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9 Understanding the Second Quantization of Fermions in Clifford and in Grassmann Space, New Way of Second Quantization of Fermions — Part II

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Abstract. We present in Part II the description of the internal degrees of freedom of fermions by the superposition of odd products of the Clifford algebra elements, either γ^a's or $\tilde{\gamma}^a$'s [1–3], which determine with their oddness the anticommuting properties of the creation and annihilation operators of the second quantized fermion fields in even d-dimensional space-time, as we do in Part I of this paper by the Grassmann algebra of θ^a's and $\frac{\partial}{\partial\theta_a}$'s. We discuss: **i.** The properties of the two kinds of the odd Clifford algebras, forming two independent spaces, both expressible with the Grassmann coordinates θ^a's and their derivatives $\frac{\partial}{\partial\theta_a}$'s [2,7,8]. **ii.** The freezing out procedure of one of the two kinds of the odd Clifford objects, enabling that the remaining Clifford objects determine with their oddness in the tensor products of the finite number of the Clifford basis vectors and the infinite number of momentum basis, the creation and annihilation operators carrying the family quantum numbers and fulfilling the anticommutation relations of the second quantized fermions: on the vacuum state, and on the whole Hilbert space defined by the sum of infinite number of "Slater determinants" of empty and occupied single fermion states. **iii.** The relation between the second quantized fermions as postulated by Dirac [19–21] and the ones following from our Clifford algebra creation and annihilation operators, what offers the explanation for the Dirac postulates.

Povzetek. V drugem delu prispevka predstavita avtorja opis notranjega prostora fermionov v sodo razsežnih prostorih s superpozicijo lihih produktov elementov Cliffordove algebre, bodisi γ^a ali $\tilde{\gamma}^a$ [1–3]. Lihi značaj teh produktov določa antikomutacijske lastnosti kreacijskih in anihilacijskih operatorjev fermionskih stanj v drugi kvantizaciji brez Diracovih postulatov. (V prvem delu prispevka sta predstavila fermionske prostotne stopnje z Grassmannovimi koordinatami θ^a in $\frac{\partial}{\partial\theta_a}$). Obravnavata: **i.** Lastnosti dveh vrst lihih Cliffordovih objektov, ki tvorita neodvisna prostora. Obe Cliffordovi algebri sta izrazljivi z Grassmannovimi koordinatami θ^a in njihovimi odvodi $\frac{\partial}{\partial\theta_a}$ [2, 7, 8]. **ii.** Pokaěta, da četudi imajo vektorski produkti končnega števila lihih Cliffordovih produktov in (zvezno) neskončnega števila bazičnih vektorjev v običajnem prostoru antikomutacijski značaj kot ga Dirac predpiše za fermione v drugi kvantizaciji v vsaki od Cliffordih algeber posebej, pa ustreže predlagani opis fermionov opazljivim lastnostim fermionov šele, ko s postulatom zagotovita, da samo ena od algeber opiše notranji prostor fermionov, operatorji preostale algebre pa določajo kvantna števila družin — na vakuumskem stanju in na celem Hilbertovem prostoru, ki ga določa vsota neskočnega števila "Slaterjevih determinant" praznih in zasedenih enofermionskih stanj, ki imajo vsa lihi značaj. **iii.** Relacijo med Diracovimi postulati za fermione v drugi kvantizaciji [19–21] in obravnavano potjo do drude kvantizacije, ki pojasni Diracove privzetke.

Keywords: Second quantization of fermion fields in Clifford and in Grassmann space, Spinor representations in Clifford and in Grassmann space, Explanation of the Dirac postulates, Kaluza-Klein-like theories, Higher dimensional spaces, Beyond the standard model

9.1 Introduction

In a long series of works we, mainly one of us N.S.M.B. ([1–3, 10–15] and the references therein), have found phenomenological success with the model named by N.S.M.B the *spin-charge-family* theory, with fermions, the internal space of which is describable as superposition of odd products of the Clifford algebra elements γ^a's in $d = (13+1)$ (may be with d infinity), interacting with only gravity. The spins of fermions from higher dimensions, $d > (3+1)$, manifest in $d = (3+1)$ as charges of the *standard model*, the gravity originating in higher dimensions manifest as the *standard model* vector gauge fields and the scalar Higgs explaining the Yukawa couplings.

There are two kinds of anticommuting algebras, the Grassmann algebra and the Clifford algebra, the later with two independent subalgebras. The Grassmann algebra, with elements θ^a, and their Hermitian conjugated partners $\frac{\partial}{\partial \theta^a}$ [3], describes fermions with the integer spins and charges in the adjoint representations, the two Clifford algebras, we call their elements γ^a and $\tilde{\gamma}^a$, can each of them be used to describe half integer spins and charges in the fundamental representations. The Grassmann algebra is expressible with the two Clifford algebras and opposite.

The two papers explain how do the oddness of the internal space of fermions manifests in the single particle wave functions, relating the oddness of the wave functions to the corresponding creation and annihilation operators of the to the second quantized fermions, in the Grassmann case and in the Clifford case, explaining therefore the postulates of Dirac for the second quantized fermions.

We learn in Part I of this paper, that in d-dimensional space 2^{d-1} superposition of odd products of d θ^a's exist, chosen to be the eigenvectors of the Cartan subalgebra, Eq. (4) of Part I, and arranged in tensor products with the momentum space to be solutions of the equation of motion for free massless "fermions", Eq. (21) of Part I.

The creation operators, defined as the tensor products of the superposition of the finite number of "basis vectors" in Grassmann space, guaranteeing the oddness of operators, and of the infinite basis in momentum space, form — applied on the vacuum state — the second quantized states of integer spin "Grassmann fermions". The creation operators fulfill together with their Hermitian conjugated partners annihilation operators (based on the internal space of odd products of $\frac{\partial}{\partial \theta_a}$'s) all the requirements of the anticommutation relations postulated by Dirac for fermions: **i.** on the simple vacuum state $|1>$ (Eqs. (7,11) of Part I), **ii.** on the Hilbert space $\mathcal{H}$ $(= \prod_{\vec{p}}^{\infty} \otimes_N \mathcal{H}_{\vec{p}}$, with the number of empty and occupied single fermion states for particular $\vec{p}$ equal to $2^{2^{d-1}}$) of infinite many "Slater determinants" of all possible empty and occupied single fermion states (with the infinite number of possibilities of moments for each of 2^{d-1} internal degrees of freedom), Eqs. (25, 34) of Part I.

While the creation and annihilation operators, which are superposition of odd products of θ^a's and $\frac{\partial}{\partial\theta_a}$'s, respectively, anticommute on the vacuum state $|\phi_o>=|1>$, Eq. (7,11), the superposition of even products of θ^a's and $\frac{\partial}{\partial\theta_a}$'s, respectively, commute, Eq. (16) of Part I.

The superposition of odd products of γ^a's and their Hermitian conjugated partners, as well as of odd products of $\tilde{\gamma}^a$'s and their Hermitian conjugated partners, on the corresponding vacuum states, Eq. (9.18), anticommute. Since the tensor products of the "basis vectors" determining the internal space of Clifford fermions and of the basis in momentum space manifest oddness of the internal space, no postulates of anticommutation relations as in the Dirac second quantization proposal is needed also for Clifford fermions with the internal space described by one of the two Clifford objects (in Subsect. 9.2.2 we make a choice of γ^a's). The oddness of the " basis vectors", defining the internal space of fermions, transfers to the creation and annihilation operators forming the second quantized single fermion states in the Clifford and the Grassmann space.

The "Grassmann fermions" have integer spins, and spins in the part with $d \geq 5$ manifesting as charges in $d = (3+1)$, in adjoint representations, Table I in Part I. There is no operator which would connect different irreducible representations of the corresponding Lorentz group. There are no elementary fermions with integer spin observed so far either.

The Clifford fermions, describing the internal space with γ^a's, have half integer spins and spins in the part with $d \geq 5$ manifesting as charges in $d = (3+1)$ in fundamental representations [10–12,15,17,18]. The operators $\tilde{S}^{ab}$ $(= \frac{i}{4}\{\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\}_-)$ connect, after the reduction of the Clifford algebra degrees of freedom by a factor of 2, Subsect. 9.2.2, different irreducible representations of the Lorentz group S^{ab} $(= \frac{i}{4}\{\gamma^a\gamma^b - \gamma^b\gamma^a\}_-)$ and determine "family" quantum numbers. All in agreement with the observed families of quarks and leptons.

In Part II the properties of the two kinds of the Clifford algebras objects, γ^a's and $\tilde{\gamma}^a$'s, are discussed. Both are expressible with θ^a's and $\frac{\partial}{\partial\theta_a}$'s ($\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})$ [2,7,8]), and both are, up to a constant $\eta^{aa} = (1,-1,-1,\ldots,-1)$, Hermitian operators. Each of these two kinds of the Clifford algebra objects of an odd Clifford character (superposition of odd number of products of either γ^a's or $\tilde{\gamma}^a$'s, respectively) has 2^{d-1} members, together again $2\cdot 2^{d-1}$ members, the same as in the case of "Grassmann fermions".

These two internal spaces, described by the two Clifford algebras, are independent, each of them with their own generators of the Lorentz transformations, Eq. (9.3), and their corresponding Cartan subalgebras, Eq. (9.4).

In each of these two internal spaces there exist $2^{\frac{d}{2}-1}$ "basis vectors" in $2^{\frac{d}{2}-1}$ irreducible representations, chosen to be ''eigenvectors" of the corresponding Cartan subalgebra elements, Eq. (9.5), and having the properties of creation and annihilation operators (the Hermitian conjugated partners of the creation operators) on the vacuum state: **i.** The application of any creation operator on the vacuum state, Eq. (9.18), gives nonzero contribution, while the application of any annihilation operator on the vacuum state gives zero contribution. **ii.** Within each of these two spaces all the annihilation operators anticommute among themselves and all the creation operators anticommute among themselves. **iii.** The vacuum state is a superposition of products of the annihilation operators with their Hermitian conjugated partners creation operators, like in the Grassmann case. The Clifford vacuum states, Eq. (9.18), are not the identity like in the Grassmann case, Eq. (19) in Part I.

However, there is not only the anticommutator of the creation operator and its Hermitian conjugated partner, which gives the nonzero contribution on the vacuum state in each of the two spaces — what in the Grassmann algebra is the case, and what the postulates of Dirac require. There are, namely, the additional $(2^{\frac{d}{2}-1}-1)$ members of the same irreducible representation, to which the Hermitian conjugated partner of the creation operator belongs, giving the nonzero anticommutator with this creation operator on the vacuum state (Eq. (9.11) in Subsect. 9.2.1 illustrates such a case).

And, there is no operators, which would connect different irreducible representations in each of the two Clifford algebras and correspondingly there is no "family" quantum number for each irreducible representation, needed to describe the observed quarks and leptons. (Let the reader be reminded that also the Grassmann algebra has no operators, which would connect different irreducible representations. The Dirac's second quantization postulates do not take care of charges and families of fermions, both can be treated and incorporated into the second quantization postulates as quantum numbers of additional groups as proposed by the *standard model*.) We solve these problems with the requirement, presented in Eq. (9.12): $\tilde{\gamma}^a B = (-)^B\, i\, B\gamma^a$, with $(-)^B = -1$, if B is (a function of) an odd product of γ^a's, otherwise $(-)^B = 1$ [8].

We present in the subsection 9.1.1 of this section a short overview of steps, which lead to the second quantized fermions in the Clifford space, offering the explanation for the Dirac's postulates. In the subsection 9.1.2 we discuss our assumption, that the oddness of the "basis vectors" in the internal space transfer to the corresponding creation and annihilation operators determining the second quantized single fermion states and correspondingly the Hilbert space of the second quantized fermions, in a generalized way.

We present in Sect. 9.2 the properties of the Clifford algebra "basis vectors" in the space of d γ^a's and in the space of d $\tilde{\gamma}^a$'s. In Subsect. 9.2.1 we discuss properties of the "basis vectors" of half integer spin. In Subsect. 9.2.2 we discuss conditions, under which operators of one of these two kinds of the Clifford algebra objects demonstrate by themselves the anticommutation relations required for the second quantized "fermions", manifesting the half integer spins, offering the explanation for the spin and charges of the observed quarks and leptons and anti-quarks and anti-leptons and also for their families [1–3, 10–15, 17].

In Subsect. 9.2.3 we generate the basis states manifesting the family quantum numbers.

In Subsect. 9.2.4 the superposition of "basis vectors", solving the Weyl equation, are constructed, forming creation operators depending on the momenta and fulfilling with their Hermitian conjugated partners the anticommutation relations for the second quantized fermions.

We illustrate in Sect. 9.2.5 properties of the Clifford odd "basis vectors" in $d = (5+1)$-dimensional space, and extending the internal space in a tensor product to momentum space, we present also the superposition solving the Weyl equation, and correspondingly present creation and annihilation operators depending on the momentum $\vec{p}$.

We present in Sect. 9.3 the Hilbert space $\mathcal{H}_{\vec{p}}$ of particular momentum $\vec{p}$ as "Slater determinants": i. with no "fermions" occupying any of the 2^{d-2} fermion states, ii. with one "fermion" occupying one of the 2^{d-2} fermion states, iii. with two "fermions" occupying the 2^{d-2} fermion states,..., up to the "Slater determinant" with all possible fermion states of a particular $\vec{p}$ occupied by "fermions". The total Hilbert space $\mathcal{H}$ is then the tensor product $\prod_{\infty} \otimes_N$ of infinite number of $\mathcal{H}_{\vec{p}}$. On $\mathcal{H}$ the tensor products of creation and annihilation operators (solving the equations of motion for free massless fermions) manifest the anticommutation relations of second quantized "fermions" without any postulates. We also illustrate the application of the tensor products of creation and annihillation operators on $\mathcal{H}$ in a simple toy model.

In Subsect. 9.3.4 the correspondence between our way and the Dirac way of second quantized fermions is presented, demonstrating that our way does explain the Dirac's postulates.

In Sect. 9.4 we note that the present work is the part of the project named the *spin-charge-family* theory of one of the two authors of this paper (N.S.M.B.).

In Sect. 9.5 we comment on what we have learned from the second quantized integer spins "fermions", with the internal degrees of freedom described with Grassmann algebra, manifesting (from the point of view of $d = (3+1)$) charges in the adjoint representations and compare these recognitions with the recognitions, which the Clifford algebra is offering for the description of fermions, appearing in families of the irreducible representations of the Lorentz group in the internal — Clifford — space, with half integer spins and charges and family quantum numbers in the fundamental representations [1–3, 10–15].

9.1.1 Steps leading to second quantized Clifford fermions

We claim that when the internal part of the single particle wave functions anticommute under the Clifford algebra product $*_A$, then the wave functions with such internal part, extended with a tensor product to momentum space, anticommute as well, and so do anticommute the creation and annihillation operators, creating and annihilating the extended fermion states, assuming that the oddness of the algebra of the wave function extends to the creation and annihilation operators as presented in Subsect. 9.1.2.

If the internal part commute with respect to $*_A$ then the corresponding wave functions and the creation operators commute as well.

Let us present steps which lead to the second quantized Clifford fermions, when using the odd Clifford algebra objects to define their internal space:

i. The superposition of an odd number of the Clifford algebra elements, either of γ^a's or of $\tilde{\gamma}^a$, each with $2 \cdot (2^{\frac{d}{2}-1})^2$ degrees of freedom, is used to describe the internal space of fermions in even dimensional spaces.

ii. The "basis vectors" — the superposition of an odd number of Clifford algebra elements — are chosen to be the "eigenvectors" of the Cartan subalgebras, Eq. (9.4), of the corresponding Lorentz algebras, Eq. (9.3), in each of the two algebras.

iii. There are two groups of $2^{\frac{d}{2}-1}$ members of $2^{\frac{d}{2}-1}$ irreducible representations of the corresponding Lorentz group, for either γ^a's or for $\tilde{\gamma}^a$ algebras, each member of one group has its Hermitian conjugated partner in another group.

Making a choice of one group of "basis vectors" (for either γ^a's or for $\tilde{\gamma}^a$) to be creation operators, the other group of "basis vectors" represents the annihilation operators. The creation operators anticommute among themselves and so do anticommute annihilation operators.

iv. The vacuum state is then (for either γ^a's or for $\tilde{\gamma}^a$'s algebras) the superposition of products of annihilation $\times$ their Hermitian conjugated partners the creation operators.

