{(3)}$ at v_H and a 2-connected cubic graph $K_{(3)}$ at v_K.*

1. *Then $\ell(G) \geq \ell(H_{(3)})$.*
2. *If there is a 2-factor F of $H_{(3)}$ so that the cycle of F through v_H has length $\ell(H_{(3)})$, then $\ell(G) \geq \ell(H_{(3)}) + 1$.*

Proof. Let F be a longest 2-factor of $H_{(3)}$ and suppose F does not use the edge e incident with v_H. There is a 2-factor F' of $K_{(3)}$ that does not contain the edge incident with v_K that corresponds to e. F and F' combine to produce a 2-factor of G, where the cycle C of F through v_H has merged with the cycle C' of F' through v_K to produce a cycle of length $|V(C)| - 1 + |V(C')| - 1 \geq |V(C)| + 1$; all other cycles of F remain unaffected, and the results follow. □

Lemma 3.6. *Let G be a cyclically 4-connected near-cubic graph with $|V(G)| \geq 10$ and a 4-cycle $C = (v_0, v_1, v_2, v_3, v_0)$. (Throughout, all indices are modulo 4.) Then:*

1. *G has no triangle;*
2. *each vertex v_i in C has a distinct neighbor w_i not in C;*
3. *for each $i \in \{1, 2\}$, $G - \{v_i v_{i+1}, v_{i+2} v_{i+3}\}$ is 2-connected;*
4. *for some $i \in \{1, 2\}$, suppressing the four degree 2 vertices v_1, v_2, v_3, and v_0 in $G - \{v_i v_{i+1}, v_{i+2} v_{i+3}\}$ produces a 2-connected near-cubic graph G' with* new *edges $E(G') - E(G) = \{w_{i+1} w_{i+2}, w_{i+3} w_i\}$.*
5. *$\ell(G) \geq \ell(G')$ and if e is a new edge, then $\ell(G) \geq \ell(G', e) + 2$.*

Proof. (1) Let H be the subgraph of G induced by the vertices in a triangle and let $K = G - V(H)$. Then $[H, K]$ is a cyclic k-cut for some $k \in \{2, 3\}$, a contradiction.

(2) $[C, G - V(C)]$ is a cyclic k-cut for some $k \leq 4$. Since G is cyclically 4-connected, $k = 4$ and the edges in the cut form a matching. Thus each v_i has a distinct neighbor w_i not in C.

(3) Let $e_j = v_j v_{j+1}$. Since e_i and e_{i+2} are not both incident with t, $G - \{e_i, e_{i+2}\}$ is connected. If it is not 2-connected, then it must have a cut-edge f, and thus $\{e_i, e_{i+2}, f\}$ is

a 3-cut in G. Since G is cyclically 4-connected, this cut must be non-cyclic, contradicting the fact that the vertices incident with e_i and e_{i+2} are all distinct and have degree 3.

(4) Suppose the graph G_i obtained from $G - \{e_i, e_{i+2}\}$ by suppressing the vertices in C has a multiple edge. Then either $w_i w_{i+3}$ or $w_{i+1} w_{i+2}$ is an edge of G and, after suppression, it is parallel to one of the paths $(w_i, v_i, v_{i+3}, w_{i+3})$ and $(w_{i+1}, v_{i+1}, v_{i+2}, w_{i+2})$. If this happens for both G_1 and G_2, then, for some $j \in \{0, 1, 2, 3\}$, w_j has the three neighbours v_j, w_{j+1} and w_{j-1}. In this case, let H be the subgraph of G induced by $V(C) \cup \{w_j, w_{j+1}, w_{j-1}\}$ and let K be the subgraph of G induced by the remaining vertices of G. The choice of H implies that $[H, K]$ is a k-cut for some $k \leq 3$, but since $|V(K)| \geq 10 - 7 = 3$ this cut is cyclic, a contradiction. Thus, we may assume that G_1 is a near-cubic graph. Suppressing degree 2 vertices does not decrease connectivity, so G_1 is 2-connected.

(5) We assume that G_1 is 2-connected and simple and let F be a 2-factor of G_1. We reconstruct a 2-factor of G in each of the three resulting cases.

