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Confidence Interval for the Process Capability Index C_p Based on the Bootstrap-*t* Confidence Interval for the Standard Deviation

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Abstract

This paper proposes a confidence interval for the process capability index based on the bootstrap-*t* confidence interval for the standard deviation. A Monte Carlo simulation study was conducted to compare the performance of the proposed confidence interval with the existing confidence interval based on the confidence interval for the standard deviation. Simulation results show that the proposed confidence interval performs well in terms of coverage probability in case of more skewed distributions. On the other hand, the existing confidence interval has a coverage probability close to the nominal level for symmetrical or less skewed distributions. The code to estimate the confidence interval in R language is provided.

1 Introduction

Statistical process quality control has been widely applied in many industries. One of the quality measurement tools used for improvement of quality and productivity is the process capability index (PCI). Process capability indices are practical tools for establishing the relationship between the actual process performance and the manufacturing specifications. Although there are many process capability indices, the most commonly used index is C_p (Kane, 1986; Zhang, 2010). In this paper, we focus on the process capability index C_p, defined by Kane (1986) as:

$$C_p = \frac{USL - LSL}{6\sigma}, \tag{1}$$

where USL is the upper specification limit, LSL is the lower specification limit, and σ is the process standard deviation. The numerator of C_p gives the size of the

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range over which the process measurements can vary. The denominator gives the size of the range over which the process actually varies (Kotz and Lovelace, 1998). Due to the fact that the process standard deviation is unknown, it must be estimated from the sample data $\{X_1,...,X_n\}$. The sample standard deviation S; $S=\left((n-1)^{-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right)^{1/2}$ is used to estimate the unknown parameter σ in Equation (1). The estimator of the process capability index C_p is therefore

$$\hat{C}_p=\frac{USL-LSL}{6S}. \tag{2}$$

Although the point estimator of the capability index C_p shown in Equation (2) can be a useful measure, the confidence interval is more useful. A confidence interval provides much more information about the population characteristic of interest than does a point estimate (e.g., Smitson, 2001; Thompson, 2002; Steiger, 2004). The confidence interval for the capability index C_p is constructed by using a pivotal quantity $Q=(n-1)S^2/\sigma^2 \sim \chi^2_{(n-1)}$. Therefore, the $(1-\alpha)100\%$ confidence interval for the capability index C_p is

$$\left(\hat{C}_p\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{n-1}},\hat{C}_p\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{n-1}}\right), \tag{3}$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the $(\alpha/2)100^{th}$ and $(1-\alpha/2)100^{th}$ percentiles of the central chi-square distribution with $n-1$ degrees of freedom.

The confidence interval for the process capability index C_p shown in Equation (3) is to be used for data that are normal. The coverage probability of this confidence interval is close to a nominal value of $1-\alpha$ when the data are normally distributed. However, the underlying process distributions are non-normal in many industrial processes. (e.g., Chen and Pearn, 1997; Bittanti et al., 1998; Wu et al., 1999; Chang et al., 2002; Ding, 2004). In these cases, the coverage probability of the confidence interval can be appreciably below $1-\alpha$. Cojbasic and Tomovic (2007) presented a nonparametric confidence interval for the population variance based on ordinary t-statistics combined with the bootstrap method for a skewed distribution. In this paper, we propose a new confidence interval for the process capability index C_p based on the bootstrap-*t* confidence interval proposed by Cojbasic and Tomovic (2007).

The paper is organized as follows. In Section 2, the theoretical background of the existing confidence interval for the C_p is discussed. In Section 3, we provide an analytical formula for the confidence interval for the C_p based on the bootstrap-*t* confidence interval for the standard deviation. In Section 4, the performance of the confidence intervals for the C_p are investigated through a Monte Carlo simulation study. Conclusions are provided in the final section.