The application of the creation operators on the vacuum state forms the "basis states" in each of the two spaces. The application of the annihilation operators on the vacuum state gives zero, Subsect. 9.1.2.

v. The requirement that application of $\tilde{\gamma}^a$ on γ^a gives $-i\eta^{aa}$, and the application of $\tilde{\gamma}^a$ on identity gives $i\eta^{aa}$ and that only γ^a's are used to determine the internal space of half integer fermions, Eq. (9.2.2), reduces the dimension of the Clifford algebra for a factor of two, enabling that the Cartan subalgebra of $\tilde{S}^{ab}$'s determines the "family" quantum numbers of each irreducible representation of S^{ab}'s, Eq. (9.3), and correspondingly also of their Hermitian conjugated partners.

vi. The tensor products of superposition of the finite number of members of the "basis vectors" and the infinite dimensional momentum basis, chosen to solve the Weyl equations for free massless half integer spin fermions, determine the creation and (their Hermitian conjugated partners) annihilation operators, which depend on the momenta $\vec{p}$, while $|p^0| = |\vec{p}|$ ($p^a = (p^0, p^1, p^2, p^3, p^5, \ldots, p^d)$), manifesting the properties of the observed fermions. These creation and annihilation operators fulfill on the Hilbert space all the requirements for the second quantized fermions, postulated by Dirac, Eq. (9.28) [19–21].

vii. The second quantized Hilbert space $\mathcal{H}_{\vec{p}}$ of a particular $\vec{p}$ is a tensor product of creation operators of a particular $\vec{p}$, defining "Slater determinants" with no single particle state occupied (with no creation operators applying on the vacuum state), with one single particle state occupied (with one creation operator applying on the vacuum state), with two single particle states occupied, and so on, defining in d-dimensional space $2^{(2^{\frac{d}{2}-1})^2}$ dimensional space for each $\vec{p}$.

viii. Total Hilbert space is the infinite product ($\otimes_N$) of $\mathcal{H}_{\vec{p}}$: $\mathcal{H} = \prod_{\vec{p}}^{\infty} \otimes_N \mathcal{H}_{\vec{p}}$. The notation $\otimes_N$ is to point out that odd algebraic products of the Clifford γ^a's operators anticommute no matter for which $\vec{p}$ they define the orthonormalized superposition of "basis vectors", solving the equations of motion as the orthonormalized plane wave solutions with $p^0 = |\vec{p}|$ and that the anticommutation character keeps also in the tensor product of internal basis and momentum basis.

Since the momentum space belonging to different $\vec{p}$ satisfy the "orthogonality" relations, the creation and annihilation operators determined by $\vec{p}$ anticommute with the creation and annihilation operators determined by any other $\vec{p}\,'$. This means that in what ever way the Hilbert space $\mathcal{H}$ is arranged, the sign is changed whenever a creation or an annihilation operator, applying on the Hilbert space $\mathcal{H}$, jumps over odd number of occupied states. No postulates for the second quantized fermions are needed in our odd Clifford space with creation and annihilation operators carrying the family quantum numbers.

x. Correspondingly the creation and annihilation operators with the internal space described by either odd Clifford or odd Grassmann algebra, since fulfilling the anticommutation relations required for the second quantized fermions without postulates, explain the Dirac's postulates for the second quantized fermions.

9.1.2 Our main assumption and definitions

(This subsection is the same as the one of Part I.)

In this subsection we clarify how does the main assumption of Part I and Part II: *the decision to describe the internal space of fermions with the "basis vectors" expressed with the superposition of odd products of the anticommuting members of the algebra*, either the Clifford one or the Grassmann one, acting algebraically, $*_A$, on the internal vacuum state $|\psi_o>$, relate to the creation and annihilation anticommuting operators of the second quantized fermion fields.

To appreciate the need for this kind of assumption, let us first have in mind that algebra with the product $*_A$ is only present in our work, usually not in other works, and thus has no well known physical meaning. It is at first a product by which you can multiply two internal wave functions B_i and B_j with each other,

$$C_k = B_i *_A B_j\,,$$
$$B_i *_A B_j = \mp B_j *_A B_i\,,$$

the sign $\mp$ depends on whether B_i and B_j are products of odd or even number of algebra elements: The sign is $-$ if both are (superposition of) odd products of algebra elements, in all other cases the sign is $+$.

Let $\mathbf{R}^{d-1}$ define the external spatial or momentum space. Then the tensor product $*_T$ extends the internal wave functions into the wave functions $\mathbf{C}_{\vec{p},\,i}$ defined in both spaces

$$\mathbf{C}_{\vec{p},\,i} = |\vec{p}> *_T |B_i>\,,$$

where again B_i represent the superposition of products of elements of the anticommuting algebras, in our case either θ^a or γ^a or $\tilde{\gamma}^a$, used in this paper.

We can make a choice of the orthogonal and normalized basis so that $<\mathbf{C}_{\vec{p},i}|\mathbf{C}_{\vec{p'},j}>= \delta(\vec{p}\vec{p'})\,\delta_{ij}$. Let us point out that either B_i or $\mathbf{C}_{\vec{p},\,i}$ apply algebraically on the vacuum state, $B_i *_A |\psi_o>$ and $\mathbf{C}_{\vec{p},\,i} *_A |\psi_o>$.

Usually a product of single particle wave functions is not taken to have any physical meaning in as far as most physicists simply do not work with such products at all.

To give to the algebraic product, $*_A$, and to the tensor product, $*_T$, defined on the space of single particle wave functions, the physical meaning, we postulate the connection between the anticommuting/commuting properties of the "basis vectors", expressed with the odd/even products of the anticommuting algebra elements and the corresponding creation operators, creating second quantized single fermion/boson states

$$\begin{aligned}
&\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},i}} *_A |\psi_o> = |\psi_{\vec{p},i}>\,,\\
&\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},i}} *_T |\psi_{\vec{p'},j}> = 0\,,\\
&\qquad \text{if}\, \vec{p} = \vec{p'}\, \text{and}\, i = j\,,\\
&\text{in all other cases} \quad \text{it follows}\\
&\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},i}} *_T \hat{b}^{\dagger}_{\mathbf{C}_{\vec{p'},j}} *_A |\psi_o> = \mp \hat{b}^{\dagger}_{\mathbf{C}_{\vec{p'},j}} *_T \hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},i}} *_A |\psi_o>\,,
\end{aligned}$$

with the sign $\pm$ depending on whether $\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},i}}$ have both an odd character, the sign is $-$, or not, then the sign is $+$.

To each creation operator $\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},i}}$ its Hermitian conjugated partner represents the annihilation operator $\hat{b}_{\mathbf{C}_{\vec{p},i}}$

$$\begin{aligned}
&\hat{b}_{\mathbf{C}_{\vec{p},i}} = (\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},i}})^{\dagger}\,,\\
&\text{with the} \quad \text{property}\\
&\hat{b}_{\mathbf{C}_{\vec{p},i}} *_A |\psi_o> = 0\,,\\
&\text{defining the} \quad \text{vacuum state as}\\
&|\psi_o>: = \sum_i (B_i)^{\dagger} *_A B_i \,|\, I>
\end{aligned}$$

where summation i runs over all different products of annihilation operator $\times$ its Hermitian conjugated creation operator, no matter for what $\vec{p}$, and $|\, I>$ represents the identity, $(B_i)^{\dagger}$ represents the Hermitian conjugated wave function to B_i.

Let the tensor multiplication $*_T$ denotes also the multiplication of any number of single particle states, and correspondingly of any number of creation operators.

What further means that to each single particle wave function we define the creation operator $\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},i}}$, applying in a tensor product from the left hand side on the second quantized Hilbert space — consisting of all possible products of any number of the single particle wave functions — adding to the Hilbert space the single particle wave function created by this particular creation operator. In the case of the second quantized fermions, if this particular wave function with the quantum numbers and $\vec{p}$ of $\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},i}}$ is already among the single fermion wave functions of a particular product of fermion wave functions, the action of the creation operator gives zero, otherwise the number of the fermion wave functions increases for one. In the boson case the number of boson second quantized wave functions increases always for one.

If we apply with the annihilation operator $\hat{b}_{\mathbf{C}_{\vec{p},i}}$ on the second quantized Hilbert space, then the application gives a nonzero contribution only if the particular products of the single particle wave functions do include the wave function with the quantum number i and $\vec{p}$.

In a Slater determinant formalism the single particle wave functions define the empty or occupied places of any of infinite numbers of Slater determinants. The creation operator $\hat{b}^{\dagger}_{c_{\vec{p},i}}$ applies on a particular Slater determinant from the left hand side. Jumping over occupied states to the place with its i and $\vec{p}$. If this state is occupied, the application gives in the fermion case zero, in the boson case the number of particles increase for one. The particular Slater determinant changes sign in the fermion case if $\hat{b}^{\dagger}_{c_{\vec{p},i}}$ jumps over odd numbers of occupied states. In the boson case the sign of the Slater determinant does not change.

When annihilation operator $\hat{b}_{c_{\vec{p},i}}$ applies on particular Slater determinant, it is jumping over occupied states to its own place, giving zero, if this space is empty and decreasing the number of occupied states, if this space is occupied. The Slater determinant changes sign in the fermion case, if the number of occupied states before its own space is odd. In the boson case the sign does not change.

Let us stress that choosing antisymmetry or symmetry is a choice which we make when treating fermions or bosons, respectively, namely the choice of using oddness or evenness of basis vectors, that is the choice of using odd products or even products of algebra anticummuting elements.

To describe the second quantized fermion states we make a choice of the basis vectors, which are the superposition of the odd numbers of algebra elements, of both Clifford and Grassmann algebras.

The creation operators and their Hermitian conjugation partners annihilation operators therefore in the fermion case anticommute. The single fermion states, which are the application of the creation operators on the vacuum state $|\psi_o>$, manifest correspondingly as well the oddness. The vacuum state, defined as the sum over all different products of annihilation $\times$ the corresponding creation operators, have an even character.

Let us end up with the recognition:

One usually means antisymmetry when talking about Slater-determinants because otherwise one would not get determinants.

In the present paper [1,2,7,10] the choice of the symmetrizing versus antisymmetrizing relates indeed the commutation versus anticommutation with respect to the a priori completely different product $*_A$, of anticommuting members of the Clifford or Grassmann algebra. The oddness or evenness of these products transfer to quantities to which these algebras extend.

9.2 Properties of Clifford algebra in even dimensional spaces

We can learn in Part I that in d-dimensional space of anticommuting Grassmann coordinates (and of their Hermitian conjugated partners — derivatives), Eqs. (2,6) of Part I, there exist two kinds of the Clifford coordinates (operators) — γ^a and $\tilde{\gamma}^a$ — both are expressible in terms of θ^a and their conjugate momenta $p^{\theta a} = i\,\frac{\partial}{\partial\theta_a}$ [2].

$$\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,,$$
$$\theta^a = \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,, \tag{9.1}$$

offering together $2 \cdot 2^d$ operators: 2^d of those which are products of γ^a and 2^d of those which are products of $\tilde{\gamma}^a$.

Taking into account Eqs. (1,2) of Part I ($\{\theta^a, \theta^b\}_+ = 0$, $\{\frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b}\}_+ = 0$, $\{\theta_a, \frac{\partial}{\partial \theta_b}\}_+ = \delta_{ab}$, $\theta^{a\dagger} = \eta^{aa} \frac{\partial}{\partial \theta_a}$ and $(\frac{\partial}{\partial \theta_a})^\dagger = \eta^{aa}\theta^a$) one finds

$$\begin{aligned}
\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \,, \\
\{\gamma^a, \tilde{\gamma}^b\}_+ &= 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,, \\
(\gamma^a)^\dagger &= \eta^{aa}\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a\,,
\end{aligned} \tag{9.2}$$

with $\eta^{ab} = diag\{1,-1,-1,\cdots,-1\}$.

It follows for the generators of the Lorentz algebra of each of the two kinds of the Clifford algebra operators, S^{ab} and $\tilde{S}^{ab}$, that:

$$\begin{aligned}
S^{ab} &= \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)\,, \quad \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,, \\
\mathbf{S}^{ab} &= S^{ab} + \tilde{S}^{ab}\,, \quad \{S^{ab}, \tilde{S}^{ab}\}_- = 0\,, \\
\{S^{ab}, \gamma^c\}_- &= i(\eta^{bc}\gamma^a - \eta^{ac}\gamma^b)\,, \\
\{\tilde{S}^{ab}, \tilde{\gamma}^c\}_- &= i(\eta^{bc}\tilde{\gamma}^a - \eta^{ac}\tilde{\gamma}^b)\,, \\
\{S^{ab}, \tilde{\gamma}^c\}_- &= 0\,, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0\,,
\end{aligned} \tag{9.3}$$

where $\mathbf{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial \theta_b} - \theta^b \frac{\partial}{\partial \theta_a})$, Eq. (3) of Part I.

Let us make a choice of the Cartan subalgebra of the commuting operators of the Lorentz algebra for each of the two kinds of the operators of the Clifford algebra, S^{ab} and $\tilde{S}^{ab}$, equivalent to the choice of Cartan subalgebra of $\mathbf{S}^{ab}$ in the Grassmann case, Eq. (4) in Part I,

$$\begin{aligned}
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\ d}\,, \\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\ d}\,.
\end{aligned} \tag{9.4}$$

Representations of γ^a and representations of $\tilde{\gamma}^a$ are independent, each with twice $2^{\frac{d}{2}-1}$ members in $2^{\frac{d}{2}-1}$ irreducible representations of an odd Clifford character and with twice $2^{\frac{d}{2}-1}$ members in $2^{\frac{d}{2}-1}$irreducible representations of an even Clifford character in even dimensional spaces.

We make a choice for the members of the irreducible representations of the two Lorentz groups to be the "eigenvectors" of the corresponding Cartan subalgebra of Eq. (9.4), taking into account Eq. (9.2),

$$\begin{aligned}
S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \\
S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \\
\tilde{S}^{ab}\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, \\
\tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,.
\end{aligned} \tag{9.5}$$

The Clifford "vectors" — nilpotents and projectors — of both algebras are normalized, up to a phase, with respect to Eq. (9.45) of 9.6. Both have half integer spins.

The "eigenvalues" of the operator S^{03}, for example, for the "vector" $\frac{1}{2}(\gamma^0 \mp \gamma^3)$ are equal to $\pm\frac{i}{2}$, respectively, for the "vector" $\frac{1}{2}(1 \pm \gamma^0\gamma^3)$ are $\pm\frac{i}{2}$, respectively, while all the rest "vectors" have "eigenvalues" $\pm\frac{1}{2}$. One finds equivalently for the "eigenvectors" of the operator $\tilde{S}^{03}$: for $\frac{1}{2}(\tilde{\gamma}^0 \mp \tilde{\gamma}^3)$ the "eigenvalues" $\pm\frac{i}{2}$, respectively, and for the "eigenvectors" $\frac{1}{2}(1 \pm \tilde{\gamma}^0\tilde{\gamma}^3)$ the "eigenvalues" $k = \pm\frac{i}{2}$, respectively, while all the rest "vectors" have $k = \pm\frac{1}{2}$.

To make discussions easier let us introduce the notation for the "eigenvectors" of the two Cartan subalgebras, Eq. (9.4), Ref. [2,7].

$$
\begin{aligned}
&\overset{ab}{(k)}:=\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b)\,, &&\overset{ab}{(k)}{}^\dagger=\eta^{aa}\overset{ab}{(-k)}\,, &&(\overset{ab}{(k)})^2=0\,,\\
&\overset{ab}{[k]}:=\frac{1}{2}(1+\frac{i}{k}\gamma^a\gamma^b)\,, &&\overset{ab}{[k]}{}^\dagger=\overset{ab}{[k]}\,, &&(\overset{ab}{[k]})^2=\overset{ab}{[k]}\,,\\
&\overset{ab}{(\tilde{k})}:=\frac{1}{2}(\tilde{\gamma}^a+\frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, &&\overset{ab}{(\tilde{k})}{}^\dagger=\eta^{aa}\overset{ab}{(\tilde{-k})}\,, &&(\overset{ab}{(\tilde{k})})^2=0\,,\\
&\overset{ab}{[\tilde{k}]}:=\frac{1}{2}(1+\frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,, &&\overset{ab}{[\tilde{k}]}{}^\dagger=\overset{ab}{[\tilde{k}]}\,, &&(\overset{ab}{[\tilde{k}]})^2=\overset{ab}{[\tilde{k}]}\,,
\end{aligned}
\tag{9.6}
$$

with $k^2 = \eta^{aa}\eta^{bb}$. Let us notice that the "eigenvectors" of the Cartan subalgebras are either projectors

$$
(\overset{ab}{[k]})^2=\overset{ab}{[k]}\,,\qquad (\overset{ab}{[\tilde{k}]})^2=\overset{ab}{[\tilde{k}]}\,,
$$

or nilpotents

$$
(\overset{ab}{(k)})^2=0\,,\qquad (\overset{ab}{(\tilde{k})})^2=0\,.
$$

We pay attention on even dimensional spaces, $d = 2(2n+1)$ or $d = 4n$, $n \geq 0$.

The "basis vectors", which are products of $\frac{d}{2}$ either of nilpotents or of projectors or of both, are "eigenstates" of all the members of the Cartan subalgebra, Eq. (9.4), of the corresponding Lorentz algebra, forming $2^{\frac{d}{2}-1}$ irreducible representations with $2^{\frac{d}{2}-1}$ members in each of the two Clifford algebras cases.