If F uses neither of the new edges $w_0 w_1$ and $w_2 w_3$, then $F \cup \{C\}$ is a 2-factor in G.

If F uses precisely one new edge, say $w_0 w_1$, then let C' be the cycle in F through this edge. In this case, we get a 2-factor of G by replacing $w_0 w_1$ in C' with $(w_0, v_0, v_3, v_2, v_1, w_1)$.

Finally, if F uses both $w_0 w_1$ and $w_2 w_3$, then we simply replace them with (w_0, v_0, v_1, w_1) and (w_2, v_2, v_3, w_3), respectively. □

Lemma 3.7. *Let G be a cyclically 4-connected cubic graph.*

1. *Suppose G has a 5-cycle $(v_0, v_1, v_2, v_3, v_4, v_0)$, and indices throughout are modulo 5. Then, for each i with $0 \leq i \leq 4$, $(G - v_i) - v_{i+2} v_{i+3}$ is 2-connected. If G has no 4-cycles, then suppressing the four degree 2 vertices v_{i+1}, v_{i+2}, v_{i+3} and v_{i+4} in $(G - v_i) - v_{i+2} v_{i+3}$ produces a 2-connected near-cubic graph.*
2. *Suppose G has a 6-cycle $(v_0, v_1, v_2, v_3, v_4, v_5, v_0)$, and indices throughout are modulo 6. Then, for each $i \in \{0, 1\}$, the graph $G - \{v_i v_{i+1}, v_{i+2} v_{i+3}, v_{i+4} v_{i+5}\}$ is 2-connected. If G has no 4-cycles, then suppressing the six degree 2 vertices in $G - \{v_i v_{i+1}, v_{i+2} v_{i+3}, v_{i+4} v_{i+5}\}$ produces a 2-connected cubic graph.*

Proof. (1) Let C be the cycle $(v_0, v_1, v_2, v_3, v_4, v_0)$ and let e_i be the edge $v_i v_{i+1}$. Since G is cubic, and triangle-free by Lemma 3.6.1, each v_i has a unique neighbor w_i not in C.

As G is 3-connected, $G - v_i$ is 2-connected, so $G - \{v_i, e_{i+2}\}$ is connected. If it is not 2-connected, then it has a cut-edge f. Let H and K be the two components of $G - \{v_i, e_{i+2}, f\}$. Choose the labelling of H and K so that v_i has at most one neighbour z in H. Consider the cut $[H, K + v_i]$ in G. Then $[H, K + v_i] \subseteq \{e_{i+2}, v_i z, f\}$; since G is cubic and cyclically 4-connected, equality holds and all 3 edges must be incident with the same vertex. Since v_i is not incident with e_{i+2} this vertex can only be z, so that $v_i z$ is a chord of C, contradicting the fact that G is triangle-free.

Finally, if G has no 4-cycles, for each $i = 0, 1, 2, 3, 4$, the w_i are distinct and there is no edge of G joining w_i and w_{i+1}, which shows suppressing the four degree 2 vertices in $G - \{v_i, e_{i+2}\}$ results in a near-cubic 2-connected graph, with degree 2 vertex w_i.

(2) Let C be the cycle $(v_0, v_1, v_2, v_3, v_4, v_5, v_0)$ and let e_i be the edge $v_i v_{i+1}$.

As G is 3-connected, if $G - \{e_i, e_{i+2}, e_{i+4}\}$ is not connected, then there is a 3-cut $[H, K]$ so that $[H, K] = \{e_i, e_{i+2}, e_{i+4}\}$. As G has no cyclic 3-cuts, one of H and K is a single vertex. However, the edges e_i, e_{i+2}, and e_{i+4} have no incident vertex in common, a contradiction. Thus, $G - \{e_i, e_{i+2}, e_{i+4}\}$ is connected.

If $G - \{e_i, e_{i+2}, e_{i+4}\}$ has a cut-edge e, then $G - \{e, e_i, e_{i+2}, e_{i+4}\}$ has precisely two (cyclic) components H and K, and $[H, K] = \{e, e_i, e_{i+2}, e_{i+4}\}$. Since G is cyclically 4-connected, the edges in this 4-cut form a matching. It follows that the edges e_{i+1}, e_{i+3} and e_{i+5} are in $G-[H, K]$ and so at least two of them are in the same one of H and K, say e_{i+1} and e_{i+3} are in H. But this is impossible, as e_{i+1} and e_{i+3} are incident with different ends of e_{i+2} and one of these is in K. Thus, $G - \{e_i, e_{i+2}, e_{i+4}\}$ is 2-connected.