2 Existing confidence interval for the process capability index

Suppose $X_i \sim N\left(\mu,\sigma^2\right), i=1,2,...,n,$ a well-known $(1-\alpha)100\%$ confidence interval for the population variance σ^2, using a pivotal quantity $Q=(n-1)S^2/\sigma^2$, is (Cojbasic and Loncar 2011)

$$\frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \tag{4}$$

where $S^2=(n-1)^{-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$, and $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the $(\alpha/2)100^{th}$ and $(1-\alpha/2)100^{th}$ percentiles of the central chi-square distribution with $n-1$ degrees of freedom, respectively. From Equation (4), we have

$$\mathrm{P}\left(\frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}\right) = 1-\alpha$$

$$\mathrm{P}\left(\frac{\chi^2_{\alpha/2,n-1}}{(n-1)S^2} < \frac{1}{\sigma^2} < \frac{\chi^2_{1-\alpha/2,n-1}}{(n-1)S^2}\right) = 1-\alpha$$

$$\mathrm{P}\left(\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{(n-1)S^2}} < \frac{1}{\sigma} < \sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{(n-1)S^2}}\right) = 1-\alpha$$

$$\mathrm{P}\left(\frac{(USL-LSL)}{6}\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{(n-1)S^2}} < \frac{(USL-LSL)}{6\sigma} < \frac{(USL-LSL)}{6}\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{(n-1)S^2}}\right) = 1-\alpha$$

$$\mathrm{P}\left(\frac{(USL-LSL)}{6S}\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{n-1}} < C_p < \frac{(USL-LSL)}{6S}\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{n-1}}\right) = 1-\alpha.$$

We obtain a $(1-\alpha)100\%$ confidence interval for the C_p based on the confidence interval for the standard deviation which is

$$CI_1 = \left(\hat{C}_p\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{n-1}}, \hat{C}_p\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{n-1}}\right). \tag{5}$$

3 Proposed confidence interval for the process capability index

The bootstrap introduced by Efron (1979) is a computer-based and resampling method for assigning measures of accuracy to statistical estimates (Efron and Tibshirani, 1993). For a sequence of independent and identically distributed (i.i.d.) random variables, the bootstrap procedure can be defined as follows (Tosasukul et al., 2009). Let $X_1, X_2, ..., X_n$ be independently and identically distributed random variables from some distribution with mean μ and variance σ^2. Let the random variables $\{X_j^*, 1 \le j \le m\}$ be the result of sampling m times with replacement from the n observations $X_1, X_2, ..., X_n$. The random variables $\{X_j^*, 1 \le j \le m\}$ are called the bootstrap samples from original data $X_1, X_2, ..., X_n$. A confidence interval for the population variance can be constructed using the aforementioned pivotal quantity $Q = (n-1)S^2/\sigma^2$. For large sample sizes, a central chi-square distribution with $n-1$ degrees of freedom can be approximated by a normal distribution with mean $n-1$ and variance $2(n-1)$ (Cojbasic and Tomovic, 2007). Therefore, the distribution of the standardized variable

$$Z = \frac{\frac{(n-1)S^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} = \frac{S^2 - \sigma^2}{\sqrt{\mathrm{var}(S^2)}}$$

converges to a standardized normal distribution as n increases to infinity. The bootstrap confidence interval for the σ^2 is calculated based on the statistic

$$T = \frac{S^2 - \sigma^2}{\sqrt{\widehat{\mathrm{var}}(S^2)}},$$

where $\widehat{\mathrm{var}}(S^2)$ is a consistent estimator of the variance of S^2. Casella and Berger (2001) have shown the estimator of $\mathrm{var}(S^2)$ for a non-normal distribution such that

$$\widehat{\mathrm{var}}(S^2) = \frac{1}{n}\left(\hat{\mu}_4 - \frac{n-3}{n-1}S^4\right) \quad \text{and} \quad \hat{\mu}_4 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^4.$$

