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THREE FIRMS ON A UNIT DISK MARKET: INTERMEDIATE PRODUCT DIFFERENTIATION

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ABSTRACT: *Irmen and Thisse (1998) demonstrate that two firms competing with multi-characteristic products differentiate them in one characteristic completely, while keeping them identical in all others. This paper shows that their* min-…-min-max *differentiation result is not robust with respect to the number of firms. A market setting that replicates their result in a duopoly, but fails to do so in a three firm oligopoly is identified. Symmetric pure strategy equilibrium with three firms differentiating their products in two dimensions, but not completely in either of them, is a novel* medium-medium *differentiation result.*

Keywords: *spatial competition, location-price game*
JEL classification: L11, L13, R39

THREE FIRMS ON A UNIT DISK MARKET: INTERMEDIATE PRODUCT DIFFERENTIATION

Models of discrete location choice commonly interpreted as modeling product differentiation, as well, have been explored in a variety of contexts.[2] Irmen and Thisse (1998) provide the most comprehensive study of two firms competing with their products in an

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[2] ReVelle and Eiselt (2005), and ReVelle, Eiselt, and Daskin (2008) provide a comprehensive survey of the field and collect an extensive bibliography covering various aspects and problems in this area, respectively. A part of the literature is interested in the extent of product differentiation that firms should employ, and the number of product dimensions they should use doing that. Pioneering work with a linear duopoly model by Hotelling (1929) offered a principle of minimum differentiation. Hotelling's contribution was revisited 50 years later, when d'Aspremont *et al*. (1979) showed that there is no price equilibrium in pure strategies when two firms are located too closely to each other. Using quadratic instead of linear transportation costs, they find unique market equilibrium with firms maximizing product differentiation. The direct demand effect that makes a firm move towards its opponent to capture its demand is followed by the opponent's price cut. The latter overrides extra profit gained with new demand from moving towards the opponent. The negative strategic effect of igniting stronger competition induces firms to differentiate their products as much as they can. As it was shown later, a maximum differentiation result rests both on the form of consumers' utility function (e.g. Economides, 1986), and the uniform distribution of their tastes within the product characteristics space (e.g. Neven, 1986, Tabuchi & Thisse, 1995).

n-dimensional product space.[3] They investigate a unit hyper-cube market that is uniformly populated by consumers and served by two firms. When consumers incur disutility that is quadratic in distance between a product variety that they would prefer the most and the variety bought, they show that it is always optimal for the two firms to differentiate their products in one dimension only, and doing so completely. This type of practice is referred to as *min-…-min-max* product differentiation.

This paper departs from the Irmen in Thisse model in that it studies a market with three firms in a two-dimensional product space. We present a unit disk market uniformly populated by consumers that are served by firms operating one store each. Firms choose respective store locations in the first stage and compete with prices in the second. There are some authors that explore how competition between more than two firms affects product differentiation. Salop (1979) and Economides (1989) look at a circular city model with location equilibria that place firms equidistantly. Economides (1993) provides price equilibrium characterizations for every location configuration in a linear city model, and Brenner (2005) presents location equilibria for different number of firms (from three to nine) in the same type of the market. Brenner shows that firms depart from a maximum differentiation result in that the store locations move towards the middle of the market.

We address two questions. First, will more than two firms in a bounded, two dimensional product characteristics space differentiate their products less than completely, as we might suspect from Brenner (2005)? Brenner shows that a firm with two neighbors does not cut prices as drastically as in a duopoly case when it is approached by one of the neighbors. The reason is that it does not want to alter its optimal revenues from the other side, where it neighbors a firm that did not deviate from an equilibrium position. Consequently, in equilibrium even the two firms on the two outskirts of the city move towards the center. The direct demand effect outweighs the strategic price effect and extent of product differentiation is reduced. There are two reinforcing effects facilitating Brenner's result. First, the area of confrontation between an intrusive firm and its victim is a single point, a marginal consumer between the two firms. Since the area of confrontation between the victim and its other side neighbor is of the same size, the incentive to cut its price to counter the intruder is offset in a large part by a lower price and suboptimal profits on the other side. Second, the other side neighbor anticipates lower prices; it reduces its price as well. Hence, a reduction of victim's price does not translate in a sizeable increase of its demand on the other neighbor's side and is not profitable. Consequently, a cut in a victim's price is not large enough to keep the intrusive firm from moving in to capture a part of its demand. Consequently, firms locate closer to each other. The role of the two effects we have described is less obvious in our case of firms competing in two dimensions. Market configuration may be such that a firm that moves its store does that in a direction of two and not just one neighbor. The firms' demands are now delineated by line segments of consumers that are indifferent between buying from any of the two

