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Three firms on a unit disk market: intermediate product differentiation

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Abstract: *Irmen and Thisse (1998) demonstrate that two firms competing with multicharacteristic products differentiate them in one characteristic completely, while keeping them identical in all others. This paper shows that their* min-…-min-max *differentiation result is not robust with respect to the number of firms. A market setting that replicates their result in a duopoly, but fails to do so in a three firm oligopoly is identified. Symmetric pure strategy equilibrium with three firms differentiating their products in two* dimensions, but not completely in either of them, is a novel medium-medium differen*tiation result.*

Keywords: *spatial competition, location-price game* **JEL classification:** L11, L13, R39

Three firms on a unit disk market: intermediate product differentiation

Models of discrete location choice commonly interpreted as modeling product differentiation, as well, have been explored in a variety of contexts.² Irmen and Thisse (1998) provide the most comprehensive study of two firms competing with their products in an

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² ReVelle and Eiselt (2005), and ReVelle, Eiselt, and Daskin (2008) provide a comprehensive survey of the field and collect an extensive bibliography covering various aspects and problems in this area, respectively. A part of the literature is interested in the extent of product differentiation that firms should employ, and the number of product dimensions they should use doing that. Pioneering work with a linear duopoly model by Hotelling (1929) offered a principle of minimum differentiation. Hotelling's contribution was revisited 50 years later, when d'Aspremont *et al*. (1979) showed that there is no price equilibrium in pure strategies when two firms are located too closely to each other. Using quadratic instead of linear transportation costs, they find unique market equilibrium with firms maximizing product differentiation. The direct demand effect that makes a firm move towards its opponent to capture its demand is followed by the opponent's price cut. The latter overrides extra profit gained with new demand from moving towards the opponent. The negative strategic effect of igniting stronger competition induces firms to differentiate their products as much as they can. As it was shown later, a maximum differentiation result rests both on the form of consumers' utility function (e.g. Economides, 1986), and the uniform distribution of their tastes within the product characteristics space (e.g. Neven, 1986, Tabuchi & Thisse, 1995).

n-dimensional product space.³ They investigate a unit hyper-cube market that is uniformly populated by consumers and served by two firms. When consumers incur disutility that is quadratic in distance between a product variety that they would prefer the most and the variety bought, they show that it is always optimal for the two firms to differentiate their products in one dimension only, and doing so completely. This type of practice is referred to as *min-…-min-max* product differentiation.

This paper departs from the Irmen in Thisse model in that it studies a market with three firms in a two-dimensional product space. We present a unit disk market uniformly populated by consumers that are served by firms operating one store each. Firms choose respective store locations in the first stage and compete with prices in the second. There are some authors that explore how competition between more than two firms affects product differentiation. Salop (1979) and Economides (1989) look at a circular city model with location equilibria that place firms equidistantly. Economides (1993) provides price equilibrium characterizations for every location configuration in a linear city model, and Brenner (2005) presents location equilibria for different number of firms (from three to nine) in the same type of the market. Brenner shows that firms depart from a maximum differentiation result in that the store locations move towards the middle of the market.

We address two questions. First, will more than two firms in a bounded, two dimensional product characteristics space differentiate their products less than completely, as we might suspect from Brenner (2005)? Brenner shows that a firm with two neighbors does not cut prices as drastically as in a duopoly case when it is approached by one of the neighbors. The reason is that it does not want to alter its optimal revenues from the other side, where it neighbors a firm that did not deviate from an equilibrium position. Consequently, in equilibrium even the two firms on the two outskirts of the city move towards the center. The direct demand effect outweighs the strategic price effect and extent of product differentiation is reduced. There are two reinforcing effects facilitating Brenner's result. First, the area of confrontation between an intrusive firm and its victim is a single point, a marginal consumer between the two firms. Since the area of confrontation between the victim and its other side neighbor is of the same size, the incentive to cut its price to counter the intruder is offset in a large part by a lower price and suboptimal profits on the other side. Second, the other side neighbor anticipates lower prices; it reduces its price as well. Hence, a reduction of victim's price does not translate in a sizeable increase of its demand on the other neighbor's side and is not profitable. Consequently, a cut in a victim's price is not large enough to keep the intrusive firm from moving in to capture a part of its demand. Consequently, firms locate closer to each other. The role of the two effects we have described is less obvious in our case of firms competing in two dimensions. Market configuration may be such that a firm that moves its store does that in a direction of two and not just one neighbor. The firms' demands are now delineated by line segments of consumers that are indifferent between buying from any of the two

³ Irmen and Thisse are not the first to explore markets where consumers care about more than one product characteristic. Neven and Thisse (1990) and Tabuchi (1994) were the first to show that in a two dimensional product characteristics space two firms will never find it optimal to differentiate their products fully. There are equilibria in which products are completely differentiated along one dimension and identical in the other.

neighboring firms. A confrontation frontier for an intrusive firm may hence be longer than is a line segment between respective neighbors fighting the intruder. This means that a price cut by the neighbors does not necessarily be as pronounced as it was in a one-dimensional case but may still keep the intrusive firm away. As a result, it might be that firms stay on the outskirts of the market. On the other hand, we might see a freerider effect, meaning that firms under attack would count on each other to counter the intruder with lower prices. This might lead to a price cut that is insufficient to keep the intruder on the edge of the market.

Another question is whether firms find it optimal to differentiate their respective products in more than one dimension in the first place. If the market was unbounded and consumers' reservation prices finite, the answer is obviously positive. With a bounded market, the answer is not imminent and might depend on the shape of the market in general. We expect that firms will find it beneficial to leave the congested competition in one product characteristic at some point and will choose to differentiate their products in another one as well. Swann (1990) explores such a process with a simple model and simulations. Whenever the field of competition becomes too dense at least one firm endogenously finds it optimal to introduce a new product attribute.

We first show that maximum differentiation in one dimension – and no differentiation in the other (Irmen and Thisse, 1998) – remains optimal in a duopoly. In our setting this means that the two firms position their stores on the perimeter of the disk, exactly opposite from each other. We then present two novel results. An oligopoly with three firms competing on the same market facilitates a pure strategy subgame perfect Nash equilibrium of our location-price game. The equilibrium has all three firms located at the same distance from the center of the disk, equidistant from each other. That means that we observe differentiation in two product characteristics, a result that extends the existent literature. Furthermore, firms do not choose full differentiation, but move inward, towards the center of the disk considerably. We find *medium-medium* type of product differentiation in a setting that yields a *min-max* differentiation result in a duopoly. This means that the conjecture based on Brenner (2005), given above, carries over to markets with more than two competitors and more than just one product characteristic. When firms are located close to the perimeter of the market, the positive demand effect of a radial deviation towards the center outweighs the negative strategic effect of rivals decreasing their prices.

Another interesting aspect of the model is that, while in a duopoly a social planner would have firms differentiating their products less extensively, the result reverses in a threefirm oligopoly. Hence, some cooperative behavior or regulation on product specifications would be beneficial both to firms and to society as a whole.

The organization of the paper is as follows. We set up our model in Section 1, and present our results for a duopoly and a three-firm oligopoly in Sections 2 and 3, respectively. Section 4 considers welfare issues, and we make our conclusions in Section 5. Proofs of Lemmas and Propositions and all necessary derivatives are deferred to the Appendices.

1. The model

We use the classical spatial model of an oligopoly. Consumers of a total mass π are uniformly distributed on a unit disk.⁴ Each consumer has a unit demand for a homogeneous good produced by *n* firms on the market. We explore configurations in which all firms are at the same radial distance from the origin, *R*. Specifically, the firms are located at *Li* =(*R*,φ*ⁱ*), *i* = 1,…, *n*. *Firm* 1 is always a counter-clockwise direction neighbor of *Firm n*, while *Firm n*−1 is a clockwise direction neighbor of *Firm n*. Firms charge *pi* , *i* = 1, …, n, per unit of the good. If a consumer residing at point x buys the good from Firm i, she derives utility: $u(p_i, L_i; x) = v - p_i - d^2(x, L_i)$.

We assume that consumers incur traveling costs that are quadratic in the Euclidean diswe assume that consumers incur traveling costs that are quadratic in the Euclidean dis-
tance, $d(x, L_i)$, traveled, and with cost per unit traveled normalized to one. The surplus v enjoyed from consuming the good is assumed to be large enough so that every consumer buys a good from one of the firms. Consumers maximize their utility by choosing the store they buy the good from optimally. stays a good from one of the firms. Sometimes maximize their during by encounty the Firms operate with symmetric constant marginal constant α ϵ from consuming the good is assumed to be large enough so that every consumer buys assume buys as

Firms operate with symmetric constant marginal costs that we normalize to zero and are allowed to operate one store each. We consider a non-cooperative two-stage location-Stage 1: Firms select locations on a unit disk. price game of the following form. is operate with symmet. \overline{a} of operation-cooperative two-stage location-price game \overline{b} Firms operation state can be consider a non-cooperative two stage focation

Stage 2: Firms observe chosen locations and compete in prices. $\frac{1}{\sqrt{2}}$ strategies in the first period is $\frac{1}{\sqrt{2}}$ Stage 1: Firms select locations on a unit disk. of the following form. Stage 1: Firms select locations on a unit disk.

Stage 2: Firms observe chosen locations and compete in prices. prices.

Formally, firm i's strategy space in the first period is $L_i = [0,1] \times [-\pi,\pi]$, $i=1,\ldots,$ $n,$ and each firm's strategy in the second stage is s $p_i : x_{i-1}^n L_i \to \mathbb{R}_+$. We seek subgame perfect
Neah equilibric of this games s^* (x^*, x^*) , (x^*, x^*) , (x^*, x^*) , A patural equilibric for each firm's strategy in the second stage is s p_i . $x_{i=1}L_i \rightarrow x_i$, we seek subgaine perfect
Nash equilibria of this game: $S^* = \{(L_1^*, p_1^*), (L_2^*, p_2^*), ..., (L_n^*, p_n^*)\}$. A natural candidate for Nash equilibrium configuration has firms positioned at an equal distance from the origin, equidistantly along the circle they occupy. Specifically, we explore locations: $L_1^* = (R, -\pi + \frac{4}{2}\pi)$, $L^* = (R, -\pi + \frac{4}{2}\pi)$, \ldots , and $L^* = (R, \pi)$. $\frac{m}{n}$ first derive matrix $\frac{m}{n}$ $+\frac{2}{n}\pi$, $L_2^* = (R, -\pi + \frac{4}{n}\pi)$, ..., and $L_n^* = (R, \pi)$. … , and Lⁿ $\frac{1}{2}$ configuration has firms positioned at an equal distance from the origin, $\frac{1}{2}$ equidistantly along the circle they occupy. Specifically, we explore locations: $L_1^* = (R, -\pi)$ τ

$\frac{1}{\sqrt{2}}$ stores on the disk, symmetrically across the origin. We search across the origin. We search across the origin. We see $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{10}}$ KMS **2. Two firms**

We first derive market equilibrium in a duopoly. Here we show that min-max result that means that firms locate their stores on the perimeter of the disk, symmetrically Figure 1 showing a possible off-equilibrium configuration $L_1^* = (R, 0)$ and $L_2 = (r, \phi)$. derived by Irmen and Thisse (1998) is also optimal strategy in our setting. In our setting across the origin. We search for an equilibrium that is symmetric in firms' locations with We first derive market equilibrium in a duopoly. Here we show that min-max result
 $\frac{1}{2}$

A=(1,α)

⁴ Polar symmetry is used to avoid non-differentiable demand functions that arise in a rectangular market when a file of consumers mumerent between buying from two neighboring stores touches that with three firms on a square. when a line of consumers indifferent between buying from two neighboring stores touches the corner of a

Figure 1: *Configuration of the market, two firms.*

Line AB in Figure 1 represents consumers that are indifferent between buying a product $\;$ from either of the two firms given their locations and product prices. AB is perpendicular to the line connecting firms' locations L_1^* and L_2 . It is closer to the firm that sets the higher of the two respective prices. AB crosses the perimeter of the disk at angles α and β. Firm 1's demand is the area between α and β , $(\alpha-\beta)/2$, reduced by the area of a triangle ABO, which is $\sin \frac{\alpha-\beta}{2} \cdot \cos \frac{\alpha-\beta}{2} = \sin(\alpha-\beta)/2$. We set $z = \alpha - \beta$ and state the firms' demands: ² ² sin cos ^α−^β ^α−^β [⋅] ⁼sin(^α [−] ^β) / ² . We set ^z [≡] ^α [−] ^β and state the firms' demands: $\frac{a_{\text{max}}}{b}$ and $\frac{a_{\text{max}}}{c}$ and $\frac{a_{\text{max}}}{d}$ to $\frac{a_{\text{max}}}{c}$ or $\frac{a_{\text{max}}}{d}$ α and α since α and α and α and state the firms of α

$$
D_1 = \frac{1}{2}(z - \sin z), \text{ and } D_2 = \pi - D_1 = \frac{1}{2}(2\pi - z + \sin z), \tag{1}
$$

Firms' second-stage profits given locations chosen in the first stage are:

$$
\Pi_1(r,\phi;R) = p_1(r,\phi;R) \cdot D_1(r,\phi;R) \text{ and } \Pi_2(r,\phi;R) = p_2(r,\phi;R) \cdot D_2(r,\phi;R). \tag{2}
$$

 \mathcal{L} and \mathcal{L} is defined by L1 and \mathcal{L} $\sum_{i=1}^{n}$ Note that p_1, p_2, α, β , and z are all functions of firms' respective locations. Second-stage profit maximization with respect to prices yields the following result.