If G has no 3- or 4-cycles, then C is an induced cycle and we let w_j be the neighbour of v_j not in C. Then w_j and w_{j+1} are distinct and not adjacent in G. This is enough to conclude that the paths $(w_j, v_j, v_{j+1}, w_{j+1})$ do not become multiple edges after suppressing the vertices in C from $G - \{e_i, e_{i+2}, e_{i+4}\}$, and hence the resulting graph is 2-connected and cubic. □

Let C be a cycle in a graph G. A *partial 2-factor of G with respect to C* is a 2-regular subgraph F of G such that $V(G) - V(F) \subseteq V(C)$, and $E(F) \cap E(C)$ is a matching.

Lemma 3.8. *Suppose a near-cubic 2-connected graph G has a partial 2-factor F with respect to a cycle C. Let $|E(C) \cap E(F)| = k$ and let s be the number of vertices in the subgraph induced by C and the cycles in F that meet C. If some longest cycle in F misses $V(C)$ (for example when $k = 0$) or $|V(C)| = 2k$, then $\ell(G) \geq \ell(F)$. If $k > 0$, then $\ell(G) \geq s/k$.*

Proof. If $|V(C)| = 2k$, then F is a 2-factor of G, and the result is obvious. Otherwise $|V(C)| > 2k$ and we consider F' such that $E(F')$ is the symmetric difference of $E(F)$ and $E(C)$: in F' each vertex in $V(F) - V(C)$ is incident with 2 edges from F and each vertex in $V(C)-V(F)$ is incident with 2 edges from C; each vertex in $V(C)\cap V(F)$ must be incident with exactly one edge from $E(C) \cap E(F)$ (since these form a matching and G is near-cubic). It follows that F' is a 2-factor of G. When $k = 0$, $F' = F \cup \{C\}$ and thus $\ell(G) \geq \ell(F') \geq \ell(F)$.

When $k > 0$, let $C^1, C^2, \dots, C^r$ be the cycles of F that meet C and let $G' = C \cup C^1 \cup C^2 \cup \dots \cup C^r$. Then G' is a 2-connected sub-cubic graph on s vertices with exactly $2k$ vertices of degree 3. Observe that $F'' = F' \cap V(G') = G' - (E(C) \cap E(F))$ is a 2-factor of G'. Since G' is 2-connected each component of F'' has at least 2 vertices of degree 3 in G', and thus F'' has at most k components. Thus one of the cycles in F'' has length at least s/k, and hence $\ell(G) \geq \ell(F') \geq \ell(F'') \geq s/k$. □

We are now ready for the proof of the main result of this section. A 2-factor F in a graph G has a *long cycle* if one of the components of F has length at least 7.

Proof of Theorem 3.1. We proceed by induction, with the base cases $|V(G)| \leq 10$ covered by Lemma 3.2. For the induction step, we may suppose $|V(G)| \geq 11$. Let t be the degree 2 vertex, if G has one.

Claim 1. *If t is in a triangle of G, then $\ell(G) \geq 7$.*

Proof. There is a $k \geq 2$ so that G is a $(2k-1)$-replacement of some edge e in a 2-connected cubic graph G'. By Lemma 3.3, $\ell(G) \geq \ell(G')$, so the result follows unless G' either is $\mathbb{P}$ or has at most 6 vertices. In these cases, Lemma 3.2 implies $\ell(G', e) = \ell(G') \geq 4$, so Lemma 3.3 implies $\ell(G) \geq \ell(G', e) + (2k - 1) \geq 4 + 3 = 7$. △

Claim 2. *If G has a cyclic 2-cut, then $\ell(G) \geq 7$.*

Proof. From Claim 1, we may assume that either G is cubic, or obtained by subdividing an edge of a cubic graph.

