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On girth-biregular graphs

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Abstract

Let Γ denote a finite, connected, simple graph. For an edge e of Γ let $n(e)$ denote the number of girth cycles containing e. For a vertex v of Γ let $\{e_1, e_2, \dots, e_k\}$ be the set of edges incident to v ordered such that $n(e_1) \le n(e_2) \le \dots \le n(e_k)$. Then $(n(e_1), n(e_2), \dots, n(e_k))$ is called the *signature* of v. The graph Γ is said to be *girth-biregular* if it is bipartite, and all of its vertices belonging to the same bipartition have the same signature.

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Let Γ be a girth-biregular graph with girth $g = 2d$ and signatures $(a_1, a_2, \ldots, a_{k_1})$ and $(b_1, b_2, \ldots, b_{k_2})$, and assume without loss of generality that $k_1 \geq k_2$. Our first result is that $\{a_1, a_2, \ldots, a_{k_1}\} = \{b_1, b_2, \ldots, b_{k_2}\}$. Our next result is the upper bound $a_{k_1} \leq M$, where $M = (k_1 - 1)^{\lfloor g/4 \rfloor}(k_2 - 1)^{\lceil g/4 \rceil}$. We describe the graphs attaining equality. For $d = 3$ or $d \geq 4$ even they are incidence graphs of Steiner systems and generalized polygons, respectively. Finally, we show that when d is even and $a_{k_1} = M - \varepsilon$ for some non-negative integer $\varepsilon < k_2 - 1$, then $\varepsilon = 0$. Similar result is valid for $d = 3$, $\varepsilon \leq 1$ and $k_2 \not| k_1$.

Keywords: Girth cycle, girth-biregular graph, Steiner system, generalized polygons.

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1 Introduction

In extremal graph theory one often considers problems of the following type: we fix some graph parameter or some graph property and want to deduce the extremal number of another parameter (in many cases the number of points or edges). Typical questions are Turán type problems, see e.g. the survey of Füredi and Simonovits [7]. The problem considered in our paper is motivated by the cage problem (and the degree/diameter problem), see [4, 12]. The cage problem was extended recently by several authors to bipartite graphs which are biregular in the sense that vertices in the same bipartition set have the same degree, see Jajcay, Ramos-Rivera and their coauthors [1, 6].

The paper by Jajcay, Kiss and Miklavič [8] defined a new type of regularity: a graph is called edge-girth regular if the number of cycles of length g (the girth) containing an edge is independent of the edge. This definition was weakened by Potočnik and Vidali [14] and in [9] it was extended to a stability theorem. One can introduce the signature $(a_1, \ldots a_k)$ of a point as the ordered sequence of the number of girth cycles containing the edges emanating from the point (see Definition 2.1). A graph is called girth-regular if all of its points have the same signature. For such graphs with valency $k \geq 3$, it was shown in [14] that $a_k \leq (k-1)^{2d}$, where $d = \lfloor g/2 \rfloor$. In [9], the upper bound was improved for $g = 2d$ in the sense that it is either $(k-1)^{2d}$ or at most $(k-1)^{2d} - (k-1)$. In the former case the graph has to be the incidence graph of a thick generalized d-gon of order $(k-1, k-1)$. In particular, we must have $d = 2, 3, 4, 6$.

The aim of the present paper is to extend some of the results of [9] to the bipartite biregular case. If the valencies in the bipartition classes are $k_1 > k_2 > 2$, then we prove that the maximum number of girth-cycles containg an edge is at most $M = (k_1 - 1)^{\lfloor g/4 \rfloor}(k_2 - 1)^{\lceil g/4 \rceil}$, see Theorem 2.6. For $g = 4$, we show that when the graph is girth regular and the largest element of the signature of a point is equal to $M - \varepsilon$, with $\varepsilon \leq k_2 - 1$, then $\varepsilon = 0$, and the graph is the complete bipartite graph K_{k_1,k_2}. In Section 3, we prove an analogous result for $g = 2d \geq 8$, d even, relating the $\varepsilon = 0$ case to incidence graphs of a finite thick generalized d-gon, see Theorem 3.4(vi). For $q = 2d$, d odd, we have partial results. In particular, similarly to the results of [1, 6], when $g = 6$, we could find a connection of $a_k = M$ and block designs. For particular k_1 and k_2, the connection is with affine planes, see Corollary 6.3.

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2 Definitions and basic properties

In this section we collect basic notation and terminology. First, for the sake of completeness, we recall some definitions from design theory and finite geometries. In the second subsection we define girth-biregular graphs and present some simple, important properties of them.

2.1 Block designs, Steiner systems, generalized polygons

Here we give only the most necessary definitions. A detailed introduction to block designs and Steiner systems we refer the reader to [2] and [3], while the concepts from finite geometries we use can be found for example in [10] and [11].

Definition 2.1. Let $v \geq k \geq t \geq 2$ and $\lambda \geq 1$ be integers. A t-(v, k, λ) *design* is a collection of k-subsets (blocks) of a v-set S (points) such that every t-subset of S is contained in exactly λ of the blocks.

A t-$(v, k, 1)$ design is called a *Steiner system*. In particular, the blocks of a Steiner system with $t = 2$ are often called lines.

A *parallelism* of a design is a partition of its blocks into classes $C_1, C_2, \ldots, C_r$ with the property that any point belongs to a unique block of each class. A design is called *resolvable*, if it has a parallelism.

Let $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a connected, finite point-line incidence geometry. The elements of $\mathcal{P}$ and $\mathcal{L}$ are called *points* and *lines*, respectively, $\mathrm{I} \subset (\mathcal{P}\times\mathcal{L})\cup(\mathcal{L}\times\mathcal{P})$ is a symmetric relation, called *incidence*. A *chain* of length h is a sequence $x_0 \,\mathrm{I}\, x_1 \,\mathrm{I}\, \ldots \mathrm{I} x_h$ where $x_i \in \mathcal{P} \cup \mathcal{L}$. The *distance* of the elements u and v, denoted by $d(u, v)$, is the length of the shortest chain joining them.

Definition 2.2. Let $n > 1$ be a positive integer. The incidence geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *thick generalized* n*-gon* if it satisfies the following axioms.

- $d(x, y) \leq n \;\forall\, x, y \in \mathcal{P} \cup \mathcal{L}$.
- If $d(x, y) = k < n$, then there is a unique chain of length k joining x and y.
- $\forall\, x \in \mathcal{P} \cup \mathcal{L} \;\exists\, y \in \mathcal{P} \cup \mathcal{L}$ such that $d(x, y) = n$.
- $\forall\, x \in \mathcal{P} \cup \mathcal{L}$ there exist at least three elements $y_i \in \mathcal{P} \cup \mathcal{L}$ such that $d(x, y_i) = 1$.

For any finite thick generalized n-gon $\mathcal{G}$ there exist integers $s, t \geq 2$ such that every line is incident with exactly $s+1$ points and every point is incident with exactly $t+1$ lines. The pair (s, t) is called the *order of* $\mathcal{G}$.

In particular, for $n = 3$, the generalized 3-gons are the finite projective planes, for $n = 4$, the generalized 4-gons are the finite generalized quadrangles (GQ-s for short). The GQ-s have an alternative definition:

Definition 2.3. Let $s > 1$ and $t > 1$ be positive integers. A *thick generalized quadrangle of order* (s, t) is a point-line incidence structure which satisfies the following axioms:

- every line is incident with exactly $s + 1$ points;
- every point is incident with exactly $t + 1$ lines;

- there exists a non-incident point-line pair;
- for every point P and every line ℓ not incident with P, there is exactly one line through P which intersects ℓ.

2.2 Girth-biregular graphs

Let Γ denote a finite, connected, simple graph with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$. Let d denote the minimal path-length distance function of Γ and let $D = \max\{d(v, w) \mid v, w \in V\}$ denote the *diameter* of Γ. For $v \in V$ and an integer i we let $\Gamma_i(v) = \{w \in V \mid d(v, w) = i\}$. We abbreviate $\Gamma(v) = \Gamma_1(v)$ and observe that $\Gamma_i(v) = \emptyset$ whenever $i < 0$ or $i > D$. For an edge uv of Γ, let $D^i_j(u, v) = \Gamma_i(u) \cap \Gamma_j(v)$.

We say that Γ is *biregular with valencies* k_1, k_2 $(k \in \mathbb{Z})$, whenever Γ is bipartite with bipartition sets A, B, and $|\Gamma(v)| = k_1$ ($|\Gamma(v)| = k_2$, respectively) for every $v \in A$ ($v \in B$, respectively). If Γ is not a tree, then the *girth* g of Γ is the length of a shortest cycle in Γ. If C is a cycle of Γ of girth length g, then we refer to C as a *girth cycle* of Γ.

The *incidence graph* (also known as *Levi graph*) of a point-line incidence geometry is a bipartite graph whose bipartition sets correspond to the set of points and lines, respectively, and there is an edge between two vertices if and only if the corresponding point is incident with the corresponding line.

The next ”folklore” statement gives an important correspondence between generalized polygons and biregular graphs. The proof can be found for example in [11, Lemma 1.3.6], or in [10, Chapter 12].

Theorem 2.4. *A finite thick generalized n-gon $\mathcal{G}$ exists if and only if there exists a connected bipartite biregular graph Γ of diameter n and girth $2n$, such that each vertex has degree at least three. In this case Γ is the incidence graph of $\mathcal{G}$.*

The following definition is a central definition of this paper.

Definition 2.5. Let Γ be a graph and let u, v be adjacent vertices of Γ. For the edge $e = uv$ of Γ let $n(e) = n(uv)$ denote the number of girth cycles containing e. For a vertex w of Γ let $\{e_1, e_2, \ldots, e_{k(w)}\}$ be the set of edges incident to w ordered such that $n(e_1) \le n(e_2) \le \cdots \le n(e_{k(w)})$. Then $(n(e_1), n(e_2), \ldots, n(e_{k_w}))$ is called the *signature* of w. The bipartite graph G is said to be *girth-biregular* if all of its vertices belonging to the same bipartition have the same signature.

Observe that girth-biregular graphs are also biregular. The following straightforward observation will be used through the rest of the paper frequently without explicitly referring to it (see also [14, Subsection 2.2] and Figure 1).

Proposition 2.6. *Let Γ be a biregular graph with valencies k_1, k_2 and girth $2d$, $d \ge 2$. Let uv be an edge of Γ, such that the valency of u is k_1 and valency of v is k_2. Let $D^i_j = D^i_j(u, v)$. Then the following hold.*

(i) *If x, y are vertices of Γ with $d(x, y) \le d - 1$, then there is a unique path of length $d(x, y)$ between x and y.*

(ii) *$D^i_i = \emptyset$ for every integer i.*

(iii) *For $1 \le i \le d-1$ and for $z \in D^i_{i+1}$ (resp. $z \in D^{i+1}_i$), we have that $|\Gamma(z) \cap D^{i-1}_i| = 1$ (resp. $|\Gamma(z) \cap D^i_{i-1}| = 1$).*

(iv) *For* $0 \le i \le d-2$ *and for* $z \in D^i_{i+1}$*, we have that* $|\Gamma(z) \cap D^{i+1}_{i+2}| = k_1 - 1$ *if* i *is even, and* $|\Gamma(z) \cap D^{i+1}_{i+2}| = k_2 - 1$ *if* i *is odd.*

(v) *For* $0 \le i \le d-2$ *and for* $z \in D^{i+1}_i$*, we have that* $|\Gamma(z) \cap D^{i+2}_{i+1}| = k_2 - 1$ *if* i *is even, and* $|\Gamma(z) \cap D^{i+1}_{i+2}| = k_1 - 1$ *if* i *is odd.*

(vi) *For* $0 \le i \le d-1$ *we have that*

$$|D^i_{i+1}| = \begin{cases} (k_1-1)^{i/2}(k_2-1)^{i/2} & \text{if } i \text{ is even,} \\ (k_1-1)^{(i+1)/2}(k_2-1)^{(i-1)/2} & \text{if } i \text{ is odd.} \end{cases}$$

$$|D^{i+1}_i| = \begin{cases} (k_1-1)^{i/2}(k_2-1)^{i/2} & \text{if } i \text{ is even,} \\ (k_1-1)^{(i-1)/2}(k_2-1)^{(i+1)/2} & \text{if } i \text{ is odd.} \end{cases}$$

(vii) *There are exactly* $n(uv)$ *edges between* D^{d-1}_d *and* D^d_{d-1}.

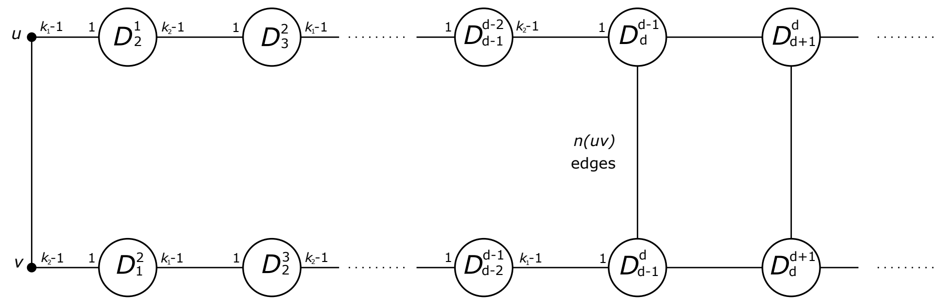

Figure 1: A biregular graph with valencies k_1, k_2 and girth $2d$, d odd. The numbers near the bubble representing the set D^i_j represent the number of neighbours that each vertex of D^i_j has in the neighbouring bubble.

3 Some properties of girth-biregular graphs

In this section we continue to study girth-biregular graphs. We prove several results about these graphs that are interesting on their own, and that will also be useful in the rest of the paper. Keeping in mind Proposition 2.6, one can calculate the number of girth cycles containing two fixed edges.

Lemma 3.1. *Let* Γ *be a girth-biregular graph with valencies* $k_1 \ge k_2$ *and girth* $g = 2d$. *Let* u_1u_2 *and* v_1v_2 *be two edges of* Γ. *Without loss of generality we may assume that* $d(u_1, v_1) = \min\{d(u_i, v_j) \colon 1 \le i, j \le 2\}$. *Let* $m = d(u_1, v_1) + 1$*, and let* c *denote the number of girth cycles containing both* u_1u_2 *and* v_1v_2. *Then* $c = 0$ *if* $m \ge d+1$ *and* $c \le 1$ *if* $m = d$. *Moreover, if* $m \le d-1$*, then*

$$c \le \begin{cases} (k_1-1)^{(d-m)/2}(k_2-1)^{(d-m)/2}, & \text{if } m \text{ and } d \text{ are of the same parity,} \\ (k_1-1)^{(d-1-m)/2}(k_2-1)^{(d+1-m)/2}, & \text{if } m \text{ is even and } d \text{ is odd,} \\ (k_1-1)^{(d-1-m)/2}(k_2-1)^{(d+1-m)/2}, & \text{if } m \text{ is odd, } d \text{ is even and valency of } v_2 \text{ is } k_2, \\ (k_1-1)^{(d+1-m)/2}(k_2-1)^{(d-1-m)/2}, & \text{if } m \text{ is odd, } d \text{ is even and valency of } v_2 \text{ is } k_1. \end{cases}$$

Proof. The statement is obvious if $m \geq d+1$. If $m = d$, then $d-1 = d(u_1, v_1) \leq d(u_2, v_2)$, so there exists a girth cycle containing both u_1u_2 and v_1v_2 if and only if $d(u_2, v_2) = d-1$, hence $c \leq 1$.

Suppose that $m \leq d-1$. Let $D^i_j = D^i_j(u_1, u_2)$ and observe that $v_1 \in D^{m-1}_m$, $v_2 \in D^m_{m+1}$. Note that there is a unique path of length $m-1$ between v_1 and u_1. Let $F = D^{d-m-1}_{d-m}(v_2, v_1)$ and note that by Proposition 2.6(iii) we have that $F \subseteq D^{d-1}_d$. Let us denote the valency of v_2 by k and let k' be the other valency of Γ. Then

$$|F| = (k-1)^{\lceil (d-m-1)/2 \rceil}(k'-1)^{\lfloor (d-m-1)/2 \rfloor},$$

and there is a unique path of length $d-m-1$ between v_2 and any element of F because the girth of Γ is $2d$. Now the number of girth cycles containing both u_1u_2 and v_1v_2 equals to the number of edges between F and D^d_{d-1}. Observe that this number is the same as the number of $(d-m)$-arcs $(v_2, x_1, \ldots, f, r)$ where $f \in F$ and $r \in D^{d-1}_d$. Observe also that the valency of f is k if $d-m-1$ is even and it is k' if $d-m-1$ is odd. Therefore, we have that $c \leq |F|(k-1)$ if $d-m-1$ is even, and $c \leq |F|(k'-1)$ if $d-m-1$ is odd. Now we distinguish four cases. If d and m are of the same parity, then $d-m-1$ is odd, and so

$$c \leq |F|(k'-1) = (k-1)^{(d-m)/2}(k'-1)^{(d-m)/2} = (k_1-1)^{(d-m)/2}(k_2-1)^{(d-m)/2}.$$

If d is odd and m is even, then $\deg(u_2) \neq \deg(v_2)$, so we may assume $\deg(v_2) = k_2$ (otherwise we interchange the roles of edges u_1u_2 and v_1v_2). Hence

$$c \leq |F|(k-1) = |F|(k_2-1) = (k_1-1)^{(d-m-1)/2}(k_2-1)^{(d-m+1)/2}.$$

Finally, if d is even and m is odd, then

$$c \leq |F|(k-1) = (k-1)^{(d-m+1)/2}(k'-1)^{(d-m-1)/2},$$

and this gives the third and fourth estimates of the statement according as $k = k_1$ or $k = k_2$. □

Proposition 3.2. *Let Γ be a girth-biregular graph with bipartition A, B and valencies $k_1 \geq k_2$. Let us denote the signature of vertices from A by $(a_1, a_2, \ldots, a_{k_1})$ and the signature of vertices from B by $(b_1, b_2, \ldots, b_{k_2})$. Then $\{a_1, a_2, \ldots, a_{k_1}\} = \{b_1, b_2, \ldots, b_{k_2}\}$.*

Proof. As Γ is bipartite, each edge e of Γ is incident with one vertex from A and with one vertex from B. It thus follows that $n(e) \in \{a_1, a_2, \ldots, a_{k_1}\}$ if and only if $n(e) \in \{b_1, b_2, \ldots, b_{k_2}\}$. This shows that $\{a_1, a_2, \ldots, a_{k_1}\} = \{b_1, b_2, \ldots, b_{k_2}\}$. □

Proposition 3.3. *Let Γ be a girth-biregular graph with bipartition A, B and valencies $k_1 \geq k_2$. Let us denote the signature of vertices from A by $(a_1, a_2, \ldots, a_{k_1})$ and the signature of vertices from B by $(b_1, b_2, \ldots, b_{k_2})$. Pick $a \in \{a_1, a_2, \ldots, a_{k_1}\} = \{b_1, b_2, \ldots, b_{k_2}\}$. Let a_A (a_B, respectively) denote the number of appearences of a in the signature $(a_1, a_2, \ldots, a_{k_1})$ ($(b_1, b_2, \ldots, b_{k_2})$, respectively). Then $k_2 a_A = k_1 a_B$.*

Proof. Let us count the number of edges of Γ that are contained in exactly a girth cycles. On the one hand, this number is equal to $|A|a_A$, and on the other hand it is equal to $|B|a_B$. Recall also that $|A|k_1 = |B|k_2$. The claim follows. □

Let Γ be a girth-biregular graph with bipartition A, B and valencies $k_1 \geq k_2$. Let us denote the signature of vertices from A by $(a_1, a_2, \ldots, a_{k_1})$ and signature of vertices from B by $(b_1, b_2, \ldots, b_{k_2})$. Let us comment on the case $k_1 = k_2$. It follows from Proposition 3.3 that in this case we have $a_A = a_B$ for every $a \in \{a_1, a_2, \ldots, a_{k_1}\} = \{b_1, b_2, \ldots, b_{k_2}\}$. Therefore, Γ is in fact girth-regular graph. As girth regular graphs were studied in details in [9] and [14], we will assume $k_1 > k_2$ for the rest of this paper.

Observe also that connected biregular graphs with valencies $k_1, k_2 = 1$ are just the star graphs, which contain no cycles at all (and are therefore girth-biregular with signatures $(0, 0, \ldots, 0)$ and (0)).

Let Γ be a girth-biregular graph with bipartition A, B and valencies $k_1 > k_2 = 2$. Then for any vertex $w \in B$ there are two edges, say $u_1 w$ and $u_2 w$ through w, hence a cycle contains $u_1 w$ if and only if it contains $u_2 w$. In particular, $n(u_1 w) = n(u_2 w)$ which implies $b_1 = b_2$. Now, define the graph Γ' in the following way: $V(\Gamma') = A$ and there is an edge between vertices u and v if and only if $d(u, v) = 2$ in Γ. Then Γ' is an edge-girth-regular graph with valency k_1. These graphs were studied in [8]. Therefore, in the rest of this paper we also assume $k_1 > k_2 > 2$.

The following theorem is a generalization of the result of *Potočnik* and *Vidali* [14, Theorem 1.3].

Theorem 3.4. *Let Γ be a girth-biregular graph with bipartition A, B, valencies $k_1 > k_2 > 2$ and girth $2d$. Let us denote the signature of vertices from A by $(a_1, a_2, \ldots, a_{k_1})$ and the signature of vertices from B by $(b_1, b_2, \ldots, b_{k_2})$. Let $M = (k_1 - 1)^{g/4}(k_2 - 1)^{g/4}$ if d is even, and $M = (k_1 - 1)^{(g-2)/4}(k_2 - 1)^{(g+2)/4}$ if d is odd. Then $a_{k_1} = b_{k_2} \leq M$.*

When the upper bound is attained, $a_{k_1} = b_{k_2} = M$, the following (i)-(vii) hold.

(i) *For every edge uv of Γ with $u \in A$ and $n(uv) = M$ we have $D_i^{i+1}(u, v) = \emptyset$ for $i \geq d$.*

(ii) *The signature of each vertex of Γ is $(M, M, \ldots, M)$, hence $n(e) = M$ for all $e \in E(\Gamma)$.*

(iii) *Every path on $d + 2$ vertices of Γ, starting in a vertex that is contained in A, is contained in a unique girth cycle;*

(iv) *If d is even and uv is an edge of Γ with $u \in A$, then $D_{i+1}^i(u, v) = \emptyset$ for $i \geq d$.*

(v) *if d is odd and uv is an edge of Γ with $u \in A$, then $D_{d+1}^d(u, v) \neq \emptyset$ and $D_{i+1}^i = \emptyset$ for $i \geq d + 1$.*

(vi) *if d is even, then Γ is the incidence graph of a generalized d-gon of order $(k_1 - 1, k_2 - 1)$;*

(vii) *if $d = 3$, then Γ is the incidence graph of a $2 - (k_1 k_2 - k_1 + 1, k_2, 1)$-design.*

Proof. Pick adjacent vertices $u \in A, v \in B$ such that $n(uv) = a_{k_1} = b_{k_2}$.

We prove the upper bound on a_{k_1} in the case when d is odd. The proof for the case when d is even is similar. By Proposition 2.6(vi) we have that $|D_{d-1}^d(u, v)| = (k_1 - 1)^{(d-1)/2}(k_2 - 1)^{(d-1)/2}$. As $D_{d-1}^d(u, v) \subseteq B$ and as every vertex from $D_{d-1}^d(u, v)$ has exactly one neighbour in $D_{d-2}^{d-1}(u, v)$, it follows that every vertex from $D_{d-1}^d(u, v)$ has at most $k_2 - 1$ neighbours in $D_d^{d-1}(u, v)$. Therefore, there are at most

$(k_1-1)^{(d-1)/2}(k_2-1)^{(d+1)/2}$ edges between $D^d_{d-1}(u,v)$ and $D^{d-1}_d(u,v)$. The result now follows from Proposition 2.6(vii).

Now, suppose that $a_{k_1}=M$.

(i): By Proposition 2.6(vii), there are M edges between $D^d_{d-1}(u,v)$ and $D^{d-1}_d(u,v)$. Recall that by Proposition 2.6(iii), every vertex from $D^d_{d-1}(u,v)$ has exactly one neighbour in $D^{d-1}_{d-2}(u,v)$. It follows that every vertex from $D^d_{d-1}(u,v)$ has all other neighbours in D^{d-1}_d, and so $D^{d+1}_d(u,v)=\emptyset$. Consequently, $D^{i+1}_i(u,v)=\emptyset$ for every $i\geq d$.

(ii): Let $w\in D^2_1(u,v)$ be any vertex. Then we have that

$$D^{d-1}_d(v,w)=D^{d-2}_{d-1}(u,v)\cup\left(D^d_{d-1}(u,v)\setminus D^{d-2}_{d-1}(w,v)\right),$$

and, as $D^{d+1}_d(u,v)=\emptyset$ by (i) above, also

$$D^d_{d-1}(v,w)=D^{d-1}_d(u,v).$$

By Proposition 2.6(iv), the number of edges between $D^{d-2}_{d-1}(u,v)$ and $D^{d-1}_d(u,v)$ is equal to $|D^{d-2}_{d-1}(u,v)|(k_2-1)$ if d is odd, and to $|D^{d-2}_{d-1}(u,v)|(k_1-1)$ if d is even. As every vertex from $D^d_{d-1}(u,v)$ has exactly one neighbour in $D^{d-1}_{d-2}(u,v)$ and as $D^{d+1}_d(u,v)=\emptyset$, the number of edges between $\left(D^d_{d-1}(u,v)\setminus D^{d-2}_{d-1}(w,v)\right)$ and $D^{d-1}_d(u,v)$ is equal to

$$\left(|D^d_{d-1}(u,v)|-|D^{d-2}_{d-1}(w,v)|\right)(k_2-1) \text{ if } d \text{ is odd,}$$

and to

$$\left(|D^d_{d-1}(u,v)|-|D^{d-2}_{d-1}(w,v)|\right)(k_1-1) \text{ if } d \text{ is even.}$$

Observe that by Proposition 2.6(vi) we have that $|D^{d-2}_{d-1}(u,v)|=|D^{d-2}_{d-1}(w,v)|$, and so Proposition 2.6(vii) and the above comments imply that $n(vw)=(k_2-1)|D^d_{d-1}(u,v)|$ if d is odd and $n(vw)=(k_1-1)|D^d_{d-1}(u,v)|$ if d is even. Finally, Proposition 2.6(vi) implies that $n(vw)=M$. Hence the signature of v is $(M,M,\dots,M)$, so the girth-biregularity of Γ implies that $n(e)=M$ for all $e\in E(\Gamma)$.

(iii): Pick any path $x_0x_1\dots x_{d+1}$ with $x_0\in A$ and consider the sets $D^i_j(x_0,x_1)$. It follows from Proposition 2.6 that $x_i\in D^i_{i-1}$ for $1\leq i\leq d$. Recall that $n(x_0x_1)=M$ by (ii) above, and so $D^{d+1}_d(x_0,x_1)=\emptyset$ by (i) above. It follows that $x_{d+1}\in D^{d-1}_d$. The result now follows from Proposition 2.6(iii).

(iv): Recall that by (ii) above we have $n(uv)=M$, and so there are exactly M edges between $D^{d-1}_d(u,v)$ and $D^d_{d-1}(u,v)$. Recall also that by Proposition 2.6(iii), every vertex from $D^{d-1}_d(u,v)$ has exactly one neighbour in $D^{d-2}_{d-1}(u,v)$. It follows that every vertex from $D^{d-1}_d(u,v)$ has all other neighbours in D^d_{d-1}, and so $D^d_{d+1}(u,v)=\emptyset$. Consequently, $D^i_{i+1}(u,v)=\emptyset$ for every $i\geq d$.

(v): By Proposition 2.6(vi) we have $|D^{d-1}_d(u,v)|=(k_1-1)^{(d-1)/2}(k_2-1)^{(d-1)/2}$. As vertices of $D^{d-1}_d(u,v)$ have valency k_1, there are therefore $k_1(k_1-1)^{(d-1)/2}(k_2-1)^{(d-1)/2}$ edges going out of $D^{d-1}_d(u,v)$. As $n(uv)=M$ by (ii) above, $M=(k_1-1)^{(d-1)/2}(k_2-1)^{(d+1)/2}$ of these edges are between $D^{d-1}_d(u,v)$ and $D^d_{d-1}(u,v)$. By Proposition 2.6(iii), $(k_1-1)^{(d-1)/2}(k_2-1)^{(d-1)/2}$ of these edges are between $D^{d-1}_d(u,v)$ and $D^{d-2}_{d-1}(u,v)$. It follows that there are exactly $(k_1-k_2)(k_1-1)^{(d-1)/2}(k_2-1)^{(d-1)/2}$ edges between $D^{d-1}_d(u,v)$ and $D^d_{d+1}(u,v)$. As $k_1>k_2\geq 3$, this number is nonzero, implying that $D^d_{d+1}(u,v)\neq\emptyset$.

Assume now that $D^{d+1}_{d+2}(u,v) \neq \emptyset$. Pick $w \in D^{d+1}_{d+2}(u,v)$ and let $ux_1x_2\dots x_dw$ be arbitrary path between u and w such that $x_i \in D^i_{i+1}$ for $1 \le i \le d$. Note that this path is not contained in a girth cycle of Γ, contradicting (iii) above. Therefore $D^{d+1}_{d+2}(u,v) = \emptyset$ and consequently $D^i_{i+1}(u,v) = \emptyset$ for every $i \ge d+1$.
(vi): Observe that (i), (ii) and (iv) above implies that the diameter of Γ is d. As $k_1 > k_2 \ge 3$, Theorem 2.4 implies that Γ is the incidence graph of a generalized d-gon.
(vii): Finally, suppose that $d = 3$. We call the vertices in A *points* and the the vertices in B *lines* and we use the geometric terminology. We claim that there is a unique line through any pair of distinct points. As the girth of Γ is 6, there is at most one line through any pair of points. Pick now distinct points $x, y \in A$. Pick an arbitrary line z through x. It follows from (i) and (v) above, that either $y \in D^2_1(x,z)$ or $y \in D^2_3(x,z)$. If $y \in D^2_1(x,z)$, then z is the unique line through x and y. If however $y \in D^2_3(x,z)$, then, by Proposition 2.6(iii), there is a unique line $w \in D^1_2(x,z)$ which is adjacent to both x and y in Γ. Therefore, in this case w is the unique line through x and y. □

In the rest of this paper we use the following notation.

Notation 3.5. Let Γ be a girth-biregular graph with bipartition A, B, valencies $k_1 > k_2 \ge 3$, girth $g = 2d$, signatures $(a_1, a_2, \dots, a_{k_1})$ and $(b_1, b_2, \dots, b_{k_2})$. Let $M = (k_1-1)^{g/4}(k_2-1)^{g/4}$ if d is even, and $M = (k_1-1)^{(g-2)/4}(k_2-1)^{(g+2)/4}$ if d is odd and suppose that $a_{k_1} = M - \varepsilon$ for some $\varepsilon < k_2 - 1$. Let uv be an edge with $u \in A, v \in B$ and $n(uv) = a_{k_1}$, and let $D^i_j = D^i_j(u,v)$. Note that $D^i_i = \emptyset$ for every i and that there are no edges between D^{i-1}_i and D^i_{i-1} for $1 \le i \le d-1$.

For every $r \in D^d_{d-1}$ ($s \in D^{d-1}_d$, respectively) we let $h(r) = |\Gamma(r) \cap D^{d+1}_d|$ ($h(s) = |\Gamma(s) \cap D^d_{d+1}|$, respectively). Let $\{r_1, r_2, \dots, r_m\} \subseteq D^d_{d-1}$ be the set of vertices of D^d_{d-1}, for which the value of the function h is positive, that is, the set of vertices of D^d_{d-1}, that have a neighbour in D^{d+1}_d. Choose the indices in such a way that $h(r_i) \le h(r_j)$ for $i < j$. Similarly, let $\{s_1, s_2, \dots, s_n\} \subseteq D^{d-1}_d$ be the set of vertices of D^{d-1}_d, for which the value of the function h is positive. Again, choose the indices in such a way that $h(s_i) \le h(s_j)$ for $i < j$. We also set $\gamma = h(r_m)$, $\sigma = h(s_n)$, $\mu = h(r_1)$ and $\nu = h(s_1)$.

Proposition 3.6. *Suppose that $g = 2d$ with d even. With reference to* Notation 3.5*, we have*

$$\sum_{r \in D^d_{d-1}} h(r) = \sum_{i=1}^{m} h(r_i) = \sum_{s \in D^{d-1}_d} h(s) = \sum_{i=1}^{n} h(s_i) = \varepsilon. \tag{3.1}$$

Proof. The first and the third of the above equalities are clear. We now prove that $\sum_{i=1}^n h(s_i) = \varepsilon$. The proof that $\sum_{i=1}^m h(r_i) = \varepsilon$ is similar. Let $\mathcal{E}$ denote the set of edges, that have one endpoint in D^{d-1}_d, and the other endpoint in D^d_{d+1}. Note that $\mathcal{E} = \sum_{i=1}^n h(s_i)$, and so it is enough to prove $|\mathcal{E}| = \varepsilon$. As d is even, it follows from Proposition 2.6(vi) that $|D^{d-1}_d| = (k_1-1)^{d/2}(k_2-1)^{(d-2)/2}$. As $D^{d-1}_d \subseteq B$, there are total $(k_1-1)^{d/2}(k_2-1)^{(d-2)/2}k_2$ edges, having one endpoint in D^{d-1}_d. By Proposition 2.6(iii), $(k_1-1)^{d/2}(k_2-1)^{(d-2)/2}$ of these edges have the other endpoint in D^{d-2}_{d-1}. Since $a_k = M - \varepsilon$, it follows from Proposition 2.6(vii) that there are $(k_1-1)^{d/2}(k_2-1)^{d/2} - \varepsilon$ edges between D^{d-1}_d and D^d_{d-1}. Combining these observations, we get the desired result. □

Lemma 3.7. *Suppose that $g = 2d$ with d even. With reference to* Notation 3.5*, we have $m \ge \sigma$ and $n \ge \gamma$.*

Proof. Set $\Gamma(u) \setminus \{v\} = \{u_1, u_2, \ldots, u_{k_1-1}\}$ and $\Gamma(v) \setminus \{u\} = \{v_1, v_2, \ldots, v_{k_2-1}\}$. Moreover, for $1 \leq i \leq k_1 - 1$ ($1 \leq i \leq k_2 - 1$, respectively) set $U_i = \Gamma_{d-2}(u_i) \cap D_d^{d-1}$ ($V_i = \Gamma_{d-2}(v_i) \cap D_{d-1}^d$, respectively). Note that as girth of Γ is $2d$, the sets U_i (V_i, respectively) are pairwise disjoint, and $|U_i| = |V_i| = (k_1 - 1)^{(d-2)/2}(k_2 - 1)^{(d-2)/2}$. Moreover, each $r \in D_{d-1}^d$ ($s \in D_d^{d-1}$, respectively) could have at most one neighbour in U_i (V_i, respectively) for each i. It is now clear that if $s \in D_d^{d-1}$ has no neighbours in V_i for some $1 \leq i \leq k_2 - 1$, then there is at least one vertex $r \in V_i$ with $h(r) \geq 1$. It follows $m \geq \sigma$. Similarly we show that $n \geq \gamma$. □

Equation (3.1) and Lemma 3.7 obviously imply the following inequalities:

$$\mu\sigma \leq \mu m \leq \varepsilon, \quad \nu\gamma \leq \nu n \leq \varepsilon. \tag{3.2}$$

If $\gamma \leq \sigma$, then observe also that it follows from the above comments that

$$\mu^2 \leq \mu\gamma \leq \mu\sigma \leq \mu m \leq \varepsilon,$$

while if $\sigma \leq \gamma$ then

$$\nu^2 \leq \nu\sigma \leq \nu\gamma \leq \nu n \leq \varepsilon.$$

This shows that if $\gamma \leq \sigma$ then $\mu \leq \sqrt{\varepsilon}$, while if $\sigma \leq \gamma$ then $\nu \leq \sqrt{\varepsilon}$.

First, we give a lower bound on a_1 using the vertex u.

Lemma 3.8. *With reference to* Notation 3.5 *we have that*

$$a_1 \geq (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\max\{(k_2-1-\sigma)(k_1-1), (k_1-1-\gamma)(k_2-1)\} - \varepsilon. \tag{3.3}$$

Proof. We prove that $a_1 \geq (k_1 - 1)^{d/2}(k_2 - 1)^{(d-2)/2}(k_2 - 1 - \sigma) - \varepsilon$. The proof of $a_1 \geq (k_1 - 1)^{(d-2)/2}(k_2 - 1)^{d/2}(k_1 - 1 - \gamma) - \varepsilon$ is similar.

Recall that $n(uv) = a_k$ and that $D_j^i = D_j^i(u, v)$. Let $s \neq v$ be a neighbour of u such that $n(us) = a_1$. Abbreviate $K = D_d^{d-1} \cap \Gamma_{d-2}(s)$. For $s' \in K$ abbreviate $L(s') = D_{d-1}^d \cap \Gamma(s')$. Note that as girth of Γ is $2d$, we have that sets $L(s')$ are pairwise disjoint, and so by (3.1) we have that

$$\sum_{s' \in K} \sum_{r' \in L(s')} h(r') \leq \varepsilon.$$

Pick $r' \in L(s')$ and observe that for each $\tilde{r} \in (\Gamma(r') \cap (D_d^{d-1} \cup D_{d-2}^{d-1})) \setminus \{s'\}$, there is a unique girth cycle containing the arc us and the 2-arc $s'r'\tilde{r}$. Note that

$|K| = (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}$, and so, by (3.1), we have

$$\begin{aligned} a_1 = n(us) &\geq \sum_{s'\in K}\sum_{r'\in L(s')}(k_1-1-h(r')) \\ &= \sum_{s'\in K}\sum_{r'\in L(s')}(k_1-1) - \sum_{s'\in K}\sum_{r'\in L(s')} h(r') \\ &\geq (k_1-1)\sum_{s'\in K}(k_2-1-h(s')) - \varepsilon \\ &\geq (k_1-1)\sum_{s'\in K}(k_2-1-\sigma) - \varepsilon \\ &= (k_1-1)(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}(k_2-1-\sigma) - \varepsilon \\ &= (k_1-1)^{d/2}(k_2-1)^{(d-2)/2}(k_2-1-\sigma) - \varepsilon. \end{aligned}$$ □

4 The case $g = 4$

In this section we consider the case $g = 4$. Throughout this section we will use Notation 3.5. Recall that m (n, respectively) denotes the number of vertices of D^d_{d-1} (D^{d-1}_d, respectively), for which the value of the function h is positive.

Lemma 4.1. *Assume that $g = 4$ and $\varepsilon \geq 1$. Pick $1 \leq i \leq n$, $1 \leq j \leq m$, $w \in \Gamma(s_i)\cap D^2_3$ and $\tilde{w} \in \Gamma(r_j)\cap D^3_2$. Then the following* (i) – (iv) *holds.*

- (i) *There are at most $(h(s_i)-1)(k_1-1)$ girth cycles of the form (w, s_i, x, y, w) such that $x \in \Gamma(s_i)\cap D^2_3$.*
- (ii) *There are at most ε girth cycles of the form (w, s_i, x, y, w) such that $x \in D^2_1$ and $y \notin D^1_2$.*
- (iii) *There are at most $(h(r_j)-1)(k_2-1)$ girth cycles of the form $(\tilde{w}, r_j, x, y, \tilde{w})$ such that $x \in \Gamma(r_j)\cap D^3_2$.*
- (iv) *There are at most ε girth cycles of the form $(\tilde{w}, r_j, x, y, \tilde{w})$ such that $x \in D^1_2$ and $y \notin D^2_1$.*

Proof. (i): Note that there are $h(s_i)-1$ choices for x, and for each such choice there are at most k_1-1 choices for y. The result follows.
(ii): As $x \in D^2_1$ and $y \notin D^1_2$, it follows that $y \in D^3_2$. It follows from Proposition 3.6 that there are at most ε choices for the edge xy. For each such edge xy there is clearly at most one girth cycle containing also the edge ws_i. The result follows.
(iii), (iv): Similar to the proofs of (i) and (ii) above. □

Lemma 4.2. *Assume that $g = 4$ and $\varepsilon \geq 1$. Then $m \geq 2$ and $n \geq 2$.*

Proof. We prove that $n \geq 2$. The proof that $m \geq 2$ is similar. Suppose on the contrary that $n = 1$. Note that in this case $\sigma = \nu = \varepsilon$, $\gamma = 1$, $m = \varepsilon$ and $h(r_i) = 1$ for $1 \leq i \leq m$. Let w be the unique neighbour of r_1 in D^3_2. Let $t = |\Gamma(w)\cap D^2_1|$ and note that $t \leq m = \varepsilon$. Note that the girth cycles containing the edge r_1w are exactly the cycles of form (w, r_1, x, y, w), where $x \in \{v\}\cup(D^1_2\setminus\{s_1\})$ and $y \in (\Gamma(w)\cap D^2_1)\setminus\{r_1\}$.

Therefore, $n(r_1w) \leq (k_1-1)(t-1) \leq (k-1)(\varepsilon-1)$. Since $\gamma = 1$ and $\sigma = \varepsilon$, we have by Lemma 3.8 that

$$a_1 \geq \max\{(k_2-1-\varepsilon)(k_1-1), (k_1-2)(k_2-1)\} - \varepsilon \geq (k_1-2)(k_2-1) - \varepsilon,$$

and so

$$(k_1-2)(k_2-1) - \varepsilon \leq a_1 \leq n(r_1w) \leq (k_1-1)(\varepsilon-1).$$

It follows that $k_1k_2 - 2k_2 + 1 \leq k_1\varepsilon$, and so

$$k_2 - 2 + \frac{1}{k_1} \leq k_2 - \frac{2k_2}{k_1} + \frac{1}{k_1} \leq \varepsilon < k_2 - 1,$$

contradicting the fact that ε is an integer. □

We now give an upper bound for a_1.

Lemma 4.3. *Assume that $g = 4$ and $\varepsilon \geq 1$. Let $\alpha = h(s_{n-1})$ and $\beta = h(r_{m-1})$. Then*

$$a_1 \leq (\alpha-1)(k_1-1) + \varepsilon + (k_2-\alpha)(\varepsilon-\alpha-\sigma+1). \tag{4.1}$$

and

$$a_1 \leq (\beta-1)(k_2-1) + \varepsilon + (k_1-\beta)(\varepsilon-\beta-\gamma+1). \tag{4.2}$$

Proof. We prove inequality (4.1). The proof of inequality (4.2) is similar. Let $\{w_1, \dots, w_\alpha\} = \Gamma(s_{n-1}) \cap D_3^2$. We estimate $n(s_{n-1}w_1)$. To do this we split the girth cycles $(w_1, s_{n-1}, x, y, w_1)$ into two types depending on the vertex x. We say that the girth cycle is of type 1 if $x \in \{w_2, \dots, w_\alpha\}$, and of type 2 if $x \in \{u\} \cup D_1^2$. By Lemma 4.1(i) there are at most $(\alpha-1)(k_1-1)$ girth cycles of type 1. To estimate the number of girth cycles of type 2, we further split these girth cycles into two subfamilies depending on the vertex y. Let us say that the girth cycle $(w_1, s_{n-1}, x, y, w_1)$ with $x \in \{u\} \cup D_1^2$ is of type 2a if $y \in D_2^1$, and of type 2b if $y \in D_2^3$.

If the girth cycle is of type 2b, then $x \in D_1^2$, and so by Lemma 4.1(ii) there are at most ε such girth cycles. To estimate the number of girth cycles of type 2a, observe that s_{n-1} has $k_2 - \alpha$ neighbours in $\{u\} \cup D_1^2$, and that w_1 has at most $\varepsilon - \alpha - \sigma + 1$ neighbours in $D_2^1 \setminus \{s_{n-1}\}$. This shows that the number of girth cycles of type 2a is at most $(k_2-\alpha)(\varepsilon - \alpha - \sigma + 1)$. As $a_1 \leq n(s_{n-1}w_1)$, the result follows. □

Lemma 4.4. *Assume that $g = 4$ and $\varepsilon \geq 1$. Then $\varepsilon = k_2 - 2$ and $k_2 - 1 \geq 2k_1/3$.*

Proof. As in Lemma 4.3, let $\alpha = h(s_{n-1})$. Then, by Lemmas 3.8 and 4.3, we get that

$$(k_1-1)(k_2-1) - \sigma(k_1-1) - \varepsilon \leq (\alpha-1)(k_1-1) + \varepsilon + (k_2-\alpha)(\varepsilon-\alpha-\sigma+1).$$

Rearranging the above inequality we find this is equivalent to

$$(k_1-1)(k_2-1) \leq (k_1-k_2-1+\alpha)(\alpha+\sigma-1) + \varepsilon(k_2-\alpha+2). \tag{4.3}$$

Taking into account that $\alpha + \sigma \leq \varepsilon$ and that $\alpha \geq 1$, inequality (4.3) implies that

$$(k_1-1)(k_2-1) \leq (k_1+1)\varepsilon - k_1 + 1 + k_2 - \alpha \leq (k_1+1)\varepsilon - k_1 + k_2,$$

and so

$$\varepsilon \geq \frac{(k_1-1)(k_2-1)+k_1-k_2}{k_1+1} = k_2 - \frac{3k_2-1}{k_1+1}. \tag{4.4}$$

As $k_1 \geq k_2$, the above inequality yields

$$\varepsilon \geq k_2 - \frac{3k_1-1}{k_1+1} = k_2 - 3 + \frac{4}{k_1+1} > k_2 - 3.$$

Recall that $\varepsilon < k_2 - 1$ by assumption, and so $\varepsilon = k_2 - 2$ as claimed. Plugging $\varepsilon = k_2 - 2$ into (4.4) we easily get that $k_2 - 1 \geq 2k_1/3$. □

Theorem 4.5. *Assume that $g = 4$. Then $\varepsilon = 0$ and Γ is the complete bipartite graph K_{k_1,k_2}.*

Proof. Suppose on the contrary that $\varepsilon \geq 1$. Recall that $\varepsilon = k_2 - 2$. As in Lemma 4.3, let $\beta = h(r_{m-1})$. Then, by Lemmas 3.8 and 4.3, we get that

$$(k_1-1)(k_2-1) - \gamma(k_2-1) - \varepsilon \leq (\beta-1)(k_2-1) + \varepsilon + (k_1-\beta)(\varepsilon-\beta-\gamma+1).$$

Rearranging the terms of the above inequality we get

$$(k_1-1)(k_2-1) \leq \varepsilon(k_1-\beta+2) + (\beta+\gamma-1)(k_2-k_1+\beta-1). \tag{4.5}$$

If $\beta = 1$, then inequality (4.5) together with $\varepsilon = k_2 - 2$ yields $k_1 - 1 \leq \gamma(k_2 - k_1)$. But this is a contradiction as $k_1 > k_2 > 0$.
If $k_2 - k_1 + \beta - 1 \leq 0$, then inequality (4.5) together with $\varepsilon = k_2 - 2$ and $\beta \geq 2$ yields

$$(k_1-1)(k_2-1) \leq (k_2-2)(k_1-\beta+2) \leq (k_2-2)k_1,$$

implying $k_1 \leq k_2 - 1$, a contradiction.
Therefore, we have that $k_2 - k_1 + \beta - 1 > 0$ and $\beta \geq 2$. Recall that $\beta + \gamma \leq \varepsilon = k_2 - 2$, and so inequality (4.5) gives us

$$(k_1-1)(k_2-1) \leq \varepsilon(k_1-\beta+2)+(\varepsilon-1)(k_2-k_1+\beta-1) = (k_2-2)(k_2+1)-k_2+k_1-\beta+1.$$

It follows that $2 \leq \beta \leq k_2^2 - k_2 - 2 + 2k_1 - k_1k_2$, or

$$k_1(k_2-2) \leq k_2^2 - k_2 - 4.$$

As $k_1 \geq k_2 + 1$ this yields $-2 \leq -4$, a contradiction. This shows that $\varepsilon = 0$ as claimed. It is now easy to see that Γ is isomorphic to the complete bipartite graph K_{k_1,k_2}. □

5 The case $g = 2d \geq 8$, where d is even

In this section we study girth-biregular graphs with girth $g = 2d \geq 8$, d even. Throughout this section we will use Notation 3.5. Assume that $g = 2d \geq 8$. For every $z \in D_1^2$ we define

$$\beta(z) = \sum_{r \in D_{d-1}^d \cap \Gamma_{d-2}(z)} h(r).$$

Note that for $z \in D_1^2$ we have $|D_{d-1}^d \cap \Gamma_{d-2}(z)| = (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}$ and that for $z, z' \in D_1^2$ $(z \neq z')$, the sets $D_{d-1}^d \cap \Gamma_{d-2}(z)$ and $D_{d-1}^d \cap \Gamma_{d-2}(z')$ are disjoint as the girth of Γ is $2d$. Therefore,

$$\sum_{z \in D_1^2} \beta(z) = \sum_{r \in D_{d-1}^d} h(r) = \varepsilon. \tag{5.1}$$

In particular, $\beta(z) \leq \varepsilon$. Recall also that for an edge e of Γ we denoted by $n(e)$ the number of girth cycles passing through e.

Lemma 5.1. *Assume that $g = 2d \geq 8$ and $\varepsilon \geq 1$. Then*

$$a_1 \geq (k_1-1)^{d/2}(k_2-1)^{d/2} - k_2\varepsilon.$$

Proof. Abbreviate $\ell = (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}$. Pick $z \in D_1^2$ with $n(vz) = a_1$ and let $w_1, w_2, \dots, w_\ell$ be the vertices of $D_{d-1}^d \cap \Gamma_{d-2}(z)$. For $1 \leq j \leq \ell$ consider the $2d$-cycles of the form $(v, z, \dots, w_j, b, r, r', \dots)$ with $b \in D_d^{d-1}$, where $(v, z, \dots, w_j)$ is the unique path from v to w_j of length $d-1$. Observe that for fixed w_j and r, there is only one such cycle (recall that as $g \geq 8$, w_j and r have a unique common neighbour), and that for fixed w_j and b, we could choose r in $k_2 - 1 - h(b)$ different ways. Therefore,

$$\begin{aligned} a_1 = n(vz) \geq & \sum_{j=1}^{\ell} \sum_{b \in \Gamma(w_j) \cap D_d^{d-1}} (k_2 - 1 - h(b)) \\ = & \sum_{j=1}^{\ell} \sum_{b \in \Gamma(w_j) \cap D_d^{d-1}} (k_2-1) - \sum_{j=1}^{\ell} \sum_{b \in \Gamma(w_j) \cap D_d^{d-1}} h(b). \end{aligned} \tag{5.2}$$

Furthermore, observe that for a fixed w_j we could choose b in $(k_1 - 1 - h(w_j))$ different ways, and so

$$\sum_{j=1}^{\ell} \sum_{b \in \Gamma(w_j) \cap D_d^{d-1}} (k_2-1) = (k_2-1) \sum_{j=1}^{\ell} (k_1-1-h(w_j)) = \ell(k_1-1)(k_2-1) - (k_2-1)\beta(z).$$

Finally, the sets $\Gamma(w_j) \cap D_d^{d-1}$ and $\Gamma(w_\ell) \cap D_d^{d-1}$ are disjoint if $j \neq \ell$ (otherwise we would get a cycle of length $2d-2$), and so

$$\sum_{j=1}^{\ell} \sum_{b \in \Gamma(w_j) \cap D_d^{d-1}} h(b) \leq \sum_{b \in D_d^{d-1}} h(b) = \varepsilon.$$

This, together with $\beta(z) \leq \varepsilon$, shows that

$$a_1 = n(vz) \geq \ell(k_1-1)(k_2-1) - (k_2-1)\beta(z) - \varepsilon \geq (k_1-1)^{d/2}(k_2-1)^{d/2} - k_2\varepsilon. \quad \square$$

Lemma 5.2. *Assume that $g = 2d \geq 8$ and $\varepsilon \geq 1$. Then*

$$a_1 < (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\big(k_1\varepsilon - k_1 + 2\big).$$

Proof. Let
$$D=\bigcup_{i=0}^{d-1}\left(D_{i+1}^{i}\cup D_{i}^{i+1}\right).$$
For vertices $x,y\in D$, let $d_D(x,y)$ denote the distance between x and y in the subgraph $\Gamma[D]$, that is, in the subgraph of Γ, that is induced by D. Observe that $d_D(x,y)\le 2d-1$ for all $x,y\in D$.

Pick a vertex $r\in D_{d-1}^{d}$ with $h(r)\ge 1$ and abbreviate $\alpha=h(r)$. Pick $w\in\Gamma(r)\cap D_d^{d+1}$ and consider the set C of $2d$-cycles $(x_0=w,x_1=r,x_2,\ldots,x_{2d-1},w)$ through wr. Note that, as $w\notin D$ at most $2d-2$ edges of such a cycle have both endpoints in D. For $1\le i\le 2d-1$ let C_i denote the subset of C defined as follows. A cycle $(x_0=w,x_1=r,x_2,\ldots,x_{2d-1},w)$ is an element of C_i if and only if $\{x_1,\ldots,x_i\}\subseteq D$ and $x_{i+1}\notin A$, where the addition in subscripts is computed modulo $2d$. For example, cycles in C_1 are those $2d$-cycles $(x_0=w,x_1=r,x_2,\ldots,x_{2d-1},w)$, for which $x_2\notin D$, while cycles in C_{2d-1} are those for which $\{x_1,x_2,\ldots,x_{2d-1}\}\subseteq D$. Note that the sets C_i are pairwise disjoint, and so
$$a_1\le n(wr)\le |C_1|+|C_2|+\cdots+|C_{2d-1}|.$$
Let us now estimate the above sum. To do this we introduce the following notation. For $i\in\{1,3,\ldots,2d-1\}$ we define
$$\varepsilon_i=\sum_{\substack{b\in D_d^{d-1}\\ d_D(r,b)=i}}h(b).$$
Note that as $\Gamma[D]$ is bipartite with diameter at most $2d-1$, we have that
$$\varepsilon_1+\varepsilon_3+\cdots+\varepsilon_{2d-1}=\varepsilon.$$
We also define
$$\kappa=|\Gamma(w)\cap(D_{d-1}^{d}\setminus\{r\})|=|\Gamma(w)\cap D_{d-1}^{d}|-1.$$
Note that $\alpha+\kappa\le\varepsilon$.

Consider a $2d$-cycle $(x_0=w,x_1=r,x_2,\ldots,x_{2d-1},w)\in C_1$. Observe that there are $\alpha-1$ choices for x_2. For each such choice of x_2, there are, by Lemma 3.1, at most $(k_1-1)^{(d-2)/2}(k_2-1)^{d/2}$ girth cycles containing both edges wr and rx_2. Therefore,
$$|C_1|\le(\alpha-1)(k_1-1)^{(d-2)/2}(k_2-1)^{d/2}.$$

Consider a $2d$-cycle $(x_0=w,x_1=r,x_2,\ldots,x_{2d-1},w)\in C_2\cup C_4\cup\cdots\cup C_{2d-2}$. Assume that this cycle is an element of C_{2j} $(1\le j\le d-1)$. Observe that in this case we have that $x_{2j}\in D_d^{d-1}$ and that $d_D(r,x_{2j})=2j-1$ (otherwise there would be a cycle of length less than $2d$). Therefore, we could choose an edge $x_{2j}x_{2j+1}$ in ε_{2j-1} different ways. For each such choice of an edge $x_{2j}x_{2j+1}$, there are, by Lemma 3.1, at most $(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}$ girth cycles containing edges wr and $x_{2j}x_{2j+1}$, and so
$$\begin{aligned}|C_2|+|C_4|+\cdots+|C_{2d-2}|&\le(\varepsilon_1+\varepsilon_3+\cdots+\varepsilon_{2d-3})(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\\&=\varepsilon(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}.\end{aligned}$$

Consider a $2d$-cycle $(x_0 = w, x_1 = r, x_2, \dots, x_{2d-1}, w) \in C_3 \cup C_5 \cup \cdots \cup C_{2d-3}$. If this cycle is an element of C_{2j+1} $(1 \le j \le d-2)$, then it is easy to see that $x_{2j+1} \in D^d_{d-1}$, and so $x_{2j+2} \in D^{d+1}_d \setminus \{w\}$. Therefore, there are at most $\varepsilon - \kappa - \alpha$ choices for an edge $x_{2j+1}x_{2j+2}$. For each such choice there are, by Lemma 3.1, at most $(k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}$ girth cycles containing edges wr and $x_{2j+1}x_{2j+2}$, and so

$$|C_3| + |C_5| + \cdots + |C_{2d-3}| \le (\varepsilon - \kappa - \alpha)(k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}.$$

Finally, consider a $2d$-cycle $(x_0 = w, x_1 = r, x_2, \dots, x_{2d-1}, w) \in C_{2d-1}$. Note that we have at most $k_1 - \alpha$ choices for a vertex x_2. For each choice of vertices $x_2, x_3, \dots, x_{i-1}$, where $i \le d$, we have at most $k_1 - 1$ choices for vertex x_i if i is even, and $k_2 - 1$ choices for x_i if i is odd. Therefore, there are at most $(k_1-\alpha)(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}$ choices for vertices $x_2, x_3, \dots, x_d$. On the other hand, there are at most κ choices for a vertex x_{2d-1}. For each such choice of vertices $x_2, x_3, \dots, x_d$ and x_{2d-1}, there is at most one girth cycle containing the edges wr, rx_2, $x_2x_3, \dots x_{d-1}x_d$ and $x_{2d-1}w$. Therefore,

$$|C_{2d-1}| \le \kappa(k_1-\alpha)(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}.$$

To further estimate the sum $|C_1| + |C_2| + \cdots + |C_{2d-1}|$, we first note that

$$\begin{aligned}
|C_1| + |C_{2d-1}| &\le (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2} \\
&\qquad \Big((\alpha-1)(k_1-1)(k_2-1) + \kappa(k_1-\alpha)(k_1-1)\Big) \\
&< (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2} \\
&\qquad \Big((\alpha-1)(k_1-1)^2 + \kappa(k_1-\alpha)(k_1-1)\Big) \\
&= (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2} \\
&\qquad \Big((\alpha-1+\kappa)(k_1-1)^2 - \kappa(\alpha-1)(k_1-1)\Big) \\
&\le (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}(\alpha-1+\kappa)(k_1-1)^2 \\
&\le (k_1-1)^{d/2}(k_2-1)^{(d-2)/2}(\varepsilon-1),
\end{aligned}$$

while

$$\begin{aligned}
|C_2| + |C_3| + \cdots + |C_{2d-2}| &\le (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}(\varepsilon(k_1-1) + (\varepsilon-\kappa-\alpha)) \\
&\le (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}\big(\varepsilon(k_1-1) + \varepsilon - 1\big) \\
&= (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}(k_1\varepsilon - 1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_1 \le n(wr) &\le |C_1| + |C_2| + \cdots + |C_{2d-1}| \\
&\le (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}\big((\varepsilon-1)(k_1-1)^2 + k_1\varepsilon - 1\big) \\
&= (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}\big(\varepsilon(k_1^2-k_1) + \varepsilon - (k_1-1)^2 - 1\big) \\
&< (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}\big(\varepsilon(k_1^2-k_1) + (k_2-1) - (k_1-1)^2\big) \\
&< (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}\big(\varepsilon(k_1^2-k_1) + (k_1-1) - (k_1-1)^2\big) \\
&= (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\big(k_1\varepsilon - k_1 + 2\big).
\end{aligned}$$

The result follows. □

Theorem 5.3. *Assume that $g = 2d \geq 8$ and d is even. Then $\varepsilon = 0$ and Γ is the incidence graph of a finite thick generalized d-gon, hence either $d = 4$ or $d = 8$.*

Proof. Suppose first that ε is positive. By Lemma 5.1 and 5.2 we have

$$(k_1-1)^{d/2}(k_2-1)^{d/2} - k_2\varepsilon \leq a_1 < (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\big(k_1\varepsilon - k_1 + 2\big).$$

This implies

$$\begin{aligned} k_2 - 1 > \varepsilon &> \frac{(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}(k_1k_2-k_2-1)}{k_2 + k_1(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}} \\ &> \frac{(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}(k_1k_2-k_2-1)}{k_1\big(1+(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\big)} \\ &= k_2 - 2 + \frac{(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}(2k_1-k_2-1) - k_1(k_2-2)}{k_1\big(1+(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\big)} \end{aligned}$$

As $k_1(k_2-2) < (k_1-1)(k_2-1) < (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}(2k_1-k_2-1)$, the above inequality implies

$$k_2 - 1 > \varepsilon > k_2 - 2,$$

contradicting the fact that ε is an integer. Therefore, $\varepsilon = 0$. □

6 The case $g = 2d$, where d is odd

In this section we consider the case $g = 2d$ with d odd, in particular the case $g = 6$ when we provide a characterization of affine planes. Unfortunately, the method we applied in the proof of Lemma 5.2 for giving an upper estimate on b_1 does not work for odd d, but we can calculate the exact value of b_1 if $\varepsilon = 1$. Throughout this section we will use Notation 3.5.

Theorem 6.1. *Assume that d is odd and suppose that $a_{k_1} = b_{k_2} = M - 1$. Then $b_1 = M - k_2 + 1$ and $b_2 = \cdots = b_{k_2} = M - 1$.*

Proof. Pick adjacent vertices $u \in A, v \in B$ such that $n(uv) = a_{k_1} = b_{k_2} = M - 1$. Let D_j^i denote $D_j^i(u,v)$ and

$$D = \bigcup_{i=0}^{d-1} \big(D_{i+1}^i \cup D_i^{i+1}\big).$$

For vertices $x, y \in D$, let $d_D(x,y)$ denote the distance between x and y in the subgraph $\Gamma[D]$, that is, in the subgraph of Γ, that is induced by D. Observe that $d_D(x,y) \leq 2d-1$ for all $x, y \in D$.

By Proposition 2.6(vi) and (vi) we have that

$$|D_d^{d-1}| = |D_{d-1}^d| = (k_1-1)^{(g-2)/4}(k_2-1)^{(g-2)/4} = \frac{M}{k_2-1},$$

and there are $M-1$ edges between D_d^{d-1} and D_{d-1}^d. Hence all but one vertices in D_{d-1}^d have $k_2 - 1$ neighbours in D_d^{d-1}. Let $p \in D_{d-1}^d$ denote the unique vertex which has only $k_2 - 2$ neighbours in D_d^{d-1}.

We claim that all but one vertices in D_d^{d-1} have $k_2 - 1$ neighbours in D_{d-1}^d, too. Let x be any vertex in D_d^{d-1}. Then for each vertex $y \in D_1^2$ there is at most one vertex $z \in D_{d-1}^d \cap \Gamma(x)$ so that $d(y, z) = d - 2$, because otherwise a cycle of length $2(d-1)$ would appear. Thus

$$|\Gamma(x) \cap D_{d-1}^d| \le |D_1^2| = k_2 - 1.$$

This implies, by the pigeonhole principle, that there is a unique vertex $r \in D_d^{d-1}$ which has only $k_2 - 2$ neighbours in D_{d-1}^d. Then r has one neighbour in D_{d-1}^{d-2} and it has $k_1 - k_2 + 1$ neighbours outside D.

Now, let w be an arbitrary vertex in D_1^2 and let $S = D_{d-1}^d \setminus D_{d-1}^{d-2}(w, v)$. Then

$$D_{d-1}^d(w, v) = D_{d-1}^{d-2} \cup S.$$

We now describe the set $D_d^{d-1}(w, v)$. Observe that

$$D_d^{d-1}(w, v) \subseteq D_d^{d-1} \cup \{p_1\}, \tag{6.1}$$

where p_1 is the unique neighbour of p outside D. There are two possibilities we have to consider, namely either w is the unique vertex of D_1^2 for which $d_D(p, w) = d - 2$, or $d_D(p, w) = d$. Let us first consider the case $d_D(p, w) = d - 2$. Note that in this case $p_1 \in D_d^{d-1}(w, v)$, so there is a unique vertex $w_1 \in D_d^{d-1}$ which is not contained in $D_d^{d-1}(w, v)$. Observe that every vertex from D_d^{d-1}, which has $k_2 - 1$ neighbours in D_{d-1}^d, is at distance $d - 1$ from w, and so $w_1 = r$. Therefore (6.1) implies

$$D_d^{d-1}(w, v) = \big(D_d^{d-1} \setminus \{r\}\big) \cup \{p_1\}.$$

We now count the number of neighbours between $D_d^{d-1}(w, v)$ and $D_{d-1}^d(w, v)$. Recall that each vertex from D_d^{d-1} has a unique neighbour in D_{d-1}^{d-2} and that each vertex from $D_d^{d-1} \setminus \{r\}$ has $k_2 - 1$ neighbours in D_{d-1}^d. Pick $x \in D_d^{d-1} \setminus \{r\}$. As $x \in D_d^{d-1}(w, v)$, x has at least one neighbour in $D_{d-1}^d \setminus S$. On the other hand, if x has more than one neighbour in $D_{d-1}^d \setminus S$, then this would imply a cycle of length $2(d-1)$, a contradiction. Using the above observations we now have

$$\begin{aligned} n(vw) &= \big(|D_d^{d-1}| - 1\big)(k_2 - 1) \\ &= \big((k_1 - 1)^{(d-1)/2}(k_2 - 1)^{(d-1)/2} - 1\big)(k_2 - 1) \\ &= M - k_2 + 1. \end{aligned}$$

In the case when $d_D(p, w) = d - 2$ we have that $d(w, p_1) = d + 1$ (note that p is the only neighbour of p_1 in D), and so by (6.1) we have $D_d^{d-1}(w, v) = D_d^{d-1}$. Observe also that $|S| = |D_{d-1}^d| - |D_{d-1}^{d-2}(w, v)| = (k_1 - 1)^{(d-1)/2}(k_2 - 1)^{(d-3)/2}(k_2 - 2)$. Similar arguments as in the previous case now show that

$$\begin{aligned} n(vw) &= |D_d^{d-1}| + |S|(k_2 - 1) - 1 \\ &= (k_1 - 1)^{(d-1)/2}(k_2 - 1)^{(d-1)/2} + (k_1 - 1)^{(d-1)/2}(k_2 - 1)^{(d-1)/2}(k_2 - 2) - 1 \\ &= M - 1. \end{aligned}$$

This proves the statement. □

Theorem 6.2. *Assume that d is odd and k_2 does not divide k_1. If $a_{k+1} = b_k = M - \varepsilon$ for a non-negative integer $\varepsilon \le 1$, then $\varepsilon = 0$ and $a_1 = \cdots = a_{k+1} = b_1 = \cdots = b_k = M$.*

Proof. We first assume $\varepsilon = 1$ and derive a contradiction. If $\varepsilon = 1$, then it follows from Theorem 6.1 that the signature of any vertex from B is $(M - k_2 + 1, M - 1, \ldots, M - 1)$. Now Proposition 3.2 yields that the signature of any vertex from A is $(M - k_2 + 1, \ldots, M - k_2 + 1, M - 1, \ldots, M - 1)$. Let $a = M - k_2 + 1$ and let a_A and a_B be as in Proposition 3.3. Observe that $a_B = 1$ and so we have $k_2 a_A = k_1$ by Proposition 3.3. Hence k_1 is divisible by k_2, a contradiction. Therefore $\varepsilon = 0$ and the result now follows from Theorem 3.4. □

In particular, we consider the case $k_1 - 1 = k_2 = k$ and $d = 3$. Then $k_1 k_2 - k_1 + 1 = k^2$ and it is well-known that a $2-(k^2, k, 1)$ design is a finite affine plane of order k. Combining Theorems 3.4(vii) and 6.2 we get the following characterization.

Corollary 6.3. *Assume that $k_1 - 1 = k_2 = k$ and that $d = 3$. If $a_{k+1} = b_k = M - \varepsilon$ for a non-negative integer $\varepsilon \le 1$, then $\varepsilon = 0$ and Γ is the incidence graph of a finite affine plane of order k.*

7 Examples

In this section we provide some examples where a_{k_1} is close to the upper bound given in Theorem 3.4. In all cases, the signatures of the points are constants, hence each edge is contained in the same number of girth cycles. So our examples are edge-girth-regular graphs, too. Let us start with the $g = 4$ case.

Example 7.1. Let $f_1 > f_2 \ge 1$ and $h > 2$ be integers and consider the complete bipartite graph $\Gamma' = K_{f_1 h, f_2 h}$ with bipartition A and B. Label the vertices so that

$$A = \bigcup_{i=1}^{f_1} \{u_{1,i}, u_{2,i}, \ldots, u_{h,i}\}, \quad B = \bigcup_{j=1}^{f_2} \{v_{1,j}, v_{2,j}, \ldots, v_{h,j}\}.$$

Let Γ denote a graph that is obtained from Γ' by deleting all edges of the form $u_{\ell,i} v_{\ell,j}$, where $\ell \in \{1, 2, \ldots, h\}$, $i \in \{1, 2, \ldots, f_1\}$ and $j \in \{1, 2, \ldots, f_2\}$. Then Γ is a bipartite biregular graph with $g = 4$, $k_1 = f_2(h-1)$ and $k_2 = f_1(h-1)$.

Take any edge $e = u_{\ell_1,i} v_{\ell_2,j}$ in Γ. Then $\ell_1 \ne \ell_2$, and there are $((f_2(h-1) - 1)$ $(f_1(h-1) - 1)$ 3-arcs of Γ which contain e. Let us now count how many of these 3-arcs are not contained in a 4-cycle. Let $\mathcal{A} = v_{\ell',j'} u_{\ell_1,i} v_{\ell_2,j} u_{\ell'',i''}$ be any 3-arc containing edge e. Note that $\ell' \ne \ell_1$ and $\ell'' \ne \ell_2$. Then $\mathcal{A}$ is not contained in a 4-cycle if and only if vertices $v_{\ell',j'}$ and $u_{\ell'',i''}$ are not adjacent in Γ, which happens if and only if $\ell' = \ell''$. As $\ell' \ne \ell_1$ and $\ell'' \ne \ell_2$, there are $h-2$ choices for $\ell' = \ell''$, hence there are $f_1 f_2(h-2)$ 3-arcs containing e, that are not contained in a 4-cycle. So the number of girth cycles through e in Γ is $((f_2(h-1) - 1)(f_1(h-1) - 1) - f_1 f_2(h-2)$. It follows that Γ is girth-biregular with

$$a_1 = \cdots = a_{k_1} = b_1 = \cdots = b_{k_2} = (k_1 - 1)(k_2 - 1) - f_1 f_2(h-2) = M - f_1 f_2(h-2).$$

For $g = 6$ we follow the examples of the paper [1].

Example 7.2. Take an affine plane of order q and remove i parallel classes. Consider the incidence graph of this structure. The lines still have size q and the points have degree $q+1-i$, so it is a bipartite biregular graph with valencies q and $q+1-i$. To count the girth cycles containing the edge corresponding to an incident point-line pair (e_0, P_0), we have to choose a point $P_0 \neq P_1 \in e_0$, and a line $e_0 \neq e_1$ through P_0 and complete it to a girth cycle (of length 6) by choosing a point $P_0 \neq P_2 \in e_1$ and a line e_2 joining P_1 and P_2. There are $q-1$ ways to choose P_1 and $q-i$ ways of choosing e_1. For e_2 we have to choose a line different from e_1, not parallel to e_0, so we have $(q-1-i)$ possibilities, since the point P_2 will just be the unique point of $e_0 \cap e_2$. So, in total there are $M' = (q-1)(q-i)(q-1-i)$ girth cycles through the edge (e_0, P_0).

In particular, when we have an affine plane of order q, its incidence graph is a bipartite biregular graph with valencies $q+1$ and q, and we have $M = (q-1)^2 q$ girth cycles through an edge. If there is an affine plane of order $q+1$ as well, then removing $i = 2$ parallel classes will also give us a bipartite biregular graph with valencies $q+1$ and q and this graph will have $M' = q(q-1)(q-2) = M - q(q-1)$ girth cycles through every edge.