Lemma 1: When firms choose locations $L_1^* = (R,0)$ and $L_2^* = (R,\pi)$ in the first stage of the Echina 1. When this choose locations $L_1 = (R_1R_2 + R_2)$ is the second. Lemma 1: When firms choose locations $L_1^* = (R,0)$ and $L_2^* = (R,\pi)$ in the first stage of the same they both charge $p^* = \pi R$ in the second

The price firms charge in the second-stage of the game increases in distance from the origin, that is in the distance between their stores or their products. This shows a common incentive dictating the need for product differentiation. \sim that is in the distance between the distance between the shows a common incentive shows a common incentive shows a common incentive stores or the shows a common incentive stores or the shows a common incentive stores

 $N_{\rm eff}$ we want to see whether the proposed structure of first structure of first stage locations can be can Next, we want to see whether the proposed structure of firms' first stage locations can be supported in equilibrium. We show there is $R \in (0,1]$, such that given Firm 1's location, L_1^* , Firm 2's location, L_2^* , is optimal. It turns out that this R equals one; firms locate their stores on the perimeter of the disk. Also, they will be positioned symmetrically across Next we want to see whether the propo

, $\frac{1}{2}$

the origin. To see this, we look at respective effects deviations in radial and polar directions have on *Firm* 2's profits. These effects are: $\frac{1}{\pi}$

supported in equilibrium. We show that given Firm 1's location, L1, such that given Firm 1's location, L1, such

$$
\frac{d\Pi_2}{dr} = \frac{\partial \Pi_2}{\partial p_2} \cdot \frac{dp_2}{dr} + p_2 \cdot \left(\frac{\partial D_2}{\partial r} + \frac{\partial D_2}{\partial p_1} \cdot \frac{dp_1}{dr}\right), \text{ and}
$$
\n(3)

$$
\frac{d\Pi_2}{d\phi} = \frac{\partial \Pi_2}{\partial p_2} \cdot \frac{dp_2}{d\phi} + p_2 \cdot \left(\frac{\partial D_2}{\partial \phi} + \frac{\partial D_2}{\partial p_1} \cdot \frac{dp_1}{d\phi}\right). \tag{4}
$$

The first term on the right hand side (RHS henceforth) of both (3) and (4) is zero due to The first term on the right hand side (RHS henceforth) of both (3) and (4) is zero due to profit maximization with respect to p_2 in the second period, so we are left with the direct (demand) effect and indirect (strategic) effect of each move, which are the first and second terms in parentheses, respectively. We first derive the demand and strategic effects of a small radial move on *Firm* 2's profit. profit may centre on the second period, $\frac{1}{2}$ in the direction of $\frac{1}{2}$ and $\frac{1}{2}$

Lemma 2: When
$$
L_1^* = (R,0)
$$
 and $L_2^* = (R,\pi)$, $R \in (0,1]$: (a) $\frac{\partial D_2}{\partial r} = -1$, (b) $\frac{\partial D_2}{\partial p_1} = \frac{1}{2R}$, and
(c) $\frac{dp_1}{dr} = \frac{\pi}{2} + \frac{2}{3}R$.

Part (a) quantifies *Firm* 2's direct gain in demand it captures from its rival by moving locate their stores on the perimeter of the disk when they are separated by angle π . adversely through a rival's price cut. These effects are such that both firms would like to towards its location, while (b) and (c) show that such a move affects *Firm* 2's demand

Lemma 3: For any pair of locations $L_1^* = (R,0)$ and $L_2^* = (R,\pi)$, $R \in (0,1]$, firms find it profitable to move farther away from the origin. able to move farther away from the origin.

Finally, we argue that the angle separating the two firms is exactly π .
Lamma μ For any R , $\phi = \pi$ is antimal for Firm 2. profitable to move farther away from the origin.

Lemma 4: For any *R*, $\phi = \pi$ is optimal for Firm 2.

Lemmas 3 and 4 prove Proposition 1.

in the second-stage is π . Furthermore, every firm's location is a local best response to perimeter of the disk symmetrically across its origin; the equilibrium price they charge Proposition 1: When in the first stage of the game two firms position their stores on the their rival's location.

Le to the complexity of the problem w We solve for the second-stage price equilibrium for every configuration numerically and calculate Firm 2's profit. Figure 2 shows that the proposed location $r=1$ and $\phi_2 = \pi$ is actually Firm 2's global best response to Firm 1's location choice. The profit attained there is $\pi^2/2$. $\epsilon = -1$, (b) $\frac{\epsilon}{\partial p_1} = \frac{1}{2R}$, and
from its rival by moving
affects *Firm* 2's demand
t both firms would like to
eparated by angle π .
 \equiv (0,1], firms find it profit-
ctly π .
osition their stores on the
libri Due to the complexity of the problem we are not able to show that each firm's location is cally. We fix *Firm* 1's location at *R*=1 and $\phi_1 = 0$, and vary *Firm* 2's location across the disk. also a global best response to an opponent's location analytically. We hence do it numeri-

Figure 2: *Firm 2*'*s profits given the location of its store.* Source: Own calculations.

We now state our first result, which links this work to the existent literature of two firms competing in a multi-characteristic product space. This gives us a valid reference point with which to compare our subsequent results.

Result 1: Two firms positioning their stores on the perimeter of the disk, symmetrically across its origin in the first stage of the game, and setting $p = \pi$ in the second is a subgame perfect Nash equilibrium of the game.

This finding is in the spirit of Neven and Thisse (1990), Tabuchi (1994), and Irmen and Thisse (1998). Two firms offer products that are fully differentiated in one characteristic, while identical in the other.

3. Three firms

We add another firm to the model and search for symmetric equilibrium. We find that firms separate their stores in both dimensions, and interestingly, do not choose to locate them on the perimeter anymore, but relocate them towards the origin noticeably.

Given locations $L_1^* = (R, -\pi/3), L_2^* = (R, \pi/3),$ and $L_3 = (r, \phi)$, firms compete in prices, p_1 , p_2 , and p_{3} . A particular market configuration is shown in Figure 3.

Figure 3: *Configuration of the market, three firms.*

Store locations and prices define the boundaries of market areas covered by firms. There are three line segments, *DA*, *DB*, and *DC*, representing buyers indifferent between buying from *Firms* 1 and 2, *Firms* 2 and 3, and *Firms* 3 and 1, respectively. These segments are needed in determining the demand functions firms face. They all join in one point, *D*, which is due to the fact that the delineating lines must be straight. Point $D = (x, y)$ represents a consumer indifferent between buying from any of the three firms. Points $A = (1, \alpha)$, $B = (1, \beta)$, and $C = (1, \gamma)$ stand for consumers on the perimeter indifferent between buying from respective firms. With the knowledge of *x*, *y*, α , β , and γ , which, as well as p_1 , p_2 , and p_3 , are all functions of *r*, ϕ , and *R*, we can write the demand functions wen as p_1 , p_2 , and p_3 , are an functions of *r*, φ , and *r*, we can write the demand functions firms face.⁵ *Firm* 1's demand equals the area of the disk between α and γ reduced for the area covered by triangles *OCD* and *ODA*. The other two firms' demands are obtained area covered by triangles *OCD* and *ODA*. The other two firms' demands are obtained similarly: δ covered by the area covered by triangles occurs of the area covered by triangles OCD. The other two firms δ $summu$

$$
D_1 = \frac{1}{2} (\alpha - \gamma - d \sin(\delta - \gamma) - d \sin(\alpha - \delta)),
$$

\n
$$
D_2 = \frac{1}{2} (\beta - \alpha + d \sin(\alpha - \delta) - d \sin(\beta - \delta)),
$$
 and
\n
$$
D_3 = \frac{1}{2} (2\pi - \beta + \gamma + d \sin(\beta - \delta) + d \sin(\delta - \gamma)).
$$

⁵ For the derivation of *x*, *y*, α, β, and γ see the proof of Lemma 5.

The *sine of the difference* rule and definitions of x and y yield:

$$
D_1 = \frac{1}{2}(\alpha - \gamma + x(\cos\alpha - \cos\gamma) - y(\sin\alpha - \sin\gamma)),
$$
\n(5)

$$
D_2 = \frac{1}{2}(\beta - \alpha + x(\cos\beta - \cos\alpha) - y(\sin\beta - \sin\alpha)),
$$
 and (6)

$$
D_3 = \frac{1}{2}(2\pi - \beta + \gamma - x(\cos\beta - \cos\gamma) + y(\sin\beta - \sin\gamma)).
$$
\n(7)

The firms' profit functions are: $\frac{1}{2}$ profits are: $T_{\rm{m}}$ $T_{\rm{m}}$ $T_{\rm{m}}$ $\sum_{i=1}^{n}$

$$
\Pi_1(r, \phi; R) = p_1(r, \phi; R) \cdot D_1(r, \phi; R), \n\Pi_2(r, \phi; R) = p_2(r, \phi; R) \cdot D_2(r, \phi; R), and \n\Pi_3(r, \phi; R) = p_3(r, \phi; R) \cdot D_3(r, \phi; R).
$$
\n(8)

Second-stage optimal prices are derived from the system of necessary conditions obtained from these profits. The system is nonlinear and its general analytical solution for any possible Firm 3's location, L_3 , is therefore out of reach. However, we are looking for symmetric equilibrium, so the derivation of optimal prices is straightforward. $f(x, \pi) = \frac{1}{1} \cdot \frac{1}{1}$ \mathcal{S} is the symmetric out of reacher. However, we are looking for symmetric for symmetric for symmetric for symmetric Second-stage optimal prices are derived from the system of necessary conditions of the origin, equidistantly from one another, the optimal Nash equilibrium price they charge in the tained from these profits. The system is nonlinear and its general analytical solution $\frac{1}{2}$ exponenting equilibrium, so the derivation of optimal prices

Lemma 5: When three firms in the first stage of the game position their stores R away Lemma 5: When three firms in the first stage of the game position their stores R away
from the origin, equidistantly from one another, the optimal Nash equilibrium price they charge in the second stage is $p^* = \frac{\pi R \sqrt{3}}{3}$. the origin, equidistantly from one another, the optimal Nash equilibrium price they charge in the Lemma 5: When three firms in the first stage of the game position their stores *away* from the origin, equid:
they charge in the secon

its location reasonably close to the proposed L_3^* ⁶. It remains to be seen whether such a configuration is optimal in the first stage of the game. The system of first-order conditions derived in the proof of Lemma 5 (A.8-A.10) completely characterizes price competition in the second stage given that *Firm 3* chooses its location reasonably close to the proposed L_3 ⁺⁶. It remains to be seen whether such a ϵ_{eq} configuration is ontimely cover to the proposed ϵ_{eq} . It is choose ϵ_{eq}