The "basis vectors" of Eq. (9.7) are "eigenvectors" of all the Cartan subalgebra members, Eq. (9.4), in $d = 2(2n+1)$-dimensional space of γ^a's. The first one is the product of nilpotents only and correspondingly a superposition of an odd products of γ^a's. The second one belongs to the same irreducible representation as the first one, if it follows from the first one by the application of S^{01}, for example.

$$
\begin{aligned}
&\overset{03}{(+i)}\overset{12}{(+)}\cdots\overset{d-1\,d}{(+)}\,,\qquad \overset{03}{[-i]}\overset{12}{[-i]}\overset{56}{(+)}\cdots\overset{d-1\,d}{(+)}\,,\\
&\overset{03}{[-i]}\overset{12}{[-]}\cdots\overset{d-1\,d}{[-]}\,.
\end{aligned}
\tag{9.7}
$$

One finds for their Hermitian conjugated partners, up to a sign,

$$
\begin{aligned}
&\overset{03}{(-i)}\overset{12}{(-)}\cdots\overset{d-1\,d}{(-)}\,,\qquad \overset{03}{[-i]}\overset{12}{[-i]}\overset{56}{(-)}\cdots\overset{d-1\,d}{(-)}\,,\\
&\overset{03}{[-i]}\overset{12}{[-]}\cdots\cdots\overset{d-1\,d}{[-]}\,.
\end{aligned}
$$

The "basis vectors" form an orthonormal basis within each of the irreducible representations or among irreducible representations, like the product of the following annihilation and the corresponding creation operator: $\stackrel{d-1\,d}{(-)} \cdots \stackrel{12}{(-)}\stackrel{03}{(-i)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)} \cdots \stackrel{d-1\,d}{(+)} = 1$, while all the algebraic products, which do not relate the annihilation operators with their Hermitian conjugated creation operators, give zero.

Usually the operators γ^a's are represented as matrices. We use γ^a's here to form the basis. One can find in Ref. [9] how does the application of γ^a's on the basis defined in $d = (3+1)$ look like.

9.2.1 Clifford "basis vectors" with half integer spin

In the Grassmann case the 2^{d-1} odd and 2^{d-1} even Grassmann operators, which are superposition of either odd or even products of θ^a's, are well distinguishable from their 2^{d-1} odd and 2^{d-1} even Hermitian conjugated operators, which are superposition of odd and even products of $\frac{\partial}{\partial \theta_a}$'s, Eq. (6) in Part I.

In the Clifford case the relation between "basis vectors" and their Hermitian conjugated partners (made of products of nilpotents ($\stackrel{ab}{(k)}$ or $\stackrel{ab}{(\tilde{k})}$) and projectors ($\stackrel{ab}{[k]}$ or $\stackrel{ab}{[\tilde{k}]}$), Eq. (9.6), are less transparent (although still easy to be evaluated). This can be noticed in Eq. (9.6), since $\frac{1}{\sqrt{2}}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)^\dagger$ is $\eta^{aa}\,\frac{1}{\sqrt{2}}(\gamma^a + \frac{\eta^{aa}}{i\,(-k)}\gamma^b)$, while $(\frac{1}{\sqrt{2}}(1 + \frac{i}{k}\gamma^a\gamma^b))^\dagger = \frac{1}{\sqrt{2}}(1 + \frac{i}{k}\gamma^a\gamma^b)$ is self adjoint. (This is the case also for representations in the sector of $\tilde{\gamma}^a$'s.)

One easily sees that in even dimensional spaces, either in $d = 2(2n+1)$ or in $d = 4n$, the Clifford odd "basis vectors" (they are products of an odd number of nilpotents and an even number of projectors) have their Hermitian conjugated partners in another irreducible representation, since Hermitian conjugation changes an odd number of nilpotents (changing at the same time the handedness of the "basis vectors"), while the generators of the Lorentz transformations change two nilpotents at the same time (keeping the handedness unchanged).

The Clifford even "basis vectors" have an even number of nilpotents and can have an odd or an even number of projectors. Correspondingly an irreducible representation of an even "basis vector" can be a product of projectors only and therefore is self adjoint.

Let us recognize the properties of the nilpotents and projectors. The relations are taken from Ref. [10].

$$
\begin{aligned}
\stackrel{ab}{(k)}\stackrel{ab}{(k)} &= 0\,, & \stackrel{ab}{(k)}\stackrel{ab}{(-k)} &= \eta^{aa}\stackrel{ab}{[k]}\,,\\
\stackrel{ab}{[k]}\stackrel{ab}{[k]} &= \stackrel{ab}{[k]}\,, & \stackrel{ab}{[k]}\stackrel{ab}{[-k]} &= 0\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[k]} &= 0\,, & \stackrel{ab}{[k]}\stackrel{ab}{(k)} &= \stackrel{ab}{(k)}\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[-k]} &= \stackrel{ab}{(k)}\,, & \stackrel{ab}{[k]}\stackrel{ab}{(-k)} &= 0\,.
\end{aligned}
\tag{9.8}
$$

The same relations are valid also if one replaces $\stackrel{ab}{(k)}$ with $\stackrel{ab}{(\tilde{k})}$ and $\stackrel{ab}{[k]}$ with $\stackrel{ab}{[\tilde{k}]}$, Eq. (9.6).

Taking into account Eq. (9.8) one recognizes that the product of annihilation and the creation operator from Eq. (9.7), $\stackrel{03}{(-i)}\stackrel{12}{(-)}\cdots\stackrel{d-1\,d}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}$, applied on a vacuum state — defined as a sum of products of all annihilation $\times$ their Hermitian conjugated partner creation operators from all irreducible representations, $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]}\stackrel{78}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \cdots$, Eq. (9.18), gives a nonzero contribution, but is not the only one for a chosen creation operator. There are several other choices, like

$$\stackrel{03}{[+i]}\stackrel{12}{[+]}\cdots\stackrel{d-1\,d}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,,$$
$$\stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{[+]}\cdots\stackrel{d-1\,d}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,,$$

which also give nonzero contributions.

Let us recognize:

i. The two Clifford spaces, the one spanned by γ^a's and the second one spanned by $\tilde{\gamma}^a$'s, are independent vector spaces, each with 2^d "vectors".

ii. The Clifford odd "vectors" (the superposition of products of odd numbers of γ^a's or $\tilde{\gamma}^a$'s, respectively) can be arranged for each kind of the Clifford algebras into two groups of $2^{\frac{d}{2}-1}$ members of $2^{\frac{d}{2}-1}$ irreducible representations of the corresponding Lorentz group. The two groups are Hermitian conjugated to each other.

iii. Different irreducible representations are indistinguishable with respect to the "eigenvalues" of the corresponding Cartan subalgebra members.

iv. The Clifford even part (made of superposition of products of even numbers of γ^a's and $\tilde{\gamma}^a$'s, respectively) splits as well into twice $2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}$ irreducible representations of the Lorentz group. One member of each Clifford even representation, the one which is the product of projectors only, is self adjoint. Members of one irreducible representation are with respect to the Cartan subalgebra indistinguishable from all the other irreducible representations.

v. The $2^{\frac{d}{2}-1}$ members of each of the $2^{\frac{d}{2}-1}$ irreducible representations are orthogonal to one another and so are orthogonal their corresponding Hermitian conjugated partners. For illustration of the orthogonality one can look at Table 9.1, and recognize that any "basis vector" of the first four multiplets of *odd I*, if multiplied from the left hand side or from the right hand side with any other "basis vector" from the rest three "families" of *odd I* get zero when taking into account Eq. (9.8). One can repeat this also for any "basis vectors" of all the "families" of *odd I*, as well as among all the "basis vectors" within *odd II*. Generalization to any even dimension d is straightforward.

vi. Denoting "basis vectors" by $\hat{b}^{m\dagger}_f$, (where f defines different irreducible representations and m a member in the representation f), and their Hermitian conjugate partners by $\hat{b}^m_f = (\hat{b}^{m\dagger}_f)^\dagger$, let us start for $d = 2(2n+1)$ with

$$\hat{b}^{m=1\dagger}_{f=1} := \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,,$$
$$(\hat{b}^{m=1\dagger}_{f=1})^\dagger = \hat{b}^{m=1}_{f=1} := \stackrel{d-1\,d}{(-)}\cdots\stackrel{12}{(-)}\stackrel{03}{(-i)}\,, \tag{9.9}$$

and making a choice of the vacuum state $|\psi_{oc}>$ as a sum of all the products of $\hat{b}^m_f \cdot \hat{b}^{m\dagger}_f$ for all $f = (1, 2, \cdots, 2^{\frac{d}{2}-1})$, one recognizes for the "basis vectors" of an odd Clifford character for each of the two Clifford algebras the properties

$$\begin{aligned}
\hat{b}^m_f {}_{*_A} |\psi_{oc}> &= 0\, |\psi_{oc}>\,, \\
\hat{b}^{m\dagger}_f {}_{*_A} |\psi_{oc}> &= |\psi^m_f>\,, \\
\{\hat{b}^m_f, \hat{b}^{m'}_{f'}\}_{*_A+} |\psi_{oc}> &= 0\, |\psi_{oc}>\,, \\
\{\hat{b}^{m\dagger}_f, \hat{b}^{m\dagger}_f\}_{*_A+} |\psi_{oc}> &= |\psi_{oc}>\ .
\end{aligned} \tag{9.10}$$

$*_A$ represents the algebraic multiplication of $\hat{b}^{m\dagger}_f$ and $\hat{b}^{m'}_{f'}$ among themselves and with the vacuum state $|\psi_{oc}>$ of Eq.(9.18), which takes into account Eq. (9.2). All the products of Clifford algebra elements are up to now the algebraic ones and so are also the products in Eq. (9.10). Since we use here anticommutation relations, we pointed out with $_{*_A}$ this algebraic character of the products, to be later distinguished from the tensor product $_{*_T}$, when the creation and annihilattion operators are defined on an extended basis, which is the tensor product of the superposition of the "basis vectors" of the Clifford space and of the momentum basis, applying on the Hilbert space of "Slater determinants". The tensor product $*_T$ is used as well as the product mapping a pair of the fermion wave functions in to two fermion wave functions and further to many fermion wave functions — that is to the extended algebra of many fermion system.

Obviously, $\hat{b}^{m\dagger}_f$ and $\hat{b}^m_f$ have on the level of the algebraic products, when applying on the vacuum state $|\psi_{oc}>$, *almost* the properties of creation and annihilation operators of the second quantized fermions in the postulates of Dirac, as it is discussed in the next items. We illustrate properties of "basis vectors" and their Hermitian conjugated partners on the example of $d = (5+1)$-dimensional space in Subsect. 9.2.5.

vii. a. There is, namely, the property, which the second quantized fermions should fulfill in addition to the relations of Eq. (9.10). The anticommutation relations of creation and annihilation operators should be:

$$\{\hat{b}^m_f, \hat{b}^{m'\dagger}_{f'}\}_{*_A+} |\psi_{oc}> = \delta^{mm'} \delta_{ff'} |\psi_{oc}>\ . \tag{9.11}$$

For any $\hat{b}^m_f$ and any $\hat{b}^{m'\dagger}_{f'}$ this is not the case; besides $\hat{b}^{m=1}_{f=1} = \overset{d-1\, d}{(-)} \cdots \overset{56}{(-)}\overset{12}{(-)}\overset{03}{(-i)}$, for example, also

$$\hat{b}^{m'}_{f'} = \overset{d-1\, d}{(-)} \cdots \overset{56}{(-)}\overset{12}{[+]}\overset{03}{[+i]}\,,$$

and several others give, when applied on $\hat{b}^{m=1\dagger}_{f=1}$, nonzero contributions. There are namely $2^{\frac{d}{2}-1} - 1$ too many annihilation operators for each creation operator, which give, applied on the creation operator, nonzero contribution.

vii. b. To use the Clifford algebra objects to describe second quantized fermions, representing the observed quarks and leptons as well as the antiquarks and antileptons [3, 10–15, 17], *the families should exist*.

vii. c. The operators should exist, which connect one irreducible representation of fermions with all the other irreducible representations.

vii. d. Two independent choices for describing the internal degrees of freedom of the observed quarks and leptons are not in agreement with the observed properties of fermions.

We solve these problems, cited in vii. a., vii. b., vii. c. and vii. d., by reducing the degrees of freedom offered by the two kinds of the Clifford algebras, γ^a's and $\tilde{\gamma}^a$'s, making a choice of one — γ^a's — to describe the internal space of fermions, and using the other one — $\tilde{\gamma}^a$'s — to describe the "family" quantum number of each irreducible representation of S^{ab}'s in space defined by γ^a's.

9.2.2 Reduction of the Clifford space by the postulate

The creation and annihilation operators of an odd Clifford algebra of both kinds, of either γ^a's or $\tilde{\gamma}^a$'s, would obviously obey the anticommutation relations for the second quantized fermions, postulated by Dirac, at least on the vacuum state, which is a sum of all the products of annihilation times, $*_A$, the corresponding creation operators, provided that each of the irreducible representations would carry a different quantum number.

But we know that a particular member m has for all the irreducible representations the same quantum numbers, that is the same "eigenvalues" of the Cartan subalgebra (for the vector space of either γ^a's or $\tilde{\gamma}^a$'s), Eq. (9.6).

The only possibility to "dress" each irreducible representation of one kind of the two independent vector spaces with a new, let us say "family" quantum number, is that we "sacrifice" one of the two vector spaces, let us make a choice of $\tilde{\gamma}^a$'s, *and use* $\tilde{\gamma}^a$'s *to define the "family" quantum number for each irreducible representation of the vector space of* γ^a's, while *keeping the relations of* Eq. (9.2) *unchanged:* $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+$, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$, $(\gamma^a)^\dagger = \eta^{aa}\gamma^a$, $(\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a$, $(a, b) = (0, 1, 2, 3, 5, \cdots, d)$.

We therefore *postulate*:

Let $\tilde{\gamma}^a$'s operate on γ^a's as follows [2,3,8,14,15]

$$\tilde{\gamma}^a B = (-)^B \, i \, B\gamma^a \,, \tag{9.12}$$

with $(-)^B = -1$, if B is (a function of) an odd product of γ^a's, otherwise $(-)^B = 1$ [8].

After this postulate the vector space of $\tilde{\gamma}^a$'s is correspondingly "frozen out". No vector space of $\tilde{\gamma}^a$'s needs to be taken into account any longer, in agreement with the observed properties of fermions. This solves the problems vii.a - vii. d. of Subsect. 9.2.1.

Taking into account Eq. (9.12) we can check that:

a. Relations of Eq. (9.2) remain unchanged [1].

b. Relations of Eq. (9.3) remain unchanged [2].

---

[1] Let us show that the relation $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab}$ remains valid when applied on B, if B is either an odd or an even product of γ^a's: $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \, \gamma^c = -i\,(\tilde{\gamma}^a\gamma^c\gamma^b + \tilde{\gamma}^b\gamma^c\gamma^a) = -i\,i\,\gamma^c(\gamma^b\gamma^a + \gamma^a\gamma^b) = 2\eta^{ab}\gamma^c$, while $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \, \gamma^c\gamma^d = i\,(\tilde{\gamma}^a\gamma^c\gamma^d\gamma^b + \tilde{\gamma}^b\gamma^c\gamma^d\gamma^a) = i(-i)\,\gamma^c\gamma^d(\gamma^b\gamma^a + \gamma^a\gamma^b) = 2\eta^{ab}\gamma^c\gamma^d$. The relation is valid for any γ^c and γ^d, even if $c = d$.