Another construction from the paper [1] is the following.

Example 7.3. Let us consider a Steiner system on v points and line size k. Delete a point P^* and all the lines through the deleted point. The incidence graph of the resulting structure will be a bipartite biregular graph with valencies k and $r-1$, again with $r = (v-1)/(k-1)$. One can more or less copy the argument in the previous example: using the same notation, the point P_1 can be chosen in $(k-1)$ ways. Now consider the line e^* in the original Steiner system that joins P_1 and P^*. If the line e_1 intersects e^*, then we have $(k-2)$ choices for P_2 and e_2, and there are $(k-2)$ such lines in the original Steiner system. So, this case gives $(k-1)(k-2)^2$ girth cycles. There remain $(r-2)-(k-2) = r-k$ lines through P_0, not intersecting e^*. If e_1 is one of them, then there are $(k-1)$ ways to extend it to a girth cycle. This is $(k-1)^2(r-k)$ possibility, so in total we have $(k-1)((k-2)^2 + (r-k)(k-1))$ girth cycles containing the edge (e_0, P_0).

It is easy to extend Example 7.2 to resolvable Steiner systems.

Example 7.4. Consider a resolvable Steiner system and denote by v the number of points, by r the degrees of points, where $r = (v-1)/(k-1)$. In this case k divides v, and the original design will have $(k-1)^2(r-1)$ girth cycles through any edge. If we remove i parallel classes of lines, then the incidence graph of the resulting structure will have degrees k and $r-i$). For determining the number of girth cycles containing an edge start from an incident point-line pair (P_0, e_0) as before. Take a point P_1 on e_0 and let U be the set of points which are on the lines through P_1 that belong to the deleted parallel classes. This implies that $|U| = i(k-1)$. Let r_j, $j = 0, \ldots, k-1$, be the number of lines through P_0 which intersect U in exactly j points. Clearly, we have $\sum_j r_j = r-1$, and $\sum_j j r_j = |U| = i(k-1)$. On a line ℓ through having j points in U, we can choose the point P_2 of the girth cycle in $(k-1-j)$ ways. This way we get in total

$$\sum_{j=0}^{k-1} (k-1-j) r_j = (k-1)(r-1) - i(k-1)$$

girth cycles for a given choice of P_1, so the total number of girth cycles will be $(k-1)^2((r-1)-i)$. For small i this is close to our upper bound.

In particular, we mention two examples arising from higher dimensional finite spaces.

1. Let $n = 2m+1$. Remove the $q^m + 1$ elements of a line spread from $\mathrm{PG}(n,q)$ and denote the correponding point-line incidence graph by Γ. Then Γ is a girth-biregular bipartite graph with $g = 6$, $k_1 = q^{2m} + \cdots + q$ and $k_2 = q+1$ and its signature is
$$a_1 = \cdots = a_{k_1} = b_1 = \cdots = b_{k_2} = q^2(q^{2m} + \cdots + q - 2) = M - q^2.$$
2. Let us remove the q^{n-1} elements of a class of parallel lines from $\mathrm{AG}(n,q)$ and denote the corresponding point-line incident graph by Γ. Then Γ is a girth-biregular bipartite graph with $g = 6$, $k_1 = q^{n-1} + \cdots + q$ and $k_2 = q$ and its signature is
$$a_1 = \cdots = a_{k_1} = b_1 = \cdots = b_{k_2} = (q-1)^2(q^{n-1} + \cdots + q - 2) = M - (q-1)^2.$$

   In both cases the magnitude of ε is only $k_1^{2/(n-1)}$.

In the case $g = 8$ our examples come from incidence graphs of generalized quadrangles. For a detailed descriptions of generalized quadrangles, their ovoids and spreads, we refer the reader to the book of Payne and Thas [13].

Example 7.5. Let $\mathcal{G}' = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized quadrangle of order (s,t) and Γ' be the Levi graph of $\mathcal{G}'$.

Suppose that $\mathcal{G}'$ admits a spread $\mathcal{S}$ (a set of $st+1$ lines, no two of which intersect). Delete the lines of $\mathcal{S}$. Then the Levi graph Γ of $\mathcal{G} = (\mathcal{P}, \mathcal{L} \setminus \mathcal{S}, \mathrm{I})$ is a bipartite graph with bipartition $|A| = (s+1)(st+1)$ and $|B| = t(st+1)$, valencies $s+1$ and t and $g = 8$. We claim that it is also girth-biregular with
$$a_1 = \cdots = a_{s+1} = b_1 = \cdots = b_t = s^2(t^2 - 3t + 2) = M - s^2(t-1).$$

Dually, if $\mathcal{G}'$ admits an ovoid $\mathcal{O}$ (a set of $st+1$ points, no two of which are collinear), then the Levi graph Γ of $\mathcal{G} = (\mathcal{P} \setminus \mathcal{O}, \mathcal{L}, \mathrm{I})$ is a girth-biregular graph with valencies s and $t+1$, and
$$a_1 = \cdots = a_{s+1} = b_1 = \cdots = b_t = t^2(s^2 - 3s + 2) = M - t^2(s-1).$$

In $\mathcal{G}$ for any incident point-line pair (P,ℓ) there are $(t-1)s$ points in $\mathcal{P}$ which are collinear with P but are not incident with ℓ, and there are $s(t-1)$ lines in which meet ℓ but are not incident with P. Let R be one of these points and e be one of these lines. Then there is a unique point-line pair (T,f) in $\mathcal{G}'$ so that $R \,\mathrm{I}\, f \,\mathrm{I}\, T \,\mathrm{I}\, e$. Thus in Γ' there are $s^2(t-1)^2$ girth cycles through the edge which corresponds to the pair (P,ℓ). For a fixed R there is a unique element $f \in \mathcal{S}$ through R. All the s other points on f determines a unique 8-cycle which contains (P,ℓ). No two elements of $\mathcal{S}$ intersect, hence there are $(t-1)s \cdot s$ deleted 8-cycles. Thus in Γ' the total number of girth cycles through the edge corresponding to (P,ℓ) is
$$s^2(t-1)^2 - s(t-1)s = s^2(t^2 - 3t + 2) = s^2(t-1)(t-2).$$

Among the known generalized quadrangles only a few admit a spread or an ovoid. In particular, the classical generalized quadrangle $\mathcal{Q}(5,q)$ admits a spread. In this case Γ has valencies $q+1$ and q^2, and the number of girth cycles through every edge is $q^2(q^2-1)(q^2-2) = M - q^2(q^2-1)$. So the magnitude of ε is $M^{2/3}$.

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An extension of the Erdős-Ko-Rado theorem to uniform set partitions

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Abstract

A (k,ℓ)-partition is a set partition which has ℓ blocks each of size k. Two (k,ℓ)-partitions P and Q are said to be *partially t-intersecting* if there exist blocks P_i in P and Q_j in Q such that $|P_i \cap Q_j| \geq t$. In this paper we prove a version of the Erdős-Ko-Rado theorem for partially 2-intersecting (k,ℓ)-partitions. In particular, we show for ℓ sufficiently large, the set of all (k,ℓ)-partitions in which a block contains a fixed pair is the largest set of 2-partially intersecting (k,ℓ)-partitions. For $k = 3$, we show this result holds for all ℓ.

Keywords: Erdős-Ko-Rado Theorem, uniform set partitions, ratio bound, clique, coclique, quotient graphs.

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1 Introduction

In 1961, Erdős, Ko, and Rado proved that if $\mathcal{F}$ is a t-intersecting family of k-subsets of $\{1,2,\dots,n\}$, then $\binom{n-t}{k-t}$ is a tight upper bound on the size of $\mathcal{F}$, provided that n is sufficiently large [7]. This result has motivated consideration of "intersecting" families of many

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other combinatorial objects using diverse proof techniques and has developed into an active and broad area of research. There are many recent results giving analogs of the EKR theorem; see, for example, [9, 13, 16, 20, 26] or [12] and the references within. In this work, we prove an extension of the EKR theorem to systems of uniform set partitions.

A (k, ℓ)-*partition* is a set partition of $\{1, 2, \dots, k\ell\}$ with exactly ℓ blocks each of size k. These are also called *uniform set partitions*. We use $\mathcal{U}_{k,\ell}$ to denote the set of all (k, ℓ)-partitions, and $u_{k,\ell} = |\mathcal{U}_{k,\ell}|$. It is easy to see that

$$u_{k,\ell} = \frac{1}{\ell!}\binom{k\ell}{k}\binom{k\ell-k}{k}\binom{k\ell-2k}{k}\cdots\binom{k}{k} = \frac{(k\ell)!}{(k!)^{\ell}\ell!}. \tag{1.1}$$

In [8], Erdős and Székely considered different types of intersection for partitions. In one of these types, and the one we consider here, two partitions P and Q are *intersecting in a pair* if there exist blocks P_i in P, and Q_j in Q such that $|P_i \cap Q_j| \geq 2$. In [20], Meagher and Moura generalized this definition: two partitions P and Q are *partially t-intersecting* if there exist P_i in P, and Q_j in Q such that $|P_i \cap Q_j| \geq t$. The work of Meagher and Moura is different than that of Erdős and Székely since only uniform partitions are considered in [20].

A set of partitions is *partially t-intersecting* if the partitions in the set are pairwise partially t-intersecting. Meagher and Moura [20] conjectured if $\mathcal{P} \subset \mathcal{U}_{k,\ell}$ is a set of partially t-intersecting partitions, with $t \leq k$, then $|\mathcal{P}| \leq \binom{k\ell-t}{k-t}u_{k,\ell-1}$. A set of this size can be formed by fixing a t-subset T and taking all (k, ℓ)-partitions with a block that contains T; such a set is called *canonically t-intersecting*. Meagher and Moura further conjectured that only the canonically t-intersecting (k, ℓ)-partitions attain this maximum size. As pointed out by Brunk in [2], this conjecture additionally requires that $k \leq \ell(t-1)$, since if $k > \ell(t-1)$, then any two (k, ℓ)-partitions are t-partially intersecting.

If $k = t = 2$, then the $(2, \ell)$-partitions are perfect matchings in the complete graph on 2ℓ vertices, and partially 2-intersecting is equivalent to intersecting (as sets of edges). The Meagher-Moura conjecture has been proven in this case in [13], so in this paper we only consider $k \geq 3$. In particular, we prove the Meagher-Moura conjecture for $t = 2$ with $k = 3$ and all values of ℓ, and for all $k \geq 4$, provided that ℓ is sufficiently large. Our approach is to define a graph in which the cocliques (also known as independent sets) are equivalent to partially 2-intersecting (k, ℓ)-partitions from $\mathcal{U}_{k,\ell}$. Then we use a version of the algebraic method from [13] to find the size of a maximum coclique in the graph. This is an approach that has been very effective in proving many EKR-type results, indeed it is the main topic of the book [12]. This method is particularly effective when considering intersecting permutations in groups and it has been applied to many families of groups, see for example [5, 6, 17, 21, 22, 23, 25].

2 Overview of method

In a graph X a *clique* is a set of vertices which induce a complete subgraph; and a *coclique* is a set of vertices which induce an empty subgraph. The size of a largest clique and a largest coclique are denoted by $\omega(X)$ and $\alpha(X)$, respectively. The *adjacency matrix* $A(X)$ of X is a matrix in which rows and columns are indexed by the vertices in X and the (i, j)-entry is 1 if i and j are adjacent, and 0 otherwise. The *eigenvalues* of X refer to the eigenvalues of its adjacency matrix. We use $\mathbf{1}$ to denote the all-ones vector; for any d-regular graph, the all-ones vector is an eigenvector with eigenvalue d.

In general, finding the largest coclique of a graph X is known to be NP-hard, but the *Delsarte-Hoffman (ratio)* bound gives an upper bound on $\alpha(X)$. This bound is based on the ratio between the largest and the smallest eigenvalue of the adjacency matrix of the graph. A proof of this result can be found in [4] or in [12, Section 2.4], we also recommend Haemers' paper [15] on the history of this bound.

Theorem 2.1 (Delsarte-Hoffman bound [4])**.** *Let A be the adjacency matrix for a d-regular graph X on vertex set $V(X)$. If the least eigenvalue of A is τ, then*

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}.$$

If equality holds for some coclique S with characteristic vector ν_S, then

$$\nu_S - \frac{|S|}{|V(X)|}\mathbf{1}$$

is an eigenvector with eigenvalue τ.

Define $X_{k,\ell}$ to be the graph with $\mathcal{U}_{k,\ell}$ as its vertex set, in which two partitions P and Q are adjacent if every pair of blocks, one from P and one from Q, have at most 1 element in common. The group $\mathrm{Sym}(k\ell)$ acts transitively on the vertices of $X_{k,\ell}$ and preserves the edges. This means the $X_{k,\ell}$ is vertex transitive and regular. We will denote the degree by $d_{k,\ell}$, or simply d when the context is clear.

A resolvable packing design on $k\ell$ points with block size k and index $\lambda = 1$ is equivalent to a clique in this graph. Further, a resolvable balanced incomplete block design on $k\ell$ points with block size k and index $\lambda = 1$, if it exists, gives a maximum clique.

For any distinct $i, j \in \{1, \dots, k\ell\}$, let $S_{i,j}$ be the subset of partitions in $\mathcal{U}_{k,\ell}$ for which the elements i and j are in the same block. Then $S_{i,j}$ is a coclique in the graph $X_{k,\ell}$ and the size of $S_{i,j}$ is

$$\frac{1}{(\ell-1)!}\binom{k\ell-2}{k-2}\binom{k\ell-k}{k}\cdots\binom{k}{k}.$$

The main goal in this paper is to prove, using the ratio bound, that $S_{i,j}$ is a maximum coclique in $X_{k,\ell}$. For the ratio bound to hold with equality, we need to prove if τ is the least eigenvalue of $X_{k,\ell}$, then

$$1 - \frac{d_{k,\ell}}{\tau} = \frac{u_{k,\ell}}{|S_{i,j}|} = \frac{k\ell - 1}{k-1}.$$

Thus we need to prove two facts: first that $\tau = -\frac{d_{k,\ell}(k-1)}{k(\ell-1)}$ is an eigenvalue of $X_{k,\ell}$; and second that τ is the least eigenvalue of $X_{k,\ell}$.

In Section 3, we show how the eigenvalues of $X_{k,\ell}$ are related to the irreducible characters of $\mathrm{Sym}(k\ell)$, and we prove some bounds on the degrees of these irreducible characters. Next, in Section 4, we calculate three of the eigenvalues of $X_{k,\ell}$; one of these eigenvalues is the τ above. Next we prove if there is an eigenvalue of $X_{k,\ell}$ that is strictly smaller than τ, there is a function that is an upper bound on the eigenvalue's multiplicity. In Section 6 we prove that this function is bounded by $\binom{k\ell}{3} - \binom{k\ell}{2}$ for ℓ sufficiently large. This uses the result from Section 5, that the limit of ratio $u_{k,\ell}/d_{k,\ell}$ is finite as ℓ goes to ∞. The bounds from Section 3 then prove that no such eigenvalues exist and this proves the

Meagher-Moura Conjecture with $t = 2$, for all values of k, provided that ℓ is sufficiently large. Finally, in Section 7, we prove a weaker bound on $u_{k,\ell}/d_{k,\ell}$ when $k = 3$ that holds for all ℓ. Thus we prove the Meagher-Moura Conjecture for $t = 2$, $k = 3$ for all values of ℓ.

3 Representations of the symmetric group

In this section we will explain the connection between the eigenvalues of the graph $X_{k,\ell}$ and the irreducible characters of the symmetric group. We also recall some results on the degree of the irreducible characters that are involved in the eigenvalues.

For any character χ of $\mathrm{Sym}(n)$, we can consider its restriction to $H \leq \mathrm{Sym}(n)$ which is denoted by $\mathrm{res}\,(\chi)_H$. Similarly if χ is a character of $H \leq \mathrm{Sym}(n)$, then its induced character on $\mathrm{Sym}(n)$ is denoted by $\mathrm{ind}\,(\chi)^{\mathrm{Sym}(n)}$. The trivial character on a group H is denoted by 1_H.

The stabilizer of a partition in $\mathcal{U}_{k,\ell}$ is the group $\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ (this is called the *wreath product* of $\mathrm{Sym}(k)$ and $\mathrm{Sym}(\ell)$). The cosets $\mathrm{Sym}(k\ell)/(\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell))$ are in one-to-one correspondence with the partitions of $\mathcal{U}_{k,\ell}$. The action of $\mathrm{Sym}(k\ell)$ on the partitions is equivalent to the action of $\mathrm{Sym}(k\ell)$ on the cosets $\mathrm{Sym}(k\ell)/(\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell))$ and this action is clearly transitive. The permutation representation of this action is

$$\mathrm{ind}\,(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}.$$

The module for this representation can be thought of as the vector space of length-$u_{k,\ell}$ vectors with the characteristic vectors of $P \in \mathcal{U}_{k,\ell}$, denoted by v_P, as its basis. The group $\mathrm{Sym}(k\ell)$ acts on this vector space by the action on the partitions, for any $\sigma \in \mathrm{Sym}(k\ell)$ the action is $\sigma(v_P) = v_{P^\sigma}$.

This representation can be decomposed as the sum of irreducible representations of $\mathrm{Sym}(k\ell)$. If the multiplicity of each irreducible representation in the decomposition is equal to 1, then the representation is called *multiplicity-free*. In general, the group $\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ is not multiplicity free in $\mathrm{Sym}(k\ell)$. In fact it is not multiplicity free unless $k = 2$, $\ell = 2$, or (k, ℓ) is one of $(3, 3)$, $(4, 3)$, $(5, 3)$ or $(3, 4)$ [11].

3.1 Orbital association scheme

The set of *orbitals* of the action of a group G on a set Ω is the set of orbits of the action of G on $\Omega \times \Omega$. Each orbital of $\mathrm{Sym}(k\ell)$ on $\mathrm{Sym}(k\ell)/(\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell))$ can be represented by an object called a *meet table*. The meet table for two (k, ℓ)-partitions is a $\ell \times \ell$ array in which the (i, j)-entry is $|P_i \cap Q_j|$. Two meet tables are *isomorphic* if one can be obtained from the other by permuting the rows and the columns. In [12, Section 15.4] it is shown that the set of non-isomorphic meet tables corresponds to the set of orbitals. For each orbital $\mathcal{O}$ there is a corresponding meet table M; this means for $P, Q \in \mathcal{U}_{k,\ell}$ the meet table of P and Q is M if and only if $(P, Q) \in \mathcal{O}$. Further, each orbital can be represented as a $u_{k,\ell} \times u_{k,\ell}$ matrix, with the (P, Q)-entry equal to 1 if and only if the meet table of P and Q is isomorphic to the table representing the orbital. The set of these $u_{k,\ell} \times u_{k,\ell}$-matrices of the orbitals forms an *association scheme* if and only if $\mathrm{ind}\,(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}$ is multiplicity-free. In general, these matrices form a *homogeneous coherent configuration*.

The graph $X_{k,\ell}$ is the union of the orbitals from the action of $\mathrm{Sym}(k\ell)$ on the cosets $\mathrm{Sym}(k\ell)/(\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell))$ that are represented by a meet table that has no entry greater than 1. This means for every permutation $\sigma \in \mathrm{Sym}(k\ell)$ its action on the partitions is an automorphism of $X_{k,\ell}$. In particular, if M_σ is the permutation representation of σ, then

$$M_{\sigma^{-1}}\, A(X_{k,\ell})\, M_\sigma = A(X_{k,\ell}).$$

Further, if v is any θ-eigenvector of $X_{k,\ell}$, then $M_\sigma v$ is also a θ-eigenvector. This implies the eigenspaces of $X_{k,\ell}$ are invariant under the action of $\mathrm{Sym}(k\ell)$ and thus a union of irreducible modules in the decomposition of

$$\mathrm{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}.$$

We say that an eigenvalue θ *belongs* to a module if the module is a subspace of the θ-eigenspace.

3.2 Degree of the irreducible characters of $\mathrm{Sym}(k\ell)$

In this section we will give some results on the irreducible representations of $\mathrm{Sym}(n)$. We refer the reader to [24], or any similar reference on this topic, for details and background. It is well-known that the irreducible representations of $\mathrm{Sym}(n)$ correspond to integer partitions on n. We will use $\lambda \vdash n$ to indicate that λ is an integer partition of n, this means that $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_j]$, each λ_i is an integer with $\sum_{i=1}^{j} \lambda_i = n$. We will use χ_λ to represent the irreducible character of $\mathrm{Sym}(n)$ corresponding to the partition λ.

From [13] we have a list of irreducible representations of the symmetric group with small degree.

Lemma 3.1. *For $n \geq 9$, let χ be a character of $\mathrm{Sym}(n)$ with degree less than $(n^2-n)/2$. If χ_λ is a constituent of χ, then λ is one of the following partitions of n:*

$$[n],\ [1^n],\ [n-1,1],\ [2,1^{n-2}],\ [n-2,2],\ [2,2,1^{n-4}],\ [n-2,1,1],\ [3,1^{n-3}].$$

This proof uses the branching rule, which we state here. For a proof of this rule see [3, Corollary 3.3.11].

Lemma 3.2. *Let $\lambda \vdash n$, then*

$$\mathrm{res}\,(\chi_\lambda)_{\mathrm{Sym}(n-1)} = \sum \chi_{\lambda^-},$$

where the sum is taken over all partitions λ^- of $n-1$ that have a Young diagram which can be obtained by the deletion of a single box from the Young diagram of λ. Further,

$$\mathrm{ind}\,(\chi_\lambda)^{\mathrm{Sym}(n+1)} = \sum \chi_{\lambda^+},$$

where the sum is taken over partitions λ^+ of $n+1$ that have a Young diagram which can be obtained by the addition of a single box to Young diagram of λ. □

Using the same approach as the proof for Lemma 3.1 we can get a second family of irreducible characters with slightly larger, but still small degree.

Lemma 3.3. *For $n \geq 13$, let χ be an irreducible character of $\mathrm{Sym}(n)$ with degree less than $\binom{n}{3} - \binom{n}{2}$. If χ_λ is a constituent of χ, then λ is one of the following partitions of n:*

$$[n], \quad [1^n], \quad [n-1,1], \quad [2,1^{n-2}], \quad [n-2,2], \quad [2,2,1^{n-4}],$$
$$[n-2,1,1], \quad [3,1^{n-3}], \quad [n-3,3], \quad [2,2,2,1^{n-6}].$$

Proof. The hook length formula confirms that each of the 10 characters above have degree less than or equal to $\binom{n}{3} - \binom{n}{2}$.

We prove this result by induction. For $n = 13$ and 14 this can be calculated directly using the GAP character table library [10]. We assume for $n \geq 14$ that the lemma holds for n and $n-1$, and we will prove that the lemma holds for $n+1$.

Assume that χ is an irreducible character of $\mathrm{Sym}(n+1)$ that has dimension less than

$$\binom{n+1}{3} - \binom{n+1}{2} = \frac{(n+1)n(n-4)}{6},$$

but is not one of the ten irreducible characters listed in the statement of the lemma. We will show that such a χ cannot exist.

If one of the ten irreducible characters of $\mathrm{Sym}(n)$ with degree less than $\binom{n}{3} - \binom{n}{2}$ is a constituent of $\mathrm{res}\,(\chi)_{\mathrm{Sym}(n)}$, then we can determine the possible constituents of χ with the branching rule.

Constituent of $\mathrm{res}\,(\chi)_{\mathrm{Sym}(n)}$	Constituents of χ
$[n]$	$[n+1]$, $[n,1]$
$[n-1,1]$	$[n,1]$, $[n-1,2]$, $[n-1,1,1]$
$[n-2,2]$	$[n-1,2]$, $[n-2,3]$, $[n-2,2,1]$
$[n-2,1,1]$	$[n-1,1,1]$, $[n-2,2,1]$, $[n-2,1,1,1]$
$[n-3,3]$	$[n-2,3]$, $[n-3,4]$, $[n-3,3,1]$
$[1^n]$	$[2,1^{n-1}]$, $[1^{n+1}]$
$[2,1^{n-2}]$	$[3,1^{n-2}]$, $[2,2,1^{n-3}]$, $[2,1^{n-1}]$
$[2,2,1^{n-4}]$	$[3,2,1^{n-4}]$, $[2,2,2,1^{n-5}]$, $[2,2,1^{n-3}]$
$[3,1^{n-3}]$	$[4,1^{n-3}]$, $[3,2,1^{n-4}]$, $[3,1^{n-2}]$
$[2,2,2,1^{n-6}]$	$[3,2,2,1^{n-6}]$, $[2,2,2,2,1^{n-7}]$, $[2,2,2,1^{n-5}]$

Table 1: Constituents of χ, if $\mathrm{res}\,(\chi)_{\mathrm{Sym}(n)}$ has a constituent with degree less than $\binom{n}{3} - \binom{n}{2}$.

By Frobenius reciprocity, for any character ϕ of $\mathrm{Sym}(n)$

$$\langle \mathrm{res}\,(\chi)_{\mathrm{Sym}(n)}, \phi \rangle_{\mathrm{Sym}(n)} = \langle \chi, \mathrm{ind}\,(\phi)^{\mathrm{Sym}(n+1)} \rangle_{\mathrm{Sym}(n+1)}.$$

Character	Degree
$[n-3,4]$	$(n+1)n(n-1)(n-7)/24$
$[n-3,3,1]$	$(n+1)n(n-2)(n-5)/8$
$[n-2,2,1]$	$(n+1)(n-1)(n-3)/3$
$[n-2,1,1,1]$	$n(n-1)(n-2)/6$
$[2,2,2,2,1^{n-8}]$	$(n+1)n(n-1)(n-7)/24$
$[3,2,2,1^{n-6}]$	$(n+1)n(n-2)(n-5)/8$
$[3,2,1^{n-4}]$	$(n+1)(n-1)(n-3)/3$
$[4,1^{n-3}]$	$n(n-1)(n-2)/6$

Table 2: Degrees of the characters from Table 1 that are larger than $\frac{(n+1)n(n-4)}{6}$ for $n \geq 13$.

This means if ϕ is a constituent of $\operatorname{res}(\chi)_{\operatorname{Sym}(n)}$, then χ is one of the constituents of $\operatorname{ind}(\phi)^{\operatorname{Sym}(n+1)}$. The possible constituents of $\operatorname{ind}(\phi)^{\operatorname{Sym}(n+1)}$ are recorded in Table 1; the second column lists the irreducible characters that, according to the branching rule, are constituents of representation of $\operatorname{Sym}(n+1)$ induced by the character in the first column.

From these lists, and the degrees of the characters given in Table 2, we see that either χ is one of the ten listed in the theorem, or the degree of χ is larger than $\binom{n}{3} - \binom{n}{2}$ (again, the degrees are calculated using the hook length formula). Thus $\operatorname{res}(\chi)_{\operatorname{Sym}(n)}$ does not contain any of the ten irreducible characters of $\operatorname{Sym}(n)$ in the statement of the theorem.

Next consider the case where the decomposition of $\operatorname{res}(\chi)_{\operatorname{Sym}(n)}$ contains at least two irreducible characters of $\operatorname{Sym}(n)$ which are not in the list of the ten irreducible characters with dimension less $\binom{n}{3} - \binom{n}{2} = n(n-1)(n-5)/6$. In this case, the degree of χ must be at least $n(n-1)(n-5)/3$. But since $n > 7$, this is strictly larger than $(n+1)n(n-4)/6$.

Finally we need to consider the case where $\operatorname{res}(\chi)_{\operatorname{Sym}(n)}$ contains exactly one irreducible character of $\operatorname{Sym}(n)$, which is not one of the ten listed in the theorem. By the branching rule the only irreducible characters of $\operatorname{Sym}(n+1)$ for which $\operatorname{res}(\chi)_{\operatorname{Sym}(n)}$ contains only one irreducible character have a rectangular Young diagram, so $\chi = \chi_{[s^t]}$ for some s and t.

Next consider $\operatorname{res}(\chi)_{\operatorname{Sym}(n-1)}$, this is the restriction of $\chi = \chi_{[s^t]}$ to $\operatorname{Sym}(n-1)$. By the branching rule, this can contain only the irreducible characters of $n-1$ that correspond to the partitions $\lambda' = [s^{t-1}, s-2]$ and $\lambda'' = [s^{t-2}, s-1, s-1]$.

If λ' is one of the ten partitions that correspond to irreducible characters of $\operatorname{Sym}(n-1)$ with degree less than $\binom{n-1}{3} - \binom{n-1}{2}$, then one of the following cases must hold:

- $t = 1$ and $\lambda' = [n-1]$ and $s = n+1$,
- $t = 2$ and $\lambda' = [n-1,1], [n-2,2]$ or $[n-3,3]$, and $s \leq 5$, or
- $2 < t < 4$ and and $s \leq 2$.

The first of these cases implies $\chi = [n+1]$, which contradicts the degree of χ, and none of the other cases can happen, since $n = st$ and n is assumed to be at least 13.

Similarly, assume $\lambda'' = [s^{t-2}, s-1, s-1]$ is one of the partitions corresponding to the ten characters of $\operatorname{Sym}(n-1)$ that have degree less than $\binom{n-1}{3} - \binom{n-1}{2}$. Then one of the following cases must hold:

- $t = 2$ and $\lambda'' = [s-1, s-1]$ and $s \leq 4$,
- $2 < t \leq 5$ and $\lambda'' = [s^{t-1}, 1, 1]$ and $s \leq 2$, or
- $s = 1$.

The first two cases imply that $n \leq 10$ and the final case implies that $\chi = [1^{(n+1)}]$ which has degree 1.

Thus $\operatorname{res}(\chi)_{\operatorname{Sym}(n-1)}$ has two characters with degree at least $\binom{n-1}{3} - \binom{n-1}{2}$, so the degree of χ is at least $(n-1)(n-2)(n-6)/3$, which is strictly greater than $(n+1)n(n-4)/6$ for $n \geq 13$. This is a contradiction, so no such χ exists. □

Next we will show that there are only three irreducible characters in the decomposition of $\operatorname{ind}(1_{\operatorname{Sym}(k)\wr\operatorname{Sym}(\ell)})^{\operatorname{Sym}(k\ell)}$ that have degree no more than $\binom{k\ell}{3} - \binom{k\ell}{2}$. To do this we will consider the action of different Young subgroups on $\mathcal{U}_{k,\ell}$. For any integer partition $\lambda \vdash n$ we will denote the Young subgroup by

$$\operatorname{Sym}(\lambda) = \operatorname{Sym}(\lambda_1) \times \operatorname{Sym}(\lambda_2) \times \cdots \times \operatorname{Sym}(\lambda_k).$$

Theorem 3.4. *Assume $k\ell \geq 13$ and $k \geq 3$. Then the only partitions in the decomposition of $\operatorname{ind}(1_{\operatorname{Sym}(k)\wr\operatorname{Sym}(\ell)})^{\operatorname{Sym}(k\ell)}$ with dimension less than or equal to $\binom{k\ell}{3} - \binom{k\ell}{2}$ are*

$$\chi_{[k\ell]}, \quad \chi_{[k\ell-2,2]}, \quad \chi_{[k\ell-3,3]}.$$

Proof. Lemma 3.3 lists the 10 irreducible representations of $\operatorname{Sym}(k\ell)$ with dimension no more than $\binom{k\ell}{3} - \binom{k\ell}{2}$. We only need to show which of these representations are in the decomposition. The tool we use is Frobenius reciprocity along with the action of different Young subgroups on $\mathcal{U}_{k,\ell}$.

By Frobenius reciprocity

$$\left\langle \operatorname{ind}(1_{\operatorname{Sym}(\lambda)})^{\operatorname{Sym}(k\ell)}, \quad \operatorname{ind}(1_{\operatorname{Sym}(k)\wr\operatorname{Sym}(\ell)})^{\operatorname{Sym}(k\ell)} \right\rangle_{\operatorname{Sym}(k\ell)} = \left\langle 1, \quad \operatorname{res}(\operatorname{ind}(1_{\operatorname{Sym}(k)\wr\operatorname{Sym}(\ell)})^{\operatorname{Sym}(k\ell)})_{\operatorname{Sym}(\lambda)} \right\rangle_{\operatorname{Sym}(\lambda)}.$$

The second inner product above gives the number of orbits of the action of $\operatorname{Sym}(\lambda)$ on the cosets $\operatorname{Sym}(k\ell)/(\operatorname{Sym}(k) \wr \operatorname{Sym}(\ell))$; or, equivalently, the number of orbits of $\operatorname{Sym}(\lambda)$ on the partitions in $\mathcal{U}_{k,\ell}$. Using this fact with different Young subgroups will allow us to determine that many of the representations with small degree do not occur in the decomposition of $\operatorname{ind}(1_{\operatorname{Sym}(k)\wr\operatorname{Sym}(\ell)})^{\operatorname{Sym}(k\ell)}$.

To start, it is clear that $\operatorname{Sym}(k\ell)$ has one orbit on the (k,ℓ)-partitions, so $\chi_{[k\ell]}$ has multiplicity 1 in the decomposition. Next consider the group $\operatorname{Sym}([k\ell-1,1])$, it is also straight-forward that this group only has one orbit on the partitions. Using the definition of the Specht modules and the labelling of the irreducible characters of the symmetric group it is straight forward to see that

$$\operatorname{ind}(1_{\operatorname{Sym}([k\ell-1,1])})^{\operatorname{Sym}(k\ell)} = \chi_{[k\ell]} + \chi_{[k\ell-1,1]},$$

so we have that

$$\left\langle \chi_{[k\ell]} + \chi_{[k\ell-1,1]}, \operatorname{ind}(1_{\operatorname{Sym}(k)\wr\operatorname{Sym}(\ell)})^{\operatorname{Sym}(k\ell)} \right\rangle = 1.$$

Since we know that $\chi_{[k\ell]}$ occurs in this decomposition with multiplicity 1, this implies that $\chi_{[k\ell-1,1]}$ does not occur in the decomposition of $\operatorname{ind}(1_{\operatorname{Sym}(k)\wr\operatorname{Sym}(\ell)})^{\operatorname{Sym}(k\ell)}$.

Next we consider the group $\mathrm{Sym}([k\ell-2,2])$. This group has two orbits on the partitions of $\mathcal{U}_{k,\ell}$. Again, from the definition of the Specht modules and the labelling of the irreducible characters, we have that

$$\left\langle \operatorname{ind}\left(1_{\mathrm{Sym}([k\ell-2,2])}\right)^{\mathrm{Sym}(k\ell)}, \quad \operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)} \right\rangle =$$
$$\left\langle \chi_{[k\ell]} + \chi_{[k\ell-1,1]} + \chi_{[k\ell-2,2]}, \quad \operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)} \right\rangle = 2.$$

This implies that $\chi_{[k\ell-2,2]}$ occurs in the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$ with multiplicity 1.

We continue this process with the group $\mathrm{Sym}([k\ell-2,1,1])$. It has two orbits on the partitions of $\mathcal{U}_{k,\ell}$. Since

$$\operatorname{ind}\left(1_{\mathrm{Sym}([k\ell-2,1,1])}\right)^{\mathrm{Sym}(k\ell)} = \chi_{[k\ell]} + \chi_{[k\ell-1,1]} + \chi_{[k\ell-2,2]} + \chi_{[k\ell-2,1,1]},$$

we conclude that $\chi_{[k\ell-2,1,1]}$ does not occur in the decomposition.

Next, we consider the group $\mathrm{Alt}(k\ell) \cap \mathrm{Sym}([k\ell-2,2])$. This group has two orbits on the partitions of $\mathcal{U}_{k,\ell}$. Again the decomposition of $\operatorname{ind}\left(1_{\mathrm{Alt}(k\ell)\cap\mathrm{Sym}([k\ell-2,2])}\right)^{\mathrm{Sym}(k\ell)}$ is well-known (a proof can be found in [11, Proposition 1.4]) and we have

$$\begin{aligned}\operatorname{ind}\left(1_{\mathrm{Alt}(k\ell)\cap\mathrm{Sym}([k\ell-2,2])}\right)^{\mathrm{Sym}(k\ell)} &= \chi_{[k\ell]} + \chi_{[k\ell-1,1]} + \chi_{[k\ell-2,2]} + \chi_{[k\ell-2,1,1]} \\ &\quad + \chi_{[1^{k\ell}]} + \chi_{[2,1^{k\ell-2}]} + \chi_{[2,2,1^{k\ell-4}]} + \chi_{[3,1^{k\ell-3}]}.\end{aligned}$$

This implies that none of $\chi_{[1^{k\ell}]}$, $\chi_{[2,1^{k\ell-2}]}$, $\chi_{[2,2,1^{k\ell-4}]}$ and $\chi_{[3,1^{k\ell-3}]}$ occur in the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$.

Next we consider the group $\mathrm{Sym}([k\ell-3,3])$. This group has three orbits on the partitions of $\mathcal{U}_{k,\ell}$ and from the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}([k\ell-3,3])}\right)^{\mathrm{Sym}(k\ell)}$ we have that

$$\langle \operatorname{ind}\left(1_{\mathrm{Sym}([k\ell-3,3])}\right)^{\mathrm{Sym}(k\ell)}, \operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)} \rangle =$$
$$\langle \chi_{[k\ell]} + \chi_{[k\ell-1,1]} + \chi_{[k\ell-2,2]} + \chi_{[k\ell-3,3]}, \operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)} \rangle = 3.$$

This implies that $\chi_{[k\ell-3,3]}$ occurs in the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$ with multiplicity 1.

Next we consider the group $\mathrm{Alt}(k\ell) \cap \mathrm{Sym}([k\ell-2,1,1])$. This group has 2 orbits on the partitions of $\mathcal{U}_{k,\ell}$ and

$$\begin{aligned}\operatorname{ind}\left(1_{\mathrm{Alt}(k\ell)\cap\mathrm{Sym}([k\ell-2,1,1])}\right)^{\mathrm{Sym}(k\ell)} &= \chi_{[k\ell]} + \chi_{[k\ell-1,1]} + \chi_{[k\ell-2,2]} + \chi_{[k\ell-2,1,1]} \\ &\quad \chi_{[1^{k\ell}]} + \chi_{[2,1^{k\ell-2}]} + \chi_{[2,2,1^{k\ell-4}]} + \chi_{[3,1^{k\ell-3}]}.\end{aligned}$$

Since $\chi_{[k\ell]}$, and $\chi_{[k\ell-2,2]}$ are in the decomposition, none of the irreducible representations $\chi_{[1^{k\ell}]}$, $\chi_{[2,1^{k\ell-2}]}$, $\chi_{[2,2,1^{k\ell-4}]}$, or $\chi_{[3,1^{k\ell-3}]}$ occur in the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$.

Finally, we consider the group $\mathrm{Alt}(k\ell) \cap \mathrm{Sym}([k\ell-3,3])$. This group has three orbits on the partitions of $\mathcal{U}_{k,\ell}$ and

$$\begin{aligned}\operatorname{ind}\left(1_{\mathrm{Alt}(k\ell)\cap\mathrm{Sym}([k\ell-3,3])}\right)^{\mathrm{Sym}(k\ell)} &= \chi_{[k\ell]} + \chi_{[k\ell-1,1]} + \chi_{[k\ell-2,2]} + \chi_{[k\ell-3,3]} \\ &\quad + \chi_{[1^{k\ell}]} + \chi_{[2,1^{k\ell-2}]} + \chi_{[2^2,1^{k\ell-4}]} + \chi_{[2^3,1^{k\ell-6}]}.\end{aligned}$$

Which shows $\chi_{[2,2,2,1^{k\ell-6}]}$ is not in the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$. $\square$

4 Eigenvalues of $X_{k,\ell}$ with $k \geq 3$

In this section we will find three of the eigenvalues of $X_{k,\ell}$. For ease of notation, we will denote the irreducible representation of χ_λ by the λ-module. Also, the number of vertices in $X_{k,\ell}$, which is equal to $u_{k,\ell}$, will be denoted simply by v and the degree of the graph $X_{k,\ell}$ will be simply written as d, rather than $d_{k,\ell}$.

Any subgroup $H \leq \mathrm{Sym}(k\ell)$ acts on the vertices of $X_{k,\ell}$ and the orbits of this action form an equitable partition. From any equitable partition, we can form a quotient graph and the eigenvalues of this quotient graph will be eigenvalues of the $X_{k,\ell}$ (details can be found in [13, Section 2.2]). The trivial case is $H = \mathrm{Sym}(k\ell)$, since this group is transitive, the equitable partition has all the vertices of $X_{k,\ell}$ in a single part. The quotient graph for this is simply the 1×1 matrix with the single entry d. The eigenvalue of this matrix is simply d, and the eigenvector is the all ones vector and the eigenspace is isomorphic to the trivial representation of $\mathrm{Sym}(k\ell)$. So d belongs to the $[k\ell]$-module.

Since the subgroup $\mathrm{Sym}([k\ell - 1, 1])$ has only one orbit on the vertices of $X_{k,\ell}$, the next subgroup we consider is the Young subgroup $\mathrm{Sym}([k\ell - 2, 2])$, considered as the stabilizer of the set $\{1, 2\}$. The action of $\mathrm{Sym}([k\ell - 2, 2])$ on the partitions will give us another eigenvalue of the graph.

Lemma 4.1. *For integers k and ℓ, with $k, \ell \geq 2$, $\tau = -\frac{(k-1)d}{k(\ell-1)}$ is an eigenvalue of $X_{k,\ell}$ with multiplicity at least $\binom{k\ell}{2} - \binom{k\ell}{1}$.*

Proof. The action of $\mathrm{Sym}([k\ell - 2, 2])$ on the (k, ℓ)-partitions has exactly 2 orbits: S_1, the set of all partitions that have 1 and 2 in the same block, and S_2, the set of all partitions in which 1 and 2 are in different blocks. The orbit S_1 is a coclique in $X_{k,\ell}$ so the quotient matrix for this partition has the form

$$\begin{pmatrix} 0 & d \\ -\tau & d+\tau \end{pmatrix}. \tag{4.1}$$

The eigenvalues of the quotient matrix (4.1) are d and τ. We can calculate the value of τ by counting edges between S_1 and S_2. Since S_1 is a coclique, each vertex in it is adjacent to exactly d vertices in S_2, and each vertex in S_2 is adjacent to $-\tau$ vertices in S_1. Using the sizes of S_1 and S_2, we have that the number of edges between S_1 and S_2 is equal to

$$|S_1|\, d = \binom{k\ell - 2}{k - 2} u_{k,\ell-1}\, d$$

and also to

$$|S_2|(-\tau) = \binom{k\ell - 2}{k - 1}\binom{k\ell - k - 1}{k - 1} u_{k,\ell-2}(-\tau).$$

Thus

$$\tau = -\frac{(k-1)d}{k(\ell - 1)} \tag{4.2}$$

is a second eigenvalue for $X_{k,\ell}$. Since this eigenvalue arises from the action of $\mathrm{Sym}([k\ell - 2, 2])$, it belongs to a module that is common between the two representations $\mathrm{ind}\left(1_{\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$ and $\mathrm{ind}\left(1_{\mathrm{Sym}([k\ell-2,2])}\right)^{\mathrm{Sym}(k\ell)}$. Thus it belongs to the module $[k\ell - 2, 2]$, as this is the only common module, and must have dimension at least $\binom{k\ell}{2} - \binom{k\ell}{1}$. □

We denote this eigenvalue by τ since in Section 6 it will be shown that τ is the least eigenvalue of $X_{k,\ell}$, provided that ℓ is sufficiently large. We also note that a second irreducible module may also have τ as the eigenvalue belonging to it, so the multiplicity of τ could be higher than the degree of the $[k\ell - 2, 2]$-module.

Next we will consider the Young subgroup $\mathrm{Sym}([k\ell - 3, 3])$, thought of as the group that stabilizes the set $\{1, 2, 3\}$. The action of this subgroup on $\mathcal{U}_{k,\ell}$ has 3 orbits: T_1, the set of all partitions with $1, 2, 3$ in the same block; T_2 the set of all partitions in which $1, 2, 3$ are in exactly two different blocks; and T_3 the set of all partitions in which $1, 2, 3$ are in three different blocks. Any vertex in T_1 is adjacent only to vertices in T_3. Similarly, a vertex in T_2 can be adjacent to vertices in T_2 and T_3. The quotient graph for this equitable partition is

$$M = \begin{pmatrix} 0 & 0 & d \\ 0 & a & d-a \\ b & c & d-b-c \end{pmatrix},$$

where a, b, c are all non-negative.

The eigenvalues for this quotient graph will be the eigenvalues that belong to modules that are both in the decomposition of $\mathrm{ind}\,(1_{\mathrm{Sym}([k\ell-3,3])})^{\mathrm{Sym}(k\ell)}$ and the decomposition of $\mathrm{ind}\,(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}$. Thus the eigenvalues will belong to the $[k\ell]$, $[k\ell-2,2]$ and $[k\ell-3,3]$ modules. We have already seen that the eigenvalue for $[k\ell]$ is d, and the eigenvalue for $[k\ell-2,2]$ is τ. We will denote the eigenvalue belonging to $[k\ell-3,3]$ by θ.

Since the trace of the matrix is the sum of the eigenvalues we have that

$$d + a - b - c = d + \tau + \theta. \tag{4.3}$$

The number of edges between T_1 and T_3 is equal to

$$d|T_1| = d\binom{k\ell-3}{k-3}u_{k,\ell-1},$$

and also to

$$b|T_3| = b\binom{k\ell-3}{k-1}\binom{k\ell-k-2}{k-1}\binom{k\ell-2k-1}{k-1}u_{k,\ell-3}.$$

Setting these equations equal to each other, then expanding the binomial coefficients and rearranging yields

$$\frac{(k-1)(k-2)}{k^2(\ell-1)(\ell-2)}d = b.$$

Replacing $d = -\frac{k(\ell-1)}{k-1}\tau$ shows that

$$b = -\frac{(k-1)(k-2)}{k^2(\ell-1)(\ell-2)}\frac{k(\ell-1)}{(k-1)}\tau = -\frac{k-2}{k(\ell-2)}\tau. \tag{4.4}$$

Putting this into Equation 4.3 produces the following formula

$$\theta = a + \frac{k-2}{k(\ell-2)}\tau - c - \tau = a - c + \frac{(k-2) - k(\ell-2)}{k(\ell-2)}\tau. \tag{4.5}$$

Similarly, counting the number of edges between T_2 and T_3 yields

$$3\binom{k\ell-3}{k-2}\binom{k\ell-k-1}{k-1}u_{k,\ell-2}(d-a) = \binom{k\ell-3}{k-1}\binom{k\ell-k-2}{k-1}\binom{k\ell-2k-1}{k-1}u_{k,\ell-3}(c).$$

Again, expanding the binomial coefficients and rearranging shows that

$$a = d - \frac{(\ell-2)k}{3(k-1)}c.$$

The characteristic polynomial of M is

$$x^3 + (-a+b+c-d)x^2 + (-ab+ad-bd-cd)x + abd.$$

Substituting in the values we have computed for b and c, and using the fact that τ is a root of the characteristic polynomial we get

$$a = \frac{2(k-1)}{k(\ell-1)}d. \tag{4.6}$$

From this we can compute that

$$c = \frac{3(k\ell-3k+2)(k-1)}{k^2(\ell-1)(\ell-2)}d. \tag{4.7}$$

Lemma 4.2. *For integers k and ℓ, with $k,\ell \geq 3$,*

$$\theta = \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}$$

is an eigenvalue of $X_{k,\ell}$ with multiplicity at least $\binom{k\ell}{3} - \binom{k\ell}{2}$.

Proof. By Equations (4.5), (4.6) and (4.7), we can calculate that

$$\theta = \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}. \tag{4.8}$$

From the comments above, $\theta = \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}$ is the eigenvalue belonging to the unique $[k\ell-3,3]$-module in $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$. Since the dimension of the irreducible representation of $[k\ell-3,3]$ is $\binom{k\ell}{3} - \binom{k\ell}{2}$, the multiplicity of θ is at least $\binom{k\ell}{3} - \binom{k\ell}{2}$. □

5 Bound on degree of $X_{k,\ell}$

In this section we will find a lower bound on the degree of $X_{k,\ell}$ for all sufficiently large ℓ. If P and Q are two partitions that are adjacent in $X_{k,\ell}$, then the meet table of P and Q is an $\ell \times \ell$ matrix with entries either 0 or 1, and further, the entries in each row and column in the meet table sum to k. We define $\mathcal{M}_{k,\ell}$ to be the set of all such meet tables, so all $\ell \times \ell$ matrices with entries either 0 or 1, and row and columns sums equal to k. To find the degree of $X_{k,\ell}$, we first state a result on the number of such meet tables. Next, for a fixed partition P and a meet table $M \in \mathcal{M}_{k,\ell}$, we count the number of partitions Q for which the meet table of P and Q is M.

Bender [1] determined the asymptotic cardinality of $\mathcal{M}_{k,\ell}$. (In fact, Bender found a much more general result, but we only state the result that we need here.)

Theorem 5.1 (Bender [1])**.** *For positive integers k, ℓ*

$$\lim_{\ell\to\infty} \frac{(k!)^{2\ell}}{(k\ell)!}|\mathcal{M}_{k,\ell}| = e^{-\frac{(k-1)^2}{2}}.$$

To get a lower bound on $d_{k,\ell}$, we fix a partition P in $\mathcal{U}_{k,\ell}$, then for each $M \in \mathcal{M}_{k,\ell}$, we will count the number of Q so that the meet table of P and Q is M, then we use Theorem 5.1 to bound the size of $\mathcal{M}_{k,\ell}$.

Lemma 5.2. *For positive integers k, ℓ with $k \leq \ell$,*

$$d_{k,\ell} = \frac{k!^{\ell}}{\ell!}|\mathcal{M}_{k,\ell}|.$$

Proof. Fix a partition $P \in \mathcal{U}_{k,\ell}$. Define a bipartite multigraph with the vertices in one part the meet tables in $\mathcal{M}_{k,\ell}$, and the vertices in the other part the partitions in the neighbourhood of P in $X_{k,\ell}$. Two vertices M and Q are adjacent if the meet table of P and Q is M. By counting the number of edges in this graph in two ways, we will determine the size of the neighbourhood of P in terms of $|\mathcal{M}_{k,\ell}|$.

For any $M \in \mathcal{M}_{k,\ell}$, with $M = [m_{i,j}]$ assume that row i corresponds to the block $P_i \in P$. Construct a partition $Q = \{Q_1, Q_2, \dots, Q_\ell\}$ so that the block Q_j corresponds to column j of M and $|P_i \cap Q_j| = m_{i,j}$. Since the entries of a row in M are either 0 or 1, and sum to k, there are $k!$ ways to select how the elements from P_i will be distributed to the blocks of Q. So for each meet table M, there are $k!^{\ell}$ partitions Q that can be constructed this way. It is possible that some of these partitions are equal, once the blocks are reordered, so this is a multigraph.

For every Q in the neighbourhood of P, there are $\ell!$ ways to order the blocks of Q, once the blocks are ordered the meet table for P and Q is uniquely defined. In the bipartite graph, Q is adjacent to each of these tables in the graph (again, these tables may not be distinct, so the graph is a multigraph). The degree of every vertex Q is $\ell!$.

Thus we have that the number of edges in the multigraph is

$$\ell! d_{k,\ell} = \sum_{M\in\mathcal{M}_{k,\ell}} k!^{\ell},$$

and the result follows. □

Using Theorem 5.1 we have the asymptotic size of $d_{k,\ell}$.

Corollary 5.3. *For a fixed integer k with $k \geq 2$,*

$$\lim_{\ell\to\infty} \frac{u_{k,\ell}}{d_{k,\ell}} = e^{\frac{(k-1)^2}{2}}.$$

Proof. From Equation (1.1) and Lemma 5.2,

$$\frac{u_{k,\ell}}{d_{k,\ell}} = \frac{(k\ell)!}{(k!)^\ell \ell!}\frac{\ell!}{k!^\ell |\mathcal{M}_{k,\ell}|} = \frac{(k\ell)!}{(k!)^{2\ell}|\mathcal{M}_{k,\ell}|}.$$

The result then follows from Theorem 5.1. □

Thus for every $\epsilon > 0$, there exists an ℓ' such that for all $\ell \geq \ell'$,

$$\frac{u_{k,\ell}}{d_{k,\ell}} \leq e^{\frac{(k-1)^2}{2}} + \epsilon.$$

6 A bound on the multiplicity of eigenvalues with large absolute value

In Section 4 we found three eigenvalues, d, τ, and θ of $X_{k,\ell}$, and the ratio between d and τ is $\frac{d}{\tau} = \frac{k(1-\ell)}{k-1}$. If τ is the least eigenvalue of the graph, then by the ratio bound any coclique will have size no more than

$$\frac{|V(X_{k,\ell})|}{1-\frac{d}{\tau}} = \frac{|V(X_{k,\ell})|}{1-\frac{k(1-\ell)}{k-1}} = u_{k,\ell-1}.$$

This is exactly the size of a set of canonically 2-intersecting (k, ℓ)-partitions. Thus our goal in this section is to show that τ is the least eigenvalue of $X_{k,\ell}$. To this end, we first show if $X_{k,\ell}$ has an eigenvalue λ with $\lambda^2 > \tau^2$, then there is a bound on the multiplicity of λ.

Let

$$\{d^{(1)}, \tau^{(m_\tau)}, \theta^{(m_\theta)}, \lambda_2^{(m_2)}, \dots, \lambda_j^{(m_j)}\}$$

be the spectrum of the matrix $X_{k,\ell}$, where the values m_i represent the multiplicities of the eigenvalues. By squaring A and taking the trace, we have

$$vd = d^2 + m_\tau \tau^2 + m_\theta \theta^2 + \sum_{i=2}^{j} m_i \lambda_i^2.$$

Hence for every $2 \leq i \leq j$ we have

$$vd - d^2 - m_\tau \tau^2 - m_\theta \theta^2 \geq m_i \lambda_i^2.$$

Assume λ_i is an eigenvalue of $X_{k,\ell}$ with $\lambda_i^2 > \tau^2$, and also that λ_i is not the eigenvalue belonging to the $[k\ell]$, $[k\ell-2, 2]$ or $[k\ell-3, 3]$ modules, then

$$\frac{vd - d^2 - m_\tau \tau^2 - m_\theta \theta^2}{\tau^2} \geq m_i.$$

Expanding θ using Equation (4.8) in the above equation produces the following equation

$$\left(\frac{v}{d} - 1\right)\frac{k^2(\ell-1)^2}{(k-1)^2} - m_\theta \frac{4(k-2)^2}{k^2(\ell-2)^2} - m_\tau \geq m_i.$$

Further, by Lemmas 4.1 and 4.2, it is known that $m_\tau \geq \binom{k\ell}{2} - \binom{k\ell}{1}$ and $m_\theta \geq \binom{k\ell}{3} - \binom{k\ell}{2}$, so this bound becomes

$$\left(\frac{v}{d} - 1\right)\frac{k^2(\ell-1)^2}{(k-1)^2} - \frac{(k\ell)(k\ell-1)(k\ell-5)}{6}\frac{4(k-2)^2}{k^2(\ell-2)^2} - \frac{(k\ell)(k\ell-3)}{2} \geq m_i.$$

Our next step is to show that this upper bound on m_i is smaller than $\binom{k\ell}{3} - \binom{k\ell}{2}$. This will be a contradiction with Theorem 3.4 since we have assumed that λ does not belong to any of the $[k\ell]$, $[k\ell - 2, 2]$, and $[k\ell - 3, 3]$ modules. In other words, we need to prove that

$$\frac{v}{d} - 1 < \frac{\ell(k-1)^2 \left(k^2(\ell-2)^2(k\ell-4)(k\ell+1) + 4(k-2)^2(k\ell-1)(k\ell-5)\right)}{6k^3(\ell-1)^2(\ell-2)^2}. \quad (6.1)$$

In the next result, we will see that this will follow from Corollary 5.3.

Theorem 6.1. *Fix an integer $k \geq 3$. For ℓ sufficiently large, the largest set of partially 2-intersecting uniform (k, ℓ)-partitions has size*

$$\binom{k\ell - 2}{k - 2} u_{k,\ell-1}.$$

Proof. For any distinct $i, j \in \{1, \dots, k\ell\}$, the set $S_{i,j}$ of all (k, ℓ)-partitions with i and j in the same block form a set of partially 2-intersecting (k, ℓ)-partitions of the size given in the theorem.

Corollary 5.3 shows that $\frac{v}{d}$ approaches a fixed constant, namely $e^{\frac{(k-1)^2}{2}}$, as ℓ goes to infinity. Since the right hand side of Equation (6.1) grows linearly in ℓ, we have that Equation (6.1) holds for ℓ sufficiently large. This implies if there is an eigenvalue λ of $X_{k,\ell}$ with $\lambda \leq \tau$, then the multiplicity of λ is less than or equal to $\binom{k\ell}{3} - \binom{k\ell}{2}$.

By Theorem 3.4, eigenspaces with dimension less than or equal to $\binom{k\ell}{3} - \binom{k\ell}{2}$ can only include the $[k\ell]$, $[k\ell - 2, 2]$ or the $[k\ell - 3, 3]$-modules. The degree, d, is the eigenvalue belonging to the $[k\ell]$-module, and Lemma 4.1 and Lemma 4.2 shows that τ is the eigenvalue belonging to the $[k\ell - 2, 2]$-module and θ belongs to the $[k\ell - 3, 3]$-module. So we can conclude that $\tau = -\frac{(k-1)d}{k(\ell-1)}$ is the least eigenvalue of $X_{k,\ell}$ and that τ belongs only to the $[k\ell - 2, 2]$-module.

By the ratio bound, Theorem 2.1, the maximum size of coclique in $X_{k,\ell}$ is

$$\frac{|V(X_{k,\ell})|}{1 - \frac{d}{\tau}} = \frac{v}{1 - \frac{d}{-\frac{(k-1)d}{k(\ell-1)}}} = \frac{v}{1 + \frac{k(\ell-1)}{k-1}} = \frac{v(k-1)}{k\ell - 1} = \binom{k\ell - 2}{k - 2} u_{k,\ell-1}. \qquad \square$$

The previous result shows that the sets $S_{i,j}$ are the largest intersecting sets. We further conjecture that these sets are the only maximum intersecting sets.

Conjecture 6.2. *For $k \geq 3$ and ℓ sufficiently large, the only sets of partially 2-intersecting (k, ℓ)-partitions with size $\binom{k\ell-2}{k-2} u_{k,\ell-1}$ are the sets $S_{i,j}$.*

We can make a step towards this conjecture with the following weaker characterization of the maximum intersecting sets. Denote the characteristic vectors of the sets $S_{i,j}$ by $v_{i,j}$.

Corollary 6.3. *For a fixed integer $k \geq 3$ and ℓ sufficiently large, let S be any maximum partially 2-intersecting set of (k, ℓ)-partitions. Then the characteristic vector of S is a linear combination of the vectors $v_{i,j}$.*

Proof. For $k \geq 3$ and ℓ sufficiently large, $S_{i,j}$ is a maximum coclique in $X_{k,\ell}$ and equality holds in the ratio bound. Let $v_{i,j}$ be the characteristic vector of $S_{i,j}$. Since we have equality in the ratio bound, this implies that

$$v_{i,j} - \frac{k-1}{k\ell - 1} \mathbf{1}$$

is a τ-eigenvector (where $\mathbf{1}$ is the all ones vector). Since no other modules have have eigenvalue τ, these vectors are in the $[k\ell-2,2]$-module. Further, the set of vectors

$$\left\{ v_{i,j} - \frac{k-1}{k\ell-1}\,\mathbf{1} \mid i,j \in \{1,\dots,k\ell\} \right\}$$

is invariant under the action of $\mathrm{Sym}(k\ell)$, so they form a module. Since the $[k\ell-2,2]$-module is irreducible, these vectors span the entire $[k\ell-2,2]$-module; this also implies that the vectors $\{v_{i,j} \mid i,j \in \{1,\dots,k\ell\}\}$ span the $[k\ell]$ and $[k\ell-2,2]$-modules.

Let S be a partially 2-intersecting set of (k,ℓ)-partition of maximum size, and let v_S denote the characteristic vector of S. Then $v_S - \frac{k-1}{k\ell-1}\mathbf{1}$ is in the $[k\ell-2,2]$-module. Thus v_S is in the span of the $[k\ell]$ and $[k\ell-2,2]$-module, so v_S is a linear combination of the vectors $v_{i,j}$. □

7 Exact result for $k = 3$

In this section we will prove Theorem 6.1 holds for all $\ell \geq 3$, when $k = 3$. It is already known, see Corollary 7.5.6 in [19], that Theorem 6.1 holds in the case where $k = 3$ and ℓ is odd; this follows from the existence of resolvable packing designs of an appropriate size.

For $k = 3$, we observed experimentally that the ratio $u_{3,\ell}/d_{3,\ell}$ converges to $e^{\frac{(k-1)^2}{2}} = e^2$ surprisingly quickly. If the sequence of $u_{3,\ell}/d_{3,\ell}$ was non-increasing this would be sufficient, but we have no proof of this. Rather, in this section we show an upper bound on the ratio $u_{3,\ell}/d_{3,\ell}$ for all ℓ, or, equivalently, a lower bound on $d_{3,\ell}$. This bound holds for $\ell > 10$, and we simply directly check the theorem for the specific graphs with smaller values of ℓ.

Lemma 7.1. *For $\ell > 10$, the degree, $d_{3,\ell}$ is greater than $u_{3,\ell}/24$.*

Proof. We will use a truncated inclusion-exclusion argument to bound the degree. Since $X_{k,\ell}$ is vertex transitive, we obtain a bound on the degree by counting the neighbours of an arbitrary partition $P \in \mathcal{U}_{3,\ell}$.

Fix a partition $P \in \mathcal{U}_{3,\ell}$ and let $\mathcal{J}$ be the set of pairs $\{x,y\}$ of elements in $\{1,2,\dots,3\ell\}$ that are contained in the same block of P. Note that $|\mathcal{J}| = 3\ell$. For a pair $\{x,y\} \in \mathcal{J}$, we define $A_{\{x,y\}}$ to be the set of all partitions which contain x and y in the same block. Further, for a subset $J \subseteq \mathcal{J}$, define

$$N(J) = |\cap_{\{x,y\}\in J} A_{\{x,y\}}|$$

and for $0 \leq j \leq 3\ell$ let $N_j = \sum_{J,|J|=j} N(J)$. By inclusion-exclusion,

$$d_{3,\ell} = \sum_{j=0}^{3\ell} -1^j N_j. \tag{7.1}$$

Next we calculate N_j. First, we note that $N_0 = N(\emptyset) = u_{3,\ell}$.

For any subset $J \subseteq \mathcal{J}$, each block in P contains either $0,1,2$ or 3 of the pairs from J. For $i = 0,1,2,3$, let n_i be the number of blocks in P that have exactly i of their pairs in J. We call the 4-tuple (n_0,n_1,n_2,n_3) the *pair distribution of* J and note that $n_0 + n_1 + n_2 + n_3 = \ell$. For each block of P its *block type relative to* J is the number of pairs in J that are in the block. With this terminology, we can find N_j.

First, fix a subset $J \subseteq \mathcal{J}$ with pair distribution (n_0, n_1, n_2, n_3) and count the number of partitions $Q \in \cap_{\{x,y\}\in J} A_{\{x,y\}}$. Each block of P with block type n_3 relative to J determines exactly which three elements are in a block of Q, as do the blocks of type n_2. Each of the blocks of type n_1 determines two of the three points in the block of Q. One more point must be chosen to complete each of these blocks, and this choice is ordered since each pair of type n_1 from J uniquely labels its corresponding block. Each of the blocks of type n_0 does not determine any points in Q. Thus the number of partitions Q which contain the pairs from J is given by the multinomial coefficient

$$\frac{1}{n_0!}\binom{3\ell - 3(n_3+n_2) - 2n_1}{1^{(n_1)}, 3^{(n_0)}}$$

(where the exponent in braces indicates the number is repeated that many times).

We now count the number of possible J which have pair distribution (n_0, n_1, n_2, n_3). The number of ways to select the type of each block in P is equal to the multinomial coefficient

$$\binom{\ell}{n_0, n_1, n_2, n_3},$$

since we are choosing the blocks from $\mathcal{P}$ that have either $0, 1, 2$ or 3 pairs in J. Each of the blocks of P with type n_3 has all of its three pairs in J, while for each block of type n_2 there are three ways to choose which two of the three possible pairs are in J. Similarly, for each block of type n_1 there is one pair in J, and there are three ways to chose this pair. Finally, each of the n_0 blocks does not contribute any pairs to J. Thus there are

$$3^{n_1+n_2}$$

different sets J in with pair distribution (n_0, n_1, n_2, n_3).

Finally, we sum the number of partitions $Q \in \cap_{\{x,y\}\in J} A_{\{x,y\}}$ over all possible pair distributions that J can have. Each pair distribution (n_0, n_1, n_2, n_3) is an ordered partition of ℓ into exactly four non-negative parts. The pair distribution (n_0, n_1, n_2, n_3) corresponds to a set J of size $n_1 + 2n_2 + 3n_3$. Define $\mathcal{C}(\ell, j)$ to be the set of compositions of ℓ into four parts with $n_1 + 2n_2 + 3n_3 = j$.

Then from our previous counting we have that

$$N_j = \sum_{(n_0,n_1,n_2,n_3)\in\mathcal{C}(\ell,j)} 3^{n_1+n_2}\binom{\ell}{n_0, n_1, n_2, n_3}\frac{1}{n_0!}\binom{3\ell - 3(n_3+n_2) - 2n_1}{1^{(n_1)}, 3^{(n_0)}}.$$

When we put this value in Equation 7.1 and truncate this sum after an odd j we will get a lower bound on $d_{3,\ell}$. Taking j up to 5 we sum over the following list of pair distributions:

$$\begin{aligned}
&\mathcal{C}(\ell,0) = \{(\ell,0,0,0)\}\\
&\mathcal{C}(\ell,1) = \{(\ell-1,1,0,0)\}\\
&\mathcal{C}(\ell,2) = \{(\ell-1,0,1,0),(\ell-2,2,0,0)\}\\
&\mathcal{C}(\ell,3) = \{(\ell-1,0,0,1),(\ell-2,1,1,0),(\ell-3,3,0,0)\}\\
&\mathcal{C}(\ell,4) = \{(\ell-2,1,0,1),(\ell-2,0,2,0),(\ell-3,2,1,0),(\ell-4,4,0,0)\}\\
&\mathcal{C}(\ell,5) = \{(\ell-2,0,1,1),(\ell-3,2,0,1),(\ell-3,1,2,0),(\ell-4,3,1,0),(\ell-5,5,0,0)\}
\end{aligned}$$

Expanding this becomes

$$d_{3,\ell} \geq \sum_{j=0}^{5} -1^j \sum_{(n_0,n_1,n_2,n_3)\in\mathcal{C}(\ell,j)} 3^{n_1+n_2} \binom{\ell}{n_0,n_1,n_2,n_3} \frac{\binom{3\ell-3(n_3+n_2)-2n_1}{1^{(n_1)},3^{(n_0)}}}{n_0!}$$

$$= \frac{\binom{\ell}{\ell}\binom{3\ell}{3^{(\ell)}}}{(\ell)!} + \frac{-3\binom{\ell}{\ell-1,1}\binom{3\ell-2}{1,3^{(\ell-1)}}}{(\ell-1)!} + \frac{3\binom{\ell}{\ell-1,1}\binom{3\ell-3}{3^{(\ell-1)}}}{(\ell-1)!} + \frac{3^2\binom{\ell}{\ell-2,2}\binom{3\ell-4}{1^{(2)},3^{(\ell-2)}}}{(\ell-2)!}$$

$$+ \frac{-\binom{\ell}{\ell-1,1}\binom{3\ell-3}{3^{(\ell-1)}}}{(\ell-1)!} + \frac{-3^2\binom{\ell}{\ell-2,1^{(2)}}\binom{3\ell-5}{1,3^{(\ell-2)}}}{(\ell-2)!} + \frac{-3^3\binom{\ell}{\ell-3,3}\binom{3\ell-6}{1^{(3)},3^{(\ell-3)}}}{(\ell-3)!}$$

$$+ \frac{3\binom{\ell}{\ell-2,1^{(2)}}\binom{3\ell-5}{1,3^{(\ell-2)}}}{(\ell-2)!} + \frac{3^2\binom{\ell}{\ell-2,2}\binom{3\ell-6}{3^{(\ell-2)}}}{(\ell-2)!} + \frac{3^3\binom{\ell}{\ell-3,2,1}\binom{3\ell-7}{1^{(2)},3^{(\ell-3)}}}{(\ell-3)!}$$

$$+ \frac{3^4\binom{\ell}{\ell-4,4}\binom{3\ell-8}{1^{(4)},3^{(\ell-4)}}}{(\ell-4)!} + \frac{-3\binom{\ell}{\ell-2,1^{(2)}}\binom{3\ell-6}{3^{(\ell-2)}}}{(\ell-2)!} + \frac{-3^2\binom{\ell}{\ell-3,2,1}\binom{3\ell-7}{1^{(2)},3^{(\ell-3)}}}{(\ell-3)!}$$

$$+ \frac{-3^3\binom{\ell}{\ell-3,1,2}\binom{3\ell-8}{1,3^{(\ell-3)}}}{(\ell-3)!} + \frac{-3^4\binom{\ell}{\ell-4,3,1}\binom{3\ell-9}{1^{(3)},3^{(\ell-4)}}}{(\ell-4)!} + \frac{-3^5\binom{\ell}{\ell-5,5}\binom{3\ell-10}{1^{(5)},3^{(\ell-5)}}}{(\ell-5)!}$$

$$= \frac{(243\ell^6 - 2997\ell^5 + 13905\ell^4 - 32355\ell^3 + 42732\ell^2 - 32728\ell + 11200)(3\ell-10)!}{80(6^{\ell-4})(\ell-10)!(\ell^6 - 39\ell^5 + 625\ell^4 - 5265\ell^3 + 24574\ell^2 - 60216\ell + 60480)}.$$

Thus

$$\frac{u_{3,\ell}}{d_{3,\ell}} \leq \frac{5(729\ell^6 - 6561\ell^5 + 23085\ell^4 - 40095\ell^3 + 35586\ell^2 - 14904\ell + 2240)}{243\ell^6 - 2997\ell^5 + 13905\ell^4 - 32355\ell^3 + 4273\ell^2 - 32728\ell + 11200}.$$

For $\ell > 10$ this gives that $u_{3,\ell}/d_{3,\ell} < 24$. □

Theorem 7.2. *For $k = 3$ and all $\ell \geq 3$ the largest set of partially 2-intersecting uniform partitions has size*

$$(3\ell-2)u_{3,\ell-1}.$$

Proof. For $\ell = 3$ all the eigenvalues of $X_{3,3}$ have long been known to be $\{36, 8, 2, -4, -12\}$ [18]. The ratio bound holds with equality, and the only irreducible representation that belongs to the least eigenvalue is $\chi_{[7,2]}$.

For $\ell = 4$, all the eigenvalues of $X_{3,4}$ are $\{1296, 96, 72, 48, 32, 0, -24, -48, -288\}$. These can be calculated by making a quotient graph of $X_{3,4}$ from the action of $\mathrm{Sym}(3) \wr \mathrm{Sym}(4)$ on the partitions. This equitable partition has a cell of size 1, so the eigenvalues of the quotient graph are exactly the eigenvalues of $X_{3,4}$. Further, the multiplicities of the eigenvalues can be calculated using the formulas in [14, Section 5.3] and the $[10, 2]$-module is the only module to which the eigenvalue -288 belongs.