After re-sampling B bootstrap samples, in each bootstrap sample we compute the value of the following statistic

$$T^* = \frac{S^{*2} - S^2}{\sqrt{\widehat{\mathrm{var}}(S^{*2})}}, \tag{6}$$

where S^{*2} is a bootstrap replication of statistic S^2, $\widehat{\mathrm{var}}(S^{*2}) = \frac{1}{n}\left(\hat{\mu}_4^* - \frac{n-3}{n-1}S^{*4}\right)$ and $\hat{\mu}_4^* = \frac{1}{m}\sum_{i=1}^{m}(X_i^* - \bar{X}^*)^4$. The $(1-\alpha)100\%$ bootstrap-*t* confidence intervals for the σ^2 is

$$\left(\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{1-\alpha/2} + \sqrt{2(n-1)}}, \frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{\alpha/2} + \sqrt{2(n-1)}}\right),$$

where $\hat{t}^*_{\alpha/2}$ and $\hat{t}^*_{1-\alpha/2}$ are the $(\alpha/2)100^{th}$ and $(1-\alpha/2)100^{th}$ percentiles of T^* shown in Equation (6). Additionally, the $(1-\alpha)100\%$ confidence interval for the standard deviation σ is

$$\left(\left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{1-\alpha/2} + \sqrt{2(n-1)}}\right]^{1/2}, \left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{\alpha/2} + \sqrt{2(n-1)}}\right]^{1/2}\right). \tag{7}$$

Then, from Equation (7), we construct the confidence interval for the C_p based on the bootstrap-*t* confidence interval for the standard deviation which is

$$\mathrm{P}\left(\left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{1-\alpha/2}+\sqrt{2(n-1)}}\right]^{1/2}<\sigma<\left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{\alpha/2}+\sqrt{2(n-1)}}\right]^{1/2}\right) \quad = \quad 1-\alpha$$

$$\mathrm{P}\left(\left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{\alpha/2}+\sqrt{2(n-1)}}\right]^{-1/2}<\frac{1}{\sigma}<\left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{1-\alpha/2}+\sqrt{2(n-1)}}\right]^{-1/2}\right) \quad = \quad 1-\alpha$$

$$\mathrm{P}\left(\frac{USL-LSL}{6}\cdot\left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{\alpha/2}+\sqrt{2(n-1)}}\right]^{-1/2}<\frac{USL-LSL}{6\sigma}<\frac{USL-LSL}{6}\cdot\left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{1-\alpha/2}+\sqrt{2(n-1)}}\right]^{-1/2}\right)$$
$$= \quad 1-\alpha$$

$$\mathrm{P}\left(\frac{USL-LSL}{6}\cdot\left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{\alpha/2}+\sqrt{2(n-1)}}\right]^{-1/2}<C_p<\frac{USL-LSL}{6}\cdot\left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{1-\alpha/2}+\sqrt{2(n-1)}}\right]^{-1/2}\right) \quad = \quad 1-\alpha.$$

Therefore, the confidence interval for the C_p based on the bootstrap-*t* confidence interval for the standard deviation is given by

$$CI_2=\left(\frac{USL-LSL}{6}\cdot\left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{\alpha/2}+\sqrt{2(n-1)}}\right]^{-1/2},\ \frac{USL-LSL}{6}\cdot\left[\frac{S^2\sqrt{2(n-1)}}{2\hat{t}^*_{1-\alpha/2}+\sqrt{2(n-1)}}\right]^{-1/2}\right). \quad (8)$$

All confidence intervals were implemented using the open source statistical package R (Ihaka and Gentleman, 1996); source code is available in Appendix.

4 Simulation study

To assess the performance of the proposed confidence interval, we conducted a Monte Carlo simulation study to estimate the coverage probabilities and expected lengths of the proposed confidence interval under different situations and compare them with the existing confidence intervals. The estimated coverage probability and the expected length (based on M replicates) are given by

$$\widehat{1-\alpha}=\frac{\#(L\le C_p\le U)}{M},$$

and

$$\widehat{Length}=\frac{\sum_{j=1}^{M}(U_j-L_j)}{M},$$

where $\#(L\le C_p\le U)$ denotes the number of simulation runs for which the true process capability index C_p lies within the confidence interval. The right-skewed data were generated with the population mean μ = 50 and the population standard deviation σ = 1 given in the Table 1.

Table 1: Probability distributions generated and the coefficient of skewness for Monte Carlo simulation.