[2] One easily checks that $\tilde{\gamma}^{a\dagger}\gamma^c = -i\gamma^c\gamma^{a\dagger} = -i\eta^{aa}\gamma^c\gamma^a = \eta^{aa}\tilde{\gamma}^a\gamma^c = -i\eta^{aa}\gamma^c\gamma^a$.

c. The eigenvalues of the operators S^{ab} and $\tilde{S}^{ab}$ on nilpotents and projectors of γ^a's are after the reduction of Clifford space

$$
\begin{aligned}
S^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}\,, & \tilde{S}^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}\,,\\
S^{ab} \stackrel{ab}{[k]} &= \frac{k}{2} \stackrel{ab}{[k]}\,, & \tilde{S}^{ab} \stackrel{ab}{[k]} &= -\frac{k}{2} \stackrel{ab}{[k]}\,,
\end{aligned}
\tag{9.13}
$$

demonstrating that the eigenvalues of S^{ab} on nilpotents and projectors of γ^a's differ from the eigenvalues of $\tilde{S}^{ab}$, so that $\tilde{S}^{ab}$ can be used to denote irreducible representations of S^{ab} with the "family" quantum number, what solves the problems vii. b. and vii. c. of Subsect. 9.2.1.

d. We further recognize that γ^a transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[-k]}$, never to $\stackrel{ab}{[k]}$, while $\tilde{\gamma}^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[k]}$, never to $\stackrel{ab}{[-k]}$

$$
\begin{aligned}
&\gamma^a \stackrel{ab}{(k)} = \eta^{aa} \stackrel{ab}{[-k]}, \quad &&\gamma^b \stackrel{ab}{(k)} = -ik \stackrel{ab}{[-k]},\\
&\gamma^a \stackrel{ab}{[k]} = \stackrel{ab}{(-k)}, \quad &&\gamma^b \stackrel{ab}{[k]} = -ik\eta^{aa} \stackrel{ab}{(-k)}\,,\\
&\tilde{\gamma^a} \stackrel{ab}{(k)} = -i\eta^{aa} \stackrel{ab}{[k]}, \quad &&\tilde{\gamma^b} \stackrel{ab}{(k)} = -k \stackrel{ab}{[k]},\\
&\tilde{\gamma^a} \stackrel{ab}{[k]} = \ i \stackrel{ab}{(k)}, \quad &&\tilde{\gamma^b} \stackrel{ab}{[k]} = -k\eta^{aa} \stackrel{ab}{(k)}\ .
\end{aligned}
\tag{9.14}
$$

e. One finds, using Eq. (9.12),

$$
\begin{aligned}
&\stackrel{ab}{\tilde{(k)}} \stackrel{ab}{(k)} = 0\,, \quad &&\stackrel{ab}{\tilde{(-k)}} \stackrel{ab}{(k)} = -i\eta^{aa} \stackrel{ab}{[k]}\,,\\
&\stackrel{ab}{\tilde{(k)}} \stackrel{ab}{[k]} = i \stackrel{ab}{(k)}\,, \quad &&\stackrel{ab}{\tilde{(k)}} \stackrel{ab}{[-k]} = 0\,,\\
&\stackrel{ab}{\tilde{[k]}} \stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, \quad &&\stackrel{ab}{\tilde{[-k]}} \stackrel{ab}{(k)} = 0\,,\\
&\stackrel{ab}{\tilde{[k]}} \stackrel{ab}{[k]} = 0\,, \quad &&\stackrel{ab}{\tilde{[-k]}} \stackrel{ab}{[k]} = \stackrel{ab}{[k]}\ .
\end{aligned}
\tag{9.15}
$$

f. From Eq. (9.14) it follows

$$
\begin{aligned}
S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{(k)} &= -\frac{i}{2}\eta^{aa}\eta^{cc} \stackrel{ab}{[-k]}\stackrel{cd}{[-k]}\,,\\
\tilde{S}^{ac} \stackrel{ab}{(k)}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{aa}\eta^{cc} \stackrel{ab}{[k]}\stackrel{cd}{[k]}\,,\\
S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{[k]} &= \frac{i}{2} \stackrel{ab}{(-k)}\stackrel{cd}{(-k)}\,,\\
\tilde{S}^{ac} \stackrel{ab}{[k]}\stackrel{cd}{[k]} &= -\frac{i}{2} \stackrel{ab}{(k)}\stackrel{cd}{(k)}\,,\\
S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa} \stackrel{ab}{[-k]}\stackrel{cd}{(-k)}\,,\\
\tilde{S}^{ac} \stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa} \stackrel{ab}{[k]}\stackrel{cd}{(k)}\,,\\
S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc} \stackrel{ab}{(-k)}\stackrel{cd}{[-k]}\,,\\
\tilde{S}^{ac} \stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc} \stackrel{ab}{(k)}\stackrel{cd}{[k]}\;.
\end{aligned}
\tag{9.16}
$$

g. Each irreducible representation has now the "family" quantum number, determined by $\tilde{S}^{ab}$ of the Cartan subalgebra of Eq. (9.4). Correspondingly the creation and annihilation operators fulfill algebraically the anticommutation relations of Dirac second quantized fermions: Different irreducible representations carry different "family" quantum numbers and to each "family" quantum member only one Hermitian conjugated partner with the same "family" quantum number belong. Also each summand of the vacuum state, Eq. (9.18), belongs to a particular "family". This solves the problem vii. a. of Subsect. 9.2.1.

The anticommutation relations of Dirac fermions are therefore fulfilled on the vacuum state, Eq. (9.18), on the algebraic level, without postulating them. They follow by themselves from the fact that the creation and annihilation operators are superposition of odd products of γ^a's.

Statement 1: The oddness of the products of γ^a's guarantees the anticommuting properties of all objects which include odd number of γ^a's.

We shall show in Subsect. 9.2.4 of this section, and in Sect. 9.3, that the same relations are valid also on the Hilbert space of all the second quantized fermions states, with the creation operators defined on the tensor product of "basis vectors" of the Clifford algebra and on the basis of the momentum space, where the Hilbert space is defined with the creation operators of all possible momenta of all possible "Slater determinants" applying on $|\psi_{oc}>$.

Let us write down the anticommutation relations of Clifford odd "basic vectors", representing the creation operators and of the corresponding annihilation operators again.

$$\begin{aligned}
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A +} |\psi_{oc}> &= \delta^{mm'}\, \delta_{ff'}\, |\psi_{oc}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'}\}_{*_A +} |\psi_{oc}> &= 0 \cdot |\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A +} |\psi_{oc}> &= 0 \cdot |\psi_{oc}>\,,\\
\hat{b}^{m\dagger}_{f} {}_{*_A} |\psi_{oc}> &= |\psi^{m}_{f}>\,,\\
\hat{b}^{m}_{f} {}_{*_A} |\psi_{oc}> &= 0 \cdot |\psi_{oc}>\,,
\end{aligned} \tag{9.17}$$

with (m, m') denoting the "family" members and (f, f') denoting "families", $*_A$ represents the algebraic multiplication of $\hat{b}^m_f$ with the vacuum state $|\psi_{oc}>$ of Eq.(9.18) and among themselves, taking into account Eq. (9.2).

h. The vacuum state for the vector space determined by γ^a's remains unchanged $|\psi_{oc}>$, Eq. (80) of Ref. [3], it is a sum of the products of any annihilation operator with its Hermitian conjugated partner of any family.

$$\begin{aligned}
|\psi_{oc}> = & \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]} \cdots \stackrel{d-1\, d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]} \cdots \stackrel{d-1\, d}{[-]}\\
+ & \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]} \cdots \stackrel{d-1\, d}{[-]} + \cdots |1>\,,\\
& \text{for } d = 2(2n+1)\,,\\
|\psi_{oc}> = & \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]} \cdots \stackrel{d-3\, d-2}{[-]}\stackrel{d-1\, d}{[+]}\\
+ & \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]} \cdots \stackrel{d-3\, d-2}{[-]}\stackrel{d-1\, d}{[+]} + \cdots |1>\,,\\
& \text{for } d = 4n\,,
\end{aligned} \tag{9.18}$$

n is a positive integer.

i. Taking into account the relation among θ^a in Eq. (9.1) and Eq. (9.12), requiring that $\tilde{\gamma}^a a_0 = i a_0 \gamma^a$, leads to $\frac{\partial}{\partial \theta_a} = 0$, and further to

$$\theta^a = \gamma^a\,. \tag{9.19}$$

Eq. (9.12)) namely requires: $\tilde{\gamma}^a(a_0 + a_b\gamma^b + a_{bc}\gamma^b\gamma^c + \cdots) = (i a_0\gamma^a + (-i) a_b \gamma^b\gamma^a + i a_{bc}\gamma^b\gamma^c\gamma^a + \cdots)$, what means that Eq. (9.19) is only one of the relations [3] The application of $\tilde{\gamma}^a$ depends on the space on which it applies.

The Hermitian conjugated part of the space in the Grassmann case is "freezed out" together with the "vector" space of $\tilde{\gamma}^a$'s.

9.2.3 Clifford fermions with families

Let us make a choice of the starting creation operator $\hat{b}^{1\dagger}_1$ of an odd Clifford character and of its Hermitian conjugated partner in $d = 2(2n+1)$ and $d = 4n$,

[3] Another relation, for example, is $\tilde{\gamma}^a\gamma^a = (-i)\gamma^a\gamma^a = -i\eta^{aa}$. One also has $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab} = \tilde{\gamma}^a\tilde{\gamma}^b + \tilde{\gamma}^b\tilde{\gamma}^a = \tilde{\gamma}^a i\gamma^b + \tilde{\gamma}^b i\gamma^a = i\gamma^b(-i)\gamma^a + i\gamma^a(-i)\gamma^b = 2\eta^{ab}$. $\{\tilde{\gamma}^a, \gamma^b\}_+ = 0 = \tilde{\gamma}^a\gamma^b + \gamma^b\tilde{\gamma}^a = \gamma^b(-i)\gamma^a + \gamma^b i\gamma^a = 0$. $\{\tilde{\gamma}^a, \gamma^a\}_+ = 0 = \tilde{\gamma}^a\gamma^a + \gamma^a\tilde{\gamma}^a = \gamma^a(-i\gamma^a + \gamma^a i\gamma^a = 0$.

respectively, as follows

$$
\begin{aligned}
\hat{b}^{1\dagger}_{1} &:= \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \cdots \stackrel{d-3\; d-2}{(+)} \stackrel{d-1\; d}{(+)} \,, \\
(\hat{b}^{1\dagger}_{1})^{\dagger} = \hat{b}^{1}_{1} &:= \stackrel{d-1\; d}{(-)} \stackrel{d-3\; d-2}{(-)} \cdots \stackrel{56}{(-)}\stackrel{12}{(-)}\stackrel{01}{(-i)} \,, \\
d &= 2(2n+1) \,, \\
\hat{b}^{1\dagger}_{1} &:= \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \cdots \stackrel{d-3\; d-2}{(+)} \stackrel{d-1\; d}{[+]} \,, \\
(\hat{b}^{1\dagger}_{1})^{\dagger} = \hat{b}^{1}_{1} &:= \stackrel{d-1\; d}{[+]} \stackrel{d-3\; d-2}{(-)} \cdots \stackrel{56}{(-)}\stackrel{12}{(-)}\stackrel{01}{(-i)} \,, \\
d &= 4n \,.
\end{aligned}
\tag{9.20}
$$

All the rest "vectors", belonging to the same Lorentz representation, follow by the application of the Lorentz generators S^{ab}'s.

The representations with different "family" quantum numbers are reachable by $\tilde{S}^{ab}$, since, according to Eq. (9.16), we recognize that $\tilde{S}^{ac}$ transforms two nilpotents $\stackrel{ab}{(k)}\stackrel{cd}{(k)}$ into two projectors $\stackrel{ab}{[k]}\stackrel{cd}{[k]}$, without changing k ($\tilde{S}^{ac}$ transforms $\stackrel{ab}{[k]}\stackrel{cd}{[k]}$ into $\stackrel{ab}{(k)}\stackrel{cd}{(k)}$, as well as $\stackrel{ab}{[k]}\stackrel{cd}{(k)}$ into $\stackrel{ab}{(k)}\stackrel{cd}{[k]}$). All the "family" members are reachable from one member of a new family by the application of S^{ab}'s.

In this way, by starting with the creation operator $\hat{b}^{1\dagger}_{1}$, Eq. (9.20), $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ "family" members follow.

Let us find the starting member of the next "family" to the "family" of Eq. (9.20) by the application of $\tilde{S}^{01}$

$$
\begin{aligned}
\hat{b}^{1\dagger}_{2} &:= \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(+)} \cdots \stackrel{d-3\; d-2}{(+)} \stackrel{d-1\; d}{(+)} \,, \\
\hat{b}^{1}_{2} &:= \stackrel{d-1\; d}{(-)} \stackrel{d-3\; d-2}{(-)} \cdots \stackrel{56}{(-)}\stackrel{12}{[+]}\stackrel{01}{[+i]} \,.
\end{aligned}
\tag{9.21}
$$

The corresponding annihilation operators, that is the Hermitian conjugated partners of $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ "family" members, following from the starting creation operator $\hat{b}^{1\dagger}_{1}$ by the application of S^{ab}'s — the family members — and the application of $\tilde{S}^{ab}$ — the same family member of another family — can be obtained by Hermitian conjugation.

The creation and annihilation operators of an odd Clifford character, expressed by nilpotents and projectors of γ^a's, obey anticommutation relations of Eq. (9.17), *without postulating the second quantized anticommutation relations* as we explain in Subsect. 9.2.2.

The even partners of the Clifford odd creation and annihilation operators follow by either the application of γ^a on the creation operators, leading to $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ members, or with the application of $\tilde{\gamma}^a$ on the creation operators, leading to another group of the Clifford even operators, again with the $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ members.

It is not difficult to recognize, that each of the Clifford even "families", obtained by the application of γ^a or by $\tilde{\gamma}^a$ on the creation operators, contains one selfadjoint operator, which is the product of projectors only, contributing as a summand to the vacuum state, Eq. (9.18).

9.2.4 Action for free massless Clifford fermions with half integer spin and solutions of Weyl equations

To relate the creation operators, expressed with the Clifford odd "basis vectors", and the creation operators, creating the second quantized fermions, we define the tensor products of the finite number of odd Clifford "basis vectors" and infinite basis of momentum space. To compare properties of our creation operators of the second quantized fermions with those of Dirac, the solution of the equations of motion of the Weyl (for massless free fermions) or of the Dirac equations are appropriate.

The Lorentz invariant action for a free massless fermion in Clifford space is well known

$$\mathcal{A} = \int d^d x \, \frac{1}{2} \, (\psi^{\dagger} \gamma^0 \, \gamma^a p_a \psi) + \text{h.c.} \,, \tag{9.22}$$

$p_a = i \, \frac{\partial}{\partial x^a}$, leading to the equation of motion

$$\gamma^a p_a |\psi> = 0 \,, \tag{9.23}$$

and to the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b |\psi> = p^a p_a |\psi>= 0 \,,$$

γ^0 appears in the action to take care of the Lorentz invariance of the action.

Our Clifford algebra "basis vectors" offer the description of only the internal degrees of freedom of fermions (in $d = (3+1)$ the "basis vectors" offers the description of only the spin and family degrees of freedom, in $d \geq 5$ also of the charges [4, 10, 11, 15] and the references therein).

We need to extend the internal degrees of freedom (offering final number — $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ — of basis vectors of the odd products of γ^a) to the momentum or coordinate space with (infinite number of) basis.

Statement 2: For deriving the anticommutation relations for the Clifford fermions, to be compared with the anticommutation relations of the second quantized fermions, we need to define the tensor product of the Clifford odd "basis vectors" and the momentum space

$$\mathbf{basis}_{(p^a, \gamma^a)} = |p^a> *_T |\gamma^a> \,.$$

The new state vector space is the tensor product of the internal space of fermions and the space of momenta or coordinates. All states have an odd Clifford character due to oddness of the internal space.

Solutions of Eq. (9.23) for free massless fermions of momentum $p^a, a = (0, 1, 2, 3, 5, \ldots, d)$ are superposition of "basis vectors" $\hat{b}^{m\dagger}_f$, expressed by operators γ^a, where f denotes a "family" and m a "family" member quantum number, Eqs. (9.20, 9.21), and of plane waves in the case of free, in our case, massless fermions. The equations of motion require that $|p^0| = |\vec{p}|$. Correspondingly it follows

$$
\begin{aligned}
< x|\psi^{\mathbf{sf}}(\tilde{\mathbf{p}},\mathbf{p^0}) > |_{\mathbf{p^0}=|\tilde{\mathbf{p}}|} &= \int dp^0 \delta(p^0 - |\vec{p}|)\, \hat{\mathbf{b}}^{sf\dagger}(\vec{p})\, e^{-ip_a x^a} \;*_A |\psi_{oc}> \\
&= (\hat{\mathbf{b}}^{sf\dagger}(\vec{p}) \cdot\; e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})})|_{p^0=|\vec{p}|} \;*_A\; |\psi_{oc}>, \\
&\quad \text{where we define}, \\
\hat{\mathbf{b}}^{sf\dagger}(\vec{p})|_{p^0=|\vec{p}|} &\stackrel{\text{def}}{=} \sum_m c^{sf}{}_m\; (\vec{p}, |p^0| = |\vec{p}|)\; \hat{b}_f^{m\dagger}, \\
|\psi^{\mathbf{sf}}(\tilde{\mathbf{x}},\mathbf{x^0}) > &= \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, (\hat{\mathbf{b}}^{sf\dagger}(\vec{p})\, e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})}|_{p^0=|\vec{p}|} \;*_A\; |\psi_{oc}>,
\end{aligned}
\tag{9.24}
$$

s represents different orthonormalized solutions of the equations of motion, $\varepsilon = \pm 1$, depending on handedness and spin of solutions, $c^{sf}{}_m(\vec{p}, |p^0| = |\vec{p}|)$ are coefficients, depending on momentum $|\vec{p}|$ with $|p^0| = |\vec{p}|$, while $*_A$ denotes the algebraic multiplication of the "basis vectors" $\hat{b}_f^{m\dagger}$ on the vacuum state $|\psi_{oc}>$, Eq. (9.17).

An illustration of $\hat{\mathbf{b}}^{sf\dagger}(\vec{p})$ is presented in Subsect. 9.2.5.