For $\ell \in \{5, \ldots, 12\}$ the only irreducible representations with dimension less then $\binom{3\ell}{3} - \binom{3\ell}{2}$ in the decomposition of $\mathrm{ind}\left(1_{\mathrm{Sym}(3)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(3\ell)}$ are the three listed in Theorem 3.4—this can be checked using GAP [10]. Thus Theorem 3.4 holds for all $5 \leq \ell \leq 12$ when $k = 3$.

For all $\ell > 10$, Lemma 7.1 shows that $u_{k,\ell}/d_{k,\ell} - 1 < 23$. In this same range, the right hand side of Equation (6.1) is at least 26. Thus the inequality from Equation (6.1) holds for all $\ell > 10$.

For $5 \leq \ell \leq 10$ the degrees $d_{3,\ell}$ can be directly computed

$$\begin{aligned} d_{3,5} &= 132192, & d_{3,7} &= 3829057920, & d_{3,9} &= 333973115062272, \\ d_{3,6} &= 19258560, & d_{3,8} &= 1001695548672, & d_{3,10} &= 138348645213579264, \end{aligned}$$

and the inequality from Equation (6.1) directly checked. □

8 Further work

In this paper we only consider partially 2-intersecting partitions, but the conjecture in [20] is for partial t-intersection sets of partitions with $k \leq \ell(t-1)$. It is possible that the approach in this paper could be applied for larger values of t, but there are some steps that we predict will be complicated.

It is straight-forward to generalize the definition of $X_{k,\ell}$ to partially t-intersecting partitions by defining the graph $X_{t,k,\ell}$. This graph will also have $\mathcal{U}_{k,\ell}$ as its vertex set, and two partitions P and Q are adjacent if and only if for all pairs of blocks $P_i \in P$ and $Q_j \in Q$ we have $|P_i \cap Q_j| < t$. A partially t-intersecting set of partitions is a coclique in $X_{t,k,\ell}$.

The conjecture is if $k < \ell(t-1)$, then the maximum cocliques in $X_{t,k,\ell}$ are exactly the canonical partially t-intersecting sets. The Young subgroup $\mathrm{Sym}([k\ell - t, t])$ is the stabilizer of a canonically partially t-intersecting set. The most significant complication is that for $t > 2$, there are more than two irreducible representations in both

$$\mathrm{ind}\left(1_{\mathrm{Sym}([k\ell-t,t])}\right)^{\mathrm{Sym}(k\ell)} \quad \text{and} \quad \mathrm{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}. \tag{8.1}$$

For the approach given in this paper to work, we believe the eigenvalues belonging to all the irreducible representations common to these two induced representations, except the trivial representation, should be the least eigenvalue of $X_{t,k,\ell}$. To make this happen we suspect that a weighted adjacency matrix of $X_{t,k,\ell}$ would be needed in the ratio bound, rather than just the adjacency matrix; the weighting would have to be chosen so that the common modules (except the trivial) in the representations in (8.1) all belong to the same eigenvalue. Another complication is that potentially more of the eigenvalues of $X_{t,k,\ell}$ would have to be calculated, at the very least all the eigenvalues belonging to the common representations would need to be known.

Bender's theorem is much more general than the version we stated here. We only state Bender's theorem for matrices with 01-entries, but the full theorem applies to matrices with entries less than t. Using the full theorem we would be able to approximate the degree of $X_{t,k,\ell}$ for $t \geq 2$.

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Almost simple groups as flag-transitive automorphism groups of symmetric designs with λ prime*

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Abstract

In this article, we study symmetric designs with λ prime admitting a flag-transitive and point-primitive automorphism group G of almost simple type with socle X. We prove that either $\mathcal{D}$ is one of the six well-known examples of biplanes and triplanes, or $\mathcal{D}$ is the point-hyperplane design of $\mathrm{PG}(n-1, q)$ with $\lambda = (q^{n-2}-1)/(q-1)$ prime and $X = \mathrm{PSL}_n(q)$.

Keywords: Almost simple group, automorphism group, flag-transitive, point-primitive, symmetric design.

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1 Introduction

Symmetric designs admitting flag-transitive automorphism groups are of most interest. Kantor [14] classified flag-transitive symmetric $(v, k, 1)$ designs known as projective planes of order n, and showed that either $\mathcal{D}$ is a Desarguesian projective plane and $\mathrm{PSL}_3(n) \trianglelefteq G$, or G is a sharply flag-transitive Frobenius group of odd order $(n^2+n+1)(n+1)$, where n is even and n^2+n+1 is prime. Regueiro gave a classification of nontrivial symmetric designs with $\lambda = 2$ (biplanes) admitting flag-transitive automorphism groups apart from those groups contained in a 1-dimensional affine group [17, 18, 19, 20, 21]. Dong, Fang and Zhou studied flag-transitive automorphism groups G of nontrivial symmetric $(v, k, 3)$ designs (triplanes), and in conclusion, excluding the case where $G \leqslant \mathrm{A\Gamma L}_1(q)$ where $q = p^m$ with $p \geqslant 5$ prime, they determined all such possible symmetric designs [13, 25, 26, 27, 28].

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Table 1: Some symmetric designs with λ prime.

Line	v	k	λ	X	H	G	Designs	References*
1	7	4	2	$\mathrm{PSL}_2(7)$	S_4	$\mathrm{PSL}_2(7)$	Complement of the Fano plane	[2, 8]
2	11	5	2	$\mathrm{PSL}_2(11)$	A_5	$\mathrm{PSL}_2(11)$	Hadamard	[2, 8]
3	11	6	3	$\mathrm{PSL}_2(11)$	A_5	$\mathrm{PSL}_2(11)$	Complement of line 2	[2, 8]
4	15	7	3	A_7	$\mathrm{PSL}_3(2)$	A_7	$\mathrm{PG}_2(3,2)$	[8, 26, 29]
5	45	12	3	$\mathrm{PSU}_4(2)$	$2.(\mathrm{A}_4{\times}\mathrm{A}_4).2$	$\mathrm{PSU}_4(2)$	-	[7, 11, 22]
6	45	12	3	$\mathrm{PSU}_4(2)$	$2.(\mathrm{A}_4{\times}\mathrm{A}_4).2{:}2$	$\mathrm{PSU}_4(2){:}2$	-	[7, 11, 22]

Note: The last column addresses to references in which a design with the parameters in the line has been constructed.

Recently, Z. Zhang, Y. Zhang and Zhou in [24] proved that if $\mathcal{D}$ is a nontrivial symmetric (v, k, λ) design with λ prime and G is a flag-transitive and point-primitive automorphism group of $\mathcal{D}$, then G must be of affine or almost simple type. In this paper, we study symmetric designs with λ prime admitting a flag-transitive and point-primitive automorphism group of almost simple type. Indeed, we have already shown in [4] that almost simple exceptional groups of Lie type give rise to no possible symmetric designs with λ prime. In addition, we investigated the case where G is an almost simple group with socle X being a finite simple classical group of Lie type, and proved that $\mathcal{D}$ is either the point-hyperplane design of a projective space $\mathrm{PG}(n-1, q)$, or it is of parameters $(7, 4, 2)$, $(11, 5, 2)$, $(11, 6, 3)$ or $(45, 12, 3)$. Here, we focus on the case where G is an almost simple group with socle X an alternating group:

Theorem 1.1. *Let $\mathcal{D}$ be a nontrivial symmetric (v, k, λ) design with λ prime and α a point of $\mathcal{D}$, and let G be a flag-transitive and point-primitive automorphism group of $\mathcal{D}$ of almost simple type. If the socle of G is an alternating group A_c with $c \geqslant 5$, then $\mathcal{D} = \mathrm{PG}_2(3, 2)$ with parameters $(15, 7, 3)$ and $G = \mathrm{A}_7$ with the point-stabiliser $G_\alpha = \mathrm{PSL}_3(2)$.*

By Theorem 1.1 and the main results of [3, 23], we consequently obtain all symmetric designs with λ prime admitting flag-transitive and point-primitive automorphism groups of almost simple type:

Corollary 1.2. *Let $\mathcal{D}$ be a nontrivial symmetric (v, k, λ) design with λ prime. Suppose that G is a flag-transitive and point-primitive automorphism group of $\mathcal{D}$ of almost simple type with socle X. Then $\lambda = 2, 3$ and $(\mathcal{D}, G)$ is as in one of the lines of* Table 1*, or $\mathcal{D}$ is the point-hyperplane design of* $\mathrm{PG}(n-1, q)$ *with $\lambda = (q^{n-2} - 1)/(q - 1)$ prime and $X = \mathrm{PSL}_n(q)$.*

In order to prove Theorem 1.1, for the case where $\lambda \leqslant 100$ or $\gcd(k, \lambda) = 1$, by [29, 30], we obtain the designs in the statement. Then we assume that $\lambda > 100$ and λ divides k (as λ is prime). In this case, we show that there is no symmetric deign with λ prime and flag-transitive and point-primitive automorphism group G. Here, we first observe that the point-stabiliser H of G has to be large, that is to say, $|G| \leqslant |H|^3$, see Corollary 2.2. The possibilities for H can be read off from [5]. In Section 3, we examine these possibilities and achieve our desired result. In Section 4, we give a detailed proof of Corollary 1.2 which follows immediately from Theorem 1.1 and the main results of [3, 23].

1.1 Definitions and notation

All groups and incidence structures in this paper are finite. A group G is said to be *almost simple* with socle X if $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$, where X is a nonabelian simple group. Symmetric and alternating groups on c letters are denoted by S_c and A_c, respectively. We write "n" for group of order n. A symmetric (v,k,λ) design $\mathcal{D}$ is a pair $(\mathcal{P},\mathcal{B})$ with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of v blocks such that each block is a k-subset of $\mathcal{P}$ and each pair of distinct points is contained in exactly λ blocks. We say that $\mathcal{D}$ is nontrivial if $2 < k < v-1$. A *flag* of $\mathcal{D}$ is a point-block pair (α, B) such that $\alpha \in B$. An *automorphism* of $\mathcal{D}$ is a permutation on $\mathcal{P}$ which maps blocks to blocks and preserving the incidence. The *full automorphism* group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ is the group consisting of all automorphisms of $\mathcal{D}$. For $G \leqslant \mathrm{Aut}(\mathcal{D})$, G is called *flag-transitive* if G acts transitively on the set of flags. The group G is said to be *point-primitive* if G acts primitively on $\mathcal{P}$. For a given positive integer n and a prime divisor p of n, we denote the p-part of n by n_p, that is to say, $n_p = p^t$ with $p^t \mid n$ but $p^{t+1} \nmid n$. Further notation and definitions in both design theory and group theory are standard and can be found, for example in [6, 9, 12, 15, 16].

2 Preliminaries

In this section, we state some useful facts in both design theory and group theory. If a group G acts on a set $\mathcal{P}$ and $\alpha \in \mathcal{P}$, the *subdegrees* of G are the length of orbits of the action of the point-stabiliser G_α on $\mathcal{P}$.

Lemma 2.1 ([2, Lemma 2.1])**.** *Let $\mathcal{D}$ be a symmetric (v,k,λ) design, and let G be a flag-transitive automorphism group of $\mathcal{D}$. If α is a point of $\mathcal{D}$, then*

(i) $k(k-1) = \lambda(v-1)$*;*

(ii) *k divides $|G_\alpha|$, and $\lambda v < k^2$;*

(iii) *$k \mid \lambda d$, for all nontrivial subdegrees d of G.*

For a point-stabiliser H of a flag-transitive automorphism group G of a design $\mathcal{D}$, by Lemma 2.1(ii), we conclude that $\lambda|G| \leqslant |H|^3$, and so we have that

Corollary 2.2. *Let $\mathcal{D}$ be a symmetric (v,k,λ) design with a flag-transitive automorphism group G and α a point of $\mathcal{D}$. Then $|G| \leqslant |G_\alpha|^3$.*

Lemma 2.3. *Suppose that s and t are positive integers. Then*

(i) *if $t > s \geqslant 9$, then $\binom{s+t}{s} > s^2t^3$;*

(ii) *if $s \geqslant 4$ and there exists $t_0 \geqslant 7$ such that $\binom{s+t_0}{s} > s^2t_0^3$, then $\binom{s+t}{s} > s^2t^3$ for all $t \geqslant t_0$.*

Proof. (i) If $t > s = 9$, then we observe that the inequality $\binom{s+t}{s} = \binom{t+9}{9} > 81t^3 = s^2t^3$ holds. If $t > s \geqslant 10$, then $10 \leqslant s \leqslant \frac{s+t}{2}$, and so that $\binom{s+t}{s} \geqslant \binom{10+t}{10} > t^5$. Since $t > s$, we have that $t^5 > s^2t^3$, and hence $\binom{s+t}{s} > t^5 > s^2t^3$.

(ii) It suffices to show that $\binom{s+t_0+1}{s} > s^2(t_0+1)^3$. Note that

$$\binom{s+t_0+1}{s} = \binom{s+t_0}{s}\frac{(s+t_0+1)}{(t_0+1)} > s^2t_0^3\frac{(s+t_0+1)}{(t_0+1)}$$
$$= s^2(t_0+1)^3\frac{(s+t_0+1)t_0^3}{(t_0+1)^4}.$$

Since $t_0 \geqslant 7$ and $s \geqslant 4$, it follows that $(s+t_0+1)t_0^3 \geqslant (t_0+5)t_0^3 > (t_0+1)^4$. Therefore, $\binom{s+t_0+1}{s} > s^2(t_0+1)^3$. □

3 Proof of Theorem 1.1

Suppose that G is a flag-transitive and point-primitive automorphism group of a symmetric (v,k,λ) design $\mathcal{D}$ with λ prime. Suppose that X is the alternating group A_c of degree $c \geqslant 5$ on $\Omega = \{1,\dots,c\}$ and that $H := G_\alpha$ with α a point of $\mathcal{D}$. Then H is maximal in G by [12, Corollary 1.5A], and since $G = HX$, we conclude that

$$v = \frac{|X|}{|H\cap X|}. \tag{3.1}$$

If $\lambda \leqslant 100$ or $\gcd(k,\lambda) = 1$, then by [29, 30], we conclude that $\mathcal{D}$ is $\mathrm{PG}_2(3,2)$ with parameters $(15,7,3)$, and $G = \mathrm{A}_7$ with the point-stabiliser $H = \mathrm{PSL}_3(2)$. Therefore, we can assume that $\lambda > 100$ and $\gcd(k,\lambda) \neq 1$. Since $k(k-1) = \lambda(v-1)$ and $\gcd(k,\lambda) \neq 1$, we conclude that $\lambda \mid k$, and so by Lemma 2.1(ii), the parameter λ divides $|H|$. In what follows, assuming that $\lambda > 100$ divides k and $\gcd(k,\lambda) \neq 1$, we show that there is no flag-transitive and point-primitive automorphism group of a symmetric (v,k,λ) design $\mathcal{D}$ with λ prime.

Let $H_0 := H \cap X$. Then by [5, Theorem 2 and Proposition 6.1], one of the following holds:

(i) H_0 is intransitive on $\Omega = \{1,\dots,c\}$;

(ii) H_0 is transitive and imprimitive on $\Omega = \{1,\dots,c\}$;

(iii) $G = \mathrm{S}_c$ and (c,H) is one of the following:

$$(5,\mathrm{AGL}_1(5)),\quad (6,\mathrm{PGL}_2(5)),\quad (7,\mathrm{AGL}_1(7)),\quad (8,\mathrm{PGL}_2(7)),$$
$$(9,\mathrm{AGL}_2(3)),\quad \left(10,\mathrm{A}_6{\cdot}2^2\right),\quad (12,\mathrm{PGL}_2(11));$$

(iv) $G = \mathrm{A}_6{\cdot}2 = \mathrm{PGL}_2(9)$ and H is D_{20} or a Sylow 2-subgroup P of G of order 16;

(v) $G = \mathrm{A}_6{\cdot}2 = \mathrm{M}_{10}$ and H is $\mathrm{AGL}_1(5)$ or a Sylow 2-subgroup P of G of order 16;

(vi) $G = \mathrm{A}_6{\cdot}2^2 = \mathrm{P\Gamma L}_2(9)$ and H is $\mathrm{AGL}_1(5){\times}2$ or a Sylow 2-subgroup P of G of order 32;

(vii) $G = \mathrm{A}_c$ and (c,H) is one of the following:

$$(5,\mathrm{D}_{10}),\quad (6,\mathrm{PSL}_2(5)),\quad (7,\mathrm{PSL}_2(7)),\quad (8,\mathrm{AGL}_3(2)),$$
$$(9,3^2{\cdot}\mathrm{SL}_2(3)),\quad (9,\mathrm{P\Gamma L}_2(8)),\quad (10,\mathrm{M}_{10}),\quad (11,\mathrm{M}_{11}),$$
$$(12,\mathrm{M}_{12}),\quad (13,\mathrm{PSL}_3(3)),\quad (15,\mathrm{A}_8),\quad (16,\mathrm{AGL}_4(2)),$$
$$(24,\mathrm{M}_{24}).$$

Since λ is a prime divisor of k, it follows from Lemma 2.1(ii) that λ is a prime divisor of $|H|$. For the possibilities recorded in (iii) – (vii), we then have $\lambda \in \{2, 3, 5, 7, 11, 13, 23\}$, and this violates our assumption that $\lambda > 100$, see Table 2. Therefore, H_0 is either intransitive, or imprimitive.

Table 2: The possibilities for λ in cases (iii) – (vii) in Section 3.

Line	H	$\|H\|$	λ
1	$\mathrm{AGL}_1(5)$	$2^2 \cdot 5$	$2, 5$
2	$\mathrm{PGL}_2(5)$	$2^3 \cdot 3 \cdot 5$	$2, 3, 5$
3	$\mathrm{AGL}_1(7)$	$2 \cdot 3 \cdot 7$	$2, 3, 7$
4	$\mathrm{PGL}_2(7)$	$2^4 \cdot 3 \cdot 7$	$2, 3, 7$
5	$\mathrm{AGL}_2(3)$	$2^4 \cdot 3^3$	$2, 3$
6	$\mathrm{A}_6 \cdot 2^2$	$2^5 \cdot 3^2 \cdot 5$	$2, 3, 5$
7	$\mathrm{PGL}_2(11)$	$2^3 \cdot 3 \cdot 5 \cdot 11$	$2, 3, 5, 11$
8	D_{10}	$2 \cdot 5$	$2, 5$
9	$\mathrm{PSL}_2(5)$	$2^2 \cdot 3 \cdot 5$	$2, 3, 5$
10	$\mathrm{PSL}_2(7)$	$2^3 \cdot 3 \cdot 7$	$2, 3, 7$
11	$\mathrm{AGL}_3(2)$	$2^6 \cdot 3 \cdot 7$	$2, 3, 7$
12	$3^2 \cdot \mathrm{SL}_2(3)$	$2^3 \cdot 3^3$	$2, 3$
13	$\mathrm{P\Gamma L}_2(8)$	$2^3 \cdot 3^3 \cdot 7$	$2, 3, 7$
14	M_{10}	$2^4 \cdot 3^2 \cdot 5$	$2, 3, 5$
15	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$2, 3, 5, 11$
16	M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	$2, 3, 5, 11$
17	$\mathrm{PSL}_3(3)$	$2^4 \cdot 3^3 \cdot 13$	$2, 3, 13$
18	A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$2, 3, 5, 7$
19	$\mathrm{AGL}_4(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7$	$2, 3, 5, 7$
20	M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$2, 3, 5, 7, 11, 23$
21	D_{20}	$2^2 \cdot 5$	$2, 5$
22	$\mathrm{AGL}_1(5)$	$2^2 \cdot 5$	$2, 5$
23	P	2^4	2
24	$\mathrm{AGL}_1(5) \times 2$	$2^3 \cdot 5$	$2, 5$
25	P	2^5	2

Note: In line 23, P is a Sylow 2-subgroup of $G = \mathrm{A}_6 \cdot 2$ of order 16.
In line 25, P is a Sylow 2-subgroup of $G = \mathrm{A}_6 \cdot 2^2$ of order 32.

(I) Suppose that $H_0 = (\mathrm{S}_s \times \mathrm{S}_{c-s}) \cap \mathrm{A}_c$ is intransitive on $\Omega = \{1, \ldots, c\}$ with $1 \leqslant s < c/2$. In this case, $H = (\mathrm{S}_s \times \mathrm{S}_{c-s}) \cap G$. Note that H is maximal in G as long as $s \neq c - s$. Since λ is an odd prime divisor of $|H|$, it follows that λ divides $s!$ or $(c-s)!$, and since $s < c - s$, we conclude that

$$\lambda \leqslant \max\{s, c-s\} = c - s. \tag{3.2}$$

Note that H_0 contains all the even permutations of H, and hence $H_0 = H$ if $G = \mathrm{A}_c$, or the index of H_0 in H is 2 if $G = \mathrm{S}_c$. Since G is flag-transitive, H is transitive on the set of blocks passing through α. Hence H fixes exactly one point in $\mathcal{P}$, and so stabilizes exactly one s-subset, say S, in Ω. Therefore, we can identify the point α of $\mathcal{P}$ with the unique s-subset S of Ω stabilized by H. Thus $v = \binom{c}{s}$. Since H_0 acting on Ω is intransitive, it has at least two orbits. According to [10, page 82], two points of $\mathcal{P}$ are in the same orbit under H_0 if and only if the corresponding s-subsets S_1 and S_2 of Ω intersect S in the same number of points. Thus G acting on $\mathcal{P}$ has rank $s+1$, and each H_0-orbit $\mathcal{O}_i$ on $\mathcal{P}$ corresponds to a possible size $i \in \{0, 1, \ldots, s\}$ and these are precisely the families of s-subsets of Ω that intersect S, see also [1, Proposition 2.5]. Then if d_i is the length of a G-orbit on $\mathcal{P}$, then $d_0 = 1$, and $d_j = \binom{s}{j-1}\binom{c-s}{s-j+1}$ when $G = \mathrm{A}_c$ or $d_j = \binom{s}{j-1}\binom{c-s}{s-j+1}/2$ when $G = \mathrm{S}_c$ for $j = 1, \ldots, s$.

By Lemma 2.1(iii), we have that k divides λd_j for all nontrivial subdegrees d_j of G. By taking $j = s$, we have that k divides $\lambda s(c-s)$, and so $k \leqslant \lambda s(c-s)$. As $\lambda v < k^2$ by Lemma 2.1(ii), it follows from (3.2) that

$$v = \binom{c}{s} < \lambda s^2(c-s)^2 \leqslant s^2(c-s)^3.$$

Set $t := c - s$. Thus

$$\binom{s+t}{s} < s^2 t^3. \tag{3.3}$$

Applying Lemma 2.3(i), we conclude that (3.3) holds only for $s \leqslant 8$.

If $s = 1$, then $v = c \geqslant 5$. Note that G is $(v-2)$-transitive on $\mathcal{P}$. Since $2 < k \leqslant v-2$, G acts k-transitively on $\mathcal{P}$. Then $\binom{c}{k} = |B^G| = |\mathcal{B}| = v = c$ for every block $B \in \mathcal{B}$. This implies that $k = 1$ or $k = c-1$. Since $k \geqslant \lambda > 100$, we conclude that $k = c-1$, that is to say, $\mathcal{D}$ is a trivial design, which is a contradiction.

If $s = 2$, then the subdegrees are 1, $\binom{c-2}{2}$, $2(c-2)$ if $G = \mathrm{A}_c$, or 1, $\binom{c-2}{2}/2$, $(c-2)$ if $G = \mathrm{S}_c$. Thus G is a primitive permutation group of rank 3. Therefore, the possibilities for $\mathcal{D}$ can be read off from [11], which gives no example with $\lambda > 100$ prime.

If $s = 3$, then by Lemma 2.1(iii), k divides $\lambda d_3 = 3\lambda(c-3)$, and so $k = 3\lambda(c-3)/u$ for some positive integer u. We apply Lemma 2.1(i) and since $v - 1 = \binom{c}{3} - 1 = (c-3)(c^2+2)/6$, we deduce that

$$\frac{3\lambda(c-3)}{u} \cdot \left(\frac{3\lambda(c-3)}{u} - 1\right) = \frac{\lambda(c-3)(c^2+2)}{6},$$

and so

$$(c^2+2)u^2 + 18u - 54(c-3)\lambda = 0, \tag{3.4}$$

for some positive integer u. Define $f(x) := (c^2+2)x^2 + 18x - 54(c-3)\lambda$ with $x \geqslant 1$. Note here that $c > 2s = 6$ and $\lambda \leqslant c-3$ by (3.2). Then $f'(x) = 2(c^2+2)x + 18 > 0$ for $x \geqslant 1$. If $x \geqslant 8$, then since $\lambda \leqslant c-3$, we have that $f(x) = (c^2+2)x^2 + 18x - 54(c-3)\lambda \geqslant 10c^2 + 324c - 214 > 0$ for $c > 6$. Therefore, we cannot find any positive integer $u \geqslant 8$ satisfying (3.4), and hence $1 \leqslant u \leqslant 7$. Thus by (3.4), we have that $54\lambda = u^2(c+3) + (11u^2+18u)/(c-3)$, and so $c-3$ divides $11u^2+18u$, where $1 \leqslant u \leqslant 7$. Moreover, $\lambda > 100$ is a prime number less than 665 as $\lambda \leqslant c-3 \leqslant 11u^2+18u \leqslant 665$. For these values of u, c and λ, it is easy to check that (3.4) does not hold.

Table 3: An upper bound and a lower bound for t and c when $4 \leqslant s \leqslant 8$.

s	Bounds for t	Bounds for c
4	$5 \leqslant t \leqslant 373$	$9 \leqslant c \leqslant 377$
5	$6 \leqslant t \leqslant 46$	$11 \leqslant c \leqslant 51$
6	$7 \leqslant t \leqslant 22$	$13 \leqslant c \leqslant 28$
7	$8 \leqslant t \leqslant 14$	$15 \leqslant c \leqslant 21$
8	$9 \leqslant t \leqslant 11$	$17 \leqslant c \leqslant 19$

If $4 \leqslant s \leqslant 8$, then we apply (3.3) and Lemma 2.3(ii), and so we can find a lower bound and an upper bound for t as in the second column of Table 3. Since also $c = t - s$, we can find a lower bound and an upper bound for c as in the third column of Table 3. For example, if $s = 4$, then we take $t_0 = 374$ and observe that $\binom{s+t_0}{s} = \binom{378}{4} = 837222750 > 837017984 = 4^2 \cdot 374^3 = s^2 t_0^3$, then Lemma 2.3(ii) implies that if $t \geqslant 374$, then (3.3) does not hold, which is a contradiction. Thus $t \leqslant 373$. Moreover, it is easy to check that (3.3) holds for $t \geqslant 5$. Thus $5 \leqslant t \leqslant 373$. Note that $t = c - 4$, and hence $9 \leqslant c \leqslant 377$. This follows the first row of Table 3. For each t, s and c as in Table 3, we note by (3.2) that $\lambda \leqslant c - s = t$. Then we obtain $v = \binom{c}{s}$ and all the possibilities for prime $\lambda > 100$. But it is easy to check that for such parameters v and λ, we cannot find any possible parameters set (v, k, λ) satisfying Lemma 2.1.

(II) Suppose now that H_0 is transitive and imprimitive on $\Omega = \{1, \ldots, c\}$. In this case, $H = (\mathrm{S}_s \wr \mathrm{S}_{c/s}) \cap G$ is imprimitive, where s divides c and $2 \leqslant s \leqslant c/2$. Indeed, H_0 is transitive and imprimitive on $\Omega = \{1, \ldots, c\}$, H_0 acting on Ω preserves a partition Σ of Ω into t classes of size s with $t \geqslant 2$, $s \geqslant 2$ and $c = st$. Thus $H_0 \leqslant G_\Sigma < G$. Since G is isomorphic to S_c or A_c and since both natural actions of G and X on Ω are primitive, we conclude that H_0 contains all the even permutations of Ω preserving the partition Σ. By the same argument as in [10, Case 2], [17, (3.2)] and [29, pages 1489-1490], the imprimitive partition Σ is the only nontrivial partition of Ω preserved by H_0. Since X acts transitively on all the partitions of Ω into t classes of size s, we can identify the points of the $\mathcal{D}$ with the partitions of Ω into t classes of size s, and so $v = \binom{ts}{s}\binom{(t-1)s}{s} \cdots \binom{3s}{s}\binom{2s}{s}/(t!)$, that is to say,

$$v = \binom{ts-1}{s-1}\binom{(t-1)s-1}{s-1} \cdots \binom{3s-1}{s-1}\binom{2s-1}{s-1}. \tag{3.5}$$

We note that the suborbits of G on Ω can be described by the notion of j-cyclics introduced in [10, page 84]. Indeed, if a partition Σ_1 of Ω is a point of $\mathcal{P}$, then for $j = 2, \ldots, t$, the set Γ_j of j-cyclic partitions with respect to Σ_1 is a union of H-orbits on $\mathcal{P}$, see [10, Case 2] and [29, pages 1490-1491]. Therefore, by Lemma 2.1(iii), k divides λd_s, where

$$d_s = \begin{cases} s^2\binom{t}{2}, & \text{if } s \geqslant 3; \\ t(t-1), & \text{if } s = 2. \end{cases} \tag{3.6}$$

Therefore, by Lemma 2.1(iii), we have that k divides λd_s, where d_s is as in (3.6). Note that λ is a prime divisor of k, and so by Lemmas 2.1(ii), we conclude that $\lambda \leqslant c = st$.

If $s = 2$, then $t \geqslant 3$ as $c = st \geqslant 5$. By (3.5), we have that $v = \prod_{i=0}^{t-2}[2t-(2i+1)]$ and since k divides $\lambda d_2 = \lambda t(t-1)$ and $\lambda \leqslant c = 2t$, it follows from Lemma 2.1(ii) that

$$\prod_{i=0}^{t-2}[2t-(2i+1)] < \lambda d_2^2 \leqslant 2t^3(t-1)^2 < 2t^5.$$

This forces $t \leqslant 6$, and hence $\lambda \leqslant 2t = 12$, which is a contradiction as $\lambda > 100$.

If $s \geqslant 3$, then since,

$$\binom{is-1}{s-1} = \frac{is-1}{s-1}\cdot\frac{is-2}{s-2}\cdots\frac{is-(s-1)}{1} > i^{s-1}$$

with $2 \leqslant i \leqslant t$, by (3.5), we conclude that $v > t^{(s-1)(t-1)}$. Since also k divides $\lambda d_s = \lambda s^2\binom{t}{2}$ and $\lambda \leqslant st$, we deduce by Lemma 2.1(ii) that

$$t^{(s-1)(t-1)} < s^5 t\binom{t}{2}^2.$$

Thus $t^{(s-1)(t-1)-5} < s^5$. Note that $s \geqslant 3$. Then this inequality holds only for

$$\begin{aligned} &t = 2 \text{ and } 3 \leqslant s \leqslant 30;\\ &t = 3 \text{ and } 3 \leqslant s \leqslant 8;\\ &t = 4 \text{ and } s = 3, 4;\\ &t = 5 \text{ and } s = 3. \end{aligned}$$

However, for such pairs (t, s), we easily observe that $\lambda \leqslant st < 100$, which is a contradiction. This completes the proof.

4 Proof of Corollary 1.2

Let $\mathcal{D}$ be a nontrivial symmetric (v, k, λ) design with λ prime. Suppose that G is a flag-transitive and point-primitive automorphism group of $\mathcal{D}$ of almost simple type with socle X. Since λ is prime, by [23], the socle X cannot be a sporadic simple group. If the socle X is a simple group of Lie type, then by [3, Theorem 1], $\mathcal{D}$ is the point-hyperplane design of $\mathrm{PG}(n-1, q)$ with $\lambda = (q^{n-2}-1)/(q-1)$ prime and $X = \mathrm{PSL}_n(q)$, or $(\mathcal{D}, G)$ is as in one of the lines 1-3 and lines 5-6 of Table 1. If the socle X is an alternating group A_c of degree $c \geqslant 5$, then Theorem 1.1 implies that $\mathcal{D}$ is the unique design $\mathrm{PG}_2(3, 2)$ with parameters $(15, 7, 3)$, and $G = \mathrm{A}_7$ with the point-stabiliser $\mathrm{PSL}_3(2)$, and this follows line 4 of Table 1. This finishes the proof.

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A classification of connected cubic vertex-transitive bi-Cayley graphs over semidihedral group*

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Abstract

A graph Γ is said to be a *bi-Cayley graph* over a group H if there exists a subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to H acting semiregularly on its vertex set with two orbits. In this paper, we give a complete classification of connected cubic vertex-transitive bi-Cayley graphs over semidihedral group. As a byproduct, we construct a family of vertex-transitive non-Cayley graphs.

Keywords: Semidihedral group, bi-Cayley graph, vertex-transitive.

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1 Introduction

Throughout this paper, all groups are assumed to be finite, and all graphs are assumed to be finite, connected, simple and undirected. For a graph Γ, let $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$ denote vertex set, edge set and arc set of Γ, respectively. A graph Γ is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* if the full automorphism group Aut (Γ) acts transitively on $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$, respectively. For other terminology related to group theory and graph theory not defined here, we refer the reader to [1, 11].

Let G be a group and S be a subset of G such that $S^{-1} = S$ and $1 \notin S$. Then the *Cayley graph* $\Gamma = \mathrm{Cay}(G, S)$ over G with respect to S is defined as the graph with vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{\{g, sg\} \mid g \in G, s \in S\}$. Similarly, for a given group H, let R, L and S be subsets of H such that $R^{-1} = R$, $L^{-1} = L$ and $R \cup L$ does not contain the identity element of H. The *bi-Cayley graph* over H denoted by BiCay(H, R, L, S) is the graph having vertex set the union of the right part $H_0 = \{h_0 \mid h \in H\}$ and the left part $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the right edges $\{\{h_0,\ g_0\} \mid gh^{-1} \in R\}$, the left edges $\{\{h_1,\ g_1\} \mid gh^{-1} \in L\}$ and the spokes $\{\{h_0,\ g_1\} \mid gh^{-1} \in S\}$. When $|R| = |L| = s$, BiCay(H, R, L, S) is said to be an *s-type bi-Cayley graph*.

The triple (R, L, S) of three subsets R, L, S of a group H is called *bi-Cayley triple* if $R = R^{-1}, L = L^{-1}$ and $1 \in S$. Two bi-Cayley triples (R, L, S) and (R', L', S') of a group H are said to be *equivalent*, denoted by $(R, L, S) \equiv (R', L', S')$, if either $(R', L', S') = (R, L, S)^{\alpha}$ or $(R', L', S') = (L, R, S^{-1})^{\alpha}$ for some automorphism α of H. By Proposition 2.1(3)–(4), the bi-Cayley graphs corresponding to two equivalent bi-Cayley triples of the same group are isomorphic.

In the study of bi-Cayley graphs, a considerable attention was given to the following problem: for a given finite group H, classify bi-Cayley graphs over H with specific symmetry properties. For example, vertex-transitive (edge-transitive) generalized Petersen graphs had been classified in [4, 9]. Marušič and Pisanski in [6] classified all cubic arc-transitive bi-Cayley graphs over dihedral group. All tetravalent edge-transitive bicirculants (bi-Cayley group over cyclic group) were characterized in [5], a classification of cubic edge-transitive bi-Cayley graphs over inner abelian p-groups were presented in [10], and all cubic vertex-transitive bi-Cayley graphs over abelian groups were classified in [13]. Recently, Zhang and Zhou in [12] gave a classification of cubic edge-transitive bi-Cayley graphs over dihedral groups.

Motivated by the works listed above, in this paper, we shall investigate cubic bi-Cayley graphs over semidihedral groups. Recall that the *semidihedral group* of order $4n$ with n an even is defined as follows:

$$SD_{4n} = \langle a, b \mid a^{2n} = b^2 = 1, b^{-1}ab = a^{n-1} \rangle.$$

Note that all cubic bi-Cayley graphs over abelian groups have been classified in [13]. So we assume that $n \geq 4$.

The Petersen graph is a bi-Cayley graph over a cyclic group of order 5, and the Petersen graph is also a vertex-transitive non-Cayley graph. There are many research focusing on the classification of vertex-transitive non-Cayley graphs, see [3, 4, 7, 8]. By Magma, we found some examples of vertex-transitive non-Cayley graph Γ, where Γ is a cubic vertex-trasntive bi-Cayley graph over SD_{4n}. So another motivation for us to consider cubic vertex-transitive bi-Cayley graphs over SD_{4n} is to construct some kind of vertex-transitive non-Cayley graphs.

In [2] a classification of cubic edge-transitive bi-Cayley graphs over semidihedral group is given. For the completeness of the results, we list the main theorem in [2] in the following. (For the definition of $CQ(t, n)$, we refer the reader to [12])

Theorem 1.1 ([2, Theorem 1]). *Let Γ be a cubic connected bi-Cayley graph over semidihedral group SD_{4n}. Then Γ is edge-transitive if and only if (R, L, S) is equivalent to one of the following triples. Furthermore, all of the corresponding graphs are arc-transitive.*

(1) $(R, L, S) \equiv (\{b\}, \{ba^4\}, \{1, a\})$ with $n = 4$ and Γ is isomorphic to F032A.

(2) $(R, L, S) \equiv (\{b\}, \{ba^2\}, \{1, a\})$ with $n = 6$ and Γ is isomorphic to F048A.

(3) $(R, L, S) \equiv (\{b\}, \{ba^{2t}\}, \{1, a\})$, with t an odd, $3 \leq t \leq n - 3$, $n \mid 2(t^2 + t + 1)$ and Γ is isomorphic to $CQ(t, n)$.

(4) $(R, L, S) \equiv (\{b, ba^2\}, \{a, a^{-1}\}, \{1\})$ with $n = 4$ and Γ is isomorphic to F032A.

(5) $(R, L, S) \equiv (\{b, ba^6\}, \{a, a^{-1}\}, \{1\})$ with $n = 10$ and Γ is isomorphic to F080A.

(6) $(R, L, S) \equiv (\{b, ba^2\}, \{a, a^{-1}\}, \{1\})$ with $n = 12$ and Γ is isomorphic to F096A.

In this paper, we determine all cubic vertex-transitive bi-Cayley graphs over semidihedral group SD_{4n}. The main results are in the following.

Theorem 1.2. *Let Γ be a 0-type cubic connected bi-Cayley graph over SD_{4n}. Then Γ is a Cayley graph and Γ is isomorphic to* $\mathrm{BiCay}(SD_{4n}, \emptyset, \emptyset, S)$, *where $S = \{1, a, b\}$, $\{1, ba, b\}$ or $\{1, ba, a\}$.*

Theorem 1.3. *Let Γ be a 1-type cubic connected bi-Cayley graph over SD_{4n}. Then Γ is a vertex-transitive graph if and only if one of the followings holds. Furthermore all of the corresponding graphs are Cayley graphs.*

(1) $(R, L, S) \equiv (\{b\}, \{ba^i\}, \{1, a^j\})$ *with i an even, j an odd and the greatest common divisor of i, j and n is equal to* 1.

(2) $(R, L, S) \equiv (\{b\}, \{ba^l\}, \{1, ba\})$ *with* $(l - 1)^2 \equiv 1, n - 1 \pmod{2n}$.

Theorem 1.4. *Let Γ be a 2-type cubic connected bi-Cayley graph over SD_{4n}. Then Γ is a vertex-transitive graph if and only if one of the followings holds:*

(1) $(R, L, S) \equiv (\{b, ba^n\}, \{ba, ba^{n+1}\}, \{1\})$;

(2) $(R, L, S) \equiv (\{a, a^{-1}\}, \{b, ba^{2l}\}, \{1\})$ *with* $2l^2 \equiv 2 \pmod{2n}$;

(3) $(R, L, S) \equiv (\{a, a^{-1}\}, \{b, ba^{2l}\}, \{1\})$ *with* $2l^2 \equiv -2 \pmod{2n}$.

Furthermore, the graphs corresponding to (1) *and* (2) *are Cayley graphs and the graph corresponding to* (3) *is a non-Cayley graph.*

Theorem 1.1 gives a classification of cubic edge-transitive and arc-transitive bi-Cayley graphs over SD_{4n}. Theorems 1.2, 1.3 and 1.4 give a classification of cubic vertex-transitive bi-Cayley graphs over SD_{4n}. As a byproduct, we construct a family of vertex-transitive non-Cayley graphs which correspond to (3) in Theorem 1.4.

2 Preliminary

In this section, we give two properties of bi-Cayley graph.

Proposition 2.1 ([14, Lemma 3.1]). *For a bi-Cayley graph* $\mathrm{BiCay}(H, R, L, S)$ *over* H*, the following hold.*

(1) H *is generated by* $R \cup L \cup S$.

(2) *Up to graph isomorphism,* S *can be chosen to contain the identity of* H.

(3) *For any automorphism* α *of* H*,* $\mathrm{BiCay}(H,\ R,\ L,\ S) \cong \mathrm{BiCay}(H,\ R^{\alpha},\ L^{\alpha},\ S^{\alpha})$.

(4) $\mathrm{BiCay}(H,\ R,\ L,\ S) \cong \mathrm{BiCay}(H,\ L,\ R,\ S^{-1})$.

Let $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ S)$. For an automorphism α of H and $x, y, g \in H$, define two permutations of $V(\Gamma) = H_0 \cup H_1$ as follows:

$$\delta_{\alpha,x,y}\colon\ h_0 \mapsto (xh^{\alpha})_1,\ h_1 \mapsto (yh^{\alpha})_0,\ \forall h \in H,$$
$$\sigma_{\alpha,g}\colon\ h_0 \mapsto (h^{\alpha})_0,\ h_1 \mapsto (gh^{\alpha})_1,\ \forall h \in H.$$

Set

$$I = \{\delta_{\alpha,x,y} \mid \alpha \in \mathrm{Aut}\,(H)\ s.t.\ R^{\alpha} = x^{-1}Lx,\ L^{\alpha} = y^{-1}Ry,\ S^{\alpha} = y^{-1}S^{-1}x\},$$
$$F = \{\sigma_{\alpha,g} \mid \alpha \in \mathrm{Aut}\,(H)\ s.t.\ R^{\alpha} = R,\ L^{\alpha} = g^{-1}Lg,\ S^{\alpha} = g^{-1}S\}.$$

Proposition 2.2 ([14, Theorem 1.1]). *Let* $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ S)$ *be a bi-Cayley graph over the group* H*. Then* $N_{\mathrm{Aut}\,(\Gamma)}(R(H)) = R(H) \rtimes F$ *if* $I = \emptyset$ *and* $N_{\mathrm{Aut}\,(\Gamma)}(R(H)) = R(H)\langle F,\ \delta_{\alpha,x,y}\rangle$ *if* $I \neq \emptyset$ *and* $\delta_{\alpha,x,y} \in I$*. Furthermore, for any* $\delta_{\alpha,x,y} \in I$*, we have the following:*

(1) $\langle R(H),\ \delta_{\alpha,x,y}\rangle$ *acts transitively on* $V(\Gamma)$*;*

(2) *if* α *has order* 2 *and* $x = y = 1$*, then* Γ *is isomorphic to the Cayley graph* $\mathrm{Cay}(\bar{H},\ R \cup \alpha S)$*, where* $\bar{H} = H \rtimes \langle\alpha\rangle$.

3 Proof of main theorems

In the beginning of this section, firstly we give some basic properties of SD_{4n} in the following lemma without proof which are needed in the proof of our main Theorems.

Lemma 3.1. *The following hold.*

(1) $SD_{4n} = \langle a\rangle \cup b\langle a\rangle$*. Where* $b\langle a\rangle = \{ba^{2i}\} \cup \{ba^{2i+1}\}$ *with* $0 \leqslant i \leqslant n-1$*, and furthermore every element of set* $\{ba^{2i}\}$ *has order* 2 *and every element of set* $\{ba^{2i+1}\}$ *has order* 4.

(2) $\mathrm{Aut}\,(SD_{4n})$ *is transitive on sets* $\{ba^{2i}\}$ *and* $\{ba^{2i+1}\}$ *with* $0 \leqslant i \leqslant n-1$.

(3) *If* $SD_{4n} = \langle x, y\rangle$*, then there exists* $\alpha \in \mathrm{Aut}\,(SD_{4n})$ *mapping* $\{x, y\}$ *to one of the following subsets:*$\{a,\ b\}$, $\{ba,\ b\}$ $\{ba,\ a\}$.

For any integers i, j satisfying $(i, 2n) = 1$ and j is even, we have $\langle a^i, ba^j\rangle = \langle a, b\rangle = SD_{4n}$ and the map

$$\psi_{i,j} : a \mapsto a^i, b \mapsto ba^j$$

can induce an automorphism of SD_{4n}. In the following of this section we shall use $\psi_{i,j}$ to denote the automorphism of SD_{4n} induced by the above map.

Proof of Theorem 1.2. Since Γ is a 0-type bi-Cayley graph, $R = L = \emptyset$. By Proposition 2.1(1) and (2), let $S = \{1, g, h\}$ with $SD_{4n} = \langle g, h\rangle$. By Lemma 3.1(3), S is equivalent to one of the following three subsets: $\{1, a, b\}, \{1, ba, b\}, \{1, ba, a\}$. It is easy to check that $\psi_{-1,0}$, $\psi_{n+1,0}$ and $\psi_{-1,n+2}$ are three automorphisms of SD_{4n} of order 2 such that $\{1, a, b\}^{\psi_{-1,0}} = \{1, a, b\}^{-1}$, $\{1, ba, b\}^{\psi_{n+1,0}} = \{1, ba, b\}^{-1}$ and $\{1, ba, a\}^{\psi_{-1,n+2}} = \{1, ba, a\}^{-1}$. By Proposition 2.2, Γ is a Cayley graph. □

In order to get the classification of 1-type graph Γ, we give the following Lemma.

Lemma 3.2 ([12, Proposition 5.1]). *Let* $K = \langle a, b \mid a^n = b^2 = (ab)^2 = 1\rangle$ *be a dihedral group of order* $2n$*, and* $\Gamma_1 = \mathrm{BiCay}(K, \{b\}, \{ba^i\}, \{1, ba^j\})$ *be a cubic bi-Cayley graph over* K*. If* Γ_1 *is vertex-transitive, then* $(j, n) = 1$ *and* $j^2 \equiv \pm(j-i)^2 \pmod{n}$.

Proof of Theorem 1.3. Assume that Γ is a 1-type cubic vertex-transitive bi-Cayley graph over SD_{4n}. By the definition of 1-type bi-Cayley graph, we can let $R = \{x\}$, $L = \{y\}$ and $S = \{1, z\}$. As $R = R^{-1}$ and $L = L^{-1}$, both x and y are involutions. Since Γ is connected, by Proposition 2.1(1), $SD_{4n} = \langle x,\ y,\ z\rangle$. We confirm that there is at least one of x and y is not contained in $\langle a\rangle$. If not, $x = y = a^n$ implies that $SD_{4n} = \langle a^n, z\rangle$ for some $z \in SD_{4n}$, a contradiction. Without loss of generality, assume that $x \in b\langle a^2\rangle$. By Lemma 3.1(2), $\mathrm{Aut}\,(SD_{4n})$ acts transitively on the set $\{ba^{2k} \mid 0 \le k \le n-1\}$. So we assume that $x = b$. Now $y = a^n$ or $y = ba^i$ for some even i.

Case 1: $y = a^n \in \langle a\rangle$.

In this case, $z = a^m$ or ba^m. Since $SD_{4n} = \langle x, y, z\rangle$, one has $(m, 2n) = 1$. The map $a^m \mapsto a, b \mapsto b$ can induce an automorphism α of SD_{4n}, such that $(R, L, S)^\alpha = (\{b\}, \{a^n\}, \{1, a\})$ or $(\{b\}, \{a^n\}, \{1, ba\})$.

If $(R, L, S) \equiv (\{b\}, \{a^n\}, \{1, a\})$, then $(1_0, 1_1, (a^n)_1, (a^n)_0, (a^{n+1})_1, a_1)$ is the unique 6-cycle passing through 1_0. On the other hand, there exists a 6-cycle $(1_1, (a^n)_1, (a^{n-1})_0, (a^{n-1})_1, (a^{-1})_1, (a^{-1})_0)$ passing through 1_1 but not passing through 1_0, contrary to the vertex-transitivity of Γ.

Suppose that $(R, L, S) \equiv (\{b\}, \{a^n\}, \{1, ba\})$. Then there is a 5-cycle $(1_1, (a^n)_1, (a^n)_0, (ba^{n+1})_1, (ba^{n+1})_0)$ passing through 1_1 but not passing through 1_0. On the other hand, $(1_0, 1_1, (a^n)_1, (ba)_0, (ba)_1)$ and $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (ba)_1)$ are all 5-cycles passing through 1_0, and these also passing through 1_1, contrary to the vertex-transitivity of Γ.

Case 2: $y = ba^i \in b\langle a^2\rangle$ for some even i.

In this case, $z = a^j$ or ba^j for some odd j.

Subcase 2.1: $z = a^j \in \langle a\rangle$ for some odd j.

In this case, it is easy to check that $\psi_{-1,i}$ is an automorphism of SD_{4n} of order 2 such that $R^{\psi_{-1,i}} = L$, $L^{\psi_{-1,i}} = R$ and $S^{\psi_{-1,i}} = S^{-1}$. By Proposition 2.2(2), Γ is a Cayley graph. If $(j, 2n) \neq 1$, then $\langle a\rangle = \langle a^i, a^j\rangle$. So the greatest common divisor of i, j and n is equal to 1. This is the graph corresponding to (1) in the theorem.

Subcase 2.2: $z = ba^j \in b\langle a\rangle$ for some odd j.

Suppose that $j = \frac{n}{2}$ with odd $\frac{n}{2}$. By the connectivity of Γ, $\langle a\rangle = \langle a^{\frac{n}{2}}, a^i\rangle$, and hence $\langle a^i\rangle = \langle a^2\rangle$ or $\langle a^4\rangle$. By Proposition 2.1(3), we may assume that $a^i = a^2$ or a^4. Now Γ is isomorphic to a bi-Cayley graph with $(R, L, S) \equiv (\{b\}, \{ba^2\}, \{1, ba^{\frac{n}{2}}\})$ or $(R, L, S) \equiv (\{b\}, \{ba^4\}, \{1, ba^{\frac{n}{2}}\})$. In the above two cases, we can find that there are two 6-cycles passing through 1_0, namely $(1_0, b_0, b_1, (a^{\frac{3n}{2}})_0, (ba^{\frac{3n}{2}})_0, 1_1)$ and $(1_0, b_0, (a^{\frac{n}{2}})_1, (a^{\frac{n}{2}})_0, (ba^{\frac{n}{2}})_0, (ba^{\frac{n}{2}})_1)$. On the other hand, $(1_1, 1_0, b_0, b_1, (a^{\frac{3n}{2}})_0, (ba^{\frac{3n}{2}})_0)$ is the unique 6-cycle passing through 1_1, contrary to the vertex-transitivity of Γ. So, we may assume that $z \neq ba^{\frac{n}{2}}$.

Suppose that $(j, 2n) \neq 1$. Since $\langle a\rangle = \langle a^i, a^j\rangle$ with i an even and j an odd, the greatest common divisor of i, j and n is equal to 1. We consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 4 respectively. Since $j \not\equiv \frac{n}{2} (\mod 2n)$, one can see that $(1_0, 1_1, (ba^{n+j})_0, (ba^{n+j})_1, (a^n)_0, (a^n)_1, (ba^j)_0, (ba^j)_1)$ is the unique 8-cycle passing through 1_0 and it is also the unique 8-cycle passing through 1_1 too.

Since $(a^n)_0$ and $(a^n)_1$ are the unique vertices that have the longest distance from 1_0 and 1_1 in the 8-cycle, respectively, $\{1_0, (a^n)_0\}$ and $\{1_1, (a^n)_1\}$ are blocks of $\mathrm{Aut}\,(\Gamma)$ on $V(\Gamma)$. Let

$$\mathcal{C}_0 = \{\{1_0, (a^n)_0\}^{R(h)} \mid h \in SD_{4n}\}, \quad \mathcal{C}_1 = \{\{1_1, (a^n)_1\}^{R(h)} \mid h \in SD_{4n}\}$$

and $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$. Then $\mathcal{C}$ is a complete block system of $\mathrm{Aut}\,(\Gamma)$.

Let $\Gamma_{\mathcal{C}}$ be the quotient graph. Let $N = \langle R(a^n)\rangle$ and let K be the kernel of $\mathrm{Aut}\,(\Gamma)$ acting on $\mathcal{C}$. Now one can show that $K = N$ and $\Gamma_{\mathcal{C}}$ has valence 3 and $K = N$ is semiregular. Since $R(SD_{4n})$ acts on $V(\Gamma)$ semiregularly with two orbits, $R(SD_{4n})/N$ acts on $\mathcal{C}$ semiregularly with two orbits $\mathcal{C}_0$ and $\mathcal{C}_1$. So the quotient graph $\Gamma_{\mathcal{C}}$ is a bi-Cayley graph over $R(SD_{4n})/N$. Let $\overline{SD_{4n}} = R(SD_{4n})/N$ and let $\bar{h} = hN$ for any $h \in R(SD_{4n})$. Now $\overline{SD_{4n}} = \langle \bar{a}, \bar{b} \mid \bar{a}^n = \bar{b}^2 = (\bar{a}\bar{b})^2 = \bar{1}\rangle \cong D_{2n}$. Also we can assume that

$$V(\Gamma_{\mathcal{C}}) = \{\bar{h}_0 \mid \bar{h} \in \overline{SD_{4n}}\} \cup \{\bar{h}_1 \mid \bar{h} \in \overline{SD_{4n}}\}.$$

Note that for any $\bar{h} \in \overline{SD_{4n}}$

$$N_{\Gamma_{\mathcal{C}}}(\bar{h}_0) = \{\overline{bh}_0, \bar{h}_1, \overline{ba^j h}_1\}, \quad N_{\Gamma_{\mathcal{C}}}(\bar{h}_1) = \{\overline{ba^i h}_1, \bar{h}_0, \overline{ba^{n+j} h}_0\}.$$

So we may view $\Gamma_{\mathcal{C}}$ as the bi-Cayley graph $\mathrm{BiCay}(\overline{SD_{4n}}, \{\bar{b}\}, \{\overline{ba^i}\}, \{\bar{1}, \overline{ba^j}\})$. Since Γ is vertex-transive, the quotient graph $\Gamma_{\mathcal{C}}$ is also vertex-transitive. Since $\Gamma_{\mathcal{C}}$ is a 1-type cubic vertex-transitive bidihedrant, $\Gamma_{\mathcal{C}}$ is a Cayley graph by [12, Proposition 5.1]. Also by Lemma 3.2, $(j, n) = 1$ and $j^2 \equiv \pm(j-i)^2 (\mod n)$. This implies that $(j, 2n) = 1$, which is contrary to $(j, 2n) \neq 1$.

So we can assume that $(j, 2n) = 1$. The map $a^j \mapsto a$, $b \mapsto b$ can induce an automorphism β of SD_{4n}, such that $(R, L, S)^\beta \equiv (\{b\}, \{ba^l\}, \{1, ba\})$. We consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 4, respectively. Considering possible 8-cycles containing 1_0 and 1_1, we see that if $l \not\equiv 2, n, n+2, \frac{n}{2}+1 \pmod{2n}$, then $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1)$ is the unique 8-cycle passing through 1_0 and it is the unique 8-cycle passing through 1_1 too. In this case, any automorphism of Γ sends spokes to spokes. Furthermore, $\{1_0, (a^n)_0\}$ and $\{1_1, (a^n)_1\}$ are blocks of $\mathrm{Aut}\,(\Gamma)$ on $V(\Gamma)$ and by a similar way, one can see that $(l-1)^2 \equiv \pm 1 \pmod{n}$. So $(l-1)^2 \equiv \pm 1 \pmod{2n}$, $(l-1)^2 \equiv n+1 \pmod{2n}$ or $(l-1)^2 \equiv n-1 \pmod{2n}$. It is easy to find that $l \equiv 2, n, n+2 \pmod{2n}$ also satisfy $(l-1)^2 \equiv \pm 1 \pmod{2n}$. In the following, we divide the proof into four subcases.

Subsubcase 2.2.1: $(R, L, S) \equiv (\{b\}, \{ba^{\frac{n}{2}+1}\}, \{1, ba\})$ with $\frac{n}{2}$ an odd.

In this case, we can find that there are two 6-cycles passing through 1_1, namely $(1_1, (ba^{\frac{n}{2}+1})_1, (ba^{\frac{n}{2}+1})_0, (a^{\frac{3n}{2}})_1, (ba)_1, 1_0)$ and $(1_1, (ba^{\frac{n}{2}+1})_1, (a^{\frac{n}{2}})_0, (a^{\frac{n}{2}})_1, (ba^{n+1})_1, (ba^{n+1})_0)$. On the other hand, $(1_1, (ba^{\frac{n}{2}+1})_1, (ba^{\frac{n}{2}+1})_0, (a^{\frac{3n}{2}})_1, (ba)_1, 1_0)$ is the unique 6-cycle passing through 1_0, contrary to the vertex-transitivity of Γ.

Subsubcase 2.2.2: $(R, L, S) \equiv (\{b\}, \{ba^l\}, \{1, ba\})$ with $(l-1)^2 \equiv \pm 1 \pmod{2n}$.

If $(l-1)^2 \equiv -1 \pmod{2n}$, then $l^2 - 2l + 2 \equiv 0 \pmod{2n}$ implies that $n | (\frac{l^2}{2} - l + 1)$. Since l is even, $\frac{l^2}{2} - l + 1$ is odd. This implies that n is also odd, a contradiction.

Assume that $(l-1)^2 \equiv 1 \pmod{2n}$. Then $((l-1)^2, 2n) = 1$, and furthermore $(n+1-l, 2n) = 1$. It is easy to check that $\psi_{n+1-l,l}$ is an automorphism of SD_{4n} of order 2 such that $R^{\psi_{n+1-l,l}} = L$, $L^{\psi_{n+1-l,l}} = R$ and $S^{\psi_{n+1-l,l}} = S^{-1}$. By Proposition 2.2(2), Γ is a Cayley graph. This is the graph of type (2) in the theorem.

Subsubcase 2.2.3: $(R, L, S) \equiv (\{b\}, \{ba^l\}, \{1, ba\})$ with $(l-1)^2 \equiv n + 1 (\mod 2n)$.

Suppose that Γ is vertex-transitive. Then there is an automorphism ω_2 of Γ such that ${1_0}^{\omega_2} = 1_1$. Note that ${1_1}^{\omega_2} = 1_0$ or ${1_1}^{\omega_2} = (ba^{n+1})_0$.

Suppose that ${1_1}^{\omega_2} = 1_0$. Then ${b_0}^{\omega_2} = (ba^l)_1$ and ${(ba^l{}_1)}^{\omega_2} = b_0$. We consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 5, respectively. It is easy to find that both $(ba^l)_0$ and $(a^{l-1})_0$ are adjacent with $(ba^l)_1$, furthermore $\{b_0, (a^{n-1})_1\}^{\omega_2} = \{(ba^l)_1, (ba^l)_0\}$ or $\{b_0, (a^{n-1})_1\}^{\omega_2} = \{(ba^l)_1, (a^{l-1})_0\}$.

Assume that $\{b_0, (a^{n-1})_1\}^{\omega_2} = \{(ba^l)_1, (ba^l)_0\}$. Then ${(a^{n-1})_1}^{\omega_2} = (ba^l)_0$ and ${(a^{n-1})_0}^{\omega_2} = (a^{n+l-1})_1$. There is a unique 10-cycle passing through vertexes $1_1, 1_0, b_0, (a^{n-1})_1$ and $(a^{n-1})_0$, that is $(1_1, 1_0, b_0, (a^{n-1})_1, (a^{n-1})_0, (ba^n)_1, (ba^n)_0, (a^n)_0, (ba^{n+1})_1, (ba^{n+1})_0)$. On the other hand, there are two 10-cycles passing through vertexes $1_0, 1_1, (ba^l)_1, (ba^l)_0$ and $(a^{n+l-1})_1$, that are $(1_0, 1_1, (ba^l)_1, (ba^l)_0, (a^{n+l-1})_1, (a^{n+l-1})_0, (ba^{n+l-1})_0, (ba^{n+l-1})_1, (a^{n-1})_1, b_0)$ and $(1_0, 1_1, (ba^l)_1, (ba^l)_0, (a^{n+l-1})_1, (a^{n+l-1})_0, (ba^{n+l})_1, (a^n)_1, (ba)_0, (ba)_1)$, a contradiction. So $\{b_0, (a^{n-1})_1\}^{\omega_2} = \{(ba^l)_1, (a^{l-1})_0\}$, which implies ${(a^{n-1})_1}^{\omega_2} = (a^{l-1})_0$. By a similar reason, one can show that $\{(ba^l)_1, (a^{l-1})_0\}^{\omega_2} = \{b_0, (a^{n-1})_1\}$, and hence ${(a^{l-1})_0}^{\omega_2} = (a^{n-1})_1$. Therefore

$$1_0 \overset{\omega_2}{—} 1_1 \overset{\omega_2}{—} 1_0, \ \ b_0 \overset{\omega_2}{—} (ba^l)_1 \overset{\omega_2}{—} b_0, \ \ (a^{l-1})_0 \overset{\omega_2}{—} (a^{n-1})_1 \overset{\omega_2}{—} (a^{l-1})_0;$$

Now let us consider the subgraphs of Γ induced by the vertices at distance from $(a^{l-1})_0$ and $(a^{n-1})_1$ at most 5, respectively. We can get

$$(a^{n-1})_0 \overset{\omega_2}{—} (a^{l-1})_1 \overset{\omega_2}{—} (a^{n-1})_0,$$

$$(ba^{n-1})_0 \overset{\omega_2}{—} (ba^{2l-1})_1 \overset{\omega_2}{—} (ba^{n-1})_0,$$

$$(a^{2(l-1)})_0 \overset{\omega_2}{—} (a^{2(n-1)})_1 \overset{\omega_2}{—} (a^{2(l-1)})_0;$$

By a similar way, we can get ${(a^{k(n-1)})_1}^{\omega_2} = (a^{k(l-1)})_0$ for any k. By inserting $k = l - 1$, we have ${(a^{(l-1)(n-1)})_1}^{\omega_2} = {(a^{n+1-l})_1}^{\omega_2} = (a^{n+1})_0$. On the other hand, $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1)$ is the unique 8-cycle passing through 1_0 and 1_1 implies that $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1)$ is mapped to $(1_1, 1_0, (ba)_1, (ba)_0, (a^n)_1, (a^n)_0, (ba^{n+1})_1, (ba^{n+1})_0)$ by ω_2. So ${(ba^{n+1})_1}^{\omega_2} = (ba)_0$, furthermore ${(a^{n+1-l})_1}^{\omega_2} = a_0$, a contradiction.

Suppose that ${1_1}^{\omega_2} = (ba^{n+1})_0$. Then ${b_0}^{\omega_2} = (ba^l)_1$ and $({ba^l}_1)^{\omega_2} = (a^{n+1})_0$. We consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 5, respectively. Since both $(a^{n+1})_1$ and $(ba^{n+2})_1$ are adjacent with $(a^{n+1})_0$, we have $\{{ba^l}_1, (a^{l-1})_0\}^{\omega_2} = \{(a^{n+1})_0, (a^{n+1})_1\}$ or $\{(ba^l)_1, (a^{l-1})_0\}^{\omega_2} = \{(a^{n+1})_0, (ba^{n+2})_1\}$.

Assume that $\{(ba^l)_1, (a^{l-1})_0\}^{\omega_2} = \{(a^{n+1})_0, (ba^{n+2})_1\}$. Then ${(a^{l-1})_0}^{\omega_2} = (ba^{n+2})_1$ and ${(a^{l-1})_1}^{\omega_2} = (ba^{n+2})_0$. There is a unique 10-cycle passing through vertexes $1_0, 1_1, (ba^l)_1, (a^{l-1})_0$ and $(a^{l-1})_1$, that is $(1_0, 1_1, (ba^l)_1, (a^{l-1})_0, (a^{l-1})_1, (ba^{n+l})_0, (ba^{n+l})_1, (a^n)_1, (ba)_0, (ba)_1)$. On the other hand, there are two 10-cycles passing through vertexes $1_1, (ba^{n+1})_0, (a^{n+1})_0, (ba^{n+2})_1$ and $(ba^{n+2})_0$, that are $(1_1, (ba^{n+1})_0, (a^{n+1})_0, (ba^{n+2})_1, (ba^{n+2})_0, a_1, a_0, (ba)_0, (ba)_1, 1_0)$ and $(1_1, (ba^{n+1})_0, (a^{n+1})_0, (ba^{n+2})_1, (ba^{n+2})_0, a_1, (ba^{l+1})_1, (a^l)_0, (ba^l)_0, (ba^l)_1)$, a contradiction. So $\{(ba^l)_1, (a^{l-1})_0\}^{\omega_2} = \{(a^{n+1})_0, (a^{n+1})_1\}$, and hence ${(a^{l-1})_0}^{\omega_2} = (a^{n+1})_1$. Therefore ω_2 maps the path $P_1\colon 1_0, 1_1, (ba^l)_1, (a^{l-1})_0$ to the path $Q_1\colon 1_1, (ba^{n+1})_0, (a^{n+1})_0, (a^{n+1})_1$ in this order, namely ${1_0}^{\omega_2} = 1_1$, ${1_1}^{\omega_2} = (ba^{n+1})_0$, ${(ba^l)_1}^{\omega_2} = (a^{n+1})_0$ and ${(a^{l-1})_0}^{\omega_2} = (a^{n+1})_1$. Similarly, we consider the subgraphs of Γ induced by the vertices at distance from $(a^{l-1})_0$ and $(a^{n+1})_1$ at most 5, respectively. One can show that ω_2 maps the path $P_2\colon (a^{l-1})_0, (a^{l-1})_1, (ba^{2l-1})_1, (a^{2(l-1)})_0$ to the path $Q_2\colon (a^{n+1})_1, (ba^2)_0, (a^2)_0, (a^2)_1$ in this order. By a similar way, we can get ${(a^{k(l-1)})_0}^{\omega_2} = (a^{k(n+1)})_1$ for any k. By inserting $k = l-1$, we have ${(a^{(l-1)(l-1)})_0}^{\omega_2} = {(a^{n+1})_0}^{\omega_2} = (a^{(l-1)(n+1)})_1 = (a^{n+l-1})_1$. On the other hand, $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1)$ is the unique 8-cycle passing through 1_0 and 1_1 implies that $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1)$ is mapped to $(1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1, 1_0)$ by ω_2. So ${(ba^{n+1})_0}^{\omega_2} = (ba^{n+1})_1$, which implies ${(a^{n+1})_0}^{\omega_2} = (a^{n+1-l})_1$, a contradiction. Therefore Γ is not vertex-transitive, a contradiction.

Subsubcase 2.2.4: $(R, L, S) \equiv (\{b\}, \{ba^l\}, \{1, ba\})$ with $(l-1)^2 \equiv n-1 \pmod{2n}$.

Note that $n \equiv 2 \pmod 4$ in this case. One can check that the map

$$\begin{array}{llllll} \omega_3\colon & (a^k)_0 & \mapsto & (a^{k(1-l)})_1, & (a^k)_1 & \mapsto & (ba^{n+1+k(1-l)})_0, \\ & (ba^k)_0 & \mapsto & (ba^{k(1-l)+l})_1, & (ba^k)_1 & \mapsto & (a^{(k-1)(1-l)})_0, \end{array}$$

with $0 \le k < 2n$ is a permutation on $V(\Gamma)$ with order 8. Furthermore, for any $0 \le k < 2n$, we have

$$N_\Gamma((a^k)_0)^{\omega_3} = \{(ba^{n+1+k(1-l)})_0, (a^{k(1-l)})_0, (ba^{k(1-l)+l})_1\} = N_\Gamma((a^{k(1-l)})_1),$$

$$N_\Gamma((a^k)_1)^{\omega_3} = \{(a^{k(1-l)})_1, (ba^{n+1+k(1-l)})_1, (a^{n+1+k(1-l)})_0\} = N_\Gamma((ba^{n+1+k(1-l)})_0),$$

$$N_\Gamma((ba^k)_0)^{\omega_3} = \{(ba^{k(1-l)+l})_0, (a^{(k-1)(1-l)})_0, (a^{k(1-l)})_1\} = N_\Gamma((ba^{k(1-l)+l})_1),$$

$$N_\Gamma((ba^k)_1)^{\omega_3} = \{(a^{(k-1)(1-l)})_1, (ba^{k(1-l)+l})_1, (ba^{(k-1)(1-l)})_0\} = N_\Gamma((a^{(k-1)(1-l)})_0).$$

So ω_3 induces an automorphism of Γ of order 8. Denote $H_{01} = \{(a^k)_0\}$, $H_{02} = \{(ba^k)_0\}$, and $H_{11} = \{(a^k)_1\}$, $H_{12} = \{(ba^k)_1\}$ with $0 \le k < 2n$. We have the following:

$$H_{01} \xrightarrow{\omega_3} H_{11} \xrightarrow{\omega_3} H_{02} \xrightarrow{\omega_3} H_{12} \xrightarrow{\omega_3} H_{01}$$

Note that $\langle R(a)\rangle$ acts transitively on the sets $H_{01}, H_{02}, H_{11}, H_{12}$, respectively. So $M_2 = \langle R(a), \omega_3\rangle$ is a vertex-transitive subgroup of $\mathrm{Aut}\,(\Gamma)$. By calculation, ${\omega_3}^4 = R(a^n)$, $\omega_3^{-1} R(a)\omega_3 = R(a)^{1-l}$. So $M_2 = \langle R(a)\rangle\langle\omega_3\rangle$ and furthermore $|M_2| = 8n$. Therefore M_2 acts regularly on $V(\Gamma)$, and hence Γ is a Cayley graph of type (2) in the theorem. □

Proof of Theorem 1.4. Let Γ be a 2-type bi-Cayley graph and let $R = \{x_1, x_2\}$, $L = \{y_1, y_2\}$ and $S = \{1\}$.

Firstly, we assume that all of x_1, x_2, y_1 and y_2 belong to $b\langle a\rangle$. By the structure of SD_{4n}, if their orders are the same, then $SD_{4n} \neq \langle x_1, x_2, y_1, y_2\rangle$. So without loss of generality, we can assume that x_1, x_2 have order 2 and y_1, y_2 have order 4. By Lemma 3.1 (2), $\mathrm{Aut}\,(SD_{4n})$ acts transitively on set $\{ba^{2i}\}$ with $0 \leqslant i \leqslant n-1$, we can let $x_1 = b$, $x_2 = ba^{2t}$, $y_1 = ba^s$ and $y_2 = ba^{n+s}$. So $(R, L, S) \equiv (\{b, ba^{2t}\}, \{ba^s, ba^{n+s}\}, \{1\})$ with s an odd. It is easy to find that $(1_1, (ba^s)_1, (a^n)_1, (ba^{n+s})_1)$ is the unique 4-cycle passing through 1_1. The vertex-transitivity of Γ implies that there is a unique 4-cycle passing through 1_0. Considering possible 4-cycles containing 1_0, we have $(a^{2t})_0 = (a^{-2t})_0$, and hence $n = 2t$. Noticing that $\langle a\rangle = \langle a^s, a^n\rangle$, we get $(s, 2n) = 1$. So the map $f\colon\ a^s \mapsto a, b \mapsto b$ can induce an automorphism of SD_{4n} such that $(R, L, S)^f \equiv (\{b, ba^n\}, \{ba, ba^{n+1}\}, \{1\})$. Hence, we can assume that $(R, L, S) = (\{b, ba^n\}, \{ba, ba^{n+1}\}, \{1\})$.