[3] Irmen and Thisse are not the first to explore markets where consumers care about more than one product characteristic. Neven and Thisse (1990) and Tabuchi (1994) were the first to show that in a two dimensional product characteristics space two firms will never find it optimal to differentiate their products fully. There are equilibria in which products are completely differentiated along one dimension and identical in the other.

neighboring firms. A confrontation frontier for an intrusive firm may hence be longer than is a line segment between respective neighbors fighting the intruder. This means that a price cut by the neighbors does not necessarily be as pronounced as it was in a one-dimensional case but may still keep the intrusive firm away. As a result, it might be that firms stay on the outskirts of the market. On the other hand, we might see a free-rider effect, meaning that firms under attack would count on each other to counter the intruder with lower prices. This might lead to a price cut that is insufficient to keep the intruder on the edge of the market.

Another question is whether firms find it optimal to differentiate their respective products in more than one dimension in the first place. If the market was unbounded and consumers' reservation prices finite, the answer is obviously positive. With a bounded market, the answer is not imminent and might depend on the shape of the market in general. We expect that firms will find it beneficial to leave the congested competition in one product characteristic at some point and will choose to differentiate their products in another one as well. Swann (1990) explores such a process with a simple model and simulations. Whenever the field of competition becomes too dense at least one firm endogenously finds it optimal to introduce a new product attribute.

We first show that maximum differentiation in one dimension – and no differentiation in the other (Irmen and Thisse, 1998) – remains optimal in a duopoly. In our setting this means that the two firms position their stores on the perimeter of the disk, exactly opposite from each other. We then present two novel results. An oligopoly with three firms competing on the same market facilitates a pure strategy subgame perfect Nash equilibrium of our location-price game. The equilibrium has all three firms located at the same distance from the center of the disk, equidistant from each other. That means that we observe differentiation in two product characteristics, a result that extends the existent literature. Furthermore, firms do not choose full differentiation, but move inward, towards the center of the disk considerably. We find *medium-medium* type of product differentiation in a setting that yields a *min-max* differentiation result in a duopoly. This means that the conjecture based on Brenner (2005), given above, carries over to markets with more than two competitors and more than just one product characteristic. When firms are located close to the perimeter of the market, the positive demand effect of a radial deviation towards the center outweighs the negative strategic effect of rivals decreasing their prices.

Another interesting aspect of the model is that, while in a duopoly a social planner would have firms differentiating their products less extensively, the result reverses in a three-firm oligopoly. Hence, some cooperative behavior or regulation on product specifications would be beneficial both to firms and to society as a whole.

The organization of the paper is as follows. We set up our model in Section 1, and present our results for a duopoly and a three-firm oligopoly in Sections 2 and 3, respectively. Section 4 considers welfare issues, and we make our conclusions in Section 5. Proofs of Lemmas and Propositions and all necessary derivatives are deferred to the Appendices.

1. THE MODEL

We use the classical spatial model of an oligopoly. Consumers of a total mass π are uniformly distributed on a unit disk.[4] Each consumer has a unit demand for a homogeneous good produced by n firms on the market. We explore configurations in which all firms are at the same radial distance from the origin, R. Specifically, the firms are located at $L_i=(R,\phi_i)$, $i = 1,\ldots, n$. *Firm* 1 is always a counter-clockwise direction neighbor of *Firm n*, while *Firm n*–1 is a clockwise direction neighbor of *Firm n*. Firms charge p_i, $i = 1, \ldots, n$, per unit of the good. If a consumer residing at point x buys the good from *Firm i*, she derives utility: $u(p_i, L_i; x) = v - p_i - d^2\left(x, L_i\right)$.