Probability Distributions	Coefficient of Skewness
$N(50,1)$	0.000
$Uniform(48.268, 51.732)$	0.000
$10 \times Beta(4.4375, 13.3125) + 47.5$	0.506
$Gamma(9,3) + 47$	0.667
$Gamma(4,2) + 48$	1.000
$Gamma(2.25,1.5) + 48.5$	1.333
$Gamma(1,1) + 49$	2.000
$Gamma(0.75, 0.867) + 49.1340$	2.309
$Gamma(0.5, 0.707) + 49.2929$	2.828
$Gamma(0.4, 0.6325) + 49.3675$	3.163
$Gamma(0.3, 0.5477) + 49.4523$	3.651
$Gamma(0.25, 0.5) + 49.5$	4.000

The true values of the process capability index C_p, LSL and USL are set in the Table 2.

Table 2: True values of C_p, LSL and USL used for Monte Carlo simulation.

True Values of C_p	LSL	USL
1.00	47.00	53.00
1.33	46.01	53.99
1.50	45.50	54.50
1.67	44.99	55.01
2.00	44.00	56.00

The sample sizes simulated were 10, 25, 50 and 100 and the number of simulation trials was set to 10,000. The number of bootstrap samples is 1,000. The nominal confidence level was fixed at 0.95. All simulations were performed using programs written in the open source statistical package R (Ihaka and Gentleman, 1996).

The simulation results are presented for four cases. As can be seen from Figures 1 and 2, the existing confidence interval (CI_1) provides more estimated coverage probabilities than the proposed confidence interval (CI_2) when the data were generated from symmetrical and less skewed distributions (coefficient of skewness between 0 and 2) for all sample sizes. Namely, CI_1 provides estimated coverage probabilities close to the nominal level 0.95, which is more than those of the CI_2 for the normal distribution. In addition, the expected lengths of CI_2 were shorter than those of CI_1 for all sample sizes (see Figures 5 and 6).

On the other hand, for more skewed distributions (coefficient of skewness between 2.309 and 4), the estimated coverage probabilities of CI_2 were greater than those of CI_1 for almost all sample sizes as shown in Figures 3 and 4. Figures 7 and 8 present the results on the expected lengths of CI_1 and CI_2 in case of right-skewed distributions. We found that the expected lengths of CI_1 were shorter than those of CI_2 for all sample sizes.

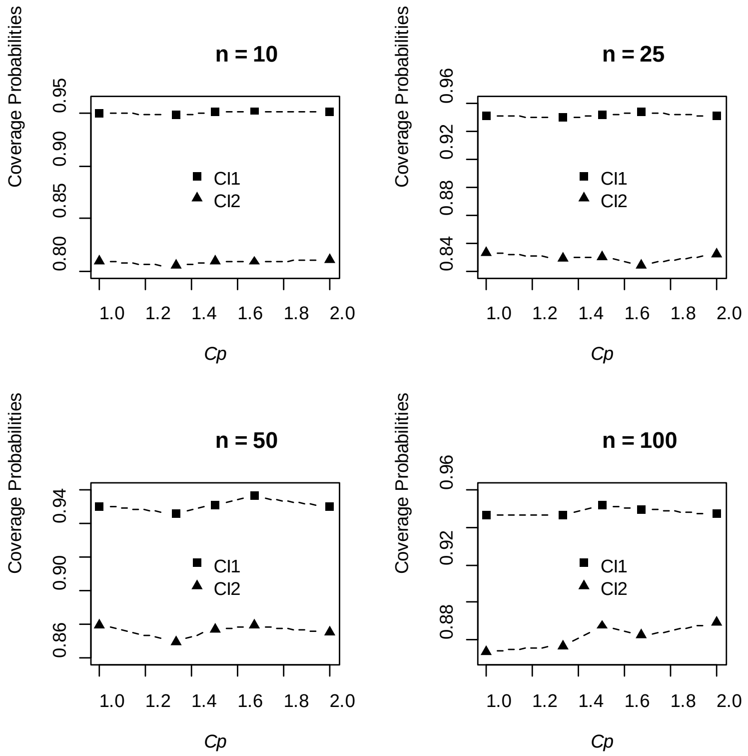

Figure 1: The estimated coverage probabilities of CI_1 and CI_2 for C_p in case of $N(50,1)$