Let $\Sigma = \mathrm{Cay}(D_{8n}, \{d, dc, dc^{2n}\})$ where $D_{8n} = \langle c, d \mid c^{4n} = d^2 = 1, dcd = c^{-1}\rangle$. Define a map from $V(\Gamma)$ to $V(\Sigma)$ as follows:

$$\phi\colon\quad \begin{array}{lllllll} (a^r)_0 & \mapsto & c^{2r}, & & (a^r)_1 & \mapsto & dc^{2r+1}, \\ (ba^r)_0 & \mapsto & dc^{2r}, & & (ba^r)_1 & \mapsto & c^{2r-1}, \end{array}$$

with $0 \le r \le 2n-1$. Furthermore, for any $r \in \mathbb{Z}_{2n}$, we have

$$\begin{aligned} N_\Gamma((a^r)_0)^\phi &= \{(a^r)_1, (ba^r)_0, (ba^{n+r})_0\}^\phi = \{dc^{2r+1}, dc^{2r}, dc^{2n+2r}\} = N_\Sigma(c^{2r}), \\ N_\Gamma((a^r)_1)^\phi &= \{(a^r)_0, (ba^{r+1})_1, (ba^{n+r+1})_1\}^\phi = \{c^{2r}, c^{2r+1}, c^{2n+2r+1}\} \\ &= N_\Sigma(dc^{2r+1}), \\ N_\Gamma((ba^r)_0)^\phi &= \{(ba^r)_1, (a^r)_0, (a^{n+r})_0\}^\phi = \{c^{2r-1}, c^{2r}, c^{2n+2r}\} = N_\Sigma(dc^{2r}), \\ N_\Gamma((ba^r)_1)^\phi &= \{(ba^r)_0, (a^{n+r-1})_1, (a^{r-1})_1\}^\phi = \{dc^{2r}, dc^{2n+2r-1}, dc^{2r-1}\} \\ &= N_\Sigma(c^{2r-1}). \end{aligned}$$

It follows that ϕ is an isomorphism from Γ to Σ. Then Γ is a Cayley graph over a dihedral group. This is the graph of type (1) in the theorem.

In the following, we assume that there is at least one of x_1, x_2, y_1, y_2 belongs to $\langle a\rangle$. Without loss of generality, let $x_1 \in \langle a\rangle$. We divide the proof into two cases:

Case 1: $|\langle x_1\rangle| = 2$.

In this case, $x_1 = a^n$. The condition $R = R^{-1}$ implies that x_2 is also an element of order 2, and hence $x_2 = ba^{2i_1}$ for some integer i_1. Since there is an automorphism of SD_{4n} sending a and ba^{2i_1} to a and b, respectively, we can assume $x_2 = b$ up to equivalence. If $y_1 \in \langle a\rangle$ and $y_2 \notin \langle a\rangle$, then $y_1 \neq {y_2}^{-1}$. So $L = L^{-1}$ implies that $y_1 = a^n$ and $y_2 = ba^{2i_2}$. Then $SD_{4n} \neq \langle b, ba^{2i_2}, a^n\rangle$, contrary to the connectivity of Γ. Similarly, if $y_2 \in \langle a\rangle$ and $y_1 \notin \langle a\rangle$, then one can get a similar contradiction. So we consider the following two subcases:

Subcase 1.1: Both y_1 and y_2 belong to $b\langle a\rangle$.

In this case, both y_1 and y_2 have order 4 by the connectivity of Γ and the structure of SD_{4n}. Let $y_1 = ba^{l_1}$ and $y_2 = ba^{n+l_1}$ for some odd integer l_1. Now it holds that $SD_{4n} = \langle b, a^n, a^{l_1}\rangle$. This implies that $(l_1, 2n) = 1$. So the map $d\colon\ a^{l_1} \mapsto a, b \mapsto b$ can induce an automorphism of SD_{4n} such that $(R, L, S)^d \equiv (\{a^n, b\}, \{ba, ba^{n+1}\}, \{1\})$. Hence, we can assume that $(R, L, S) \equiv (\{a^n, b\}, \{ba, ba^{n+1}\}, \{1\})$ up to equivalence. Let us consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 3, respectively. There are two 7-cycles passing through 1_0 but not passing through 1_1 that is $(1_0, b_0, b_1, (a^{n-1})_1, (ba^n)_1, (ba^n)_0, (a^n)_0)$ and $(1_0, b_0, b_1, (a^{-1})_1, (ba^n)_1, (ba^n)_0, (a^n)_0)$. On the other hand, $(1_1, (ba)_1, (ba)_0, a_0, (a^{n+1})_0, (ba^{n+1})_0, (ba^{n+1})_1)$ is the unique 7-cycle passing through 1_1 but not passing through 1_0, contrary to the vertex-transitivity of Γ.

Subcase 1.2: Both y_1 and y_2 belong to $\langle a\rangle$.

In this case, we can let $y_1 = a^{i_3}, y_2 = a^{-i_3}$. Now it holds that $SD_{4n} = \langle b, a^n, a^{i_3}\rangle$. This implies that $(i_3, 2n) = 1$. So the map $e\colon\ a^{i_3} \mapsto a, b \mapsto b$ can induce an automorphism of SD_{4n} such that $(R, L, S)^e \equiv (\{a^n, b\}, \{a, a^{-1}\}, \{1\})$. Hence we can assume that $(R, L, S) \equiv (\{a^n, b\}, \{a, a^{-1}\}, \{1\})$. It is easy to find that there is a unique 4-cycle $(1_0, b_0, (ba^n)_0, (a^n)_0)$ passing through 1_0. The vertex-transitivity of Γ implies that there is a unique 4-cycle passing through 1_1. Considering possible 4-cycles containing 1_1, we have $(a^2)_1 = (a^{-2})_1$, and hence $n = 2$, contrary to $n \geq 4$.

Case 2: $|\langle x_1\rangle| \neq 2$

In this case, we can let $x_1 = a^i, x_2 = a^{-i}$. By the connectivity of Γ and the structure of SD_{4n}, if $y_1 \in \langle a\rangle$, then $y_2 \notin \langle a\rangle$ and $y_1 \neq {y_2}^{-1}$. By the condition $L = L^{-1}$, we can assume that $L = \{a^n, b\}$ up to equivalence. In this case, Γ is isomorphic to a graph of subcase 1.2, by Proposition 2.1(4). So we can assume that both y_1 and y_2 belong to $b\langle a\rangle$. We divide the proof into the following two subcases:

Subcase 2.1: The orders of y_1 and y_2 are 2.

By Lemma 3.1(2), Aut (SD_{4n}) acts transitively on the set $\{ba^{2i}\}$ for $0 \leqslant i \leqslant n-1$, we can let $y_1 = b$, $y_2 = ba^{2j}$. So $(R, L, S) \equiv (\{a^i, a^{-i}\}, \{b, ba^{2j}\}, \{1\})$. The vertex-transitivity of Γ implies that there must exist an automorphism α in $\mathrm{Aut}(\Gamma)$ such that $1_0^{\alpha} = 1_1$. We consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 5, respectively. There are six 10-cycles passing through edge $\{1_0, (a^i)_0\}$, $\{1_0, (a^{-i})_0\}$, $\{1_1, b_1\}$ and $\{1_1, (ba^{2j})_1\}$ separately. In the same time, there are eight 10-cycles passing through edge $\{1_0, 1_1\}$. This implies that $\{1_0, (a^i)_0\}^{\alpha} \neq \{1_0, 1_1\}$. Hence $\{1_0, (a^i)_0\}^{\alpha} = \{1_1, b_1\}$ or $\{1_1, (ba^{2j})_1\}$ and $\{1_0, (a^{-i})_0\}^{\alpha} = \{1_1, b_1\}$ or $\{1_1, (ba^{2j})_1\}$. So ${(a^i)_0}^{\alpha} = b_1$ or $(ba^{2j})_1$ and ${(a^{-i})_0}^{\alpha} = b_1$ or $(ba^{2j})_1$. Without loss of generality, let ${(a^i)_0}^{\alpha} = b_1$. We consider the subgraphs of Γ induced by the vertices at distance from $(a^i)_0$ and b_1 at most 5, respectively. Similarly, we can find that there exists six 10-cycles passing through edge $\{(a^i)_0, (a^{2i})_0\}$ and $\{b_1, (a^{-2j})_1\}$ separately. There are eight 10-cycles passing through edge $\{(a^i)_0, (a^i)_1\}$ and $\{b_1, b_0\}$ separately. So by the vertex-transitivity of Γ, $\{(a^i)_0, (a^{2i})_0\}^{\alpha} = \{b_1, (a^{-2j})_1\}$, and hence ${(a^{2i})_0}^{\alpha} = (a^{-2j})_1$. In a similar way, one can see that the cycle $(1_0, (a^i)_0, (a^{2i})_0, (a^{3i})_0, \cdots, (a^{-i})_0)$ is mapped to the cycle $(1_1, b_1, (a^{-2j})_1, (ba^{-2j})_1, \cdots, (ba^{2j})_1)$ by α. So the lengthes of the cycles $(1_0, (a^i)_0, (a^{2i})_0, (a^{3i})_0, \cdots, (a^{-i})_0)$ and $(1_1, b_1, (a^{-2j})_1, (ba^{-2j})_1, \cdots, (ba^{2j})_1)$ are the same, and hence $\langle a^i\rangle = \langle a^j\rangle$. Furthermore, $SD_{4n} = \langle a^i, a^{2j}, b\rangle = \langle a^i, b\rangle$ implies that $(i, 2n) = (j, 2n) = 1$. It is easy to find that the map $g\ :\ a^i \mapsto a, b \mapsto b$ can induce an automorphism of SD_{4n} such that $(R, L, S)^g \equiv (\{a, a^{-1}\}, \{b, ba^{2l}\}, \{1\})$ with $(l, 2n) = 1$. So we can assume that $(R, L, S) = (\{a, a^{-1}\}, \{b, ba^{2l}\}, \{1\})$ with $(l, 2n) = 1$ up to equivalence.

Now the cycle $(1_0, a_0, (a^2)_0, (a^3)_0, \cdots, (a^{-1})_0)$ is mapped to the cycle $(1_1, b_1, (a^{-2l})_1, (ba^{-2l})_1, \cdots, (a^{2l})_1, (ba^{2l})_1)$ by α. So we can get

$$ {(a^{2k})_0}^{\alpha} = (a^{-2kl})_1, {(a^{2k+1})_0}^{\alpha} = (ba^{-2kl})_1 \tag{I} $$

where $0 \leq k \leq n-1$. Note that $(a^{2k})_0$ and $(a^{2k+1})_0$ are adjacent with $(a^{2k})_1$ and $(a^{2k+1})_1$, respectively, and $(a^{-2kl})_1$ and $(ba^{-2kl})_1$ are adjacent with $(a^{-2kl})_0$ and $(ba^{-2kl})_0$, respectively. So we can get

$$(a^{2k})_1{}^{\alpha} = (a^{-2kl})_0, (a^{2k+1})_1{}^{\alpha} = (ba^{-2kl})_0 \tag{II}$$

where $0 \leq k \leq n-1$. By (I) and (II), it is easy to get

$$\alpha\colon\ (a^{2k})_0 \mapsto (a^{-2kl})_1 \mapsto (a^{2kl^2})_0. \tag{III}$$

Since $(a^{2k})_0$ is adjacent with $(a^{2k+1})_0$, we have $(a^{2k+1})_0{}^{\alpha^2} = (a^{2kl^2\pm1})_0$. Similarly, by $\{(a^{2k+1})_0, (a^{2k+2})_0\}^{\alpha^2} = \{(a^{2kl^2\pm1})_0, (a^{2kl^2+2l^2})_0\}$, it holds that $(a^{2kl^2+2l^2})_0 = (a^{2kl^2\pm2})_0$ or $(a^{2kl^2+2l^2})_0 = (a^{2kl^2})_0$. If $(a^{2kl^2+2l^2})_0 = (a^{2kl^2})_0$, then $2l^2 \equiv 0 \pmod{2n}$, which implies that $n|l^2$, contrary to l is odd. So $(a^{2kl^2+2l^2})_0 = (a^{2kl^2\pm2})_0$, and hence, $2l^2 \equiv \pm 2 \pmod{2n}$.

If $2l^2 \equiv 2 \pmod{2n}$, then the map

$$\begin{array}{llllll}\beta\colon & (a^{2k})_0 & \mapsto (a^{-2kl})_1, & (a^{2k})_1 & \mapsto (a^{-2kl})_0,\\ & (a^{2k+1})_0 & \mapsto (ba^{-2kl})_1, & (a^{2k+1})_1 & \mapsto (ba^{-2kl})_0,\\ & (ba^{2k})_0 & \mapsto (a^{-2kl+1})_1, & (ba^{2k})_1 & \mapsto (a^{-2kl+1})_0,\\ & (ba^{2k+1})_0 & \mapsto (ba^{n+1-2kl})_1, & (ba^{2k+1})_1 & \mapsto (ba^{n+1-2kl})_0,\end{array}$$

with $0 \leq k < n$ is a permutation on $V(\Gamma)$ with order 2. Furthermore, for any $0 \leq k < n$ we have

$$\begin{aligned}
&N_\Gamma((a^{2k})_0)^\beta = \{(ba^{-2kl})_1, (ba^{2(1-k)l})_1, (a^{-2kl})_0\} = N_\Gamma((a^{-2kl})_1),\\
&N_\Gamma((a^{2k})_1)^\beta = \{(a^{-2kl+1})_0, (a^{-2kl-1})_0, (a^{-2kl})_1\} = N_\Gamma((a^{-2kl})_0),\\
&N_\Gamma((a^{2k+1})_0)^\beta = \{((a^{-2kl})_1, (a^{-2(k+1)l})_1, (ba^{-2kl})_0\} = N_\Gamma((ba^{-2kl})_1),\\
&N_\Gamma((a^{2k+1})_1)^\beta = \{(ba^{n-1-2kl})_0, (ba^{n+1-2kl})_0, (ba^{-2kl})_1\} = N_\Gamma((ba^{-2kl})_0),\\
&N_\Gamma((ba^{2k})_0)^\beta = \{(ba^{-2kl+1})_1, (ba^{2(1-k)l+1})_1, (a^{-2kl+1})_0\} = N_\Gamma((a^{-2kl+1})_1),\\
&N_\Gamma((ba^{2k})_1)^\beta = \{(a^{-2kl+2})_0, (a^{-2kl})_0, (a^{-2kl+1})_1\} = N_\Gamma((a^{-2kl+1})_0)\\
&N_\Gamma((ba^{2k+1})_0)^\beta = \{(a^{n+1-2(k+1)l})_1, (a^{n+1-2kl})_1, (ba^{n+1-2kl})_0\} = N_\Gamma((ba^{n+1-2kl})_1),\\
&N_\Gamma((ba^{2k+1})_1)^\beta = \{(ba^{-2kl})_0, (ba^{2-2kl})_0, (ba^{n+1-2kl})_1\} = N_\Gamma((ba^{n+1-2kl})_0).
\end{aligned}$$

So β induces an automorphism of Γ of order 2. Denote $H_{01} = \{(a^{2i})_0\}$, $H_{02} = \{(a^{2i+1})_0\}$, $H_{03} = \{(ba^{2i})_0\}$, $H_{04} = \{(ba^{2i+1})_0\}$, $H_{11} = \{(a^{2i})_1\}$, $H_{12} = \{(a^{2i+1})_1\}$, $H_{13} = \{(ba^{2i})_1\}$, $H_{14} = \{(ba^{2i+1})_1\}$, with $0 \leq i < n$. Let $H_0 = H_{01} \bigcup H_{02} \bigcup H_{03} \bigcup H_{04}$; $H_1 = H_{11} \bigcup H_{12} \bigcup H_{13} \bigcup H_{14}$; We have the following:

$H_{01} \overset{\beta}{\text{—}} H_{11} \overset{\mathrm{R(b)}}{\text{—}} H_{13} \overset{\beta}{\text{—}} H_{02} \overset{\mathrm{R(b)}}{\text{—}} H_{04} \overset{\beta}{\text{—}} H_{14} \overset{\mathrm{R(b)}}{\text{—}} H_{12} \overset{\beta}{\text{—}} H_{03} \overset{\mathrm{R(b)}}{\text{—}} H_{01}$.
$R(a^2)$ acts transitively on H_{ij} where $i = 0, 1$; $j = 1, 2, 3, 4$; So $M = \langle R(a^2), R(b), \beta\rangle$ is a vertex-transitive subgroup of Γ.

By calculation, $\beta R(a^2)\beta = R(a^{-2l}) \in \langle R(a^2)\rangle$, so β normalizes $\langle R(a^2)\rangle$. Noticing that $R(b)$ also normalizes $\langle R(a^2)\rangle$, we see that $\langle R(a^2)\rangle \trianglelefteq M$. We consider the group $M/\langle R(a^2)\rangle = \langle \overline{\beta}, \overline{R(b)}\rangle = \langle \overline{\beta R(b)}, \overline{R(b)}\rangle$. By calculation $(\beta R(b))^4 = R(a^{-2l-2}) \in \langle R(a^2)\rangle$ and $R(b)^2 = 1$ imply that $\overline{\beta R(b)}^4 = \overline{R(b)}^2 = \overline{1}$. It is easy to find that $\overline{R(b)}^{-1}\overline{\beta R(b)}\,\overline{R(b)} = \overline{\beta R(b)}^{-1}$. So $\overline{M} \simeq D_8$, and $|M| = 8n$. Therefore M acts regularly on $V(\Gamma)$. So Γ is a Cayley graph. This is the graph of type (2) in the theorem.

Let $2l^2 \equiv -2 \pmod{2n}$. Now the map

$$\begin{array}{lllllll} \gamma: & (a^{2k})_0 & \mapsto & (a^{-2kl})_1, & (a^{2k})_1 & \mapsto & (a^{-2kl})_0, \\ & (a^{2k+1})_0 & \mapsto & (ba^{-2kl})_1, & (a^{2k+1})_1 & \mapsto & (ba^{-2kl})_0, \\ & (ba^{2k})_0 & \mapsto & (a^{-2kl-1})_1, & (ba^{2k})_1 & \mapsto & (a^{-2kl-1})_0, \\ & (ba^{2k+1})_0 & \mapsto & (ba^{n-1-2kl})_1, & (ba^{2k+1})_1 & \mapsto & (ba^{n-1-2kl})_0, \end{array}$$

with $0 \le k < n$ is a permutation on $V(\Gamma)$. Furthermore, for any $0 \le k < n$ we have

$$N_\Gamma((a^{2k})_0)^\gamma = \{(ba^{-2kl})_1, (ba^{2(1-k)l})_1, (a^{-2kl})_0\} = N_\Gamma((a^{-2kl})_1),$$
$$N_\Gamma((a^{2k})_1)^\gamma = \{(a^{-2kl+1})_0, (a^{-2kl-1})_0, (a^{-2kl})_1\} = N_\Gamma((a^{-2kl})_0),$$
$$N_\Gamma((a^{2k+1})_0)^\gamma = \{((a^{-2kl})_1, (a^{-2(k+1)l})_1, (ba^{-2kl})_0\} = N_\Gamma((ba^{-2kl})_1),$$
$$N_\Gamma((a^{2k+1})_1)^\gamma = \{(ba^{n-1-2kl})_0, (ba^{n+1-2kl})_0, (ba^{-2kl})_1\} = N_\Gamma((ba^{-2kl})_0),$$
$$N_\Gamma((ba^{2k})_0)^\gamma = \{(ba^{-2kl-1})_1, (ba^{2(1-k)l-1})_1, (a^{-2kl-1})_0\} = N_\Gamma((a^{-2kl-1})_1),$$
$$N_\Gamma((ba^{2k})_1)^\gamma = \{(a^{-2kl-2})_0, (a^{-2kl})_0, (a^{-2kl-1})_1\} = N_\Gamma((a^{-2kl-1})_0)$$
$$N_\Gamma((ba^{2k+1})_0)^\gamma = \{(a^{n-1-2kl})_1, (a^{n-1-2(k+1)l})_1, (ba^{n-1-2kl})_0\} = N_\Gamma((ba^{n-1-2kl})_1),$$
$$N_\Gamma((ba^{2k+1})_1)^\gamma = \{(ba^{-2kl})_0, (ba^{-2-2kl})_0, (ba^{n-1-2kl})_1\} = N_\Gamma((ba^{n-1-2kl})_0).$$

Therefore γ induces an automorphism of Γ mapping 1_0 to 1_1. So Γ is a vertex-transitive graph. This is the graph of type (3) in the theorem. Now we aim to show that Γ is a non-Cayley graph. Let H be a vertex-transitive subgroup of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$. Then there exists an automorphism $\varphi \in H$ such that ${1_0}^\varphi = 1_1$. It is easy to find that ${1_1}^\varphi = 1_0$. So $\varphi^2 \in H_{1_0}$. Similar with proof of (III), we can get ${(a^{2k})_0}^{\varphi^2} = (a^{2kl^2})_0 = (a^{-2k})_0$. Since $(a^{2k+1})_0$ is adjacent with $(a^{2k})_0$ and $(a^{2k+2})_0$, we have ${(a^{2k+1})_0}^{\varphi^2} = (a^{-2k-1})_0$, where $0 \le k \le n-1$. The condition ${(a^{2k+1})_0}^{\varphi^2} = (a^{-2k-1})_0$ implies that ${a_0}^{\varphi^2} = a_0^{-1}$. So $\varphi^2 \ne 1$ and $|H_{1_0}| \ge 2$. This implies that H does not act regularly on $V(\Gamma)$. Since we choose an arbitrary vertex-transitive subgroup H, Γ is a non-Cayley graph.

Subcase 2.2: The orders of y_1 and y_2 are 4.

By Lemma 3.1(2), $\mathrm{Aut}\,(SD_{4n})$ acts transitively on the set $\{ba^{2i+1}\}$ for $0 \leqslant i \leqslant n-1$, we can let $y_1 = ba$, $y_2 = ba^{n+1}$. The condition $SD_{4n} = \langle a^i, ba\rangle$ implies that $(i, 2n) = 1$. So we can assume $(R, L, S) = (\{a, a^{-1}\}, \{ba, ba^{n+1}\}, \{1\})$ up to equivalence. It is easy to find that $(1_1, (ba)_1, (a^n)_1, (ba^{n+1})_1)$ is the unique 4-cycle passing through 1_1. The vertex-transitivity of Γ implies that there is a unique 4-cycle through 1_0. Considering possible 4-cycles containing 1_0, we have $(a^2)_0 = (a^{-2})_0$, and hence $n = 2$, contrary to $n \ge 4$. The proof is now complete. □

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Domination and independence numbers of large 2-crossing-critical graphs*

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Abstract

After 2-crossing-critical graphs were characterized in 2016, their most general subfamily, large 3-connected 2-crossing-critical graphs, has attracted separate attention. This paper presents sharp upper and lower bounds for their domination and independence numbers.

Keywords: Crossing-critical graphs, domination number, independence number.

Math. Subj. Class. (2020): 05C10, 05C62, 05C69

1 Introduction

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the smallest number of edge crossings in a drawing of G in the plane. The topic has been widely studied, see for example [7, 8, 17, 18, 20]. A graph G is *k-crossing-critical* if $\mathrm{cr}(G) \geq k$, but every proper subgraph H of G has $\mathrm{cr}(H) < k$. Note that subdividing an edge or its inverse operation (suppressing a vertex)

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do not affect the crossing number of a graph. Thus we can restrict our studies to graphs without degree 2 vertices. Under this restriction, Kuratowski's Theorem tells us that the only 1-crossing-critical graphs are K_5 and $K_{3,3}$. The classification of 2-crossing-critical graphs has been of interest since the 1980s. Partial results on the topic have been reported in [2, 9, 12, 16, 19], and some related results can be found in [1, 10, 13]. Crossing numbers of graphs with a tile structure have been studied in [14, 15]. Finally, Bokal, Oporewski, Richter, and Salazar [6] provided an almost complete characterization of 2-crossing-critical graphs. In particular, they describe a tile structure of large 3-connected 2-crossing-critical graphs (i.e., all but finitely many 3-connected 2-crossing-critical graphs). Recently, the degree properties of crossing-critical graphs have been studied in [3, 5, 11].

The above-mentioned large 3-connected 2-crossing-critical graphs have since attracted separate attention, see [4, 21, 22]. In [21, 22], the Hamiltonicity of these graphs is discussed, and the number of all Hamiltonian cycles is determined. In [4], several additional properties of large 3-connected 2-crossing-critical graphs have been studied. In particular, the number of vertices and edges can be determined from the signature of a graph, and several results regarding their chromatic number, chromatic index, and tree-width are presented. In the present paper, we extend the studies of large 3-connected 2-crossing-critical graphs to their domination and independence numbers.

The rest of the paper is organized as follows. In the next section, necessary definitions and known results are listed. In Section 3, the sharp upper and lower bounds for the domination number of large 3-connected 2-crossing-critical graphs are given, while in Section 4 analogous results are proved for their independence number.

2 Preliminaries

Let G be a graph. Its vertex set is denoted by $V(G)$ and its edge set by $E(G)$. The *(open) neighborhood* of a vertex $v \in V(G)$ is $N(v) = \{u \in V(G);\ uv \in E(G)\}$ and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. Similarly, for $D \subseteq V(G)$, $N[D] = \bigcup_{v \in D} N[v]$ is the closed neighborhood of D. Note also that $[n] = \{1, \dots, n\}$ and that the reversed sequence of a sequence a is denoted by $\overline{a}$.

We now recall the definitions of the domination number and the independence number.

Definition 2.1. Let G be a graph. A subset $D \subseteq V(G)$ *dominates* the set of vertices $X \subseteq V(G)$ if $X \subseteq N[D]$. If $N[D] = V(G)$, then D is a *dominating set* of G. The *domination number* $\gamma(G)$ of a graph G is the size of a smallest dominating set in G.

Definition 2.2. Let G be a graph. A subset $X \subseteq V(G)$ is *independent* if none of the vertices from X are adjacent. The *independence number* $\alpha(G)$ of the graph G is the size of a largest independent set.

In the rest of the section, we recall the characterization of 2-crossing-critical graphs and provide the necessary definitions which help us describe large 3-connected 2-crossing-critical graphs, i.e., graphs studied in this paper. Note that vertices of degrees 1 and 2 do not affect the crossing number, thus the assumption that the minimum degree is at least 3 is reasonable. Note also that V_{10} is the graph obtained from C_{10} by adding the five diagonal edges. We quote the following theorem from [6]. The detailed explanation of the terminology used in this theorem is given afterwards.

Theorem 2.3 ([6, Theorem 1.1])**.** *Let G be a 2-crossing-critical graph with a minimum degree of at least 3. Then one of the following holds.*

(i) *G is 3-connected, contains a subdivision of V_{10}, and has a very particular twisted Möbius band tile structure, with each tile isomorphic to one of 42 possibilities.*

(ii) *G is 3-connected, does not have a subdivision of V_{10}, and has at most 3 million vertices.*

(iii) *G is not 3-connected and is one of 49 particular examples.*

(iv) *G is 2- but not 3-connected and is obtained from a 3-connected 2-crossing-critical graph by replacing digons with digonal paths.*

In the present paper, we study graphs from (i), i.e., 3-connected 2-crossing-critical graphs that contain a subdivision of V_{10}. Since 3-connected 2-crossing-critical graphs that do not contain a subdivision of V_{10} have at most 3 million vertices, we may call graphs from Theorem 2.3(i) *large 3-connected 2-crossing-critical graphs* or *large 3-con 2-cc graphs* for short. This abbreviation is used throughout the paper. Note that it would also be interesting to study other subclasses of graphs, especially graphs from (iv). However, like in [4], we restrict our studies to graphs from (i).

To understand the tile structure of large 3-con 2-cc graphs, we need the following definitions that first appeared in [14, 15].

Definition 2.4. 1. A *tile* is a triplet $T = (G, \lambda, \rho)$, where G is a graph and λ, ρ are sequences of pairwise distinct vertices of G, where no vertex of G appears in both λ and ρ. The *left wall* of T is λ, and the *right wall* of T is ρ.

2. A *tile drawing* is a drawing D of G in the unit square $[0, 1] \times [0, 1]$ for which the intersection of the boundary of the square with D contains precisely the images of the left wall λ and the right wall ρ, and these are drawn in $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$, respectively, such that the y-coordinates of the vertices are increasing with respect to their orders in the sequences λ and ρ.

3. The tiles $T = (G, \lambda, \rho)$ and $T' = (G', \lambda', \rho')$ are *compatible* if $|\rho| = |\lambda'|$.

4. A sequence $(T_0, \dots, T_m)$ of tiles is *compatible* if T_{i-1} is compatible with T_i for every $i \in [m]$.

5. The *join* of compatible tiles (G, λ, ρ) and (G', λ', ρ') is the tile, denoted as $(G, \lambda, \rho) \otimes (G', \lambda', \rho')$, whose graph is obtained from G and G' by identifying the sequence ρ term by term with the sequence λ'. The left wall of the obtained tile is λ and the right wall is ρ'.

6. The *join* $\otimes \mathcal{T}$ of a compatible sequence $\mathcal{T} = (T_0, \dots, T_m)$ of tiles is defined as $T_0 \otimes \cdots \otimes T_m$.

7. A tile T is *cyclically-compatible* if T is compatible with itself. For a cyclically-compatible tile T, the *cyclization* of T is the graph $\circ T$ obtained by identifying the respective vertices of the left wall with the right wall. Cyclization of a cyclically-compatible sequence of tiles is $\circ \mathcal{T} = \circ(\otimes \mathcal{T})$.

8. Let $T = (G, \lambda, \rho)$ be a tile. The *right-inverted* tile of T is $T^{\updownarrow} = (G, \lambda, \overline{\rho})$. The *left-inverted* tile of T is ${}^{\updownarrow}T = (G, \overline{\lambda}, \rho)$. The *inverted* tile is ${}^{\updownarrow}T^{\updownarrow} = (G, \overline{\lambda}, \overline{\rho})$. The *reversed* tile is $T^{\leftrightarrow} = (G, \rho, \lambda)$.

Note that $\otimes\mathcal{T}$ in Definition 2.4, 6. is well-defined since $\otimes$ is associative.

Definition 2.5. The set $\mathcal{S}$ of tiles consists of tiles obtained as a combination of one of the two *frames* shown in Figure 1 and one of the 13 *pictures* shown in Figure 2 in such a way that a picture is inserted into a frame by identifying the two geometric squares. (This can mean subdividing the frame's square.) A given picture can be inserted into a frame either with the given orientation or with a $180°$ rotation.

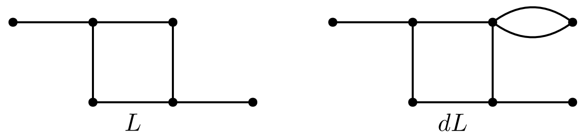

Figure 1: Both possible frames.

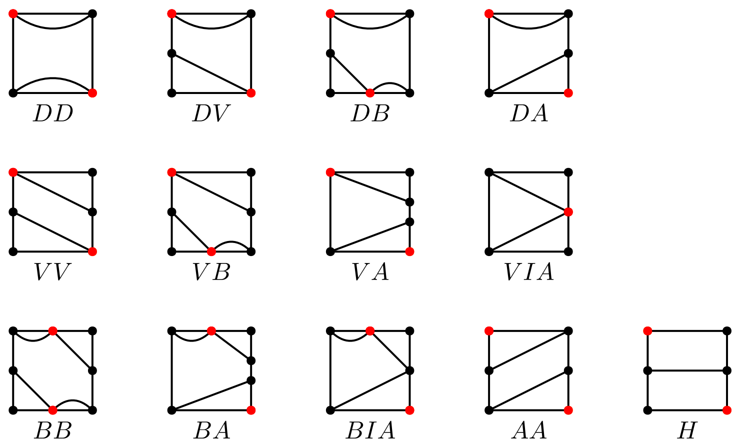

Figure 2: All possible pictures. For later need, the red vertices mark a dominating set of each of them.

Note that each picture yields either two or four tiles in $\mathcal{S}$. Altogether the set $\mathcal{S}$ contains 42 different tiles. For example, in Figure 3 we see that picture VIA yields four different tiles.

We can now define the tile structure of graphs that are of our interest. Their definition first appeared in [6].

Definition 2.6. The set $\mathcal{T}(\mathcal{S})$ consists of all graphs of the form $\circ((\otimes\mathcal{T})^{\updownarrow})$, where $\mathcal{T}$ is a sequence $(T_0, {}^{\updownarrow}T_1^{\updownarrow}, T_2, \dots, {}^{\updownarrow}T_{2m-1}^{\updownarrow}, T_{2m})$, where $m \geq 1$ and $T_i \in \mathcal{S}$ for every $i \in \{0, \dots, 2m\}$. The obtained vertices of degree 2 are suppressed.

Suppressing a vertex of degree 2 is the inverse operation to subdividing an edge, and it does not affect the crossing number of a graph. Note that for the case of calculating the domination and independence numbers of graphs, double edges can be replaced with single ones without changing the invariant.

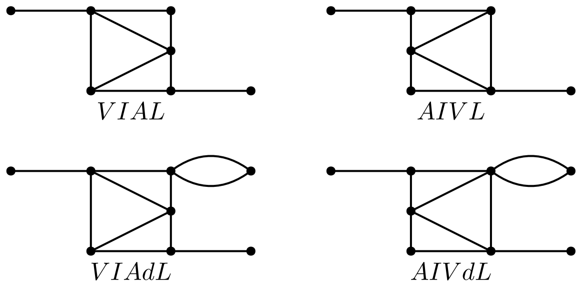

Figure 3: All possible tiles that can be obtained from picture VIA.

Theorem 2.7 ([6, Theorems 2.18 and 2.19])**.** *Each graph from $\mathcal{T}(\mathcal{S})$ is 3-connected and 2-crossing-critical. Moreover, all but finitely many 3-connected 2-crossing-critical graphs are contained in $\mathcal{T}(\mathcal{S})$.*

Theorem 2.7 gives a nice representation of large 3-con 2-cc graphs, i.e., graphs from Theorem 2.3(i).

Graphs from the set $\mathcal{T}(\mathcal{S})$ can be described as sequences over the alphabet $\Sigma = \{L, d, A, B, D, H, I, V\}$ (see [22]). A *signature* of a tile T is

$$\mathrm{sig}(T) = P_t\, Id\, P_b\, Fr,$$

where $P_t \in \{A, B, D, H, V\}$ describes the top path of the picture, $Id \in \{I, \emptyset\}$ indicates a possible identifier of the picture, $P_b \in \{A, B, D, V, \emptyset\}$ describes the bottom path of the picture, and $Fr \in \{L, dL\}$ describes the frame. Here, $\emptyset$ labels the empty word. See Figure 1 for possible signatures of frames (Fr), Figure 2 for all possible signatures of pictures ($P_t\, Id\, P_b$), and Figure 3 for an additional example of how to describe a tile with its signature.

For a graph $G \in \mathcal{T}(\mathcal{S})$, $G = \circ((\otimes\mathcal{T})^{\updownarrow}) = (T_0, {}^{\updownarrow}T_1^{\updownarrow}, T_2, \dots, {}^{\updownarrow}T_{2m-1}^{\updownarrow}, T_{2m})$, a signature is defined as

$$\mathrm{sig}(G) = \mathrm{sig}(T_0)\,\mathrm{sig}(T_1)\cdots\mathrm{sig}(T_{2m}).$$

Additionally, $\#X$ denotes the number of occurrences of X in $\mathrm{sig}(G)$, where $X \in \Sigma$. Given a tile T, the join of a sequence of k tiles, starting with T and then alternating between T and $T^{\updownarrow}$, is denoted by $k \cdot T$.

3 Domination number

In this section, we present an upper bound and a lower bound for the domination number of large 3-con 2-cc graphs, including equality cases for both bounds.

3.1 Upper bound

Theorem 3.1. *If G is a large 3-con 2-cc graph, then*

$$\gamma(G) \leq \#A + \#B + \#D + \#V + 2 \cdot \#H - \#AIV - \#VIA.$$

Proof. Each vertex lies on at least one picture. Thus, if $D \subseteq V(G)$ dominates all vertices in each picture, then D is a dominating set of G. Inside each picture we have at least one path (by path we mean the top and the bottom path as in the definition of the signature of a tile). We can see that domination of A, B, D, and V requires at least one vertex, while domination of H requires two vertices. The only exceptions are pictures AIV and VIA, where domination of the picture only requires one vertex and not two, which would be the result of the summation of domination numbers of paths A and V. Figure 2 shows all possible pictures with marked smallest dominating sets.

Edges between pictures only add edges between vertices and lower the domination number. This means that the domination number has an upper bound of the sum of domination numbers for individual paths. □

The upper bound from Theorem 3.1 is sharp, which can be seen in the following two examples. They also show that the number of frames L and dL does not affect the upper bound.

Example 3.2. Let $G_1 = n \cdot VBdL$, where $n \geq 3$ is an odd number. Figure 4 shows a dominating set of size $2n$, meaning $\gamma(G_1) \leq 2n$. The formula from Theorem 3.1 shows the same, as $\#A + \#B + \#D + \#V + 2 \cdot \#H - \#AIV - \#VIA = 2n$.

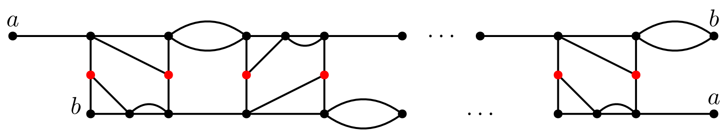

Figure 4: Graph G_1 with a marked dominating set of size $2n$. Note that to obtain the desired graph, vertices a are identified, vertices b are identified, and after this vertices of degree 2 are suppressed. The same simplification of drawings is used for the rest of the paper.

Assume $\gamma(G_1) < 2n$. The Pigeonhole principle says that there exists at least one picture, which is dominated by at most one vertex. Vertices in the corners of the picture can be dominated by vertices from neighboring pictures. The remaining three inner vertices, which we get from B and V and are painted orange in Figure 5, are yet to be dominated. Since these three vertices cannot be dominated by one vertex, we need at least two vertices to dominate this picture, which leads to a contradiction. Therefore $\gamma(G_1) \geq 2n$.

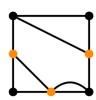

Figure 5: Picture VB, where we require that the inner vertices, marked orange, are dominated by one vertex.

From this, it follows that $\gamma(G_1) = 2n$.

Example 3.3. Let $G_2 = n \cdot AIVL$, where $n \geq 3$ is an odd number. We can find a dominating set of size n (see Figure 6), thus $\gamma(G_2) \leq n$. This also follows from the formula in Theorem 3.1, as $\#A + \#B + \#D + \#V + 2 \cdot \#H - \#AIV - \#VIA = n$.

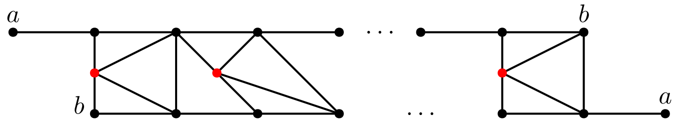

Figure 6: Graph G_2 with a marked dominating set of size n.

We next show that $\gamma(G_2) \geq n$. Divide the graph G_2 into n disjoint subgraphs, as shown in Figure 7. Each subgraph is induced on the closed neighborhood of the degree 3 vertex and is isomorphic to the paw graph. The position of degree 3 vertices in G_2 ensures that the obtained n subgraphs are all pairwise disjoint. We notice that the middle vertex of each subgraph (the vertex of degree 3) can only be dominated by one of the vertices in the same subgraph. Hence we must choose at least one vertex from each one of the n disjoint subgraphs, which means that $\gamma(G_2) \geq n$.

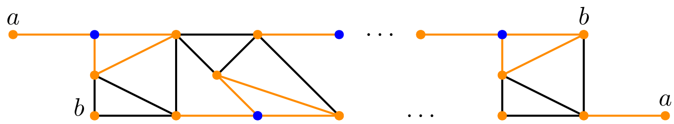

Figure 7: Graph G_2 with disjoint subgraphs marked orange and the middle vertex of each subgraph marked blue. Recall that when vertex a is identified, the obtained vertex of degree 2 is suppressed.

It follows that $\gamma(G_2) = n$.

3.2 Lower bound

Theorem 3.4. *If G is a large 3-con 2-cc graph, then*

$$\gamma(G) \geq \left\lceil \frac{2}{3} \cdot \#L \right\rceil .$$

Before proving the result, we list two useful observations. Let G be a large 3-con 2-cc graph.

1. Every vertex of G lies on at least one and at most two tiles.

2. All vertices of a picture P can be dominated by a single vertex v only if the picture is VIA. Moreover, in this case, the vertex v only dominates vertices within picture P.

Proof of Theorem 3.4. Let G be a 3-con 2-cc graph. We can assume that G only has L frames, since replacing dL frames with L frames means that we contract some edges, which can only decrease the domination number. Replacing dL frames with L frames doesn't change the number $\#L$.

From Observation 1 we know that every vertex of G lies either on only one tile or on two consecutive tiles. If it lies on only one tile, we say that it belongs to that tile. If it lies on two tiles, we say that it belongs to the tile on the right. Hence every vertex belongs to exactly one tile.

Let D be a smallest dominating set of G. First we show that for every trinity of consecutive tiles, there exist at least two vertices from the set D that belong to one of the tiles in the trinity. Figure 8 shows frames of a trinity of tiles (without the pictures). Vertices that are marked red belong to one of the tiles in the trinity. Vertices that are marked with a green circle can only be dominated from one of the vertices that belong to the trinity of tiles.

From Observation 2 it follows that we need at least two elements from the set D to dominate all green vertices. Hence there exist at least two vertices from the set D that belong to the trinity of tiles.

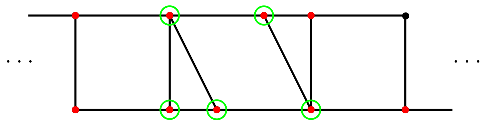

Figure 8: A trinity of consecutive tiles. Vertices that are marked with a green circle can only be dominated from one of the vertices that belong to this trinity of tiles.

Now we can prove that $|D| \geq \frac{2}{3}\#L$. There are $\#L$ trinities of consecutive tiles in the graph G, we denote them $1, 2, \ldots, \#L$. Let

$$D' = \{(d, k) \mid d \in D, d \text{ belongs to one of the tiles in the trinity } k\}.$$

Since each vertex belongs to exactly three trinities of consecutive tiles, it follows that $|D'| = 3 \cdot |D|$. For every trinity of tiles k, there exist at least two vertices from D that belong to tiles in the trinity k. Hence there exist at least two elements in D' with second component k for every $k = 1, 2, \ldots, \#L$. Therefore $|D'| \geq 2 \cdot \#L$, hence it holds that $3 \cdot |D| \geq 2 \cdot \#L$. Because the domination number of G is an integer, it follows that $\gamma(G) \geq \lceil \frac{2}{3} \cdot \#L \rceil$. □

The lower bound from Theorem 3.4 is sharp, which can be seen in the following example.

Example 3.5. Let $G_3 = n \cdot DDLDDLAIVL$, where $n \geq 1$ is an odd number. Figure 9 shows the dominating set of size $\frac{2}{3} \cdot 3 \cdot n$, meaning $\gamma(G_3) \leq 2n$. Our formula from Theorem 3.4 shows the same, as $\lceil \frac{2}{3} \cdot \#L \rceil = \lceil \frac{2}{3} \cdot 3 \cdot n \rceil = 2n$.

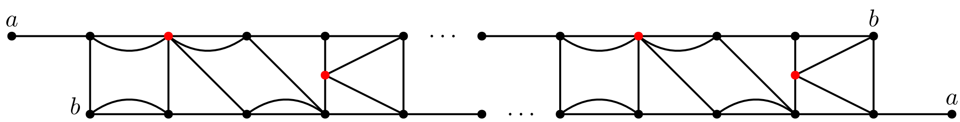

Figure 9: Graph G_3 with a marked dominating set of size $2n$.

Every trinity of consecutive pictures $DDLDDLAIVL$ requires at least two vertices from the dominating set to dominate all the inner vertices of the trinity, which are marked orange in Figure 10. This means that at least two vertices are needed to dominate this trinity of pictures. Therefore $\gamma(G_3) \geq 2n$.

From this follows that $\gamma(G_3) = 2n$.

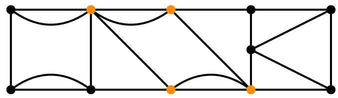

Figure 10: Trinity of consecutive pictures $DDLDDLAIVL$, where we want the inner vertices, marked orange, to be dominated by one vertex. Note that none of these vertices can be dominated by a vertex outside of this trinity of tiles.

4 Independence number

In this section, we present sharp upper and lower bounds for the independence number of large 3-con 2-cc graphs.

4.1 Upper bound

Theorem 4.1. *If G is a large 3-con 2-cc graph, then*

$$\alpha(G) \leq \left\lfloor \frac{|\mathrm{V}(G)|}{2} \right\rfloor .$$

Proof. Since all large 3-con 2-cc graphs are Hamiltonian [22], and the independence number of Hamiltonian graphs is at most $\frac{1}{2}|\mathrm{V}(G)|$, we obtain the desired upper bound. □

The following example shows that the upper bound from Theorem 4.1 is sharp.

Example 4.2. Let $G_4 = n \cdot HdL$, where $n \geq 3$ is an odd number. Then $|\mathrm{V}(G_4)| = 6n$. Figure 11 shows that we can choose $3n$ independent vertices from the graph G_4, meaning $\alpha(G_4) \geq 3n$.

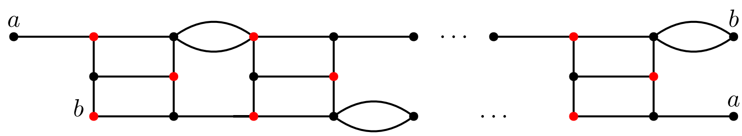

Figure 11: Graph G_4 with a marked independent set of size $3n$.

Every vertex of graph G_4 lies in exactly one picture. Since we can choose at most three independent vertices in each of the n pictures, $\alpha(G_4) \leq 3n$, which is also the result of Theorem 4.1. Therefore $\alpha(G_4) = 3n = \left\lfloor \frac{|\mathrm{V}(G_4)|}{2} \right\rfloor$.

4.2 Lower bound

Theorem 4.3. *If G is a large 3-con 2-cc graph, then*

$$\alpha(G) \geq \min\{\#L + \#d, 2 \cdot \#L - 1\}.$$

Proof. For every large 3-con 2-cc graph G we can construct the graph G' from the same frames used for G, without using the pictures. We notice that if we add pictures into the frames in G' to get the original graph G, we only add vertices and do not connect any vertices that were previously not connected, thus any picture we add can only increase the independence number, therefore $\alpha(G) \geq \alpha(G')$. Note that the graph G' will have the same number of frames L and dL as the initial graph G. Therefore it suffices to prove the proposed lower bound for the graph G'.

We distinguish two cases, the first case is if there are only dL frames and the second if there is at least one L frame.

Case 1 If there are only dL frames, we can find $2 \cdot \#L - 1$ independent vertices, as is shown in Figure 12. Note that in this case $\min\{\#L + \#d, 2 \cdot \#L - 1\} = 2 \cdot \#L - 1$.

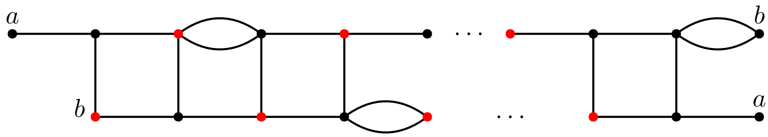

Figure 12: Graph G' from Case 1 with a marked independent set of size $2 \cdot \#L - 1$.

Case 2 If there is at least one L frame, then we can choose the independent set based on the following method. Note that double edges can be ignored when studying the independence number. The graph G' is then composed of 3- and 4-cycles, which are connected with additional edges (marked orange in Figure 13). These additional edges come from where the top and bottom paths of the pictures were in G. To obtain an independent set of appropriate size, we select one vertex from each 3-cycle and two vertices from each 4-cycle. For every 3-cycle, we select the vertex of degree 3 on its left side. If we have two consecutive 3-cycles, the vertices we chose from them are independent. When selecting vertices in the 4-cycles, we consider all consecutive 4-cycles between two 3-cycles and select vertices for the independent set in these 4-cycles from right to left. The 3-cycle on the right of the consecutive 4-cycles determines how we choose the independent set in the right-most 4-cycle, which in turn uniquely determines how we select two independent vertices in each of these 4-cycles (in the same manner as in Figure 12). Notice that the 3-cycle on the left of these 4-cycles gives no restriction on the selected vertices.

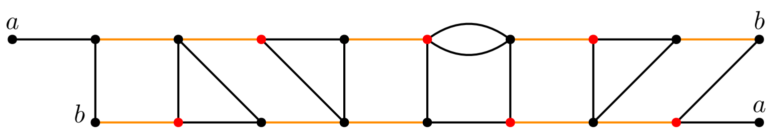

Figure 13: An example of the graph G' from Case 2 with a marked independent set of size $\#L + \#d$. The edges that connect the 3- and 4-cycles are marked orange.

We have thus chosen two vertices in each 4-cycle and one vertex in each 3-cycle. Since the number of 3-cycles is $\#L - \#d$ and the number of 4-cycles is $\#d$, we have found an independent set of size $(\#L - \#d) + 2 \cdot \#d = \#L + \#d$. Note that in this case $\min\{\#L + \#d, 2 \cdot \#L - 1\} = \#L + \#d$. □

The following two examples show that the lower bound from Theorem 4.3 is sharp. The first example naturally follows from the proof of Theorem 4.3, while the second example provides a non-trivial family of sharpness examples. Additionally, examples are selected in such a way that different parts of the minimum are attained.

Example 4.4. Let G_5 be a large 3-con 2-cc graph built from tiles $DDdL$ and DDL, so that not all of the tiles are $DDdL$.

From Theorem 4.3 we know that $\alpha(G_5) \geq \#L + \#d$. Similarly as in the proof, we can find $\#d$ 4-cycles and $\#L - \#d$ 3-cycles in G_5, so that every vertex lies on exactly one of them. Every 4-cycle is formed by the two vertices on the right of a $DDdL$ tile and the two vertices on the left of the next tile to the right. Every 3-cycle is formed by the two vertices on the right of a DDL tile and the two vertices on the left of the next tile. Two of those vertices are identified, thus giving us a 3-cycle. The 3-cycles and 4-cycles are marked in Figure 14.

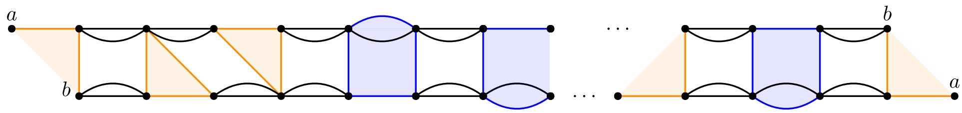

Figure 14: Graph G_5 with marked 3-cycles and 4-cycles.

We can choose at most one independent vertex from every 3-cycle and at most two independent vertices from every 4-cycle, therefore $\alpha(G_5) \leq 2 \cdot \#d + 1 \cdot (\#L - \#d) = \#L + \#d$.

From this it follows that $\alpha(G_5) = \#L + \#d$.

Example 4.5. Let G_6 be a large 3-con 2-cc graph that is built from $DDdL$, $VIAdL$, and $AIVdL$ tiles, but not all tiles are $VIAdL$, and not all tiles are $AIVdL$.

From Theorem 4.3 we know that $\alpha(G_6) \geq 2 \cdot \#L - 1$. We can find at most two independent vertices in each of the tiles $DDdL$, $VIAdL$, and $AIVdL$, therefore we can find at most $2 \cdot \#L$ independent vertices in G_6.

For contradiction suppose that $\alpha(G_6) \neq 2 \cdot \#L - 1$, meaning $\alpha(G_6) = 2 \cdot \#L$. We try to construct an independent set A with $2 \cdot \#L$ vertices. Set A must include exactly two vertices from every tile because otherwise set A would have to include at least 3 vertices from one tile, which is impossible.

There are two different ways in which we can choose two independent vertices from a $DDdL$ tile, and three different ways for tiles $VIAdL$ and $AIVdL$. All options are shown in Figure 15.

Even though tiles $VIAdL$ and $AIVdL$ have a third option for the choice of two independent vertices (where the selected vertices are not diagonal), we can't choose the vertices in set A in this way, since we know that we have to choose two independent vertices from every tile. If we choose the top and bottom right vertex in a $VIAdL$ tile, then the only way to choose two vertices in the next tile is if that tile is also a $VIAdL$ tile and we choose the top and bottom right vertices. We continue this for all tiles, but since not all tiles are $VIAdL$, at some point we are not able to choose two independent vertices in the next tile. For the same reason, we also cannot choose the two vertices on the left of an $AIVdL$ tile.

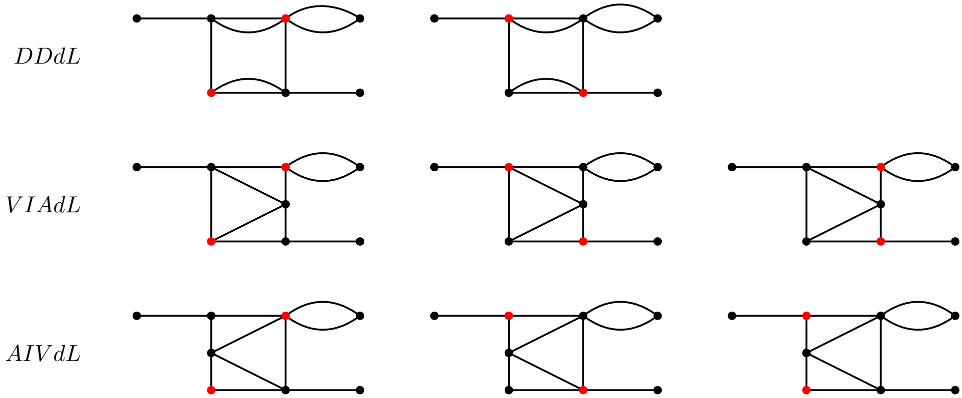

Figure 15: Tiles $DDdL$, $VIAdL$, and $AIVdL$ with two independent vertices marked.

This means that for all tiles, the two vertices that are included in set A are the diagonal ones, without loss of generality we can assume that those diagonal vertices in the first tile are the bottom left and the top-right vertex. This choice determines which vertices we must choose in the tile to the right and so on, as is shown in Figure 16.

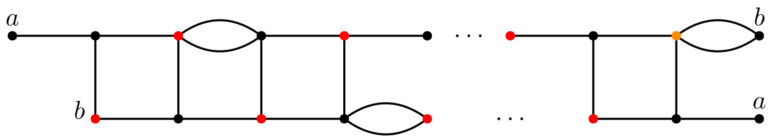

Figure 16: Graph G'_6 was constructed from the same frames used for G_6, without using the pictures. The first tile of graph G'_6 determines which two vertices are included in set A for all other tiles. When we get to the last tile, we get a contradiction (marked orange).

When we get to the last tile we get a contradiction. Because of the tile to the left, the only possible vertices from the last tile that can be included in A are the bottom left and the top-right vertex. But the top-right vertex is connected to a vertex in the first tile that is already included in set A, therefore set A cannot include two vertices from the last tile.

This means that the independent set A that has $2 \cdot \#L$ elements cannot exist and $\alpha(G_6) = 2 \cdot \#L - 1$.

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Enumerating symmetric pyramids in Motzkin paths*

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Abstract

A path in the first quadrant of the xy-plane that starts at the origin having North-East steps (X), Horizontal steps (Z), South-East steps (Y), and that ends on the x-axis is called *Motzkin*. A maximal *pyramid* is a subpath of the form $X^hZ^mY^h$ that cannot be extended to $X^{h+1}Z^mY^{h+1}$. It is *symmetric* if it cannot be extended to any of these subpaths: $X^{h+1}Z^mY^h$ or $X^hZ^mY^{h+1}$. We use generating functions to enumerate symmetric pyramids and give the asymptotic behavior of the number of symmetric pyramids. Additionally, we give combinatorial arguments to count some of the mentioned aspects.

Keywords: Motzkin path, generating function, symmetric pyramid.

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1 Introduction

A path in the first quadrant of the xy-plane that starts at the origin having North-East steps (X), Horizontal steps (Z), South-East steps (Y), and that ends on the x-axis is called *Motzkin*. Whoever is familiar with Dyck paths may verify that Motzkin paths are a generalization of them.

The set of all Motzkin paths of length n is denoted by $\mathcal{M}_n$ and the set $\cup_{n=0}^{\infty}\mathcal{M}_n$ is denoted by $\mathcal{M}$. A maximal *pyramid* of *weight* h in a Motzkin path is a subpath of the

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form $X^hZ^mY^h$ that cannot be extended to $X^{h+1}Z^mY^{h+1}$, with $h \geq 1$ and $m \geq 0$. It is *symmetric* if it can not be extended to any of these subpaths: $X^{h+1}Z^mY^h$ or $X^hZ^mY^{h+1}$. There are two types of distinguishable maximal pyramids: if $m = 0$, the pyramid is called a *triangular pyramid*, denoted by $\triangle_h$ or by $\triangle$ if there is no ambiguity, see the left-hand side of Figure 1; and if $m > 0$, the pyramid is called *truncated*, denoted by $⏢_h$ or by $⏢$ if there is no ambiguity, see the right-hand side of Figure 1. The *symmetric weight* of a path is the sum of the weights of its symmetric pyramids. The *height* of a pyramid is the y-coordinate of its highest point. That is, the height is measured from the x-axis to the highest point. The *symmetric height* of a path is the sum of the heights of its symmetric pyramids. A *hump* of a Motzkin path is a subpath of the form XZ^kY for any non-negative integer k. If $k = 0$, it is called *sharp hump* (or *sharp peak*). Notice that every pyramid has a hump.

Motivated in part by the work done by Asakly [1], Flórez and Ramírez [8] introduced the concept of symmetric and asymmetric peaks in Dyck paths, see also Elizalde et al. [5], Flórez et al. [7], and Sun et al. [14]. Following up these works we generalize the symmetric concept to Motzkin paths. Using generating functions we enumerate symmetric pyramids and give the asymptotic behavior of the number of symmetric pyramids. Additionally, we give combinatorial arguments to count some of the mentioned aspects. Our results here, in particular, apply to recover the known results for Dyck paths.

Figure 1: Triangular symmetric pyramid and truncated symmetric pyramid.

2 Motzkin paths, humps, and valleys

If $c_k = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number, then the number of Motzkin paths of length n, $|\mathcal{M}_n|$, is given by the nth Motzkin number defined as

$$m_n := \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1}\binom{n}{2k}\binom{2k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} c_k \binom{n}{2k}. \tag{2.1}$$

See, for example, the sequence A001006 of [12].

In this section we give a bivariate generating function enumerating Motzkin paths with respect to the length and the number of humps. The results here are, in fact, lemmas for the main results studied in the upcoming sections. For example, in Section 3 we compare the number of symmetric pyramids with respect to the number of humps. In particular, we prove that these two quantities are in a proportion of $1/2$ (see Corollary 3.8).

We denote by $\texttt{humps}(M)$ the number of humps in a Motzkin path M. We now introduce a bivariate generating function to count the number of humps with respect to the length of a Motzin path:

$$H(q, x) := \sum_{M \in \mathcal{M}} q^{\texttt{humps}(M)} x^{|M|}.$$

Proposition 2.1. *The generating function for Motzkin paths with respect to the number of humps is given by*

$$H(q,x)=\frac{1-2x+2x^2-qx^2-\sqrt{(1-4x+(4-q)x^2)(1-qx^2)}}{2x^2(1-x)}.$$

Proof. A Motzkin path M can be decomposed as —using first return decomposition— λ (the empty path), ZM, or XM_1YM_2, where M_1 and M_2 are Motzkin paths, possibly empty. Notice that for the last decomposition, $\texttt{humps}(M)=\texttt{humps}(M_1)+\texttt{humps}(M_2)$ —unless M_1 is the path Z^k, for any $k\geq 0$, in which case $\texttt{humps}(M)=1+\texttt{humps}(M_2)$. Therefore, the generating function $H(q,x)$ satisfies this functional equation

$$H(q,x)=1+xH(q,x)+x^2\Big(H(q,x)-\frac{1}{1-x}+\frac{q}{1-x}\Big)H(q,x).$$

Solving this functional equation for $H(q,x)$ we obtain the desired result. □

Let us denote by hm_n the total number of humps in $\mathcal{M}_n$. Then we have

$$\frac{\partial}{\partial q}H(q,x)\mid_{q=1}=\sum_{n\geq 0}hm_nx^n=\frac{1-2x+x^2-(1-x)\sqrt{1-2x-3x^2}}{2(1-x)^2\sqrt{1-2x-3x^2}}.$$

The sequence hm_n is given by the combinatorial sum (cf. [13, 3])

$$hm_n=\frac{1}{2}\sum_{j\geq 0}\binom{n}{j}\binom{n-j}{j}-\frac{1}{2}.$$

Notice that if $\binom{n,3}{n}$ denotes the central trinomial coefficient, that is, the nth coefficient of the expansion of $(x^2+x+1)^n$, then $hm_n=\frac{1}{2}\binom{n,3}{n}-1$. The central trinomial coefficient has the asymptotic approximation (cf. [11])

$$\binom{n,3}{n}\sim\frac{3^n}{2}\sqrt{\frac{3}{\pi n}}.$$

Therefore, the sequence hm_n has this asymptotic approximation

$$hm_n\sim\frac{3^n}{4}\sqrt{\frac{3}{\pi n}}. \tag{2.2}$$

A *weak valley* of a Motzkin path is any subpath of the form ZZ, YX, ZX, or YZ, see Figure 2. Let $\texttt{wv}(M)$ be the number of weak valleys in a Motzkin path M. Let us denote by $e(M)$ the number of horizontal steps in M. We now introduce a multivariate generating function to count the number of weak valleys and horizontal steps, with respect to the length of a Motzin path:

$$M_{\texttt{wv}}(q,t,x):=\sum_{M\in\mathcal{M}}q^{\texttt{wv}(M)}t^{e(M)}x^{|M|}.$$

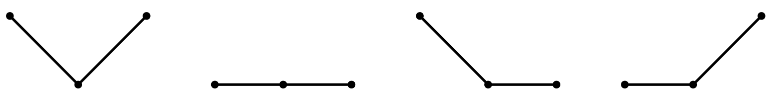

Figure 2: Weak valleys in a Motzkin path.

We need this proposition for our main theorem of Section 3. See [2] for more results about valleys of Motzkin paths.

Proposition 2.2. *The generating function for Motzkin paths with respect to the number of weak valleys and horizontal steps is given by:*

$$M_{\mathtt{wv}}(q,t,x) = \frac{1 - tqx - x^2 + qx^2 - \sqrt{(1 - tqx - x^2 + qx^2)^2 - 4qx^2(1 + tx - tqx)}}{2qx^2}.$$

Proof. From the first return decomposition, a Motzkin path M can be decomposed as λ, ZM_1 or XM_2YM_3, where $M_i \in \mathcal{M}$, for $i = 1, 2, 3$. Notice that

$$\mathtt{humps}(M) = \mathtt{humps}(M_1) + 1 \quad \text{and} \quad \mathtt{humps}(M) = \mathtt{humps}(M_2) + \mathtt{humps}(M_3) + 1,$$

unless M_1 and M_3 are the empty paths, in this case, we have that $\mathtt{humps}(M) = 0$ and $\mathtt{humps}(M) = \mathtt{humps}(M_1)$, respectively. Therefore, the generating function $M_{\mathtt{wv}}(q,t,x)$ satisfies the functional equation

$$M_{\mathtt{wv}}(q,t,x) = 1 + tx(qM_{\mathtt{wv}}(q,t,x) - q + 1) + x^2 M_{\mathtt{wv}}(q,t,x)(qM_{\mathtt{wv}}(q,t,x) - q + 1).$$

Solving this functional equation for $M_{\mathtt{wv}}(q,t,x)$ we obtain the desired result. □

The series expansion of the generating function $M_{\mathtt{wv}}(q,1,x)$ is

$$\begin{aligned} &1 + x + (1+q)x^2 + (1+2q+q^2)x^3 + (1+4q+3q^2+q^3)x^4 \\ &+ (1+6q+9q^2+4q^3+q^4)x^5 + (1+9q+19q^2+16q^3+5q^4+q^5)x^6 + O(x^7). \end{aligned}$$

The array coefficients of the above series correspond to the sequence A110470.

3 Symmetric pyramids of Motzkin paths

In this section we give a multivariate generating function to count the number of symmetric pyramids, the number of triangular, and truncated symmetric pyramids with respect to the length of a Motzkin path. We give an asymptotic behavior of symmetric pyramids. In the end of the section we analyze the proportion existing between the total number of symmetric pyramids and the total number of humps in M_n, when $n \to \infty$.

Let us denote by $\mathtt{sp}(M)$, $\mathtt{stp}(M)$, and $\mathtt{strp}(M)$ the number of symmetric pyramids, the number of symmetric triangular pyramids, and the number symmetric truncated pyramids, respectively, of a Motzkin path M. It is clear that $\mathtt{sp}(M) = \mathtt{stp}(M) + \mathtt{strp}(M)$. We now introduce a multivariate generating function with respect to the defined parameters:

$$M_{\mathtt{sp}}(q,p,t,x) := \sum_{M \in \mathcal{M}} q^{\mathtt{stp}(M)} p^{\mathtt{strp}(M)} t^{e(M)} x^{|M|}.$$

Theorem 3.1 gives an expression for this multivariate generating function. The result of this theorem is based on *marking* or *distinguishing* elements within the generating function. For more details about the method used in the theorem see, for example, these books: Goulden and Jackson [9, page 128] or Flajolet and Sedgewick [6, page 209].

Theorem 3.1. *The generating function for Motzkin paths with respect to the number of symmetric triangular pyramids, the number symmetric truncated pyramids, and the number of horizontal steps is given by:*

$$M_{sp}(q,p,t,x) = \frac{(1-x^2)(1-tx)\left(p_1 - \sqrt{p_2}\right)}{2x^2(1-tx-qx^2+(1-p+q)tx^3)^2},$$

where p_1 and p_2 are these multivariate polynomials

$$p_1 = 1 - 2tx - (q-t^2)x^2 + (2-p+q)tx^3 + (1-q-t^2)x^4 - (p-q)tx^5$$

and

$$p_2 = (1-x^2)(1-(1+2t)x-(1+q-t^2)x^2-(1-q-2t+pt-qt-t^2)x^3+(p-q)tx^4) \times (1+(1-2t)x-(1+q-t^2)x^2+(1-q+2t-pt+qt-t^2)x^3-(p-q)tx^4).$$

Proof. Consider the set $\mathcal{G}$ of all Motzkin paths in which an arbitrary number of symmetric pyramids (some, possibly none, possibly all) have been marked. Any path in G can be obtained from a Motzkin path M by inserting pyramids (triangular or truncated) with a marked hump. These pyramids can be inserted in the node of a weak valley, in the initial or final node of M. These kinds of nodes are called *insertion nodes* (see [4]). Let $\mathtt{ins}(M)$ be the number of insertion nodes of M. It is clear that $\mathtt{ins}(M) = \mathtt{wv}(M) + 2$ —unless M is the empty path, in which case $\mathtt{ins}(M) = 1$. Consider the generating function

$$M_{\mathtt{ins}}(q,t,x) = \sum_{M \in \mathcal{M}} q^{\mathtt{ins}(M)} t^{e(M)} x^{|x|}.$$

Then, it is clear that

$$M_{\mathtt{ins}}(q,t,x) = q^2(M_{\mathtt{wv}}(q,t,x) - 1) + q.$$

Let M be a marked Motzkin path in $\mathcal{G}$. We denote by $\mathtt{mark}_1(M)$ and $\mathtt{mark}_2(M)$ the number of marked triangular pyramids and truncated pyramids of M, respectively. Consider the generating function

$$G(q,p,t,x) = \sum_{M \in \mathcal{G}} q^{\mathtt{mark}_1(M)} p^{\mathtt{mark}_2(M)} t^{e(M)} x^{|M|}.$$

Since the elements of $\mathcal{G}$ are obtained from elements of $\mathcal{M}$ by inserting marked pyramids in the insertions nodes, it translates in terms of generating functions as

$$G(q,p,t,x) = M_{\mathtt{ins}}\Big(\frac{1}{1-qx^2/(1-x^2)-ptx^3/((1-x^2)(1-tx))}, t, x\Big).$$

Finally, the Motzkin paths with marked symmetric pyramids are generated by these substitutions: $q \mapsto q+1$ and $p \mapsto p+1$ within $M_{\mathtt{sp}}(q,p,t,x)$. That is, we obtain the equality

$$G(q,p,t,x) = M_{\mathtt{sp}}(q+1,p+1,t,x)$$

or equivalently

$$M_{\mathtt{sp}}(q,p,t,x) = G(q-1,p-1,t,x).$$

Using the above equality and Proposition 2.2 we obtain the desired result. □

The series expansion of the generating function $M_{\mathrm{sp}}(q,p,t,x)$ is

$$1 + tx + (q + t^2)x^2 + (pt + 2qt + t^3)x^3 + \boldsymbol{(q + q^2 + 3pt^2 + 3qt^2 + t^4)x^4} + (2t + pt + 2qt + 2pqt + 3q^2t + 6pt^3 + 4qt^3 + t^5)x^5 + O(x^6).$$

Figure 3 shows some Motzkin paths. Their corresponding weights are shown in the previous expansion in boldface.

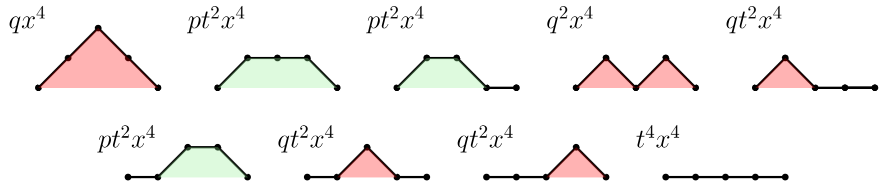

Figure 3: Motzkin paths and the symmetric pyramid statistic.

We use s_n to denote the total number of symmetric pyramids in $\mathcal{M}_n$.

Corollary 3.2. *The generating function for Motzkin paths with respect to the number of symmetric pyramids is given by:*

$$\frac{(1-x)^2(1+x)\left(1 - 2x + (1-q)x^2 + 2x^3 - qx^4 - (1-x^2)\sqrt{p(x,q)}\right)}{2x^2(1 - x - qx^2 + x^3)^2},$$

where $p(x,q) = 1 - 4x + 2(2-q)x^2 + 4qx^3 - (4-q^2)x^4$*. Moreover, the sequence* s_n *has the generating function*

$$\frac{\partial}{\partial q} M_{sp}(q,q,1,x)\mid_{q=1} = \frac{1 - 4x^3 + 4x + x^4 + (1 - x^2 - 2x)(1-x)\sqrt{1 - 2x - 3x^2}}{2(1-x)^3(1+x)\sqrt{1 - 2x - 3x^2}}.$$

These are the first 11 values of the sequence s_n for $n = 1, 2, \ldots, 11$:

$$0, \quad 1, \quad 3, \quad 9, \quad 23, \quad 60, \quad 156, \quad 415, \quad 1121, \quad 3076, \quad 8540.$$

We use s_k^*, g_k^*, and t_k^* to denote the number of all first symmetric pyramids, all first symmetric triangular pyramids, and all first truncated symmetric pyramids at a ground level, respectively, in $\mathcal{M}_k$. Note that given a path $P \in \mathcal{M}_k$, there is another path $Q \in \mathcal{M}_k$ that is symmetric to P —the first seen from left-to-right and the second seen from from right-to-left. Therefore, using this symmetry of paths in $\mathcal{M}_k$ we have that s_k^*, g_k^*, and t_k^* also count the number of all last symmetric pyramids, all last symmetric triangular pyramids, and all last truncated symmetric pyramids at ground level, respectively.