We assume that consumers incur traveling costs that are quadratic in the Euclidean distance, $d(x, L_i)$, traveled, and with cost per unit traveled normalized to one. The surplus v enjoyed from consuming the good is assumed to be large enough so that every consumer buys a good from one of the firms. Consumers maximize their utility by choosing the store they buy the good from optimally.

Firms operate with symmetric constant marginal costs that we normalize to zero and are allowed to operate one store each. We consider a non-cooperative two-stage location-price game of the following form.

Stage 1: Firms select locations on a unit disk.

Stage 2: Firms observe chosen locations and compete in prices.

Formally, firm i's strategy space in the first period is $L_i = [0,1]\times[-\pi,\pi]$, $i = 1,\ldots, n$, and each firm's strategy in the second stage is s $p_i : \times_{i=1}^{n} L_i \rightarrow \Re_+$. We seek subgame perfect Nash equilibria of this game: $S^* = \left\{(L_1^*, p_1^*), (L_2^*, p_2^*), \ldots, (L_n^*, p_n^*)\right\}$. A natural candidate for equilibrium configuration has firms positioned at an equal distance from the origin, equidistantly along the circle they occupy. Specifically, we explore locations: $L_1^* = (R, -\pi + \frac{2}{n}\pi)$, $L_2^* = (R, -\pi + \frac{4}{n}\pi))$, … , and $L_n^* = (R, \pi)$.

2. TWO FIRMS

We first derive market equilibrium in a duopoly. Here we show that min-max result derived by Irmen and Thisse (1998) is also optimal strategy in our setting. In our setting that means that firms locate their stores on the perimeter of the disk, symmetrically across the origin. We search for an equilibrium that is symmetric in firms' locations with Figure 1 showing a possible off-equilibrium configuration $L_1^* = (R,0)$ and $L_2 = (r,\phi)$.

[4] Polar symmetry is used to avoid non-differentiable demand functions that arise in a rectangular market when a line of consumers indifferent between buying from two neighboring stores touches the corner of a market. Moreover, there is no symmetry to exploit in the market with three firms on a square.

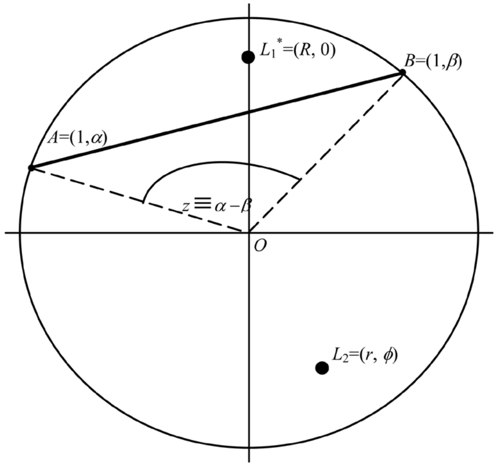

Figure 1: *Configuration of the market, two firms.*

Line AB in Figure 1 represents consumers that are indifferent between buying a product from either of the two firms given their locations and product prices. AB is perpendicular to the line connecting firms' locations L_1^* and L_2. It is closer to the firm that sets the higher of the two respective prices. AB crosses the perimeter of the disk at angles α and β. *Firm* 1's demand is the area between α and β, $(\alpha-\beta)/2$, reduced by the area of a triangle ABO, which is $\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} = \sin(\alpha-\beta)/2$. We set $z \equiv \alpha - \beta$ and state the firms' demands:

$$D_1 = \frac{1}{2}\left(z - \sin z\right), \text{ and } \quad D_2 = \pi - D_1 = \frac{1}{2}\left(2\pi - z + \sin z\right), \tag{1}$$

Firms' second-stage profits given locations chosen in the first stage are:

$$\Pi_1(r,\phi;R) = p_1(r,\phi;R)\cdot D_1(r,\phi;R) \quad \text{and} \quad \Pi_2(r,\phi;R) = p_2(r,\phi;R)\cdot D_2(r,\phi;R)\,. \tag{2}$$

Note that p_1, p_2, α, β, and z are all functions of firms' respective locations. Second-stage profit maximization with respect to prices yields the following result.