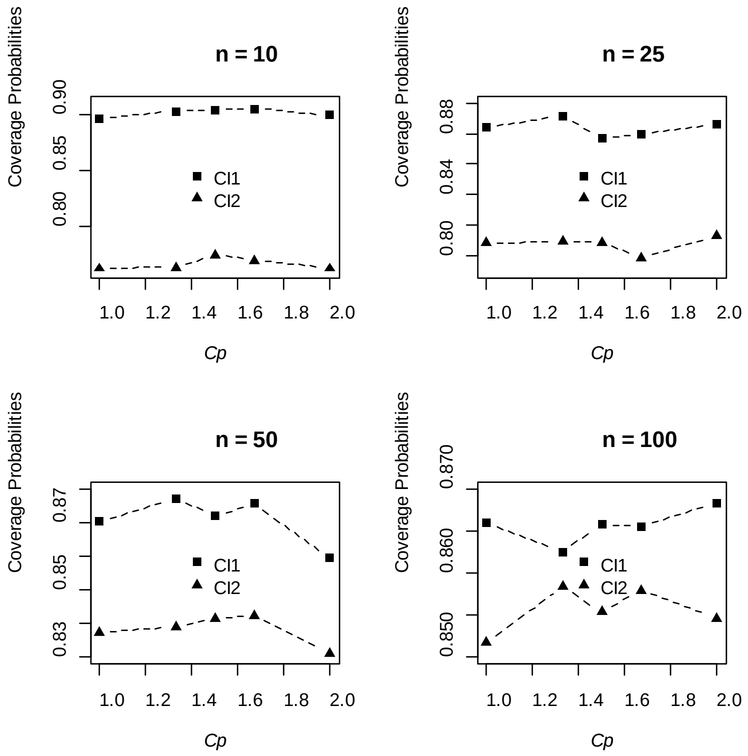

Figure 2: The estimated coverage probabilities of CI_1 and CI_2 for C_p in case of $Gamma(4,2)+48$

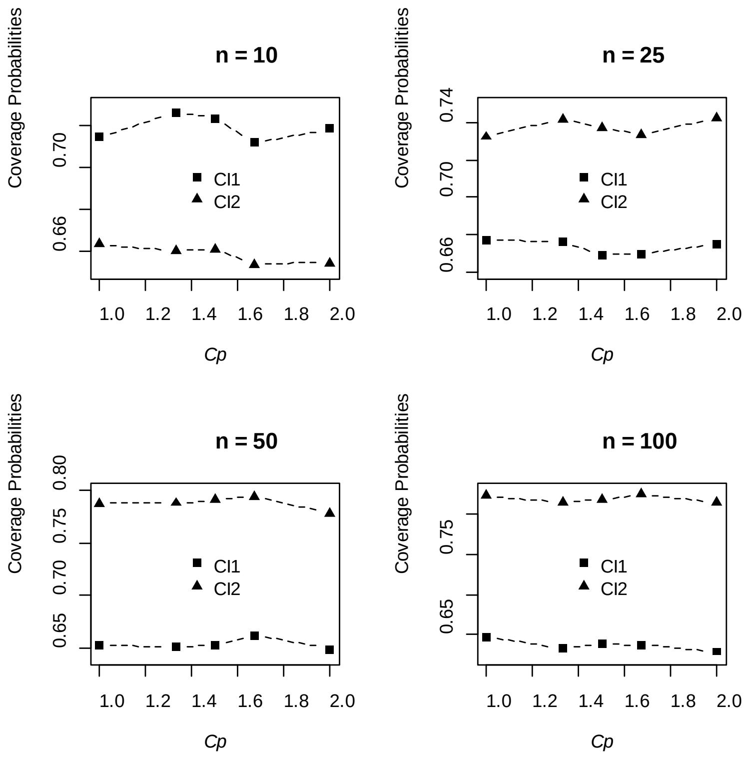

Figure 3: The estimated coverage probabilities of CI_1 and CI_2 for C_p in case of $Gamma(0.75,0.867)+49.1340$