Lemma 3.3. *If* c_k *is the kth Catalan number, then* $s_k^* = g_k^* + t_k^*$*, where*

$$g_k^* = 2\sum_{i=1}^{k-1} (-1)^{i+1} \binom{i}{\lfloor \frac{i-1}{2} \rfloor} \binom{k-1}{i+1}$$

and

$$t_k^* = 2\sum_{i=1}^{k-1} \left(\sum_{j=0}^{\lfloor \frac{k-i}{2} \rfloor} c_j \left\lfloor \frac{i-1}{2} \right\rfloor \binom{k-i}{2j} \right).$$

Proof. Let us find g_k^*. A Motzkin path μ_k can be decomposed as $\triangle_i\mu_{k-i}$ for $i = 0, 1, \ldots, k-1$, where $\triangle_0 = \emptyset$. This implies that the number of that type of pyramid is given by

$$g_k^* = \begin{cases} \displaystyle\sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} m_{2i}, & \text{if } k \text{ is even;} \\ \displaystyle\sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} m_{2i-1}, & \text{if } k \text{ is odd.} \end{cases}$$

This formula coincides with the sequence A082397 in OEIS multiplied by 2. Therefore, the expression given in the statement is from A082397.

We now find t_k^*. A Motzkin path of the form μ_k can be decomposed as ⏢$_i\mu_{k-i}$, for $i = 0, 1, \ldots, k-1$, where ⏢$_0 = \emptyset$. This implies that the number of that type of pyramids, in the set of all paths of the form ⏢$_i\mu_{k-i}$, is given by $t_k^* = \sum_{i=1}^{k-1} \left\lfloor \frac{i-1}{2} \right\rfloor m_{k-i}$. This and (2.1) give the result in the statement of this lemma.

Clearly, the number of first symmetric pyramids is the sum of these two quantities. □

From the first return decomposition of a Motzkin path we have this particular expression

$$Z\mu_{n-1} \quad \text{and} \quad X\mu_kY\mu_{n-2-k}, \quad \text{with} \quad \mu_i \in \mathcal{M}_i \quad \text{and} \quad \mu_0 = \lambda. \tag{3.1}$$

Here we say that $X\mu_kY$ is the first primitive subpath. We need this decomposition again for the proof of Theorem 4.4.

Theorem 3.4. *Let s_n be the total number of symmetric pyramids in $\mathcal{M}_n$. Then s_n satisfies $s_1 = 0$, $s_2 = 1$, $s_3 = 3$, and for $n > 3$*

$$s_n = s_{n-1} + s_{n-2} + s_{n-3} + \sum_{k=0}^{n-2} m_{n-2} + \sum_{k=2}^{n-2} \Big((s_k - 2s_k^*)m_{n-k-2} + s_{n-k-2}m_k \Big).$$

Proof. From the decomposition given in (3.1) the number of symmetric pyramids in a path of the form $Z\mu_{n-1}$ is given by the number of symmetric pyramids in μ_{n-1}, that is counted by s_{n-1}.

We now analyze $X\mu_kY\mu_{n-1-k}$, for $k = 0, 1, \ldots, n-1$. If $k = 0$ or 1, then $\triangle_1$ or ⏢$_1$ becomes the first symmetric pyramid in the path. These give that the total number of symmetric pyramids in the set of all paths of the form $XY\mu_{n-2}$ and $XZY\mu_{n-3}$ is given by $m_{n-2} + s_{n-2} + m_{n-3} + s_{n-3}$.

Let $0 < k < n$ be a fixed number. Then the number of symmetric pyramids in the set of paths of the form $X\mu_kY\mu_{n-1-k}$ is given by $(s_k - 2s_k^*)m_{n-2-k} + s_{n-2-k}m_k$. Thus, $s_{n-2-k}m_k$ counts all symmetric pyramids after the first primitive subpath and $(s_k - s_k^*)m_{n-2-k}$ counts all symmetric pyramids in the first primitive subpath. Note that s_km_{n-2-k} counts all symmetric pyramids in μ_k. However, the first and the last pyramids at the ground level in μ_k are counted by s_k as symmetric, but in $X\mu_kY$ they are not symmetric. So, we must subtract all of them; and that is what s_k^* does (see Lemma 3.3).

Therefore, for $k = 1, 2, \ldots, n-1$ we have that the number of symmetric pyramids in paths of the form $X\mu_kY\mu_{n-1-k}$ is given by $\sum_{k=1}^{n-1}((s_k - 2s_k^*)m_{n-k-2} + s_{n-k-2}m_k)$. Now adding the results from each decomposition and varying k from 0 to $n-1$, we obtain the desired result. □

Remark 3.5. Notice that the generating function of the number of symmetric pyramids in $\mathcal{M}_n$ is algebraic, then the counting sequence s_n satisfies a recurrence relation with polynomial coefficients. This can be automatically solved with Kauers's algorithm [10]. In particular we obtain that s_n satisfies the recurrence relation

$$p_0(n)s_n + p_1(n)s_{n+1} + p_2(n)s_{n+2} + p_3(n)s_{n+3} + p_4(n)s_{n+4} + p_5(n)s_{n+5} = 0,$$

for $n \geq 5$, where

$$\begin{aligned} p_0(n) &= n^3 - 3n^2 - 3n; & p_1(n) &= -3n^3 + 9n^2 + 15n - 26; \\ p_2(n) &= -2n^3 + 12n^2 - 6n - 34; & p_3(n) &= 6n^3 - 24n^2 - 18n + 86; \\ p_4(n) &= n^3 - 9n^2 + 9n + 34; & p_5(n) &= -3n^3 + 15n^2 + 3n - 60. \end{aligned}$$

Theorem 3.6. *An asymptotic approximation for s_n is given by:*

$$s_n \sim \frac{3^n}{8}\sqrt{\frac{3}{\pi n}}.$$

Proof. The generating function of the sequence s_n (see Corollary 3.2) can be written as $A(x) + B(x)$, where

$$A(x) = \sum_{n\geq 0} a_n x^n = \frac{1 - 2x - x^2}{2(1-x)^2(1+x)} \quad \text{and}$$
$$B(x) = \sum_{n\geq 0} b_n x^n = \frac{-1 + 3x + 3x^2 - x^3}{2(1-x)^2(1+x)\sqrt{1 - 2x - 3x^2}}.$$

The sequence a_n satisfies $a_0 = 1$ and $a_n = -2\lfloor (n-1)/2 \rfloor + 1$, for all $n \geq 1$. Then, $a_n \sim -n/2$. On the other hand, the main singularity of $B(x)$ is 1/3. From the singularity analysis described in [6] we obtain that $b_n \sim \frac{3^n}{8}\sqrt{\frac{3}{\pi n}}$. Therefore, $s_n \sim b_n$. □

Comparing the nth coefficients of the generating functions $A(x)$ and $B(x)$ defined in Theorem 3.6 we obtain this corollary.

Corollary 3.7. *For all $n \geq 0$, we have*

$$s_n = \frac{1}{2}\sum_{i=0}^{n} \binom{n,3}{i} b_{n-i} - \left\lfloor \frac{n-1}{2} \right\rfloor - \frac{1}{2},$$

where $b_0 = -1$ and $b_n = (n((n+1) \bmod 2) + 2(n \bmod 2))(-1)^{n(n-1)/2}$, for all $n \geq 1$.

The trinomial coefficient $\binom{n,3}{i}$ can be calculated using the established combinatorial sum $\binom{n,3}{i} = \sum_{k=0}^{n} \binom{n}{k}\binom{k}{i-k}$. From Theorem 3.6 and (2.2) we can also conclude the following asymptotic expression.

Corollary 3.8. *The asymptotic proportion between the number of symmetric pyramids and the number of humps is*

$$\lim_{n\to\infty} \frac{s_n}{hm_n} = \frac{1}{2}.$$

From Theorem 3.1, setting $t = 0$, we obtain the distribution of symmetric pyramids (symmetric peaks) over the Dyck lattice paths. A *Dyck path* is a Motzkin path without horizontal steps. The number of Dyck paths of semi-length n is given by the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

The generating function of the Dyck paths with respect to the number of symmetric peaks and semi-length n is given by $M_{\mathrm{sp}}(q, 0, 0, x^{1/2})$. The explicit expression is

$$\frac{(1-x)\left(1+x^2-qx(1+x)-\sqrt{t(x,q)}\right)}{2x(1-qx)^2},$$

where $t(x,q) = (1-x)(1-(3+2q)x-(1-4q-q^2)x^2-(1-q)^2x^3)$, see [4, Theorem 2.1]. Moreover, the generating function of the total number of symmetric peaks over all Dyck paths of semi-length n is (see [8, Theorem 2.3])

$$\frac{\partial}{\partial q} M_{\mathrm{sp}}(q,0,0,x)\mid_{q=1} = \frac{-1+5x+(1-x)\sqrt{1-4x}}{2(1-x)\sqrt{1-4x}}.$$

4 Symmetric triangular pyramids

In this section we give asymptotic approximations for the proportions between the different aspects that we have been studying in this paper. Thus, we give the proportion between the number of symmetric triangular pyramids and these quantities: the number of pyramids, the number of humps, and the number of sharp peaks. We also discuss the combinatorial behavior of triangular pyramids —symmetric pyramids, symmetric weight, and symmetric height.

We use g_n to denote the number of symmetric triangular pyramids in $\mathcal{M}_n$.

Theorem 4.1. *The generating function for Motzkin paths with respect to the number of symmetric triangular pyramids is*

$$G(x) := \frac{\partial}{\partial q} M_{sp}(q,1,1,x)\mid_{q=1}.$$

More precisely, $G(x)$ is given by

$$\frac{1-6x+9x^2+4x^3-13x^4+2x^5+3x^6-(1-4x+x^2)(1-x)^2(1+x)\sqrt{p(x)}}{2(1-x)^3(1+x)^2(1-3x)},$$

where $p(x) = 1-2x-3x^2$.

These are the first 11 values of the sequence g_n for $n = 1, 2, \ldots, 11$:

$$0,\quad 1,\quad 2,\quad 6,\quad 14,\quad 37,\quad 96,\quad 259,\quad 706,\quad 1955,\quad 5464.$$

Theorem 4.2. *The sequence g_n has this asymptotic approximation*

$$g_n \sim \frac{3^n}{4}\sqrt{\frac{3}{\pi n}}.$$

Let sp_n be the number of sharp humps in $\mathcal{M}_n$. Brennan and Mavhung [2] proved that $sp_n \sim \frac{3^n}{6}\sqrt{\frac{3}{\pi n}}$.

Corollary 4.3. *If s_n, g_n, hm_n, and sp_n represent the number of symmetric pyramids, triangular pyramids, humps, and sharp humps of $\mathcal{M}_n$, respectively, then these hold:*

$$\lim_{n\to\infty} \frac{g_n}{s_n} = \frac{2}{3}; \qquad \lim_{n\to\infty} \frac{g_n}{hm_n} = \frac{1}{3}; \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{g_n}{sp_n} = \frac{1}{2}.$$

We recall that m_i is the number of Motzkin paths of length i. This theorem gives a recursive relation for the sequence g_n.

Theorem 4.4. *If g_k^* is as in* Lemma 3.4*, then the sequence g_n satisfies $g_1 = 0$, $g_2 = 1$, and for $n \geq 3$*

$$g_n = g_{n-1} + g_{n-2} + m_{n-2} + \sum_{k=1}^{n-1} \Big((g_k - 2g_k^*) m_{n-k-2} + g_{n-k-2} m_k \Big).$$

Proof. This proof is similar to the proof of Theorem 3.4. Replace s_k by g_k and instead of s_k^* use Lemma 3.4 with g_k^*. □

4.1 Weight and height of the symmetric triangular pyramids

We use w_n to denote the total weight of symmetric triangular pyramids in $\mathcal{M}_n$. In Proposition 4.5 we give a recursive relation to find the values of w_n. Its proof is based on a similar argument as in Theorem 4.4. So, we omit it.

Proposition 4.5. *The sequence w_n satisfies that $w_1 = 0$, $w_2 = 1$, and for $n \geq 3$*

$$\begin{aligned} w_n &= w_{n-1} + w_{n-2} + m_{n-2} \\ &\quad + \sum_{k=1}^{n-2} \Bigg(\Big(w_k + (k+1 \bmod 2) - 2 \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} i m_{k-2i} \Big) m_{n-k-2} + m_k w_{n-k-2} \Bigg). \end{aligned}$$

These are the first 11 values of the sequence w_n, for $n = 1, 2, \ldots, 11$:

$$0, \quad 1, \quad 2, \quad 7, \quad 16, \quad 44, \quad 112, \quad 303, \quad 818, \quad 2258, \quad 6282.$$

We use h_n to denote the symmetric height of triangular pyramids in $\mathcal{M}_n$.

Proposition 4.6. *Let g_k be number of symmetric triangular pyramids in $\mathcal{M}_k$. The sequence h_n satisfies $h_1 = 0$, $h_2 = 1$, $h_3 = 2$, and for $n \geq 4$*

$$\begin{aligned} h_n &= h_{n-1} + h_{n-2} + m_{n-2} \\ &\quad + \sum_{k=1}^{n-2} \Bigg(\Big(h_k + g_k - 2 \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} (i+1) m_{k-2i} \Big) m_{n-k-2} + m_k h_{n-k-2} \Bigg). \end{aligned}$$

These are the first 11 values of the sequence h_n, for $n = 1, 2, \ldots, 11$:

$$0, \quad 1, \quad 2, \quad 7, \quad 16, \quad 45, \quad 118, \quad 331, \quad 930, \quad 2673, \quad 7744.$$

5 Symmetric truncated pyramids

In this section we give asymptotic approximations for the proportions between the different aspects that we have been studying in this paper. Thus, we give the proportion between the number of symmetric truncated pyramids and these quantities: the number of pyramids and the number of humps. We also discuss the combinatorial behavior of truncated pyramids —symmetric pyramids, symmetric weight, and symmetric height.

We use t_n to denote the number of symmetric truncated pyramids in the family of Motzkin paths of length n.

Theorem 5.1. *The generating function for Motzkin paths with respect to the number of symmetric truncated pyramids is*

$$T(x) := \frac{\partial}{\partial p} M_{sp}(1, p, 1, x) \mid_{p=1} .$$

In particular, if $G(x)$ denotes the generating function of the symmetric triangular pyramids, see Theorem 4.1*, then*

$$T(x) = \frac{x}{1-x} G(x).$$

These are the first 11 values of the sequence t_n for $n = 1, 2, \ldots, 11$:

$$0, \quad 0, \quad 1, \quad 3, \quad 9, \quad 23, \quad 60, \quad 156, \quad 415, \quad 1121, \quad 3076.$$

From the equality given in Theorem 5.1 we have the relation $t_n = \sum_{i=0}^{n-1} g_i$.

Theorem 5.2. *The sequence t_n has this asymptotic approximation*

$$t_n \sim \frac{3^n}{8} \sqrt{\frac{3}{\pi n}}.$$

Corollary 5.3. *If s_n, t_n, and hm_n represent the number of symmetric pyramids, symmetric truncated pyramids, and humps of $\mathcal{M}_n$, respectively, then these hold:*

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \frac{1}{3} \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{t_n}{hm_n} = \frac{1}{6}.$$

In Theorem 5.4 we give a recursive relation to calculate the values of the sequence t_n. Its proof uses the same technique used in the proof of Theorem 4.4.

Theorem 5.4. *If t_k^* is as in* Lemma 3.4*, then the sequence t_n satisfies $t_0 = t_1 = t_2 = 0$, $t_3 = 1$, and for $n \geq 4$*

$$t_n = 2t_{n-1} - t_{n-4} + m_{n-3} + \sum_{k=2}^{n-2} ((t_k - 2t_k^*)(m_{n-k-2} - m_{n-k-3}) + m_k(t_{n-k-2} - t_{n-k-3})).$$

We use v_n to denote the total weight of symmetric truncated pyramids in $\mathcal{M}_n$. We define $\mathrm{Ob}(i, j) := \lfloor \frac{i-1}{2} \rfloor \left(\lfloor \frac{i-1}{2} \rfloor + j \right)$. The proofs of the following two results are similar to the proofs of Theorems 4.4 and 5.4. Therefore, we omit them.

Proposition 5.5. *The sequence* v_n *satisfies* $v_1 = v_2 = 0$*,* $v_3 = 1$*, and for* $n \geq 4$

$$v_n = v_{n-3} + v_{n-2} + v_{n-1} + \sum_{k=0}^{n-3} m_k + \sum_{k=2}^{n-2} \left(\left(v_k + \left\lfloor \frac{k-1}{2} \right\rfloor - \sum_{i=2}^{k-1} \mathrm{Ob}(i,1) m_{k-i} \right) m_{n-k-2} + v_{n-k-2} m_k \right).$$

These are the first 11 values of the sequence w_n for $n = 1, 2, \ldots, 11$:

$$0, \quad 1, \quad 3, \quad 10, \quad 26, \quad 70, \quad 182, \quad 485, \quad 1303, \quad 3561, \quad 9843.$$

The *symmetric truncated height* in $\mathcal{M}_n$ is the sum of the heights of all symmetric truncated pyramids in $\mathcal{M}_n$.

Proposition 5.6. *Let* k_n *be the symmetric truncated height in* $\mathcal{M}_n$ *and let* t_i *be the number of symmetric truncated pyramids in* $\mathcal{M}_i$*. Then the sequence* k_n *satisfies* $k_1 = k_2 = 0$*,* $k_3 = 1$*, and for* $n \geq 4$

$$k_n = k_{n-1} + k_{n-2} + k_{n-3} + \sum_{j=0}^{n-3} m_j + \sum_{j=2}^{n-2} \left(\left(k_j + t_j - \sum_{i=2}^{j-1} \mathrm{Ob}(i,3) m_{j-i} \right) m_{n-j-2} + m_j k_{n-j-2} \right).$$

These are the first 11 values of the sequence k_n for $n = 1, 2, \ldots, 11$:

$$0, \quad 0, \quad 1, \quad 3, \quad 10, \quad 26, \quad 71, \quad 189, \quad 520, \quad 1450, \quad 4123.$$

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The core of a vertex-transitive complementary prism

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Abstract

The complementary prism $\Gamma\bar{\Gamma}$ is obtained from the union of a graph Γ and its complement $\bar{\Gamma}$ where each pair of identical vertices in Γ and $\bar{\Gamma}$ is joined by an edge. It generalizes the Petersen graph, which is the complementary prism of the pentagon. The core of a vertex-transitive complementary prism is studied. In particular, it is shown that a vertex-transitive complementary prism $\Gamma\bar{\Gamma}$ is a core, i.e. all its endomorphisms are automorphisms, whenever Γ is a core or its core is a complete graph.

Keywords: Graph homomorphism, core, complementary prism, self-complementary graph, vertex-transitive graph.

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1 Introduction

In the study of graph homomorphism a basic object is a *core* (a.k.a. *unretractive graph*), which is a graph such that all its endomorphisms are automorphisms. A subgraph Γ' of a graph Γ is its core, if there exists some graph homomorphism $\varphi\colon \Gamma \to \Gamma'$ and Γ' is a core. Equivalently, Γ' is the minimal retract of Γ (cf. [17]). Despite that each graph has its core, which is unique up to isomorphism, it can be often very difficult to determine if a given graph is a core or not (cf. [6, 16, 30]). From this point of view, graphs that have either high degree of symmetry (i.e. 'large' automorphism group) or some 'nice' combinatorial properties are the most interesting. Many of such classes of graphs are *core-complete*, which

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means that they are either cores or their cores are complete graphs. Among them we can find all non-edge-transitive graphs [6, Corollary 2.2], connected regular graphs with the automorphism group that acts transitively on unordered pairs of vertices at distance two [16, Theorem 4.1], and all primitive strongly regular graphs [36, Corollary 3.6]. Given a core-complete graph, it can be extremely complicated to decide if the graph is a core or its core is complete. For some graphs, this task is equivalent to some of the longstanding open problems in finite geometry (see [6, 30]). A well known core is the Petersen graph, which has both 'large' automorphism group and 'nice' combinatorial properties. Given a family of graphs that (naturally) generalize the Petersen graph it is interesting to study if its members are cores or not. Kneser graphs $K(n, r)$, with $2r < n$, are all cores [15, Theorem 7.9.1]. The graph $HGL_n(\mathbb{F}_4)$, whose vertex set is formed by all $n \times n$ invertible hermitian matrices over the field with four elements and with the edge set $\{\{A, B\} : \text{rank}(A - B) = 1\}$, is a core whenever $n \geq 2$ [29]. The core of a generalized Petersen graph $G(n, k)$ was studied very recently in [13]. The complementary prism $\Gamma\bar{\Gamma}$, whose definition in full details is given in Section 2, is another generalization of the Petersen graph, which is obtained if Γ is the 5-cycle C_5[1]. Graph $\Gamma\bar{\Gamma}$ was introduced in [18] and is the main mater of research in several papers (see for example [1, 3, 7, 10, 19, 26]). Recall that C_5 is strongly regular vertex-transitive self-complementary graph. The core of $\Gamma\bar{\Gamma}$ was recently studied by the author [28] (see also the arXiv version [31]). In particular, it was shown that $\Gamma\bar{\Gamma}$ is a core whenever Γ is strongly regular and self-complementary. In this paper we build on a result from [28] and investigate vertex-transitive self-complementary graphs Γ. These are precisely the graphs that provide vertex-transitive complementary prisms [31, Corollary 3.8]. For such graphs we prove that $\Gamma\bar{\Gamma}$ is a core whenever Γ is core-complete (see Theorem 3.3), and state an open problem, which asks if there exists a vertex-transitive self-complementary graph Γ such that $\Gamma\bar{\Gamma}$ is not a core (Problem 3.6). The main results are presented in Section 3. In Section 2 we recall some tools and definitions that we need in what follows.

2 Preliminaries

All graphs in this paper are finite and simple. The vertex set and the edge set of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. A subset of pairwise adjacent vertices in $V(\Gamma)$ is a *clique*, while a set of pairwise nonadjacent vertices in $V(\Gamma)$ is an *independent set*. The *clique number* $\omega(\Gamma)$ and the *independence number* $\alpha(\Gamma)$ are the orders of the largest clique and the largest independent set in Γ, respectively. In particular, $\alpha(\Gamma) = \omega(\bar{\Gamma})$, where $\bar{\Gamma}$ is the complement of the graph Γ. The chromatic number of a graph is denoted by $\chi(\Gamma)$. It is well known that $\chi(\Gamma) \geq \omega(\Gamma)$ and $\chi(\Gamma) \geq \frac{n}{\alpha(\Gamma)}$, where $n = |V(\Gamma)|$ (cf. [5]). A *graph homomorphism* between graphs Γ_1, Γ_2 is a map $\varphi\colon V(\Gamma_1) \to V(\Gamma_2)$ such that $\{\varphi(u), \varphi(v)\} \in E(\Gamma_2)$ whenever $\{u, v\} \in E(\Gamma_1)$. If in addition φ is bijective and $\{u, v\} \in E(\Gamma_1) \iff \{\varphi(u), \varphi(v)\} \in E(\Gamma_2)$, then φ is a *graph isomorphism* and graphs Γ_1, Γ_2 are *isomorphic*, which we denote by $\Gamma_1 \cong \Gamma_2$. If $\Gamma_1 = \Gamma_2$, then a graph homomorphism/graph isomorphism is a graph *endomorphism/automorphism*, respectively. A graph Γ is *self-complementary* if there exists a graph isomorphism $\sigma\colon \Gamma \to \bar{\Gamma}$, which is referred to as *antimorphism* or a *complementing permutation*. Observe that in this case σ is also an antimorphism as a map $\bar{\Gamma} \to \Gamma$. A graph Γ is regular if each vertex has the same number of neighbors. If this number equals k, then we say that it is *k-regular*.

[1]Graphs $K(5, 2)$, $HGL_2(\mathbb{F}_4)$, $G(5, 2)$, $C_5\overline{C_5}$ are all isomorphic to the Petersen graph.

If for each pair of vertices $u, v \in V(\Gamma)$ there exists an automorphism φ of Γ such that $u = \varphi(v)$, then Γ is a *vertex-transitive* graph. Clearly, each vertex-transitive graph is regular. Lemma 2.1 can be found in [14, Corollaries 2.1.2, 2.1.3], where it is stated in a more general settings.

Lemma 2.1. *If a graph Γ is vertex-transitive, then $\alpha(\Gamma)\omega(\Gamma) \leq |V(\Gamma)|$. If the equality holds, then $|C \cap I| = 1$ for each clique C and each independent set I that provide the equality.*

If a graph on n vertices is $(\frac{n-1}{2})$-regular, then n must be odd. Moreover, it follows from the hand-shaking lemma that $n = 4m + 1$ for some integer $m \geq 0$. In particular, this is true for all regular self-complementary graphs. By a result of Sachs [37] or Ringel [35], each cycle in an antimorphism of a self-complementary graph has the length divisible by four, except for one cycle of length one, in the case the order of the graph equals 1 modulo 4 (a proof in English can be found in [11, page 12]). Lemma 2.2 is a special case of this fact, and is crucial in the proof of Theorem 3.3.

Lemma 2.2. *If σ is an antimorphism of a regular self-complementary graph Γ, then there exists a unique vertex $v \in V(\Gamma)$ such that $\sigma(v) = v$.*

A graph Γ is a *core* if each its endomorphism is an automorphism. Given a graph Γ, we use $\text{core}(\Gamma)$ to denote any subgraph of Γ that is a core and such that there exists some graph homomorphism $\varphi\colon \Gamma \to \text{core}(\Gamma)$. Graph $\text{core}(\Gamma)$ is referred to as the *core of* Γ. It is always an induced subgraph and unique up to isomorphism [15, Lemma 6.2.2]. Clearly, a graph Γ is a core if and only if $\Gamma = \text{core}(\Gamma)$. On the other hand, $\text{core}(\Gamma)$ is a complete graph if and only if $\chi(\Gamma) = \omega(\Gamma)$. We remark that there always exists a *retraction* $\psi\colon \Gamma \to \text{core}(\Gamma)$, i.e. a graph homomorphism that fixes each vertex in $\text{core}(\Gamma)$. In fact, if $\varphi\colon \Gamma \to \text{core}(\Gamma)$ is any graph homomorphism, then the restriction $\varphi|_{V(\text{core}(\Gamma))}$ is invertible and the composition $(\varphi|_{V(\text{core}(\Gamma))})^{-1} \circ \varphi$ is the required retraction. Lemma 2.3 is proved in [39, Theorem 3.2], where it is stated in an old terminology. Its proof can be found also in [15, 17].

Lemma 2.3. *If graph Γ is vertex-transitive, then $\text{core}(\Gamma)$ is vertex-transitive.*

Let Γ be a graph with the vertex set $V(\Gamma) = \{v_1, \ldots, v_n\}$. The *complementary prism* of Γ is the graph $\Gamma\bar{\Gamma}$, which is obtained from the disjoint union of Γ and its complement $\bar{\Gamma}$, by adding an edge between each vertex in Γ and its copy in $\bar{\Gamma}$. In this paper we use the following notation. The vertex set of the complementary prism of graph Γ is the set $V(\Gamma\bar{\Gamma}) = W_1 \cup W_2$, where

$$W_1 = W_1(\Gamma\bar{\Gamma}) = \{(v_1, 1), \ldots, (v_n, 1)\} \text{ and } W_2 = W_2(\Gamma\bar{\Gamma}) = \{(v_1, 2), \ldots, (v_n, 2)\}.$$

The edge set $E(\Gamma\bar{\Gamma})$ is the union of the sets

$$\big\{\{(u,1),(v,1)\} : \{u,v\} \in E(\Gamma)\big\},$$
$$\big\{\{(u,2),(v,2)\} : \{u,v\} \in E(\bar{\Gamma})\big\},$$
$$\big\{\{(u,1),(u,2)\} : u \in V(\Gamma)\big\}.$$

It follows from the definition that a complementary prism $\Gamma\bar{\Gamma}$ is regular if and only if Γ is $\left(\frac{n-1}{2}\right)$-regular (see also [7, Theorem 3.6]). The core of a complementary prism for general graph Γ was recently studied in [28] (see also the arXiv version [31]). For regular case, the following result was proved.

Lemma 2.4 ([28, Corollary 3.4]). *Let* Γ *be any graph on* n *vertices that is* $\left(\frac{n-1}{2}\right)$*-regular. Then one of the following three possibilities is true.*

(i) *Graph* $\Gamma\bar{\Gamma}$ *is a core.*

(ii) *All vertices of core*$(\Gamma\bar{\Gamma})$ *are contained in* W_1*, in which case*

$$core(\Gamma\bar{\Gamma}) \cong core(\Gamma).$$

(iii) *All vertices of core*$(\Gamma\bar{\Gamma})$ *are contained in* W_2*, in which case*

$$core(\Gamma\bar{\Gamma}) \cong core(\bar{\Gamma}).$$

The same conclusion can be obtained also if the core of $\Gamma\bar{\Gamma}$ is regular and we exclude some small graphs. Below, K_2 is a complete graph on two vertices, and P_3 is a path on three vertices.

Lemma 2.5 ([28, Corollary 3.6]). *Let* Γ *be any graph, which is not isomorphic to* K_2, $\overline{K_2}$, P_3*, or* $\overline{P_3}$*. If core*$(\Gamma\bar{\Gamma})$ *is regular, then one of the three possibilities in* Lemma 2.4 *is true.*

Clearly, Lemma 2.4 is valid for each regular self-complementary graph Γ. The study of such graphs and their vertex-transitive counterparts has origins in the papers [21, 37, 40], which influenced a lot of research related to vertex-transitive self-complementary graphs (see for example [2, 4, 8, 9, 12, 20, 22, 23, 24, 25, 27, 33, 34, 38, 41] and the references therein). In this paper the aim is to to study the core of a complementary prism $\Gamma\bar{\Gamma}$, where Γ is vertex-transitive and self-complementary graph. The following observation is obtained for free, with a double proof.

Corollary 2.6. *If graph* Γ *is vertex-transitive and self-complementary graph, then one of the three possibilities in* Lemma 2.4 *is true.*

Proof. The claim follows directly from Lemma 2.4. The same claim is deduced also if we combine Lemmas 2.5 and 2.3. □

In [28] some examples of regular self-complementary graphs Γ are provided such that the statement (ii) or (iii) in Lemma 2.4 is true. In this paper we show that this is not possible for a large class of vertex-transitive self-complementary graphs. It should be mentioned that it was recently proved that a complementary prism $\Gamma\bar{\Gamma}$ is vertex-transitive if and only if Γ is vertex-transitive and self-complementary [31, Corollary 3.8]. Despite our proofs do not rely on this result, it means that this paper studies the core of vertex-transitive complementary prisms.

3 Main results

The main result of this paper is Theorem 3.3. Propositions 3.1 and 3.2 are the stepping stones towards its proof.

Proposition 3.1. *Let* Γ *be a regular self-complementary graph. If* Γ *is a core, then* $\Gamma\bar{\Gamma}$ *is a core.*

Proof. Let $\mathrm{core}(\Gamma\bar{\Gamma})$ be any core of Γ and let $n = |V(\Gamma)|$. Since Γ is $\left(\frac{n-1}{2}\right)$-regular, one of the statements (i), (ii), (iii) in Lemma 2.4 is true. Suppose that (iii) is correct, that is,

$$V(\mathrm{core}(\Gamma\bar{\Gamma})) \subseteq W_2 \tag{3.1}$$

and

$$\mathrm{core}(\Gamma\bar{\Gamma}) \cong \mathrm{core}(\bar{\Gamma}). \tag{3.2}$$

Since $\bar{\Gamma}$ is a core, (3.2) implies that $\mathrm{core}(\Gamma\bar{\Gamma}) \cong \bar{\Gamma}$. Hence, (3.1) yields

$$V(\mathrm{core}(\Gamma\bar{\Gamma})) = W_2. \tag{3.3}$$

Let $\psi_1(v) = (v, 1)$, for $v \in V(\Gamma)$, be the canonical isomorphism between Γ and the subgraph of $\Gamma\bar{\Gamma}$, which is induced by the set W_1. Similarly, let $\psi_2(v) = (v, 2)$, for $v \in V(\Gamma)$, be the canonical isomorphism between $\bar{\Gamma}$ and the subgraph induced by W_2. If Ψ is any retraction from $\Gamma\bar{\Gamma}$ onto $\mathrm{core}(\Gamma\bar{\Gamma})$, and σ is any antimorphism between $\bar{\Gamma}$ and Γ, then the composition $\sigma \circ \psi_2^{-1} \circ (\Psi|_{W_1}) \circ \psi_1$ is an endomorphism of Γ. Since Γ is a core, the restriction $\Psi|_{W_1}$ is an isomorphism between the subgraphs in $\Gamma\bar{\Gamma}$ that are induced by the sets W_1 and W_2, respectively. Consequently $\psi_2^{-1} \circ (\Psi|_{W_1}) \circ \psi_1 \colon \Gamma \to \bar{\Gamma}$ is an antimorphism. By Lemma 2.2, there exists $v \in V(\Gamma)$ such that $\left(\psi_2^{-1}\circ(\Psi|_{W_1})\circ\psi_1\right)(v) = v$. Consequently, $\Psi(v, 1) = (\Psi|_{W_1})(v, 1) = (v, 2)$. Since Ψ is a retraction, (3.3) implies that $\Psi(v, 2) = (v, 2)$. Since $\{(v, 1), (v, 2)\}$ is an edge in $\Gamma\bar{\Gamma}$, we have a contradiction.

In the same way we see that (ii) in Lemma 2.4 is not possible, which means that $\Gamma\bar{\Gamma}$ is a core. □

Proposition 3.2. *If Γ is a vertex-transitive self-complementary graph on $n > 1$ vertices, then core$(\Gamma\bar{\Gamma})$ is not a complete graph.*

Proof. We need to prove that $\chi(\Gamma\bar{\Gamma}) > \omega(\Gamma\bar{\Gamma})$. Since $n > 1$ and Γ is both self-complementary and vertex-transitive, Lemma 2.1 implies that

$$\omega(\Gamma\bar{\Gamma}) = \max\{\alpha(\Gamma), \omega(\Gamma)\} = \omega(\Gamma) \leq \sqrt{n}.$$

Let I be any independent set in $\Gamma\bar{\Gamma}$. Then I is a disjoint union of some sets $I_1 \subseteq W_1$ and $I_2 \subseteq W_2$. If we write $I_i = \{(u, i) : u \in J_i\}$ for $i \in \{1, 2\}$, where $J_1, J_2 \subseteq V(\Gamma)$, then

$$J_1 \cap J_2 = \emptyset. \tag{3.4}$$

Since J_1 and J_2 are an independent set and a clique in Γ, respectively, we have $|J_1| \leq \alpha(\Gamma) = \omega(\Gamma) \leq \sqrt{n}$ and $|J_2| \leq \omega(\Gamma) = \sqrt{n}$, while Lemma 2.1 and (3.4) imply that $|J_1| \cdot |J_2| < n$. Hence, $|I| = |J_1| + |J_2| < 2\sqrt{n}$, and therefore

$$\chi(\Gamma\bar{\Gamma}) \geq \frac{|V(\Gamma\bar{\Gamma})|}{\alpha(\Gamma\bar{\Gamma})} > \frac{2n}{2\sqrt{n}} = \sqrt{n} \geq \omega(\Gamma\bar{\Gamma}).$$

□

Theorem 3.3. *Let Γ be a vertex-transitive self-complementary graph. If Γ is either a core or its core is a complete graph, then $\Gamma\bar{\Gamma}$ is a core.*

Proof. If Γ is a core, then the claim follows from Proposition 3.1. Hence, we may assume that Γ has a complete core and more than one vertex. By Corollary 2.6, one of the statements (i), (ii), (iii) in Lemma 2.4 is true for $\mathrm{core}(\Gamma\bar{\Gamma})$. If (ii) or (iii) is correct, then the self-complementarity implies that $\mathrm{core}(\Gamma\bar{\Gamma})$ is complete, which contradicts Proposition 3.2. □

Remark 3.4. The claims in Proposition 3.2 and Theorem 3.3 are not true for some regular self-complementary graphs. In fact, there exists a regular self-complementary graph Γ with a complete core such that $\mathrm{core}(\Gamma\bar{\Gamma})$ is complete (see [28, Example 3.5] or [31, Example 5.5]).

Recall that a k-regular graph on n vertices is *strongly regular* with parameters (n, k, λ, μ) if each pair of adjacent vertices has λ common neighbors and each pair of distinct non-adjacent vertices has μ common neighbors. Theorem 3.5 was very recently proved in [28] (see also the arXiv version [31, Theorem 5.7]). The proof relied on application of Lemma 2.4 together with several properties of the Lovász theta function and the graph spectrum. Here we provide a sketch of an alternative proof that essentially copies the proofs in this section and applies a remarkable result that Roberson recently proved [36].

Theorem 3.5 ([28])**.** *If Γ is a strongly regular self-complementary graph, then $\Gamma\bar{\Gamma}$ is a core.*

Sketch of a proof. To see that $\Gamma\bar{\Gamma}$ does not have a complete core, we copy the proof of Proposition 3.2, where we replace the application of Lemma 2.1 by its analog that can be found in [14, Corolarry 3.8.6 and Theorem 3.8.4] and is valid for all strongly regular graphs. From [36, Corollary 3.6] it follows that Γ is either a core or its core is complete. Then we just copy the proof of Theorem 3.3, where we rely directly on Lemma 2.4 instead on Corollary 2.6. □

Recall from the introduction that many 'nice' graphs are either cores or their cores are complete. Hence it is expected that many vertex-transitive self-complementary graphs fulfil the assumptions in Theorem 3.3. Consequently, we state the following open problem.

Problem 3.6. Does there exist a vertex-transitive self-complementary graph Γ such that $\Gamma\bar{\Gamma}$ is not a core?

Note that edge-transitive self-complementary graphs are also arc-transitive (see [11]). Since self-complementary graphs are always connected (cf. [11]), it follows that each edge-transitive self-complementary graph is also vertex-transitive. However, such graphs are always strongly regular (cf. [11]) and therefore their complementary prisms are cores by Theorem 3.5. Despite the orders of vertex-transitive self-complementary graphs were fully determined in [27], there is a major gap between the understanding of vertex-transitive self-complementary graphs and the understanding of edge-transitive self-complementary graphs. In fact, the later were completely characterized in [33]. Moreover, the first non-Cayley vertex-transitive self-complementary graph was constructed only in 2001 [24], and the construction is highly nontrivial. We believe that all these facts indicate that Problem 3.6 may be challenging.

In a distinct paper [32], we consider the only families of vertex-transitive self-complementary graphs the author is aware of, which are neither cores nor their cores are complete graphs (i.e. they do not satisfy the assumption in Theorem 3.3). Unfortunately they do not solve Problem 3.6.

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Normal Cayley digraphs of dihedral groups with the CI-property*

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Abstract

A Cayley (di)graph $\mathrm{Cay}(G,S)$ of a group G with respect to a set $S \subseteq G$ is said to be normal if the image of G under its right regular representation is normal in the automorphism group of $\mathrm{Cay}(G,S)$, and is called a CI-(di)graph if for every $T \subseteq G$ with $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$, there is $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$. A finite group G is called a DCI-group or an NDCI-group if all Cayley digraphs or all normal Cayley digraphs of G are CI-digraphs, respectively, and is called a CI-group or an NCI-group if all Cayley graphs or all normal Cayley graphs of G are CI-graphs, respectively.

Motivated by a conjecture proposed by Ádám in 1967, CI-groups and DCI-groups have been actively studied during the last fifty years by many researchers in algebraic graph theory. It took about thirty years to obtain the classification of cyclic CI-groups and DCI-groups, and recently, the first two authors, among others, classified cyclic NCI-groups and NDCI-groups. Even though there are many partial results on dihedral CI-groups and DCI-groups, their classification is still elusive. In this paper, we prove that a dihedral group of order $2n$ is an NCI-group or an NDCI-group if and only if $n = 2, 4$ or n is odd. As a direct consequence, we have that if a dihedral group D_{2n} of order $2n$ is a DCI-group then $n = 2$ or n is odd-square-free, and that if D_{2n} is a CI-group then $n = 2, 9$ or n is odd-square-free, throwing some new light on classification of dihedral CI-groups and DCI-groups. As a byproduct, we construct a non-CI Cayley graph of the dihedral group D_8, but Holt and Royle in 2020 claimed that D_8 is a CI-group by an algorithm there.

Keywords: Dihedral group, CI-group, DCI-group, NCI-group, NDCI-group.

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1 Introduction

Graphs and digraphs considered in this paper are finite and simple, and groups are finite. For a (di)graph Γ, we use $V(\Gamma)$, $E(\Gamma)$, $\mathrm{Arc}(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote the vertex set, edge set, arc set, and automorphism group of Γ, respectively, where an arc means a directed edge in a digraph and an ordered pair of adjacent vertices in a graph.

Let G be a group and S be a subset of G with $1 \notin S$. A digraph with vertex set G and arc set $\{(g, sg) \mid g \in G, s \in S\}$, denoted by $\mathrm{Cay}(G, S)$, is called the *Cayley digraph* of G with respect to S. If S is inverse-closed, that is, $S = S^{-1} := \{x^{-1} \mid x \in S\}$, then for two adjacent vertices u and v in $\mathrm{Cay}(G, S)$, both (u, v) and (v, u) are arcs, and in this case, we view $\mathrm{Cay}(G, S)$ as a graph by identifying the two arcs with one edge $\{u, v\}$.

Two Cayley (di)graphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are called *Cayley isomorphic* if there is $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$. Cayley isomorphic Cayley (di)graphs are isomorphic, but the converse is not true. A subset S of G with $1 \notin S$ is said to be a *CI-subset* if $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$, for some $T \subseteq G$ with $1 \notin T$, implies that $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are Cayley isomorphic. In this case, $\mathrm{Cay}(G, S)$ is called a *CI-digraph*, or a *CI-graph* when $S = S^{-1}$. A group G is called a *DCI-group* if all Cayley digraphs of G are CI-digraphs, and a *CI-group* if all Cayley graphs of G are CI-graphs.

DCI-groups and CI-groups have been widely investigated over last fifty years, and they have been reduced to some special restricted groups in [12, 28]. However, it is still very difficult to determine whether a particular group is a DCI-group or a CI-group; see [2, 5, 11, 20, 26, 31, 32, 37, 38, 39, 40] for examples. In fact, it is even an open problem to classify dihedral DCI-groups or CI-groups.

Ádám [1] conjectured that every finite cyclic group is a CI-group. Due to contributions of many researchers like Elspas and Turner [16], Djoković [10], Turner [41], Babai [4], Alspach and Parsons [3], Godsil [22] and Pálfy [33], the cyclic DCI-groups and CI-groups were classified finally by Muzychuk [29, 30]. It follows that a cyclic group of order n is a DCI-group if and only if $n = mk$ where $m = 1, 2, 4$ and k is odd-square-free, and is a CI-group if and only if either $n = 8, 9, 18$, or $n = mk$ where $m = 1, 2, 4$ and k is odd-square-free. The generalised dihedral group $\mathrm{Dih}(A)$ over an abelian group A is the group $\langle A, b \mid b^2 = 1, a^b = a^{-1}, \forall a \in A\rangle$, and in particular, if A is a cyclic group of order n, $\mathrm{Dih}(A)$ is the dihedral group D_{2n} of order $2n$. There are also some partial results on dihedral DCI-groups or CI-groups. Babai [4] proved that the dihedral group D_{2p} for a prime p is a CI-group. Conder and Li [8] proved that D_{18} is a CI-group. In [12], it was further proved that D_{6p} (p a prime) is a DCI-group if and only if $p \geq 5$, and is a CI-group if and only if $p \geq 3$. Recently, Dobson et al. [13] proved that if R is a generalised dihedral CI-group, then for every odd prime p, the Sylow p-subgroup of R has order p, or 9, which reduces dihedral DCI-groups to D_{2n} with $n = 2^m k$ $(m = 0, 1)$ and k odd-square-free and dihedral CI-groups to D_{18} or D_{2n} with $n = 2^m k$ $(m = 0, 1)$ and k odd-square-free.

Let $\mathrm{Cay}(G, S)$ be a Cayley (di)graph of G with respect to a set $S \subseteq G$. For a given $g \in G$, the right multiplication $R(g)\colon x \mapsto xg$, $x \in G$, is an automorphism of $\mathrm{Cay}(G, S)$, and $R(G) := \{R(g) \mid g \in G\}$ is a regular group of automorphisms of $\mathrm{Cay}(G, S)$, that is, the image of G under its right regular representation. A Cayley (di)graph $\mathrm{Cay}(G, S)$ is said to be normal if $R(G)$ is a normal subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Normality of Cayley (di)graphs is very important because the automorphism groups of normal Cayley (di)graph are actually known; see Godsil [21] or Proposition 2.2. Furthermore, the study of normality of Cayley (di)graphs is currently a hot topic in algebraic graph theory, and we refer to [14, 15, 17, 18, 19, 34, 45, 46] for examples.

A group G is called an *NDCI-group* or an *NCI-group* if all normal Cayley digraphs or graphs of G are CI-digraphs or CI-graphs, respectively. Obviously, a DCI-group is an NDCI-group and a CI-group is an NCI-group. Li [27] constructed some normal Cayley digraphs of cyclic groups of order 2-powers that are not CI-graphs, and proposed the following problem: characterize normal Cayley digraphs which are not CI-digraphs. Similar to DCI-groups and CI-groups, a natural problem is to classify finite NDCI-groups and NCI-groups. Recently, Xie, Feng, Ryabov and Liu [43] classified cyclic NDCI-groups and NCI-groups, and in this paper, we classify dihedral NDCI-groups and NCI-groups.

Theorem 1.1. *Let $n \geq 2$ be an integer and let D_{2n} be the dihedral group of order $2n$. Then the following statements are equivalent:*

(1) D_{2n} *is an NDCI-group;*

(2) D_{2n} *is an NCI group;*

(3) *Either $n = 2, 4$ or n is odd.*

Li [27, Problem 6.3(3)]) proposed the following problem: are all connected cubic Cayley graphs CI-graphs? To prove Theorem 1.1, an infinite family of connected cubic normal non-CI Cayley graphs are constructed in Lemma 4.1, which gives a negative answer to Li's problem. Classification of NDCI-groups and NCI-groups can be helpful for classification of DCI-groups and CI-groups. In fact, by Theorem 1.1, together with [13], we have the following corollary.

Corollary 1.2. *If a dihedral group D_{2n} of order $2n$ is a* DCI-*group then $n = 2$ or n is odd-square-free, and if D_{2n} is a* CI-*group then $n = 2, 9$ or n is odd-square-free.*

We believe that the converse of Corollary 1.2 is true.

Conjecture 1.3. *A dihedral group D_{2n} of order $2n$ is a* DCI-*group if and only if $n = 2$ or n is odd-square-free, and D_{2n} is a* CI-*group if and only if $n = 2, 9$ or n is odd-square-free.*

Note that Holt and Royle [24, 6.2] claimed that D_8 is a CI-group, and this is not true by Lemma 4.1, where a non-CI Cayley graph of D_8 is constructed and its non-CI-property is also checked by using MAGMA [7].

All the notation and terminologies used in this paper are standard, and for group and graph concepts not defined here, we refer to [6, 9, 35, 36].

2 Preliminaries

In this section, we give some basic concepts and facts that will be needed later.

Let F be a free group of rank r and let G be a group generated by a set X of r elements, say $X = \{x_1, x_2, \ldots, x_r\}$. Then there is a standard free presentation from F to G induced by a bijective mapping from the free generators of F to X. Let G have a defining relation set S, that is, $G = \langle X \mid S \rangle$. Let H be a group that can be generated by a set Y of r elements, say $Y = \{y_1, y_2, \ldots, y_r\}$. The following is the well-known von Dyck's Theorem [35, 2.2.1].

Proposition 2.1. *Assume that all generators y_i of H satisfy every relation in S by replacing x_i with y_i. Then there is an epimorphism from G to H induced by $x_i \mapsto y_i$ for all $1 \leq i \leq r$.*

Let $\mathrm{Cay}(G, S)$ be a Cayley digraph of a group G with respect to S, and let $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$. Then $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. By Godsil [21], the normalizer of $R(G)$ in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is the semiproduct $R(G) \rtimes \mathrm{Aut}(G, S)$, where $R(g)^\alpha = R(g^\alpha)$ for all $g \in G$ and $\alpha \in \mathrm{Aut}(G, S)$, and by [44, Propositions 1.3 and 1.5], we have the following.

Proposition 2.2. *Let* $\mathrm{Cay}(G, S)$ *be a Cayley digraph of a group G with respect to S and let* $A = \mathrm{Aut}(\mathrm{Cay}(G, S))$. *Then* $N_A(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S)$ *and* $\mathrm{Cay}(G, S)$ *is normal if and only if* $A_1 = \mathrm{Aut}(G, S)$.

Babai [4] gave a well-known criterion for a Cayley digraph to be a CI-digraph, that is, a Cayley digraph $\mathrm{Cay}(G, S)$ is a CI-digraph if and only if every regular group of automorphisms of $\mathrm{Cay}(G, S)$ isomorphic to G, is conjugate to $R(G)$ in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Based on this, the following proposition is straightforward (also see [27, Corollary 6.9]).

Proposition 2.3. *Let* $\mathrm{Cay}(G, S)$ *be a normal Cayley digraph of a group G with respect to S. Then* $\mathrm{Cay}(G, S)$ *is a* CI-*digraph if and only if* $\mathrm{Aut}(\mathrm{Cay}(G, S))$ *has a unique regular subgroup isomorphic to G, that is, $R(G)$.*

Let G be a finite group and let $L \subseteq \mathrm{Aut}(G)$. Write $F_G(L) = \{g \mid g^\ell = g$ for every $\ell \in L\}$, the fixed-points of L in G. Clearly, $F_G(L) \leq G$. The following proposition was given in [42, Theorem 2.2], which is about non-normal Cayley digraphs.

Proposition 2.4. *Let* $\mathrm{Cay}(G, S)$ *be a Cayley digraph. Let* $1 \neq L \leq \mathrm{Aut}(G, S)$ *and* $H \trianglelefteq G$ *be such that for every right coset Hg in G, either L fixes Hg pointwise, or L fixes Hg setwise and is transitive on Hg. Suppose that one of the following holds:*

(1) $|G : F_G(L)| > 2$*;*

(2) $|G : F_G(L)| = 2$*, and there is* $g \in G \backslash F_G(L)$ *and* $h \in H$ *such that* $h^g \neq h^{-1}$*;*

(3) $|G : F_G(L)| = 2$*, and there is* $1 \neq \gamma \in \mathrm{Aut}(G, S)$ *such that* $F_G(\langle\gamma\rangle) \neq F_G(L)$ *and* γ *fixes every coset of H in G setwise.*

Then $\mathrm{Cay}(G, S)$ *is non-normal.*

3 Automorphisms and holomorphs of dihedral groups

In this section we collect some details about automorphism groups and holomorphs of dihedral groups, which are crucial for the proof of Theorem 1.1.

Let G be a finite group and let $g \in G$. Denote by $o(g)$ the order of g in G. Let p be a prime and π a set of primes. Denote by G_p a Sylow p-subgroup of G. An element g of G is called a *π-element* if all prime factors of $o(g)$ belong to π, and a *p'-element* if $o(g)$ has no factor p. If G is soluble, denote by G_π a Hall π-subgroup of G, and by $G_{p'}$ a Hall p'-subgroup of G.

Let n be a positive integer. We first make the following convention:

$$n = \prod\nolimits_{i=1}^{m} p_i^{r_i} \geq 2,\ C_n = C_{p_1^{r_1}} \times \cdots \times C_{p_m^{r_m}} = \langle a_1\rangle \times \cdots \times \langle a_m\rangle,\ a = a_1 \ldots a_m, \quad (3.1)$$

where $p_1, \ldots, p_m$ are all distinct prime factors of n, $C_{p_i^{r_i}} = \langle a_i\rangle \cong \mathbb{Z}_{p_i^{r_i}}$ for each $1 \leq i \leq m$, and $C_n = \langle a\rangle \cong \mathbb{Z}_n$. By [36, Theorem 7.3], we have

$$\mathrm{Aut}(C_n) = \mathrm{Aut}(C_{p_1^{r_1}}) \times \mathrm{Aut}(C_{p_2^{r_2}}) \times \cdots \times \mathrm{Aut}(C_{p_m^{r_m}}), \quad (3.2)$$

where $\mathrm{Aut}(C_{p_i^{r_i}})$ is viewed as the subgroup of $\mathrm{Aut}(C_n)$ by identifying $\alpha_i \in \mathrm{Aut}(C_{p_i^{r_i}})$ as the automorphism of C_n induced by $a_i \mapsto a_i^{\alpha_i}$ and $a_j \mapsto a_j$ for all $j \neq i$. Further we set

$$\mathrm{D}_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle = C_n \rtimes \langle b \rangle. \tag{3.3}$$

By Proposition 2.1, for $a^i \in \langle a \rangle$ with $i \in \mathbb{Z}_n$, there is an automorphism of D_{2n}, denoted by θ_{a^i}, which is induced by

$$\theta_{a^i} : a \mapsto a, b \mapsto ba^i. \tag{3.4}$$

Then $o(\theta_{a^i}) = o(a^i)$ and $\langle \theta_a \rangle \cong \mathbb{Z}_n$. Write $\overline{a}_i = a_1 \dots a_{i-1}a_{i+1} \dots a_m$. Then $o(\overline{a}_i) = n/p_i^{r_i}$, and hence $\langle \theta_a \rangle = \langle \theta_{a_i} \rangle \times \langle \theta_{\overline{a}_i} \rangle$. In fact, $\langle \theta_{a_i} \rangle$ and $\langle \theta_{\overline{a}_i} \rangle$ are the unique Sylow p_i-subgroup and the unique Hall p_i'-subgroup of $\langle \theta_a \rangle$, respectively. Clearly,

$$\theta_{a^i}\theta_{a^j} = \theta_{a^{i+j}} \text{ and } \theta_{a^i}^k = \theta_{a^{ik}} \text{ for all } i, j, k \in \mathbb{Z}_n. \tag{3.5}$$

Again by Propostion 2.1, we also view $\mathrm{Aut}(C_n)$ as the subgroup of $\mathrm{Aut}(\mathrm{D}_{2n})$ by identifying $\beta \in \mathrm{Aut}(C_n)$ as the automorphism of $\mathrm{Aut}(\mathrm{D}_{2n})$ induced by $a \mapsto a^\beta$ and $b \mapsto b$. Then for each $1 \leq i \leq m$, we have $\mathrm{Aut}(C_{p_i^{r_i}}) \leq \mathrm{Aut}(\mathrm{D}_{2n})$.

Lemma 3.1. *Let $n \geq 3$. Then $\mathrm{Aut}(\mathrm{D}_{2n})$ has the following properties.*

(1) $\mathrm{Aut}(\mathrm{D}_{2n}) = \langle \theta_a \rangle \rtimes \mathrm{Aut}(C_n)$*, and $\langle \theta_a \rangle$ is the kernel of the natural action of* $\mathrm{Aut}(\mathrm{D}_{2n})$ *on C_n. Furthermore, $\theta_x^\beta = \theta_{x^\beta}$ for all $x \in \langle a \rangle$ and $\beta \in \mathrm{Aut}(C_n)$;*

(2) *For every $1 \leq i \leq m$, $\langle \theta_a \rangle \rtimes \mathrm{Aut}(C_{p_i^{r_i}}) = (\langle \theta_{a_i} \rangle \rtimes \mathrm{Aut}(C_{p_i^{r_i}})) \times \langle \theta_{\overline{a}_i} \rangle$, where $\overline{a}_i = a_1 \dots a_{i-1}a_{i+1} \dots a_m$;*

(3) *Assume $p_1 > p_2 > \cdots > p_m$. For every $1 \leq i \leq m$, set $\pi_i = \{p_1, \dots, p_i\}$. Then* $\mathrm{Aut}(\mathrm{D}_{2n})$ *has a normal Hall π_i-subgroup and* $\mathrm{Aut}(\mathrm{D}_{2n})_{\pi_i} = \langle \theta_{a_1 \dots a_i} \rangle \rtimes \mathrm{Aut}(C_n)_{\pi_i}$, *where* $\mathrm{Aut}(C_n)_{\pi_i} = \mathrm{Aut}(C_{p_1^{r_1}})_{\pi_i} \times \cdots \times \mathrm{Aut}(C_{p_i^{r_i}})_{\pi_i}$.

Proof. It is easy to prove that $\mathrm{Aut}(\mathrm{D}_{2n}) = \langle \theta_a \rangle \rtimes \mathrm{Aut}(C_n)$ (also see [25, Theorem 7.2]). Let K be the kernel of $\mathrm{Aut}(\mathrm{D}_{2n})$ acting on C_n. Then $a^\gamma = a$ for all $\gamma \in K$, which implies $\langle \theta_a \rangle \leq K$. Furthermore, K is transitive on $b\langle a \rangle$ as $\langle \theta_a \rangle$ is transitive. Since $\mathrm{D}_{2n} = \langle a, b \rangle$ and $K \leq \mathrm{Aut}(\mathrm{D}_{2n})$, we have $K_b = 1$, implying that K is regular on $b\langle a \rangle$. It follows that $|K| = |b\langle a \rangle| = n$, and hence $K = \langle \theta_a \rangle$.

For $\beta \in \mathrm{Aut}(C_n)$, we have $b^\beta = b$, and for $x \in \langle a \rangle$, θ_x fixes $\langle a \rangle$ pointwise. Since β fixes $\langle a \rangle$ setwise,

$$a^{\theta_x{}^\beta} = (a^{\beta^{-1}})^{\theta_x \beta} = a^{\beta^{-1}\beta} = a = a^{\theta_{x^\beta}},$$

and

$$b^{\theta_x{}^\beta} = (b^{\beta^{-1}})^{\theta_x \beta} = b^{\theta_x \beta} = (bx)^\beta = bx^\beta = b^{\theta_{x^\beta}}.$$

Since $\mathrm{D}_{2n} = \langle a, b \rangle$, we obtain $\theta_x^\beta = \theta_{x^\beta}$. This completes the proof of part (1).

Recall that $\langle \theta_a \rangle = \langle \theta_{a_i} \rangle \times \langle \theta_{\overline{a}_i} \rangle$. Let $\beta \in \mathrm{Aut}(C_{p_i^{r_i}})$. Since $\overline{a}_i = a_1 \dots a_{i-1}a_{i+1} \dots a_m$, we have $\overline{a}_i^\beta = \overline{a}_i$, and $\theta_{\overline{a}_i}^\beta = \theta_{(\overline{a}_i)^\beta} = \theta_{\overline{a}_i}$, that is, $\theta_{\overline{a}_i}\beta = \beta\theta_{\overline{a}_i}$. Thus,

$$\langle \theta_a \rangle \rtimes \mathrm{Aut}(C_{p_i^{r_i}}) = (\langle \theta_{a_i} \rangle \rtimes \mathrm{Aut}(C_{p_i^{r_i}})) \times \langle \theta_{\overline{a}_i} \rangle.$$

This completes the proof of part (2).

Let $1 \leq i \leq m$. Note that $\mathrm{Aut}(C_{p_i^{r_i}}) \cong \mathbb{Z}_{p_i^{r_i-1}(p_i-1)}$ if p_i is odd, and $\mathrm{Aut}(C_{p_i^{r_i}}) \cong \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i^{r_i-2}}$ if $p_i = 2$ and $r_i \geq 2$. Since $p_1 > p_2 > \cdots > p_m$, we have $\mathrm{Aut}(C_{p_k^{r_k}})_{\pi_i} = 1$ for $i < k \leq m$, and hence $\mathrm{Aut}(C_n)_{\pi_i} = \mathrm{Aut}(C_{p_1^{r_1}})_{\pi_i} \times \cdots \times \mathrm{Aut}(C_{p_i^{r_i}})_{\pi_i}$.

It is easy to see that $\theta_{a_1\ldots a_i} = \theta_{a_1}\ldots\theta_{a_i}$ has order $p_1^{r_1}\ldots p_i^{r_i}$. Then $\langle\theta_{a_1\ldots a_i}\rangle$ is a Hall π_i-subgroup of $\langle\theta_a\rangle$, and hence $\langle\theta_{a_1\ldots a_i}\rangle$ is characteristic in $\langle\theta_a\rangle$. Since $\langle\theta_a\rangle \trianglelefteq \mathrm{Aut}(\mathrm{D}_{2n})$, we have $\langle\theta_{a_1\ldots a_i}\rangle \trianglelefteq \mathrm{Aut}(\mathrm{D}_{2n})$, and $\langle\theta_{a_1\ldots a_i}\rangle \rtimes \mathrm{Aut}(C_n)_{\pi_i}$ is a Hall π_i-subgroup of $\mathrm{Aut}(\mathrm{D}_{2n})$.

Recall that $\mathrm{Aut}(\mathrm{D}_{2n}) = \langle\theta_a\rangle \rtimes \mathrm{Aut}(C_n)$. To prove $\langle\theta_{a_1\ldots a_i}\rangle \rtimes \mathrm{Aut}(C_n)_{\pi_i} \trianglelefteq \mathrm{Aut}(\mathrm{D}_{2n})$, it suffices to show that $\mathrm{Aut}(C_n)_{\pi_i}^{\theta_a} \leq \langle\theta_{a_1\ldots a_i}\rangle \rtimes \mathrm{Aut}(C_n)_{\pi_i}$, or alternatively, $\mathrm{Aut}(C_{p_j^{r_j}})_{\pi_i}^{\theta_a} \leq \langle\theta_{a_1\ldots a_i}\rangle \rtimes \mathrm{Aut}(C_n)_{\pi_i}$ for every $1 \leq j \leq i$. Now take $\alpha \in \mathrm{Aut}(C_{p_j^{r_j}})_{\pi_i}$. Then $a_k^\alpha = a_k$ for every $1 \leq k \leq m$ with $k \neq j$, and $a_j^\alpha \in \langle a_j\rangle$. It follows that

$$\begin{aligned}\alpha^{\theta_a} &= \theta_a^{-1}\alpha\theta_a = \theta_a^{-1}\theta_a^{\alpha^{-1}}\alpha = \theta_{a^{-1}}\theta_{a^{\alpha^{-1}}}\alpha = \theta_{a^{-1}a^{\alpha^{-1}}}\alpha \\ &= \theta_{a_j^{-1}a_j^{\alpha^{-1}}}\alpha \in \langle\theta_{a_j}\rangle\mathrm{Aut}(C_{p_j^{r_j}})_{\pi_i},\end{aligned}$$

where

$$a^{-1}a^{\alpha^{-1}} = a_1^{-1}a_2^{-1}\ldots a_m^{-1}a_1\ldots a_{j-1}a_j^{\alpha^{-1}}a_{j+1}\ldots a_m = a_j^{-1}a_j^{\alpha^{-1}} \in \langle a_j\rangle.$$

Since $1 \leq j \leq i$, we have

$$\langle\theta_{a_j}\rangle\mathrm{Aut}(C_{p_j^{r_j}})_{\pi_i} \leq \langle\theta_{a_1\ldots a_i}\rangle \rtimes \mathrm{Aut}(C_n)_{\pi_i},$$

and hence $\mathrm{Aut}(C_{p_j^{r_j}})_{\pi_i}^{\theta_a} \leq \langle\theta_{a_1\ldots a_i}\rangle \rtimes \mathrm{Aut}(C_n)_{\pi_i}$, as required. Thus, $\mathrm{Aut}(\mathrm{D}_{2n})$ has a normal Hall π_i-subgroup, that is, $\mathrm{Aut}(\mathrm{D}_{2n})_{\pi_i} = \langle\theta_{a_1\ldots a_i}\rangle \rtimes \mathrm{Aut}(C_n)_{\pi_i}$. This complete the proof of part (3). □

For a finite group G, the right regular representation $R(G)$ and the automorphism group $\mathrm{Aut}(G)$ are permutation groups on G. Furthermore, $R(G) \cap \mathrm{Aut}(G) = 1$, and $R(G)\mathrm{Aut}(G) = R(G) \rtimes \mathrm{Aut}(G)$, where $R(g)^\alpha = R(g^\alpha)$ for all $g \in G$ and $\alpha \in \mathrm{Aut}(G)$. The normalizer of $R(G)$ in the symmetric group S_G on G is called the *holomorph* of G, denoted by $\mathrm{Hol}(G)$, and by [36, Lemma 7.16], $\mathrm{Hol}(G) = R(G) \rtimes \mathrm{Aut}(G)$. Now we have

$$\mathrm{Hol}(\mathrm{D}_{2n}) = R(\mathrm{D}_{2n}) \rtimes \mathrm{Aut}(\mathrm{D}_{2n}) = R(\mathrm{D}_{2n}) \rtimes (\langle\theta_a\rangle \rtimes \mathrm{Aut}(C_n)). \tag{3.6}$$

Lemma 3.2. *Let n be odd. Using the notations and formulae in* Equations (3.1) – (3.6), *we have the following.*

(1) *For all $d \in \mathrm{D}_{2n}$ and $\alpha \in \mathrm{Aut}(\mathrm{D}_{2n})$, $\langle R(a)\rangle\langle\theta_a\rangle = \langle R(a)\rangle \times \langle\theta_a\rangle \trianglelefteq \mathrm{Hol}(\mathrm{D}_{2n})$;*

(2) *Assume $p_1 > p_2 > \cdots > p_m$. For each $1 \leq i \leq m$, set $\pi_i = \{p_1, \ldots, p_i\}$. Then $\mathrm{Hol}(\mathrm{D}_{2n})_{\pi_i} \leq (\langle R(a)\rangle \times \langle\theta_a\rangle) \rtimes \mathrm{Aut}(C_n)_{\pi_i} \trianglelefteq \mathrm{Hol}(\mathrm{D}_{2n})$, where $\mathrm{Aut}(C_n)_{\pi_i} = \mathrm{Aut}(C_{p_1^{r_1}})_{\pi_i} \times \cdots \times \mathrm{Aut}(C_{p_i^{r_i}})_{\pi_i}$. Furthermore, $\langle R(b)\rangle\mathrm{Aut}(C_n)_2 = \langle R(b)\rangle \times \mathrm{Aut}(C_n)_2$ is a Sylow 2-subgroup of $\mathrm{Hol}(\mathrm{D}_{2n})$, where $\mathrm{Aut}(C_n)_2 = \mathrm{Aut}(C_{p_1^{r_1}})_2 \times \mathrm{Aut}(C_{p_2^{r_2}})_2 \times \cdots \times \mathrm{Aut}(C_{p_m^{r_m}})_2$;*

(3) *Let $1 \leq i, j \leq m$ with $i \neq j$. Then $\mathrm{Aut}(C_{p_i^{r_i}})$, under conjugacy, fixes each p_j-element in $\langle R(a)\rangle \times \langle\theta_a\rangle$, and if $\alpha \in \mathrm{Aut}(C_{p_i^{r_i}})$ fixes an element of order $p_i^{r_i}$ in $\langle R(a)\rangle \times \langle\theta_a\rangle$, then $\alpha = 1$.*

Proof. Recall that $R(d)^\alpha = R(d^\alpha)$ for all $d \in \mathrm{D}_{2n}$ and $\alpha \in \mathrm{Aut}(\mathrm{D}_{2n})$. Then $R(a)^{\theta_a} = R(a^{\theta_a}) = R(a)$, that is, $R(a)$ commutes with θ_a. Since $\langle R(a)\rangle \cap \langle \theta_a\rangle = 1$, we have $\langle R(a)\rangle\langle \theta_a\rangle = \langle R(a)\rangle \times \langle \theta_a\rangle$. Since n is odd, $\langle R(a)\rangle \times \langle \theta_a\rangle$ is a Hall $2'$-subgroup of $R(\mathrm{D}_{2n}) \rtimes \langle \theta_a\rangle$, and hence $\langle R(a)\rangle \times \langle \theta_a\rangle$ is characteristic in $R(\mathrm{D}_{2n}) \rtimes \langle \theta_a\rangle$. It follows that $\langle R(a)\rangle \times \langle \theta_a\rangle \trianglelefteq \mathrm{Hol}(\mathrm{D}_{2n})$, as $R(\mathrm{D}_{2n}) \rtimes \langle \theta_a\rangle \trianglelefteq \mathrm{Hol}(\mathrm{D}_{2n})$ by Equation (3.6). This completes the proof of part (1).

Now we prove part (2). By part (1), $\langle R(a)\rangle \times \langle \theta_a\rangle \trianglelefteq \mathrm{Hol}(\mathrm{D}_{2n})$, and by Equation (3.6),

$$\mathrm{Hol}(\mathrm{D}_{2n}) = R(\mathrm{D}_{2n}) \rtimes (\langle \theta_a\rangle \rtimes \mathrm{Aut}(C_n)) = (R(\mathrm{D}_{2n}) \rtimes \langle \theta_a\rangle) \rtimes \mathrm{Aut}(C_n).$$

Write $A = \langle R(a)\rangle \times \langle \theta_a\rangle$. Then $A \cap \mathrm{Aut}(C_n) = 1$, and by Equation (3.6), $(R(\mathrm{D}_{2n}) \rtimes \langle \theta_a\rangle)/A$ is a normal subgroup of order 2 in $\mathrm{Hol}(\mathrm{D}_{2n})/A$, and hence lies in the center of $\mathrm{Hol}(\mathrm{D}_{2n})/A$. Thus,

$$\mathrm{Hol}(\mathrm{D}_{2n})/A = (R(\mathrm{D}_{2n}) \rtimes \langle \theta_a\rangle)/A \times \mathrm{Aut}(C_n)A/A,$$

where $\mathrm{Aut}(C_n)A/A \cong \mathrm{Aut}(C_n)$. Since $\mathrm{Aut}(C_n)$ is abelian, $\mathrm{Hol}(\mathrm{D}_{2n})/A$ is abelian, and therefore, $(\langle R(a)\rangle \times \langle \theta_a\rangle) \rtimes \mathrm{Aut}(C_n)_{\pi_i} = A \rtimes \mathrm{Aut}(C_n)_{\pi_i} \trianglelefteq \mathrm{Hol}(\mathrm{D}_{2n})$.

Since n is odd, all p_i are odd and hence $2 \notin \pi_i$ for each $1 \le i \le m$. Clearly,

$$\mathrm{Hol}(\mathrm{D}_{2n})/((R(\mathrm{D}_{2n}) \rtimes \langle \theta_a\rangle) \rtimes \mathrm{Aut}(C_n)_{\pi_i}) \cong \mathrm{Aut}(C_n)/\mathrm{Aut}(C_n)_{\pi_i}.$$

Then for every Hall π_i-subgroup $\mathrm{Hol}(\mathrm{D}_{2n})_{\pi_i}$, $\mathrm{Hol}(\mathrm{D}_{2n})_{\pi_i} \le (R(\mathrm{D}_{2n}) \rtimes \langle \theta_a\rangle) \rtimes \mathrm{Aut}(C_n)_{\pi_i}$, and it follows that $\mathrm{Hol}(\mathrm{D}_{2n})_{\pi_i} \le (\langle R(a)\rangle \times \langle \theta_a\rangle) \rtimes \mathrm{Aut}(C_n)_{\pi_i}$, as $|(R(\mathrm{D}_{2n}) \rtimes \langle \theta_a\rangle) : (\langle R(a)\rangle \times \langle \theta_a\rangle)| = 2$. By Lemma 3.1(3), $\mathrm{Aut}(C_n)_{\pi_i} = \mathrm{Aut}(C_{p_1^{r_1}})_{\pi_i} \times \cdots \times \mathrm{Aut}(C_{p_i^{r_i}})_{\pi_i}$.

Since $\mathrm{Aut}(C_n)$ fixes b, we have $\langle R(b)\rangle \mathrm{Aut}(C_n) = \langle R(b)\rangle \times \mathrm{Aut}(C_n)$, and therefore, $\langle R(b)\rangle \mathrm{Aut}(C_n)_2 = \langle R(b)\rangle \times \mathrm{Aut}(C_n)_2$. By Equation (3.6), $|\mathrm{Hol}(\mathrm{D}_{2n})_2| = 2|\mathrm{Aut}(C_n)_2|$, so $\langle R(b)\rangle \times \mathrm{Aut}(C_n)_2$ is a Sylow 2-subgroup of $\mathrm{Hol}(\mathrm{D}_{2n})$. Clearly, $\mathrm{Aut}(C_n)_2 = \mathrm{Aut}(C_{p_1^{r_1}})_2 \times \mathrm{Aut}(C_{p_2^{r_2}})_2 \times \cdots \times \mathrm{Aut}(C_{p_m^{r_m}})_2$. This completes the proof of part (2).