Lemma 1: When firms choose locations $L_1^* = (R,0)$ and $L_2^* = (R,\pi)$ in the first stage of the game, they both charge $p^* = \pi\cdot R$ in the second.

The price firms charge in the second-stage of the game increases in distance from the origin, that is in the distance between their stores or their products. This shows a common incentive dictating the need for product differentiation.

Next, we want to see whether the proposed structure of firms' first stage locations can be supported in equilibrium. We show there is $R \in (0,1]$, such that given *Firm* 1's location, L_1^*, *Firm* 2's location, L_2^*, is optimal. It turns out that this R equals one; firms locate their stores on the perimeter of the disk. Also, they will be positioned symmetrically across the origin. To see this, we look at respective effects deviations in radial and polar directions have on *Firm* 2's profits. These effects are:

$$\frac{d\Pi_2}{dr} = \frac{\partial \Pi_2}{\partial p_2} \cdot \frac{dp_2}{dr} + p_2 \cdot \left(\frac{\partial D_2}{\partial r} + \frac{\partial D_2}{\partial p_1} \cdot \frac{dp_1}{dr} \right), \quad \text{and} \tag{3}$$

$$\frac{d\Pi_2}{d\phi} = \frac{\partial \Pi_2}{\partial p_2} \cdot \frac{dp_2}{d\phi} + p_2 \cdot \left(\frac{\partial D_2}{\partial \phi} + \frac{\partial D_2}{\partial p_1} \cdot \frac{dp_1}{d\phi} \right). \tag{4}$$

The first term on the right hand side (RHS henceforth) of both (3) and (4) is zero due to profit maximization with respect to p_2 in the second period, so we are left with the direct (demand) effect and indirect (strategic) effect of each move, which are the first and second terms in parentheses, respectively. We first derive the demand and strategic effects of a small radial move on *Firm* 2's profit.

Lemma 2: When $L_1^* = (R,0)$ and $L_2^* = (R,\pi)$, $R \in (0,1]$: (a) $\frac{\partial D_2}{\partial r} = -1$, (b) $\frac{\partial D_2}{\partial p_1} = \frac{1}{2R}$, and (c) $\frac{dp_1}{dr} = \frac{\pi}{2} + \frac{2}{3}R$.

Part (a) quantifies *Firm* 2's direct gain in demand it captures from its rival by moving towards its location, while (b) and (c) show that such a move affects *Firm* 2's demand adversely through a rival's price cut. These effects are such that both firms would like to locate their stores on the perimeter of the disk when they are separated by angle π.

Lemma 3: For any pair of locations $L_1^* = (R,0)$ and $L_2^* = (R,\pi)$, $R \in (0,1]$, firms find it profitable to move farther away from the origin.

Finally, we argue that the angle separating the two firms is exactly π.

Lemma 4: For any R, $\phi = \pi$ is optimal for Firm 2.

Lemmas 3 and 4 prove Proposition 1.

Proposition 1: When in the first stage of the game two firms position their stores on the perimeter of the disk symmetrically across its origin; the equilibrium price they charge in the second-stage is π. Furthermore, every firm's location is a local best response to their rival's location.

Due to the complexity of the problem we are not able to show that each firm's location is also a global best response to an opponent's location analytically. We hence do it numerically. We fix *Firm* 1's location at R=1 and $\phi_1 = 0$, and vary *Firm* 2's location across the disk. We solve for the second-stage price equilibrium for every configuration numerically and calculate *Firm* 2's profit. Figure 2 shows that the proposed location r=1 and $\phi_2 = \pi$ is actually *Firm* 2's global best response to *Firm* 1's location choice. The profit attained there is $\pi^2/2$.