We rewrite Firm 3's profit function with rivals' store positions being L_i^* and L_2^* , and firms charging equilibrium prices in the second stage: charging equilibrium prices in the second stage: $\mathcal{L} \left(\mathcal{L} \right) = \mathcal{L} \left(\mathcal{L} \right)$ (,), $\frac{1}{2}$ r $\frac{1$ L_1 and L_2 , and mini examging equilibrium prices in the second stage: reasonably close to the proposed L³ . It remains to be seen whether such a configuration is we rewrite *firm 5* s profit funct.

$$
\Pi_{3}(r,\phi;R) = p_{3}(r,\phi;R) \cdot D_{3}(r,\phi;R,p_{1}(r,\phi;R),p_{2}(r,\phi;R),p_{3}(r,\phi;R)).
$$

3 3

We are interested in two derivatives: $π$ ∂Π ⁼ ^Π \mathfrak{m} and \mathfrak{m} we are interested in two derivatives

3

3

1

$$
\frac{d\Pi_3}{dr} = \frac{\partial \Pi_3}{\partial p_3} \cdot \frac{dp_3}{dr} + p_3 \cdot \left(\frac{\partial D_3}{\partial r} + \frac{\partial D_3}{\partial p_1} \cdot \frac{dp_1}{dr} + \frac{\partial D_3}{\partial p_2} \cdot \frac{dp_2}{dr}\right), \text{ and } (9)
$$

2

he top of the disk, the configuration of the demands would have c
∞ould not be valid anymore. ī If *Firm 3* located its store at the top of the disk, the configuration of the demands would have quantities defined in Figure 3 would not be valid anymore. Ĭ D .
. \overline{a} p ⁶ If *Firm* 3 located its store at the top of the disk, the configuration of the demands would have changed and
quantities defined in Figure 3 would not be valid anymore. in both equations represents the direct or demand effect or demand effect of the deviation in a respective variable, \mathcal{L} f 1 3 $A_{\rm 2D}$ and (10) and (10) equals zero. The first term in parentheses σ and σ ľ δ If Firm 3 located its store at the top of the disk, the configuration of the demands would have
quantities defined in Figure 3 would not be valid anymore.

$$
\frac{d\Pi_3}{d\phi} = \frac{\partial \Pi_3}{\partial p_3} \cdot \frac{dp_3}{d\phi} + p_3 \cdot \left(\frac{\partial D_3}{\partial \phi} + \frac{\partial D_3}{\partial p_1} \cdot \frac{dp_1}{d\phi} + \frac{\partial D_3}{\partial p_2} \cdot \frac{dp_2}{d\phi}\right).
$$
(10)

[∂] ⁺ [∂]

D

D

[∂] [⋅] ⁺ [⋅] [∂]

d p

Again, the first term in RHS of both (9) and (10) equals zero. The first term in parentheses Again, the first term in RHS of both (9) and (10) equals zero. The first term in parentheses in both equations represents the direct or demand effect of the deviation in a respective in both equations represents the direct or demand effect of the deviation in a respective variable, while the last two represent the indirect or strategic effects of such a deviation through competitors' prices. We show that there exists a distance from the origin, *R*, the necessary conditions for symmetric equilibrium for symmetric equilibrium for symmetric equilibrium for symmetric equilibrium for s such that if firms locate there equidistantly from one another, the necessary conditions for symmetric equilibrium are satisfied. The derivatives we need to determine the demand and strategic effects in (9) are collected in Lemma 6. $\frac{1}{2}$ in both equations represents the uncer or uch and encer or the deviation in a

Lemma 6: When
$$
L_1^* = (R, -\pi/3), L_2^* = (R, \pi/3)
$$
 and $L_3^* = (R, \pi)$: (a) $\frac{\partial D_3}{\partial r} = -\frac{4R - 1}{2R\sqrt{3}}$,
\n(b) $\frac{\partial D_3}{\partial p_1} = \frac{\partial D_3}{\partial p_2} = \frac{1}{2R\sqrt{3}}$, and (c) $\frac{dp_1}{dr} = \frac{dp_2}{dr} = \frac{-9 + 9\pi\sqrt{3} - \pi^2 + (36 - 2\pi\sqrt{3})R}{90 - 3\pi\sqrt{3}}$.

nents' demand around the center of the disk. For $R < 0.25$ the effect reverses, Firm 3 loses ϵ to the origin as far as the origin as the origin as the origin ℓ arriand to compenions and would fike to move away from the origin. P demand around the dence of the disk, For R \times 0.25 file effect reverses, Firm 5 loses concerned to competitive and as far as far as the original this congress concerned to the origin as $\frac{1}{2}$ as the origin $\frac{1}{2}$ as the origin $\frac{1}{2}$ as the opponents' demand to competitors and would like to move over the origin $\frac{1}{2}$ demand to competitors and would like to move away from the origin.⁷ Part (a) considers the direct demand effect. If *R* > 0.25 *Firm* 3 would like to position its store closer to the origin as far as this effect is concerned. This way it captures the oppo-

Part (b) quantifies the positive effect competitors' prices have on the demand captured $\frac{1}{2}$ by Firm 3 \mathcal{F} Firm 3. Part (b) quantifies the positive effect competitors' prices have on the demand captured $\frac{1}{3}$ by *Firm* 3.

The most demanding task is to evaluate the effect Firm 3's radial deviation has on opponents' prices (part (c)). In order to simplify a very complex exercise we exploit the symmetry of the problem extensively. It is clear that replies in prices of both competitors must be identical when *Firm* 3 moves along the vertical axis. We therefore use only necessary conditions for profit maximization with respect to own prices for Firms 2 and 3 (A.9-A.10) in the proof of part (c). It can be readily verified that dp_1/dr , dp_2/dr , and dp_3 are positive for any $R \in [0,1]$, that is, when Firm 3 moves towards the origin, prices decrease as competition toughens, and *vice versa*. part (c). It can be readily verified that dp dr ¹ , dp dr ² , and dp dr ³ are positive for any

These results, when compared to those for a duopoly (Lemma 2), offer some idea for what follows. Suppose all three firms were located on the perimeter of the disk, and Firm 3 contemplated a small radial move towards the origin. The line of marginal consumers affected by this move is of length two $(BO \text{ and } CO)$; see Figure 3). The same is true for the duonaly gase. Hones, direct domand offects should be very similar in both gases. They duopoly case. Hence, direct demand effects should be very similar in both cases. They respect to its radial distribution of α is respectively. The first the period of the period of the period of α is α in the distribution of α is α in the distribution of α is α in the distribution of are −1 in the duopoly and −0.87 with three firms.⁸ Furthermore, we derive the elasticity respect to its respect to its radial distance from the origin when all the periodic states of the periodic the periodic theory of are −1 in the duopoly and −0.87 with three firms.⁸ Furthermore, we derive the elasticity

this move is of length two (BO and CO; see Figure 3). The same is true for the duopoly case.

current demands, a firm in a three-firm in a three-firm case gains relatively more than in the duopoly when it

current demands, a firm in a firm in a three-firm in three-firm case gains relatively when it than in the duop

⁷ Point *D* moves along the vertical axes towards the top of the disk, so *Firm* 3 gains some new customers from the oponents (see Figure 3). At the same time line segments DC and DB rotate toward each other, which means that *Firm* 3 loses some customers on the outskirts of the market. The total effect is negative from the oponents (see Figure 5). At the same time time segments *DC* and *DB* rotate toward each other,
which means that *Firm* 3 loses some customers on the outskirts of the market. The total effect is negative for $\kappa < 0.25$. for $R < 0.25$. \leq 0.25. for $R < 0.25$.

 s From part (a) in Lemmas 2 and 6.

of the firm's demand with respect to its radial distance from the origin when all the firms are on the perimeter of the disk. The results are −0.64 and −0.83 for the duopoly and three-firm oligopoly, respectively. At current demands, a firm in a three-firm case gains relatively more than in the duopoly when it moves its store towards the origin ($\Delta R < 0$). We also derive the elasticity of opponents' prices with respect to the firm's radial distance from the origin.⁹ It is 0.71 in the duopoly and 0.41 in the three-firm oligopoly. Rivals' price cut response to a firm moving towards the origin in the first stage of the game will be weaker in the three-firm case than in the duopoly. This is because a price cut by one of the two firms that have not moved would not only affect the aggressive rival, but the other neighbor as well. This would provoke a response from a peaceful rival and would lead to lower profits made on consumers not affected by the aggressive firm. We have illustrated the incentive a firm in a three-firm oligopoly has when *R* = 1, when compared to the duopoly. It will gain relatively more demand directly and will be punished by relatively less severe a price cut by rivals in the second stage. If a firm residing at $R = 1$ in the duopoly had an incentive to move even farther away from the origin we expect this not to be the case with three firms anymore. Lemma 7 presents an interior radial distance from the origin for the three firms positioned equidistantly from each other, which does not make them want to relocate in radial direction.

Lemma 7: When $R^* = \frac{72 + 15\pi\sqrt{3} - 2\pi^2}{288 - 8\pi\sqrt{3}}$ (≈ 0.5476) $\frac{72 + 15\pi\sqrt{3} - 2\pi^2}{288 - 8\pi\sqrt{3}} \quad (\approx$ $\frac{\pi\sqrt{3}-2\pi^2}{-8\pi\sqrt{3}}$ (≈ 0.5476) and firms are located equidistantly, firm finds it profitable to locally deviate from it. no firm finds it profitable to locally deviate from it.

Proof of Lemma 7 yields another result. Symmetric configuration with maximum distance Proof of Lemma 7 yields another result. Symmetric configuration with maximum distance between three firms, i.e. maximum differentiation, is never optimal.

Corollary 1: The three firms located on the perimeter of the disk, equidistantly from each other, is not a subgame perfect Nash equilibrium of the game.

Corollary 1: The three firms located on the perimeter of the disk, equidistantly from each Lemma 5 shows that locating at the perimeter of the disk will still yield the highest pos-Lemma 5 shows that locating at the perimeter of the disk will still yield the highest possible sible prices and profits in a symmetric configuration, but these are not sustainable since capturing the opponents' demand around the origin is too tempting; a classic prisoner dilemma on an oligopoly market.

Next, we show that, when firms are positioned equidistantly, none of them has an incen- $\frac{1}{2}$ demand around the origin is too tempting; a classical prior direction tive to locally move along a polar direction.

Lemma 8: For any *R*, $\phi = \pi$ is locally optimal for Firm 3. to locally move along a polar direction.

Results from Lemmas 5, 7, and 8 are summarized in Proposition 2.

⁹ From part (c) in Lemmas 2 and 6.

Proposition 2: When in the first stage of the game the three firms locate their stores $R^* = \frac{72 + 15\pi\sqrt{3}}{288 - 8\pi\sqrt{3}}$ (≈ 0.5476) $\frac{72 + 15\pi\sqrt{3} - 2\pi^2}{288 - 8\pi\sqrt{3}}$ (\approx $\frac{\pi\sqrt{3}-2\pi^2}{-8\pi\sqrt{3}}$ (≈ 0.5476) away from origin, equidistantly from each other, the allibrium price they charge in the second stage is $\frac{3}{\sqrt{3}}$. Furthermore, every firm s
ation is a local best response to rivals' locations. away from origin, equidistantly from each other, the equilibrium price they charge in the second stage is when in the first stage of the
 $\frac{1}{2}$ = $2\pi^2$ (≈ 0.5476) away from $\pi\sqrt{3}$ ice they charge in the second stage is $\frac{\pi R^* \sqrt{3}}{3}$. Furthermore, every firm's location is a local best response to rivals' locations.

response to rivals' locations. locations given the price competition in the second stage of the game. We therefore solve the system of first-order conditions for second-stage profit maximization with respect to We lack analytical proof that every firm's location is in fact a global best reply to rivals' We lack analytical proof that every firm's location is in fact a global best reply to rivals' prices (A.8-A.10) for an array of Firm 3's locations numerically to derive optimal profits. We find (see Figure 4) that locations we propose are globally optimal, and state our next result.