To prove part (3), let $1 \le i, j \le m$ with $j \ne i$. Recall that p_i is odd. By part (1), $\langle R(a)\rangle \times \langle \theta_a\rangle \trianglelefteq \mathrm{Hol}(\mathrm{D}_{2n})$.

Let $\alpha \in \mathrm{Aut}(C_{p_i^{r_i}})$ and let x be a p_j-element in $\langle R(a)\rangle \times \langle \theta_a\rangle$. Since $\langle R(a_j)\rangle \times \langle \theta_{a_j}\rangle$ is a normal Sylow p_j-subgroup of $\langle R(a)\rangle \times \langle \theta_a\rangle$, we may take $x = R(a_j)^s\theta_{a_j}^t$ for some $s, t \in \mathbb{Z}_{p_j^{r_j}}$. Since $j \ne i$, $a_j^\alpha = a_j$ and hence

$$x^\alpha = (R(a_j)^s\theta_{a_j}^t)^\alpha = R(a_j^\alpha)^s\theta_{a_j^\alpha}^t = R(a_j)^s\theta_{a_j}^t = x.$$

Thus, $\mathrm{Aut}(C_{p_r^{r_i}})$ fixes each p_j-element in $\langle R(a)\rangle \times \langle \theta_a\rangle$.

Now assume that $\alpha \in \mathrm{Aut}(C_{p_i^{r_i}})$ fixes an element of order $p_i^{r_i}$ in $\langle R(a)\rangle \times \langle \theta_a\rangle$. Since $\langle R(a_i)\rangle \times \langle \theta_{a_i}\rangle$ is a normal Sylow p_i-subgroup of $\langle R(a)\rangle \times \langle \theta_a\rangle$, α fixes an element of order $p_i^{r_i}$ in $\langle R(a_i)\rangle \times \langle \theta_{a_i}\rangle$, say $R(a_i)^k\theta_{a_i}^\ell$ for some $k, \ell \in \mathbb{Z}_{p_i^{r_i}}$. Since $o(R(a_i)) = o(\theta_{a_i}) = p_i^{r_i}$, we have $(k, p_i) = 1$ or $(\ell, p_i) = 1$. Furthermore,

$$R(a_i^k)\theta_{a_i^\ell} = R(a_i)^k\theta_{a_i}^\ell = (R(a_i)^k\theta_{a_i}^\ell)^\alpha = R((a_i^k)^\alpha)\theta_{(a_i^\ell)^\alpha},$$

and since $\langle R(a)\rangle \cap \langle \theta_a\rangle = 1$, we have $(a_i^k)^\alpha = a_i^k$ and $(a_i^\ell)^\alpha = a_i^\ell$, which implies that $(a_i)^\alpha = a_i$ as $(k, p_i) = 1$ or $(\ell, p_i) = 1$. Thus, $\alpha = 1$, completing the proof of part (3). □

4 Proof of Theorem 1.1

First we construct normal Cayley graphs on dihedral groups which are not CI-graphs. Recall that $\mathrm{D}_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$, as given in Equation (3.3).

Lemma 4.1. *Let* $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, \{a, a^{-1}, b\})$. *Then we have the following.*

(1) *If* $n = 4$ *then* Γ *is non-normal, and if* $n \geq 5$ *then* Γ *is normal;*

(2) *If* $n \geq 4$ *is even, then* Γ *is not a CI-graph.*

Proof. Let $n \geq 4$. Write $A = \mathrm{Aut}(\Gamma)$ and $S = \{a, a^{-1}, b\}$. It is easy to see that $\mathrm{Aut}(\mathrm{D}_{2n}, S) = \langle\alpha\rangle$, where α is the automorphism of D_{2n} induced by $a \mapsto a^{-1}$ and $b \mapsto b$.

Since $S = S^{-1}$ and $\langle S\rangle = \mathrm{D}_{2n}$, Γ is a connected cubic graph of order $2n$. Clearly, $\Gamma \cong \mathcal{C}_n \times K_2$ is the ladder graph of order $2n$, where $\mathcal{C}_n$ is the cycle of length n and K_2 is the complete graph of order 2 (the Cartesian product $\Gamma_1 \times \Gamma_2$ of two graphs Γ_1 and Γ_2 have vertex set $\{(u_1, u_2) \mid u_i \in V(\Gamma_i), i = 1, 2\}$ and edge set $\{\{(u_1, u_2), (v_1, v_2)\} \mid$ either $u_1 = v_1$ and $(u_2, v_2) \in E(\Gamma_2)$, or $u_2 = v_2$ and $(u_1, v_1) \in E(\Gamma_1)\}$). Note that $\mathcal{C}_4 \cong K_2 \times K_2$, and for $n \geq 5$, $\mathcal{C}_n$ and K_2 are relatively prime, that is, $\mathcal{C}_n$ cannot be a Cartesian product of K_2 and a graph because $\mathcal{C}_n$ contains no cycle of length 4 (see [23, Lemma 6.3]).

Assume $n = 4$. Then $\Gamma \cong \mathcal{C}_4 \times K_2 \cong K_2 \times K_2 \times K_2 \cong K_{4,4} - 4K_2$, the complete bipartite graph $K_{4,4}$ minus one factor. Then $A \cong S_4 \times C_2$, where S_4 is the symmetric group of degree 4. Since $A \neq R(\mathrm{D}_8) \rtimes \mathrm{Aut}(\mathrm{D}_8, S) = R(\mathrm{D}_8) \rtimes \langle\alpha\rangle$, Γ is non-normal.

Assume that $n \geq 5$. Since $\mathcal{C}_n$ and K_2 are relatively prime, by [23, Corollary 6.12] we have $A \cong \mathrm{Aut}(\mathcal{C}_n) \times \mathrm{Aut}(K_2) \cong \mathrm{D}_{2n} \times \mathbb{Z}_2$. It follows that $|A : R(\mathrm{D}_{2n})| = 2$, and so $R(\mathrm{D}_{2n}) \trianglelefteq A$, that is, Γ is a normal Cayley graph. This completes the proof of part (1).

To prove that Γ is not a CI-graph for $n \geq 4$ and n even, by the well-known Babai criterion (see [4]), we only need to show that A has a regular dihedral subgroup, which is not conjugate to $R(\mathrm{D}_{2n})$ in A.

Note that $R(a)^{R(b)} = R(a^{-1})$, $R(b)^\alpha = R(b)$ and $R(a)^\alpha = R(a^{-1})$. Thus, $R(b)\alpha$ is an involution and $R(a)^{(R(b)\alpha)} = R(a^{-1})^\alpha = R(a)$, that is, $R(a)$ commutes with $R(b)\alpha$. Since $R(a)$ has order n and n is even, $R(ab)\alpha = R(a)(R(b)\alpha)$ has order n. Furthermore,

$$(R(ab)\alpha)^{R(b)} = R(ab)^{R(b)}\alpha = R(ba)\alpha = \alpha R(ba)^\alpha = \alpha R(ba^{-1}) = (R(ab)\alpha)^{-1}.$$

Thus, $\langle R(ab)\alpha, R(b)\rangle$ is a dihedral group of order $2n$.

Note that $(R(ab)\alpha)^2 = R(ab)\alpha R(ab)\alpha = R(ab)R(ab)^\alpha = R(ab)R(a^{-1}b) = R(a^2)$. Clearly, $R(a^2)$ has order $n/2$ and $\langle R(a^2)\rangle \trianglelefteq \langle R(ab)\alpha, R(b)\rangle$. Since $\langle R(a)\rangle$ is semiregular on D_{2n} with two orbits, $\langle R(a^2)\rangle$ has four orbits on D_{2n}, that is, $\langle a^2\rangle$, $a\langle a^2\rangle$, $b\langle a^2\rangle$ and $ba\langle a^2\rangle$. The involution $R(b)$ interchanges $\langle a^2\rangle$ and $b\langle a^2\rangle$, and $a\langle a^2\rangle$ and $ba\langle a^2\rangle$. Furthermore, $R(ab)\alpha$ interchanges $\langle a^2\rangle$ and $ba\langle a^2\rangle$, and $a\langle a^2\rangle$ and $b\langle a^2\rangle$. It follows that $\langle R(ab)\alpha, R(b)\rangle$ is transitive on D_{2n}, and hence regular as $|\langle R(ab)\alpha, R(b)\rangle| = 2n$.

By MAGMA [7], $\langle R(ab)\alpha, R(b)\rangle$ is not conjugate to $R(D_8)$ for $n = 4$, and so Γ is not a CI-graph. Let $n \geq 5$. Suppose Γ is a CI-graph. By part (1), Γ is normal, and by Proposition 2.3, $R(\mathrm{D}_{2n}) = \langle R(ab)\alpha, R(b)\rangle$, forcing $\alpha \in R(\mathrm{D}_{2n})$, a contradiction. Thus, Γ is not a CI-graph. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $n \geq 2$. First we prove (1) and (3) are equivalent, that is, D_{2n} is an NDCI-group if and only if either $n = 2, 4$ or n is odd. The necessity follows from Lemma 4.1.

To prove the sufficiency, let n be odd, or $n = 2, 4$, and we only need to prove that D_{2n} is an NDCI-group. Let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be a normal Cayley digraph. It suffices to show that Γ is a CI-digraph. Note that we use the notations or formulae from Equations (3.1) – (3.6). Let $A = \mathrm{Aut}(\Gamma)$. By Proposition 2.2, $A = R(\mathrm{D}_{2n})\mathrm{Aut}(\mathrm{D}_{2n}, S) \leq \mathrm{Hol}(\mathrm{D}_{2n})$ and $A_1 = \mathrm{Aut}(\mathrm{D}_{2n}, S) \leq \mathrm{Aut}(\mathrm{D}_{2n})$.

Assume $n = 2$. Then $\mathrm{D}_4 \cong C_2 \times C_2$, and D_4 is an NDCI-group because $C_2 \times C_2$ is a DCI-group. Assume $n = 4$. By Lemma 4.1, D_8 is a non-DCI-group. However, with the help of MAGMA [7], one may easily check that D_8 is an NDCI-group; it also can be proved by restricting the valency of Γ to be no more than 3 because the complement of Γ is isomorphic to $\mathrm{Cay}(\mathrm{D}_8, \mathrm{D}_8 \backslash (S \cup \{1\}))$ and by using the fact $|A|$ is a divisor of $|\mathrm{Hol}(\mathrm{D}_8)| = 64$.

Assume that n is odd. Without loss of generality, we may further assume that $p_1 > p_2 > \cdots > p_m$, where p_i's are all prime factors of n. For $1 \leq i \leq m$, we may set $\pi_i = \{p_1, p_2, \ldots, p_i\}$. Recall that $A = R(\mathrm{D}_{2n})\mathrm{Aut}(\mathrm{D}_{2n}, S) \leq \mathrm{Hol}(\mathrm{D}_{2n})$ and $A_1 = \mathrm{Aut}(\mathrm{D}_{2n}, S) \leq \mathrm{Aut}(\mathrm{D}_{2n})$. Since C_n is characteristic in D_{2n}, A_1 fixes C_n setwise. By Lemma 3.1(1), $\langle \theta_a \rangle$ is the kernel of $\mathrm{Aut}(\mathrm{D}_{2n})$ on C_n, and therefore, the kernel of A_1 on C_n is $\langle \theta_a \rangle \cap A_1$. Then $A_1/(\langle \theta_a \rangle \cap A_1)$ induces an subgroup of $\mathrm{Aut}(C_n)$, say B, and $|A_1| = |\langle \theta_a \rangle \cap A_1| \cdot |B|$. Note that $\mathrm{Aut}(C_n)$ is viewed as a subgroup of $\mathrm{Aut}(\mathrm{D}_{2n})$, and so is B, too. Then $A_1 \leq \langle \langle \theta_a \rangle \cap A_1, B \rangle$. On the other hand, $\langle \theta_a \rangle \cap A_1$ is characteristic in $\langle \theta_a \rangle$, and hence normal in $\mathrm{Aut}(\mathrm{D}_{2n})$, forcing that $(\langle \theta_a \rangle \cap A_1)B = (\langle \theta_a \rangle \cap A_1) \rtimes B$. It follows that $\langle \langle \theta_a \rangle \cap A_1, B \rangle = (\langle \theta_a \rangle \cap A_1) \rtimes B$, and since $|A_1| = |(\langle \theta_a \rangle \cap A_1)| \cdot |B|$, we have $A_1 = (\langle \theta_a \rangle \cap A_1) \rtimes B$. Thus,

$$A = R(\mathrm{D}_{2n}) \rtimes A_1,\ A_1 = (\langle \theta_a \rangle \cap A_1) \rtimes B \text{ with } B \leq \mathrm{Aut}(C_n).$$

Let G be a regular subgroup of A such that $G \cong \mathrm{D}_{2n}$. To prove that Γ is a CI-digraph, by Proposition 2.3 it suffices to show that $G = R(\mathrm{D}_{2n})$. We argue by contradiction, and we suppose that $G \neq R(\mathrm{D}_{2n})$.

By Lemma 3.2(2),

$$\langle R(b) \rangle \times \mathrm{Aut}(C_n)_2 = \langle R(b) \rangle \times \mathrm{Aut}(C_{p_1^{r_1}})_2 \times \mathrm{Aut}(C_{p_2^{r_2}})_2 \times \cdots \times \mathrm{Aut}(C_{p_m^{r_m}})_2$$

is a Sylow 2-subgroup of $\mathrm{Hol}(\mathrm{D}_{2n})$, denote by HD_2.

Let B_2 be a Sylow 2-subgroup of B. Then we have $B_2 \leq \mathrm{Aut}(C_n)_2$, and hence $\langle R(b) \rangle \times B_2$ is a Sylow 2-subgroup of A. Since Sylow 2-subgroups of A are conjugate by the second Sylow theorem and $|G_2| = 2$, there is $d \in A$ such that $G \cap (\langle R(b) \rangle \times B_2)^d \neq 1$. Thus, $G^{d^{-1}} \cap (\langle R(b) \rangle \times B_2) \neq 1$. Write $H = G^{d^{-1}}$. Then $H \leq A$ is regular on $V(\Gamma)$, and since $R(\mathrm{D}_{2n}) \trianglelefteq A$ and $R(\mathrm{D}_{2n}) \neq G$, we have

$$H \neq R(\mathrm{D}_{2n}) \text{ and } |H \cap HD_2| = 2.$$

By Equation (3.6) and Lemma 3.2(1),

$$\mathrm{Hol}(\mathrm{D}_{2n}) = R(\mathrm{D}_{2n}) \rtimes (\langle \theta_a \rangle \rtimes \mathrm{Aut}(C_n)) = (((\langle R(a) \rangle \times \langle \theta_a \rangle) \rtimes \langle R(b) \rangle) \rtimes \mathrm{Aut}(C_n).$$

Since $H \cong \mathrm{D}_{2n}$, we may assume that

$$H = \langle v, w \rangle \cong \mathrm{D}_{2n}, o(v) = n, o(w) = 2, v^w = v^{-1}, \text{ and } w \in HD_2.$$

Claim 1. $v \in \langle R(a)\rangle \times \langle \theta_a\rangle$.

Proof of Claim 1. Since $o(v) = n$, we have $o(v^{n/p_i^{r_i}}) = p_i^{r_i}$. Write $T_0 = \{1\}$, $T_1 = \{1, v^{n/p_1^{r_1}}\}$, $T_2 = \{1, v^{n/p_1^{r_1}}, v^{n/p_2^{r_2}}\}$, ..., $T_m = \{1, v^{n/p_1^{r_1}}, v^{n/p_2^{r_2}}, \ldots, v^{n/p_m^{r_m}}\}$. Clearly, $\langle v\rangle = \langle T_m\rangle$. To finish the proof of Claim 1, it suffices to show that $T_m \subseteq \langle R(a)\rangle \times \langle \theta_a\rangle$. To do this, we proceed by induction on k to show $T_k \subseteq \langle R(a)\rangle \times \langle \theta_a\rangle$, where $0 \leq k \leq m$.

Clearly, $T_0 \subseteq \langle R(a)\rangle \times \langle \theta_a\rangle$, and we may let $k > 0$. By induction hypothesis, we may assume that $T_j \subseteq \langle R(a)\rangle \times \langle \theta_a\rangle$ for all $0 \leq j < k$ and aim to show $T_k \subseteq \langle R(a)\rangle \times \langle \theta_a\rangle$. Since $T_k = T_{k-1} \cup \{v^{n/p_k^{r_k}}\}$, we only need to show that $v^{n/p_k^{r_k}} \in \langle R(a)\rangle \times \langle \theta_a\rangle$.

Since $o(v^{n/p_k^{r_k}}) = p_k^{r_k}$, $v^{n/p_k^{r_k}}$ is a π_k-element. By Lemma 3.2(2), $v^{n/p_k^{r_k}} \in ((\langle R(a)\rangle \times \langle \theta_a\rangle) \rtimes (\mathrm{Aut}(C_{p_1^{r_1}})_{\pi_k} \times \cdots \times \mathrm{Aut}(C_{p_k^{r_k}})_{\pi_k})$. Then we may write $v^{n/p_k^{r_k}} = x\beta_1\beta_2 \ldots \beta_k$, where $x \in \langle R(a)\rangle \times \langle \theta_a\rangle$ and $\beta_j \in \mathrm{Aut}(C_{p_j^{r_j}})_{\pi_k}$ for $1 \leq j \leq k$.

Clearly, v commutes with every element in T_{k-1}, and so does $v^{n/p_k^{r_k}}$. Since $\langle R(a)\rangle \times \langle \theta_a\rangle$ is abelian, x commutes with every element in T_{k-1} because $T_{k-1} \subseteq \langle R(a)\rangle \times \langle \theta_a\rangle$. For every $1 \leq \ell \leq k-1$, by Lemma 3.2(3) we have that if $j \neq \ell$ then $\beta_j \in \mathrm{Aut}(C_{p_j^{r_j}})_{\pi_k}$ commutes with every element of order $p_\ell^{r_\ell}$ in T_{k-1}, and then β_ℓ commutes with every element of order $p_\ell^{r_\ell}$ in T_{k-1} because $\beta_\ell = \beta_{\ell-1}^{-1} \ldots \beta_1^{-1} x^{-1} v^{n/p_k^{r_k}} \beta_k^{-1} \ldots \beta_{\ell+1}^{-1}$, which implies that β_ℓ commutes with an element of order $p_\ell^{r_\ell}$ in $\langle R(a)\rangle \times \langle \theta_a\rangle$ as $T_{k-1} \subseteq \langle R(a)\rangle \times \langle \theta_a\rangle$. Again by Lemma 3.2(3), we obtain $\beta_\ell = 1$.

It follows $v^{n/p_k^{r_k}} = x\beta_k = R(y)\theta_z\beta_k$ for some $y, z \in \langle a\rangle$ and $\beta_k \in \mathrm{Aut}(C_{p_k^{r_k}})_{\pi_k}$. Since $R(y) \in A$, we have $\theta_z\beta_k \in A$, so $\theta_z\beta_k \in A_1 = (\langle \theta_a\rangle \cap A_1) \rtimes B$, where $B \leq \mathrm{Aut}(C_n)$. Then there exist $\theta_{z'} \in \langle \theta_a\rangle \cap A_1$ and $\beta_{k'} \in B$ such that $\theta_z\beta_k = \theta_{z'}\beta_{k'}$. It follows that $\theta_{z'}^{-1}\theta_z = \beta_{k'}\beta_k^{-1} \in \langle \theta_a\rangle \cap \mathrm{Aut}(C_n) = 1$, that is, $\theta_z = \theta_{z'} \in \langle \theta_a\rangle \cap A_1 \leq A_1$ and $\beta_k = \beta_{k'} \in B \leq A_1$.

Suppose $\beta_k \neq 1$. Recall that $p_1 > p_2 > \cdots > p_k$, $\pi_k = \{p_1, p_2, \ldots, p_k\}$ and $\mathrm{Aut}(C_{p_k^{r_k}}) \cong \mathbb{Z}_{p_k^{r_k-1}(p_k-1)}$. Then every π_k-element in $\mathrm{Aut}(C_{p_k^{r_k}})$ is a p_k-element, and since $\beta_k \in \mathrm{Aut}(C_{p_k^{r_k}})_{\pi_k}$, $o(\beta_k)$ has order p_k-power. Let β be the automorphism of D_{2n} induced by

$$\beta\colon\ a_k \mapsto a_k^{p_k^{r_k-1}+1},\ b \mapsto b,\ a_i \mapsto a_i \text{ for every } i \neq k.$$

Let $L = \langle \beta\rangle$. Then L is the unique subgroup of order p_k of $\langle \beta_k\rangle$. It follows that $r_k \geq 2$ and $L \leq \langle \beta_k\rangle \leq A_1 = \mathrm{Aut}(\mathrm{D}_{2n}, S)$. Write

$$M = \langle a_1, \ldots, a_{k-1}, a_k^{p_k}, a_{k+1}, \ldots, a_m\rangle, \text{ and } K = \langle a_k^{p_k^{r_k-1}}\rangle.$$

Clearly, $1 < K \trianglelefteq \mathrm{D}_{2n}$. Note that every element of D_{2n} has the form $(a_1a_2\cdots a_m)^t$ or $b(a_1a_2\cdots a_m)^t$ for some $t \in \mathbb{Z}_n$. It is easy to see that the set of fixed-points of L in D_{2n} is $M \cup bM$, that is, $F_{\mathrm{D}_{2n}}(L) = M \cup bM$. Furthermore, L is transitive on Kc for every $c \in \mathrm{D}_{2n} \setminus (M \cup bM)$, and $|\mathrm{D}_{2n} : F_{\mathrm{D}_{2n}}(L)| \geq p_k > 2$ as p_k is odd. By Proposition 2.4(1), Γ is non-normal, a contradiction.

Then $\beta_k = 1$. It follows that $v^{n/p_k^{r_k}} = R(y)\theta_z\beta_k = R(y)\theta_z \in \langle R(a)\rangle \times \langle \theta_a\rangle$ and $T_k \subseteq \langle R(a)\rangle \times \langle \theta_a\rangle$, as required. This completes the proof of Claim 1. □

By Claim 1, we may assume that $v = R(a)^k\theta_a^\ell$ for some $k, \ell \in \mathbb{Z}_n$. Since θ_a fixes $\langle a\rangle$ pointwise, we have $1^{\langle v\rangle} = \langle a^k\rangle$, the orbit of $\langle v\rangle$ containing 1 in D_{2n}, and since $\langle v\rangle \leq H$ is semiregular, $|1^{\langle v\rangle}| = |\langle a^k\rangle| = o(v) = n$, forcing $o(a^k) = n$. Thus, $\langle a^k\rangle = \langle a\rangle$ and hence $(k, n) = 1$. Since $v \in H$, if necessary we replace v by a power of v, and then one may let

$$v = R(a)\theta_a^\ell \quad \text{for some } \ell \in \mathbb{Z}_n.$$

Recall that $w \in H$ with $o(w) = 2$ and $w \in HD_2 = \langle R(b)\rangle \times \mathrm{Aut}(C_n)_2$. If $w \in \mathrm{Aut}(C_n)_2 = \mathrm{Aut}(C_{p_1^{r_1}})_2 \times \mathrm{Aut}(C_{p_2^{r_2}})_2 \times \cdots \times \mathrm{Aut}(C_{p_m^{r_m}})_2$, then w fixes 1, contradicting the regularity of H. Thus, $w \in R(b)\mathrm{Aut}(C_n)_2$, and by Lemma 3.2(2), we have

$$w = R(b)\varepsilon_1\varepsilon_2 \ldots \varepsilon_m, \ \text{ where } \varepsilon_i \in \mathrm{Aut}(C_{p_i^{r_i}})_2 \ \text{ and } \varepsilon_i^2 = 1 \text{ for every } \ 1 \le i \le m.$$

Claim 2. For every $1 \le k \le m$, either $p_k^{r_k} \mid \ell$ and $\varepsilon_k = 1$, or $(\ell, p_k) = 1$ and $\varepsilon_k \ne 1$.

Proof of Claim 2. Write $\varepsilon = \varepsilon_1 \ldots \varepsilon_m \in \mathrm{Aut}(C_n)$. Since $H \cong \mathrm{D}_{2n}$, we have $v^{R(b)\varepsilon} = v^w = v^{-1} = R(a^{-1})\theta_{a^{-\ell}}$. By Lemma 3.2(1), we get

$$\begin{aligned}\theta_a^{R(b)} &= \theta_a R(b)^{\theta_a} R(b) = \theta_a R(b^{\theta_a} b) = \theta_a R(a^{-1}) \text{ and } (\theta_a^\ell)^{R(b)} = \theta_a^\ell R(a^{-\ell}) \\ &= R(a^{-\ell})\theta_{a^\ell}.\end{aligned}$$

Since $v^{R(b)} = (v^{-1})^\varepsilon = R((a^\varepsilon)^{-1})\theta_{(a^\varepsilon)^{-\ell}}$, we have

$$R(a^{-(\ell+1)})\theta_{a^\ell} = R(a^{-1})R(a^{-\ell})\theta_{a^\ell} = R(a)^{R(b)}(\theta_a^\ell)^{R(b)} = v^{R(b)} = R((a^\varepsilon)^{-1})\theta_{(a^\varepsilon)^{-\ell}}.$$

Since $\langle R(a)\rangle \cap \langle\theta_a\rangle = 1$, we deduce $R(a^{\ell+1}) = R(a^\varepsilon)$ and $\theta_{a^\ell} = \theta_{(a^\varepsilon)^{-\ell}}$. It follows that $a^\varepsilon = a^{\ell+1}$ and $(a^\varepsilon)^\ell = a^{-\ell}$. This yields $a^{\ell(\ell+2)} = 1$, so $n \mid \ell(\ell+2)$ and $p_k^{r_k} \mid \ell(\ell+2)$ for every $1 \le k \le m$. If $p_k \mid \ell$ and $p_k \mid \ell + 2$, then $p_k \mid 2$, which is impossible because n is odd. It follows that either $p_k^{r_k} \mid \ell$ and $(p_k, \ell+2) = 1$, or $p_k^{r_k} \mid (\ell+2)$ and $(p_k, \ell) = 1$.

Assume $p_k^{r_k} \mid \ell$ and $(p_k, \ell+2) = 1$. Since $a^\varepsilon = a^{\ell+1}$, we have

$$a^\varepsilon = (a_1a_2\ldots a_m)^\varepsilon = a_1^{\varepsilon_1}a_2^{\varepsilon_2}\ldots a_m^{\varepsilon_m} = a^{\ell+1} = a_1^{\ell+1}\cdot a_{k-1}^{\ell+1}a_k a_{k+1}^{\ell+1}\ldots a_m^{\ell+1},$$

implying $a_k^{\varepsilon_k} = a_k$, that is, $\varepsilon_k = 1$.

Assume $p_k^{r_k} \mid (\ell+2)$ and $(p_k, \ell) = 1$. Since $(a^\varepsilon)^\ell = a^{-\ell}$, we have $(a_k^\ell)^{\varepsilon_k} = (a_k^\ell)^{-1}$, and so $\varepsilon_k \ne 1$ because $(p_k, \ell) = 1$ implies $o(a_k^\ell) = p_k^{r_k}$. This completes the proof of Claim 2. □

If $p_k^{r_k} \mid \ell$ for all $1 \le k \le m$, by Claim 2 we have $\varepsilon_k = 1$, and hence $v = R(a)\theta_a^\ell = R(a)$ and $w = R(b)\varepsilon_1\varepsilon_2\ldots\varepsilon_m = R(b)$, yielding $H = R(\mathrm{D}_{2n})$, a contradiction. Thus, $\varepsilon_k \ne 1$ for some k. To simplify notation, from now on we do not assume $p_1 > p_2 > \cdots > p_m$ any more, which has no confusion. Then we may assume that there exists $1 \le s \le m$ such that $(\ell, p_1\ldots p_s) = 1$, $\varepsilon_j \ne 1$ for all $1 \le j \le s$, $p_{s+1}^{r_{s+1}}\ldots p_m^{r_m} \mid \ell$, and $\varepsilon_j = 1$ for all $s+1 \le j \le m$. It follows that $o(a^\ell) = p_1^{r_1}\ldots p_s^{r_s}$, forcing $\langle a^\ell\rangle = \langle a_1a_2\ldots a_s\rangle$ and $\langle\theta_a^\ell\rangle = \langle\theta_{a_1a_2\ldots a_s}\rangle$. Let $L = \langle\theta_{a_1a_2\ldots a_s}\rangle$.

Recall that $v = R(a)\theta_a^\ell \in H \le A$. It follows from $R(\mathrm{D}_{2n}) \le A$ that $\theta_a^\ell \in A$, implying $\theta_a^\ell \in A_1 = \mathrm{Aut}(\mathrm{D}_{2n}, S)$. Then $L = \langle\theta_a^\ell\rangle \le \mathrm{Aut}(\mathrm{D}_{2n}, S)$. Let $K = \langle a_1a_2\ldots a_s\rangle$. Note that every coset of K has the form $K(a_{s+1}\cdots a_m)^t$ or $Kb(a_{s+1}\cdots a_m)^t$ with $t \in \mathbb{Z}_{p_{s+1}^{r_{s+1}}\ldots p_m^{r_m}}$. Since $\theta_{a_1a_2\ldots a_s}$ is the automorphism of D_{2n} induced by $a \mapsto a$ and $b \mapsto ba_1a_2\ldots a_s$, we have that $F_{\mathrm{D}_{2n}}(L) = \langle a\rangle$, and for every coset of Kc in $b\langle a\rangle$, L is transitive on Kc. Clearly, $K \trianglelefteq \mathrm{D}_{2n}$ as K is characteristic in $\langle a\rangle$, and $|\mathrm{D}_{2n} : F_{\mathrm{D}_{2n}}(L)| = 2$.

Write $\gamma = \varepsilon_1 \dots \varepsilon_s$. Then $\gamma \neq 1$ and γ is the automorphism of D_{2n} induced by

$$a_i^\gamma = a_i^{-1} \text{ for } 1 \leq i \leq s, \;\; a_j^\gamma = a_j \text{ for } s+1 \leq j \leq m, \;\; b^\gamma = b.$$

Then γ fixes each coset of K in D_{2n} setwise, and $F_{\mathrm{D}_{2n}}(\langle\gamma\rangle) \neq \langle a \rangle = F_{\mathrm{D}_{2n}}(L)$. Noting that $w = R(b)\varepsilon_1\varepsilon_2 \dots \varepsilon_s\varepsilon_{s+1} \dots \varepsilon_m = R(b)\gamma \in H \leq A$ and $R(b) \in R(\mathrm{D}_{2n}) \leq A$, we have $\gamma \in A$, and hence $\gamma \in A_1 = \mathrm{Aut}(\mathrm{D}_{2n}, S)$. By Proposition 2.4(3), Γ is non-normal, a contradiction.

Thus, $\varepsilon_k = 1$ for all $1 \leq k \leq m$. By Claim 2, $\varepsilon = \varepsilon_1 \dots \varepsilon_m = 1$ and $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m} \mid \ell$. It follows that $v = R(a)\theta_a^\ell = R(a)$ and $w = R(b)\varepsilon = R(b)$, implying $H = \langle v, w \rangle = \langle R(a), R(b) \rangle = R(\mathrm{D}_{2n})$, a contradiction. This completes the proof of the the equivalence of (1) and (3).

Now we are ready to finish the proof of Theorem 1.1 by proving that (2) and (3) are equivalent, that is, D_{2n} is an NCI-group if and only if either $n = 2, 4$ or n is odd. The sufficiency follows from the fact that an NDCI-group is an NCI-group, and the necessity follows from Lemma 4.1. □

Proof of Corollary 1.2. Let the dihedral group D_{2n} be a DCI-group. Then D_{2n} is an NDCI-group. By Theorem 1.1, n is $2, 4$ or odd. By Lemma 4.1, D_8 is a non-CI-group and so a non-DCI-group. If n is odd, by [13, Theorem 1.2], n is square-free. Thus, $n = 2$ or n is odd-square-free. Now let D_{2n} be a CI-group. Then D_{2n} is an NCI-group. Again by Theorem 1.1, either n is $2, 4$, or n is odd. By [13, Theorem 1.2], if n is odd, then $n = 9$ or n is square-free. Since D_8 is a non-CI-group, we have $n = 2, 9$, or n is odd-square-free. □

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On cubic bi-Cayley graphs of p-groups*

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Abstract

A graph is called a *Cayley graph* (or *bi-Cayley graph*, respectively) of a group G if it has a group G of automorphisms acting semiregularly on the vertices with exactly one orbit (or two orbits, respectively). It is known every Cayley graph is vertex-transitive. In this paper, we first present a classification of connected cubic non-Cayley vertex-transitive bi-Cayley graphs of a finite p-group H, where $p > 3$ is a prime and the derived subgroup of H is either cyclic or isomorphic to $Z_p \times Z_p$. This is then used to give a classification of connected cubic non-Cayley vertex-transitive graphs of order $2p^4$ for each prime p.

Keywords: Bi-Cayley graph, vertex-transitive, non-Cayley.

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1 Introduction

In this paper, we shall describe an investigation of non-Cayley vertex-transitive bi-Cayley graphs of finite p-groups. To explain this, we first introduce some terminology.

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. The stabilizer of α in G is the subgroup $G_\alpha = \{g \in G \mid \alpha^g = \alpha\}$. If $G_\alpha = 1$ for every $\alpha \in \Omega$, then G is said to be *semiregular* on Ω, and if G is transitive and semiregular on Ω, then we say that G is *regular* on Ω.

Let Γ be a finite simple connected graph with vertex set $V(\Gamma)$ and $E(\Gamma)$. We use $\mathrm{Aut}(\Gamma)$ to denote the full automorphism group of Γ. Note that $\mathrm{Aut}(\Gamma)$ is a permutation group on $V(\Gamma)$. A graph Γ is *vertex-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on $V(\Gamma)$, and a vertex-transitive graph Γ is called *non-Cayley vertex-transitive* if $\mathrm{Aut}(\Gamma)$ has no regular subgroups.

Let R, L and S be subsets of a group H such that $R = R^{-1}$, $L = L^{-1}$ and $R \cup L$ does not contain the identity element of H. The *bi-Cayley graph* $\mathrm{BiCay}(H, R, L, S)$ of H relative to R, L, S is a graph having vertex set the union of the following two copies of H:

$$H_0 = \{h_0 \mid h \in H\} \text{ and } H_1 = \{h_1 \mid h \in H\},$$

and edge set

$$\{\{h_0, g_0\} \mid gh^{-1} \in R\} \cup \{\{h_1, g_1\} \mid gh^{-1} \in L\} \cup \{\{h_0, g_1\} \mid gh^{-1} \in S\}.$$

In the study of bi-Cayley graphs, much work has been devoted to construct and classify non-Cayley vertex-transitive graphs, see, for example, [19, 20, 23, 25]. In [25], the following problem was proposed:

Problem A ([25, Problem 1]). *Characterize cubic non-Cayley vertex-transitive bi-Cayley graphs of a p-group for an odd prime p.*

In [25], Problem A was partially solved for cubic bi-Cayley graphs of regular p-groups. In this paper, we make a progress towards Problem A by classifying non-Cayley vertex-transitive bi-Cayley graphs of a p-group H such that the derived group of H is cyclic or isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, where $p > 3$ is prime. The following is our first main result.

Theorem 1.1. *Let $p > 3$ be a prime and let H be a finite p-group such that the derived subgroup of H is either cyclic or isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Let Γ be a connected cubic bi-Cayley graph of H. Then Γ is non-Cayley vertex-transitive if and only if Γ is isomorphic to one of the graphs given in* Constructions I – V (*see* Section 4).

Note that every cubic non-Cayley vertex-transitive graph of order $2p$ or $2p^2$ (p a prime) is a generalized Petersen graph, see [13, 21]. In [25], all connected cubic non-Cayley vertex-transitive graphs of order $2p^3$ were classified for each prime p. Applying the above theorem, our next main theorem gives a classification of connected cubic non-Cayley vertex-transitive graphs of order $2p^4$ for each prime p.

Theorem 1.2. *Let p be a prime. Then a connected cubic graph Γ of order $2p^4$ is a non-Cayley vertex-transitive graph if and only if Γ is isomorphic to $\Gamma(p, m, n, s; t)(m+n+3 = 4, m > n \geq 0, s \geq 0)$, $\Gamma(p, 1; k, i)$ or $\Delta(p, 1; k)$* (*see* Section 4 *for the construction of these graphs*).

2 Preliminaries

We first introduce some notation about groups. For a positive integer, let $\mathbb{Z}_n$ be the cyclic group of order n and $\mathbb{Z}_n^*$ be the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n. A semidirect product of a group N by a group M is denoted by $N \rtimes M$. If H is a subgroup of a group G, the centralizer and normalizer of H in G are denoted by $C_G(H)$ and $N_G(H)$, respectively. The automorphism group, the center, the derived subgroup and the Frattini subgroup of a group G will be represented by Aut(G), $Z(G)$, G' and $\Phi(G)$, respectively.

2.1 Cayley graphs

Given a finite group G and a self-inverse subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\Gamma = \text{Cay}(G, S)$ on G relative to S is a graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Let $\text{Aut}(G, S) := \{\alpha \in \text{Aut}(G) | S^\alpha = S\}$. Then $\text{Aut}(G, S) \leq \text{Aut}(\Gamma)_1$. For any $g \in G$, $R(g)$ is the permutation of G defined by $R(g)\colon x \mapsto xg$ for $x \in G$. Set $R(G) := \{R(g) \mid g \in G\}$. It is well-known that $R(G)$ is a regular subgroup of $\text{Aut}(\text{Cay}(G, S))$. By [9, Lemma 2.1], we have $N_{\text{Aut}(\Gamma)} = R(G) \rtimes \text{Aut}(G, S)$. Clearly, every Cayley graph is vertex-transitive. In general, we have the following proposition.

Proposition 2.1 ([1, Lemma 16.3]). *A graph Γ is isomorphic to a Cayley graph of a group G if and only if* Aut(Γ) *has a subgroup isomorphic to G, acting regularly on the vertex set of* Γ.

The next two propositions about Cayley graphs will be used in the sequel.

Proposition 2.2 ([22, Theorem 1.1]). *Let G be a finite p-group with an odd prime p, and let Γ = Cay(G, S) be connected and of valency 4. Assume that Γ is X-edge-transitive, where $G \leq X \leq$ Aut(Γ). Then either G is normal in X, or X has a normal subgroup R such that $R \leq G$, and Γ is a normal cover of Γ_R, and moreover, the quotient graph Γ_R and the quotient group $\bar{X}$ satisfy one of the following conditions* (1) *and* (2) (*Here, for any $R \leq Y \leq X$ and $x \in X$, let $\bar{Y} = Y/R$ and $\bar{x} = xR$.*)*:*

(1) $\Gamma_R \cong \mathbf{K}_5$(*the complete graph of order* 5) *and* $\mathrm{A}_5 \leq \bar{X} \leq \mathrm{S}_5$.

(2) *$\bar{X}$ has a unique minimal normal subgroup $\bar{N} \cong Z_p^n$ with $n = p^m$ such that*

(i) *$\bar{G} = \bar{N} \rtimes \bar{M}$, where $\bar{M} \cong Z_{p^m}$ with $m \geqslant 1$,*

(ii) *$\bar{X} = \bar{N} \rtimes ((\bar{H} \rtimes \bar{M}).\bar{O})$ and $\bar{X}_{1_{\bar{G}}} = \bar{H}.\bar{O}$, where $\bar{H} \cong Z_2^l$ with $2 \leq l \leq n$ and $\bar{O} \cong Z_t$ with $t = 1$ or 2, such that there exist $\bar{b}_1, ..., \bar{b}_n \in \bar{N}$ and $\bar{h}_1, ..., \bar{h}_n \in \bar{H}$ such that $\bar{N} = \langle \bar{b}_1, ..., \bar{b}_n \rangle$, $\langle \bar{b}_i, \bar{h}_i \rangle \cong D_{2p}$ and $\bar{H} = \langle \bar{h}_i \rangle \times C_{\bar{H}}(\bar{b}_i)$ for $1 \leq i \leq n$.*

A graph Γ is called *symmetric* if Aut(Γ) is transitive on the arcs of Γ. By [8, Corollary 3.4], we have the following result.

Proposition 2.3. *Every connected cubic symmetric graph of order $2p^n$ is a Cayley graph whenever $n \geq 1$ and $p \geq 7$ is a prime.*

2.2 Basic properties of bi-Cayley graphs

In this subsection, we let Γ be a connected bi-Cayley graph $\mathrm{BiCay}(H, R, L, S)$ of a group H. The following lemma provides some basic properties of Γ (see [24, Lemma 3.1]).

Lemma 2.4. *The following hold.*

(1) *H is generated by $R \cup L \cup S$.*

(2) *Up to graph isomorphism, S can be chosen to contain the identity of H.*

For each $g \in H$, let

$$\mathcal{R}(g)\colon h_i \mapsto (hg)_i, \quad \forall i \in \mathbb{Z}_2,\ h \in H. \tag{2.1}$$

Set $\mathcal{R}(H) = \{\mathcal{R}(g) \mid g \in H\}$. Then $\mathcal{R}(H)$ is a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ with H_0 and H_1 as its two orbits. In general, a graph Γ is isomorphic to a bi-Cayley graph of a group H if and only if Γ admits a group of automorphisms which is isomorphic to H and acts semiregularly on the vertices with two orbits.

For $\alpha \in \mathrm{Aut}(H)$ and $x, y, g \in H$, let

$$\begin{aligned} \delta_{\alpha,x,y}\colon\ & h_0 \mapsto (xh^\alpha)_1,\ h_1 \mapsto (yh^\alpha)_0,\ \forall h \in H, \\ \sigma_{\alpha,g}\colon\ & h_0 \mapsto (h^\alpha)_0,\ h_1 \mapsto (gh^\alpha)_1,\ \forall h \in H. \end{aligned} \tag{2.2}$$

Set

$$\begin{aligned} I &= \{\delta_{\alpha,x,y} \mid \alpha \in \mathrm{Aut}(H)\ s.t.\ R^\alpha = x^{-1}Lx,\ L^\alpha = y^{-1}Ry,\ S^\alpha = y^{-1}S^{-1}x\}, \\ F &= \{\sigma_{\alpha,g} \mid \alpha \in \mathrm{Aut}(H)\ s.t.\ R^\alpha = R,\ L^\alpha = g^{-1}Lg,\ S^\alpha = g^{-1}S\}. \end{aligned} \tag{2.3}$$

The following theorem determines the normalizer of $\mathcal{R}(H)$ in $\mathrm{Aut}(\Gamma)$.

Theorem 2.5 ([24, Theorem 1.1])**.** *Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$ be a connected bi-Cayley graph of a group H. Then $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H)) = \mathcal{R}(H) \rtimes F$ if $I = \emptyset$ and $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H)) = \mathcal{R}(H)\langle F,\ \delta_{\alpha,x,y}\rangle$ if $I \neq \emptyset$ and $\delta_{\alpha,x,y} \in I$. Furthermore, if $I \neq \emptyset$, then for any $\delta_{\alpha,x,y} \in I$, we have*

(1) *$\langle \mathcal{R}(H),\ \delta_{\alpha,x,y}\rangle$ acts transitively on $V(\Gamma)$;*

(2) *if α has order 2 and $x = y = 1$, then Γ is isomorphic to the Cayley graph $\mathrm{Cay}(\bar{H}, R \cup \alpha S)$, where $\bar{H} = H \rtimes \langle \alpha \rangle$.*

2.3 Normal covers

Let Γ be a graph. Assume that $G \leq \mathrm{Aut}(\Gamma)$ is vertex-transitive on Γ. Let N be a normal subgroup of G such that N is intransitive on $V(\Gamma)$. The *normal quotient graph* Γ_N of Γ relative to N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two different orbits adjacent if there exists an edge in Γ between the vertices lying in those two orbits. If Γ_N and Γ have the same valency, then we say that Γ is a *normal N-cover* of Γ_N. By [12, Lemma 2.5], we have the following lemma.

Lemma 2.6. *Let Γ be a connected cubic graph such that $G \leq \mathrm{Aut}(\Gamma)$ is arc-transitive on Γ. Suppose that there exists $N \trianglelefteq G$ such that N has at least three orbits on $V(\Gamma)$. Then*

(1) Γ *is a normal* N*-cover of the quotient graph* Γ_N *of* Γ *relative to* N.

(2) N *acts semiregularly on* $V(\Gamma)$, N *is the kernel of* G *acting on* $V(\Gamma_N)$ *and* $G/N \leq \mathrm{Aut}(\Gamma_N)$.

(3) G/N *is arc-transitive on* Γ_N.

3 Cubic bi-Cayley graphs of groups of odd order

In this section, we shall give a description of cubic bi-Cayley graphs of groups of odd order, and prove a sufficient condition for such graphs being non-Cayley vertex-transitive.

Lemma 3.1. *Let* Γ *be a connected cubic non-Cayley vertex-transitive graph. Let* $A \leq \mathrm{Aut}(\Gamma)$ *be vertex- but not arc-transitive on* Γ. *Take any vertex* v *of* Γ. *Then there exists a unique neighbor* v^* *such that* $A_v = A_{v^*}$. *In particular,* $\{v, v^*\}$ *is a block of imprimitivity of* A *on* $V(\Gamma)$.

Proof. Since Γ is not a Cayley graph, one has $A_v > 1$. Assume that $|A_v|$ is divisible by p, where $p > 3$ is an odd prime. Then A_v contains an element α of order p. Note that each orbit of $\langle\alpha\rangle$ has length either 1 or p. Since $\langle\alpha\rangle$ fixes v and Γ has valency 3, the connectedness of Γ implies that each orbit of $\langle\alpha\rangle$ has length 1, and so $\alpha = 1$, a contradiction. Since Γ is not arc-transitive, it follows that $3 \nmid |A_v|$. Thus A_v is a nontrivial 2-group for every $v \in V(\Gamma)$. Let A_v^* be the kernel of A_v acting on the neighborhood of v in Γ. Then $A_v/A_v^* \leq \mathbb{Z}_2$. If $A_v/A_v^* = 1$, then $A_v = A_v^*$ fixes all neighbors of v, and then since Γ is connected and A is vertex-transitive on Γ, A_v would fix all vertices of Γ, forcing $A_v = 1$, a contradiction. Thus, $A_v/A_v^* \cong \mathbb{Z}_2$. So there exists a unique neighbor, say v^*, of v such that $A_v = A_{v^*}$.

For any $g \in A$, it is easily verified that $A_{v^g} = A_v^g = A_{v^*}^g = A_{(v^*)^g}$. It follows that either $\{v, v^*\} = \{v^g, (v^*)^g\}$ or $\{v, v^*\} \cap \{v^g, (v^*)^g\} = \emptyset$. So $\{v, v^*\}$ is a block of imprimitivity of A on $V(\Gamma)$. □

Definition 3.2. Let Γ be a connected cubic non-Cayley vertex-transitive graph. Let $A \leq \mathrm{Aut}(\Gamma)$ be vertex- but not arc-transitive on Γ. Set

$$\mathcal{M}(\Gamma) = \{\{v, v^*\} \mid v, v^* \in V(\Gamma), v \sim v^*, A_v = A_{v^*}\}.$$

The *matching quotient graph* $\Gamma_{\mathcal{M}}$ of Γ is the graph with vertex set $\mathcal{M}(\Gamma)$ and with two elements in $\mathcal{M}(\Gamma)$ adjacent in $\Gamma_{\mathcal{M}}$ if there exists an edge of Γ between them.

In the above definition, it is easy to see that $\mathcal{M}(\Gamma)$ is A-invariant and A induces a group of automorphisms of $\Gamma_{\mathcal{M}}$. By [14, Lemma 9 & Theorem 10], we have the following lemma.

Lemma 3.3. *Let* Γ *be a connected cubic non-Cayley vertex-transitive graph, and let* $A \leq \mathrm{Aut}(\Gamma)$ *be vertex- but not arc-transitive on* Γ. *Then* $\Gamma_{\mathcal{M}}$ *is a connected tetravalent* A*-arc-transitive graph, that is,* A *acts faithfully on* $\mathcal{M}(\Gamma)$ *and is arc-transitive* $\Gamma_{\mathcal{M}}$.

The following proposition gives a description of connected cubic bi-Cayley graphs of groups of odd order.

Proposition 3.4. *Let* $\Gamma = \mathrm{BiCay}(H, R, L, S)$ *be a connected cubic bi-Cayley graph of a group* H *of odd order. Let* $A \leq \mathrm{Aut}(\Gamma)$ *be such that* $\mathcal{R}(H) \leq A$*. If* Γ *is a non-Cayley vertex-transitive graph and* A *is vertex- but not arc-transitive on* Γ*, then one of the following holds.*

(1) $\Gamma \cong \mathrm{BiCay}(H, \emptyset, \emptyset, \{1, x, y\})$*, and* $\Gamma_{\mathcal{M}} \cong \mathrm{Cay}(H, \{x, y, x^{-1}, y^{-1}\})$*, where* $H = \langle x, y\rangle$*. Furthermore,* H *has no automorphism* α *such that* $x^\alpha = x^{-1}$ *and* $y^\alpha = y^{-1}$*. In particular,* $\mathcal{R}(H)$ *is not normal in* A*.*

(2) $\Gamma \cong \mathrm{BiCay}(H, \{a, a^{-1}\}, \{b, b^{-1}\}, \{1\})$*, and* $\Gamma_{\mathcal{M}} \cong \mathrm{Cay}(H, \{a, b, a^{-1}, b^{-1}\})$*, where* $H = \langle a, b\rangle$*. Furthermore,* H *has no automorphism* α *such that* $a^\alpha = a^{-1}$ *and* $b^\alpha = b^{-1}$*.*

Proof. Assume that Γ is a non-Cayley vertex-transitive graph. Clearly, Γ has $2|H|$ vertices as Γ is a bi-Cayley of the group H. Since $A \leq \mathrm{Aut}(\Gamma)$ is vertex- but not arc-transitive on Γ, by Lemma 3.3, A acts faithfully on $\mathcal{M}(\Gamma)$ (see Definition 3.2). As H has odd order, for any $\{u, u^*\} \in \mathcal{M}(\Gamma)$, we have $|\{u, u^*\} \cap H_i| = 1$ with $i = 1, 2$, where $H_i (i = 0, 1)$ are the two orbits of $\mathcal{R}(H)$ on $V(\Gamma)$. It follows that $\mathcal{R}(H)$ acts regularly on $\mathcal{M}(\Gamma)$, and so $\Gamma_{\mathcal{M}}$ is a Cayley graph of H. By Lemma 2.4, we may assume that S contains the identity 1 of H. Then $\{1_0, 1_1\}$ is an edge of Γ. Since H has odd order, we have $|R| = |L| = 0$ or 2.

If $|R| = |L| = 0$, then we may let $S = \{1, x, y\}$. Since Γ is connected, by Lemma 2.4 we have $H = \langle S\rangle = \langle x, y\rangle$. We may assume that $\{1_0, s_1\} \in \mathcal{M}(\Gamma)$ with $s \in S$. Then $A_{1_0} = A_{s_1}$. Define a map on $V(\Gamma)$ as follows:

$$L(s)\colon h_0 \mapsto h_0, h_1 \mapsto (s^{-1}h)_1, \forall h \in H.$$

It is easy to verify that $L(s)$ is a permutation on $V(\Gamma)$. Let

$$\Gamma' = \mathrm{BiCay}(H, \emptyset, \emptyset, s^{-1}S).$$

For each edge $\{h_0, g_1\}$ of Γ, we have $gh^{-1} \in S$ and so $s^{-1}gh^{-1} \in s^{-1}S$. It follows that $\{h_0, (s^{-1}g)_1\}$ is an edge of Γ'. Clearly, both Γ and Γ' have valency 3, so $L(s)$ is an isomorphism between Γ and Γ'. For any edge e of Γ' and for any $\alpha \in A$, we have $e^{L(s)^{-1}\alpha L(s)} \in E(\Gamma')$. This implies that $L(s)^{-1}AL(s) \leq \mathrm{Aut}(\Gamma')$. Let $B = L(s)^{-1}AL(s)$. Since $L(s)$ fixes 1_0, one has $B_{1_0} = L(s)^{-1}A_{1_0}L(s)$. It is easy to see that $L(s)^{-1}A_{s_1}L(s) = B_{1_1}$. Since $A_{1_0} = A_{s_1}$, one has $B_{1_0} = B_{1_1}$. Then $\{1_1, 1_0\} \in \mathcal{M}(\Gamma')$. Therefore, without loss of generality, we may assume that $\{1_1, 1_0\} \in \mathcal{M}(\Gamma)$. Then we have $\mathcal{M}(\Gamma) = \{\{h_0, h_1\} \mid h \in H\}$. By Lemma 3.3, $\Gamma_{\mathcal{M}}$ has valency 4. So $\{x_0, x_1\}, \{y_0, y_1\}, \{(x^{-1})_0, (x^{-1})_1\}, \{(y^{-1})_0, (y^{-1})_1\}$ are just the four neighbors of $\{1_1, 1_0\}$ in $\Gamma_{\mathcal{M}}$. Clearly, $\{1_1, 1_0\}^{R(h)} = \{h_0, h_1\}$ for each $h \in H$, so we have $\Gamma_{\mathcal{M}} \cong \mathrm{Cay}(H, \{x, y, x^{-1}, y^{-1}\})$.

If H has an automorphism, say α, such that $x^\alpha = x^{-1}$ and $y^\alpha = y^{-1}$, then α has order 2 and $S^\alpha = S^{-1}$, and then by Theorem 2.5, $\mathcal{R}(H) \rtimes \langle \delta_{\alpha,1,1}\rangle$ acts regularly on $V(\Gamma)$. This is contrary to the fact that Γ is a non-Cayley vertex-transitive graph.

By Lemma 3.3, $\Gamma_{\mathcal{M}}$ is A-arc-transitive. If $\mathcal{R}(H)$ is normal in A, then $\mathrm{Aut}(H, \{x, y, x^{-1}, y^{-1}\})$ is transitive on $\{x, y, x^{-1}, y^{-1}\}$. So $\mathrm{Aut}(H)$ must have an involution interchanging the two pairs (x, y) and (x^{-1}, y^{-1}), which is impossible by the argument in the above paragraph. Thus, we obtain part (1).

Now assume that $|R| = |L| = 2$. In this case, we must have $\mathcal{M}(\Gamma) = \{\{h_0, h_1\} \mid h \in H\}$. Assume that $R = \{a, a^{-1}\}$ and $L = \{b, b^{-1}\}$. Again, by Lemma 3.3, $\Gamma_{\mathcal{M}}$

has valency 4. So $\{a_0, a_1\}$, $\{b_0, b_1\}$, $\{(a^{-1})_0, (a^{-1})_1\}$, $\{(b^{-1})_0, (b^{-1})_1\}$ are just the four neighbors of $\{1_1, 1_0\}$ in $\Gamma_{\mathcal{M}}$. As $\{1_1, 1_0\}^{R(h)} = \{h_0, h_1\}$ for each $h \in H$, we have $\Gamma_{\mathcal{M}} \cong \mathrm{Cay}(H, \{a, b, a^{-1}, b^{-1}\})$. If H has an automorphism, say α, such that $a^\alpha = a^{-1}$ and $b^\alpha = b^{-1}$, then α has order 2 and $R^\alpha = L$, and then by Theorem 2.5, $\mathcal{R}(H) \rtimes \langle \delta_{\alpha,1,1} \rangle$ acts regularly on $V(\Gamma)$. This is contrary to the fact that Γ is a non-Cayley vertex-transitive graph. Then part (2) happens. □

We now give a sufficient condition for a cubic bi-Cayley graph of a group of odd order in (2) of Proposition 3.4 being non-Cayley vertex-transitive.

Theorem 3.5. *Let* $\Gamma = \mathrm{BiCay}(H, \{a, a^{-1}\}, \{b, b^{-1}\}, \{1\})$ *be a connected cubic bi-Cayley of a group* H *of odd order, where* $a, b \in H$*. If* $\mathrm{Aut}(H, \{a, b, a^{-1}, b^{-1}\}) \cong \mathbb{Z}_4$*, then* Γ *is a non-Cayley vertex-transitive graph.*

Proof. Assume that $\mathrm{Aut}(H, \{a, b, a^{-1}, b^{-1}\}) \cong \mathbb{Z}_4$. Then H has an automorphism α such that $a^\alpha = b$ and $b^\alpha = a^{-1}$. So α swaps $\{a, a^{-1}\}$ and $\{b, b^{-1}\}$. By Theorem 2.5, $\langle \mathcal{R}(H), \delta_{\alpha,1,1} \rangle$ acts transitively on $V(\Gamma)$.

Suppose on the contrary that Γ is a Cayley graph. First, assume that Γ is symmetric. Let N be the largest normal subgroup of $\mathrm{Aut}(\Gamma)$ contained in $\mathcal{R}(H)$. Note that each orbit $H_i (i = 1$ or $2)$ of $\mathcal{R}(H)$ on $V(\Gamma)$ induces a subgraph which is a union of cycles of odd length. If $N = \mathcal{R}(H)$, then $\mathcal{R}(H) \trianglelefteq \mathrm{Aut}(\Gamma)$ and then each H_i will be a block of imprimitivity of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$. Since Γ is symmetric and connected, it follows that $H_1 = H_2 = V(\Gamma)$, a contradiction. This implies that $N < \mathcal{R}(H)$ and Γ is non-bipartite. By Lemma 2.6, the quotient graph Γ_N of Γ relative to N is a cubic graph with $\mathrm{Aut}(\Gamma)/N$ as an arc-transitive group of automorphisms. Clearly, $\mathcal{R}(H)/N \leq \mathrm{Aut}(\Gamma)/N$ acts semiregularly on $V(\Gamma_N)$ with two orbits. So Γ_N is a bi-Cayley graph of $\mathcal{R}(H)/N$. If $\mathcal{R}(H)/N$ has a subgroup, say M/N, which is normal in $\mathrm{Aut}(\Gamma_N)$, then $M/N \trianglelefteq \mathrm{Aut}(\Gamma)/N$. Since N is the largest normal subgroup of $\mathrm{Aut}(\Gamma)$ contained in $\mathcal{R}(H)$, it follows that M/N is trivial. Thus, the only proper subgroup of $\mathcal{R}(H)/N$ which is normal in $\mathrm{Aut}(\Gamma_N)$ is the identity subgroup. As Γ_N has order $2|H|/|N|$ and $|H|$ is odd, by [7, Theorem 7.1], either Γ_N is 3-arc-regular and has order $6, 10, 110$ or 506, or Γ_N is 4-arc-regular and has order $14, 506$ or 2162. (One may the definition of s-arc-regular graphs in [7, page 145].) Furthermore, since Γ is not bipartite, by inspecting the list of cubic symmetric graphs of order at most 10000 [6], we get that either $\Gamma_N \cong \mathrm{F}10$ or $\mathrm{F}506A$, or $\Gamma_N \cong \mathrm{F}2162A$. However, by Magma [3], each of $\mathrm{F}10, \mathrm{F}506A, \mathrm{F}2162A$ is a non-Cayley graph, a contradiction. (For an integer n, if there is a unique cubic symmetric graph of order n up to graph isomorphism, then we use $\mathrm{F}n$ to denote this graph, and if there are more than one cubic symmetric graph of order n up to graph isomorphism, then we use $\mathrm{F}nA$, $\mathrm{F}nB$, etc. to denote the corresponding graphs (see [4]).)

Now assume that Γ is nonsymmetric. Then $\mathrm{Aut}(\Gamma)_{1_0}$ is a 2-group. Since $|\mathrm{Aut}(\Gamma)| = 2|H||\mathrm{Aut}(\Gamma)_{1_0}|$, $\mathcal{R}(H)$ is $2'$-Hall subgroup of $\mathrm{Aut}(\Gamma)$. Since Γ is supposed to be a Cayley graph, $\mathrm{Aut}(\Gamma)$ has a subgroup, say G, acting regularly on $V(\Gamma)$. Then $|G| = 2|H|$, and since $|H|$ is odd, G is solvable, and so G has a $2'$-Hall subgroup, say B. By [10, Theorem A], $\mathcal{R}(H)$ and B are conjugate in $\mathrm{Aut}(\Gamma)$. Without loss of generality, we may assume $\mathcal{R}(H) \leq G$. Then $\mathcal{R}(H) \trianglelefteq G$. Then $\{H_0, H_1\}$ is G-invariant. As 1_1 is the unique neighbor of 1_0 contained in H_1, for each $b \in G$, either $\{1_0, 1_1\}^b = \{1_0, 1_1\}$ or $\{1_0, 1_1\}^b \cap \{1_0, 1_1\} = \emptyset$. Since G is regular on $V(\Gamma)$, there exists $b \in G$ mapping 1_0 to 1_1, and then $\{1_0, 1_1\}^b = \{1_0, 1_1\}$. It follows that b interchanges 1_0 and 1_1, and hence b is an involution.

By Theorem 2.5, there exists an automorphism, say α, of H and $x, y \in H$ such that $\delta_{\alpha,x,y}$ swaps 1_0 and 1_1. By the definition of $\delta_{\alpha,x,y}$ (see Eqs. 2.2–2.3), we have $x = y = 1$ and α swaps $\{a, a^{-1}\}$ and $\{b, b^{-1}\}$. However, since $\mathrm{Aut}(H, \{a, b, a^{-1}, b^{-1}\}) \cong \mathbb{Z}_4$, the only involution in $\mathrm{Aut}(H, \{a, b, a^{-1}, b^{-1}\})$ must fix both $\{a, a^{-1}\}$ and $\{b, b^{-1}\}$ setwise, a contradiction. □

4 Five families of cubic VNC-graphs

In this section, we shall apply Theorem 3.5 to construct five families of cubic VNC graphs.

Construction I Let p be an odd prime, and $m > n \geq 0$ and let t be an integer such that $t^2 \equiv -1 (\mathrm{mod}\, p^{m-n})$. Let

$$H(p, m, n, s; t) = \langle x, y, z \mid x^{p^m} = y^{p^m} = z^{p^s} = 1, z = [x, y], [x, z] = [y, z] = 1, x^{p^n} = y^{p^n} \rangle,$$

and $\Gamma(p, m, n, s; t) = \mathrm{BiCay}(H(p, m, n, s; t), \{x, x^{-1}\}, \{y^t, y^{-t}\}, \{1\})$.

Lemma 4.1. *The graph $\Gamma(p, m, n, s; t)$ is a connected cubic non-Cayley vertex-transitive graph with $2p^{m+n+s}$ vertices.*

Proof. Let $G = H(p, m, n, s; t)$ and $\Sigma = \Gamma(p, m, n, s; t)$. Clearly, $G' = \langle z \rangle \cong \mathbb{Z}_{p^s}$ and $G/G' = \langle xG', yG' \rangle$. By the definition of $H(p, m, n, s; t)$, we have $\langle x \rangle \cap \langle z \rangle = \langle y \rangle \cap \langle z \rangle = 1$. Since $x^{p^n} = y^{p^n}$, one has $G/G' = \langle xG' \rangle \times \langle xy^{-1}G' \rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$. So G has order p^{m+n+s}, and so Σ has $2p^{m+n+s}$ vertices.

Let $u = y^t$, $v = x^{-r}$ and $w = [u, v]$, where $r \in \mathbb{Z}_{p^m}^*$ is such that $rt \equiv 1 (\mathrm{mod}\, p^m)$. Note that $t^2 \equiv -1 (\mathrm{mod}\, p^{m-n})$. Clearly, $u^{p^m} = v^{p^m} = 1$. Since $z = [x, y] \in Z(G)$, one has $w = [u, v] = [y^t, x^{-r}] = [y, x]^{-tr} = z^{tr}$, and so $w^{p^s} = 1$. Also, $[u, w] = [v, w] = 1$. Since $rt \equiv 1 (\mathrm{mod}\, p^m)$ and $t^2 \equiv -1 (\mathrm{mod}\, p^{m-n})$, one has $r = -t (\mathrm{mod}\, p^{m-n})$. So $u^{p^n} = y^{tp^n} = x^{tp^n} = x^{-rp^n} = v^{p^n}$. Thus, u, v, w have the same relations as do x, y, z. So G has an automorphism, say α, such that $x^\alpha = y^t$ and $y^\alpha = x^{-r}$. Then $(y^t)^\alpha = x^{-tr} = x^{-1}$, and then $\alpha \in \mathrm{Aut}(G, \{x, x^{-1}, y^t, y^{-t}\})$. To complete the proof, by Proposition 3.4, it suffices to prove that $\mathrm{Aut}(G, \{x, x^{-1}, y^t, y^{-t}\}) = \langle \alpha \rangle$.

By way of contradiction, assume that $\mathrm{Aut}(G, \{x, x^{-1}, y^t, y^{-t}\}) > \langle \alpha \rangle$. Then there would exist $\beta \in \mathrm{Aut}(G, \{x, x^{-1}, y^t, y^{-t}\})$ such that $x^\beta = y^t$ and $(y^t)^\beta = x$. Then $(xH')^\beta = y^t H'$ and $(y^t H')^\beta = xH'$. Since $yH' = x \cdot (x^{-1}y)H'$, one has $(xH')^\beta = y^t H' = x^t \cdot (x^{-1}y)^t H'$. It follows that

$$xH' = (y^t H')^\beta = (x^t H')^\beta \cdot ((x^{-1}y)^t H')^\beta = x^{t^2} H' \cdot (x^{-1}y)^{t^2} H' \cdot ((x^{-1}y)^t)^\beta H'.$$

As $x^{p^n} = y^{p^n}$, one has $(x^{-1}y)^{p^n} H' = H'$, and hence $x^{p^n} H' = x^{t^2 p^n} H'$. It follows that $x^{(t^2-1)p^n} \in H'$, and since $\langle x \rangle \cap H' = 1$, one has $x^{(t^2-1)p^n} = 1$. This implies that $t^2 \equiv 1 (\mathrm{mod}\, p^{m-n})$. However, it is assumed that $t^2 \equiv -1 (\mathrm{mod}\, p^{m-n})$ and $m > n$. This forces that $p^{m-n} \mid 2$, a contradiction. □

Construction II Let p be an odd prime, and m be an integer. Take $i, k \in \mathbb{Z}_p^*$ and assume that $k^2 \equiv -1 (\mathrm{mod}\, p)$. Let

$$H(p, m; k, i) = \langle a, b, c \mid a^{p^{m+1}} = c^p = 1, a^{p^m} = b^{kp^m}, c = [a, b], [a, c] = a^{ip^m}, [b, c] = b^{ip^m} \rangle$$

and let $\Gamma(p,m;k,i) = \text{BiCay}(H(p,m;k,i), \{a,a^{-1}\}, \{b,b^{-1}\}, \{1\})$.

Lemma 4.2. *The graph $\Gamma(p,m;k,i)$ is a connected cubic non-Cayley vertex-transitive graph with $2p^{2m+2}$ vertices.*

Proof. Let $G = H(p,m;k,i)$ and let $\Sigma = G(p,m;k,i)$. Clearly, $G' = \langle c\rangle \times \langle a^{p^m}\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $a^{p^m} \in Z(G)$. Letting $N = \langle a^{p^m}\rangle$, we have

$$G/N = \langle aN, bN, cN \mid a^{p^m}N = b^{p^m}N = c^pN = N, cN = [a,b]N, [cN, aN] = [cN, bN] = N\rangle.$$

So $|G/N| = p^{2m+1}$ and so $|G| = p^{2m+2}$. Thus, Σ has $2p^{2m+2}$ vertices.

Let $u = b$, $v = a^{-1}$ and $w = [u,v]$. Since $a^{p^m} = b^{kp^m}$ and $k^2 \equiv -1(\text{mod}\, p)$, one has $b^{p^m} = a^{-kp^m}$. So we have $u^{p^m} = v^{kp^m}$ and $u^{p^{m+1}} = 1$. As $w = [u,v] = [b,a^{-1}] = [a,b]^{a^{-1}} = c^{a^{-1}}$, one has $w^p = 1$. As $a^{p^m} = b^{kp^m}$, one has $[a,c] = a^{ip^m} \in Z(G)$ and $[b,c] = b^{ip^m} \in Z(G)$. It follows that $[u,w] = [b, c^{a^{-1}}] = [b,[a,c]c] = [b,c] = b^{ip^m} = u^{ip^m}$, and $[v,w] = [a^{-1}, c^{a^{-1}}] = [a^{-1},[a,c]c] = [a^{-1},c] = [a,c]^{-1} = a^{-ip^m} = v^{ip^m}$. Thus u,v,w have the same relations as do a,b,c. So H has an automorphism, say α, such that $a^\alpha = b$ and $b^\alpha = a^{-1}$. This implies $\alpha \in \text{Aut}(G, \{a,a^{-1},b,b^{-1}\})$. To complete the proof, by Proposition 3.4, it suffices to prove that $\text{Aut}(G, \{a,a^{-1},b,b^{-1}\}) = \langle\alpha\rangle$.

Suppose on the contrary that $\text{Aut}(G, \{a,a^{-1},b,b^{-1}\}) > \langle\alpha\rangle$. Then there would exist $\beta \in \text{Aut}(G, \{a,a^{-1},b,b^{-1}\})$ such that $a^\beta = b$ and $b^\beta = a$. Then $b^{p^m} = (a^{p^m})^\beta = (b^{kp^m})^\beta = a^{kp^m}$. It follows that $a^{p^m} = a^{k^2p^m}$ and hence $k^2 \equiv 1(\text{mod}\, p)$. However, since it is assumed that $k^2 \equiv -1(\text{mod}\, p)$, one has $p \mid 2$, a contradiction. □

Construction III Let p be an odd prime, and $m > n > 0$ be integers. Take $i \in \mathbb{Z}_p^*$ and $s \in \mathbb{Z}_{p^n}^*$ such that $s^2 \equiv -1(\text{mod}\, p^n)$. Let $H(p,m,n;s,i)$ be a group with the follow presentation:

$$\langle a,b,c \mid a^{p^{m+1}} = c^p = 1, a^{p^{m-n}} = b^{sp^{m-n}}, c = [a,b], [a,c] = a^{ip^m}, [b,c] = b^{ip^m}\rangle$$

and $\Gamma(p,m,n;s,i) = \text{BiCay}(H(p,m,n;s,i), \{a,a^{-1}\}, \{b,b^{-1}\}, \{1\})$.

Lemma 4.3. *The graph $\Gamma(p,m,n;s,i)$ is a connected cubic non-Cayley vertex-transitive graph with $2p^{2m-n+2}$ vertices.*

Proof. Let $G = H(p,m,n;s,i)$ and let $\Sigma = \Gamma(p,m,n;s,i)$. Clearly, $G' = \langle c\rangle \times \langle a^{p^m}\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $a^{p^m} \in Z(G)$ and $G/G' = \langle aG', bG'\rangle$. Since $a^{p^m}, b^{p^m} \in G'$, one has $G' \cap \langle a\rangle = G' \cap \langle b\rangle = \langle a^{p^m}\rangle$. Since $a^{p^{m-n}} = b^{sp^{m-n}}$, one has $\langle aG'\rangle \cap \langle bG'\rangle = \langle a^{p^{m-n}}G'\rangle \cong \mathbb{Z}_{p^n}$, and so $|G/G'| = p^{2m-n}$. It follows that $G/G' = \langle aG'\rangle \times \langle ab^{-s}G'\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{m-n}}$, and hence $|G| = p^{2m-n+2}$. Thus, Σ has $2p^{2m-n+2}$ vertices.