Since G has a cyclic 2-cut, there are disjoint subgraphs H and K of G and a ladder L so that $G = H \cup L \cup K$, and t, if it exists, is in H. If $\ell(H_{(2)}) \geq 7$, then Lemma 3.4.2 shows that $\ell(G) \geq 7$, as required. Thus, we may assume $\ell(H_{(2)}) < 7$.

We know that $H_{(2)}$ is either $\mathbb{P}$ or $\mathbb{P}^*$, or has at most 6 vertices. In any of these cases Lemma 3.2 implies that $\ell(H_{(2)}, e_H) = \ell(H_{(2)}) \geq 4$. Thus Lemma 3.4.1 implies $\ell(G) \geq \ell(H_{(2)}, e_H) + \ell(K_{(2)}, e_K) + 2k - 2 \geq 4 + 3 + 0 = 7$. △

Claim 3. *If G is not cyclically 4-connected, then $\ell(G) \geq 7$.*

Proof. From Claims 1 and 2, we may assume G is cyclically 3-connected. Since G is not cyclically 4-connected, G has a cyclic 3-cut $[H, K]$, with the labelling chosen so that t, if it exists, is in H. Since $[H, K]$ is a minimum size cyclic cut it must be a matching, and thus the graph $H_{(3)}$ obtained from H by adding a new degree 3 vertex v_H adjacent to the vertices in H that are incident with edges in $[H, K]$ is simple near-cubic and 2-connected. Similarly the graph $K_{(3)}$ obtained from K by introducing a new vertex v_K is cubic 2-connected, and G is the 3-merge of $H_{(3)}$ and $K_{(3)}$.

If $\ell(H_{(3)}) \geq 7$, then Lemma 3.5.1 implies that $\ell(G) \geq 7$, as required. Thus, we may assume $\ell(H_{(3)}) < 7$. If $H_{(3)}$ is either $\mathbb{P}^*$, or has 6 vertices, then Lemma 3.5.2 implies (in the case of $\mathbb{P}^*$ because any vertex and any edge of $\mathbb{P}$ are in a 5-cycle in $\mathbb{P}$) $\ell(G) \geq 6+1 = 7$. Thus, $H_{(3)}$ is $\mathbb{P}$, K_4 or K_4^*. If $H_{(3)}$ is cubic, then the roles of H and K can be reversed, so that $K_{(3)}$ must also be K_4 or $\mathbb{P}$. So, since $|V(G)| \geq 11$, it follows that G is the 3-merge of $\mathbb{P}$ and either K_4 or $\mathbb{P}$, whence $\ell(G) \geq 9$. Hence it remains to consider the case $H_{(3)} = K_4^*$.

In this case, let u_1, u_2, u_3 be the neighbors of v_H in $H_{(3)}$, and let v_1, v_2, v_3 be their corresponding neighbors in K. Suppose the neighbors of v_H in a Hamilton cycle D in $H_{(3)}$ are u_i, u_j. Let F_k be a 2-factor of $K_{(3)}$ avoiding $v_K v_k$, where $k \in \{1, 2, 3\} \setminus \{i, j\}$ and let C_k be the cycle of F_k through v_K. Then $F_k - C_k$ together with the cycle $C = (D - v_H) \cup (C_k - v_K) \cup \{u_i v_i, u_j v_j\}$ is a 2-factor of G. As long as C_k has length at least 4, C has length at least 7. Thus, we may assume C_k is the 3-cycle (v_K, v_i, v_j, v_K). In particular, $v_i v_j$ is an edge of G.

However $H_{(3)}$ has two different Hamilton cycles D, so we can assume that, say $v_1 v_2$ and $v_2 v_3$ are edges of G. There are at least three cubic vertices of G not in $H \cup \{v_1, v_2, v_3\}$, so the 2-cut $[H \cup \{v_1, v_2, v_3\}, K - \{v_1, v_2, v_3\}]$ is cyclic, contradicting the assumption that G is cyclically 3-connected. △

From now on we may assume that G is cyclically 4-connected. In particular, G is triangle-free by Lemma 3.6.1.