Result 2: Three firms positioning their stores at $R^* = \frac{72 + 15\pi\sqrt{3} - 2\pi^2}{288 - 8\pi\sqrt{3}}$ (≈ 0.5476) $\frac{\pi\sqrt{3}-2\pi^2}{-8\pi\sqrt{3}}$ (≈ 0.5476), 6 distantly from each other in the first stage of the game, and setting $p = \frac{1}{3}$ in second, is a subgame perfect Nash equilibrium of the game. , equidistantly from each other in the first stage of the game, and setting :
t 2: Three firms positioning their stores at $R^* = \frac{72+15\pi\sqrt{3}-2\pi^2}{\pi} \quad (\approx 0.5476)$ equ $\frac{1}{1-2\pi^2}$ (≈ 0.5476) equi $e^{288-8\pi\sqrt{3}}$ istantly from each other in the first stage of the game, and setting $p^* = \frac{\pi R^* \sqrt{3}}{3}$ in the second, is a subgame perfect Nash equilibrium of the game. second, is a subgame perfect Nash equilibrium of the game.

There are two novel perspectives on product differentiation to this result. First, in a sistent with the existing literature, the three firms find it optimal to differentiate their products in two characteristics. Second, firms do not differentiate their products as much as they could have. In a setting that yields familiar a min-max differentiation result in the duopoly, a medium-medium type of product differentiation is observed. When all three firms are located on the perimeter of the disk, a radial deviation towards the center of the disk by a firm is followed by rivals' price cut that is less severe than the one observed in duopoly. Therefore, positive direct demand effect of such a cual the one observed in duopoly. Therefore, positive direct demand effect of steam α move outweighs the negative strategic effect of rivals' price cuts, and firms move cit together. setting that leads to differentiation in one characteristic only in duopoly, which is con- α such a move outweights the negative strategic effect of rivals' price cuts, and firms move closes. move outweighs the negative strategic effect of rivals' price cuts, and firms move closer

We can speculate on whether the min-min-...-min part of the product differentiation result by Irmen and Thisse (1998) could be observed in a three-firm market with adcount by finith and finsst (1990) count of observed in a time finite manner characteristics, in this may not want to differentiate their product any additional characteristics, since they find $max-max$ differentiation to be excessive in two dimensions, already. This suggests that firms may have no need to differentiate their products in another, third, dimension, since even the possibilities in two were not exhausted. We therefore predict that the min-min-...-min part of a product differentiation result is going to hold in our setting as well, which is what Feldin (2001) formally shows. This hypothesis rests on the market being uniformly populated by consumers. If there were some areas with higher population density there would be obviously more characteristics, since the wave the characteristics, since the maximum maximum maximum and the differentiation that the differentiation that the differentiation that the differentiation the differentiation that the differe ditional product characteristics. Firms may not want to differentiate their products in another, in another, in another, in another, in another, in another, in another characteristics. differentiation, since firms would try to tailor their products to meet the tastes of these different groups of customers.

Figure 4: *Firm 3's profits with respect to location of its store and given equilibrium locations of opponents.* Source: Own calculations.

4. Welfare analysis

In this section we compare the extent of product differentiation in our competitive market to a social optimum for a duopoly and a three-firm oligopoly. A social planner who cares about well-being of all agents in the market would simply minimize the traveling costs borne by buyers. Each of *n* firms on the market will satisfy π/*n* of the market demand in a symmetric configuration. Every such piece of the pie can be split into two halves symmetrically over the line segment connecting the firm's location and the center of the disk. One such part of the disk is presented in Figure 5. To find the social optimum, we look at where a particular firm should have been in order to minimize the total traveling costs that consumers, from such a half of the disk, bear.

Figure 5: *Derivation of socially optimal product differentiation.*

The cost that a consumer residing at (ρ, ϕ) is faced with when buying from the firm located at R is cated at *R* is: $\frac{1}{2}$ is faced with when $\frac{1}{2}$ is faced with when $\frac{1}{2}$ is faced with when $\frac{1}{2}$ α cated at R is: cated at R is: $\frac{1}{\sqrt{2}}$ is faced with when buying from the firm located with when buying from the firm located with when $\frac{1}{\sqrt{2}}$ is faced with when $\frac{1}{\sqrt{2}}$ is faced with when $\frac{1}{\sqrt{2}}$ is faced with whe

$$
c(\rho, \phi; R) = \rho^{2} \sin^{2} \phi + (R - \rho \cos \phi)^{2} = R^{2} - 2\rho R \cos \phi + \rho^{2}.
$$

The total cost borne by buyers from this portion of the disk is then: $c(\rho, \phi; R) = \rho^2 \sin^2 \phi + (R - \rho \cos \phi)^2 = R^2 - 2\rho R \cos \phi + \rho^2$.
The total cost borne by buyers from this portion of the disk is then

$$
C(R) = \int_{0}^{\frac{\pi}{n}} \int_{0}^{1} c(\rho, \phi, R) \cdot \rho \, d\rho \, d\phi.
$$

This integrates to:
$$
C(R) = \frac{2R^2 + 1}{4} \cdot \frac{\pi}{n} - \frac{2}{3} R \sin \frac{\pi}{n}
$$

This integrates to: $C(R) = \frac{2R^2 + 1}{4} \cdot \frac{\pi}{n} - \frac{2}{3}R \sin \frac{\pi}{l}$ 2 $\overline{}$ $\frac{2R^2+1}{4}\cdot\frac{\pi}{n}-\frac{2}{3}R\sin\frac{\pi}{n}$. This integrates to: $C(R) = \frac{2R + 1}{4} \cdot \frac{n}{n} - \frac{2}{3}R \sin \frac{n}{n}$ $\frac{2}{\pi}$ 4 This integrates to: $C(R) = \frac{2R^2 + 1}{4} \cdot \frac{\pi}{n} - \frac{2}{3}R \sin \frac{\pi}{n}$. 4 $\frac{2R^2+1}{4} \cdot \frac{\pi}{2} - \frac{2}{3} R \sin \frac{\pi}{2}$. $2R^2+1 \pi^2-2 p_{\text{sin}}$ 3 2 \overline{a} $\frac{2R^2+1}{4} \cdot \frac{\pi}{r} - \frac{2}{3} R \sin \frac{\pi}{r}$

Minimize this expression with respect to R to get the socially optimal distance from the contar of the disk for the firms P . center of the disk for the firms, R_s :

$$
R_{S}=\frac{2n\sin\frac{\pi}{n}}{3\pi}.
$$

We compare socially optimal locations with competitive equilibrium ones derived in Sections 3 and 4 in Table 1. Sections 3 and 4 in Table 1. Sections 3 and 4 in Table 1. Sections 3 Sections 3 and 4 in Table 1.

Table 1: Social optimum vs. competitive equilibrium. Table 1: Social optimum vs. competitive equilibrium. Table 1: Social Table 1: *Social optimum vs. competitive equilibrium.*

enner (2005) finds. Intense pri drives firms to the edge of the market, and that is too far apart from a social standtoo weak when compared to the social optimum. Price competition becomes less intense and firms move in on each other's demands. This gives rise to an interesting situation. It is in the social planner's interest to induce cooperation among firms. She might pass a regulation demanding the firms locate R_s away from the origin. This would help the the social planner's demanding the many because λ_s and from the social measurement of the social planner's dilemma. They would now be able to charge high prices; therefore, they would be willing to accept such a regulation. firms to partly resolve their prisoner's dilemma. They would now be able to charge higher prices; therefore, they would be willing to accept such a regulation. 2 0.4244 1 Source: Own calculations $R_s = \frac{25.0 \text{ m}}{3\pi}$.

Setimal locations with competitive equilibrium ones derived in e 1.

1: Social optimum vs. competitive equilibrium.
 $\frac{n}{3}$ Concluding remarks and the section of the section of the section of th This is analogous to what Brenner (2005) finds. Intense price competition in a duopoly point. In a three-firm oligopoly the situation reverses. Product differentiation becomes

5. CONCLUDING REMARK

We have presented some novel findings on the extent of product differentiation that firms in an oligopoly employ. Contrary to the literature on duopoly markets, we show that three firms will use two product characteristics to separate their products from rivals'. in an oligopoly employ. Contrary to the literature on duopoly markets, we show that the first state of the firms of the first state of the first s In a setting that usually yields *min-max* differentiation we find three firms to utilize a *medium-medium* type of differentiation. It remains to be seen how even a larger number of firms affects product differentiation in multi-characteristic space.

The demand side of the model is the sole driver of our results. If firms had to cover some R&D costs to introduce new varieties of products or improve some of the characteristics, which certainly is the case in reality, differentiation would be even weaker. We therefore note that firms deciding to introduce new characteristics to their products might want to be careful as not to engage in excessive differentiation that would not yield the maximum possible profits. Also, the fact that differentiation possibilities are not exhausted leads us to think that three firms will not want to use any new product characteristics to differentiate themselves from competitors. The first step towards such a *min-…-min-mediummedium* differentiation result is presented in Feldin (2001).

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Appendix A: Proofs of Lemmas and Propositions Appendix A: Proofs of Lemmas and Propositions Appendix A: Proofs of Lemmas and Propositions

Proof of Lemma 1: We first determine α and β that we need for the demand functions. They are given by the indifference conditions for the two consumers residing in points A and *B*: $p_1 + \sin^2 x + (R - \cos x)^2 = p_2 + (r \sin \phi - \sin x)^2 + (r \cos \phi - \cos x)^2$, where $x = \alpha, \beta$. T_{max} by the simplifies to $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ in $\frac{1}{\sqrt{2}}$ in summer $\frac{1}{\sqrt{2}}$ (if $\frac{1}{\sqrt{2}}$ is $\frac{1}{\sqrt{2}}$ in $\frac{1}{\sqrt{2}}$ in $\frac{1}{\sqrt{2}}$ in $\frac{1}{\sqrt{2}}$ is $\frac{1}{\sqrt{2}}$ in $\frac{1}{\sqrt{2}}$ in $\frac{1$ 2 T_1 and $($ and $\frac{1}{2}$ using the fact that in symmetric equation $\frac{1}{2}$ $\frac{P(\mathbf{r}, \mathbf{r})}{\mathbf{r} \cdot \mathbf{r}} = \frac{P(\mathbf{r}, \mathbf{r})}{\mathbf{r} \cdot \mathbf$ are given by the indicate $p_1 + \sin^2 x + (R - \cos x)^2 = p_2 + (r \sin \phi - \sin x)^2 + (r \cos \phi - \cos x)^2$, where $x = \alpha, \beta$. nature conditions for the two consumers residents. Proof of Lemma 1: We first determine α and β that we need for the demand functions. They

This equation simplifies to $(A.1)$ and by using the fact that in symmetric equilibrium (if $2, 10$ (A.2): exists) $r = R$ and $\phi = \pi$, to (A.2): exists) $r = R$ and $\phi = \pi$, to (A.2): $2 \times 2 \times 2 \times 4$ $\Gamma(x) = \Gamma(x) + \Gamma(x) + \Gamma(x)$ and $\Gamma(x) = \Gamma(x) + \Gamma(x) + \Gamma(x) + \Gamma(x) + \Gamma(x)$ This equation simplifies to $(A.1)$ and by using the fact that in symmetric equilibrium (if

$$
2r\sin\phi\cdot\sin x + 2(r\cos\phi - R)\cdot\cos x = p_2 - p_1 + r^2 - R^2.
$$
 (A.1)

$$
-4R \cdot \cos x = p_2 - p_1 \tag{A.2}
$$

We state the necessary conditions for profit maximization with respect to own prices for Firm 1 and Firm 2 (from (2)): *Firm* 1 and *Firm* 2 (from (2)): p − p + r − R . (A.1) $\overline{}$ ¹ − + ⋅ $\begin{array}{ccc} \n 1 & & & & \n \end{array}$ $\begin{array}{ccc} 1 & & p \\ & & p \end{array}$ Ve state the necessary conditions for profit maximization with respect to own prices for (2) ² ¹ − 4R ⋅ cos x = p − p (A.2)

$$
\frac{1}{2}(z - \sin z) + \frac{p_1}{2} \cdot (z_{p_1} - \cos z \cdot z_{p_1}) = 0 \text{ and}
$$

$$
\frac{1}{2}(2\pi - z + \sin z) + \frac{p_2}{2} \cdot (-z_{p_2} + \cos z \cdot z_{p_2}) = 0.
$$
(A.3)

Firms' behavior in prices will be symmetric, hence, it is enough to use only first equation in (A.3), the necessary condition for *Firm* 1. Since in equilibrium $z = \pi$, we get: $\frac{1}{2}$, $\frac{1}{2}$, the necessary condition for Firm 1. Since in equilibrium $\frac{1}{2}$

$$
\pi + 2p_1 z_{p_1} = 0.
$$

All derivatives we need in this section are collected in Appendix B. Using (B.1) completes $O.E.D.$ the proof. $Q.E.D.$ All derivatives we need in this section are collected in Appendix B. Using (B.I) completes
the proof. (O.E.D. A l derivatives we need in this section are collected in A All derivatives we need in this section are collected in Appendix B. Using (B.1) completes All derivatives we need in this section are collected in Appendix B. Using (B.1) completes P , P is a $=$ P $=$ the proof. Q.E.D.