Since the "basis vectors" in internal space of fermions are orthogonal according to Eq. (9.10) ($\{\hat{b}_f^m{}_{*_A}, \hat{b}_{f'}^{m'\dagger}{}_{*_A}\}_+|\psi_{oc}>= \hat{b}_f^m{}_{*_A}\, \hat{b}_{f'}^{m'\dagger}{}_{*_A}|\psi_{oc}>$),

$$
\begin{aligned}
&\hat{b}_f^m{}_{*_A}\, \hat{b}_{f'}^{m'\dagger}{}_{*_A}|\psi_{oc}>= \delta^{mm'}\delta_{ff'}\,|\psi_{oc}>, \\
&\text{it follows for particular } \vec{p}\,, p^0 = |\vec{p}|, \text{ that} \\
&\sum_m c^{sf*}{}_m(\vec{p}, |p^0| = |\vec{p}|)\;\; c^{s'f'}{}_m(\vec{p}, |p^0| = |\vec{p}|) = \delta^{ss'}\delta_{ff'}\,, \\
&\text{leading to} \\
&\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}}\; < \psi^{s'\mathbf{f}'}(\vec{p'}, p'^0 = |\vec{p'}|)|x>< x||\,\psi^{\mathbf{sf}}(\vec{p}, p^0 = |\vec{p}|)>= \\
&\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}} e^{ip'_a x^a}|_{p'^0=|\vec{p'}|}\, e^{-ip_a x^a}|_{p^0|=|\vec{p}|} \\
&\cdot < \psi_{oc}|\,(\hat{\mathbf{b}}^{s'f'}(\vec{p'})\; \hat{\mathbf{b}}^{sf\dagger}(\vec{p}))_{*_A}\,|\psi_{oc}>=\; \delta ss'\delta^{ff'}\delta(\vec{p}-\vec{p'}),
\end{aligned}
\tag{9.25}
$$

while we take into account that $\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}}\, e^{ip'_a x^a}\, e^{-ip_a x^a} = \delta(\vec{p}-\vec{p'})$.

Let us now evaluate the scalar product $< \psi^{\mathbf{sf}}(\vec{x}, x^0)\,|\,\psi^{\mathbf{s'f'}}(\vec{x}\,', x^0)>$, taking into account that the scalar product is evaluated at a time x^0 and correspondingly using the relation

$$
\begin{aligned}
&< \psi^{\mathbf{sf}}(\vec{x}, x^0)|\,\psi^{\mathbf{s'f'}}(\vec{x}\,', x^0)>= \delta^{ss'}\;\delta_{ff'}\,\delta(\vec{x}-\vec{x}\,') = \\
&\int \frac{dp^0}{\sqrt{2\pi}} \int \frac{dp'^0}{\sqrt{2\pi}} \delta(p^0-p'^0) \int_{-\infty}^{+\infty} \frac{d^{d-1}p'}{(\sqrt{2\pi})^{d-1}} \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} \delta(p^0-|\vec{p}|)\,\delta(p'^0-|\vec{p'}|) \\
&< \psi_{oc}|(\hat{\mathbf{b}}^{s'f'}(\vec{p'}, p'^0)\,\hat{\mathbf{b}}^{sf\dagger}(\vec{p}, p^0))_{*_A}\,|\psi_{oc}>e^{ip'_a x'^a}\; e^{-ip_a x^a} = \\
&\int \frac{dp^0}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d^{d-1}p'}{(\sqrt{2\pi})^{d-1}}\; \delta(p^0-|\vec{p'}|) \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\; \delta(p^0-|\vec{p}|) \\
&< \psi_{oc}|(\hat{\mathbf{b}}^{sf}(\vec{p}, p^0)\,\hat{\mathbf{b}}^{s'f'\dagger}(\vec{p'}, p^0))_{*_A}\,|\psi_{oc}>e^{i(p^0x^0-\vec{p}\cdot\vec{x})}\; e^{-i(p^0x'^0-\vec{p'}\cdot\vec{x})}\,.
\end{aligned}
\tag{9.26}
$$

The scalar product $< \psi^{\mathbf{sf}}(\vec{x}, x^0) \,|\, \psi^{\mathbf{s'f'}}(\vec{x}\,', x^0) >$ has obviously the desired properties of the second quantized states.

Let us define the creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, which determine, when applying on the vacuum state, the fermion states, Eq. (9.24),

$$\begin{aligned}
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) &\stackrel{\text{def}}{=} \hat{\mathbf{b}}^{sf\dagger}(\vec{p})\, e^{-i(p^0x^0-\vec{p}\cdot\vec{x})}\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) &= (\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}))^{\dagger} = \hat{\mathbf{b}}^{sf}(\vec{p})\, e^{i(p^0x^0-\vec{p}\cdot\vec{x})}\,,\\
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\,|\psi_{oc}> &= |\,\psi^{\mathbf{sf}}(\vec{p}, p^0 = |\vec{p}|) > \,.
\end{aligned} \tag{9.27}$$

In Eq. (9.27) $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ creates on the vacuum state $|\psi_{oc}>$ the single fermion states. We can multiply, using the tensor product $*_T$ multiplication this time, an arbitrary number of such single particle states, what means that we multiply an arbitrary number of creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p'}) *_T \cdots *_T \hat{\underline{\mathbf{b}}}^{s''f''\dagger}_{tot}(\vec{p''})$, applying on $|\psi_{oc}>$, which gives nonzero contributions, provided that they distinguish among themselves in at least one of the properties $(s, f, \vec{p})$, in the internal space quantum numbers (s, f) or in momentum part $\vec{p}$, due to the orthonormal property of plane waves.

The space of all such functions, which one can form - including the identity - represents the second quantized Hilbert space. We present these tensor products as "Slater determinants" of occupied and empty states in Section 9.3.

Due to anticommutation relations of any two of creation operators

$$\{\hat{\mathbf{b}}^{sf\dagger}(\vec{p})\,,\, \hat{\mathbf{b}}^{s'f'}(\vec{p})\}_{+}\,|\psi_{oc}> = \delta^{ff'}\delta^{ss'}\,|\psi_{oc}>,$$

Eqs. (9.17, 9.24), while plane waves form the orthonormal basis in the momentum representation, Eq. (9.25), the new creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, which are are generated on the tensor products of both spaces, internal and momentum, fulfill the anticommutation relations when applied on $|\psi_{oc}>$.

$$\begin{aligned}
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,, \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,')\}_{+\,*_T}|\psi_{oc}> &= \delta^{ss'}\,\delta_{ff'}\,\delta(\vec{p}-\vec{p'})\,|\psi_{oc}>\,,\\
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,, \hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p'})\}_{+\,*_T}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\,, \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,')\}_{+\,*_T}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\,{*_T}|\psi_{oc}> &= |\psi^{sf}(\vec{p})>\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,{*_T}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
|p^0| &= |\vec{p}|\,.
\end{aligned} \tag{9.28}$$

It is not difficult to show that $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ manifest the same anticommutation relations also on tensor products of an arbitrary chosen set of single fermion states, what we discuss in Sect. 9.3.

Therefore, with the choice of the Clifford odd "basis states" to describe the internal space of fermions (we can proceed equivalently in the Grassmann case) and using the tensor product of the internal space and the momentum or coordinate space to solve the equations of motion, we derive the anticommutation relations among creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and their Hermitian conjugated partners annihilation operators $(\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}))^{\dagger} = \hat{\mathbf{b}}^{sf}(\vec{p})\, e^{i(p^0x^0-\vec{p},\cdot\vec{x})} = \hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$, with $|p^0| = |\vec{p}|$. While application of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ on $|\psi_{oc}>$ generates the single fermion state, the application of $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ gives zero.

We shall demonstrate in Sect. 9.3 that there is $\{\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p'})\,,\,\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\}_{+}$, which when applied on the Hilbert space of the second quantized fermions (that is on tensor products of all single fermion states, or equivalently on all possible "Slater determinants"), gives zero when at least one of $(s', f', \vec{p'})$ differ from $(s, f, \vec{p})$, while $\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,,\,\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\}_{+}$ applied on the Hilbert space, gives the whole Hilbert space back.

Taking into account the last line of Eq. (9.24) and Eqs. (9.26,9.27), the creation operators $\underline{\Psi}^{\dagger}$ follow, which determine, when applying on the vacuum state $|\psi_{oc}>$, the fermion fields $|\psi^{\mathbf{sf}}(\tilde{\mathbf{x}}, \mathbf{x^0})>$, depending on coordinates at particular time x^0

$$
\begin{aligned}
\underline{\Psi}^{sf\dagger}(\vec{x}, x^0) &\stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})_{|p^0|=|\vec{p}|}\,,\\
\underline{\Psi}^{sf\dagger}(\vec{x}, x^0)\,|\psi_{oc}> &= |\,\psi^{\mathbf{sf}}(\vec{x}, x^0)>\,,\\
\{\underline{\Psi}^{sf\dagger}(\vec{x}, x^0)\,,\underline{\Psi}^{s'f'}(\vec{x'}, x^0)\}_{+}\,|\psi_{oc}> &= \delta^{ss'}\delta^{ff'}\delta(\vec{x}-\vec{x'})|\psi_{oc}>\,,\\
\{\underline{\Psi}^{sf}(\vec{x}, x^0)\,,\underline{\Psi}^{s'f'}(\vec{x'}, x^0)\}_{+}\,|\psi_{oc}> &= 0\,.\\
\{\underline{\Psi}^{sf\dagger}(\vec{x}, x^0)\,,\underline{\Psi}^{s'f'\dagger}(\vec{x'}, x^0)\}_{+}\,|\psi_{oc}> &= 0\,,
\end{aligned} \tag{9.29}
$$

where $\underline{\Psi}^{\dagger}(\vec{x}, x^0)$ and $\underline{\Psi}^{sf}(\vec{x}, x^0)$ are creation and annihilation partners, respectively, Hermitian conjugated to each other, in the coordinate representation, presenting the creation and annihilation operators of the second quantized fields.

The application of the creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})_{|p^0|=|\vec{p}|}$ and $\underline{\Psi}^{\dagger}(\vec{x}, x^0)$ and their Hermitian conjugated partners on the Hilbert space of fermion fields will be discussed in Sect. 9.3.

Dirac uses the Lagrange and Hamilton formalism for fermion fields and assuming that the second quantized states should anticommute to describe fermions, he derives the anticommuting creation and annihillation operators. In Subsect. 9.3.4 we compare the Dirac anticommutation relations with our way of deriving anticommutation relations for second quantized fields in details.

In Subsect. 9.2.5 the properties of creation and annihilation operators, $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$, respectively, described by the odd Clifford algebra objects in $d = (5+1)$-dimensional space are discussed.

9.2.5 Illustration of Clifford fermions with families in $d = (5+1)$-dimensional space

We illustrate properties of the Clifford odd, and correspondingly anticommuting, creation and their Hermitian conjugated partners annihilation operators, belonging to $2^{\frac{6}{2}-1} = 4$ "families", each with $2^{\frac{6}{2}-1} = 4$ members in $d = (5+1)$-dimensional space. The spin in the fifth and the sixth dimension manifests as the charge in $d = (3+1)$.

In Table 9.1 the "basis vectors" of odd and even Clifford character are presented. They are "eigenvectors" of the Cartan subalgebras, Eq. (9.4).

Half of the Clifford odd "basis vectors" are (chosen to be) creation operators $\hat{b}^{m\dagger}_{f}$, denoted in table by *odd I*, appearing in four "families", $f = (1(a), 2(b), 3(c), 4(d))$. The rest half of the Clifford odd "basis vectors" are their Hermitian conjugated partners $\hat{b}^{m}_{f}$, presented in *odd II* part and denoted with the corresponding "family" and family members (a_m, b_m, c_m, d_m) quantum numbers.

The normalized vacuum state is the product of $\hat{b}^{m}_{f} \cdot \hat{b}^{m\dagger}_{f}$ — this product is the same for each member of a particular family and different for different families — summed over four families

$$
|\psi_{oc}> = \frac{1}{\sqrt{2^{\frac{6}{2}-1}}} \Big(\stackrel{03}{[-i]}\stackrel{12}{[-1]}\stackrel{56}{[-1]} + \stackrel{03}{[+i]}\stackrel{12}{[+1]}\stackrel{56}{[-1]}
+ \stackrel{03}{[+i]}\stackrel{12}{[-1]}\stackrel{56}{[+1]} + \stackrel{03}{[-i]}\stackrel{12}{[+1]}\stackrel{56}{[+1]} \Big). \tag{9.30}
$$

One easily checks, by taking into account Eq. (9.15), that the creation operators $\hat{b}^{m\dagger}_{f}$ and the annihilation operators $\hat{b}^{m}_{f}$ fulfill the anticommutation relations of Eq (9.17).

The summands of the vacuum state $|\psi_{oc}>$ appear among selfadjoint members of *even I* part of Table 9.1, each of summands belong to different "family" [4].

All the Clifford even "families" with "family" members of Table 9.1 can be obtained as algebraic products, $*_A$, of the Clifford odd "vectors" of the same table.

Let us find the solutions of the Weyl equation, Eq. (9.23), taking into account four basis creation operators of the first family, $f = 1(a)$, in Table 9.1. Assuming that moments in the fifth and the sixth dimensions are zero, $p^a = (p^0, p^1, p^2, p^3, 0, 0)$, the following four plane wave solutions for positive energy, $p^0 = |\vec{p}|$, can be found, two with the positive charge $\frac{1}{2}$ and with spin S^{12} either equal to $\frac{1}{2}$ or to $-\frac{1}{2}$, and two with the negative charge $-\frac{1}{2}$ and again with S^{12} either $\frac{1}{2}$ or $-\frac{1}{2}$.

$$
\begin{aligned}
&\text{Clifford odd creation operators in } d = (5+1) \\
&p^0 = |p^0|\,,\;\; S^{56} = \frac{1}{2}\,,\;\; \Gamma^{(3+1)} = 1\,, \\
&\left(\underline{\hat{\mathbf{b}}}^{11\dagger}_{tot}(\vec{p}) = \beta \left(\stackrel{03}{(+i)}\stackrel{12}{(+)} | \stackrel{56}{(+)} + \frac{p^1 + ip^2}{|p^0| + |p^3|} \stackrel{03}{[-i]}\stackrel{12}{[-]} | \stackrel{56}{(+)}\right)\right) \cdot \\
&\quad e^{-i(|p^0|x^0 - \vec{p}\cdot\vec{x})}\,, \\
&\left(\underline{\hat{\mathbf{b}}}^{21\dagger}_{tot}(\vec{p}) = \beta^* \left(\stackrel{03}{[-i]}\stackrel{12}{[-]} | \stackrel{56}{(+)} - \frac{p^1 - ip^2}{|p^0| + |p^3|} \stackrel{03}{(+i)}\stackrel{12}{(+)} | \stackrel{56}{(+)}\right)\right) \cdot \\
&\quad e^{-i(|p^0|x^0 + \vec{p}\cdot\vec{x})}\,,
\end{aligned}
$$

---

[4] If we would make a choice for creation operators the "families" with the "family" members of *odd II* of Table 9.1, instead of "families" with the "family" members of *odd I*, then their Hermitian conjugated partners would be the "families" with the "family" members in *odd I*. The vacuum state would be the sum of products of annihilation operators of *odd I* times the creation operators of *odd II* and would be the sum of selfadjoint members appearing in *even II*.

odd I	m	f = 1(a) $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	f = 2(b) $(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	f = 3(c) $(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	f = 4(d) $(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	03 12 56 $(+i)(+)(+)$	03 12 56 $[+i][+](+)$	03 12 56 $[+i](+)[+]$	03 12 56 $(+i)[+][+]$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
	2	$[-i][-](+)$	$(-i)(-)(+)$	$(-i)[-][+]$	$[-i](-)[+]$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	1
	3	$[-i](+)[-]$	$(-i)[+][-]$	$(-i)(+)(-)$	$[-i][+](-)$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	−1
	4	$(+i)[-][-]$	$[+i](-)[-]$	$[+i][-](-)$	$(+i)(-)(-)$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	−1
odd II		03 12 56 $_{f\,m}$	03 12 56 $_{f\,m}$	03 12 56 $_{f\,m}$	03 12 56 $_{f\,m}$	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
		$(-i)(+)(+)_{d_4}$	$[-i][+](+)_{d_3}$	$[-i](+)[+]_{d_2}$	$(-i)[+][+]_{d_1}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−1	−1
		$[+i][-](+)_{c_4}$	$(+i)(-)(+)_{c_3}$	$(+i)[-][+]_{c_2}$	$[+i](-)[+]_{c_1}$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	−1	−1
		$[+i](+)[-]_{b_4}$	$(+i)[+][-]_{b_3}$	$(+i)(+)(-)_{b_2}$	$[+i][+](-)_{b_1}$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	−1	1
		$(-i)[-][-]_{a_4}$	$[-i](-)[-]_{a_3}$	$[-i][-](-)_{a_2}$	$(-i)(-)(-)_{a_1}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	1

even I	m	$(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	$[-i](+)(+)$	$(-i)[+](+)$	$[-i][+][+]$	$(-i)(+)[+]$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−1	−1
	2	$(+i)[-](+)$	$[+i](-)(+)$	$(+i)(-)[+]$	$[+i][-][+]$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	−1	−1
	3	$(+i)(+)[-]$	$[+i][+][-]$	$(+i)[+](-)$	$[+i](+)(-)$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	−1	1
	4	$[-i][-][-]$	$(-i)(-)[-]$	$[-i](-)(-)$	$(-i)[-](-)$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	1
even II	m	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	$[+i](+)(+)$	$(+i)[+](+)$	$[+i][+][+]$	$(+i)(+)[+]$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
	2	$(-i)[-](+)$	$[-i](-)(+)$	$(-i)(-)[+]$	$[-i][-][+]$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	1
	3	$(-i)(+)[-]$	$[-i][+][-]$	$(-i)[+](-)$	$[-i](+)(-)$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	−1
	4	$[+i][-][-]$	$(+i)(-)[-]$	$[+i](-)(-)$	$(+i)[-](-)$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	−1