Claim 4. *If G has a 4-cycle, then $\ell(G) \geq 7$.*

Proof. Let C be a 4-cycle $(v_0, v_1, v_2, v_3, v_0)$ in G. Since G is cyclically 4-connected, Lemma 3.6.2 implies that, for every $i \in \{0, 1, 2, 3\}$, v_i has a unique neighbor w_i not in C. Note that, for $i = 0, 1, 2, 3$, $w_i \neq w_{i+1}$ (indices being read modulo 4). By Lemma 3.6.4, for some $i \in \{0, 1\}$, the result of suppressing the four degree 2 vertices v_0, v_1, v_2, and v_3 in $G - \{v_i v_{i+1}, v_{i+2} v_{i+3}\}$ is a 2-connected graph G'.

Obviously $|V(G')| = |V(G)| - 4 \geq 7$. If $\ell(G') \geq 7$, then the result follows from the first part of Lemma 3.6.5. Otherwise G' is $\mathbb{P}$ or $\mathbb{P}^*$, and Lemmas 3.2.2 and 3.6.5 imply $\ell(G) \geq \ell(G') + 2 \geq 7$. △

Perhaps surprisingly, we next treat two cases in which G has a 6-cycle. Because there are no 3- or 4-cycles, any 6-cycle is induced.

Claim 5. *If there is a 6-cycle C in G containing t, then $G = \mathbb{P}^*$ or $\ell(G) \geq 7$.*

Proof. Let $C = (v_0, v_1, v_2, v_3, v_4, v_5, v_0)$, with the labelling chosen so that $v_0 = t$. For $1 \leq i \leq 5$, let w_i be the neighbour of v_i not in C. Since G has no 3- or 4-cycles, if $w_i = w_j$, then either $i = j$ or $i = j \pm 3$. Furthermore, we cannot have both $w_1 = w_4$ and $w_2 = w_5$, as then there is a cyclic 3-cut with $V(H) = V(C) \cup \{w_1, w_2\}$. So we may assume $w_2 \neq w_5$. Also, for $i = 1, 2, 3, 4$, w_i and w_{i+1} are not adjacent in G.

Let $G'' = G - \{v_0, v_3\}$. Suppressing v_0 in G produces a graph to which Lemma 3.7.1 applies, showing G'' is 2-connected. Furthermore, suppressing the four degree 2 vertices v_1, v_2, v_4, v_5 in G'' yields a 2-connected near-cubic graph G' (since consecutive w_i's are not adjacent) with degree 2 vertex w_3.

We note that $|V(G')| = |V(G)| - 6 \geq 5$. As G' is not cubic, by induction either $G' = K_4^*$ or $G' = \mathbb{P}^*$ or $\ell(G') \geq 7$.

Suppose first that $|V(G')| = 5$. If $w_1 \neq w_4$, then the five vertices of G' are w_1, w_2, w_3, w_4, and w_5. As w_3 is the degree 2 vertex and w_3 is not adjacent to either w_2 or w_4, the edges in $G - V(C)$ must be w_3w_5, w_3w_1, w_4w_2, w_4w_1 and w_2w_5, and thus $G = \mathbb{P}^*$. If $w_1 = w_4$, then neither $w_1 = w_4$ nor w_2 is adjacent to w_3 in G'. Thus, the neighbors of w_3 are w_5 and some other vertex x. But then $w_4 = w_1$ is adjacent to w_5 in G, a contradiction.

In the remaining case, either $G' = \mathbb{P}^*$ or $\ell(G') \geq 7$. If the latter, then let F' be any longest 2-factor of G'. If $G' = \mathbb{P}^*$, then we claim there is a 2-factor F' of G' with a cycle C' through exactly one of w_4w_5 and w_1w_2. Indeed, if $w_1 = w_4$, then observe that any F' containing the third edge incident with w_1 will avoid exactly one of these edges altogether. If $w_1 \neq w_4$, then w_1w_2 and w_4w_5 are not incident with the same vertex of G' and so it is possible either to find a 6-cycle containing exactly one of these and w_3, or a 5-cycle containing only one of these together with a 6-cycle containing w_3.