Proof of Lemma 2: Since firms will charge symmetric prices: $\alpha = \pi/2$, $\beta = -\pi/2$, and $z = \pi$. All derivatives we need in this section are collected in $\mathcal{L}_\mathcal{B}$. Using (B.1) completes $\mathcal{L}_\mathcal{B}$ Proof of Lemma 2: Since firms will charge symmetric prices: $\alpha = \pi/2$, $\beta = -\pi/2$, and $z = \pi$. of I ammo 2. Since firms will share symmetric prices: $\alpha = \pi/2$, $\beta = \pi/2$ and $\tau = \pi$ $\frac{d}{dx}$ original $\frac{d}{dx}$ or $\frac{d}{dx}$ are $\frac{d}{dx}$. The area 2×dr $\frac{d}{dx}$ in the area 2×dr $\frac{d}{dx}$ of of Lemma 2: Since firms will charge symmetric prices: $\alpha = \pi/2$, $\beta = -\pi/2$, and $z = \pi$. $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ in the origin, the origin,

- (a) In equilibrium line *AB* crosses the origin, therefore, any firm's infinitesimal radial move towards origin. $-dr$ yields new customers in the area $2 \times dr/2$, hence, raising move towards origin, $-dr$, yields new customers in the area $2 \times dr/2$, hence, raising
demand by dr. ∂_t by dr. demand by dr . towards origin, −dr, yields new customers in the area 2×dr/2, hence, raising demand by dr. α using α and α and α \liminf or θ and θ . α . α . (a) in equilibrium line AB crosses the origin, therefore, any firms infinitesimal radial \int demand by dr
	- appendix. The derivative in $\mathcal{O}(\mathcal{A})$ is obtained from the system of first-order conditions (A.3) (b) Using (1) and derivative (B.1) the partial derivative in question is:
 $\frac{\partial D_2}{\partial t} = \frac{1}{2}(-z + \cos z \cdot z) = -z = \frac{1}{2}$ Using (1) and derivative (B.1) the partial ϵ

$$
\frac{\partial D_2}{\partial p_1} = \frac{1}{2}(-z_{p_1} + \cos z \cdot z_{p_1}) = -z_{p_1} = \frac{1}{2R}.
$$

 $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{A}}$ (from $\mathcal{F}_{\mathcal{A}}$): $\mathcal{F}_{\mathcal{A}}$ (from (2)):

(c) Denote the total derivative of any variable x with respect to r by x' for the length of this appendix. The derivative in question is obtained from the system of first-order conditions (A.5) we used to derive equinorium prices. Since this system must note to
any pair of locations, the total differentiation with respect to r yields p'_1 . The total
derivatives are (Note sin $z = 0$ and $\cos z = -1$ this appendix. The derivative in question is obtained from the system of first-order
conditions (A.3) we used to derive equilibrium prices. Since this system must hold for derivatives are (Note: $\sin z = 0$ and $\cos z = -1$ in equilibrium): we use to derive equilibrium prices. Since this system must hold for any pair of locations, the locations, the (c) Denote the total derivative of any variable x with respect to r by x' for the length of $T_{\rm c}$ above system are collected in above system are collected in (B.1)-(B.3). It is becomes:

$$
z' + p'_1 \cdot z_{p_1} + p \cdot z'_{p_1} = 0
$$
 and $z' + p'_2 \cdot z_{p_2} + p \cdot z'_{p_2} = 0$.

The derivatives needed in above system are collected in $(B.1)$ - $(B.3)$. It becomes: The derivatives needed in above system are collected in $\overline{}$ above system are collected in $\overline{1}$ e system are collected in (B.1)

$$
-4p'_1 + 2p'_2 + 4R + \pi = 0, \text{ and } 2p'_1 - 4p'_2 - 4R + \pi = 0,
$$

with the unique solution $p'_1 = \frac{\pi}{2} + \frac{2}{3}R$ and $p'_2 = \frac{\pi}{2} - \frac{2}{3}R$. Q.E.D.

 $2\frac{3}{4}$
Proof of Lemma 3: Use the results from Lemma 2 and put them in parentheses in RHS of (3) $\frac{3}{3}$ and $\frac{4R}{\sqrt{2}}$ or $\frac{1}{2}$ of $\frac{1}{2}$ and $\frac{1}{2}$ to get the sign of the effect a radial move has on Firm 2's profits: $-\frac{2}{3} + \frac{\pi}{4R} > 0$, $\forall R \in (0,1]$. *Q.E.D.* Q.E.D. Proof of Lemma 3: Use the results from Lemma 2 and put them in parentheses in RHS of (3) of of Lemma 3: Use the results from Lemma 2 and put them in parentheses in RHS of (3) to get the sign of the effect a radial move has on Firm 2's profits: $-\frac{2}{3} + \frac{\pi}{4R} > 0$, $\forall R \in (0.1]$ to get the sign of the effect a radial move has on *Firm* 2's profits: $-\frac{2}{3} + \frac{\pi}{4R} > 0$, $\forall R \in (0.1]$.
Q.E.D. to get the sign of the effect a radial move has on *Firm* 2's profits: $-\frac{2}{3} + \frac{\pi}{4R} > 0$, $\forall R \in (0.1]$.
Q.E.D. to get the sign of the effect a radial move has on $FHM \geq s$ profits: $-\frac{1}{3} + \frac{1}{4R} > 0$, $\forall K \in (0.1]$.
Q.E.D. $2 + N$ Q

Proof of Lemma 4: The demand effect of any move in a polar direction is zero, since the new line of indifferent consumers (line *AB*) must pass through the origin (firms will new line of indifferent consumers (line *AB*) must pass through the origin (firms will expansive intervention corresponds to the expansive intervention components.
Q.E.D. charge identical prices). The demands firms face are unaffected by such a move, but, any such move that brings both firms closer together makes them more competitive and low- \mathbf{r} beth firms close them more competitive and lowers them more competitive and lowers the price theorem more competitive and lowers the price theorem more competitive and lowers they have been defined as \mathbf{r} new line of indifferent consumers (line \overrightarrow{AB}) must pass through the origin (firms will $\overline{\mathbf{C}}$ $\frac{1}{\sqrt{E}}$ or the price they charge. The result follows $\frac{1}{\sqrt{E}}$ or $\frac{1}{\sqrt{E}}$ or $\frac{1}{\sqrt{E}}$ cho the price they enlarge. $\frac{1}{\sqrt{1-\frac{1$ ers the price they charge. The result follows. Q.E.D. Proof of the demand and the demand and the demand of any move in a provider of the sero, since the single single since the single single single single sin such move that brings both firms closer together makes them more competitive and leadership. $\tau_{\rm eff}$ the sign of the effect a radial move has on $\tau_{\rm eff}$

Proof of Lemma 5: First we characterize x, y, α , β , and γ , which define the firms' demands. Coordinates of point D , x and y solve the system: Proof of Lemma 5: First we characterize x, y, α , β , and γ which define the firms' that brings both first both firms closer together makes the price them more competitive and lowers they represent

$$
p_1 + (R\sin(-\frac{\pi}{3}) - x)^2 + (R\cos(-\frac{\pi}{3}) - y)^2 = p_2 + (R\sin\frac{\pi}{3} - x)^2 + (R\cos\frac{\pi}{3} - y)^2,
$$

$$
p_3 + (r\sin\phi - x)^2 + (r\cos\phi - y)^2 = p_2 + (R\sin\frac{\pi}{3} - x)^2 + (R\cos\frac{\pi}{3} - y)^2,
$$

³ (sinφ sin β) (cosφ cosβ) (sin sin β) (cos cosβ) ^π ^π p + r − + r − = p + R − + R − .

We note that $\sin \alpha = x$ and we proceed with β . It solves: We note that $\sin \alpha = x$ and we proceed with β . It solves: $W_{\rm eff} = 0.001$ and with $R_{\rm eff} = 0.001$ and with $R_{\rm eff} = 0.001$ We note that sin $\alpha = x$ and we proceed with β . It solves: We note that sin $\alpha = x$ and we proceed with β . It solves:

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

2 2

$$
p_3 + (r\sin\phi - \sin\beta)^2 + (r\cos\phi - \cos\beta)^2 = p_2 + (R\sin\frac{\pi}{3} - \sin\beta)^2 + (R\cos\frac{\pi}{3} - \cos\beta)^2.
$$

And, γ solves: $\overline{3}$ (sin sin sin sin sin sin sin $\overline{3}$) π $\overline{1}$ + $\overline{$ $\overline{3}$ (sing coseful cosefu Δ nd γ colver Δ and, γ solves:

And,
$$
\gamma
$$
 solves:
\n
$$
p_1 + (R\sin{\frac{\pi}{3}} - \sin{\gamma})^2 + (R\cos{\frac{\pi}{3}} - \cos{\gamma})^2 = p_3 + (r\sin{\phi} - \sin{\gamma})^2 + (r\cos{\phi} - \cos{\gamma})^2.
$$

 $A = \frac{1}{2}$ 2 2 \overline{Y} solves: st four equations simplify and lead to: \sim \sim 1.6 \mathbf{p} \mathbf{p} approximation \mathbf{p} and \mathbf{p} and lead to: Last four equations simplify and lead to:

$$
x = \sin \alpha = \frac{p_2 - p_1}{2R\sqrt{3}},\tag{A.4}
$$

$$
y = \frac{R\sqrt{3}(2p_3 - p_1 - p_2 + 2r^2 - 2R^2) - 2r\sin\phi(p_2 - p_1)}{2R\sqrt{3}(2r\cos\phi - R)},
$$
\n(A.5)

$$
(2r\sin\phi - R\sqrt{3})\sin\beta + (2r\cos\phi - R)\cos\beta = p_3 - p_2 + r^2 - R^2,
$$
\n(A.6)

$$
(2r\sin\phi + R\sqrt{3})\sin\gamma + (2r\cos\phi - R)\cos\gamma = p_3 - p_1 + r^2 - R^2.
$$
 (A.7)

z = −1 in equilibrium):

z = −1 in equilibrium):

The system of first-order conditions for profit maximization derived from (8), which we mainly need for future reference, is: $(\alpha \cdot \alpha)$ (sin sin) (cos cos cos cos cos α) (cos $x \in \mathbb{R}^n$ y p $x \in \mathbb{R}^n$, we can expect the set of $x \in \mathbb{R}^n$ − + − − − + − + −

$$
\alpha - \gamma + x(\cos\alpha - \cos\gamma) - y(\sin\alpha - \sin\gamma) + p_1(\alpha_{p_1} - \gamma_{p_1} + x_{p_1}(\cos\alpha - \cos\gamma))
$$