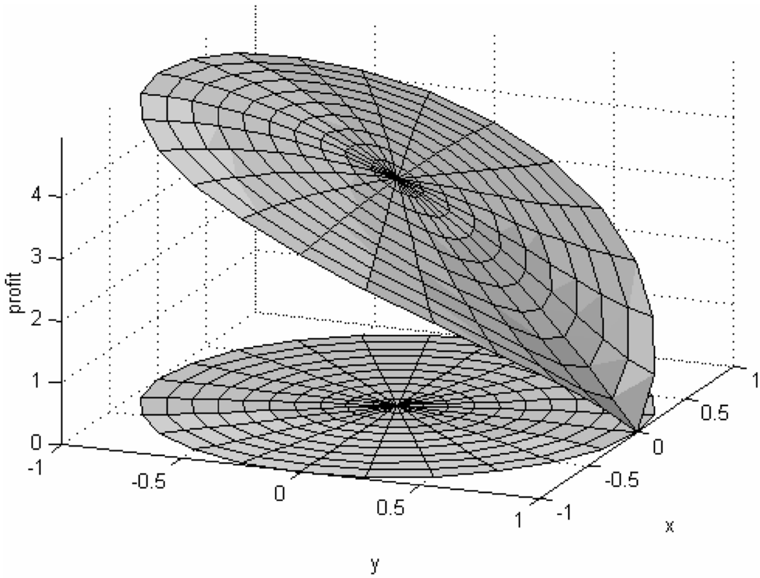

Figure 2: *Firm 2's profits given the location of its store.*
Source: Own calculations.

We now state our first result, which links this work to the existent literature of two firms competing in a multi-characteristic product space. This gives us a valid reference point with which to compare our subsequent results.

Result 1: Two firms positioning their stores on the perimeter of the disk, symmetrically across its origin in the first stage of the game, and setting $p=\pi$ in the second is a subgame perfect Nash equilibrium of the game.

This finding is in the spirit of Neven and Thisse (1990), Tabuchi (1994), and Irmen and Thisse (1998). Two firms offer products that are fully differentiated in one characteristic, while identical in the other.

3. THREE FIRMS

We add another firm to the model and search for symmetric equilibrium. We find that firms separate their stores in both dimensions, and interestingly, do not choose to locate them on the perimeter anymore, but relocate them towards the origin noticeably.

Given locations $L_1^* = (R,-\pi/3)$, $L_2^* = (R,\pi/3)$, and $L_3 = (r,\phi)$, firms compete in prices, p_1, p_2, and p_3. A particular market configuration is shown in Figure 3.