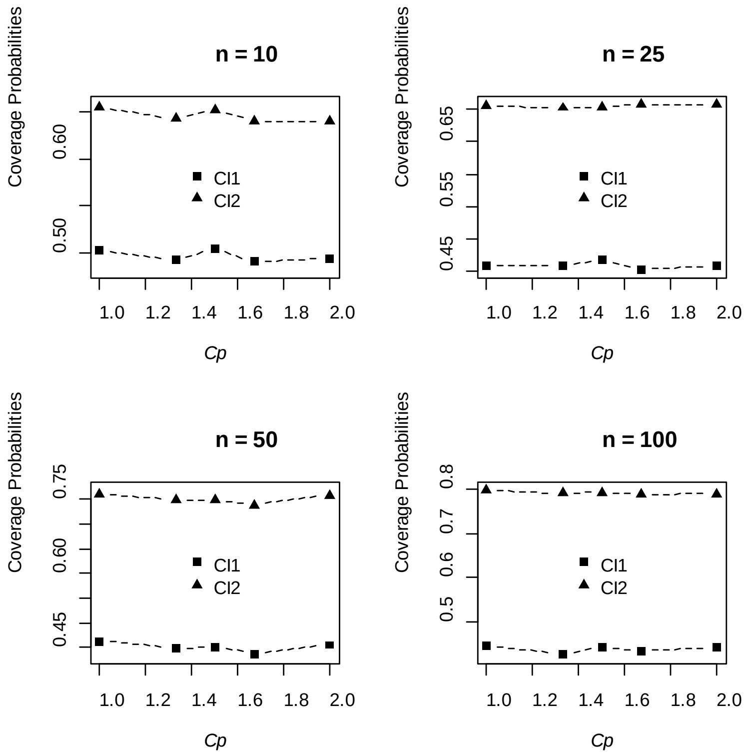

Figure 4: The estimated coverage probabilities of CI_1 and CI_2 for C_p in case of $Gamma(0.25, 0.5) + 49.5$