- x(\sin\alpha \cdot \alpha_{p_1} - \sin\gamma \cdot \gamma_{p_1}) - y_{p_1}(\sin\alpha - \sin\gamma) - y(\cos\alpha \cdot \alpha_{p_1} - \cos\gamma \cdot \gamma_{p_1})) = 0' (A.8)

$$
\beta - \alpha + x(\cos \beta - \cos \alpha) - y(\sin \beta - \sin \alpha) + p_2(\beta_{p_2} - \alpha_{p_2} + x_{p_2}(\cos \beta - \cos \alpha) \n- x(\sin \beta \cdot \beta_{p_2} - \sin \alpha \cdot \alpha_{p_2}) - y_{p_2}(\sin \beta - \sin \alpha) - y(\cos \beta \cdot \beta_{p_2} - \cos \alpha \cdot \alpha_{p_2}) = 0
$$
\n(A.9)

$$
2\pi - \beta + \gamma - x(\cos\beta - \cos\gamma) + y(\sin\beta - \sin\gamma) + p_3(-\beta_{p_3} + \gamma_{p_3} - x_{p_3}(\cos\beta - \cos\gamma)+ x(\sin\beta \cdot \beta_{p_3} - \sin\gamma \cdot \gamma_{p_3}) + y_{p_3}(\sin\beta - \sin\gamma) + y(\cos\beta \cdot \beta_{p_3} - \cos\gamma \cdot \gamma_{p_3}) = 0
$$
 (A.10)

use the assumed symmetry of equilibrium configuration to find the optimal prices. We first note that in equilibrium, $x = 0$, $y = 0$, $\alpha = 0$, $\beta = 2\pi/3$, and $\gamma = -2\pi/3$, which immediately simplifies (A.10): We use the assumed symmetry of equilibrium configuration to find the optimal prices. We use the assumed symmetry of equilibrium configuration to find the optimal prices.
We first note that in equilibrium, $x = 0$, $y = 0$, $\alpha = 0$, $\beta = 2\pi/3$, and $\gamma = -2\pi/3$, which imfirst note that in equilibrium, $x = 0$, $y = 0$, $\alpha = 0$, $p = 2\pi/3$, and $y = -2\pi/3$, which mediately simplifies (A 10). We first note that in equilibrium, $x = 0$, $y = 0$, $\alpha = 0$, $\beta = 2\pi/3$, and $\gamma = -2\pi/3$, which in mediately simplifies (A.10): We first note that in equilibrium, $x = 0$, $y = 0$, $\alpha = 0$, $\beta = 2\pi/3$, and $\gamma = -2\pi/3$, which im-
mediately simplifies (A 10).

$$
\frac{2\pi}{3} + p_3 \left(-\beta_{p_3} + \gamma_{p_3} + y_{p_3} \sqrt{3} \right) = 0.
$$

Next, symmetry yields $\gamma_{p_3} = -\beta_{p_3}$. Remaining two partial derivatives are in (B.8) and N_e yields particle pa (B.14). We get: Next, symmetry $\begin{array}{ccc} 3 \overline{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$ and in (B.8) and Next, symmetry yields $\gamma_{p_3} = -\beta_{p_3}$. Remaining two partial derivatives are in (B.8) (B.14). We get: Next, symmetry yields $\gamma_{p_3} = -\beta_{p_3}$. Remaining two partial derivatives are in (B.8) and $\gamma_{p_3} = -\beta_{p_3}$.

$$
\frac{2\pi}{3} + p_3 \left(-\frac{2}{2R\sqrt{3}} - \frac{1}{3R} \sqrt{3} \right) = 0
$$
, which gives the result.
Proof of Lemma 6: (a) Partially differentiate (7) with respect to *r* to get:¹⁰

$$
D_r = \frac{1}{2} \left(-\beta_r + \gamma_r - x_r (\cos \beta - \cos \gamma) + y_r (\sin \beta - \sin \gamma) \right)
$$

Again symmetry and partials (B.4). (B.9) and (B.15) give:

 \sim
n, symmetry, and partials (B.4), (B.9), and (B.15) give: in, symmetry, and partials (B.4), (B.9), and (B.15) give: $\mathcal{A}_{\mathcal{A}}$, symmetry, and $\mathcal{A}_{\mathcal{A}}$, and (B.15), and (B.15) gives: α gain, symmetry, and partials $(B,1)$, $(B,9)$, and $(B,15)$ B^{11} Again, symmetry, and partials (B.4), (B.9), and (B.15) give: Again, symmetry, and partials (B.4), (B.9), and (B.15) give: Again, symmetry, and partials (B.4), (B.9), and (B.15) give: σ ^a $\mathcal{A}_{\mathcal{A}}$, symmetry, and $\mathcal{A}_{\mathcal{A}}$, and (B.15), and (B.15) gives:

$$
D_r = \frac{1}{2} \left(-2\frac{2R - 1}{2R\sqrt{3}} - \frac{2}{3}\sqrt{3} \right) = \frac{1}{2} \left(-\frac{4R - 2}{2R\sqrt{3}} - \frac{4R}{2R\sqrt{3}} \right) = -\frac{4R - 1}{2R\sqrt{3}}.
$$

(b) We proceed similarly as in part (a), to get:

 $\sum_{i=1}^{n}$ we proceed similarly as in part (a), to get: (b) We proceed similarly as in part (a), to get: $% \mathcal{N}$

$$
D_r = \frac{1}{2} \left(-2\frac{2R}{2R\sqrt{3}} - \frac{2}{3}\sqrt{3} \right) = \frac{1}{2} \left(-\frac{4R}{2R\sqrt{3}} - \frac{4R}{2R\sqrt{3}} \right) = -\frac{4R}{2R\sqrt{3}}.
$$

\n(b) We proceed similarly as in part (a), to get:
\n
$$
D_{p_2} = \frac{1}{2} \left(-\beta_{p_2} + \gamma_{p_2} - x_{p_2} (\cos \beta - \cos \gamma) + y_{p_2} (\sin \beta - \sin \gamma) \right).
$$

\nFrom (A.7) we note that $\gamma_{p_2} = 0$ (the indifference line *CD* is unaffected by *Firm* 2's pri change). With partials (B.4), (B.7) and (B.14) we obtain:
\n
$$
D_{p_2} = \frac{1}{2} \left(\frac{1}{2R\sqrt{3}} + \frac{1}{6R} \sqrt{3} \right) = \frac{1}{2R\sqrt{3}}.
$$

m (A.7) we note that $\gamma_{p_2} = 0$ (the indifference line CD is unaffected by Firm 2's price
200) With partials (B.4) (B.7) and (B.14) we obtain. σ , σ , (B.7) and (B.7), (B.7) and (B.7) we obtain: From (A.7) we note that $\gamma_{p_2} = 0$ (the indifference line CD is unaffected by Firm 2's price change). With partials (B.4), (B.7) and (B.14) we obtain change). With partials (B.4), (B.7) and (B.14) we of change). With partials $(B.4)$, $(B.7)$ and $(B.14)$ we obtain:

$$
D_{p_2} = \frac{1}{2} \left(\frac{1}{2R\sqrt{3}} + \frac{1}{6R} \sqrt{3} \right) = \frac{1}{2R\sqrt{3}}.
$$

¹⁰ We denote partial derivatives with the variable with respect to which we differentiate in subscript and omit the index of the firm, since we always work with *Firm* 3.

,

,

(c) The reaction in first two firms' prices to a radial move by *Firm* 3 will be identical; hence, we can disregard Firm 1's behavior. Totally differentiate (A.9) and (A.10) with $\frac{1}{2}$ respect to r to get: c first two first two first two first two first two firms $\frac{1}{\sqrt{2}}$ $\langle \sigma \rangle$ the reaction in first two first two firms

$$
\beta' - \alpha' + x'(\cos \beta - \cos \alpha) - y'(\sin \beta - \sin \alpha)
$$

+ $p'_2(\beta_{p_2} - \alpha_{p_2} + x_{p_2}(\cos \beta - \cos \alpha) - y_{p_2}(\sin \beta - \sin \alpha)) + p_2(\beta'_{p_2} - \alpha'_{p_2} + x'_{p_2}(\cos \beta - \cos \alpha) - x_{p_2}(\sin \beta \cdot \beta' - \sin \alpha \cdot \alpha') - x'(\sin \beta \cdot \beta_{p_2} - \sin \alpha \cdot \alpha_{p_2})$
- $y'_{p_2}(\sin \beta - \sin \alpha) - y_{p_2}(\cos \beta \cdot \beta' - \cos \alpha \cdot \alpha') - y'(\cos \beta \cdot \beta_{p_2} - \cos \alpha \cdot \alpha_{p_2}) = 0$
 $\beta' + y' - x'(\cos \beta - \cos \gamma) + y'(\sin \beta - \sin \gamma)$
+ $p'_3(-\beta_{p_3} + \gamma_{p_3} - x_{p_3}(\cos \beta - \cos \gamma) + y_{p_3}(\sin \beta - \sin \gamma)) + p_3(-\beta'_{p_3} + \gamma'_{p_3} - x'_{p_3}(\cos \beta - \cos \gamma) + x_{p_3}(\sin \beta \cdot \beta' - \sin \gamma \cdot \gamma') + x'(\sin \beta \cdot \beta_{p_3} - \sin \gamma \cdot \gamma_{p_3})$
+ $y'_{p_3}(\sin \beta - \sin \gamma) + y_{p_3}(\cos \beta \cdot \beta' - \cos \gamma \cdot \gamma') + y'(\cos \beta \cdot \beta_{p_3} - \cos \gamma \cdot \gamma_{p_3}) = 0$

Next, we assume the first equality from part (c) of present Lemma and observe from (B.5) and (A.4) that $x' = 0$ and $\alpha' = 0$. We also use the second part of (B.5), which says that the total derivative with respect to r of partial derivative of x, and, hence by $(A.5)$ also of α , with respect to any of the three prices is 0. Note again that $\alpha = 0$ in equilibrium, use (B.4) nmetry of the problem to simplify the above system:
 \overline{z} Next, we assume the first equality from part (c) of present Lemma and observe from $(B.5)$ and symmetry of the problem to simplify the above system: and (A.4) that $x' = 0$ and $\alpha' = 0$. We also use the second part of (B.5), which says that the and (A.4) that $x' = 0$ and $\alpha' = 0$. We also use the second part of (B.5), which says that the
total derivative with reconcer to x of portial derivative of x and hance by (A.5) also of α total derivative with respect to r of partial derivative of x, and, hence by $(A.5)$ also of α , with respect to any of the three prices is 0. Note excipt that $\alpha = 0$ in equilibrium, use (R 4). with respect to any of the three prices is 0. Note again that α = 0 in equilibrium, use (B.4) and symmetry of the another to simplify the charge system. Next, we assume the first equality from part (c) of present Lemma and observed $\frac{1}{2}$ with respect to any of the three prices is 0. Note again that $\alpha = 0$ in equilibrium, use (B.4) the problem to simplify the above system: $\frac{1}{2}$ total derivative with respect to r of partial derivative of x, and, hence by $(A.5)$ also of α
asith says that the total observe from part (c) of the total observe from particle firms and $(A.5)$ Next, we assume the first equality from part (c) of present Lemma and obse

$$
2\beta' - \sqrt{3}y' + p'_2\left(2\beta_{p_2} - 2\alpha_{p_2} - 3x_{p_2} - \sqrt{3}y_{p_2}\right) + p\left(2\beta'_{p_2} - \sqrt{3}x_{p_2}\beta' - \sqrt{3}y'_{p_2} + y_{p_2}\beta' - y'\left(-\beta_{p_2} - 2\alpha_{p_2}\right)\right) = 0 - 2\beta' + \sqrt{3}y' + p'_3\left(-2\beta_{p_3} + \sqrt{3}y_{p_3}\right) + p\left(-2\beta'_{p_3} + \sqrt{3}y'_{p_3} - y_{p_3}\beta' - y'\beta_{p_3}\right) = 0.
$$

We next note that by (A, E) , \cdots (executive in equilibrium The nome integrals) We next note that by $(A.5) \alpha_{p_2} = x_{p_2}/\cos \alpha = x_{p_2}$ in equilibrium. The remaining derivatives are collected in (B 7) (B 8) (B 11) (B 12) (B 13) (B 14) (B 17) and (B 18) Applying all these we get the final form of the system: tives are collected in (B.7), (B.8), (B.11), (B.12), (B.13), (B.14), (B.17), and (B.18). Applying
all these we get the final form of the system: all these we get the final form of the system: all these we get the final form of the system: \mathbf{t} We next note that by (A.5) $\alpha_n = x_n / \cos \alpha = x_n$ in equilibrium. The rem $\overline{}$, (B.8), (B.8), (B.12), (B.12), (B.11), and (B.12), and (B.12), and (B.17), and (B.18). Applying all $\overline{}$

$$
-54p'_2 + 18p'_3 + 36R - 9 + \pi\sqrt{3}(p'_2 - p'_3 - 2R + 3) = 0,
$$

$$
18p'_2 - 36p'_3 - 36R + 9 + \pi\sqrt{3}(-p'_2 + p'_3 + 2R + 3) = 0,
$$

with a solution: ′ with a solution: with a solution: with a solution: with a solution: with a solution:

with a solution:
\n
$$
p'_{2} = \frac{-9 + 9\pi\sqrt{3} - \pi^{2} + (36 - 2\pi\sqrt{3})R}{90 - 3\pi\sqrt{3}}, \text{ and}
$$
\n
$$
p'_{3} = \frac{18 + 12\pi\sqrt{3} - \pi^{2} - (72 - 4\pi\sqrt{3})R}{90 - 3\pi\sqrt{3}}.
$$
 Q.E.D.