Expanding F' back to G, we obtain a partial 2-factor F of G with respect to C. Set $k = |E(C) \cap E(F)|$ and let s be the number of vertices in the subgraph induced by C and the cycles in F that meet C. Then $k = 1$ when $G' = \mathbb{P}^*$. By Lemma 3.8, $\ell(G) \geq \ell(F) \geq \ell(F') \geq 7$, unless $1 \leq k \leq 2$ and a longest cycle C' of F' contains a new edge. Since $s \geq |V(C)| + |V(C')|$, it follows that, for $\ell(G') \geq 7$, we obtain $\ell(G) \geq (6+7)/k > 6$, as desired. If $G' = \mathbb{P}^*$, then $\ell(G) \geq (6+5)/1 = 11$. △

Claim 6. *If G is cubic and has a 6-cycle, then $\ell(G) \geq 7$.*

Proof. Let C be the 6-cycle $(v_0, v_1, v_2, v_3, v_4, v_5, v_0)$ in G. Note that each v_i has a neighbour w_i not in C. For $0 \leq i \leq 5$, let e_i be the edge v_iv_{i+1} (all indices being read modulo 6). Let G' be the graph obtained from $G - \{e_1, e_3, e_5\}$ by suppressing the 6 vertices in C which are now of degree 2. By Lemma 3.7.2, G' is 2-connected and cubic, in particular, $G' \neq \mathbb{P}^*$. As G is cubic and $|V(G)| \geq 11$, we see that $|V(G)| \geq 12$ and $|V(G')| = |V(G)| - 6 \geq 6$. It is important to note that, since G has no 3- or 4-cycles, if i and j are distinct and $w_i = w_j$, then $j = i+3$. Thus, the three new edges w_0w_1, w_2w_3, w_4w_5 can't all be incident with the same vertex in G'.

If $\ell(G') \geq 7$, then let F' be a 2-factor of G' having a cycle of length $\ell(G')$. If $\ell(G') \leq 6$, then G' is one of $K_{3,3}$, $K_3\Box K_2$ or $\mathbb{P}$. When G' is $\mathbb{P}$, F' may be chosen to contain exactly one or two new edges, and no more than one in each component. Since w_0w_1, w_2w_3, w_4w_5 don't have a common vertex, when $G' = K_{3,3}$, we can let F' be a 6-cycle containing all three of them.

When $G' = K_3\Box K_2$ we observe that, since G is triangle-free, each of the triangles in G' must contain one of the new edges. However, no triangle can contain two new edges, say w_0w_1 and w_2w_3, since then $w_0 = w_3$, making w_1 and w_2 adjacent. Up to symmetry, there are two ways to pick one edge from each triangle. In both cases, there is (up to symmetry) only one way to choose the third edge not from the triangles to also eliminate all 4-cycles. In either case, we can let F' be a 6-cycle containing all 3 new edges.

Expanding F' back to G, we obtain a partial 2-factor F of G with respect to C. Set $k = |E(C) \cap E(F)|$ and let s be the number of vertices in the subgraph induced by C and the cycles in F that meet C. If $|V(G')| = 6$, then $|V(C)| = 6 = 2k$ and F consists of one cycle, so $\ell(G) \geq \ell(F) = 12$. If $G' = \mathbb{P}$, then either $k = 1$ and $s = 6+5$, or $k = 2$ and $s = 6+10$, and in either case $\ell(G) \geq 8$. If $\ell(G') \geq 7$, then, by Lemma 3.8, $\ell(G) \geq \ell(F) \geq \ell(F') \geq 7$, unless $1 \leq k \leq 2$ and a longest cycle C' of F' contains a new edge. Since $s \geq |V(C)| + |V(C')| \geq 6+7$, we get $\ell(G) \geq 13/k > 6$, as desired. △

The next two claims will now complete the proof.

Claim 7. *If G has a vertex of degree 2, then $G = \mathbb{P}^*$ or $\ell(G) \geq 7$.*

Proof. If the vertex t of degree 2 in G is not in a 5- or 6-cycle in G, then any 2-factor F of G suffices, as the cycle of F through t is long. If t is in a 6-cycle, then Claim 5 shows either $G = \mathbb{P}^*$ or $\ell(G) \geq 7$. The remaining case is that t is in a 5-cycle $C = (v_0, v_1, v_2, v_3, v_4, v_0)$, with the labelling chosen so that $v_0 = t$. For $1 \leq i \leq 4$, let w_i be the neighbour of v_i not in C. As G has no 3- or 4-cycles, these four w_i are distinct.