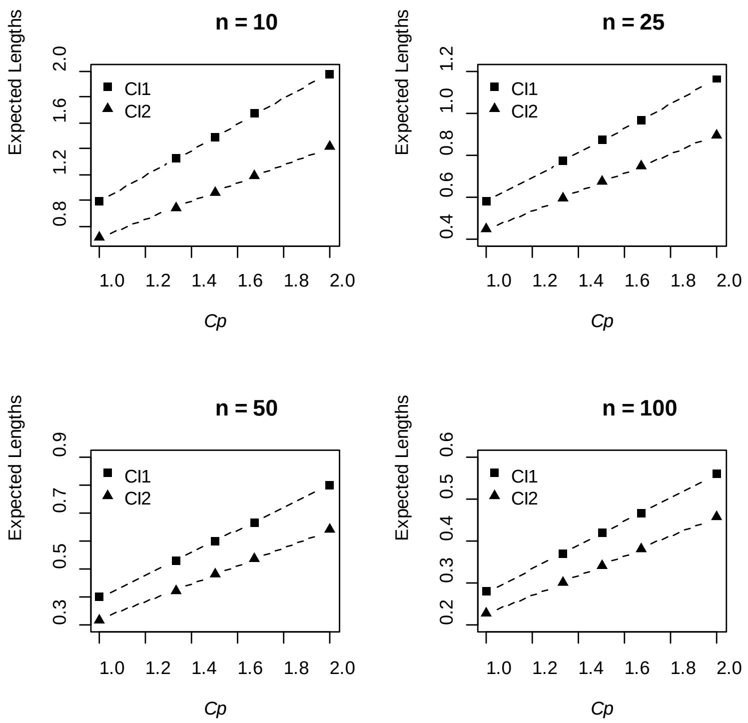

Figure 5: The expected lengths of CI_1 and CI_2 for C_p in case of $N(50,1)$

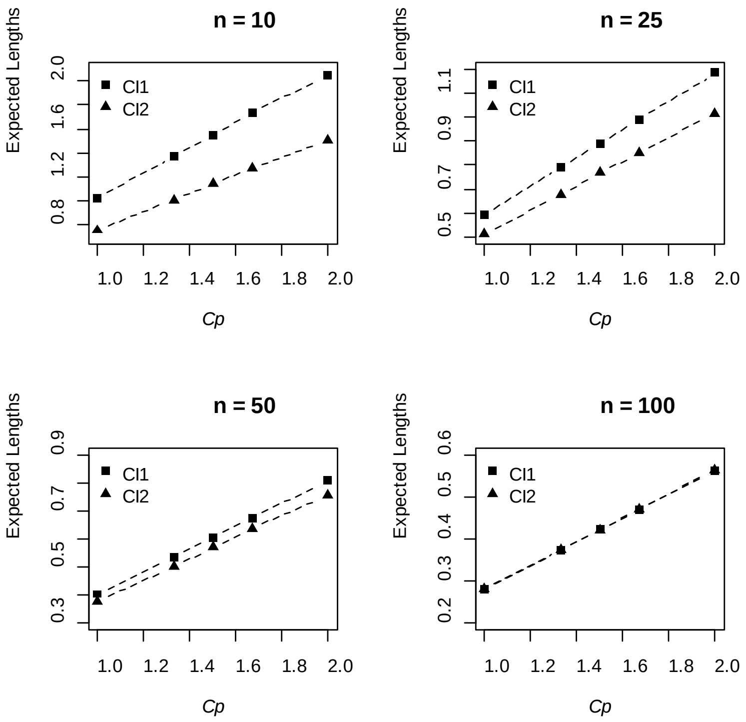

Figure 6: The expected lengths of CI_1 and CI_2 for C_p in case of $Gamma(4,2)+48$

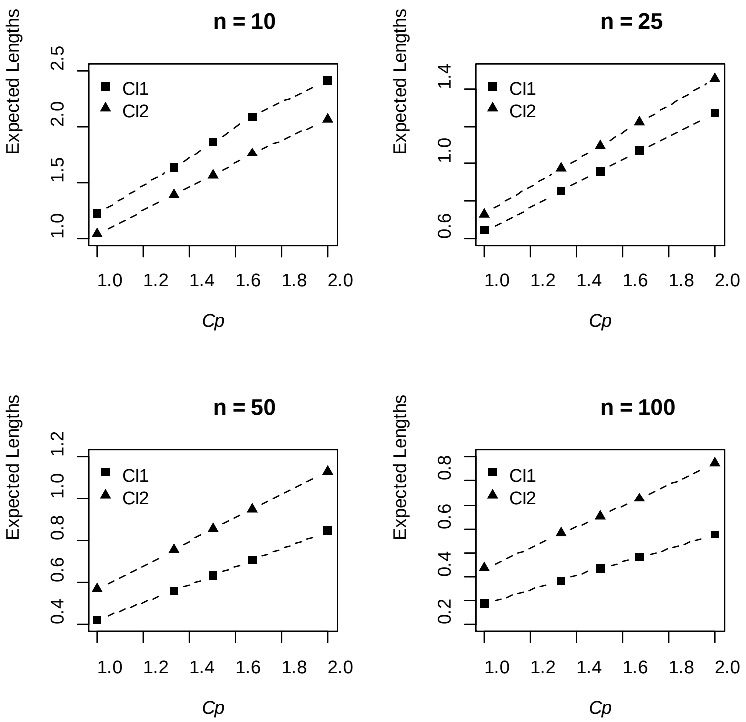

Figure 7: The expected lengths of CI_1 and CI_2 for C_p in case of $Gamma(0.75, 0.867)+49.1340$