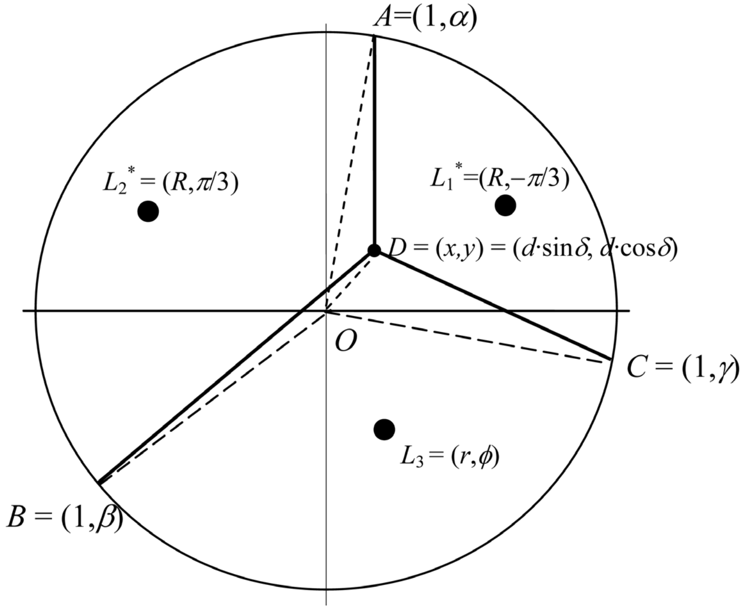

Figure 3: *Configuration of the market, three firms.*

Store locations and prices define the boundaries of market areas covered by firms. There are three line segments, *DA*, *DB*, and *DC*, representing buyers indifferent between buying from *Firms* 1 and 2, *Firms* 2 and 3, and *Firms* 3 and 1, respectively. These segments are needed in determining the demand functions firms face. They all join in one point, *D*, which is due to the fact that the delineating lines must be straight. Point $D = (x, y)$ represents a consumer indifferent between buying from any of the three firms. Points $A = (1, \alpha)$, $B = (1, \beta)$, and $C = (1, \gamma)$ stand for consumers on the perimeter indifferent between buying from respective firms. With the knowledge of x, y, α, β, and γ, which, as well as p_1, p_2, and p_3, are all functions of r, ϕ, and R, we can write the demand functions firms face.[5] *Firm* 1's demand equals the area of the disk between α and γ reduced for the area covered by triangles *OCD* and *ODA*. The other two firms' demands are obtained similarly:

$$D_1 = \frac{1}{2}\left(\alpha - \gamma - d\sin(\delta - \gamma) - d\sin(\alpha - \delta)\right),$$

$$D_2 = \frac{1}{2}\left(\beta - \alpha + d\sin(\alpha - \delta) - d\sin(\beta - \delta)\right), \text{ and}$$

$$D_3 = \frac{1}{2}\left(2\pi - \beta + \gamma + d\sin(\beta - \delta) + d\sin(\delta - \gamma)\right).$$

[5] For the derivation of x, y, α, β, and γ see the proof of Lemma 5.

The *sine of the difference* rule and definitions of x and y yield:

$$D_1 = \frac{1}{2}\left(\alpha - \gamma + x(\cos\alpha - \cos\gamma) - y(\sin\alpha - \sin\gamma)\right), \tag{5}$$

$$D_2 = \frac{1}{2}\left(\beta - \alpha + x(\cos\beta - \cos\alpha) - y(\sin\beta - \sin\alpha)\right), \text{ and} \tag{6}$$

$$D_3 = \frac{1}{2}\left(2\pi - \beta + \gamma - x(\cos\beta - \cos\gamma) + y(\sin\beta - \sin\gamma)\right). \tag{7}$$

The firms' profit functions are:

$$\begin{aligned} &\Pi_1(r,\phi;R) = p_1(r,\phi;R)\cdot D_1(r,\phi;R), \\ &\Pi_2(r,\phi;R) = p_2(r,\phi;R)\cdot D_2(r,\phi;R), \text{ and} \\ &\Pi_3(r,\phi;R) = p_3(r,\phi;R)\cdot D_3(r,\phi;R). \end{aligned} \tag{8}$$

Second-stage optimal prices are derived from the system of necessary conditions obtained from these profits. The system is nonlinear and its general analytical solution for any possible *Firm* 3's location, L_3, is therefore out of reach. However, we are looking for symmetric equilibrium, so the derivation of optimal prices is straightforward.

Lemma 5: When three firms in the first stage of the game position their stores R away from the origin, equidistantly from one another, the optimal Nash equilibrium price they charge in the second stage is $p^* = \frac{\pi R\sqrt{3}}{3}$.

The system of first-order conditions derived in the proof of Lemma 5 (A.8-A.10) completely characterizes price competition in the second stage given that *Firm* 3 chooses its location reasonably close to the proposed L_3^*.[6] It remains to be seen whether such a configuration is optimal in the first stage of the game.

We rewrite *Firm* 3's profit function with rivals' store positions being L_1^* and L_2^*, and firms charging equilibrium prices in the second stage:

$$\Pi_3(r,\phi;R) = p_3(r,\phi;R)\cdot D_3(r,\phi;R,p_1(r,\phi;R),p_2(r,\phi;R),p_3(r,\phi;R)).$$

We are interested in two derivatives:

$$\frac{d\Pi_3}{dr} = \frac{\partial\Pi_3}{\partial p_3}\cdot\frac{dp_3}{dr} + p_3\cdot\left(\frac{\partial D_3}{\partial r} + \frac{\partial D_3}{\partial p_1}\cdot\frac{dp_1}{dr} + \frac{\partial D_3}{\partial p_2}\cdot\frac{dp_2}{dr}\right), \text{ and} \tag{9}$$