Contracting v_0v_1 to apply Lemma 3.6.3, we deduce that the graph $(G - v_0) - v_2v_3$ is 2-connected. As G has no 4-cycles, suppressing the four vertices v_1, v_2, v_3, v_4 of degree 2 in $(G - v_0) - v_2v_3$ results in a graph G' with new edges w_1w_2 and w_3w_4. Obviously, G' is 2-connected and cubic and $|V(G')| = |V(G)| - 5 \geq 6$. If $G' = K_{3,3}$, then G' contains a 4-cycle avoiding the new edges, which is thus a 4-cycle in G, a contradiction. Similarly, if $G' = K_3\Box K_2$, then G' contains either a 3-cycle or a 4-cycle without a new edge, a contradiction. Thus $|V(G')| \geq 8$, so by induction $G' = \mathbb{P}$ or $\ell(G') \geq 7$.

If G' is $\mathbb{P}$, then it has a 2-factor F' that contains exactly one of the two new edges w_1w_2 and w_3w_4. If $\ell(G') \geq 7$, we let F' be any longest 2-factor of G'.

Expanding F' back to G, we obtain a partial 2-factor F of G with respect to C. Set $k = |E(C) \cap E(F)|$ and let s be the number of vertices in the subgraph induced by C and the cycles in F that meet C. Then $k = 1$ when $G' = \mathbb{P}$. By Lemma 3.8, $\ell(G) \geq \ell(F) \geq \ell(F') \geq 7$, unless $1 \leq k \leq 2$ and a longest cycle C' of F' contains a new edge. Since $s \geq |V(C)| + |V(C')|$ it follows that for $\ell(G') \geq 7$ we obtain $\ell(G) \geq 12/k \geq 6$, but observe that equality can only hold if $k = 2$ and $V(C) \cup V(C')$ yields two 6-cycles. One of these 6-cycles contains $v_0 = t$, and the result follows by Claim 5. If $G' = \mathbb{P}$, then $\ell(G) \geq (5+5)/1 = 10$. △

Claim 8. *If G is cubic, then $\ell(G) \geq 7$.*

Proof. Again, we may assume G is cyclically 4-connected and has no 3- or 4-cycles. Claim 6 allows us to assume that G has no 6-cycles. We may further assume $(v_0, v_1, v_2, v_3, v_4, v_0)$ is a 5-cycle C in G, as otherwise we are done: every 2-factor has a long cycle.

Let w_i be the neighbour of v_i that is not in C, and let $W = \{w_0, w_1, w_2, w_3, w_4\}$. As G has no 3- or 4-cycles, the vertices $v_0, v_1, \dots, v_4, w_0, w_1, \dots, w_4$ are distinct. Furthermore, the vertices in W are pairwise non-adjacent, since G has no 4- or 6-cycles.

Now let G' be obtained from $(G - v_0) - v_2v_3$ by suppressing the four vertices v_1, v_2, v_3, v_4 of degree 2. By Lemma 3.7.1, G' is a 2-connected near-cubic graph with degree 2 vertex w_0. Clearly $|V(G')| \geq 6$, so by induction either $G' = \mathbb{P}^*$ or $\ell(G') \geq 7$. There are four different 6-cycles through w_0 in $\mathbb{P}^*$; no set of two edges not incident with either w_0 or its neighbors meets all four of these. Thus, if $G' = \mathbb{P}^*$, then we can find a 6-cycle of G' containing w_0 that avoids w_1w_2 and w_3w_4. This is a 6-cycle in G, contradicting the fact that G has no 6-cycles. Therefore $G' \neq \mathbb{P}^*$, and it remains to consider $\ell(G') \geq 7$.

Let F' be a longest 2-factor of G'. Expanding F' back to G, we obtain a partial 2-factor F of G with respect to C. Set $k = |E(C) \cap E(F)|$ and let s be the number of vertices in the subgraph induced by C and the cycles in F that meet C. By Lemma 3.8, $\ell(G) \geq \ell(F) \geq \ell(F') \geq 7$, unless $1 \leq k \leq 2$ and a longest cycle C' of F' contains a new edge. Since $s \geq |V(C)| + |V(C')| \geq 5 + 7$ we obtain $\ell(G) \geq 12/k \geq 6$, but observe that equality can only hold if G contains a 6-cycle, in which case Claim 6 finishes the proof. △

Obviously Theorem 3.1 follows from Claims 7 and 8. □

4 Infinite Graphs

In this section, we indicate the connections to 2-factors in infinite cubic graphs. The graphs $J(k)$ from Section 2 have a natural limit version $J(\infty)$ (let k go to infinity). This graph is 3-connected, 1-ended, and cubic and every 2-factor in $J(\infty)$ contains only cycles of length at most 16.