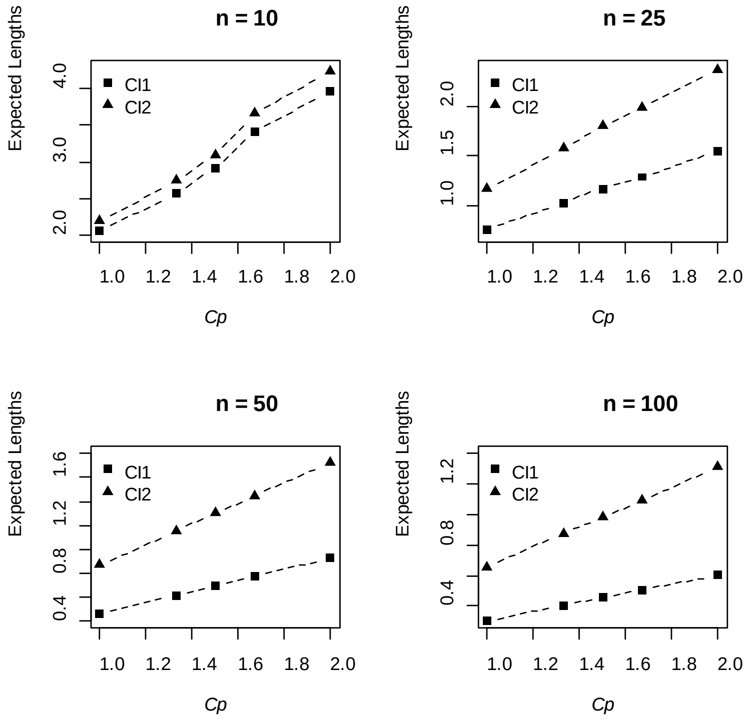

Figure 8: The expected lengths of CI_1 and CI_2 for C_p in case of $Gamma(0.25, 0.5) + 49.5$

5 Conclusions

The existing confidence interval for the capability index C_p based on the confidence interval for the standard deviation was based on a normal distribution. However, the underlying distribution may be non-normal or skewed in some circumstances. A confidence interval for the capability index C_p based on the bootstrap-*t* confidence interval for the standard deviation was developed. The proposed confidence intervals were compared with the existing confidence interval through a Monte Carlo simulation study. The proposed confidence interval proved to be better than the existing confidence interval in terms of the coverage probability when the data have a coefficient of skewness > 2. On the other hand, when the data are symmetrical or have a coefficient of skewness ≤ 2, the estimated coverage probability of the existing confidence interval can be close to the nominal level.

Appendix: Source R code for all confidence intervals

```
CI1 <- function (x,LSL,USL,alpha)
{
        n <- length(x)
        S <- sd(x)
        chisq1 <- qchisq(alpha/2,df=n-1)
        chisq2 <- qchisq(1-alpha/2,df=n-1)
        K <- (USL-LSL)/(6*S)
        ci.low <- K*sqrt(chisq1/(n-1))
        ci.up  <- K*sqrt(chisq2/(n-1))
        out <- cbind(ci.low,ci.up)
        return(out)
}

CI2 <- function (x,LSL,USL,alpha)
{
        n  <- length(x)
        s2 <- var(x)
        percentile.T.S <- percentile.T.star(x,alpha)
        T1 <- percentile.T.S[1]
        T2 <- percentile.T.S[2]
        K1 <- (USL-LSL)/6
        K2 <- s2*sqrt(2*(n-1))
        ci.low <- K1*(K2/(2*T1+sqrt(2*(n-1))))^(-1/2)
        ci.up  <- K1*(K2/(2*T2+sqrt(2*(n-1))))^(-1/2)
        out <- cbind(ci.low,ci.up)
        return(out)
}

percentile.T.star <- function (x,alpha)
{
        B  <- 1000
        n  <- length(x)
        S2 <- var(x)
        T.star <- numeric(B)
                for (i in 1:B){
                        xs <- sample(x,n,replace=TRUE)
                        s2.star <- var(xs)
                        T.star[i] <- sqrt((n-1)/2)*((s2.star/S2)-1)
                }
        T1 <- quantile(T.star,probs=alpha)
        T2 <- quantile(T.star,probs=1-alpha)
        out <- cbind(T1,T2)
        return(out)
}
```

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