Let $u = b$, $v = a^{-1}$ and $w = [u,v]$. Since $a^{p^{m-n}} = b^{sp^{m-n}}$ and $s^2 \equiv -1(\text{mod}\, p^n)$, one has $b^{p^{m-n}} = a^{-sp^{m-n}}$. So we have $u^{p^{m-n}} = v^{sp^{m-n}}$ and $u^{p^{m+1}} = 1$. As $w = [u,v] = [b,a^{-1}] = [a,b]^{a^{-1}} = c^{a^{-1}}$, one has $w^p = 1$. As $a^{p^{m-n}} = b^{sp^{m-n}}$, one has $[a,c] = a^{ip^m} \in Z(G)$ and $[b,c] = b^{ip^m} \in Z(G)$. It follows that $[u,w] = [b,c^{a^{-1}}] = [b,[a,c]c] = [b,c] = b^{ip^m} = u^{ip^m}$, and $[v,w] = [a^{-1},c^{a^{-1}}] = [a^{-1},[a,c]c] = [a^{-1},c] = [a,c]^{-1} = a^{-ip^m} = v^{ip^m}$. Thus u,v,w have the same relations as do a,b,c. So G has an automorphism, say α, such that $a^\alpha = b$ and $b^\alpha = a^{-1}$, and hence $\alpha \in \text{Aut}(G, \{a,a^{-1},b,$

$b^{-1}\}$). To complete the proof, by Proposition 3.4, it suffices to prove that $\mathrm{Aut}(G, \{a, a^{-1}, b, b^{-1}\}) = \langle\alpha\rangle$.

Suppose on the contrary that $\mathrm{Aut}(G, \{a, a^{-1}, b, b^{-1}\}) > \langle\alpha\rangle$. Then there would exist $\beta \in \mathrm{Aut}(G, \{a, a^{-1}, b, b^{-1}\})$ such that $a^\beta = b$ and $b^\beta = a$. Then $b^{p^{m-n}} = (a^{p^{m-n}})^\beta = (b^{sp^{m-n}})^\beta = a^{sp^{m-n}}$. It follows that $a^{p^{m-n}} = a^{s^2p^{m-n}}$ and hence $s^2 \equiv 1(\mathrm{mod}\, p^n)$. However, since it is assumed that $s^2 \equiv -1(\mathrm{mod}\, p^n)$, one has $p \mid 2$, a contradiction. □

Construction IV Let p be an odd prime, and m be an integer. Take $k \in \mathbb{Z}_p^*$ such that $k^2 \equiv -1(\mathrm{mod}\, p)$. Let $H(p, m; k)$ be a group with the following representation:

$$\langle a, b, c, d \mid a^{p^m} = b^{p^m} = c^p = d^p = 1, c = [a, b], [a, c] = d, [b, c] = d^k, [a, d] = [b, d] = 1\rangle$$

and let $\Delta(p, m; k) = \mathrm{BiCay}(H(p, m; k), \{a, a^{-1}\}, \{b, b^{-1}\}, \{1\})$.

Lemma 4.4. *The graph $\Delta(p, m; k)$ is a connected cubic non-Cayley vertex-transitive graph with $2p^{2m+2}$.*

Proof. Let $G = H(p, m; k)$ and let $\Sigma = \Delta(p, m; k)$. Clearly, $G' = \langle c\rangle \times \langle d\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $d \in Z(G)$ and $G/G' = \langle aG', bG'\rangle$. Letting $N = \langle d\rangle$, we have

$$G/N = \langle aN, bN, cN \mid a^{p^m}N = b^{p^m}N = c^pN = N, cN = [a, b]N, [cN, aN] = [cN, bN] = N\rangle.$$

So $|G/N| = p^{2m+1}$ and so $|G| = p^{2m+2}$. Thus, Σ has $2p^{2m+2}$ vertices.

Let $u = b$, $v = a^{-1}$, $x = [u, v]$ and $y = [u, x]$. Clearly, $u^{p^m} = v^{p^m} = 1$. Note that

$$x = [u, v] = [b, a^{-1}] = [a, b]^{a^{-1}} = aca^{-1}c^{-1}ac(ac)^{-1}c = d^{(ac)^{-1}}c = dc.$$

Then $y = [u, x] = [b, dc] = [b, c] = d^k$. It follows that $x^p = y^p = 1$. Furthermore,

$$[v, x] = [a^{-1}, c] = [c, a]^{a^{-1}} = d^{-1} = d^{k^2} = y^k.$$

Clearly, $[u, y] = [v, y] = 1$ due to $[a, d] = [b, d] = 1$. Thus u, v, x, y have the same relations as do a, b, c, d. So G has an automorphism, say α, such that $a^\alpha = b$ and $b^\alpha = a^{-1}$. This implies $\alpha \in \mathrm{Aut}(G, \{a, a^{-1}, b, b^{-1}\})$. To complete the proof, by Proposition 3.4, it suffices to prove that $\mathrm{Aut}(G, \{a, a^{-1}, b, b^{-1}\}) = \langle\alpha\rangle$.

Suppose on the contrary that $\mathrm{Aut}(G, \{a, a^{-1}, b, b^{-1}\}) > \langle\alpha\rangle$. Then there would exist $\beta \in \mathrm{Aut}(G, \{a, a^{-1}, b, b^{-1}\})$ such that $a^\beta = b$ and $b^\beta = a$. Then $c^\beta = [b, a] = c^{-1}$ and $d^\beta = [b, c^{-1}] = [c, b]^{c^{-1}} = d^{-k}$. So $(d^k)^\beta = d^{-k^2} = d$ since $k^2 \equiv -1(\mathrm{mod}\, p)$. On the other hand, $(d^k)^\beta = [b^\beta, c^\beta] = [a, c^{-1}] = d^{-1}$. It follows that $d = d^{-1}$ which is impossible since d has order $p > 2$. □

Construction V Let p be an odd prime, and let $m > n > 0$ be integers. Take $k \in \mathbb{Z}_p^*$ and $t \in \mathbb{Z}_{p^n}^*$ such that $t^2 \equiv -1(\mathrm{mod}\, p^n)$ and $t \equiv k(\mathrm{mod}\, p)$. Take $j \in \mathbb{Z}_p$. Let $G(p, m, n; t, k, j)$ be a group with the following representation:

$$\langle a, b, c, d \mid a^{p^m} = c^p = d^p = 1, a^{p^{m-n}} = b^{tp^{m-n}}d^j, c = [a, b], [a, c] = d, [b, c] = d^k, [a, d] = [b, d] = 1\rangle$$

and let $\Theta(p, m, n; t, k, j) = \mathrm{BiCay}(G(p, m, n; t, k, j), \{a, a^{-1}\}, \{b, b^{-1}\}, \{1\})$.

Lemma 4.5. *The graph $\Theta(p,m,n;t,k,j)$ is a connected cubic non-Cayley vertex-transitive graph with $2p^{2m-n+2}$ vertices.*

Proof. Let $G = G(p,m,n;t,k)$ and let $\Sigma = \Theta(p,m,n;t,k)$. Clearly, $G' = \langle c\rangle \times \langle d\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $d \in Z(G)$ and $G/G' = \langle aG', bG'\rangle$. Furthermore, $G' \cap \langle a\rangle = G' \cap \langle b\rangle = 1$. Since $a^{p^{m-n}} = b^{tp^{m-n}} d^j$, one has $\langle aG'\rangle \cap \langle bG'\rangle = \langle a^{p^{m-n}} G'\rangle \cong \mathbb{Z}_{p^n}$, and so $|G/G'| = p^{2m-n}$. It follows that $G/G' = \langle aG'\rangle \times \langle ab^{-t}G'\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{m-n}}$, and hence $|G| = p^{2m-n+2}$. Thus, Σ has $2p^{2m-n+2}$ vertices.

Let $u = b$, $v = a^{-1}$, $x = [u,v]$ and $y = [u,x]$. Note that

$$x = [u,v] = [b,a^{-1}] = [a,b]^{a^{-1}} = aca^{-1}c^{-1}ac(ac)^{-1}c = d^{(ac)^{-1}}c = dc.$$

Then $y = [u,x] = [b,dc] = [b,c] = d^k$. It follows that $x^p = y^p = 1$.

As $a^{p^{m-n}} = b^{tp^{m-n}} d^j$ and $t^2 \equiv -1 (\mathrm{mod}\, p^n)$, one has $u^{p^{m-n}} = b^{p^{m-n}} = a^{-tp^{m-n}} d^{jt} = v^{tp^{m-n}} y^j$ (due to $t \equiv k (\mathrm{mod}\, p)$), and hence $u^{p^m} = 1$. Furthermore,

$$[v,x] = [a^{-1},c] = [c,a]^{a^{-1}} = d^{-1} = d^{k^2} = y^k.$$

Clearly, $[u,y] = [v,y] = 1$ due to $[a,d] = [b,d] = 1$. Thus u,v,x,y have the same relations as do a,b,c,d. So G has an automorphism, say α, such that $a^\alpha = b$ and $b^\alpha = a^{-1}$, and hence $\alpha \in \mathrm{Aut}(G, \{a,a^{-1},b,b^{-1}\})$. To complete the proof, by Proposition 3.4, it suffices to prove that $\mathrm{Aut}(G,\{a,a^{-1},b,b^{-1}\}) = \langle\alpha\rangle$.

Suppose on the contrary that $\mathrm{Aut}(G,\{a,a^{-1},b,b^{-1}\}) > \langle\alpha\rangle$. Then there would exist $\beta \in \mathrm{Aut}(G,\{a,a^{-1},b,b^{-1}\})$ such that $a^\beta = b$ and $b^\beta = a$. Then $c^\beta = [b,a] = c^{-1}$ and $d^\beta = [b,c^{-1}] = [c,b]^{c^{-1}} = d^{-k}$. So $(d^k)^\beta = d^{-k^2} = d$ since $k^2 \equiv -1 (\mathrm{mod}\, p)$. On the other hand, $(d^k)^\beta = [b^\beta, c^\beta] = [a,c^{-1}] = d^{-1}$. It follows that $d = d^{-1}$ which is impossible since d has order $p > 2$. □

5 Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. Let G be a finite p-group with p a prime. By [18, 5.3.2], if $|G : \Phi(G)| = p^r$ then every generating set of G has a subset of r elements which also generates G. In particular, $G/\Phi(G) \cong \mathbb{Z}_p^r$. This implies that all minimal generating sets of a finite p-group G have the same cardinality, called the *rank* of G and denoted by $d(G)$.

Lemma 5.1. *Let p be an odd prime and let H be a finite p-group such that $d(H') \leq 2$. Let Γ be a connected cubic bi-Cayley of H. Let $A \leq \mathrm{Aut}(\Gamma)$ be such that $\mathcal{R}(H) \leq A$. If Γ is a non-Cayley vertex-transitive graph and A is vertex- but not arc-transitive on Γ, then either $p = 3$ and $d(H') = 2$, or $\mathcal{R}(H) \trianglelefteq A$.*

Proof. Assume that Γ is a non-Cayley vertex-transitive graph and A is vertex- but not arc-transitive on Γ. Assume further that $\mathcal{R}(H)$ is not normal in A. We shall prove that $p = 3$ and $d(H') = 2$. By Proposition 3.4, $\Gamma_{\mathcal{M}}$ is a connected tetravalent Cayley graph of H, and by Lemma 3.3, $\Gamma_{\mathcal{M}}$ is A-arc-transitive and $\mathcal{R}(H) \leq A$. For convenience, we may identify $\mathcal{R}(H)$ with H and let $\Sigma = \Gamma_{\mathcal{M}}$. Applying Proposition 2.2 to the A-arc-transitive Cayley graph Σ of odd order group H, we see that A has a normal subgroup R such that $R \leq H$, Σ is a normal cover of Σ_R, and moreover, Σ_R and A/R satisfy

(1) or (2) of Proposition 2.2. Since Γ is non-symmetric, for each $v \in V(\Gamma)$, A_v is a 2-group, and so A is a $\{2, p\}$-group. It follows that A is solvable. If Proposition 2.2(1) happens, then $\mathrm{A}_5 \leq A/R \leq \mathrm{S}_5$, contradicting that A is solvable. If Proposition 2.2(2) happens, then H/R has two subgroups N/R and M/R such that $H/R = N/R \rtimes M/R$ with $M/R \cong \mathbb{Z}_{p^m}$ and $N/R \cong \mathbb{Z}_p^n$, where $n = p^m$. Then $H' \leq N$. Since $M \cap N = R$, one has $M \cap H'R = R$, and so $MH'R/H'R \cong M/R \cong \mathbb{Z}_{p^m}$. It follows that $H/H'R = N/H'R \times MH'/H'R$. If $N = H'R$, then $H'/(H' \cap R) \cong N/R \cong \mathbb{Z}_p^n$. As $d(H') \leq 2$, one has $n = p^m \leq 2$, contrary to that p is an odd prime. Thus, $N > H'R$. By Proposition 3.4, we have $d(H) \leq 2$, and so $N/H'R \cong \mathbb{Z}_p$. Then $H'/(H' \cap R) \cong H'R/R \cong \mathbb{Z}_p^{n-1}$. Again, as $d(H') \leq 2$, one has $n - 1 = p^m - 1 \leq 2$, and hence $p^m \leq 3$. Thus, $d(H') = 2$ and $p^m = 3$. This completes the proof. □

We shall fulfill our task of the proof of Theorem 1.1 by the following three lemmas.

Lemma 5.2. *Let $p > 3$ be a prime and let H be a finite p-group such that either $H' \cong \mathbb{Z}_p \times \mathbb{Z}_p$ or H' is cyclic. Let Γ be a connected cubic bi-Cayley of H. If Γ is a symmetric non-Cayley vertex-transitive graph, then $\Gamma \cong \Gamma(5, 1, 0, 0; 2)$ or $\Gamma(5, 2; 2, 4)$ (see* Constructions I & II).

Proof. Assume that Γ is symmetric and non-Cayley. Since $p > 3$, by Proposition 2.3, we have $p = 5$. Let $|H| = 5^n$. Then $\mathcal{R}(H)$ is a Sylow 5-subgroup of $\mathrm{Aut}(\Gamma)$. Let P be the maximal normal 5-subgroup of $\mathrm{Aut}(\Gamma)$. By [16, Lemma 18], we have $|P| = 5^n$ or 5^{n-1}. If $|P| = 5^n$, then $P = \mathcal{R}(H)$. Let Q be a Sylow 2-subgroup of $\mathrm{Aut}(\Gamma)$. Then $A = \mathcal{R}(H) \rtimes Q$ is vertex- but not arc-transitive on Γ. Since $\mathcal{R}(H) \trianglelefteq A$, by Proposition 3.4, we have $\Gamma \cong \mathrm{BiCay}(H, \{a, a^{-1}\}, \{b, b^{-1}\}, \{1\})$ with $\langle a, b\rangle = H$. However, as $\mathcal{R}(H) = P \trianglelefteq \mathrm{Aut}(\Gamma)$ and since Γ is symmetric, the two orbits H_0 and H_1 of $\mathcal{R}(H)$ do not contain edges of Γ, a contradiction.

If $|P| = 5^{n-1}$, then by Proposition 2.6, Γ is a normal cover of the quotient graph Γ_P of Γ relative to P, and $\mathrm{Aut}(\Gamma)/P$ is arc-transitive on Γ_P. Clearly, Γ_P has order 10. By inspecting the list of cubic symmetric graphs of order at most 10000 (see [6]), Γ_P is isomorphic to the Petersen graph. It follows that Γ is not bipartite. So we have $\Gamma = \mathrm{BiCay}(H, R, L, S)$ with $|R| = |L| > 0$. By Proposition 2.4, we may assume that $1 \in S$. Since $|H|$ is odd and Γ has valency 3, one has $R = \{a, a^{-1}\}$ and $L = \{b, b^{-1}\}$. Since Γ is connected, we have $H = \langle a, b\rangle$ by Proposition 2.4. Thus, $d(H) \leq 2$. For convenience of the statement, in the following proof of this lemma, we shall identify $\mathcal{R}(H)$ with H. Clearly, P is maximal in H, so $H' \leq P$. It follows that $P/H' \leq H/H'$ and $P' \leq H'$. If $H' = 1$, then by [11, Theorem 1.1] or [23, Proposition 5.1], we have $H \cong \mathbb{Z}_5$ and Γ is just the Petersen graph. By Magma [3], $\Gamma(5, 1, 0, 0; 2)$ is a symmetric cubic graph of order 10. So $\Gamma \cong \Gamma(5, 1, 0, 0; 2)$.

Now assume that $H' > 1$. Then $P > 1$. Since $P/H' \leq H/H'$, one has $d(P/H') \leq 2$. As $P/H' \cong (P/P')/(H'/P')$ and since either $H' \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ or H' is cyclic, one has $d(P/P') \leq 4$. Since P' is characteristic in P and $P \trianglelefteq \mathrm{Aut}(\Gamma)$, one has $P' \trianglelefteq \mathrm{Aut}(\Gamma)$. Consider the quotient graph $\Gamma_{P'}$ of Γ relative to P'. By Proposition 2.6, Γ is a normal cover of $\Gamma_{P'}$, and $\mathrm{Aut}(\Gamma)/P'$ is arc-transitive on $\Gamma_{P'}$. As $P/P' \trianglelefteq \mathrm{Aut}(\Gamma)/P'$, the quotient graph, denote by Δ, of $\Gamma_{P'}$ relative to P/P' is isomorphic to Γ_P and so isomorphic to the Petersen graph. So $\Gamma_{P'}$ is a normal P/P'-cover of the Petersen graph. Note that P/P' is abelian. By [5, Theorem 7.1], we have $P/P' \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ as $d(P/P') \leq 4$. Since $d(P/H') \leq 2$, one has $P' < H'$. Since either $H' \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ or H' is cyclic, it follows that either $H'/P' \cong \mathbb{Z}_5$ or $P' = 1$ and $H' \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. In the former case, H/P' is an inner abelian 5-group. Clearly, $\Gamma_{P'}$ is a bi-Cayley graph of H/P'. By [16, Lemma 19] and [17, Lemma 4.7], we have $H/P' \trianglelefteq \mathrm{Aut}(\Gamma_{P'})$, and hence $H/P' \trianglelefteq \mathrm{Aut}(\Gamma)/P'$. It follows that $H \trianglelefteq \mathrm{Aut}(\Gamma)$, contrary to the maximality of P. For the latter case, we have $P' = 1$ and $P \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$. So Γ is a cubic symmetric non-Cayley vertex-transitive graph of order 1250. By inspecting the list of cubic symmetric graphs of order at most 10000 (see [6]), up to isomorphism, there exists a unique cubic symmetric non-Cayley graph of order 1250, and by Magam [3], we see that $\Gamma(5,2;2,4)$ (see Construction II) is a cubic symmetric non-Cayley graph of order 1250. Thus, $\Gamma \cong \Gamma(5,2;2,4)$. □

The next lemma will classify cubic non-Cayley vertex-transitive bi-Cayley graphs of a p-group H in case that H' is cyclic, where $p > 3$ is a prime.

Lemma 5.3. *Let $p > 3$ be a prime and let H be a finite p-group such that H' is cyclic. Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$ be a connected cubic bi-Cayley of H. Then Γ is a non-Cayley vertex-transitive graph if and only if Γ is isomorphic to the graphs given in* Construction I.

Proof. From Lemma 4.1 we obtain the sufficiency.

Next we prove the necessity. If Γ is symmetric, then by Lemma 5.2, we have $\Gamma \cong \Gamma(5,1,0,0;2)$. Now assume that Γ is not symmetric. Let $A = \mathrm{Aut}(\Gamma)$. By Lemma 5.1, we have $\mathcal{R}(H) \trianglelefteq A$, and then by Proposition 3.4, we may let $\Gamma = \mathrm{BiCay}(H, \{a, a^{-1}\}, \{b, b^{-1}\}, \{1\})$, and $\Gamma_{\mathcal{M}} \cong \mathrm{Cay}(H, \{a, b, a^{-1}, b^{-1}\})$, where $H = \langle a, b\rangle$. Furthermore, H has no automorphisms interchanging the two pairs (a, a^{-1}) and (b, b^{-1}). As $\Gamma_{\mathcal{M}}$ is A-arc-transitive by Lemma 3.3 and since $\mathcal{R}(H) \trianglelefteq A$, it follows that $\mathrm{Aut}(H, \{a, b, a^{-1}, b^{-1}\})$ is transitive on $\{a, b, a^{-1}, b^{-1}\}$. Consequently, we have $\mathrm{Aut}(H, \{a, b, a^{-1}, b^{-1}\}) \cong \mathbb{Z}_4$. This implies that H has an automorphism, say α, such that $a^\alpha = b$ and $b^\alpha = a^{-1}$. Then $[a,b]^\alpha = [b, a^{-1}] = [a,b]^{a^{-1}}$ and $[a,b]^{\alpha^2} = [a^{-1}, b^{-1}] = [a,b]^{(ab)^{-1}}$.

Since $H = \langle a, b\rangle$, one has $H' = \langle [a,b]^h \mid h \in H\rangle$, and since $d(H') = 1$, one has $H' = \langle [a,b]\rangle$. It follows that $[a,b]^\alpha = [a,b]^k$ and $[a,b]^{\alpha^2} = [a,b]^{k^2}$, where k is an integer coprime with p. Since α has order 4, one has $k^4 \equiv 1 (\mathrm{mod}\ |H'|)$. On the other hand, as we already have $[a,b]^\alpha = [a,b]^{a^{-1}}$. It follows that $k^{p^r} \equiv 1 (\mathrm{mod}\ |H'|)$, where p^r is the order of a. Thus, $k \equiv 1 (\mathrm{mod}\ |H'|)$. This implies that $[a,b] = [a,b]^\alpha = [a,b]^{a^{-1}}$ and then $[a,b] = [a,b]^{\alpha^2} = [a^{-1}, b^{-1}] = [a,b]^{(ab)^{-1}}$. Consequently, we obtain that $[a,b]$ commutes with both a and b, and so $[a,b]$ is contained in the center $Z(H)$ of H. This implies that $H' = \langle [a,b]\rangle$.

Assume that $H/H' \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ with $m \geq n$. Then $a^{p^m}, b^{p^m} \in H'$. Let $c = [a,b]$. Assume that c has order p^s. Since $c \in Z(H)$, one has $c^{p^m} = [a^{p^m}, b] = 1$ and hence $m \geq s$. Suppose that $a^{p^m} = c^{p^t}$ with $t \leq s$. Then $b^{p^m} = (a^{p^m})^\alpha = (c^{p^t})^\alpha = c^{p^t} = a^{p^m}$, implying that α fixes a^{p^m}. Since $a^{\alpha^2} = a^{-1}$, one has $a^{-p^m} = a^{p^m}$ and so $a^{p^m} = 1$. Thus, $a^{p^m} = b^{p^m} = 1$. Then $\langle a\rangle \cap \langle c\rangle = \langle b\rangle \cap \langle c\rangle = 1$. As $H/H' \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, one has $|\langle aH'\rangle\langle bH'\rangle| = p^{m+n}$. It follows that $|\langle aH'\rangle \cap \langle bH'\rangle| = p^{m-n}$, and hence $\langle a^{p^n}H'\rangle = \langle b^{p^n}H'\rangle$. Then $a^{p^n} = b^{\ell p^n}c^\lambda$, with $\ell \in \mathbb{Z}^*_{p^{m-n}}$ and $\lambda \in \mathbb{Z}_{p^s}$. Now we have $b^{p^n} = (a^{p^n})^\alpha = (b^{\ell p^n}c^\lambda)^\alpha = a^{-\ell p^n}c^\lambda$. Then $a^{p^n} = b^{\ell p^n}c^\lambda = a^{-\ell^2 p^n}c^{2\lambda}$. Since $\langle a\rangle \cap \langle c\rangle = 1$, one has $c^{2\lambda} = 1$ and hence $c^\lambda = 1$. So $a^{p^n} = a^{-\ell^2 p^n}$. Hence, either $m = n$ or $\ell^2 \equiv -1 (\mathrm{mod}\ p^{m-n})$. If $m = n$, then we have

$$H = \langle a, b, c \mid a^{p^m} = b^{p^m} = c^{p^s} = 1, c = [a,b], [a,c] = [b,c] = 1\rangle.$$

It is easy to see that $\beta\colon a \mapsto b, b \mapsto a$ induces an automorphism of H. Clearly, $\beta \in \mathrm{Aut}(H, \{a, b, a^{-1}, b^{-1}\})$. This, however, is impossible since $\mathrm{Aut}(H, \{a, b, a^{-1}, b^{-1}\} = \langle\alpha\rangle \cong \mathbb{Z}_4$.

Now assume that $m > n$ and let $x = a$, $y = b^\ell$ and $z = [x, y]$. Then $z = c^{p^\ell}$, and

$$H = H(p, m, n, s; t) = \langle x, y, z \mid x^{p^m} = z^{p^s} = 1, z = [x, y], [x, z] = [y, z] = 1, \\ x^{p^n} = y^{p^n}\rangle.$$

Let $t \in \mathbb{Z}^*_{p^{m-n}}$ be such that $t\ell \equiv 1 (\mathrm{mod}\, p^{m-n})$. Then $b = y^t$ and hence we obtain that $\Gamma = \mathrm{BiCay}(H, \{x, x^{-1}\}, \{y^t, y^{-t}\}, \{1\}) = \Gamma(p, m, n, s; t)$. □

Finally, we shall classify cubic non-Cayley vertex-transitive bi-Cayley graphs of a p-group H in case $H' \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $p > 3$ is prime.

Lemma 5.4. *Let $p > 3$ be a prime and let H be a finite p-group such that $H' \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$ be a connected cubic bi-Cayley of H. Then Γ is a non-Cayley vertex-transitive graph if and only if Γ is isomorphic to one of the graphs given* Constructions II – V.

Proof. The sufficiency can be obtained from Lemmas 4.2 – 4.5.

In the following, we prove the necessity. Let $A = \mathrm{Aut}(\Gamma)$. If Γ is symmetric, then by Lemma 5.2, we have $\Gamma \cong \Gamma(5, 2; 2, 4)$ (see Construction II). Now assume that Γ is not symmetric. By Lemma 5.1, we have $\mathcal{R}(H) \trianglelefteq A$, and then by Proposition 3.4, we may let $\Gamma = \mathrm{BiCay}(H, \{a, a^{-1}\}, \{b, b^{-1}\}, \{1\})$, and $\Gamma_{\mathcal{M}} \cong \mathrm{Cay}(H, \{a, b, a^{-1}, b^{-1}\})$, where $H = \langle a, b\rangle$. Furthermore, H has no automorphisms interchanging the two pairs (a, a^{-1}) and (b, b^{-1}). As $\Gamma_{\mathcal{M}}$ is A-arc-transitive by Lemma 3.3 and since $\mathcal{R}(H) \trianglelefteq A$, it follows that $\mathrm{Aut}(H, \{a, b, a^{-1}, b^{-1}\})$ is transitive on $\{a, b, a^{-1}, b^{-1}\}$. Consequently, we have $\mathrm{Aut}(H, \{a, b, a^{-1}, b^{-1}\}) \cong \mathbb{Z}_4$. This implies that H has an automorphism, say α, such that $a^\alpha = b$ and $b^\alpha = a^{-1}$.

Since $H' \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $H' = \langle [a, b]^h \mid h \in H\rangle$, $[a, b]$ has order p and $[a, b]$ is not in the center of H. Let

$$H_3 = \langle [[g, h], k]^r \mid g, h, k \in \{a, b\}, r \in H\rangle.$$

Then $H_3 \trianglelefteq H$. Since H is a p-group, one has $H_3 < H'$. So $H_3 \cong \mathbb{Z}_p$ as $H' \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Since H is a p-group, one has $H_3 \cap Z(H) > 1$, implying that $H_3 \leq Z(H)$. As $[a, b]$ is not in the center of H, one has $H' \cap Z(H) = H_3$. Let $c = [a, b]$. We shall first prove the following claim:

Claim 1 There exists $k \in \mathbb{Z}^*_p$ such that $k^2 \equiv -1 (\mathrm{mod}\, p)$ and $[b, c] = [a, c]^k$.

A direct computation shows that $c^\alpha = c^{a^{-1}}$ and $c^{\alpha^2} = c^{(ab)^{-1}}$. Using the fact that $H_3 \leq Z(H)$, we obtain the following:

$$[a, c]^\alpha = [b, c^{a^{-1}}] = [b, [a^{-1}, c^{-1}]c] = [b, c],$$
$$[a, c]^{\alpha^2} = [b, c]^\alpha = [a^{-1}, c^{a^{-1}}] = [a^{-1}, [a^{-1}, c^{-1}]c] = [a, c]^{-1},$$
$$[a, c]^{\alpha^3} = [b, c]^{-1}.$$

Then $[a, c], [b, c] \neq 1$, and since $H_3 \cong \mathbb{Z}_p$, one has $H_3 = \langle [a, c]\rangle = \langle [b, c]\rangle$. As α is an automorphism of H of order 4, one has $H_3^\alpha = H_3$. We may let $[a, c]^\alpha = [a, c]^k$ with

$k \in \mathbb{Z}_p^*$. Since $[a,c]^\alpha = [b,c]$, we have $[b,c] = [a,c]^k$, and since $[a,c]^{\alpha^2} = [a,c]^{-1}$, one has $k^2 \equiv -1 (\mathrm{mod}\, p)$, as claimed.

Assume that $a^{p^m} \in H'$ for some integer $m > 0$. Then $b^{p^m} = (a^{p^m})^\alpha \in H'$. Then $(H/H_3)' = H'/H_3 = \langle cH_3 \rangle \cong \mathbb{Z}_p$. Clearly, $\Phi(H') = 1$. If H/H_3 is metacyclic, then by [2, Theorem 2.3], H is metacyclic. This, however is impossible since $H' \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Thus, H/H_3 is not metacyclic. Since $H'/H_3 \cong \mathbb{Z}_p$, it follows that H/H_3 is inner abelian (see, for example, [2, Lemma 2.5]). Clearly, $H/H_3 = \langle aH_3 \rangle (\langle bH_3, cH_3 \rangle)$. Assume that $\langle aH_3 \rangle \cap \langle bH_3, cH_3 \rangle \cong \mathbb{Z}_{p^n}$ with $n < m$. We shall consider the following two cases:

Case 1: $n = 0$.

In this case, H/H_3 is generated by aH_3, bH_3, cH_3 with the following defining relations:

$$a^{p^m} H_3 = b^{p^m} H_3 = c^p H_3 = H_3, cH_3 = [aH_3, bH_3], [cH_3, aH_3] = [cH_3, bH_3] = H_3.$$

If $\langle a \rangle \cap H_3 \neq 1$, then a has order p^{m+1} and $H_3 = \langle a^{p^m} \rangle$. Then $[a,c] = a^{ip^m}$ for some $i \in \mathbb{Z}_p^*$. As $[a,c]^\alpha = [b,c]$ and $a^\alpha = b$, one has $[b,c] = b^{ip^m}$. Again, since $a^\alpha = b$, b also has order p^{m+1} and $H_3 = \langle b^{p^m} \rangle$. By Claim 1, we have $a^{p^m} = b^{kp^m}$ with $k \in \mathbb{Z}_p^*$ and $k^2 \equiv -1 (\mathrm{mod}\, p)$. This implies that H is isomorphic to the following group:

$$H(p,m;k,i) = \langle a,b,c \mid a^{p^{m+1}} = c^p = 1, a^{p^m} = b^{kp^m}, c = [a,b], [a,c] = a^{ip^m}, [b,c] = b^{ip^m} \rangle.$$

This is just the group given in Construction II. So $\Gamma \cong \Gamma(p,m;k,i)$.

If $\langle a \rangle \cap H_3 = 1$, then a has order p^m. Since $a^\alpha = b$, b also has order p^m. Let $d = [a,c]$. By Claim 1, we have $d^\alpha = [b,c] = d^k$ with $k \in \mathbb{Z}_p^*$ and $k^2 \equiv -1 (\mathrm{mod}\, p)$. This implies that H is isomorphic to the following group as given in Construction IV:

$$\langle a,b,c,d \mid a^{p^m} = b^{p^m} = c^p = d^p = 1, c = [a,b], [a,c] = d, [b,c] = d^k, [a,d] = [b,d] = 1 \rangle.$$

It follows that $\Gamma \cong \Delta(p,m;k)$.

Case 2: $n > 0$.

In this case, we have $a^{p^{m-n}} H_3 = b^{tp^{m-n}} dH_3$ with $t \in \mathbb{Z}_{p^n}^*$ and $d \in \langle c \rangle$. Then $dH_3 = a^{p^{m-n}} b^{-tp^{m-n}} H_3 \in \langle ab^{-t} H_3 \rangle$ as H/H_3 is inner abelian. If $dH_3 \neq H_3$, then we have $H'/H_3 = \langle dH_3 \rangle \leq \langle ab^{-t} H_3 \rangle$ and then $\langle ab^{-t} H_3 \rangle \trianglelefteq H/H_3$. This implies that H/H_3 is metacyclic since $H/H_3 = \langle ab^{-t} H_3, bH_3 \rangle$. This is a contradiction. Thus, $dH_3 = H_3$ and so $d = 1$ as $d \in \langle c \rangle$. It follows that $a^{p^{m-n}} H_3 = b^{tp^{m-n}} H_3$ with $t \in \mathbb{Z}_{p^n}^*$. In particular, $\langle a^{p^{m-n}}, H_3 \rangle = \langle b^{p^{m-n}}, H_3 \rangle$.

Since $a^\alpha = b$ and $b^\alpha = a^{-1}$, it follows that $b^{p^{m-n}} H_3 = (a^{p^{m-n}} H_3)^\alpha = (b^{tp^{m-n}} H_3)^\alpha = a^{-tp^{m-n}} H_3 = b^{-t^2 p^{m-n}} H_3$. Consequently, we obtain that $t^2 \equiv -1 (\mathrm{mod}\, p^n)$.

Now we see that H/H_3 is generated by aH_3, bH_3, cH_3 with the following defining relations:

$$b^{p^m} H_3 = c^p H_3 = H_3, a^{p^{m-n}} H_3 = b^{tp^{m-n}} H_3, cH_3 = [aH_3, bH_3], [cH_3, aH_3] = [cH_3, bH_3] = H_3.$$

If $\langle a \rangle \cap H_3 \neq 1$, then a has order p^{m+1} and $H_3 = \langle a^{p^m} \rangle$. Since $0 < n < m$, one has $H_3 \leq \langle a^{p^{m-n}} \rangle \cap \langle b^{tp^{m-n}} \rangle$. As $\langle a^{p^{m-n}}, H_3 \rangle = \langle b^{p^{m-n}}, H_3 \rangle$, one has $\langle a^{p^{m-n}} \rangle = \langle b^{p^{m-n}} \rangle$.

Then $a^{p^{m-n}} = b^{sp^{m-n}}$ for some $s \in \mathbb{Z}_{p^n}^*$. As $(a^{p^{m-n}})^\alpha = b^{p^{m-n}}$ and $(b^{p^{m-n}})^\alpha = a^{-p^{m-n}}$, we have $s^2 \equiv -1 (\mathrm{mod}\, p^n)$. Since $H_3 = \langle [a,c] \rangle$, $[a,c] = a^{ip^m}$ for some $i \in \mathbb{Z}_p^*$. As $[a,c]^\alpha = [b,c]$ and $a^\alpha = b$, one has $[b,c] = b^{ip^m}$.This implies that H is isomorphic to the following group:

$$H(p,m,n;s,i) = \langle a,b,c \mid a^{p^{m+1}} = c^p = 1, a^{p^{m-n}} = b^{sp^{m-n}}, c = [a,b], [a,c] = a^{ip^m}, [b,c] = b^{ip^m} \rangle.$$

This is just the group given in Construction III, and so $\Gamma \cong \Gamma(p,m,n;s,i)$.

If $\langle a \rangle \cap H_3 = 1$, then a has order p^m. Since $a^\alpha = b$, b also has order p^m. Let $d = [a,c]$. By Claim 1, we have $d^\alpha = [b,c] = d^k$ with $k \in \mathbb{Z}_p^*$ and $k^2 \equiv -1 (\mathrm{mod}\, p)$. As $a^{p^{m-n}} H_3 = b^{tp^{m-n}} H_3$, one has $a^{p^{m-n}} = b^{tp^{m-n}} d^j$ for some $j \in \mathbb{Z}_p$. Considering the image of $a^{p^{m-n}} = b^{tp^{m-n}} d^j$ under α, we have $t \equiv k (\mathrm{mod}\, p)$.

This implies that H is isomorphic to the following group which is just the group given in Construction V:

$$\begin{aligned} G(p,m,n;t,k,j) = \ & \langle a,b,c,d \mid a^{p^m} = c^p = d^p = 1, a^{p^{m-n}} = b^{tp^{m-n}} d^j, \\ & c = [a,b], [a,c] = d, [b,c] = d^k, [a,d] = [b,d] = 1 \rangle. \end{aligned}$$

Consequently, we obtain that $\Gamma \cong \Theta(p,m,n;t,k,j)$. This completes the proof. □

6 Proof of Theorem 1.2

In this final section, we shall prove Theorem 1.2 which gives a classification of cubic VNC-graphs of order p^4 for each prime p.

Proof of Theorem 1.2. By Lemmas 4.1, 4.2 and 4.4, we see that the graphs $\Gamma(p,m,n,s;t)$ $(m+n+3 = 4, m > n \geq 0, s \geq 0)$, $\Gamma(p,1;k,i)$ and $\Delta(p,1;k)$ are connected cubic non-Cayley vertex-transitive graphs of order $2p^4$. So we only need to prove the sufficiency.

Let Γ be a connected cubic non-Cayley vertex-transitive graph of order $2p^4$. Assume first that Γ is symmetric. By [8, Corollary 3.4], every connected cubic symmetric graph of order $2p^n$ is a Cayley graph whenever $p \geq 7$ and $n \geq 1$, and by [24, Theorem 1.2], every connected cubic symmetric graph order a 2-power is a Cayley graph. So we have $p = 3$ or 5. Then by inspecting the list of cubic symmetric graphs of order at most 10000 (see [6]), we obtain that $p = 5$, and up to isomorphism, there exists a unique cubic symmetric non-Cayley graph of order 1250. Thus, $\Gamma \cong \Gamma(5,2;2,4)$ by Lemma 5.2.

Assume now that Γ is not symmetric. If $p \leq 3$, then by inspecting the list of cubic vertex-transitive graphs of order up to 1280 (see [15]), we see that every cubic vertex-transitive graph of order 32 or 162 is a Cayley graph, a contradiction. Thus, we may assume that $p > 3$. Since Γ is not symmetric, the stabilizer $\mathrm{Aut}(\Gamma)_v$ of any vertex $v \in V(\Gamma)$ in $\mathrm{Aut}(\Gamma)$ is a 2-group. Let P be a Sylow p-subgroup. Then P acts semiregularly on $V(\Gamma)$ with two orbits. So Γ is a bi-Cayley graph of P. By Lemma 5.1, we have $P \trianglelefteq \mathrm{Aut}(\Gamma)$, and then by Proposition 3.4, we may let $\Gamma \cong \mathrm{BiCay}(P, \{a, a^{-1}\}, \{b, b^{-1}\}, \{1\})$, where $P = \langle a, b \rangle$. Then $|H'| \leq p^2$, and so either H' is cyclic or $H' \cong \mathbb{Z}_p^2$. By Lemmas 5.3 and 5.4, we have Γ is isomorphic to $\Gamma(p,m,n,s;t)(m+n+3 = 4, m > n \geq 0, s \geq 0)$, $\Gamma(p,1;k,i)$ or $\Delta(p,1;k)$ since Γ has order $2p^4$. □

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Using a q-shuffle algebra to describe the basic module $V(\Lambda_0)$ for the quantized enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$*

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Abstract

We consider the quantized enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and its basic module $V(\Lambda_0)$. This module is infinite-dimensional, irreducible, integrable, and highest-weight. We describe $V(\Lambda_0)$ using a q-shuffle algebra in the following way. Start with the free associative algebra $\mathbb{V}$ on two generators x, y. The standard basis for $\mathbb{V}$ consists of the words in x, y. In 1995 M. Rosso introduced an associative algebra structure on $\mathbb{V}$, called a q-shuffle algebra. For $u, v \in \{x, y\}$ their q-shuffle product is $u \star v = uv + q^{(u,v)}vu$, where $(u, v) = 2$ (resp. $(u, v) = -2$) if $u = v$ (resp. $u \neq v$). Let $\mathbb{U}$ denote the subalgebra of the q-shuffle algebra $\mathbb{V}$ that is generated by x, y. Rosso showed that the algebra $\mathbb{U}$ is isomorphic to the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$. In our first main result, we turn $\mathbb{U}$ into a $U_q(\widehat{\mathfrak{sl}}_2)$-module. Let $\mathbf{U}$ denote the $U_q(\widehat{\mathfrak{sl}}_2)$-submodule of $\mathbb{U}$ generated by the empty word. In our second main result, we show that the $U_q(\widehat{\mathfrak{sl}}_2)$-modules $\mathbf{U}$ and $V(\Lambda_0)$ are isomorphic. Let $\mathbf{V}$ denote the subspace of $\mathbb{V}$ spanned by the words that do not begin with y or xx. In our third main result, we show that $\mathbf{U} = \mathbb{U} \cap \mathbf{V}$.

Keywords: Quantized enveloping algebra, q-Serre relations, basic module, q-shuffle algebra.

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1 Introduction

The quantized enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is associative, noncommutative, and infinite dimensional. A presentation by generators and relations is given in Appendix B below. The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ has a subalgebra U_q^+, called the positive part. The algebra U_q^+ has a presentation involving two generators A, B and two relations, called the q-Serre relations:

$$[A,[A,[A,B]_q]_{q^{-1}}]=0, \qquad [B,[B,[B,A]_q]_{q^{-1}}]=0.$$

Both U_q^+ and $U_q(\widehat{\mathfrak{sl}}_2)$ are well known in algebraic combinatorics [4, 21, 26], representation theory [11, 12, 15, 47], and mathematical physics [6, 7, 17, 27]. In the following paragraphs, we describe a few situations in which U_q^+ and $U_q(\widehat{\mathfrak{sl}}_2)$ play a role.

There is an object in algebraic combinatorics called a tridiagonal pair [21]. Roughly speaking, this is a pair of diagonalizable linear maps on a finite-dimensional vector space, that each act on the eigenspaces of the other one in a block-tridiagonal fashion. According to [21, Example 1.7], for a finite-dimensional irreducible U_q^+-module V on which the generators A, B are not nilpotent, the pair A, B act on V as a tridiagonal pair. The resulting tridiagonal pair is said to have q-geometric type or q-Serre type. This type of tridiagonal pair is described in [1–3, 22–26, 36, 48].

Another object in algebraic combinatorics is a partially ordered set Y called the Young lattice [35, page 288]. The elements of Y are the Young diagrams (partitions), and the partial order is given by diagram inclusion. Define a vector space V consisting of the formal linear combinations of Y. In the Hayashi realization [4, Theorem 10.6] the vector space V becomes an integrable $U_q(\widehat{\mathfrak{sl}}_2)$-module with the following features. Each Young diagram λ is a weight vector. The $U_q(\widehat{\mathfrak{sl}}_2)$-generators E_0, E_1, F_0, F_1 act on λ as follows. Color the boxes of λ alternating blue and red, with the top left box colored blue. The generator E_0 (resp. E_1) sends λ to a linear combination of the Young diagrams μ obtained from λ by removing a blue box (resp. red box). In this linear combination the μ-coefficient is a power of q that depends on the location of the box λ/μ. Similarly, F_0 (resp. F_1) sends λ to a linear combination of the Young diagrams μ obtained from λ by adding a blue box (resp. red box). In this linear combination the μ-coefficient is a power of q that depends on the location of the box μ/λ. The $U_q(\widehat{\mathfrak{sl}}_2)$-submodule of V generated by the empty Young diagram is denoted $V(\Lambda_0)$ and called the basic representation. The $U_q(\widehat{\mathfrak{sl}}_2)$-module $V(\Lambda_0)$ is infinite-dimensional, irreducible, integrable, and highest-weight. For more detail about $V(\Lambda_0)$ see [4, Chapter 10], [20, Section 9], [27, Chapter 5].

Next we recall an embedding, due to M. Rosso [33, 34] of U_q^+ into a q-shuffle algebra. Start with a free associative algebra $\mathbb{V}$ on two generators x, y. These generators are called letters. For $n \geq 0$, a word of length n in $\mathbb{V}$ is a product of letters $\ell_1\ell_2\cdots\ell_n$. We interpret the word of length 0 to be the multiplicative identity in $\mathbb{V}$; this word is called trivial and denoted by $\mathbf{1}$. The words in $\mathbb{V}$ form a basis for the vector space $\mathbb{V}$; this basis is called standard. In [33, 34] M. Rosso introduced an associative algebra structure on $\mathbb{V}$, called a q-shuffle algebra. For letters u, v their q-shuffle product is $u \star v = uv + q^{(u,v)}vu$, where $(u, v) = 2$ (resp. $(u, v) = -2$) if $u = v$ (resp. $u \neq v$). In [34, Theorem 15] Rosso gave an injective algebra homomorphism $\natural$ from U_q^+ into the q-shuffle algebra $\mathbb{V}$, that sends $A \mapsto x$ and $B \mapsto y$.

We mention some applications of the map $\natural\colon U_q^+ \to \mathbb{V}$. In [16] I. Damiani obtained a Poincaré-Birkhoff-Witt (or PBW) basis for U_q^+ whose elements are defined recursively. In [11, Proposition 6.1] J. Beck obtained another PBW basis for U_q^+ by adjusting some of the elements in the Damiani PBW basis. In [40] (resp. [45]) we applied the map $\natural$ to the Damiani (resp. Beck) PBW basis, and expressed the images in the standard basis for $\mathbb{V}$. We gave the images in closed form [40, Theorem 1.7], [45, Theorem 7.1]. The map $\natural$ is used in [39] to define the alternating elements of U_q^+. In [39, Theorem 10.1] a set of alternating elements is shown to form a PBW basis for U_q^+. This PBW basis is said to be alternating [39, Definition 10.3]. In [38] we used the alternating elements to obtain a central extension $\mathcal{U}_q^+$ of U_q^+. The algebra $\mathcal{U}_q^+$ is defined by generators and relations. These generators, said to be alternating, are in bijection with the alternating elements of U_q^+. By [38, Lemma 3.3] there exists a surjective algebra homomorphism $\mathcal{U}_q^+ \to U_q^+$ that sends each alternating generator of $\mathcal{U}_q^+$ to the corresponding alternating element in U_q^+. In [38, Lemma 3.6] this homomorphism is adjusted to obtain an algebra isomorphism $\mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$ where $\mathbb{F}$ is the ground field and $\{z_n\}_{n=1}^{\infty}$ are mutually commuting indeterminates. By [38, Theorem 10.2] the alternating generators form a PBW basis for $\mathcal{U}_q^+$. The algebra $\mathcal{U}_q^+$ is called the alternating central extension of U_q^+ [38, 46]. We remark that $\mathcal{U}_q^+$ is related to the work of Baseilhac, Koizumi, Shigechi concerning the q-Onsager algebra and integrable lattice models [8, 10]. See [5–7, 9, 37, 41–44, 46] for related work.

Turning to the present paper, our goal is to describe the basic $U_q(\widehat{\mathfrak{sl}}_2)$-module $V(\Lambda_0)$ using the q-shuffle algebra $\mathbb{V}$. We have three main results, which are summarized below. Let $\mathrm{End}(\mathbb{V})$ denote the algebra consisting of the linear maps from $\mathbb{V}$ to $\mathbb{V}$. We now define some maps X, Y, K in $\mathrm{End}(\mathbb{V})$. The map X (resp. Y) is the automorphism of the free algebra $\mathbb{V}$ that sends $x \mapsto qx$ and $y \mapsto y$ (resp. $x \mapsto x$ and $y \mapsto qy$). Define $K = X^2Y^{-2}$. Define the maps A_L^*, B_L^*, A_R^*, B_R^* in $\mathrm{End}(\mathbb{V})$ that send $\mathbf{1} \mapsto 0$ and for a nontrivial word $w = \ell_1\ell_2 \cdots \ell_n$ in $\mathbb{V}$,

$$A_L^* w = \ell_2 \cdots \ell_n \delta_{\ell_1,x}, \qquad B_L^* w = \ell_2 \cdots \ell_n \delta_{\ell_1,y},$$
$$A_R^* w = \ell_1 \cdots \ell_{n-1} \delta_{\ell_n,x}, \qquad B_R^* w = \ell_1 \cdots \ell_{n-1} \delta_{\ell_n,y}.$$

Here $\delta_{r,s}$ is the Kronecker delta. Define the maps A_ℓ, B_ℓ, A_r, B_r in $\mathrm{End}(\mathbb{V})$ such that for $v \in \mathbb{V}$,

$$A_\ell v = x \star v, \qquad B_\ell v = y \star v, \qquad A_r v = v \star x, \qquad B_r v = v \star y.$$

Let $\mathbb{U}$ denote the subalgebra of the q-shuffle algebra $\mathbb{V}$ that is generated by x, y. By construction the map $\natural : U_q^+ \to \mathbb{U}$ is an algebra isomorphism. Our first main result is that $\mathbb{U}$ becomes a $U_q(\widehat{\mathfrak{sl}}_2)$-module on which the $U_q(\widehat{\mathfrak{sl}}_2)$-generators act as follows:

generator	E_0	F_0	$K_0^{\pm 1}$	E_1	F_1	$K_1^{\pm 1}$	$D^{\pm 1}$
action on $\mathbb{U}$	A_R^*	$\frac{qA_rK^{-1} - q^{-1}A_\ell}{q - q^{-1}}$	$q^{\pm 1}K^{\mp 1}$	B_R^*	$\frac{B_rK - B_\ell}{q - q^{-1}}$	$K^{\pm 1}$	$X^{\mp 1}$

Let $\mathbf{U}$ denote the submodule of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$ that is generated by the vector $\mathbf{1}$. Our second main result is that the $U_q(\widehat{\mathfrak{sl}}_2)$-modules $\mathbf{U}$ and $V(\Lambda_0)$ are isomorphic. Let $\mathbf{V}$ denote the intersection of the kernel of B_L^* and the kernel of $(A_L^*)^2$. The vector space $\mathbf{V}$ has a basis consisting of the words in $\mathbb{V}$ that do not begin with y or xx. Note that the sum $\mathbf{V} = \mathbb{F}\mathbf{1} + \mathbb{F}x + xy\mathbb{V}$ is direct. Our third main result is that $\mathbf{U} = \mathbb{U} \cap \mathbf{V}$.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we recall the algebra U_q^+ and discuss its basic properties. In Section 4 we describe the free algebra $\mathbb{V}$. In Section 5 we describe the maps X, Y, K in $\mathrm{End}(\mathbb{V})$. In Section 6 we describe the maps $A^*_L, B^*_L, A^*_R, B^*_R$ in $\mathrm{End}(\mathbb{V})$. In Section 7 we describe the q-shuffle algebra $\mathbb{V}$. In Section 8 we describe the maps A_ℓ, B_ℓ, A_r, B_r in $\mathrm{End}(\mathbb{V})$. In Section 9 we describe the subalgebra $\mathbb{U}$ of the q-shuffle algebra $\mathbb{V}$. In Sections 10, 11 we give our main results, which are Theorems 10.1, 10.7, 11.11. In Section 12 we describe some variations on Theorem 10.1. In Appendix A we display some relations that are satisfied by the maps from the main body of the paper. In Appendix B we give a presentation of $U_q(\widehat{\mathfrak{sl}}_2)$. In Appendix C we give a basis for some of the weight spaces of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbf{U}$. In Appendix D we show how the $U_q(\widehat{\mathfrak{sl}}_2)$-generators act on the bases in Appendix C. In Appendix E we discuss a linear algebraic situation that comes up in Section 11.

2 Preliminaries

We now begin our formal argument. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Let $\mathbb{F}$ denote a field with characteristic zero. Throughout this paper, every vector space we discuss is understood to be over $\mathbb{F}$. Every algebra we discuss is understood to be associative, over $\mathbb{F}$, and have a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra. Let $\mathcal{A}$ denote an algebra. An *automorphism* of $\mathcal{A}$ is an algebra isomorphism $\mathcal{A} \to \mathcal{A}$. The *opposite algebra* $\mathcal{A}^{\mathrm{opp}}$ consists of the vector space $\mathcal{A}$ and the multiplication map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(a, b) \mapsto ba$. An *antiautomorphism* of $\mathcal{A}$ is an algebra isomorphism $\mathcal{A} \to \mathcal{A}^{\mathrm{opp}}$.

We recall a few concepts from linear algebra. Let V denote a vector space, and consider an $\mathbb{F}$-linear map $T\colon V \to V$. The map T is said to be *nilpotent* whenever there exists a positive integer n such that $T^n = 0$. The map T is said to be *locally nilpotent* whenever for all $v \in V$ there exists a positive integer n such that $T^n v = 0$. If T is nilpotent then T is locally nilpotent. If T is locally nilpotent and the dimension of V is finite, then T is nilpotent.

Throughout the paper, fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n \in \mathbb{N}.$$

3 The positive part of $U_q(\widehat{\mathfrak{sl}}_2)$

Later in the paper, we will discuss the quantized enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$. For now, we consider a subalgebra U_q^+ of $U_q(\widehat{\mathfrak{sl}}_2)$, called the positive part. Shortly we will give a presentation of U_q^+ by generators and relations.

For elements $\mathcal{X}, \mathcal{Y}$ in any algebra, define their commutator and q-commutator by

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}, \qquad [\mathcal{X}, \mathcal{Y}]_q = q\mathcal{X}\mathcal{Y} - q^{-1}\mathcal{Y}\mathcal{X}.$$

Note that

$$[\mathcal{X}, [\mathcal{X}, [\mathcal{X}, \mathcal{Y}]_q]_{q^{-1}}] = \mathcal{X}^3\mathcal{Y} - [3]_q\mathcal{X}^2\mathcal{Y}\mathcal{X} + [3]_q\mathcal{X}\mathcal{Y}\mathcal{X}^2 - \mathcal{Y}\mathcal{X}^3. \tag{3.1}$$

Definition 3.1 (See [30, Corollary 3.2.6]). Define the algebra U_q^+ by generators A, B and relations

$$[A, [A, [A, B]_q]_{q^{-1}}] = 0, \qquad [B, [B, [B, A]_q]_{q^{-1}}] = 0. \tag{3.2}$$

We call U_q^+ the *positive part of* $U_q(\widehat{\mathfrak{sl}}_2)$. The relations (3.2) are called the *q-Serre relations*.

We mention some symmetries of U_q^+.

Lemma 3.2. *There exists an automorphism σ of U_q^+ that sends $A \leftrightarrow B$. Moreover $\sigma^2 =$ id, where id denotes the identity map.*

Lemma 3.3 (See [40, Lemma 2.2])**.** *There exists an antiautomorphism $\dagger$ of U_q^+ that fixes each of A, B. Moreover $\dagger^2 =$ id.*

Lemma 3.4 (See [41, Lemma 3.4])**.** *The maps σ, $\dagger$ commute.*

Definition 3.5. Let τ denote the composition of σ and $\dagger$. Note that τ is an antiautomorphism of U_q^+ that sends $A \leftrightarrow B$. We have $\tau^2 = \text{id}$.

Next we describe a grading for the algebra U_q^+. The q-Serre relations are homogeneous in both A and B. Therefore, the algebra U_q^+ has a $\mathbb{N}^2$-grading for which A and B are homogeneous, with degrees $(1,0)$ and $(0,1)$ respectively. For $(r,s) \in \mathbb{N}^2$ let $U_q^+(r,s)$ denote the (r,s)-homogeneous component of the grading. The dimension of $U_q^+(r,s)$ is described by a generating function, as we now discuss. Let t and u denote commuting indeterminates.

Definition 3.6. Define the generating function

$$\Phi(t,u) = \prod_{n=1}^{\infty} \frac{1}{1-t^n u^{n-1}} \frac{1}{1-t^n u^n} \frac{1}{1-t^{n-1}u^n}.$$

Using $(1-z)^{-1} = 1 + z + z^2 + \cdots$ we expand the above generating function as a power series:

$$\Phi(t,u) = \sum_{(r,s)\in\mathbb{N}^2} d_{r,s} t^r u^s, \qquad\qquad d_{r,s} \in \mathbb{N}.$$

For notational convenience, define $d_{r,-1} = 0$ and $d_{-1,s} = 0$ for $r, s \in \mathbb{N}$.

Example 3.7 (See [39, Example 3.4])**.** For $0 \le r, s \le 6$ we display $d_{r,s}$ in the (r,s)-entry of the matrix below:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 & 3 & 3 & 3 \\ 1 & 3 & 6 & 8 & 9 & 9 & 9 \\ 1 & 3 & 8 & 14 & 19 & 21 & 22 \\ 1 & 3 & 9 & 19 & 32 & 42 & 48 \\ 1 & 3 & 9 & 21 & 42 & 66 & 87 \\ 1 & 3 & 9 & 22 & 48 & 87 & 134 \end{pmatrix}$$

We have $\Phi(t,u) = \Phi(u,t)$. Moreover $d_{r,s} = d_{s,r}$ for $(r,s) \in \mathbb{N}^2$.

Lemma 3.8 (See [39, Definition 3.2, Corollary 3.7])**.** *For $(r,s) \in \mathbb{N}^2$ we have*

$$d_{r,s} = \dim U_q^+(r,s).$$

Our next goal is to show that $d_{r,s-1} \leq d_{r,s}$ and $d_{r-1,s} \leq d_{r,s}$ for $(r, s) \in \mathbb{N}^2$. To reach the goal, we modify the generating function $\Phi(t, u)$ in the following way.

Definition 3.9. Define the generating function

$$\Delta(t, u) = \prod_{n=1}^{\infty} \frac{1}{1 - t^n u^{n-1}} \frac{1}{1 - t^n u^n} \frac{1}{1 - t^n u^{n+1}}. \tag{3.3}$$

Lemma 3.10. *We have* $\Delta(t, u) = \Phi(t, u)(1 - u)$ *and* $\Delta(u, t) = \Phi(t, u)(1 - t)$. *Moreover*

$$\Delta(t, u) = \sum_{(r,s)\in\mathbb{N}^2} \big(d_{r,s} - d_{r,s-1}\big) t^r u^s, \qquad \Delta(u, t) = \sum_{(r,s)\in\mathbb{N}^2} \big(d_{r,s} - d_{r-1,s}\big) t^r u^s.$$

Proof. Use Definitions 3.6, 3.9. □

Lemma 3.11. *For* $(r, s) \in \mathbb{N}^2$ *we have* $d_{r,s-1} \leq d_{r,s}$ *and* $d_{r-1,s} \leq d_{r,s}$.

Proof. Expand the right-hand side of (3.3) as a power series. In this power series, the coefficient of $t^r u^s$ is nonnegative for $(r, s) \in \mathbb{N}^2$. The result follows in view of Lemma 3.10. □

Our next general goal is to compute $\max\{d_{r,s} | s \in \mathbb{N}\}$ for $r \in \mathbb{N}$, and $\max\{d_{r,s} | r \in \mathbb{N}\}$ for $s \in \mathbb{N}$. To reach the goal, we will use the concept of a partition.
For $n \in \mathbb{N}$, a *partition of* n is a sequence $\lambda = \{\lambda_i\}_{i=1}^{\infty}$ of natural numbers such that $\lambda_i \geq \lambda_{i+1}$ for $i \geq 1$ and $n = \sum_{i=1}^{\infty} \lambda_i$. Let p_n denote the number of partitions of n. For example,

n	0	1	2	3	4	5	6
p_n	1	1	2	3	5	7	11

Define the generating function for partitions:

$$p(t) = \sum_{n\in\mathbb{N}} p_n t^n. \tag{3.4}$$

The following result is well known; see for example [13, Theorem 8.3.4].

$$p(t) = \prod_{n=1}^{\infty} \frac{1}{1 - t^n}. \tag{3.5}$$

We expand the generating function $\big(p(t)\big)^3$ as a power series:

$$\big(p(t)\big)^3 = \sum_{n\in\mathbb{N}} \mu_n t^n, \qquad \mu_n \in \mathbb{N}. \tag{3.6}$$

Consider the coefficients $\{\mu_n\}_{n\in\mathbb{N}}$. For example,

n	0	1	2	3	4	5	6
μ_n	1	3	9	22	51	108	221

Proposition 3.12. *For $r \in \mathbb{N}$ we have*

$$\mu_r = \max\{d_{r,s} | s \in \mathbb{N}\}. \tag{3.7}$$

For $s \in \mathbb{N}$ we have

$$\mu_s = \max\{d_{r,s} | r \in \mathbb{N}\}. \tag{3.8}$$

Proof. First, for $r \in \mathbb{N}$ we verify (3.7). Let μ'_r denote right-hand side of (3.7). We show that $\mu_r = \mu'_r$. By Lemma 3.11, we may view

$$\mu'_r = \sum_{s\in\mathbb{N}} (d_{r,s} - d_{r,s-1}).$$

By this and Lemma 3.10,

$$\Delta(t,1) = \sum_{(r,s)\in\mathbb{N}^2} \big(d_{r,s} - d_{r,s-1}\big)t^r = \sum_{r\in\mathbb{N}} \mu'_r t^r. \tag{3.9}$$

Set $u = 1$ in (3.3), and evaluate the result using (3.5), (3.6). This yields

$$\Delta(t,1) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^3} = \big(p(t)\big)^3 = \sum_{r\in\mathbb{N}} \mu_r t^r. \tag{3.10}$$

Comparing (3.9), (3.10) we obtain $\mu_r = \mu'_r$ for $r \in \mathbb{N}$. We have verified (3.7). The second assertion in the proposition statement follows from the first assertion in the proposition statement and the comment above Lemma 3.8. □

Our next general goal is to embed U_q^+ into a q-shuffle algebra. For this q-shuffle algebra the underlying vector space is a free algebra on two generators. This free algebra is described in the next section.

4 The free algebra $\mathbb{V}$

Let x, y denote noncommuting indeterminates. Let $\mathbb{V}$ denote the free algebra with generators x, y. By a *letter* in $\mathbb{V}$ we mean x or y. For $n \in \mathbb{N}$, a *word of length* n in $\mathbb{V}$ is a product of letters $\ell_1\ell_2\cdots\ell_n$. We interpret the word of length 0 to be the multiplicative identity in $\mathbb{V}$; this word is called *trivial* and denoted by $\mathbf{1}$. The vector space $\mathbb{V}$ has a basis consisting of its words; this basis is called *standard*.

We mention some symmetries of the free algebra $\mathbb{V}$. For the next four lemmas, the proofs are routine and omitted.

Lemma 4.1. *There exists an automorphism σ of the free algebra $\mathbb{V}$ that sends $x \leftrightarrow y$. Moreover $\sigma^2 = \mathrm{id}$.*

Lemma 4.2. *There exists an antiautomorphism † of the free algebra $\mathbb{V}$ that fixes each of x, y. Moreover $†^2 = \mathrm{id}$.*

Lemma 4.3. *The map σ from* Lemma 4.1 *commutes with the map † from* Lemma 4.2.

Lemma 4.4. *There exists an antiautomorphism τ of the free algebra $\mathbb{V}$ that sends $x \leftrightarrow y$. The map τ is the composition the map σ from* Lemma 4.1 *and the map $\dagger$ from* Lemma 4.2. *We have $\tau^2 = \mathrm{id}$.*

Example 4.5. The automorphism σ sends

$$xxx \leftrightarrow yyy, \qquad xxyy \leftrightarrow yyxx, \qquad xyxxyy \leftrightarrow yxyyxx.$$

The antiautomorphism $\dagger$ sends

$$xxx \leftrightarrow xxx, \qquad xxyy \leftrightarrow yyxx, \qquad xyxxyy \leftrightarrow yyxxyx.$$

The antiautomorphism τ sends

$$xxx \leftrightarrow yyy, \qquad xxyy \leftrightarrow xxyy, \qquad xyxxyy \leftrightarrow xxyyxy.$$

The free algebra $\mathbb{V}$ has a $\mathbb{N}^2$-grading for which x and y are homogeneous, with degrees $(1,0)$ and $(0,1)$ respectively. For $(r,s) \in \mathbb{N}^2$ let $\mathbb{V}(r,s)$ denote the (r,s)-homogeneous component of the grading. These homogeneous components are described as follows. Let $w = \ell_1\ell_2\cdots\ell_n$ denote a word in $\mathbb{V}$. The x*-degree* of w is the cardinality of the set $\{i|1 \leq i \leq n,\ \ell_i = x\}$. The y*-degree* of w is the cardinality of the set $\{i|1 \leq i \leq n,\ \ell_i = y\}$. For $(r,s) \in \mathbb{N}^2$ the subspace $\mathbb{V}(r,s)$ has a basis consisting of the words in $\mathbb{V}$ that have x-degree r and y-degree s. The dimension of $\mathbb{V}(r,s)$ is equal to the binomial coefficient $\binom{r+s}{r}$. By construction $\mathbb{V}(0,0) = \mathbb{F}\mathbf{1}$. By construction, the sum $\mathbb{V} = \sum_{(r,s)\in\mathbb{N}^2} \mathbb{V}(r,s)$ is direct.

Example 4.6. The following is a basis for the vector space $\mathbb{V}(2,3)$:

$$\begin{array}{ccccc} xxyyy, & xyxyy, & xyyxy, & xyyyx, & yxxyy, \\ yxyxy, & yxyyx, & yyxxy, & yyxyx, & yyyxx. \end{array}$$

Let $\mathrm{End}(\mathbb{V})$ denote the algebra consisting of the $\mathbb{F}$-linear maps from $\mathbb{V}$ to $\mathbb{V}$. Let I denote the identity in $\mathrm{End}(\mathbb{V})$.

5 The maps X, Y, K

In this section we describe some maps X, Y, K in $\mathrm{End}(\mathbb{V})$ that will be used in our main results.

Definition 5.1. Let X denote the automorphism of the free algebra $\mathbb{V}$ that sends $x \mapsto qx$ and $y \mapsto y$. Let Y denote the automorphism of the free algebra $\mathbb{V}$ that sends $x \mapsto x$ and $y \mapsto qy$.

Example 5.2. The map X sends

$$xxx \mapsto q^3xxx, \qquad xxyy \mapsto q^2xxyy, \qquad xyxxyy \mapsto q^3xyxxyy.$$

The map Y sends

$$xxx \mapsto xxx, \qquad xxyy \mapsto q^2xxyy, \qquad xyxxyy \mapsto q^3xyxxyy.$$

Lemma 5.3. *For $(r,s) \in \mathbb{N}^2$ the maps X and Y act on $\mathbb{V}(r,s)$ as q^rI and q^sI, respectively.*

Proof. By the description of $\mathbb{V}(r,s)$ above Example 4.6. □

By construction the maps X, Y are invertible, and they commute.

Definition 5.4. Define $K = X^2Y^{-2}$. Thus K is the automorphism of the free algebra $\mathbb{V}$ that sends $x \mapsto q^2x$ and $y \mapsto q^{-2}y$.

Example 5.5. The map K sends

$$xxx \mapsto q^6xxx, \qquad xxyy \mapsto xxyy, \qquad xyxxyy \mapsto xyxxyy.$$

Lemma 5.6. *For $(r,s) \in \mathbb{N}^2$ the map K acts on $\mathbb{V}(r,s)$ as $q^{2r-2s}I$.*

Proof. By Lemma 5.3 and Definition 5.4. □

Lemma 5.7. *The following diagrams commute:*

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{X^{\pm1}} & \mathbb{V} \\ \sigma\downarrow & & \downarrow\sigma \\ \mathbb{V} & \xrightarrow[Y^{\pm1}]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{Y^{\pm1}} & \mathbb{V} \\ \sigma\downarrow & & \downarrow\sigma \\ \mathbb{V} & \xrightarrow[X^{\pm1}]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{K^{\pm1}} & \mathbb{V} \\ \sigma\downarrow & & \downarrow\sigma \\ \mathbb{V} & \xrightarrow[K^{\mp1}]{} & \mathbb{V} \end{array}$$

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{X^{\pm1}} & \mathbb{V} \\ \dagger\downarrow & & \downarrow\dagger \\ \mathbb{V} & \xrightarrow[X^{\pm1}]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{Y^{\pm1}} & \mathbb{V} \\ \dagger\downarrow & & \downarrow\dagger \\ \mathbb{V} & \xrightarrow[Y^{\pm1}]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{K^{\pm1}} & \mathbb{V} \\ \dagger\downarrow & & \downarrow\dagger \\ \mathbb{V} & \xrightarrow[K^{\pm1}]{} & \mathbb{V} \end{array}$$

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{X^{\pm1}} & \mathbb{V} \\ \tau\downarrow & & \downarrow\tau \\ \mathbb{V} & \xrightarrow[Y^{\pm1}]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{Y^{\pm1}} & \mathbb{V} \\ \tau\downarrow & & \downarrow\tau \\ \mathbb{V} & \xrightarrow[X^{\pm1}]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{K^{\pm1}} & \mathbb{V} \\ \tau\downarrow & & \downarrow\tau \\ \mathbb{V} & \xrightarrow[K^{\mp1}]{} & \mathbb{V} \end{array}$$

Proof. Routine. □

6 The maps A_L^*, B_L^*, A_R^*, B_R^*

In this section we recall from [32] some maps A_L^*, B_L^*, A_R^*, B_R^* in $\mathrm{End}(\mathbb{V})$ that will be used in our main results. First we mention some notation. The Kronecker delta $\delta_{r,s}$ is equal to 1 if $r = s$, and 0 if $r \neq s$.

Definition 6.1 (See [32, Lemma 4.3])**.** Define the maps A_L^*, B_L^*, A_R^*, B_R^* in $\mathrm{End}(\mathbb{V})$ as follows. For a nontrivial word $w = \ell_1\ell_2\cdots\ell_n$ in $\mathbb{V}$,

$$A_L^*w = \ell_2\cdots\ell_n\delta_{\ell_1,x}, \qquad B_L^*w = \ell_2\cdots\ell_n\delta_{\ell_1,y},$$
$$A_R^*w = \ell_1\cdots\ell_{n-1}\delta_{\ell_n,x}, \qquad B_R^*w = \ell_1\cdots\ell_{n-1}\delta_{\ell_n,y}.$$

Moreover

$$A_L^*\mathbf{1} = 0, \qquad B_L^*\mathbf{1} = 0, \qquad A_R^*\mathbf{1} = 0, \qquad B_R^*\mathbf{1} = 0. \tag{6.1}$$

Example 6.2. The maps A_L^*, B_L^*, A_R^*, B_R^* are illustrated in the table below.

w	x	y	xx	xy	yx	yy
A_L^*w	$\mathbf{1}$	0	x	y	0	0
B_L^*w	0	$\mathbf{1}$	0	0	x	y
A_R^*w	$\mathbf{1}$	0	x	0	y	0
B_R^*w	0	$\mathbf{1}$	0	x	0	y

Lemma 6.3. *For* $v \in \mathbb{V}$,

$$A_L^*(xv) = v, \qquad A_L^*(yv) = 0, \qquad B_L^*(xv) = 0, \qquad B_L^*(yv) = v,$$
$$A_R^*(vx) = v, \qquad A_R^*(vy) = 0, \qquad B_R^*(vx) = 0, \qquad B_R^*(vy) = v.$$

Proof. Use Definition 6.1. □

For notational convenience, define $\mathbb{V}(r,-1) = 0$ and $\mathbb{V}(-1,s) = 0$ for $r, s \in \mathbb{N}$.

Lemma 6.4. *For* $(r,s) \in \mathbb{N}^2$ *we have*

$$A_L^*\mathbb{V}(r,s) \subseteq \mathbb{V}(r-1,s), \qquad B_L^*\mathbb{V}(r,s) \subseteq \mathbb{V}(r,s-1),$$
$$A_R^*\mathbb{V}(r,s) \subseteq \mathbb{V}(r-1,s), \qquad B_R^*\mathbb{V}(r,s) \subseteq \mathbb{V}(r,s-1).$$

Proof. By Definition 6.1 or Lemma 6.3. □

Lemma 6.5. *The maps* A_L^*, B_L^*, A_R^*, B_R^* *are locally nilpotent on the vector space* $\mathbb{V}$.