Proof of Lemma 7: Collect results from Lemma 6 to calculate the expression in parentheses of (9) and equate it with zero:

$$
-\frac{4R^*-1}{2R^*\sqrt{3}}+\frac{2}{2R^*\sqrt{3}}\cdot\frac{-9+9\pi\sqrt{3}-\pi^2+\left(36-2\pi\sqrt{3}\right)R^*}{90-3\pi\sqrt{3}}=0.
$$

Simple algebra yields the result. \mathbf{H} to establish that we have really found a firm \mathbf{H} Simple algebra yields the result. Simple algebra yields the result. 2R* 3 2R* 3 2R* 3 2R* 3 2R* 3 90 − 3π 3π
3π 3π 3π 3π Simple algebra yields the result.

Simple algebra yields the result.

However, to establish that we have really found a firm's local best response to the rival locations we must check the second order condition also. The second order derivative of Firm 3's profit with respect to r is derived from (9): 2R* 3 2R* 3 90 − 3π 3 2R* 3 2R* 3 90 − 3π 3 2R* 3 2R* 3 90 − 3π 3 Simple algebra yields the result. locations we must check the second order condition also. The second order derivative or

$$
\frac{d^2\Pi_3}{dr^2} = \frac{d}{dr} \left(\frac{\partial \Pi_3}{\partial p_3} \cdot \frac{dp_3}{dr} \right) + \frac{dp_3}{dr} \cdot \left(\frac{\partial D_3}{\partial r} + \frac{\partial D_3}{\partial p_1} \cdot \frac{dp_1}{dr} + \frac{\partial D_3}{\partial p_2} \cdot \frac{dp_2}{dr} \right)
$$

+ $p_3 \cdot \left(\left(\frac{\partial D_3}{\partial r} \right)' + \left(\frac{\partial D_3}{\partial p_1} \right)' \cdot \frac{dp_1}{dr} + \left(\frac{\partial D_3}{\partial p_2} \right)' \cdot \frac{dp_2}{dr} + \frac{\partial D_3}{\partial p_1} \cdot \frac{d^2 p_1}{dr^2} + \frac{\partial D_3}{\partial p_2} \cdot \frac{d^2 p_2}{dr^2} \right)$ (A.11)
At $r = R$ ' both terms in the first row of (A.11) are zero. First one due to optimizing behavior.

 $\alpha = \frac{1}{2}$ At $r = R^*$ both terms in the first row of (A.11) are zero. First one due to optimizing behavior in the second stage of the game, and the second one due to the expression in parentheses being zero at $r = R^*$ (F.O.C.). It remains to evaluate the sign of the big expression in parentheses in the second row of (A.11). We prove a series of claims. second row of λ 11). Prove a series of claims. $\frac{1}{2}$ are $\frac{1}{2}$ at r $\frac{1}{2}$ (F.O.C.). It remains to evaluate the sign of the big expression in parentheses in the second row of (A 11). We prove a series of claims second row of (A111). We prove $\frac{1}{2}$ are computed at $\frac{1}{2}$. It is remainded to evaluate the sign of the sign of $\frac{1}{2}$. The sign of the sign of $\frac{1}{2}$ are computed in the second row of (A 11). We prove a series of claims second row of (A.11). We prove a series of claims. parentheses in the second row of (A.11). We prove a series of claims. in the second stage of the game, and the second one due to the expression in parentheses being in the second stage of the game, and the second one due to the expression in parentheses being in the second stage of the game, and the second one due to the expression in parentheses being

$$
\text{Claim 1: } \left(\frac{\partial D_3}{\partial r}\right)' < 0 \, .
$$

We go back to the proof of Lemma 6(a) to obtain the relevant part of the partial de-
rivative of interest Using the symmetry of the problem and the fact that $\mu = 0$ we get. we go back to the proof of Lemma $o(a)$ to botain the relevant part of the partial de-
rivative of interest. Using the symmetry of the problem and the fact that $x_r \equiv 0$ we get: $D_r = -\beta_r + y_r \sin \beta$. Hence, interest. Using the symmetry of the problem and the fact that Ξ rivative of interest. Using the symmetry of the problem and the fact that \bar{z}

$$
(D_r)' = -\beta'_r + \frac{\sqrt{3}}{2} y'_r - \frac{1}{2} y_r \cdot \beta'.
$$

The total derivatives are in $(B.9)$, $(B.10)$ $\overline{1}$ $\overline{2}$ $\overline{$ p_0 p_1 p_2 activatives at The total derivatives are in (B.9), (B.10), (B.16)
rields: \mathbf{y} ields: rne total de
yields: The total derivatives are in (B.9), (B.10), (B.16) and (B.18). Putting them all together (3.3) S: $\sqrt{2}$ $\sqrt{2}$ $\sqrt{2}$ \int $\frac{1}{2}$ \int $\frac{1}{$ $\frac{1}{2}$ $\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}})$ and $\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}})$ $T_{\rm g}$ are in (B.9), (B.10), $B_{\rm g}$. Putting them all to get here $T_{\rm g}$. Putting them all together yields: $\mathcal{L}^{\mathcal{L}}$ $T_{\rm eff}$, (B.9), (B.9), (B.10), (B.16), (B.16), (B.16), (B.16). Putting them all together yields: $\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}$ $\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}$ $\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}$ $\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}$ $\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}$ T_{S} , (B.10), B_{S} r de la provincia de la provin

$$
(D_r)' = -\frac{2R+1}{4R\sqrt{3}} - \frac{\sqrt{3}}{2} \frac{2}{9R} + \frac{1}{2} \frac{2}{3} \frac{p'_3 - p'_2 + 2R - 1}{2R\sqrt{3}}.
$$

Evaluating this at $R = R^*$ and us: ן
ו ∂ this at $R = R^*$ and using re \overline{a} \mathfrak{u} ∂ $= R^*$ and using resul Ì. J. using result from Lemma 6 Ĭ. luating this at $R = R^*$ and using result from Lemma 6(c Ŕ is at $R = R^*$ and using result ^r Evaluating this at $R = R^*$ and using result f Solution at $R = R^*$ and using represent to the set of th Evaluating this at $R = R^*$ and using result from Lemma 6(c) yields $(D_r)' \approx -1.3167$. $\overline{}$ $\text{dist } R = R^* \text{ and using result}$ using result f

Claim 2:
$$
\left(\frac{\partial D_3}{\partial p_2}\right)' < 0
$$
 and $\left(\frac{\partial D_3}{\partial p_1}\right)' < 0$.

We take the partial derivative from Lemma 6(b) and proceed similarly as in previous claim: $D_{p_2} = -\beta_{p_2} + y_{p_2} \sin \beta$. So,

 $\overline{3}$

 \overline{a} ∂

$$
\left(D_{p_2}\right)'=-\beta'_{p_2}+\frac{\sqrt{3}}{2}\,y'_{p_2}-\frac{1}{2}\,y_{p_2}\cdot\beta'.
$$

Claim 2: 0

 \overline{a} ∂

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Derivatives needed are in (B.7), (B.11), (B.17), and (B.18). We get:

$$
\left(D_{p_2}\right)' = -\frac{1}{4R^2\sqrt{3}} - \frac{\sqrt{3}}{2} \frac{1}{9R^2} - \frac{1}{2} \frac{1}{6R} \cdot \frac{p_3' - p_2' + 2R - 1}{2R\sqrt{3}}.
$$

Evaluating this at $R = R^*$ and using result from Lemma 6(c) yields $\left(D_{p_2}\right)' \approx -0.8121$. Due Evaluating this at $R = R$ and using result from Lemma 6(c) yields $(D_{p_2}) \approx -0.8121$. Due
to the symmetry it must also be $(D_{p_1}) \approx -0.8121$ Evaluating this at $R = R^*$ and using result from u
7 $\ddot{ }$ $\ddot{}$ 2 $\frac{2}{1}$ $\frac{11}{1}$ Evaluating this at $R = R^*$ and using result from Lemma 6(c) yields $\left(D_{p_2}\right)' \approx -0$
o the symmetry it must also be $\left(D_{p_2}\right)' \approx -0.8121$

Claim 3:
$$
\frac{d^2 p_2}{dr^2} < 0
$$
 and $\frac{d^2 p_1}{dr^2} < 0$.

To prove this claim analytically we would have to solve the system $(A.8)$ - $(A.10)$ without asserting $r = R$ to get p'_1 , p'_2 and p'_3 as functions of r. The dependence of relevant derivatives on r from Appendix B shows that such analytical solution is out of reach. We proceed as follows. We solve the system of the first-order conditions (A.8)-(A.10) for Firms ceed as follows. We solve the system of the first-order conditions (A.8)-(A.10) for *Firms*
1 and 2 located R^{*} away from the origin, while varying *Firm* 3's location, r, around R^{*},
numerically. The results are: numerically. The results are: α and α and α and α and α and α are to solve the system (A, δ) - (A, α) without $\frac{W}{\sigma}$ son τ from Appendix B shows that such analytical solution is out of reach. We pro-To prove this claim analytically we would have to solve the system (A.8)-(A.10) without To prove this claim analytically we would have to solve the system $(A.8)$ - $(A.10)$ without tives on r from Appendix B shows that such analytical solution is out of reach. We protives on the interpretation in shows that such analytical solution is out of reach. We pro-
cool as follows Me solve the exitem of the first-order conditions (A.8)-(A.10) for Firms ceed as follows. We solve the system of the first-order conditions $(A.8)-(A.10)$ for Firms
dend 2 legated P_1^* wave from the spirits sphill segming F_2^* in the setting a spared P_1^* α shows that such analytical solution is out of α . We proceed as follows.

$$
p_1(r = R^* - 0.005) = p_2(r = R^* - 0.005) = 0.991583,
$$

\n
$$
p_1(r = R^*) = p_2(r = R^*) = 0.993237
$$
, and
\n
$$
p_1(r = R^* + 0.005) = p_2(r = R^* + 0.005) = 0.994882.
$$

We are interested in two differences representing the first-order derivatives of p respect to r at $R^*-0.0025$ and $R^*+0.0025$: We are interested in two differences representing the first-order derivatives of p_2 with respect to *r* at *R*'–0.0025 and *R*'+0.0025:

$$
p_2(r = R^*) - p_2(r = R^* - 0.005) = 1.654 \times 10^{-3}
$$

$$
p_2(r = R^* + 0.005) - p_2(r = R^*) = 1.645 \times 10^{-3}.
$$

we see that in vierinty of K the mst-order derivative of fivals prices with respect to
decreasing in r hence, the second order derivative must be negative. This completes necessary in 3 1 and 3 2 2 ∂p are also positive and 3 2 ∂p and 3 2 ∂p are also proof of Claim 3 c see that in vierinty of it the first order derivative of fivals prices with respect to re-
creasing in r-hence-the second order derivative must be negative. This completes th decreasing in r, hence, the second order derivative must be negative. This completes the
proof of Claim 3 We see that in vicinity of R^* the first-order derivative of rivals' prices with respect to r is $\frac{1}{2}$ single summand in the big parameter in the second row of $\frac{1}{2}$ proof of Claim 3.
One can verify that p'_1 and p'_2 presented in Lemma 6 (c) are positive when $r=R^*$, while proof of Claim 3. proof of Claim 3. ² (0.005) () 1.645 10 [−] p r = R + − p r = R = × . $\frac{d}{dt}$ r, hence, the second of $\frac{d}{dt}$ in $\frac{d}{dt}$ $\frac{d}{dt}$ is that in vicinity of R the first-order derivative of rivals' prices with respect to r is $\frac{d}{dt}$ decrease in reduced the second order derivative must be negative. This complete the

see that every single summand in the big parentheses in the second row of (A.11) is negative.