Table 9.1. $2^d = 64$ "eigenvectors" of the Cartan subalgebra, Eq. (9.4), of the Clifford odd and even algebras in $d = (5+1)$ are presented, divided into four groups, each group with four "families", each "family" with four "family" members. Two of four groups are sums of an odd number of γ^a's. The "basis vectors", $\hat{b}^{m\dagger}_f$, Eqs. (9.20, 9.21), in *odd I* group, belong to four "families" $(f = 1(a), 2(b), 3(c), 4(d))$ with four members $(m = 1, 2, 3, 4)$, having their Hermitian conjugated partners, $\hat{b}^{m}_f$, among "basis vectors" of the *odd II* part, denoted by the corresponding "family" and "family" members (a_m, b_m, c_m, d_m) quantum numbers. The "family" quantum numbers, the eigenvalue of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, of $\hat{b}^{m\dagger}_f$ are written above each "family". The two groups with the even number of γ^a's, *even I* and *even II*, have their Hermitian conjugated partners within their own group each. There are members in each group, which are products of projectors only. Numbers — 03 12 56 — denote the indexes of the corresponding Cartan subalgebra members $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, Eq. (9.4). In the columns $(7, 8, 9)$ the eigenvalues of the Cartan subalgebra members (S^{03}, S^{12}, S^{56}), Eq. (9.4), are presented. The last two columns tell the handedness of $d = (5+1)$, $\Gamma^{(5+1)}$, and of $d = (3+1)$, $\Gamma^{(3+1)}$, respectively, defined in Eq.(9.48).

$$
\begin{aligned}
&\text{Clifford odd creation operators in } d = (5+1)\\
&p^0 = |p^0|\,, \quad S^{56} = -\frac{1}{2}\,, \quad \Gamma^{(3+1)} = -1\,,\\
&\left(\underline{\hat{\mathbf{b}}}^{31\dagger}_{tot}(\vec{p}) = -\beta \left(\overset{03}{[-i]}\,\overset{12}{(+)} \mid \overset{56}{[-]} + \frac{p^1 + ip^2}{|p^0| + |p^3|}\,\overset{03}{(+i)}\,\overset{12}{[-]} \mid \overset{56}{[-]}\right)\right) \cdot\\
&\qquad e^{-i(|p^0|x^0 + \vec{p}\cdot\vec{x})}\,,\\
&\left(\underline{\hat{\mathbf{b}}}^{41\dagger}_{tot}(\vec{p}) = -\beta^* \left(\overset{03}{(+i)}\,\overset{12}{[-]} \mid \overset{56}{[-]} - \frac{p^1 - ip^2}{|p^0| + |p^3|}\,\overset{03}{[-i]}\,\overset{12}{(+)} \mid \overset{56}{[-]}\right)\right) \cdot\\
&\qquad e^{-i(|p^0|x^0 - \vec{p}\cdot\vec{x})}\,,
\end{aligned}
\tag{9.31}
$$

Index $^{s=(1,2,3,4)}$ counts different solutions of the Weyl equations, index $^{f=1}$ denotes the family quantum number, all solutions belong to the same family, while $\beta^*\beta = \frac{|p^0|+|p^3|}{2|p^0|}$ takes care that the corresponding states are normalized.

All four superposition of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})|_{p^0=|\vec{p}|} = \sum_m c^{sf=1}{}_m(\vec{p}, |p^0| = |\vec{p}|)\ \hat{b}^{m\dagger}_{f=1}\ e^{-i(p^0x^0 - \varepsilon\vec{p}\cdot\vec{x})}$, with $m = (1,2)$ for the first two states, and with $m = (3,4)$ for the second two states, Table 9.1, $s = (1,2,3,4)$, are orthogonal and correspondingly normalized, fulfilling Eq. (9.25).

9.3 Hilbert space of Clifford fermions

The Clifford odd creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, with $|p^0| = |\vec{p}|$, are defined in Eq. (9.27) on the tensor products of the $(2^{\frac{d}{2}-1})^2$ "basis vectors" (describing the internal space of fermion fields) and of the (continuously) infinite number of basis in the momentum space. The solutions of the Weyl equation, Eq. (9.23), are plane waves of particular momentum $\vec{p}$ and with energy related to momentum, $|p^0| = |\vec{p}|$.

The creation operator $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ defines, when applied on the vacuum state $|\psi_{oc}>$, the s^{th} of the $2^{\frac{d}{2}-1}$ plane wave solutions of a particular momentum $\vec{p}$ belonging to the f^{th} of the $2^{\frac{d}{2}-1}$ "families". They fulfill together with the Hermitian conjugated partners annihilation operators $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ the anticommutation relations of Eq. (9.28).

These creation operators form the Hilbert space of "Slater determinants", defining for each "Slater determinant" the "space" for any of the single particle fermion states of an odd Clifford character, due to the oddness of the "basis vector" of an odd Clifford character. Each of these "spaces" can be empty or occupied. Correspondingly there is the "Slater determinant" with all the "spaces" empty, the "Slater determinants" with only one of the "spaces" occupied, any one, and all the rest empty, the "Slater determinants" with two "spaces" occupied, any two, and all the rest empty, and so on.

These "Slater determinant" of all possible occupied and empty states can be explained as well if introducing the tensor multiplication of single fermion states of any quantum number and any momentum, with the constant included.

Statement 3: Introducing the tensor product multiplication $*_T$ of any number of Clifford odd fermion states of all possible internal quantum numbers and all possible momenta (that is of any number of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$) of any $(s, f, \vec{p})$ we generate the Hilbert space of Clifford fermions.

The Hilbert space of a particular momentum $\vec{p}$, $\mathcal{H}_{\vec{p}}$, contains the finite number of "Slater determinants". The number of "Slater determinants" is in d-dimensional space equal to

$$N_{\mathcal{H}_{\vec{p}}} = 2^{2^{d-2}}\,. \tag{9.32}$$

The total Hilbert space of anticommuting fermions is the product $\otimes_N$ of the Hilbert spaces of particular $\vec{p}$

$$\mathcal{H} = \prod_{\vec{p}}^{\infty} \otimes_N \mathcal{H}_{\vec{p}}\,. \tag{9.33}$$

The total Hilbert space $\mathcal{H}$ is correspondingly infinite and contains $N_{\mathcal{H}}$ "Slater determinants"

$$N_{\mathcal{H}} = \prod_{\vec{p}}^{\infty} 2^{2^{d-2}} \,. \tag{9.34}$$

Before starting to comment the application of the creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and annihilation $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ operators on the Hilbert space $\mathcal{H}$ (described with all possible "Slater determinants" of all possible occupied and empty fermion states of all possible $(s, f, \vec{p})$, or by the tensor products of all possible single fermion states of all possible $(s, f, \vec{p})$, with the identity included) let us discuss properties of creation and annihilation operators, the anticommutation relations of which are presented in Eq. (9.28).

The creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and the annihilation operators $\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p'})$, having an odd Clifford character, anticommute, manifesting the properties as follows

$$\begin{aligned}
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,,\\
\text{if at least one of}\,(s,f,\vec{p}) &\quad \text{is different from}\,(s',f',\vec{p}\,')\,,\\
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) &= 0\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) &= 0\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) &= 1\,(\text{identity})\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})|\psi_{oc}> &= 0\,.
\end{aligned} \tag{9.35}$$

The above relations, leading from the commutation relations of Eq. (9.28), determine the rules of the application of creation and annihilation operators on "Slater determinants":

i. The creation operator $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ jumps over the creation operators determining the occupied state of another kind (that is over the occupied state distinguishing from the jumping creation one in any of the internal quantum numbers (s, f) or in $\vec{p}$) up to the last step when it comes to its own empty state with the quantum numbers (f, s) and $\vec{p}$, occupying this empty state, or, if this state is already occupied, gives zero. Whenever $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ jumps over an occupied state changes the sign of the "Slater determinant".

ii. The annihilation operator changes the sign whenever jumping over the occupied state carrying different internal quantum numbers (s, f) or $\vec{p}$, unless it comes to the occupied state with its own internal quantum numbers (s, f) and its own $\vec{p}$, emptying this state, or, if this state is empty, gives zero.

Let us point out that the Clifford odd creation operators, $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, and annihilation operators, $\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p'})$, fulfill the anticommutation relations of Eq. (9.28) for any $\vec{p}$ and any (s, f) due to the anticommuting character (the Clifford oddness) of the "basis vectors", $\hat{b}^{m\dagger}_{f}$ and their Hermitian conjugated partners $\hat{b}^{m}_{f}$, Eqs. (9.20, 9.21), what means that the anticommuting character of creation and annihilation operators is not postulated.

The total Hilbert space $\mathcal{H}$ has infinite number of degrees of freedom (of "Slater determinants") due to the infinite number of Hilbert spaces $\mathcal{H}_{\vec{p}}$ of particular $\vec{p}$, $\mathcal{H} = \prod_{\vec{p}}^{\infty} \otimes_N \mathcal{H}_{\vec{p}}$, while the Hilbert space $\mathcal{H}_{\vec{p}}$ of particular momentum $\vec{p}$ has the finite dimension $2^{2^{d-2}}$.

In Subsects. 9.3.1, 9.3.2, 9.3.3 the properties of Hilbert spaces are discussed in more details.

9.3.1 Application of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ on Hilbert space of Clifford fermions of particular $\vec{p}$

The 2^{d-2} Clifford odd creation operators of particular momentum $\vec{p}$, $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}, p^0)$, with the property $|p^0| = |\vec{p}|$, each representing the s^{th} solution of Eq. (9.23) for a particular family f, fulfill together with the (Hermitian conjugated partners) annihilation operators $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ the anticommutation relations of Eq. (9.28), the application of which on the Hilbert space of "Slater determinants" are discussed in Eq. (9.35) and in the text below this equation.

The Hilbert space $\mathcal{H}_{\vec{p}}$ of a particular momentum $\vec{p}$ consists correspondingly of $2^{2^{d-2}}$ "Slater determinants". Let us write down explicitly these $2^{2^{d-2}}$ contributions to the Hilbert space $\mathcal{H}_{\vec{p}}$ of a particular momentum $\vec{p}$, using the notation that $\mathbf{0^{sf}}_{\vec{p}}$ represents the unoccupied state $|\psi^{sf}(\vec{p}, p^0) > |_{|p^0|=|\vec{p}|} = \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})|_{|p^0|=|\vec{p}|}\, |\psi_{oc}>$ of the s^{th} solution of the equations of motion for the f^{th} family and the momentum $|p^0| = |\vec{p}|$), Eq. (9.24), while $\mathbf{1^{sf}}_{\vec{p}}$ represents the corresponding occupied state.

The number operator is defined as

$$
\begin{aligned}
N^{sf}_{\vec{p}} &= \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,, \\
N^{sf}_{\vec{p}}\, |\psi_{oc}> &= 0 \cdot |\psi_{oc}>\,, \qquad N^{sf}_{\vec{p}} *_T \mathbf{0^{sf}}_{\vec{p}} = 0\,, \\
N^{sf}_{\vec{p}} *_T \mathbf{1^{sf}}_{\vec{p}} &= 1 \cdot \mathbf{1^{sf}}_{\vec{p}}\,, \quad N^{sf}_{\vec{p}} *_T N^{sf}_{\vec{p}} *_T \mathbf{1^{sf}}_{\vec{p}} = 1 \cdot \mathbf{1^{sf}}_{\vec{p}}\,.
\end{aligned}
\tag{9.36}
$$

One can check the above relations on the example of $d = (5+1)$, with the "basis vectors" for $f = 1$ presented in Table 9.2 and with the solution for Weyl equation, Eq. (9.23), presented in Eq. (9.31).

i	$f = 1(a)$	Her. con. $f = 1(a)$
1	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}$	$\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{(-)}$
2	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}$	$\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}$
3	$\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{[-]}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{[-]}$
4	$\stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{[-]}$	$\stackrel{03}{(-i)}\stackrel{12}{[-]}\stackrel{56}{[-]}$

Table 9.2. The four creation operators of the irreducible representation *odd I* from Table 9.1, $d = (5+1)$, $f = 1(a)$. together with their Hermitian conjugated partners are presented (up to a phase).

Let us write down the Hilbert space of second quantized fermions $\mathcal{H}_{\vec{p}}$, using the simplified notation as in Part I, Sect. III.A., counting for $f = 1$ empty states as $\mathbf{0_{rp}}$, and occupied states as $\mathbf{1_{rp}}$, with $r = (1, \dots, 2^{\frac{d}{2}-1})$, for $f = 2$ we count $r = 2^{\frac{d}{2}-1} + 1, \cdots, 2^{d-2}$. Correspondingly we can represent $\mathcal{H}_{\vec{p}}$ as follows

$$\begin{aligned}
&|\mathbf{0_{1p}}, \mathbf{0_{2p}}, \mathbf{0_{3p}}, \dots, \mathbf{0_{2^{d-2}p}} >|_1 \quad ,\\
&|\mathbf{1_{1p}}, \mathbf{0_{2p}}, \mathbf{0_{3p}}, \dots, \mathbf{0_{2^{d-2}p}} >|_2 \quad ,\\
&|\mathbf{0_{1p}}, \mathbf{1_{2p}}, \mathbf{0_{3p}}, \dots, \mathbf{0_{2^{d-2}p}} >|_3 \quad ,\\
&|\mathbf{0_{1p}}, \mathbf{0_{2p}}, \mathbf{1_{3p}}, \dots, \mathbf{0_{2^{d-2}p}} >|_4 \quad ,\\
&\vdots\\
&|\mathbf{1_{1p}}, \mathbf{1_{2p}}, \mathbf{0_{3p}}, \dots, \mathbf{0_{2^{d-2}p}} >|_{2^{d-2}+2} \quad ,\\
&\vdots\\
&|\mathbf{1_{1p}}, \mathbf{1_{2p}}, \mathbf{1_{3p}}, \dots, \mathbf{1_{2^{d-2}p}} >|_{2^{2^{d-2}}} \quad ,
\end{aligned} \tag{9.37}$$

with a part with none of states occupied ($N_{rp} = 0$ for all $r = 1, \dots, 2^{d-2}$), with a part with only one of states occupied ($N_{rp} = 1$ for a particular $r = (1, \dots, 2^{d-2})$, while $N_{r'p} = 0$ for all the others $r' \neq r$), with a part with only two of states occupied ($N_{rp} = 1$ and $N_{r'p} = 1$, where r and r' run from $(1, \dots, 2^{d-2})$, and so on. The last part has all the states occupied.

It is not difficult to see that the creation and annihilation operators, when applied on this Hilbert space $\mathcal{H}_{\vec{p}}$, fulfill the anticommutation relations for the second quantized Clifford fermions.

$$\begin{aligned}
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,, \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p})\}_{*_T+}\mathcal{H}_{\vec{p}} &= \delta^{ss'}\,\delta^{ff'}\mathcal{H}_{\vec{p}}\,,\\
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p})\}_{*_T+}\, \mathcal{H}_{\vec{p}} &= 0\, \cdot \mathcal{H}_{\vec{p}}\,,\\
\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\,, \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p})\}_{*_T+}\, \mathcal{H}_{\vec{p}} &= 0\, \cdot \mathcal{H}_{\vec{p}}\,.
\end{aligned} \tag{9.38}$$

The proof for the above relations easily follows if one takes into account that whenever the creation or annihilation operator jumps over an odd products of occupied states the sign of the "Slater determinant" changes due to the oddness of the occupied states, while states, belonging to different $\vec{p}$ are orthogonal [5], see Eq. (9.35) and the text below this equation. Then one sees that the contribution of the application of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T\, \hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p}) *_T$ on $\mathcal{H}_{\vec{p}}$ has the opposite sign than the contribution of $\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p})\, *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T$ on $\mathcal{H}_{\vec{p}}$.