Proof. We mentioned above Example 4.6 that the sum $\mathbb{V} = \sum_{(r,s)\in\mathbb{N}^2} \mathbb{V}(r,s)$ is direct. The result follows from this and Lemma 6.4. □

Next we describe how the maps X, Y are related to the maps A_L^*, B_L^*, A_R^*, B_R^*.

Lemma 6.6. *We have*

$$XA_L^* = q^{-1}A_L^*X, \qquad XB_L^* = B_L^*X, \qquad XA_R^* = q^{-1}A_R^*X, \qquad XB_R^* = B_R^*X,$$
$$YA_L^* = A_L^*Y, \qquad YB_L^* = q^{-1}B_L^*Y, \qquad YA_R^* = A_R^*Y, \qquad YB_R^* = q^{-1}B_R^*Y.$$

Proof. By Lemmas 5.3, 6.4. □

The next result is about A_L^* and B_L^*; a similar result holds for A_R^* and B_R^*. Observe that the sum $\mathbb{V} = \mathbb{F}\mathbf{1} + x\mathbb{V} + y\mathbb{V}$ is direct.

Lemma 6.7. *The following* (i) – (v) *hold:*

(i) $\ker A_L^*$ *has a basis consisting of the words in* $\mathbb{V}$ *that do not begin with* x*;*

(ii) $\ker A_L^* = \mathbb{F}\mathbf{1} + y\mathbb{V}$*;*

(iii) $\ker B_L^*$ *has a basis consisting of the words in* $\mathbb{V}$ *that do not begin with* y*;*

(iv) $\ker B_L^* = \mathbb{F}\mathbf{1} + x\mathbb{V}$*;*

(v) $\ker A_L^* \cap \ker B_L^* = \mathbb{F}\mathbf{1}$.

Proof. Use Definition 6.1 and the observation above the lemma statement. □

The following result appears in [32]; we give a short proof for the sake of completeness.

Lemma 6.8 (See [32, Lemma 4.6])**.** *Let W denote a nonzero subspace of $\mathbb{V}$ that is closed under A^*_L and B^*_L. Then $\mathbf{1} \in W$.*

Proof. For $n \in \mathbb{N}$ define $\mathbb{V}_n = \sum_{r+s\le n} \mathbb{V}(r,s)$. Note that $\mathbb{V}_0 = \mathbb{F}\mathbf{1}$. We have $\mathbb{V}_{n-1} \subseteq \mathbb{V}_n$ for $n \ge 1$, and $\mathbb{V} = \cup_{n\in\mathbb{N}}\mathbb{V}_n$. For $n \ge 1$ we have $A^*_L \mathbb{V}_n \subseteq \mathbb{V}_{n-1}$ and $B^*_L \mathbb{V}_n \subseteq \mathbb{V}_{n-1}$, in view of Lemma 6.4. Since $W \neq 0$, there exists $n \in \mathbb{N}$ such that $W \cap \mathbb{V}_n \neq 0$. Assume for the moment that $n = 0$. Then $\mathbf{1} \in W$ and we are done. Next assume that $n \ge 1$. Without loss, we may assume that $W \cap \mathbb{V}_{n-1} = 0$. Pick $0 \neq v \in W \cap \mathbb{V}_n$. We have $A^*_L v \in W \cap \mathbb{V}_{n-1} = 0$ and $B^*_L v \in W \cap \mathbb{V}_{n-1} = 0$, so $v \in \mathbb{F}\mathbf{1}$ in view of Lemma 6.7(v). By construction $0 \neq v \in W$, so $\mathbf{1} \in W$. □

Lemma 6.9. *Let W denote a nonzero subspace of $\mathbb{V}$ that is closed under A^*_R and B^*_R. Then $\mathbf{1} \in W$.*

Proof. The †-image $W^\dagger$ is a subspace of $\mathbb{V}$ that is invariant under A^*_L and B^*_L. We have $\mathbf{1} \in W^\dagger$ by Lemma 6.8, and $\mathbf{1}^\dagger = \mathbf{1}$ by construction, so $\mathbf{1} \in W$. □

Lemma 6.10. *The following diagrams commute:*

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{A^*_L} & \mathbb{V} \\ \sigma\downarrow & & \downarrow\sigma \\ \mathbb{V} & \xrightarrow[B^*_L]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{B^*_L} & \mathbb{V} \\ \sigma\downarrow & & \downarrow\sigma \\ \mathbb{V} & \xrightarrow[A^*_L]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{A^*_R} & \mathbb{V} \\ \sigma\downarrow & & \downarrow\sigma \\ \mathbb{V} & \xrightarrow[B^*_R]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{B^*_R} & \mathbb{V} \\ \sigma\downarrow & & \downarrow\sigma \\ \mathbb{V} & \xrightarrow[A^*_R]{} & \mathbb{V} \end{array}$$

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{A^*_L} & \mathbb{V} \\ \dagger\downarrow & & \downarrow\dagger \\ \mathbb{V} & \xrightarrow[A^*_R]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{B^*_L} & \mathbb{V} \\ \dagger\downarrow & & \downarrow\dagger \\ \mathbb{V} & \xrightarrow[B^*_R]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{A^*_R} & \mathbb{V} \\ \dagger\downarrow & & \downarrow\dagger \\ \mathbb{V} & \xrightarrow[A^*_L]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{B^*_R} & \mathbb{V} \\ \dagger\downarrow & & \downarrow\dagger \\ \mathbb{V} & \xrightarrow[B^*_L]{} & \mathbb{V} \end{array}$$

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{A^*_L} & \mathbb{V} \\ \tau\downarrow & & \downarrow\tau \\ \mathbb{V} & \xrightarrow[B^*_R]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{B^*_L} & \mathbb{V} \\ \tau\downarrow & & \downarrow\tau \\ \mathbb{V} & \xrightarrow[A^*_R]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{A^*_R} & \mathbb{V} \\ \tau\downarrow & & \downarrow\tau \\ \mathbb{V} & \xrightarrow[B^*_L]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} \mathbb{V} & \xrightarrow{B^*_R} & \mathbb{V} \\ \tau\downarrow & & \downarrow\tau \\ \mathbb{V} & \xrightarrow[A^*_L]{} & \mathbb{V} \end{array}$$

Proof. Routine. □

7 The q-shuffle algebra $\mathbb{V}$

In the previous sections we discussed the free algebra $\mathbb{V}$. There is another algebra structure on $\mathbb{V}$, called the q-shuffle algebra. This algebra was introduced by Rosso [33, 34] and described further by Green [18]. We will adopt the approach of [18], which is suited to our purpose. The q-shuffle product is denoted by $\star$. To describe this product, we first consider some special cases. We have $\mathbf{1} \star v = v \star \mathbf{1} = v$ for $v \in \mathbb{V}$. For letters u, v we have $u \star v = uv + vuq^{(u,v)}$, where

(,)	x	y
x	2	−2
y	−2	2

Thus

$$x \star x = (1+q^2)xx, \qquad x \star y = xy + q^{-2}yx,$$
$$y \star x = yx + q^{-2}xy, \qquad y \star y = (1+q^2)yy.$$

For a letter u and a nontrivial word $v = v_1 v_2 \cdots v_n$ in $\mathbb{V}$,

$$u \star v = \sum_{i=0}^{n} v_1 \cdots v_i u v_{i+1} \cdots v_n q^{(v_1,u)+(v_2,u)+\cdots+(v_i,u)}, \tag{7.1}$$
$$v \star u = \sum_{i=0}^{n} v_1 \cdots v_i u v_{i+1} \cdots v_n q^{(v_n,u)+(v_{n-1},u)+\cdots+(v_{i+1},u)}. \tag{7.2}$$

For example

$$y \star (xxx) = yxxx + q^{-2}xyxx + q^{-4}xxyx + q^{-6}xxxy,$$
$$(xxx) \star y = q^{-6}yxxx + q^{-4}xyxx + q^{-2}xxyx + xxxy.$$

For nontrivial words $u = u_1 u_2 \cdots u_r$ and $v = v_1 v_2 \cdots v_s$ in $\mathbb{V}$,

$$u \star v = u_1\big((u_2 \cdots u_r) \star v\big) + v_1\big(u \star (v_2 \cdots v_s)\big)q^{(u_1,v_1)+(u_2,v_1)+\cdots+(u_r,v_1)},$$
$$u \star v = \big(u \star (v_1 \cdots v_{s-1})\big)v_s + \big((u_1 \cdots u_{r-1}) \star v\big)u_r q^{(u_r,v_1)+(u_r,v_2)+\cdots+(u_r,v_s)}.$$

For example, assume $r = 2$ and $s = 2$. Then

$$u \star v = u_1u_2v_1v_2 + u_1v_1u_2v_2q^{(u_2,v_1)} + u_1v_1v_2u_2q^{(u_2,v_1)+(u_2,v_2)} + v_1u_1u_2v_2q^{(u_1,v_1)+(u_2,v_1)}$$
$$+ v_1u_1v_2u_2q^{(u_1,v_1)+(u_2,v_1)+(u_2,v_2)} + v_1v_2u_1u_2q^{(u_1,v_1)+(u_1,v_2)+(u_2,v_1)+(u_2,v_2)}.$$

The map σ from Lemma 4.1 is an automorphism of the q-shuffle algebra $\mathbb{V}$. The map † from Lemma 4.2 is an antiautomorphism of the q-shuffle algebra $\mathbb{V}$. The map τ from Lemma 4.4 is an antiautomorphism of the q-shuffle algebra $\mathbb{V}$. Above Example 4.6 we mentioned an $\mathbb{N}^2$-grading of the free algebra $\mathbb{V}$. This is also an $\mathbb{N}^2$-grading for the q-shuffle algebra $\mathbb{V}$.

See [19, 29, 31, 32, 38–40] for more information about the q-shuffle algebra $\mathbb{V}$.

8 The maps A_ℓ, B_ℓ, A_r, B_r

In this section we recall from [32] some maps A_ℓ, B_ℓ, A_r, B_r in $\text{End}(\mathbb{V})$ that will be used in our main results.

Definition 8.1 (See [32, Definition 7.1])**.** Define the maps A_ℓ, B_ℓ, A_r, B_r in $\text{End}(\mathbb{V})$ as follows. For $v \in \mathbb{V}$,

$$A_\ell v = x \star v, \qquad B_\ell v = y \star v, \qquad A_r v = v \star x, \qquad B_r v = v \star y.$$

Example 8.2. The maps A_ℓ, B_ℓ, A_r, B_r are illustrated in the table below.

v	$\mathbf{1}$	x	y	xy
$A_\ell v$	x	$q[2]_q xx$	$xy + q^{-2}yx$	$q[2]_q xxy + xyx$
$B_\ell v$	y	$q^{-2}xy + yx$	$q[2]_q yy$	$q^{-1}[2]_q xyy + yxy$
$A_r v$	x	$q[2]_q xx$	$q^{-2}xy + yx$	$q^{-1}[2]_q xxy + xyx$
$B_r v$	y	$xy + q^{-2}yx$	$q[2]_q yy$	$q[2]_q xyy + yxy$

Lemma 8.3. *For $(r, s) \in \mathbb{N}^2$ we have*

$$A_\ell \mathbb{V}(r,s) \subseteq \mathbb{V}(r+1,s), \qquad B_\ell \mathbb{V}(r,s) \subseteq \mathbb{V}(r,s+1),$$
$$A_r \mathbb{V}(r,s) \subseteq \mathbb{V}(r+1,s), \qquad B_r \mathbb{V}(r,s) \subseteq \mathbb{V}(r,s+1).$$

Proof. By Definition 8.1 and the description of $\mathbb{V}(r,s)$ above Example 4.6. □

Next we describe how the maps X, Y are related to the maps A_ℓ, B_ℓ, A_r, B_r.

Lemma 8.4. *We have*

$$XA_\ell = qA_\ell X, \qquad XB_\ell = B_\ell X, \qquad XA_r = qA_r X, \qquad XB_r = B_r X,$$
$$YA_\ell = A_\ell Y, \qquad YB_\ell = qB_\ell Y, \qquad YA_r = A_r Y, \qquad YB_r = qB_r Y.$$

Proof. By Lemmas 5.3, 8.3. □

Lemma 8.5. *The following diagrams commute:*

$$\begin{CD} \mathbb{V} @>A_\ell>> \mathbb{V} \\ @V\sigma VV @VV\sigma V \\ \mathbb{V} @>>B_\ell> \mathbb{V} \end{CD} \qquad \begin{CD} \mathbb{V} @>B_\ell>> \mathbb{V} \\ @V\sigma VV @VV\sigma V \\ \mathbb{V} @>>A_\ell> \mathbb{V} \end{CD} \qquad \begin{CD} \mathbb{V} @>A_r>> \mathbb{V} \\ @V\sigma VV @VV\sigma V \\ \mathbb{V} @>>B_r> \mathbb{V} \end{CD} \qquad \begin{CD} \mathbb{V} @>B_r>> \mathbb{V} \\ @V\sigma VV @VV\sigma V \\ \mathbb{V} @>>A_r> \mathbb{V} \end{CD}$$

$$\begin{CD} \mathbb{V} @>A_\ell>> \mathbb{V} \\ @V\dagger VV @VV\dagger V \\ \mathbb{V} @>>A_r> \mathbb{V} \end{CD} \qquad \begin{CD} \mathbb{V} @>B_\ell>> \mathbb{V} \\ @V\dagger VV @VV\dagger V \\ \mathbb{V} @>>B_r> \mathbb{V} \end{CD} \qquad \begin{CD} \mathbb{V} @>A_r>> \mathbb{V} \\ @V\dagger VV @VV\dagger V \\ \mathbb{V} @>>A_\ell> \mathbb{V} \end{CD} \qquad \begin{CD} \mathbb{V} @>B_r>> \mathbb{V} \\ @V\dagger VV @VV\dagger V \\ \mathbb{V} @>>B_\ell> \mathbb{V} \end{CD}$$

$$\begin{CD} \mathbb{V} @>A_\ell>> \mathbb{V} \\ @V\tau VV @VV\tau V \\ \mathbb{V} @>>B_r> \mathbb{V} \end{CD} \qquad \begin{CD} \mathbb{V} @>B_\ell>> \mathbb{V} \\ @V\tau VV @VV\tau V \\ \mathbb{V} @>>A_r> \mathbb{V} \end{CD} \qquad \begin{CD} \mathbb{V} @>A_r>> \mathbb{V} \\ @V\tau VV @VV\tau V \\ \mathbb{V} @>>B_\ell> \mathbb{V} \end{CD} \qquad \begin{CD} \mathbb{V} @>B_r>> \mathbb{V} \\ @V\tau VV @VV\tau V \\ \mathbb{V} @>>A_\ell> \mathbb{V} \end{CD}$$

Proof. Routine. □

9 The subspace $\mathbb{U}$

In this section we discuss a subspace $\mathbb{U} \subseteq \mathbb{V}$ that will be used in our main results.

Definition 9.1. Let $\mathbb{U}$ denote the subalgebra of the q-shuffle algebra $\mathbb{V}$ that is generated by x, y.

The algebra $\mathbb{U}$ is described as follows. By [33, Theorem 13] or [18, page 10],

$$x \star x \star x \star y - [3]_q x \star x \star y \star x + [3]_q x \star y \star x \star x - y \star x \star x \star x = 0,$$
$$y \star y \star y \star x - [3]_q y \star y \star x \star y + [3]_q y \star x \star y \star y - x \star y \star y \star y = 0.$$

So in the q-shuffle algebra $\mathbb{V}$ the elements x, y satisfy the q-Serre relations. Therefore, there exists an algebra homomorphism $\natural$ from U_q^+ to the q-shuffle algebra $\mathbb{V}$, that sends $A \mapsto x$ and $B \mapsto y$. The map $\natural$ has image $\mathbb{U}$ by Definition 9.1, and is injective by [34, Theorem 15]. Consequently $\natural\colon U_q^+ \to \mathbb{U}$ is an algebra isomorphism. By construction the following diagrams commute:

$$\begin{array}{ccc} U_q^+ & \xrightarrow{\natural} & \mathbb{V} \\ \sigma\downarrow & & \downarrow\sigma \\ U_q^+ & \xrightarrow[\natural]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} U_q^+ & \xrightarrow{\natural} & \mathbb{V} \\ \dagger\downarrow & & \downarrow\dagger \\ U_q^+ & \xrightarrow[\natural]{} & \mathbb{V} \end{array} \qquad \begin{array}{ccc} U_q^+ & \xrightarrow{\natural} & \mathbb{V} \\ \tau\downarrow & & \downarrow\tau \\ U_q^+ & \xrightarrow[\natural]{} & \mathbb{V} \end{array} \tag{9.1}$$

Consequently $\mathbb{U}$ is invariant under each of σ, $\dagger$, τ. Earlier we mentioned an $\mathbb{N}^2$-grading for both the algebra U_q^+ and the q-shuffle algebra $\mathbb{V}$. These gradings are related as follows. The algebra $\mathbb{U}$ has an $\mathbb{N}^2$-grading inherited from U_q^+ via $\natural$. With respect to this grading, for $(r, s) \in \mathbb{N}^2$ the (r, s)-homogeneous component of $\mathbb{U}$ is the $\natural$-image of the (r, s)-homogeneous component of U_q^+. We denote this homogeneous component by $\mathbb{U}(r, s)$. By construction,

$$\mathbb{U}(r, s) = \mathbb{V}(r, s) \cap \mathbb{U}, \qquad (r, s) \in \mathbb{N}^2. \tag{9.2}$$

By construction $\mathbb{U}(0, 0) = \mathbb{F}\mathbf{1}$. By construction, the sum $\mathbb{U} = \sum_{(r,s)\in\mathbb{N}^2} \mathbb{U}(r, s)$ is direct. By Lemma 3.8 and the construction,

$$d_{r,s} = \dim \mathbb{U}(r, s), \qquad (r, s) \in \mathbb{N}^2. \tag{9.3}$$

Lemma 9.2. *For $(r, s) \in \mathbb{N}^2$ the following hold on $\mathbb{U}(r, s)$:*

$$X = q^r I, \qquad Y = q^s I, \qquad K = q^{2r-2s} I.$$

Proof. By Lemmas 5.3, 5.6 and since $\mathbb{U}(r, s) \subseteq \mathbb{V}(r, s)$. □

Lemma 9.3. *The vector space $\mathbb{U}$ is invariant under each of*

$$X^{\pm 1}, \qquad Y^{\pm 1}, \qquad K^{\pm 1}.$$

Proof. By Lemma 9.2 and since $\mathbb{U} = \sum_{(r,s)\in\mathbb{N}^2} \mathbb{U}(r, s)$. □

Lemma 9.4 (See [32, Proposition 9.1]). *The vector space $\mathbb{U}$ is invariant under each of*

$$A^*_L, \qquad B^*_L, \qquad A^*_R, \qquad B^*_R.$$

For notational convenience, define $\mathbb{U}(r,-1)=0$ and $\mathbb{U}(-1,s)=0$ for $r,s\in\mathbb{N}$.

Lemma 9.5. *For $(r,s)\in\mathbb{N}^2$ we have*

$$A^*_L\mathbb{U}(r,s)\subseteq\mathbb{U}(r-1,s), \qquad B^*_L\mathbb{U}(r,s)\subseteq\mathbb{U}(r,s-1),$$
$$A^*_R\mathbb{U}(r,s)\subseteq\mathbb{U}(r-1,s), \qquad B^*_R\mathbb{U}(r,s)\subseteq\mathbb{U}(r,s-1).$$

Proof. By (9.2) and Lemmas 6.4, 9.4. □

Lemma 9.6. *The subspace $\mathbb{U}$ is invariant under each of*

$$A_\ell, \qquad B_\ell, \qquad A_r, \qquad B_r.$$

Proof. By Definitions 9.1, 8.1. □

Lemma 9.7. *For $(r,s)\in\mathbb{N}^2$ we have*

$$A_\ell\mathbb{U}(r,s)\subseteq\mathbb{U}(r+1,s), \qquad B_\ell\mathbb{U}(r,s)\subseteq\mathbb{U}(r,s+1),$$
$$A_r\mathbb{U}(r,s)\subseteq\mathbb{U}(r+1,s), \qquad B_r\mathbb{U}(r,s)\subseteq\mathbb{U}(r,s+1).$$

Proof. By (9.2) and Lemmas 8.3, 9.6. □

In [32, Propositions 9.1, 9.3] there are many relations satisfied by the maps

$$K, \quad K^{-1}, \quad A^*_L, \quad B^*_L, \quad A^*_R, \quad B^*_R, \quad A_\ell, \quad B_\ell, \quad A_r, \quad B_r.$$

For convenience we reproduce these relations in Appendix A. These relations will be used in our main results.

10 The $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$ and its submodule $\mathbf{U}$

We now bring in the algebra $U_q(\widehat{\mathfrak{sl}}_2)$. The definition of this algebra can be found in Appendix B. In the present section, we turn the vector space $\mathbb{U}$ into a $U_q(\widehat{\mathfrak{sl}}_2)$-module, and describe the submodule $\mathbf{U}$ generated by the vector $\mathbf{1}$.

The following is our first main result.

Theorem 10.1. *The vector space $\mathbb{U}$ becomes a $U_q(\widehat{\mathfrak{sl}}_2)$-module on which the $U_q(\widehat{\mathfrak{sl}}_2)$-generators act as follows:*

generator	E_0	F_0	$K_0^{\pm1}$	E_1	F_1	$K_1^{\pm1}$	$D^{\pm1}$
action on $\mathbb{U}$	A^*_R	$\frac{qA_rK^{-1}-q^{-1}A_\ell}{q-q^{-1}}$	$q^{\pm1}K^{\mp1}$	B^*_R	$\frac{B_rK-B_\ell}{q-q^{-1}}$	$K^{\pm1}$	$X^{\mp1}$

Proof. This is routinely checked using Lemmas 9.3, 9.4, 9.6 along with the relations in Lemmas 6.6, 8.4 and Appendix A. Among the things to check, is that $qA_rK^{-1}-q^{-1}A_\ell$ and B_rK-B_ℓ satisfy the q-Serre relations. This can be checked easily using [32, Lemma 10.3, Corollary 10.4]. □

Consider the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$ from Theorem 10.1. Recall the $\mathbb{N}^2$-grading of $\mathbb{U}$ from around (9.2). Next we describe how the $U_q(\widehat{\mathfrak{sl}}_2)$-generators act on the homogeneous components of this grading.

Lemma 10.2. *For $(r, s) \in \mathbb{N}^2$ the following hold on $\mathbb{U}(r, s)$:*

$$K_0 = q^{2s-2r+1} I, \qquad K_1 = q^{2r-2s} I, \qquad D = q^{-r} I. \tag{10.1}$$

Moreover

$$E_0 \mathbb{U}(r, s) \subseteq \mathbb{U}(r-1, s), \qquad F_0 \mathbb{U}(r, s) \subseteq \mathbb{U}(r+1, s), \tag{10.2}$$

$$E_1 \mathbb{U}(r, s) \subseteq \mathbb{U}(r, s-1), \qquad F_1 \mathbb{U}(r, s) \subseteq \mathbb{U}(r, s+1). \tag{10.3}$$

Proof. By Lemmas 9.2, 9.5, 9.7 and the data in Theorem 10.1. □

In this paragraph we recall a few concepts about $U_q(\widehat{\mathfrak{sl}}_2)$-modules; see for example [20, Section 3.2]. Let W denote a $U_q(\widehat{\mathfrak{sl}}_2)$-module. A *weight space* for W is a common eigenspace for the action of K_0, K_1, D on W. The sum of these weight spaces is direct. We call W a *weight module* whenever W is equal to the sum of its weight spaces. If W is a weight module, then every submodule of W is a weight module [20, Proposition 3.2.1]. We return our attention to the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$ from Theorem 10.1. We mentioned above Lemma 9.2 that the sum $\mathbb{U} = \sum_{(r,s)\in\mathbb{N}^2} \mathbb{U}(r, s)$ is direct. By this and (10.1), the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$ is a weight module, and its weight spaces are the nonzero subspaces among $\mathbb{U}(r, s)$ ($r, s \in \mathbb{N}$). Note that these weight spaces have finite dimension.
It turns out that the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$ is not irreducible. Next we consider its submodules.

Definition 10.3. Let $\mathbf{U}$ denote the submodule of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$ that is generated by the vector $\mathbf{1}$.

Lemma 10.4. *For the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbf{U}$,*

(i) $\mathbf{U}$ *is contained in every nonzero submodule of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$;*

(ii) $\mathbf{U}$ *is the unique irreducible submodule of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$.*

Proof. (i) Let W denote a nonzero submodule of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$. The vector space W is invariant under A^*_R, B^*_R by Theorem 10.1, so $\mathbf{1} \in W$ by Lemma 6.9. The $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbf{U}$ is generated by $\mathbf{1}$, so $\mathbf{U} \subseteq W$.
(ii) By (i) above. □

Next we consider how the $U_q(\widehat{\mathfrak{sl}}_2)$-generators act on the vector $\mathbf{1}$.

Lemma 10.5. *For the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$,*

$$K_0 \mathbf{1} = q\mathbf{1}, \qquad K_1 \mathbf{1} = \mathbf{1}, \qquad D\mathbf{1} = \mathbf{1},$$

$$E_0 \mathbf{1} = 0, \qquad F_0^2 \mathbf{1} = 0, \qquad E_1 \mathbf{1} = 0, \qquad F_1 \mathbf{1} = 0.$$

Proof. This is routinely checked using the data in Theorem 10.1 and Lemma 10.2. □

There is a well known $U_q(\widehat{\mathfrak{sl}}_2)$-module $V(\Lambda_0)$ that is said to be basic; see [20, page 221]. The module $V(\Lambda_0)$ is highest weight, integrable, and level one; see [4, Chapter 10] and [27, Chapter 5]. The module $V(\Lambda_0)$ is characterized as follows.

Lemma 10.6 (See [27, pages 63, 64])**.** *There exists a $U_q(\widehat{\mathfrak{sl}}_2)$-module $V(\Lambda_0)$ with the following property: $V(\Lambda_0)$ is generated by a nonzero vector $\mathbf{v}$ such that*

$$K_0\mathbf{v} = q\mathbf{v}, \qquad K_1\mathbf{v} = \mathbf{v}, \qquad D\mathbf{v} = \mathbf{v},$$
$$E_0\mathbf{v} = 0, \qquad F_0^2\mathbf{v} = 0, \qquad E_1\mathbf{v} = 0, \qquad F_1\mathbf{v} = 0.$$

Moreover $V(\Lambda_0)$ is irreducible, infinite-dimensional, and unique up to isomorphism of $U_q(\widehat{\mathfrak{sl}}_2)$-modules.

The following is our second main result.

Theorem 10.7. *The $U_q(\widehat{\mathfrak{sl}}_2)$-modules $\mathbf{U}$ and $V(\Lambda_0)$ are isomorphic.*

Proof. By Definition 10.3 and Lemmas 10.5, 10.6. □

Descriptions of $V(\Lambda_0)$ can be found in [4, Chapter 10] and [20, Section 9] and [27, Chapter 5]; see also [14, Section 20.4] and [28, Chapter 14]. Our next general goal is to describe $V(\Lambda_0)$ from the point of view of $\mathbf{U}$.

Definition 10.8. For $(r,s) \in \mathbb{N}^2$ define $\mathbf{U}(r,s) = \mathbf{U} \cap \mathbb{U}(r,s)$.

The $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbf{U}$ is a weight module, and its weight spaces are the nonzero subspaces among $\mathbf{U}(r,s)$ $(r, s \in \mathbb{N})$. More detail is given in the next result.

Lemma 10.9. *The following* (i) – (iv) *hold for the $U_q(\widehat{\mathfrak{sl}}_2)$-module* $\mathbf{U}$.

(i) $\mathbf{U}(0,0) = \mathbb{F}\mathbf{1}$.

(ii) *The sum $\mathbf{U} = \sum_{(r,s)\in\mathbb{N}^2} \mathbf{U}(r,s)$ is direct.*

(iii) *For $(r,s) \in \mathbb{N}^2$ the following hold on $\mathbf{U}(r,s)$:*

$$K_0 = q^{2s-2r+1}I, \qquad K_1 = q^{2r-2s}I, \qquad D = q^{-r}I.$$

(iv) *For $(r,s) \in \mathbb{N}^2$,*

$$E_0\mathbf{U}(r,s) \subseteq \mathbf{U}(r-1,s), \qquad F_0\mathbf{U}(r,s) \subseteq \mathbf{U}(r+1,s),$$
$$E_1\mathbf{U}(r,s) \subseteq \mathbf{U}(r,s-1), \qquad F_1\mathbf{U}(r,s) \subseteq \mathbf{U}(r,s+1),$$

where $\mathbf{U}(r,-1) = 0$ and $\mathbf{U}(-1,s) = 0$.

Proof. (i) By Definition 10.8 and since $\mathbb{U}(0,0) = \mathbb{F}\mathbf{1}$.
(ii) Since $\mathbf{U}$ is a weight module.
(iii), (iv) By Lemma 10.2 and Definition 10.8. □

Our next goal is to describe the weight space dimensions for the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbf{U}$. Recall the partition numbers $\{p_n\}_{n\in\mathbb{N}}$ from Section 3.

Proposition 10.10. *For* $(r, s) \in \mathbb{N}^2$ *the vector space* $\mathbf{U}(r, s) \neq 0$ *if and only if* $r \geq (r-s)^2$. *In this case* $\dim \mathbf{U}(r, s) = p_n$, *where* $n = r - (r - s)^2$.

Proof. For the $U_q(\widehat{\mathfrak{sl}}_2)$-module $V(\Lambda_0)$ the weight space dimensions are described in [20, pages 221, 222]. The result follows from that description and Theorem 10.7 above. □

Example 10.11. For $0 \leq r, s \leq 6$ the dimension of $\mathbf{U}(r, s)$ is given in the (r, s)-entry of the matrix below:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 5 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 7 & 5 \\ 0 & 0 & 0 & 0 & 2 & 7 & 11 \end{pmatrix}$$

Compare the above matrix with the one in Example 3.7.

Next we describe the generating function $\sum_{(r,s)\in\mathbb{N}^2} \dim \mathbf{U}(r, s) t^r u^s$.

Define the generating function

$$\phi(t, u) = \sum_{n\in\mathbb{Z}} t^{n^2} u^{n^2-n}. \tag{10.4}$$

Note that

$$\phi(t, u) = 1 + t + tu^2 + t^4u^2 + t^4u^6 + \cdots .$$

Proposition 10.12. *We have*

$$\sum_{(r,s)\in\mathbb{N}^2} \dim\mathbf{U}(r, s)t^r u^s = p(tu)\phi(t, u),$$

where $p(t)$ *is from* (3.4) *and* $\phi(t, u)$ *is from* (10.4).

Proof. This is a reformulation of Proposition 10.10. □

In Appendix C, we give a basis for each nonzero $\mathbf{U}(r, s)$ such that $r + s \leq 10$.

11 A characterization of the $U_q(\widehat{\mathfrak{sl}}_2)$-module U

In order to motivate this section, we glance at the basis vectors displayed in Appendix C. Each displayed vector is a linear combination of some words in $\mathbb{V}$ that do not begin with y or xx. Consequently, each displayed vector is contained in the kernel of B^*_L and the kernel of $(A^*_L)^2$. Using this observation, we will characterize the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbf{U}$.

Definition 11.1. Let **V** denote the intersection of the kernel of B^*_L and the kernel of $(A^*_L)^2$. Note that **V** is a subspace of the vector space $\mathbb{V}$.

We have several comments about **V**.

Lemma 11.2. *The vector space* $\mathbf{V}$ *has a basis consisting of the words in* $\mathbb{V}$ *that do not begin with* y *or* xx.

Proof. By Definitions 6.1, 11.1. □

Lemma 11.3. *The sum* $\mathbf{V} = \mathbb{F}\mathbf{1} + \mathbb{F}x + xy\mathbb{V}$ *is direct.*

Proof. By Lemma 11.2. □

We are going to show that $\mathbf{U} = \mathbb{U} \cap \mathbf{V}$. We will do this in several steps. In the first step, we show that $\mathbb{U} \cap \mathbf{V}$ is a submodule of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$.

Lemma 11.4. *The vector space* $\mathbf{V}$ *is invariant under each of*

$$X^{\pm 1}, \qquad Y^{\pm 1}, \qquad K^{\pm 1}, \qquad A_R^*, \qquad B_R^*.$$

Proof. Use Definitions 5.1, 5.4 and Lemma 11.3. □

Lemma 11.5. *The vector space* $\mathbf{V}$ *is invariant under each of*

$$qA_rK^{-1} - q^{-1}A_\ell, \qquad\qquad B_rK - B_\ell.$$

Proof. We will use Lemma 11.3. The map $qA_rK^{-1} - q^{-1}A_\ell$ sends $\mathbf{1} \mapsto (q - q^{-1})x$ and $x \mapsto 0$. The map $B_rK - B_\ell$ sends $\mathbf{1} \mapsto 0$ and

$$x \mapsto q^2 x \star y - y \star x = q^2(xy + q^{-2}yx) - (yx + q^{-2}xy) = (q^2 - q^{-2})xy.$$

Pick $(r, s) \in \mathbb{N}^2$ and a word $w \in \mathbb{V}(r, s)$. The map $qA_rK^{-1} - q^{-1}A_\ell$ sends

$$xyw \mapsto q^{1+2s-2r}(xyw) \star x - q^{-1}x \star (xyw). \tag{11.1}$$

Using (7.2) we obtain

$$(xyw) \star x = xy(w \star x) + [2]_q q^{2r-2s-1}xxyw. \tag{11.2}$$

Using (7.1) we obtain

$$x \star (xyw) = q[2]_q xxyw + xy(x \star w). \tag{11.3}$$

By (11.1) – (11.3) the map $qA_rK^{-1} - q^{-1}A_\ell$ sends

$$xyw \mapsto xy\big(q^{1+2s-2r}w \star x - q^{-1}x \star w\big).$$

The map $B_rK - B_\ell$ sends

$$xyw \mapsto q^{2r-2s}(xyw) \star y - y \star (xyw). \tag{11.4}$$

Using (7.2) we obtain

$$(xyw) \star y = xy(w \star y) + q^{2s-2r+2}xyyw + q^{2s-2r}yxyw. \tag{11.5}$$

Using (7.1) we obtain

$$y \star (xyw) = yxyw + q^{-2}xyyw + xy(y \star w). \tag{11.6}$$

By (11.4) – (11.6) the map $B_rK - B_\ell$ sends

$$xyw \mapsto xy\big((q^2 - q^{-2})yw + q^{2r-2s}w \star y - y \star w\big).$$

The result follows from the above comments. □

Lemma 11.6. *The vector space $\mathbb{U} \cap \mathbf{V}$ is a submodule of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$.*

Proof. By Theorem 10.1 and Lemmas 11.4, 11.5. □

Lemma 11.7. *The vector space $\mathbf{U}$ is a submodule of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U} \cap \mathbf{V}$.*

Proof. We have $\mathbf{1} \in \mathbb{U}$ by the comment below (9.2). We have $\mathbf{1} \in \mathbf{V}$ by Lemma 11.3. So $\mathbf{1} \in \mathbb{U} \cap \mathbf{V}$. The result follows in view of Definition 10.3 and Lemma 11.6. □

The $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U} \cap \mathbf{V}$ is a weight module, and its weight spaces are the nonzero subspaces among $\mathbb{U}(r, s) \cap \mathbf{V}$ $(r, s \in \mathbb{N})$. We will return to the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U} \cap \mathbf{V}$ after some comments about $\mathbb{U}$.

Lemma 11.8. *The $U_q(\widehat{\mathfrak{sl}}_2)$-generators E_0, E_1 are locally nilpotent on the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$.*

Proof. By Lemma 6.5 and Theorem 10.1. □

Lemma 11.9. *The $U_q(\widehat{\mathfrak{sl}}_2)$-generators F_0, F_1 are not locally nilpotent on the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$.*

Proof. The words xx and y are contained in $\mathbb{U}$, but $F_0^n(xx) \neq 0$ and $F_1^n y \neq 0$ for all $n \in \mathbb{N}$. □

Lemma 11.10. *The $U_q(\widehat{\mathfrak{sl}}_2)$-generators F_0, F_1 are locally nilpotent on the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U} \cap \mathbf{V}$.*

Proof. First consider F_0. Assume that there exists $v \in \mathbb{U} \cap \mathbf{V}$ such that $F_0^{n+1} v \neq 0$ for all $n \in \mathbb{N}$. We will get a contradiction. By our comments below Lemma 11.7, we may assume without loss of generality that $v \in \mathbb{U}(r, s)$ for some $(r, s) \in \mathbb{N}^2$. We have $r \geq 1$ and $s \geq 1$, by Lemma 11.3 and $F_0^2 \mathbf{1} = 0$ and $F_0 x = 0$. Let $n \in \mathbb{N}$. By (10.2) and Lemma 11.6, we have $F_0^{n+1} v \in \mathbb{U}(r + n + 1, s) \cap \mathbf{V}$. In particular $F_0^{n+1} v \in \mathbf{V}$, so $(A_L^*)^2 F_0^{n+1} v = 0$ and $B_L^* F_0^{n+1} v = 0$ in view of Definition 11.1. We have $A_L^* F_0^{n+1} v \neq 0$, by Lemma 6.7(v) and since $B_L^* F_0^{n+1} v = 0$. We have $A_L^* F_0^{n+1} v \in \mathbb{U}(r + n, s)$ by Lemma 9.5. By these comments $0 \neq A_L^* F_0^{n+1} v \in \ker(A_L^*) \cap \mathbb{U}(r+n, s)$. Therefore $0 \neq \ker(A_L^*) \cap \mathbb{U}(r+n, s)$. The map A_L^* is locally nilpotent by Lemma 6.5. In Appendix A we find $A_L^* A_\ell - q^2 A_\ell A_L^* = I$. The vector space $\mathbb{U}$ is invariant under A_L^* and A_ℓ. By these comments, we may apply Appendix E with $S = A_L^*$ and $T = A_\ell$ and $V = \mathbb{U}$. By Lemma 17.8 the map A_L^* is surjective on $\mathbb{U}$. By this and Lemma 9.5, $A_L^* \mathbb{U}(r + n, s) = \mathbb{U}(r + n - 1, s)$. By this and $0 \neq \ker(A_L^*) \cap \mathbb{U}(r + n, s)$, we obtain $\dim \mathbb{U}(r + n - 1, s) < \dim \mathbb{U}(r + n, s)$. Since $n \in \mathbb{N}$ is arbitrary,

$$\dim \mathbb{U}(r - 1, s) < \dim \mathbb{U}(r, s) < \dim \mathbb{U}(r + 1, s) < \dim \mathbb{U}(r + 2, s) < \cdots$$

This contradicts (3.8) and (9.3), so F_0 is locally nilpotent.

Next we consider F_1. Assume that there exists $v \in \mathbb{U} \cap \mathbf{V}$ such that $F_1^{m+1} v \neq 0$ for all $m \in \mathbb{N}$. We will get a contradiction. By our comments below Lemma 11.7, we may assume without loss of generality that $v \in \mathbb{U}(r, s)$ for some $(r, s) \in \mathbb{N}^2$. We have $r \geq 1$ and $s \geq 1$, by Lemma 11.3 and $F_1 \mathbf{1} = 0$ and $F_1^3 x = 0$. Let $m \in \mathbb{N}$. By (10.3) and Lemma 11.6, we have $F_1^{m+1} v \in \mathbb{U}(r, s + m + 1) \cap \mathbf{V}$. In particular $F_1^{m+1} v \in \mathbf{V}$, so $B_L^* F_1^{m+1} v = 0$ in view of Definition 11.1. By these comments $0 \neq F_1^{m+1} v \in \ker(B_L^*) \cap \mathbb{U}(r, s + m + 1)$.

Therefore $0 \neq \ker(B^*_L) \cap \mathbb{U}(r, s+m+1)$. The map B^*_L is locally nilpotent by Lemma 6.5. In Appendix A we find $B^*_L B_\ell - q^2 B_\ell B^*_L = I$. The vector space $\mathbb{U}$ is invariant under B^*_L and B_ℓ. By these comments, we may apply Appendix E with $S = B^*_L$ and $T = B_\ell$ and $V = \mathbb{U}$. By Lemma 17.8 the map B^*_L is surjective on $\mathbb{U}$. By this and Lemma 9.5, $B^*_L \mathbb{U}(r, s+m+1) = \mathbb{U}(r, s+m)$. By this and $0 \neq \ker(B^*_L) \cap \mathbb{U}(r, s+m+1)$, we obtain $\dim \mathbb{U}(r, s+m) < \dim \mathbb{U}(r, s+m+1)$. Since $m \in \mathbb{N}$ is arbitrary,

$$\dim \mathbb{U}(r,s) < \dim \mathbb{U}(r,s+1) < \dim \mathbb{U}(r,s+2) < \dim \mathbb{U}(r,s+3) < \cdots$$

This contradicts (3.7) and (9.3), so F_1 is locally nilpotent. $\square$

The following is our third main result.

Theorem 11.11. *We have* $\mathbf{U} = \mathbb{U} \cap \mathbf{V}$.

Proof. By Lemmas 11.8, 11.10 the $U_q(\widehat{\mathfrak{sl}}_2)$-generators E_0, E_1, F_0, F_1 are locally nilpotent on the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U} \cap \mathbf{V}$. Therefore, the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U} \cap \mathbf{V}$ is integrable in the sense of [4, Definition 4.2]. By this and [20, Theorem 3.5.4], the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U} \cap \mathbf{V}$ is completely reducible. By Lemma 11.7, $\mathbf{U}$ is a submodule of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U} \cap \mathbf{V}$. By these comments, there exists a submodule W of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U} \cap \mathbf{V}$ such that the sum $\mathbb{U} \cap \mathbf{V} = \mathbf{U} + W$ is direct. Assume for the moment that $W \neq 0$. Then $\mathbf{U} \subseteq W$ by Lemma 10.4(i), for a contradiction. Consequently $W = 0$, so $\mathbf{U} = \mathbb{U} \cap \mathbf{V}$. $\square$

12 Variations on the theme

In Theorem 10.1 we turned the vector space $\mathbb{U}$ into a $U_q(\widehat{\mathfrak{sl}}_2)$-module. In this section we describe three more ways to do this. Each way yields a $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$ that is isomorphic to the one in Theorem 10.1.

Proposition 12.1. *For each row in the table below, the vector space $\mathbb{U}$ becomes a $U_q(\widehat{\mathfrak{sl}}_2)$-module on which the $U_q(\widehat{\mathfrak{sl}}_2)$-generators act as indicated:*

generator	E_0	F_0	$K_0^{\pm 1}$	E_1	F_1	$K_1^{\pm 1}$	$D^{\pm 1}$
action on $\mathbb{U}$	B^*_R	$\frac{qB_rK - q^{-1}B_\ell}{q-q^{-1}}$	$q^{\pm 1}K^{\pm 1}$	A^*_R	$\frac{A_rK^{-1} - A_\ell}{q-q^{-1}}$	$K^{\mp 1}$	$Y^{\mp 1}$
action on $\mathbb{U}$	A^*_L	$\frac{qA_\ell K^{-1} - q^{-1}A_r}{q-q^{-1}}$	$q^{\pm 1}K^{\mp 1}$	B^*_L	$\frac{B_\ell K - B_r}{q-q^{-1}}$	$K^{\pm 1}$	$X^{\mp 1}$
action on $\mathbb{U}$	B^*_L	$\frac{qB_\ell K - q^{-1}B_r}{q-q^{-1}}$	$q^{\pm 1}K^{\pm 1}$	A^*_L	$\frac{A_\ell K^{-1} - A_r}{q-q^{-1}}$	$K^{\mp 1}$	$Y^{\mp 1}$

The above three $U_q(\widehat{\mathfrak{sl}}_2)$-modules $\mathbb{U}$ are isomorphic to the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbb{U}$ in Theorem 10.1. *For row 1 (resp. row 2) (resp. row 3), a $U_q(\widehat{\mathfrak{sl}}_2)$-module isomorphism is given by the restriction of σ (resp. $\dagger$) (resp. τ) to $\mathbb{U}$.*

Proof. Below (9.1) we mentioned that $\mathbb{U}$ is invariant under each of σ, $\dagger$, τ. The result follows from this along with Lemmas 9.3, 9.4, 9.6 and Lemmas 5.7, 6.10, 8.5. $\square$

13 Appendix A: Some relations

In this appendix we list some relations satisfied by the maps

$$K, \quad K^{-1}, \quad A^*_L, \quad B^*_L, \quad A^*_R, \quad B^*_R, \quad A_\ell, \quad B_\ell, \quad A_r, \quad B_r.$$

Proposition 13.1 (See [32, Proposition 9.1]). *We have*

$$KA_L^* = q^{-2}A_L^*K, \qquad KB_L^* = q^2B_L^*K,$$
$$KA_R^* = q^{-2}A_R^*K, \qquad KB_R^* = q^2B_R^*K,$$

$$KA_\ell = q^2A_\ell K, \qquad KB_\ell = q^{-2}B_\ell K,$$
$$KA_r = q^2A_rK, \qquad KB_r = q^{-2}B_rK,$$

$$A_L^*A_R^* = A_R^*A_L^*, \qquad B_L^*B_R^* = B_R^*B_L^*,$$
$$A_L^*B_R^* = B_R^*A_L^*, \qquad B_L^*A_R^* = A_R^*B_L^*,$$

$$A_\ell A_r = A_rA_\ell, \qquad B_\ell B_r = B_rB_\ell,$$
$$A_\ell B_r = B_rA_\ell, \qquad B_\ell A_r = A_rB_\ell,$$

$$A_L^*B_r = B_rA_L^*, \qquad B_L^*A_r = A_rB_L^*,$$
$$A_R^*B_\ell = B_\ell A_R^*, \qquad B_R^*A_\ell = A_\ell B_R^*,$$

$$A_L^*B_\ell = q^{-2}B_\ell A_L^*, \qquad B_L^*A_\ell = q^{-2}A_\ell B_L^*,$$
$$A_R^*B_r = q^{-2}B_rA_R^*, \qquad B_R^*A_r = q^{-2}A_rB_R^*,$$

$$A_L^*A_\ell - q^2A_\ell A_L^* = I, \qquad A_R^*A_r - q^2A_rA_R^* = I,$$
$$B_L^*B_\ell - q^2B_\ell B_L^* = I, \qquad B_R^*B_r - q^2B_rB_R^* = I,$$

$$A_L^*A_r - A_rA_L^* = K, \qquad B_L^*B_r - B_rB_L^* = K^{-1},$$
$$A_R^*A_\ell - A_\ell A_R^* = K, \qquad B_R^*B_\ell - B_\ell B_R^* = K^{-1},$$

$$A_\ell^3B_\ell - [3]_qA_\ell^2B_\ell A_\ell + [3]_qA_\ell B_\ell A_\ell^2 - B_\ell A_\ell^3 = 0,$$
$$B_\ell^3A_\ell - [3]_qB_\ell^2A_\ell B_\ell + [3]_qB_\ell A_\ell B_\ell^2 - A_\ell B_\ell^3 = 0,$$
$$A_r^3B_r - [3]_qA_r^2B_rA_r + [3]_qA_rB_rA_r^2 - B_rA_r^3 = 0,$$
$$B_r^3A_r - [3]_qB_r^2A_rB_r + [3]_qB_rA_rB_r^2 - A_rB_r^3 = 0.$$

Proposition 13.2 (See [32, Proposition 9.3]). *The following relations hold on* $\mathbb{U}$*:*

$$(A_L^*)^3B_L^* - [3]_q(A_L^*)^2B_L^*A_L^* + [3]_qA_L^*B_L^*(A_L^*)^2 - B_L^*(A_L^*)^3 = 0,$$
$$(B_L^*)^3A_L^* - [3]_q(B_L^*)^2A_L^*B_L^* + [3]_qB_L^*A_L^*(B_L^*)^2 - A_L^*(B_L^*)^3 = 0,$$
$$(A_R^*)^3B_R^* - [3]_q(A_R^*)^2B_R^*A_R^* + [3]_qA_R^*B_R^*(A_R^*)^2 - B_R^*(A_R^*)^3 = 0,$$
$$(B_R^*)^3A_R^* - [3]_q(B_R^*)^2A_R^*B_R^* + [3]_qB_R^*A_R^*(B_R^*)^2 - A_R^*(B_R^*)^3 = 0.$$

14 Appendix B: The algebra $U_q(\widehat{\mathfrak{sl}}_2)$

In this appendix we recall the quantized enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$. We will generally follow the approach of Ariki [4, Section 3.3]. We will refer to the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

We index the rows and columns of $\mathbf{A}$ by $0, 1$.

Definition 14.1 (See [4, Definition 3.16])**.** Define the algebra $U_q(\widehat{\mathfrak{sl}}_2)$ by generators

$$K_i^{\pm 1}, \quad D^{\pm 1}, \quad E_i, \quad F_i, \qquad\qquad i \in \{0, 1\}$$

and the following relations. For $i, j \in \{0, 1\}$,

$$\begin{aligned}
&K_i K_i^{-1} = K_i^{-1} K_i = 1, \qquad DD^{-1} = D^{-1}D = 1, \\
&[K_i, K_j] = 0, \qquad [D, K_i] = 0, \\
&K_i E_j K_i^{-1} = q^{\mathbf{A}_{i,j}} E_j, \qquad K_i F_j K_i^{-1} = q^{-\mathbf{A}_{i,j}} F_j, \\
&DE_0D^{-1} = qE_0, \qquad DF_0D^{-1} = q^{-1}F_0, \\
&[D, E_1] = 0, \qquad [D, F_1] = 0, \\
&[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
&[E_i, [E_i, [E_i, E_j]_q]_{q^{-1}}] = 0, \qquad [F_i, [F_i, [F_i, F_j]_q]_{q^{-1}}] = 0, \qquad i \neq j.
\end{aligned}$$

Note 14.2. The Ariki notation is related to our notation as follows.

Ariki notation	our notation
v	q
t_i	K_i
v^d	D
$\alpha_j(h_i)$	$\mathbf{A}_{i,j}$

Note 14.3 (See [4, Section 3.3])**.** The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is sometimes called the quantum algebra of type $A_1^{(1)}$.

15 Appendix C: The subspaces $\mathbf{U}(r, s)$

In this appendix, we give a basis for each nonzero $\mathbf{U}(r, s)$ such that $r + s \leq 10$.

r	s	basis for $\mathbf{U}(r,s)$
0	0	$\mathbf{1}$

r	s	basis for $\mathbf{U}(r,s)$
1	0	x

r	s	basis for $\mathbf{U}(r,s)$
1	1	xy

r	s	basis for $\mathbf{U}(r,s)$
2	1	xyx
1	2	xyy

r	s	basis for $\mathbf{U}(r,s)$
2	2	$xyxy, xyyx$

r	s	basis for $\mathbf{U}(r,s)$
3	2	$xyxyx, xyyxx$
2	3	$xyxyy + xyyxy$

r	s	basis for $\mathbf{U}(r,s)$
4	2	$xyxyxx + [3]_q xyyxxx$
3	3	$xyxyxy, xyyxxy, xyyxyx + xyxyyx$

r	s	basis for $\mathbf{U}(r,s)$
4	3	$xyxyxyx, xyyxyxx + xyxyyxx + xyyxxyx,$
		$xyxyxxy + [3]_q xyyxxxy + xyyxxyx$
3	4	$xyyxxyy, xyyxyxy + xyxyyxy + xyxyxyy$

r	s	basis for $\mathbf{U}(r,s)$
5	3	$[3]_q xyyxyxxx + [3]_q xyxyyxxx + [2]_q^2 xyyxxyxx$
		$+2xyxyxyxx + xyxyxxyx + [3]_q xyyxxxyx$
4	4	$xyxyxyxy, xyyxxyyx,$
		$xyyxyxxy + xyxyyxxy + xyyxxyxy,$
		$xyxyxxyy + [3]_q xyyxxxyy + xyyxxyxy,$
		$xyyxyxyx + xyxyyxyx + xyxyxyyx$

r	s	basis for $\mathbf{U}(r,s)$
5	4	$[3]_q xyyxyxxxy + [3]_q xyxyyxxxy + [2]_q^2 xyyxxyxxy$
		$+2xyxyxyxxy + xyxyxxyxy + [3]_q xyyxxxyxy$
		$+xyyxyxxyx + xyxyyxxyx + xyyxxyxyx,$
		$xyxyxyxyx, xyyxxyyxx,$
		$xyxyxxyyx + [3]_q xyyxxxyyx + xyyxxyxyx,$
		$xyyxyxyxx + xyxyyxyxx + xyxyxyyxx$
		$+xyyxyxxyx + xyxyyxxyx + xyyxxyxyx$
4	5	$xyyxyxxyy + xyxyyxxyy + xyyxxyxyy + xyyxxyyxy,$
		$xyxyxyyxy + xyxyyxyxy + xyyxyxyxy + xyxyxyxyy,$
		$[3]_q xyxyxxyyy + [3]_q^2 xyyxxxyyy + [3]_q xyyxxyxyy + xyxyxyxyy$

r	s	basis for $\mathbf{U}(r,s)$
6	4	$[3]_q xyyxyxxxyx + [3]_q xyxyyxxxyx + [2]_q^2 xyyxxyxxyx$
		$+2xyxyxyxxyx + xyxyxxyxyx + [3]_q xyyxxxyxyx$
		$+xyyxyxxyxx + xyxyyxxyxx + xyyxxyxyxx$
		$+3xyxyxyxyxx + [3]_q xyyxyxyxxx + [3]_q xyxyyxyxxx$
		$+[3]_q xyxyxyyxxx + [3]_q xyyxyxxyxx + [3]_q xyxyyxxyxx + [3]_q xyyxxyxyxx,$
		$[3]_q xyyxxyyxxx + xyxyxxyyxx + [3]_q xyyxxxyyxx + xyyxxyxyxx$
5	5	$xyxyxyxyxy,$
		$xyxyxyxyyx + xyxyxyyxyx + xyxyyxyxyx + xyyxyxyxyx,$
		$xyxyxyyxxy + xyxyyxyxxy + xyyxyxyxxy + xyxyyxxyxy$
		$+xyyxxyxyxy + xyyxyxxyxy,$
		$xyyxxyyxyx + xyyxxyxyyx + xyyxyxxyyx + xyxyyxxyyx,$
		$xyyxxyyxxy,$
		$[3]_q xyyxxyxyyx + xyxyxxyyxy + [3]_q xyyxxxyyxy + [3]_q^2 xyyxxxyyyx$
		$+[3]_q xyxyxxyyyx + xyxyxyxyyx + xyyxxyxyxy,$
		$xyxyyxxyxy + [3]_q xyxyyxxxyy + 2xyxyxyxxyy + xyyxyxxyxy$
		$+[3]_q xyyxyxxxyy + [2]_q^2 xyyxxyxxyy + 2xyyxxyxyxy + [3]_q xyyxxxyxyy$
		$+[3]_q xyyxxxyyxy + xyxyxxyxyy + xyxyxxyyxy$
4	6	$[2]_q[3]_q xyyxyxxyyy + [2]_q[3]_q xyxyyxxyyy + [2]_q[3]_q xyyxxyxyyy$
		$+[2]_q[3]_q xyyxxyyxyy + [2]_q xyxyxyyxyy + [2]_q xyxyyxyxyy$
		$+[2]_q xyyxyxyxyy + [2]_q[3]_q xyxyxyxyyy + [3]_q[4]_q xyxyxxyyyy$
		$+[3]_q^2[4]_q xyyxxxyyyy + [3]_q[4]_q xyyxxyxyyy$

16 Appendix D: Some matrix representations

In this appendix we consider the $U_q(\widehat{\mathfrak{sl}}_2)$-module $\mathbf{U}$ from Definition 10.3. We display the matrices that represent the actions of $E_0, F_0, K_0, E_1, F_1, K_1, D$ on the bases in Appendix C.

On $\mathbf{U}(0,0)$:

$$K_0 : \big(q\big), \qquad K_1 : \big(1\big), \qquad D : \big(1\big).$$

From $\mathbf{U}(1,0)$ to $\mathbf{U}(0,0)$:

$$E_0 : \big(1\big), \qquad E_1 : \big(0\big)$$

From $\mathbf{U}(0,0)$ to $\mathbf{U}(1,0)$:

$$F_0 : \big(1\big), \qquad F_1 : \big(0\big)$$

On $\mathbf{U}(1,0)$:

$$K_0 : \big(q^{-1}\big), \qquad K_1 : \big(q^2\big), \qquad D : \big(q^{-1}\big).$$

From $\mathbf{U}(1,1)$ to $\mathbf{U}(1,0)$:

$$E_0 : \big(0\big), \qquad E_1 : \big(1\big)$$

From $\mathbf{U}(1,0)$ to $\mathbf{U}(1,1)$:

$$F_0 : \big(0\big), \qquad F_1 : \big([2]_q\big)$$

On $\mathbf{U}(1,1)$:

$$K_0 : \begin{pmatrix} q \end{pmatrix}, \qquad K_1 : \begin{pmatrix} 1 \end{pmatrix}, \qquad D : \begin{pmatrix} q^{-1} \end{pmatrix}.$$

From $\mathbf{U}(2,1) + \mathbf{U}(1,2)$ to $\mathbf{U}(1,1)$:

$$E_0 : \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad E_1 : \begin{pmatrix} 0 & 1 \end{pmatrix}$$

From $\mathbf{U}(1,1)$ to $\mathbf{U}(2,1) + \mathbf{U}(1,2)$:

$$F_0 : \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad F_1 : \begin{pmatrix} 0 \\ [2]_q \end{pmatrix}$$

On $\mathbf{U}(2,1) + \mathbf{U}(1,2)$:

$$K_0 : \mathrm{diag}(q^{-1}, q^3), \qquad K_1 : \mathrm{diag}(q^2, q^{-2}), \qquad D : \mathrm{diag}(q^{-2}, q^{-1}).$$

From $\mathbf{U}(2,2)$ to $\mathbf{U}(2,1) + \mathbf{U}(1,2)$:

$$E_0 : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

From $\mathbf{U}(2,1) + \mathbf{U}(1,2)$ to $\mathbf{U}(2,2)$:

$$F_0 : \begin{pmatrix} 0 & 1 \\ 0 & [3]_q \end{pmatrix}, \quad F_1 : \begin{pmatrix} [2]_q & 0 \\ [2]_q & 0 \end{pmatrix}$$

On $\mathbf{U}(2,2)$:

$$K_0 : \mathrm{diag}(q, q), \qquad K_1 : \mathrm{diag}(1, 1), \qquad D : \mathrm{diag}(q^{-2}, q^{-2}).$$

From $\mathbf{U}(3,2) + \mathbf{U}(2,3)$ to $\mathbf{U}(2,2)$:

$$E_0 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_1 : \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

From $\mathbf{U}(2,2)$ to $\mathbf{U}(3,2) + \mathbf{U}(2,3)$:

$$F_0 : \begin{pmatrix} 1 & 1 \\ 0 & [2]_q^2 \\ 0 & 0 \end{pmatrix}, \quad F_1 : \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ [2]_q & 0 \end{pmatrix}$$

On $\mathbf{U}(3,2) + \mathbf{U}(2,3)$:

$$K_0 : \mathrm{diag}(q^{-1}, q^{-1}, q^3), \qquad K_1 : \mathrm{diag}(q^2, q^2, q^{-2}), \qquad D : \mathrm{diag}(q^{-3}, q^{-3}, q^{-2}).$$

From $\mathbf{U}(4,2) + \mathbf{U}(3,3)$ to $\mathbf{U}(3,2) + \mathbf{U}(2,3)$:

$$E_0 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ [3]_q & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_1 : \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From $\mathbf{U}(3,2)+\mathbf{U}(2,3)$ to $\mathbf{U}(4,2)+\mathbf{U}(3,3)$:

$$F_0:\begin{pmatrix}0&1&0\\0&0&2\\0&0&[2]_q^2\\0&0&[3]_q\end{pmatrix},\quad F_1:\begin{pmatrix}0&0&0\\ [2]_q&0&0\\0&[2]_q&0\\ [2]_q&0&0\end{pmatrix}$$

On $\mathbf{U}(4,2)+\mathbf{U}(3,3)$:

$$K_0:\mathrm{diag}(q^{-3},q,q,q),\qquad K_1:\mathrm{diag}(q^4,1,1,1),\qquad D:\mathrm{diag}(q^{-4},q^{-3},q^{-3},q^{-3}).$$

From $\mathbf{U}(4,3)+\mathbf{U}(3,4)$ to $\mathbf{U}(4,2)+\mathbf{U}(3,3)$:

$$E_0:\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\0&1&1&0&0\\0&1&0&0&0\end{pmatrix},\quad E_1:\begin{pmatrix}0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$$

From $\mathbf{U}(4,2)+\mathbf{U}(3,3)$ to $\mathbf{U}(4,3)+\mathbf{U}(3,4)$:

$$F_0:\begin{pmatrix}0&1&0&2\\0&0&0&[2]_q^2\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad F_1:\begin{pmatrix}[2]_q&0&0&0\\ [2]_q&0&0&0\\ [4]_q&0&0&0\\0&0&[2]_q&0\\0&[2]_q&0&0\end{pmatrix}$$

On $\mathbf{U}(4,3)+\mathbf{U}(3,4)$:

$$K_0:\mathrm{diag}(q^{-1},q^{-1},q^{-1},q^3,q^3),\qquad K_1:\mathrm{diag}(q^2,q^2,q^2,q^{-2},q^{-2}),$$
$$D:\mathrm{diag}(q^{-4},q^{-4},q^{-4},q^{-3},q^{-3}).$$

From $\mathbf{U}(5,3)+\mathbf{U}(4,4)$ to $\mathbf{U}(4,3)+\mathbf{U}(3,4)$:

$$E_0:\begin{pmatrix}2&0&0&0&0&0\\ [3]_q&0&0&0&0&0\\1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{pmatrix},\quad E_1:\begin{pmatrix}0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$$

From $\mathbf{U}(4,3)+\mathbf{U}(3,4)$ to $\mathbf{U}(5,3)+\mathbf{U}(4,4)$:

$$F_0:\begin{pmatrix}0&1&0&0&0\\0&0&0&0&3\\0&0&0&[3]_q&0\\0&0&0&0&[2]_q^2\\0&0&0&1&0\\0&0&0&0&[3]_q\end{pmatrix},\quad F_1:\begin{pmatrix}0&0&0&0&0\\ [2]_q&0&[2]_q&0&0\\0&[2]_q&[2]_q&0&0\\0&[2]_q&[2]_q&0&0\\0&0&[2]_q[3]_q&0&0\\ [2]_q&0&0&0&0\end{pmatrix}$$

On $\mathbf{U}(5,3)+\mathbf{U}(4,4)$:

$$K_0:\mathrm{diag}(q^{-3},q,q,q,q,q),\qquad K_1:\mathrm{diag}(q^4,1,1,1,1,1),$$
$$D:\mathrm{diag}(q^{-5},q^{-4},q^{-4},q^{-4},q^{-4},q^{-4}).$$

From $\mathbf{U}(5,4)+\mathbf{U}(4,5)$ to $\mathbf{U}(5,3)+\mathbf{U}(4,4)$:

$$E_0:\begin{pmatrix}0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\end{pmatrix},\quad E_1:\begin{pmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&[3]_q\\0&0&0&0&0&0&1&0\end{pmatrix}$$

From $\mathbf{U}(5,3)+\mathbf{U}(4,4)$ to $\mathbf{U}(5,4)+\mathbf{U}(4,5)$:

$$F_0:\begin{pmatrix}0&0&0&1&0&0\\0&1&0&0&0&3\\0&0&[2]_q^2&0&0&0\\0&0&1&0&1&0\\0&0&0&0&0&[2]_q^2\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\quad F_1:\begin{pmatrix}[4]_q&0&0&0&0&0\\3[2]_q&0&0&0&0&0\\[2]_q^3&0&0&0&0&0\\[2]_q[3]_q&0&0&0&0&0\\2[2]_q&0&0&0&0&0\\0&0&0&[2]_q&[2]_q&0\\0&[2]_q&0&0&0&0\\0&0&0&0&[2]_q&0\end{pmatrix}$$

On $\mathbf{U}(5,4)+\mathbf{U}(4,5)$:

$K_0:\mathrm{diag}(q^{-1},q^{-1},q^{-1},q^{-1},q^{-1},q^{3},q^{3},q^{3}), K_1:\mathrm{diag}(q^{2},q^{2},q^{2},q^{2},q^{2},q^{-2},q^{-2},q^{-2})$,
$D:\mathrm{diag}(q^{-5},q^{-5},q^{-5},q^{-5},q^{-5},q^{-4},q^{-4},q^{-4})$.

From $\mathbf{U}(6,4)+\mathbf{U}(5,5)+\mathbf{U}(4,6)$ to $\mathbf{U}(5,4)+\mathbf{U}(4,5)$:

$$E_0:\begin{pmatrix}1&0&0&0&0&0&0&0&0&0\\3&0&0&0&0&0&0&0&0&0\\0&[3]_q&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\[3]_q&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\end{pmatrix},$$

$$E_1:\begin{pmatrix}0&0&0&0&0&0&0&0&1&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&1&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&[2]_q[3]_q\\0&0&0&0&0&0&0&0&0&[2]_q\\0&0&0&0&0&0&0&0&0&[4]_q\end{pmatrix}$$

From $\mathbf{U}(5,4)+\mathbf{U}(4,5)$ to $\mathbf{U}(6,4)+\mathbf{U}(5,5)+\mathbf{U}(4,6)$:

$$F_0:\begin{pmatrix}0&0&0&0&1&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&4&1\\0&0&0&0&0&0&[3]_q&0\\0&0&0&0&0&0&[2]_q^2&0\\0&0&0&0&0&[3]_q&0&0\\0&0&0&0&0&[2]_q^2&0&0\\0&0&0&0&0&0&0&[3]_q\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\end{pmatrix},$$

$$F_1:\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\3[2]_q&[2]_q&0&0&0&0&0&0\\0&[2]_q&0&0&0&0&0&0\\2[2]_q&0&0&0&[2]_q&0&0&0\\[2]_q&0&0&[2]_q&[2]_q&0&0&0\\[2]_q^3&0&[2]_q&0&0&0&0&0\\0&0&0&[2]_q&0&0&0&0\\[2]_q[3]_q&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\end{pmatrix}$$

On $\mathbf{U}(6,4)+\mathbf{U}(5,5)+\mathbf{U}(4,6)$:

$$K_0:\operatorname{diag}(q^{-3},q^{-3},q,q,q,q,q,q,q,q^5),\qquad K_1:\operatorname{diag}(q^4,q^4,1,1,1,1,1,1,1,q^{-4}),$$
$$D:\operatorname{diag}(q^{-6},q^{-6},q^{-5},q^{-5},q^{-5},q^{-5},q^{-5},q^{-5},q^{-5},q^{-4}).$$

17 Appendix E: Some linear algebra

In this appendix we consider the following situation. Let V denote an infinite-dimensional vector space. Let $S\colon V\to V$ and $T\colon V\to V$ denote $\mathbb{F}$-linear maps. Assume that S is locally nilpotent and

$$ST-q^2TS=I. \tag{17.1}$$

We will show that S is surjective and T is injective. We remark that the surjectivity of S is used in the proof of Lemma 11.10, and the injectivity of T is used to obtain the surjectivity of S.

Lemma 17.1. *The map T is injective.*

Proof. Let $v\in V$ such that $Tv=0$. We show that $v=0$. For $n\geq 1$, use (17.1) and induction on n to obtain

$$TS^nv=-q^{-n-1}[n]_qS^{n-1}v. \tag{17.2}$$

Since S is locally nilpotent, there exists $n\geq 1$ such that $S^nv=0$. By applying (17.2) repeatedly, we see that each of $S^nv,S^{n-1}v,\ldots Sv,v$ is equal to 0. In particular $v=0$. $\square$

For $n \in \mathbb{N}$, we adjust (17.1) to obtain

$$ST - q^n[n+1]_q I = q^2\big(TS - q^{n-1}[n]_q I\big). \tag{17.3}$$

Therefore, the kernel of $ST - q^n[n+1]_q I$ is equal to the kernel of $TS - q^{n-1}[n]_q I$. Let V_n denote this common kernel. By construction

$$\big(ST - q^n[n+1]_q I\big)V_n = 0, \qquad \big(TS - q^{n-1}[n]_q I\big)V_n = 0. \tag{17.4}$$

Note that the sum $\sum_{n\in\mathbb{N}} V_n$ is direct. For notational convenience define $V_{-1} = 0$.

Lemma 17.2. *We have* $\ker(S) = V_0$.

Proof. By the discussion below (17.3), we obtain $\ker(TS) = V_0$. The map T is injective by Lemma 17.1, so $\ker(S) = \ker(TS)$. Therefore $\ker(S) = V_0$. □

Lemma 17.3. *For* $n \in \mathbb{N}$ *we have*

$$SV_n \subseteq V_{n-1}, \qquad TV_n \subseteq V_{n+1}.$$

Proof. First we verify $SV_n \subseteq V_{n-1}$. For $n = 0$ this holds by Lemma 17.2. For $n \geq 1$ we use (17.4) to obtain

$$\big(ST - q^{n-1}[n]_q I\big)SV_n = S\big(TS - q^{n-1}[n]_q I\big)V_n = S0 = 0,$$

so $SV_n \subseteq V_{n-1}$. Next we verify $TV_n \subseteq V_{n+1}$. For $n \geq 0$ we have

$$\big(TS - q^n[n+1]_q I\big)TV_n = T\big(ST - q^n[n+1]_q I\big)W_n = T0 = 0,$$

so $TV_n \subseteq V_{n+1}$. □

Lemma 17.4. *For* $n \geq 1$ *the following maps are inverses:*

$$S\colon V_n \to V_{n-1}, \qquad q^{1-n}[n]_q^{-1}T\colon V_{n-1} \to V_n. \tag{17.5}$$

Proof. By (17.4) we have $(ST_n - I)V_{n-1} = 0$ and $(T_nS - I)V_n = 0$, where $T_n = q^{1-n}[n]_q^{-1}T$. □

Lemma 17.5. *For* $n \geq 1$ *the maps*

$$S\colon V_n \to V_{n-1}, \qquad T\colon V_{n-1} \to V_n$$

are bijections.

Proof. By Lemma 17.4. □

Lemma 17.6. *For* $n \in \mathbb{N}$,

$$\ker(S^{n+1}) = V_0 + V_1 + \cdots + V_n. \tag{17.6}$$

Proof. We use induction on n. First assume that $n = 0$. Then (17.6) holds by Lemma 17.2. Next assume that $n \geq 1$. The inclusion $\supseteq$ in (17.6) holds by Lemma 17.3. We next obtain the inclusion $\subseteq$ in (17.6). Let $v \in \ker(S^{n+1})$. We will show that $v \in V_0 + V_1 + \cdots + V_n$. We have $0 = S^{n+1}v = S^nSv$, so by induction $Sv \in V_0 + V_1 + \cdots + V_{n-1}$. By Lemma 17.5, there exists $w \in V_1 + V_2 + \cdots + V_n$ such that $Sw = Sv$. Therefore $S(w - v) = 0$, so $w - v \in V_0$. By these comments $v \in V_0 + V_1 + \cdots + V_n$. □

Lemma 17.7. *We have $V = \sum_{n\in\mathbb{N}} V_n$.*

Proof. Since S is locally nilpotent, we have $V = \cup_{n\in\mathbb{N}}\ker(S^{n+1})$. The result follows from this and Lemma 17.6. □

Lemma 17.8. *The map S is surjective.*

Proof. By Lemma 17.7 and since $V_n = SV_{n+1}$ for $n \in \mathbb{N}$. □

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