In a slightly different direction, we show how to construct a 3-connected, 1-ended, cubic graph so that every 2-factor is a 2-way infinite Hamilton path. The construction starts by noting that the edges of $\mathbb{P}$ partitions into five sets of three edges each with the property that every 2-factor of $\mathbb{P}$ contains precisely two edges from each of the five sets, and these two edges are in different components of the 2-factor.

Let $\mathbb{P}^+$ be the 3-connected cubic graph obtained from $\mathbb{P}$ by subdividing the three edges in one of the five sets and then adding a new vertex n adjacent to the three vertices of subdivision. Let u be any vertex of $\mathbb{P}$ not incident with any of the three original edges.

The key properties of $\mathbb{P}^+$, easily proved from properties of $\mathbb{P}$, are: its only 2-factors are Hamilton cycles; and $\mathbb{P}^+ - n$ has no 2-factor.

Let $G_1, G_2, \dots$ be disjoint copies of $\mathbb{P}^+$, with n_i and u_i being the copies of n and u, respectively, in G_i. Iteratively 3-merge the G_i, starting with G_1 at n_1 and G_2 at u_2. Then this new graph is 3-merged, at n_2, with G_3 at u_3, and so on. The infinite graph thus created is easily seen to have 2-way infinite Hamilton paths as its only 2-factors.

5 Open Questions

Several open questions naturally arise. We mention only a few here.

Naturally, we wonder what the exact values of $L_2(n)$ and $L_3(n)$ are and we conjecture that $L_k(n)$ is unbounded for sufficiently large k. This can be viewed as a weaker version of the following unpublished question Thomassen first asked in the 1970's: Is every graph in $\mathcal{C}_k(n)$ Hamiltonian when k is sufficiently large?

Kochol (personal communication, based on [4, 5]) has examples of cyclically 6-connected cubic graphs on $2n$ vertices for which every 2-factor has cn components, for some $c > 0$. Of course this does not imply that $L_6(n)$ is bounded.

On the other hand, Fleischner [1] conjectures that every cyclically 4-connected Class 2 graph has a dominating cycle. This might suggest that there is always a 2-factor with a long cycle for cyclically 4-connected cubic graphs.

Two other natural questions are: what is the average value of $\ell(G)$ over all graphs in either $C_2(n)$ or $C_3(n)$? and what is $\ell(R)$, where R is the random cubic graph?

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References

[1] H. Fleischner, *Cycle decompositions, 2-coverings, removable cycles, and the four-color-disease*, Progress in graph theory, Waterloo, Ont., 1982, 233–246, Academic Press, Toronto, ON, 1984.

[2] D. A. Holton and J. Sheehan, *The Petersen Graph*, Australian Mathematical Society Lecture Series, **7**, Cambridge University Press, Cambridge, 1993.

[3] B. Jackson and K. Yoshimoto, Spanning even subgraphs of 3-edge-connected graphs, *J. Graph Theory* **62** (2009), 37–47.

[4] M. Kochol, Superposition and constructions of graphs without nowhere-zero k-flows, *Europ. J. Combin.* **23** (2002), 281–306.

[5] M. Kochol, Equivalences between hamiltonicity and flow conjectures, and the sublinear defect property, *Discrete Math.* **254** (2002), 221–230.

[6] J. Petersen, Die Theorie der regulären Graphen, *Acta Math.* **15** (1891), 193–220.

[7] T. Schönberger, Ein Beweis des Petersenschen Graphensatzes, *Acta Scientia Mathematica Szeged* **7** (1934), 51–57.

[8] W. T. Tutte, The factors of graphs, *Canad. J. Math.* **4** (1952), 314–328.

[9] D. B. West, *Introduction to Graph Theory*, 2nd ed., Prentice-Hall, Upper Saddle River, NJ, 2001.