If the creation and annihilation operators are Hermitian conjugated to each other, the result follows

$$(\,\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\, *_T\, \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) + \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\, *_T\, \hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,)\, *_T\, \mathcal{H}_{\vec{p}} = \mathcal{H}_{\vec{p}}\,,$$

manifesting that this application of $\mathcal{H}_{\vec{p}}$ gives the whole $\mathcal{H}_{\vec{p}}$ back. Each of the two summands operates on their own half of $\mathcal{H}_{\vec{p}}$. Jumping together over an even

[5] The orthogonality of the states are even easier to be visualized since the two delta functions at $\vec{x}$ and at $\vec{x}\,'$, $\vec{x} \neq \vec{x}\,'$ are obviously orthogonal.

number of occupied states, $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ and $\hat{\mathbf{b}}^{sf\dagger}_{tot}(\vec{p})$ do not change the sign of the particular "Slater determinant". (Let us add that $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ reduces for the particular s and f the Hilbert space $\mathcal{H}_{\vec{p}}$ for the factor $\frac{1}{2}$, and so does $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$. The sum of both, applied on $\mathcal{H}_{\vec{p}}$, reproduces the whole $\mathcal{H}_{\vec{p}}$.)

Let us repeat that the number of "Slater determinants" in the Hilbert space of particular momentum $\vec{p}$, $\mathcal{H}_{\vec{p}}$, in d-dimensional space is finite and equal to $N_{\mathcal{H}_{\vec{p}}} = 2^{2^{d-2}}$.

9.3.2 Application of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ on total Hilbert space $\mathcal{H}$ of Clifford fermions

The total Hilbert space of anticommuting fermions is the infinite product of the Hilbert spaces of particular $\vec{p}$, Eq. (9.33), $\mathcal{H} = \prod^{\infty}_{\vec{p}} \otimes_N \mathcal{H}_{\vec{p}}$.

Due to the Clifford odd character of creation and annihilation operators, Eq. (9.28), and the orthogonality of the plane waves belonging to different momenta $\vec{p}$, it follows that $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,') *_T \mathcal{H} \neq 0, \vec{p} \neq \vec{p}\,'$, while $\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,') + \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\} *_T \mathcal{H} = 0, \vec{p} \neq \vec{p}\,'$. This can be proven if taking into account Eq. (9.35). For "plane wave solutions" of equations of motion in a box the momentum $\vec{p}$ is discretized, otherwise is continuous. The number of "Slater determinants" in the Hilbert space $\mathcal{H}$ in d-dimensional space is infinite (in both cases) $N_{\mathcal{H}} = \prod^{\infty}_{\vec{p}} 2^{2^{d-2}}$.

Since the creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and the annihilation operators $\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p}\,')$ fulfill for particular $\vec{p}$ the anticommutation relations on $\mathcal{H}_{\vec{p}}$, Eq. (9.38), and since the momentum states, the plane wave solutions, are orthogonal, and correspondingly the creation and annihilation operators defined on the tensor products of the internal basis and the momentum basis, representing fermions, anticommute, Eq. (9.28) (the Clifford odd objects $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ demonstrate their oddness also with respect to $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,')$), the anticommutation relations follow also for the application of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ on $\mathcal{H}$

$$\begin{aligned}
&\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,, \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\, \mathcal{H} = \delta^{ss'}\, \delta_{ff'}\, \delta(\vec{p} - \vec{p}\,')\, \mathcal{H}\,,\\
&\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}), \hat{\mathbf{b}}^{s'f'\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\, \mathcal{H} = 0\, \cdot \mathcal{H}\,,\\
&\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}), \hat{\mathbf{b}}^{s'f'\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\, \mathcal{H} = 0\, \cdot \mathcal{H}\,.
\end{aligned} \tag{9.39}$$

9.3.3 Illustration of $\mathcal{H}$ in $d = (1+1)$

Let us illustrate the properties of $\mathcal{H}$ and the application of the creation operators on $\mathcal{H}$ in $d = (1+1)$ dimensional space in a toy model with two discrete momenta (p^1_1, p^1_2). Generalization to many momenta is straightforward.

The internal space of fermions contains only one creation operator, one "basis vector" $\hat{b}^1_1 = \overset{01}{(+i)}$, one family member $m = 1$ of the only family $f = 1$. Correspondingly the creation operators $\hat{\underline{\mathbf{b}}}^{11\dagger}_{tot}(\vec{p}^1_i)|_{|p^0|=|p^1_i|} : = \overset{01}{(+i)}\; e^{-i(p^0x^0 - p^1_i x^1)}|_{|p^1_i|=|p^0_i|}$ distinguish only in momentum space of the fermion degrees of freedom. Their Hermitian conjugated annihilation operators are $\hat{\underline{\mathbf{b}}}^{11}_{tot}(\vec{p}^{1}_{i})_{|p^0|=|p^1_i|}$, while the vacuum state is $|\psi_{oc}> = \stackrel{01}{(-i)} \cdot \stackrel{01}{(+i)} = \stackrel{01}{[-i]}$.

The whole Hilbert space for this toy model has correspondingly four "Slater determinants", numerated by $| \quad >_i, i = (1,2,3,4)$

$$(|\mathbf{0}_{p_1}\mathbf{0}_{p_2} > |_1\,,\ |\mathbf{1}_{p_1}\mathbf{0}_{p_2} > |_2\,,\ |\mathbf{0}_{p_1}\mathbf{1}_{p_2} > |_3\,,\ |\mathbf{1}_{p_1}\mathbf{1}_{p_2} > |_4)\,,$$

$\mathbf{0}_{p^1_i}$ represents an empty state and $\mathbf{1}_{p^1_i}$ the occupied state. Let us evaluate the application of $\{\hat{\underline{\mathbf{b}}}^{11}_{tot}(\vec{p}^{1}_{1})\,, \hat{\underline{\mathbf{b}}}^{11\dagger}_{tot}(\vec{p}^{1}_{2})\}_{*_T+}$ on the Hilbert space $\mathcal{H}$. It follows

$$\begin{aligned}
&\{\hat{\underline{\mathbf{b}}}^{11}_{tot}(\vec{p}^{1}_{1})\,, \hat{\underline{\mathbf{b}}}^{11\dagger}_{tot}(\vec{p}^{1}_{2})\}_{*_T+}\mathcal{H} = \\
&\hat{\underline{\mathbf{b}}}^{11}_{tot}(\vec{p}^{1}_{1})\ *_T\ (|\mathbf{0}_{p_1}\mathbf{1}_{p_2} > |_{1\to 3}\,,\ -|\mathbf{1}_{p_1}\mathbf{1}_{p_2} > |_{2\to 4}) + \\
&\hat{\underline{\mathbf{b}}}^{11\dagger}_{tot}(\vec{p}^{1}_{2})\ *_T\ (|\mathbf{0}_{p_1}\mathbf{0}_{p_2} >_{2\to 1}\,,\ +|\mathbf{0}_{p_1}\mathbf{1}_{p_2} >_{4\to 3}) = \\
&(-|\mathbf{0}_{p_1}\mathbf{1}_{p_2} >_{2\to 4\to 3} + |\mathbf{0}_{p_1}\mathbf{1}_{p_2} >_{2\to 1\to 3})\ = 0\,.
\end{aligned}$$

9.3.4 Relation between second quantized fermions of Dirac and second quantized fermions originated in odd Clifford algebra

The Clifford odd creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and their Hermitian conjugated partners annihilation operators $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ obey the anticommutation relations of Eq. (9.39) — on the vacuum state $|\psi_{oc}>$, Eq. (9.18), and on the whole Hilbert space $\mathcal{H}$, Eq. (9.39). Creation operators, $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, operating on a vacuum state, as well as on the whole Hilbert space, define second quantized fermion states.

Let us relate here the Dirac's second quantization relations and the relations between creation operators $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ and their Hermitian conjugated partners annihilation operators, without paying attention on the charges and family quantum numbers, since Dirac's creation operators do not pay attention on these two kinds of quantum numbers. We shall relate vectors in $d = (3+1)$ of both origins.

In the Dirac case the second quantized field operators are in $d = (3+1)$ dimensions postulated as follows

$$\underline{\Psi}^{hs\dagger}(\vec{x}, x^0) = \sum_{m,\vec{p}_k} \hat{\mathbf{a}}^{h\dagger}_{m}(\vec{p}_k)\, v^{hs}_{m}(\vec{p}_k)\,. \tag{9.40}$$

$v^{hs}_{m}(\vec{p}_k) = u^{hs}_{m}(\vec{p}_k)\, e^{-i(p^0x^0 - \varepsilon\vec{p}_k\cdot\vec{x})}$ are the two left handed ($\Gamma^{(3+1)} = -1 = h$) and the two right handed ($\Gamma^{(3+1)} = 1 = h$, Eq. (B.3)) two-component column matrices, $m = (1,2)$, representing the twice two solutions s of the Weyl equation for free massless fermions of particular momentum $|\vec{p}_k| = |p^0_k|$ ([20], Eqs. (20-49) - (20-51)), the factor $\varepsilon = \pm 1$ depends on the product of handedness and spin.

$\hat{\mathbf{a}}^{h\dagger}_{m}(\vec{p}_k)$ are by Dirac postulated creation operators, which together with the annihilation operators $\hat{\mathbf{a}}^{h}_{m}(\vec{p}_k)$, fulfill the anticommutation relations ([20], Eqs. (20-49) - (20-51))

$$\begin{aligned}
&\{\hat{\mathbf{a}}^{h\dagger}_{m}(\vec{p}_k),\ \hat{\mathbf{a}}^{h'\dagger}_{n}(\vec{p}_l)\}_{*_T+} = 0 = \{\hat{\mathbf{a}}^{h}_{m}(\vec{p}_k),\ \hat{\mathbf{a}}^{h'}_{n}(\vec{p}_l)\}_{*_T+}\,, \\
&\{\hat{\mathbf{a}}^{h}_{m}(\vec{p}_k),\ \hat{\mathbf{a}}^{h'\dagger}_{n}(\vec{p}_l)\}_{*_T+} = \delta_{mn}\delta^{hh'}\delta_{\vec{p}_k\vec{p}_l}
\end{aligned} \tag{9.41}$$

in the case of discretized momenta for a fermion in a box. (Massive fermions are represented by four vectors which are the superposition of both handedness.)

Let us present the two "basis vectors" $\hat{b}^{h\dagger}_m$, $m = (1,2)$, h representing left and right handedness, in the internal space of fermions in $d = (3+1)$, described by the Clifford odd algebra, representing the creation operators of one particular family (f not shown in this case), without charges, of one handedness and with spins $\pm\frac{1}{2}$, respectively, operating on the vacuum state $|\psi_{oc}> = \stackrel{03}{[+i]}\stackrel{12}{[-]}$: $\hat{b}^{h\dagger}_1 = \stackrel{03}{[+i]}\stackrel{12}{(+)}$ and $\hat{b}^{h\dagger}_2 = \stackrel{03}{(-i)}\stackrel{12}{[-]}$, Eq. (9.20, 9.21) [6], with $h = 1$, representing the right handedness. These two "basis vectors" should be compared with the two vectors, one corresponding to the spin $\frac{1}{2}$ and the other to the spin $-\frac{1}{2}$ in the Dirac case.

Since Dirac did not postulate such creation operators on the level of $\hat{b}^{h\dagger}_m$, let us postulate them now on the level of $\hat{b}^{h\dagger}_m$, to be able to compare in this paper presented creation operators for this particular case, $\hat{a}^{h\dagger}_\uparrow$ and $\hat{a}^{h\dagger}_\downarrow$, of right handedness h and spin up and down $(\uparrow,\downarrow)$ as follows

$$\hat{b}^{h\dagger}_1 = \stackrel{03}{[+i]})\stackrel{12}{(+)} \;\text{ to be related to }\; \hat{a}^{h\dagger}_\uparrow\,, \qquad \hat{b}^{h\dagger}_2 = \stackrel{03}{(-i)})\stackrel{12}{[-]} \;\text{ to be related to }\; \hat{a}^{h\dagger}_\downarrow\,.$$

One should repeat this also for left handedness $h = -1$. But these creation operators $\hat{a}^{h\dagger}_m$, $m = (1,2) = (\uparrow,\downarrow)$, still can not be compared with the Dirac's ones.

Let us make the superposition of both creation operators of particular handedness h, $\mathbf{\hat{a}}^{hs\dagger}(\vec{p}_k) := \alpha^{hs}_\uparrow(\vec{p}_k)\,\hat{a}^{h\dagger}_\uparrow + \alpha^{hs}_\downarrow(\vec{p}_k)\,\hat{a}^{h\dagger}_\downarrow$, with the coefficients $\alpha^{hs}_\uparrow(\vec{p}_k)$ and $\alpha^{hs}_\downarrow(\vec{p}_k)$ chosen so that $\underline{\mathbf{\hat{a}}}^{hs\dagger}_{tot}(\vec{p}_k) := \mathbf{\hat{a}}^{hs\dagger}(\vec{p}_k)\, e^{-i(p^0x^0-\vec{p}_k\cdot\vec{x})}$ solves the equations of motion, Eq. (9.23) [7], for a plane wave $e^{i\varepsilon\vec{p}_k\cdot\vec{x}}$ for $|\vec{p_k}| = |p^0_k|$, then it follows

$$\underline{\mathbf{\hat{a}}}^{hs\dagger}_{tot}(\vec{p}_k) := (\alpha^{hs}_\uparrow(\vec{p}_k)\,\hat{a}^{h\dagger}_\uparrow + \alpha^{hs}_\downarrow(\vec{p}_k)\,\hat{a}^{h\dagger}_\downarrow)\, e^{-i(p^0x^0-\vec{p}_k\cdot\vec{x})} = \sum_m \mathbf{\hat{a}}^{h\dagger}_m(\vec{p}_k)v^{hs}_m(\vec{p}_k)\,, \tag{9.42}$$

where the summation runs over m up and down spin m of the chosen handedness h.

Since $v^{hs}_m(\vec{p}_k) = u^{hs}_m(\vec{p}_k)\, e^{-i(p^0x^0-\vec{p}_k\cdot\vec{x})}$ it follows also that

$$\mathbf{\hat{a}}^{hs\dagger}(\vec{p}_k) = \sum_m u^{hs}_m\,\hat{a}^{h\dagger}_m,$$

and $u^{hs}_m(\vec{p}_k) = \alpha^{hs}_m(\vec{p}_k)$. We conclude that $\underline{\mathbf{\hat{a}}}^{hs\dagger}_{tot}(\vec{p}_k)$ obviously determine

$$\mathbf{\hat{a}}^{h\dagger}_m(\vec{p}_k)v^{hs}_m(\vec{p}_k) = \mathbf{\hat{a}}^{h\dagger}_m(\vec{p}_k)\,u^{hs}_m(\vec{p}_k)e^{-i(p^0x^0-\vec{p}_k\cdot\vec{x})}.$$

Anticommutation relations of Eq. (9.41), postulated by Dirac, ensure the equivalent anticommutation relations also for $\mathbf{\hat{a}}^{hs\dagger}(\vec{p}_k)$ and $\mathbf{\hat{a}}^{hs}(\vec{p}_k)$.

[6] We choose in the Clifford case the first two members of the third family in Table 9.1, since they manifest in $d = (3+1)$ the Clifford odd character.

[7] The equations of motion read in the Dirac case: $\{\hat{p}^0 + (-2iS^{0i}\hat{p}_i)\}(\alpha^s_1(\vec{p}_k)\,\mathbf{\hat{a}}^\dagger_1 + \alpha^s_2(\vec{p}_k)\,\mathbf{\hat{a}}^\dagger_2)e^{-i(p^0x^0-\vec{p}_k\cdot\vec{x})} = 0$. To solve them we need to recognize that the matrices in the chiral representation S^{0i}, $i = (1,2)$, transform $\mathbf{\hat{a}}^\dagger_1$ into $\mathbf{\hat{a}}^\dagger_2$, and opposite.

Now we are able to relate creation and annihilation operators in both cases, the Dirac case and our case of using the odd Clifford algebra to represent the internal space of fermions.

$$
\begin{aligned}
\hat{b}_1^{h\dagger} &= \overset{03}{[+i]}\overset{12}{(+)} \quad \text{to be related to} \quad \hat{a}_{\uparrow}^{h\dagger}\,,\\
\hat{b}_2^{h\dagger} &= \overset{03}{(-i)}\overset{12}{[-]} \quad \text{to be related to} \quad \hat{a}_{\downarrow}^{h^h\dagger}\\
\hat{b}_1^{h} &= -\,\overset{03}{[+i]}\overset{12}{(-)} \quad \text{to be related to} \quad \hat{a}_{\uparrow}^{h}\,,\\
\hat{b}_2^{h} &= \quad \overset{03}{(+i)}\overset{12}{[-]} \quad \text{to be related to} \quad \hat{a}_{\downarrow}^{h}\,,
\end{aligned}
\tag{9.43}
$$

both sides representing the creation operators, with $S^{12} = \frac{1}{2}$ and handedness $\Gamma^{(3+1)} = 1$, Eq. (9.48), in the first row, and with $S^{12} = -\frac{1}{2}$ and handedness $\Gamma^{(3+1)} = 1 = h$, in the second row [8].