One can verify that p'_1 and p'_2 presented in Lemma 6 (c) are positive when $r=R^*$, while and verify that p_1 and p_2 presence in Eximina 6 (c) are positive with r=R; with
ma 6(b) says that $\partial D_1/\partial p_1$ and $\partial D_2/\partial p_2$ are also positive. Using the three claims above, we see that every single summand in the big parentheses in the second row of of Firm 3's profits with respect to r. $Q.E.D.$ α is obvious with respect to α . α ^{D3} β ^{D3} β ² ll y $\frac{1}{dr^2} < v$ are going to show that the expression in the parentheses in the parentheses in $\frac{1}{dr^2}$ One can verify that p'_1 and p'_2 presented in Lemma 6 (c) are positive when $r=R^*$, while
Lemma 6(b) says that $\partial D_3/\partial p_1$ and $\partial D_3/\partial p_2$ are also positive. Using the three claims
above we see that every single summ (A.11) is negative. Hence, $\frac{d^2 \Pi_3}{\pi}$ < 0 at $r = R^*$ at $r = R^*$. We have found a local maximum (A.11) is negative. Hence, $\frac{d^2 \Pi_3}{dr^2} < 0$ at $r = R^*$ at $r = R^*$. We have found a local max dr at $r = R^*$. We have found a local maximum $\frac{d^2 \Pi_3}{dt^2} < 0$ at $r = R^*$ at $r = R^*$. We have found a local maximum dr (A.11) is negative. Hence, $\frac{d^2\Pi_3}{dt^2}$ < 0 at $r = R^*$ at $r = R^*$. We have found a local maximum dr $\frac{d^2\Pi_3}{dt^2}$ < 0 at $r = R^*$ at $r = R^*$. We have found a local maximum with respect to r . $Q.E.D.$

Proof of Lemma 8: We are going to show that the expression in the parentheses in (10) is Proof of Lemma 8: We are going to show that the expression in the parentheses in (10) is zero. First, it is obvious that $\partial D_3/\partial \phi = 0$, since the same number of customers won by a move along polar direction from one neighbor is lost to the other one.

Second, we claim $dp_1/d\phi = -dp_2/d\phi$. Clearly, the neighbor who becomes closer by such a move becomes more aggressive and reduces its price and vice versa, the neighbor that is now farther away becomes less aggressive and raises its price. The magnitude of the two derivatives is the same due to symmetry of the proposed configuration. The result follows. *Q.E.D.*

Appendix B: Derivatives

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We collect all the derivatives needed in preceding Appendix in subsections B.1 and B.2 for duopoly and three-firm oligopoly markets, respectively. ω collect all the derivatives needed in preceding Appendix in subsections B.1 and B.2 led in preceding Appendix in subsections B.1 and B.2
oly markets, respectively. μ collect all the derivatives needed in preceding μ for duopoly and three-firm ongopoly matrices, respect luopoly and three-firm oligopoly markets, respectively. duopoly and three-firm oligopoly markets, respectively. duopoly and three-firm oligopoly markets, respectively. duopoly and three-firm oligopoly markets, respectively.

B.1 *Two firms* \mathcal{L} B.1 Two firms \overline{a} and the firm oligopoly matrix \overline{a} .

In equilibrium $r = R$, $\phi = \pi$, and $\alpha = \pi/2$. \mathbf{I}^{max} = \mathbf{I}^{max} and \mathbf{I}^{max} and \mathbf{I}^{max} equilibrium $r = R$, $\phi = \pi$, and $\alpha = \pi/2$.

From (A.2):
$$
\alpha_{p_1} = -\alpha_{p_2} = -\frac{1}{2(r+R) \cdot \sin \alpha} = -\frac{1}{4R}
$$
 and $z_{p_1} = -z_{p_2} = -\frac{1}{2R}$ (B.1)

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\n**APPENDIX B:** DERIVATIVES
\nWe collect all the derivatives needed in preceding Appendix in subsections B.1 and B.2
\nfor duopoly and three-firm oligopoly markets, respectively.
\nB.1 Two firms
\nIn equilibrium
$$
r = R
$$
, $\phi = \pi$, and $\alpha = \pi/2$.
\nFrom (A.2): $\alpha_{p_1} = -\alpha_{p_2} = -\frac{1}{2(r+R) \cdot \sin \alpha} = -\frac{1}{4R}$ and $z_{p_1} = -z_{p_2} = -\frac{1}{2R}$ (B.1)
\nFrom (B.1): $\alpha'_{p_1} = -\alpha'_{p_2} = \frac{2 \sin \alpha}{4(r+R)^2 \cdot \sin^2 \alpha} = \frac{1}{2(r+R)^2} = \frac{1}{8R^2}$ and $z'_{p_1} = -z'_{p_2} = \frac{1}{4R^2}$ (B.2)
\nFrom (A.1): $\alpha' = \frac{p'_2 - p'_1 + 2r}{4} = \frac{p'_2 - p'_1 + 2R}{4} = \frac{p'_2 - p'_1 + 2R}{4R^2}$ (B.3)

From (A.1):
$$
\alpha' = \frac{p'_2 - p'_1 + 2r}{2(r+R) \cdot \sin \alpha} = \frac{p'_2 - p'_1 + 2R}{4R}
$$
 and $z' = \frac{p'_2 - p'_1 + 2R}{2R}$ (B.3)
B.2 Three firms

B.2 Three firms From (A.1): ^R B.2 Three firms B.2 Three firms B.2 *Three firms* B.2 Three firms B.2 Three firms B.2 Three firms B.2 Three firms

From (A.4):
$$
x_{p_1} = -x_{p_2} = -\frac{1}{2R\sqrt{3}}
$$
, $x_{p_3} = 0$, $x_r = 0$, (B.4)

From (A.4) and (B.4):
$$
x' = \frac{p'_2 - p'_1}{2R\sqrt{3}}
$$
, $x'_{p_1} = x'_{p_2} = x'_{p_3} = 0$, (B.5)

From (A.4) and (B.4):
$$
x' = \frac{P_2 - P_1}{2R\sqrt{3}}
$$
, $x'_{p_1} = x'_{p_2} = x'_{p_3} = 0$, (B.5)
\nFrom (A.5): $y_{p_1} = \frac{-R\sqrt{3} + 2r\sin\phi}{2R\sqrt{3}(2r\cos\phi - R)} = \frac{1}{2(2r + R)} = \frac{1}{6R}$, (B.6)
\nFrom (A.5): $y_{p_2} = \frac{-R\sqrt{3} - 2r\sin\phi}{2R\sqrt{3}(2r\cos\phi - R)} = \frac{1}{2(2r + R)} = \frac{1}{6R}$, (B.7)

From (A.5):
$$
y_{p_2} = \frac{-R\sqrt{3} - 2r\sin\phi}{2R\sqrt{3}(2r\cos\phi - R)} = \frac{1}{2(2r + R)} = \frac{1}{6R},
$$

\nFrom (A.5): $y_{p_3} = \frac{2R\sqrt{3}}{2R\sqrt{3}(2r\cos\phi - R)} = -\frac{1}{2r + R} = -\frac{1}{3R},$ (B.8)

From (A.5):
$$
y_{p_2} = \frac{-R\sqrt{3} - 2r\sin\phi}{2R\sqrt{3}(2r\cos\phi - R)} = \frac{1}{2(2r + R)} = \frac{1}{6R}
$$
, (B.7)
From (A.5): $y_{p_3} = \frac{2R\sqrt{3}}{2R\sqrt{3}(2r\cos\phi - R)} = -\frac{1}{2r + R} = -\frac{1}{3R}$, (B.8)

From (A.5):
$$
y_{p_3} = \frac{2R\sqrt{3}}{2R\sqrt{3}(2r\cos\phi - R)} = -\frac{1}{2r + R} = -\frac{1}{3R}
$$
, (B.8)
\nFrom (A.5): $y_r = \frac{R\sqrt{3} \cdot 4r}{2R\sqrt{3}(2r\cos\phi - R)} = -\frac{2r}{2r + R} = -\frac{2}{3}$, (B.9)
\nFrom (B.9): $y'_r = -\frac{2(2r + R) - 4r}{(2r + R)^2} = -\frac{2}{9R}$, (B.10)

From (B.9):
$$
y'_r = -\frac{2(2r+R)-4r}{(2r+R)^2} = -\frac{2}{9R}
$$
, (B.10)

From (B.6) and (B.7):
$$
y'_{p_1} = y'_{p_2} = \frac{R\sqrt{3} \cdot 2\cos\phi}{2R\sqrt{3}(2r\cos\phi - R)^2} = \frac{-1}{(2r+R)^2} = -\frac{1}{9R^2}
$$
, (B.11)

From (B.8):
$$
y'_{p_3} = -\frac{2\cos\phi}{(2r\cos\phi - R)^2} = \frac{2}{(2r+R)^2} = \frac{2}{9R^2}
$$
, (B.12)

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From (A.5):
$$
y' = \frac{R\sqrt{3}(2p'_3 - p'_1 - p'_2 + 4r)}{2R\sqrt{3}(2r\cos\phi - R)} = \frac{2p'_3 - p'_1 - p'_2 + 4r}{2(2r\cos\phi - R)} = -\frac{2p'_3 - p'_1 - p'_2 + 4R}{6R}
$$
, (B.13)

$$
\beta_{p_2} = -\beta_{p_3} = -\frac{1}{(2r\sin\phi - R\sqrt{3})\cos\beta - (2r\cos\phi - R)\sin\beta}
$$
\n
$$
= -\frac{1}{R\frac{\sqrt{3}}{2} + (2r + R)\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}(r + R)} = -\frac{1}{2R\sqrt{3}}
$$
\n(B.14)

From (A.6):
$$
\beta_r = \frac{2r - 2\cos\phi\cos\beta}{(2r\sin\phi - R\sqrt{3})\cos\beta - (2r\cos\phi - R)\sin\beta} = \frac{2r - 1}{R\frac{\sqrt{3}}{2} + (2r + R)\frac{\sqrt{3}}{2}} = \frac{2R - 1}{2R\sqrt{3}}
$$

(B.15)

From (B.15):
$$
\beta'_{r} = \left(\frac{2r-1}{\sqrt{3}(r+R)}\right)' = \frac{2\sqrt{3}\cdot 2R - (2R-1)\sqrt{3}}{12R^2} = \frac{2R+1}{4R\sqrt{3}},
$$
 (B.16)

⁺ [⋅] ′ [−] [−] [−] [⋅] ′ ′

2sin cos 3 sin 2cos sin (2 cos) cos

2sin cos 3 sin 2cos sin (2 cos) cos

$$
\beta'_{p_2} = \beta'_{p_3} = \frac{2\sin\phi\cos\beta + R\sqrt{3}\sin\beta \cdot \beta' - 2\cos\phi\sin\beta - (2r\cos\phi - R)\cos\beta \cdot \beta'}{\left((2r\sin\phi - R\sqrt{3})\cos\beta - (2r\cos\phi - R)\sin\beta\right)^2},
$$

=
$$
\frac{\sqrt{3}}{3(r+R)^2} = \frac{1}{4R^2\sqrt{3}}
$$
(B.17)

 From (A.6): From $(A.6)$: From $(A.6)$: From $(A.6)$: From (A.6): From (A.6): From (A.6):

From (A.6):
\n
$$
\beta' = \frac{p'_3 - p'_2 + 2r - 2\cos\phi\cos\beta}{(2r\sin\phi - R\sqrt{3})\cos\beta - (2r\cos\phi - R)\sin\beta} = \frac{p'_3 - p'_2 + 2r - 1}{\sqrt{3}(r + R)} = \frac{p'_3 - p'_2 + 2R - 1}{2R\sqrt{3}}.
$$
\n(B.18)