None of the creation operators, $\hat{a}_m^{h\dagger}$, $m = (\uparrow, \downarrow)$ and $\hat{b}_m^{h\dagger}$, $m = (1, 2)$, depend on momenta, but $\hat{\mathbf{a}}^{hs\dagger}(\vec{p}_k)$ and $\hat{\mathbf{b}}^{sf\dagger}(\vec{p}_k)$ as well as $\hat{\underline{\mathbf{a}}}_{tot}^{hs\dagger}(\vec{p}_k)$ and $\hat{\underline{\mathbf{b}}}_{tot}^{sf\dagger}(\vec{p}_k)$ do depend on momenta.

The creation operators $\hat{\underline{\mathbf{a}}}_{tot}^{s\dagger}(\vec{p}_k)$ fulfill the anticommutation relations of Eqs. (9.28, 9.38, 9.39), the same as $\hat{\underline{\mathbf{b}}}_{tot}^{sf\dagger}(\vec{p})$ do. We can just replace $\hat{\underline{\mathbf{a}}}_{tot}^{s\dagger}(\vec{p}_k)$ by $\hat{\underline{\mathbf{b}}}_{tot}^{sf\dagger}(\vec{p})$ for any of families (for plane waves solutions with continuous $\vec{p}$).

We can conclude:

$$
\begin{aligned}
\hat{\underline{\mathbf{a}}}_{tot}^{hs\dagger}(\vec{p}) \quad &\text{is to be related to} \quad \hat{\underline{\mathbf{b}}}_{tot}^{hs\dagger}(\vec{p})\,,\\
\hat{a}_m^{h\dagger}, m = (\uparrow, \downarrow) \quad &\text{is to be related to} \quad \hat{b}_m^{h\dagger}\; m = (1, 2)\,,
\end{aligned}
\tag{9.44}
$$

with h representing the handedness. This can be done for any chosen family in the Clifford case. In all the relations with $\hat{\underline{\mathbf{b}}}_{tot}^{hs\dagger}(\vec{p})$ the handedness is not written explicitly and is included in the index m and in the index s, while the index f represents the family quantum number. Only in this chapter we introduce handedness in addition to clarify the relations.h

In the Clifford case the charges origin in spins $d \geq 6$. In $d = (13 + 1)$ all the charges of quarks and leptons and antiquarks and antileptons can be explained, as well as the families of quarks and leptons and antiquarks and antileptons. In the Dirac case charges come from additional groups and so do families.

Let us add: The odd Clifford algebra influences the algebra of the associated creation and annihilation operators acting on the second quantized Hilbert space $\mathcal{H}$; Due to oddness of the Clifford algebra, which determines internal degrees of freedom of fermions, the creation operators and their Hermitian conjugated annihilation partners, determined on the tensor products of internal and momentum space, make the creation and annihilation operators to anticommute.

We conclude: The by Dirac postulated creation operators, $\hat{\mathbf{a}}_m^{h\dagger}(\vec{p})$, and their annihilation partners, $\hat{\mathbf{a}}_m^{h}(\vec{p})$, Eqs. (9.40, 9.42), related in Eq. (9.44) to the Clifford

[8] The vacuum state is on the left hand side equal to $\overset{03}{[+i]}\overset{12}{[-]}$, while on the right hand side the corresponding vacuum state can be defined, if we follow our way of defining the vacuum state, to be proportional to $(\hat{a}_{\uparrow}\hat{a}_{\uparrow}^{\dagger} + \hat{a}_{\downarrow}\hat{a}_{\downarrow}^{\dagger})$.

odd creation and annihilation operators, manifest that the odd Clifford algebra offers the explanation for the second quantization postulates of Dirac.

9.4 Creation and annihilation operators in $d = (13 + 1)$-dimensional space

The *standard model* offered an elegant new step in understanding elementary fermion and boson fields by postulating:

i. Massless family members of (coloured) quarks and (colourless) leptons, the left handed fermions as the weak charged doublets and the weak chargeless right hand members, the left handed quarks distinguishing in the hyper charge from the left handed leptons, each right handed member having a different hyper charge. All fermion charges are in the fundamental representation of the corresponding groups. Antifermions carry the corresponding anticharges and opposite handedness. The massless families to each family member exist.

ii. The existence of the massless vector gauge fields to the observed charges of quarks and leptons, carrying charges in the corresponding adjoint representations.

iii. The existence of a massive scalar Higgs, gaining at some step of the expanding universe the nonzero vacuum expectation value, responsible for masses of fermions and heavy bosons and for the Yukawa couplings. The Higgs carries a half integer weak charge and hyper charge.

iv. Fermions and bosons are second quantized fields.

The *standard model* assumptions have in the literature several explanations, mostly with many new not explained assumptions. The most successful seem to be the grand unifying theories [22–36, 38–40], if postulating in addition the family group and the corresponding gauge scalar fields.

The *spin-charge-family* theory, the project of one of the authors of this paper (N.S.M.B. [1–3, 10–15, 17]), is offering the explanation for all the assumptions of the *standard model*, unifying in $d = (13 + 1)$-dimensional space not only charges, but also charges and spins and families [2, 7], explaining the appearance of families [8, 10, 15], the appearance of the vector gauge fields [12, 14], of the scalar field and the Yukawa couplings [13]. Theory offers the explanation for the dark matter [4, 5], for the matter-antimatter asymmetry [11], and makes several predictions [4, 6, 11].

The *spin-charge-family* theory is a kind of the Kaluza-Klein like theories [17, 42, 44–50] due to the assumption that in $d \geq 5$ (in the *spin-charge-family* theory $d \geq (13+1)$) fermions interact with the gravity only (vielbeins and two kinds of the spin connection fields). Correspondingly this theory shares with the Kaluza-Klein like theories their weak points, at least:

a. Not yet solved the quantization problem of the gravitational field.

b. The spontaneous break of the starting symmetry, which would at low energies manifest the observed almost massless fermions [44].

c. The appearance of gravitational anomalies, what makes the theory not well defined [53], but in the low energy limit the fields manifest in $d = (3 + 1)$ properties of the observed vector and scalar gauge fields.

d. And other problems.

In the *spin-charge-family* theory fermions interact in $d = (13 + 1)$ with the gravity only: with the spin connections (the gauge fields of S^{ab} and of $\tilde{S}^{ab}$) and vielbeins (the gauge fields of momenta), with fermions as a condensate present, breaking the symmetry (and with no other gauge fields present), manifesting at low energies in $d = (3 + 1)$ as the ordinary gravity and all the observed vector gauge fields.

It is proven in Refs. [51, 52], that one can have massless spinors even after breaking the starting symmetry. Ref. [12] proves, that at low enough energies, after breaking the staring symmetry, the two spin connections manifest in $d = (3 + 1)$ as the observed vector gauge fields, as well as the scalar fields, which offer the explanation for the Higgs and the Yukawa couplings. Ref. [11] offers the explanation for the matter-antimatter asymmetry due to the existence of the scalar fields with the "colour charges" in the fundamental representations. In Ref. [17] the *spin-charge-family* theory explains the *standard model* triangle anomaly cancellation better than the $SO(10)$ theory [23].

The working hypotheses of the authors of this paper (in particular of N.S.M.B.) is, since the higher dimensions used in the *spin-charge-family* theory offer in an elegant (simple) way explanations for the so many observed phenomena, that they should not be excluded by the renormalization and anomaly arguments. At least the low energy behavior of the spin connections and vielbeins as vector and scalar gauge fields manifest as the known and more or less well defined theories.

In this paper we present that using the half of the odd Clifford algebra objects to explain the internal degrees of freedom of fermions (the other half represent the Hermitian conjugated partners), as suggested by the *spin-charge-family* theory, leads to the second quantized fermions without postulates of Dirac [9].

9.5 Conclusions

We present in Part I and Part II of this paper that the description of the internal space of fermions with the odd elements of the anticommuting algebra defines the creation and annihilation operators, which anticommute when applied on the corresponding vacuum state. The internal space, described by the odd Clifford algebra, extends its oddness to creation and annihilation operators generated on the tensor products of the internal basis with finite numbers of elements and the momentum basis with infinite number of elements. The application of these creation and annihilation operators on the Hilbert space, determined by the tensor multiplication of all possible creation operators of any numbers, formally observed

[9] The authors of Ref. [43] let us know that their path integral formulation enabled them to see a great deal of what we present in this paper. We went through their paper noting that they did a lot concerning path integral formulation of quantum mechanics, offering ways to treat anomalies. But we couldn't recognize that they propose some replacement for the Dirac postulates of creation and annihilation operators. We also could not found whether our proposal for explaining the Dirac postulates would bring any new light on path integral formulations and anomalies cancelations. To clarify this topics the discussions with authors would be needed.

in this paper and in [3] in the Clifford algebra, manifests the same anticommutation relations as the creation and annihilation of the second quantized fermions, explaining therefore the Dirac postulates of the second quantized fermion fields.

In the subsection 9.1.2 we clarify the relation between our description of the internal space of fermions with "basis vectors", manifesting oddness and transferring the oddness to the corresponding creation and annihilation operators of second quantized fermions, to the ordinary second quantized creation and annihilation operators from a generalized point of view.

We learn in Part I of this paper, that odd products of superposition of θ^a's, Eqs. (8-11,13,22) in Part I, exist forming the odd algebra "basis vectors" in the internal space of "Grassmann fermions" with integer spin, which together with their Hermitian conjugated partners fulfill on the algebraic level on the vacuum state all the requirements for the anticommutation relations for the Dirac fermions. The creation and annihilation operators, defined on the tensor products of the superposition of the Grassmann odd algebra "basis vectors" and the momentum space basis, and manifesting correspondingly the oddness of the "basis vectors", fulfill the anticommutation relations of the second quantized Dirac's fermions on the vacuum state, as well as on the "Slater determinants" of all possibilities of occupied and empty single particle "Grassmann fermion" states of integer spins of any number. These "Slater deerminants", representing the Hilbert space of second quantized "Grassmann fermions", can be represented as well with the tensor product multiplication of any possible choice of single "Grasmann fermion states" of all possible numbers of states, started with none (that is with the identity), distinguishing at least either in one of the quantum numbers of the "basic vectors" or in momentum basis.

In Part II we learn, that the creation and annihilation operators exist in the Clifford odd algebra, defining the internal space of half integer fermions, which applying on the vacuum state fulfill the anticommutation relations postulated by Dirac. Creation operators, defined on the tensor products of the superposition of the finite "basis vectors" of the internal space described with the Clifford algebra and of the infinite momentum basis, fulfill as well together with their Hermitian conjugated annihilation operators the anticommutation relations postulated by Dirac, on the vacuum state and on the Hilbert space of infinite number of the single particle fermion states, $N_{\mathcal{H}} = \prod_{\vec{p}}^{\infty} 2^{2^{d-2}}$, Eqs. (9.33, 9.34), creating "Slater determinants" (Eqs. (9.37, 9.39)), but only after the reduction of the degrees of freedom of the Clifford algebras for a factor of two, Eq. (9.12).

The reduction of the Clifford algebras for the factor of two leaves the anticommutation relations of Eqs. (9.2, 9.3) unchanged, enabling the appearance of family quantum numbers. The Clifford fermions carry half integer spins, families and charges in fundamental representations, Eq. (9.5).

The reduction of Clifford space causes the reduction also in Grassmann space, what leads to the disappearance of integer spin fermions, Eq. (9.19).

The Clifford algebra oddness of the "basis vectors", describing the internal space of fermions, makes odd also the corresponding fermion states defined on the tensor products of the internal and momentum space. Correspondingly any two states fulfill the anticommutation relations and so do any tensor products of odd numbers of fermion states, forming the Hilbert second quantized space.

We present the creation operators, defined on the tensor products of ”basis vectors” and plane waves, solve the equations of motion, in our case for free massless fermions, Eq. (9.23), derived from the action, Eq. (9.22).

Anticommutation relations are not postulated, as it is in the Dirac case, they follow from the oddness of the Clifford objects, and correspondingly explain the second postulates of Dirac (what is stressed in several places in Part I and Part II and in a short way also in Subsect. 9.1.2).

The relation between the Dirac's creation and annihilation operators and the ones offered by the odd Clifford algebra, discussed in In Subsect. 9.3.4 demonstrates that the basic differences between these two descriptions is on the level of the single particle creation operators: While the odd Clifford algebra offers the creation and annihilation operators, which fulfill the anticommutation relations, already on the level of the ”basis vectors” determining the internal space of fermions, Eq. (9.17), when applied on the vacuum state, Dirac postulates the anticommutation relations on the level of second quantized objects, following the procedure of Lagrange and Hamilton.

The final result is in both cases equivalent, leading to the Hilbert space of second quantized fields. However, our way not only explains the Dirac postulates but demonstrates in addition, that also the single particle states in the first quantization do anticommute due to the oddness of the ”basis vectors” defining the internal space. The oddness of the Clifford objects of creation and annihilation operators is transmitted from the ”basis vectors” of internal space to the tensor products of the superposition of the ”basis vectors” and the momentum or coordinate space.

Correspondingly the odd Clifford algebra, equipped with the family quantum numbers, and fulfilling the anticommutation relations already on the level of the single particle creation operators applying on the vacuum state, as well as on the level of the whole Hilbert space, offers the explanation for the anticommutation relations, postulated by Dirac.

The Hilbert space of all ”Slater determinants” with any number of occupied or empty states of an odd character, follows in all three cases, the Dirac one (with postulated creation and annihilation operators and offering no families and no charges), the Grassmann one (offering spins and charges in adjoint representations, and no families) and the Clifford one (offering spins, families and charges), in an equivalent way: due to the anticommuting creation and annihilation operators, representing ”basis vectors” and their Hermitian conjugated partners. One can see this in Sect. 9.3.4.

Let us repeat: Internal space contributes the final number of states, the infinity of number of states is due to momentum/coordinate space [10].

The anticommuting single fermion states manifest correspondingly the oddness already on the level of the first quantization. Correspondingly these odd

[10] Let us add that the single particle vacuum state is the sum of products of annihilation $\times$ creation operators: In the Grassmann case it is just an identity, in the Clifford case is the sum of products of projectors for each family.

fermion states form in the tensor products $*_T$ the Hilbert space H of second quantized states.

9.6 APPENDIX: Norms in Grassmann space and Clifford space

Let us define the integral over the Grassmann space [2] of two functions of the Grassmann coordinates $<\mathbf{B}|\theta><\mathbf{C}|\theta>$, $<\mathbf{B}|\theta>=<\theta|\mathbf{B}>^{\dagger}$,

$$<\mathbf{b}|\theta>=\sum_{k=0}^{d} b_{a_1\ldots a_k}\theta^{a_1}\cdots\theta^{a_k},$$

by requiring

$$\begin{aligned}
&\{d\theta^a,\theta^b\}_+ = 0\,, \quad \int d\theta^a = 0\,, \quad \int d\theta^a\theta^a = 1\,,\\
&\int d^d \quad \theta\ \theta^0\theta^1\cdots\theta^d = 1\,,\\
&d^d\theta = d\theta^d\ldots d\theta^0\,, \quad \omega = \Pi^d_{k=0}(\frac{\partial}{\partial\theta_k}+\theta^k)\,,
\end{aligned} \tag{9.45}$$

with $\frac{\partial}{\partial\theta_a}\theta^c=\eta^{ac}$. We shall use the weight function [2] $\omega=\Pi^d_{k=0}(\frac{\partial}{\partial\theta_k}+\theta^k)$ to define the scalar product in Grassmann space $<\mathbf{B}|\mathbf{C}>$

$$\begin{aligned}
<\mathbf{B}|\mathbf{C}> &= \int d^d\theta^a\ \omega <\mathbf{B}|\theta><\theta|\mathbf{C}>\\
&= \sum_{k=0}^{d} b^*_{b_1\ldots b_k} c_{b_1\ldots b_k}\,.
\end{aligned} \tag{9.46}$$

To define norms in Clifford space Eq. (9.45) can be used as well.

9.7 APPENDIX: Handedness in Grassmann and Clifford space

The handedness $\Gamma^{(d)}$ is one of the invariants of the group $SO(d)$, with the infinitesimal generators of the Lorentz group S^{ab}, defined as

$$\Gamma^{(d)} = \alpha\varepsilon_{a_1a_2\ldots a_{d-1}}\,{}_{a_d}\,S^{a_1a_2}\cdot S^{a_3a_4}\cdots S^{a_{d-1}a_d}\,, \tag{9.47}$$

with α, which is chosen so that $\Gamma^{(d)}=\pm 1$.

In the Grassmann case S^{ab} is defined in Eq. (9.3), while in the Clifford case Eq. (9.47) simplifies, if we take into account that $S^{ab}|_{a\neq b}=\frac{i}{2}\gamma^a\gamma^b$ and $\tilde{S}^{ab}|_{a\neq b}=\frac{i}{2}\tilde{\gamma}^a\tilde{\gamma}^b$, as follows

$$\Gamma^{(d)} := (i)^{d/2}\prod_a (\sqrt{\eta^{aa}}\gamma^a), \quad \text{if} \quad d=2n\,. \tag{9.48}$$

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