[6] If *Firm* 3 located its store at the top of the disk, the configuration of the demands would have changed and quantities defined in Figure 3 would not be valid anymore.

$$\frac{d\Pi_3}{d\phi}=\frac{\partial\Pi_3}{\partial p_3}\cdot\frac{dp_3}{d\phi}+p_3\cdot\left(\frac{\partial D_3}{\partial\phi}+\frac{\partial D_3}{\partial p_1}\cdot\frac{dp_1}{d\phi}+\frac{\partial D_3}{\partial p_2}\cdot\frac{dp_2}{d\phi}\right). \tag{10}$$

Again, the first term in RHS of both (9) and (10) equals zero. The first term in parentheses in both equations represents the direct or demand effect of the deviation in a respective variable, while the last two represent the indirect or strategic effects of such a deviation through competitors' prices. We show that there exists a distance from the origin, *R*, such that if firms locate there equidistantly from one another, the necessary conditions for symmetric equilibrium are satisfied. The derivatives we need to determine the demand and strategic effects in (9) are collected in Lemma 6.

Lemma 6: When $L_1^* = (R,-\pi/3)$, $L_2^* = (R,\pi/3)$ and $L_3^* = (R,\pi)$: (a) $\frac{\partial D_3}{\partial r}=-\frac{4R-1}{2R\sqrt{3}}$, (b) $\frac{\partial D_3}{\partial p_1}=\frac{\partial D_3}{\partial p_2}=\frac{1}{2R\sqrt{3}}$, and (c) $\frac{dp_1}{dr}=\frac{dp_2}{dr}=\frac{-9+9\pi\sqrt{3}-\pi^2+\left(36-2\pi\sqrt{3}\right)R}{90-3\pi\sqrt{3}}$.

Part (a) considers the direct demand effect. If $R > 0.25$ *Firm* 3 would like to position its store closer to the origin as far as this effect is concerned. This way it captures the opponents' demand around the center of the disk. For $R < 0.25$ the effect reverses, *Firm* 3 loses demand to competitors and would like to move away from the origin.[7]

Part (b) quantifies the positive effect competitors' prices have on the demand captured by *Firm* 3.

The most demanding task is to evaluate the effect *Firm* 3's radial deviation has on opponents' prices (part (c)). In order to simplify a very complex exercise we exploit the symmetry of the problem extensively. It is clear that replies in prices of both competitors must be identical when *Firm* 3 moves along the vertical axis. We therefore use only necessary conditions for profit maximization with respect to own prices for *Firms* 2 and 3 (A.9-A.10) in the proof of part (c). It can be readily verified that dp_1/dr, dp_2/dr, and dp_3 are positive for any $R\in[0,1]$, that is, when *Firm* 3 moves towards the origin, prices decrease as competition toughens, and *vice versa*.

These results, when compared to those for a duopoly (Lemma 2), offer some idea for what follows. Suppose all three firms were located on the perimeter of the disk, and *Firm* 3 contemplated a small radial move towards the origin. The line of marginal consumers affected by this move is of length two (*BO* and *CO*; see Figure 3). The same is true for the duopoly case. Hence, direct demand effects should be very similar in both cases. They are –1 in the duopoly and –0.87 with three firms.[8] Furthermore, we derive the elasticity

[7] Point *D* moves along the vertical axes towards the top of the disk, so *Firm* 3 gains some new customers from the oponents (see Figure 3). At the same time line segments *DC* and *DB* rotate toward each other, which means that *Firm* 3 loses some customers on the outskirts of the market. The total effect is negative for $R < 0.25$.

[8] From part (a) in Lemmas 2 and 6.

of the firm's demand with respect to its radial distance from the origin when all the firms are on the perimeter of the disk. The results are −0.64 and −0.83 for the duopoly and three-firm oligopoly, respectively. At current demands, a firm in a three-firm case gains relatively more than in the duopoly when it moves its store towards the origin ($\Delta R < 0$). We also derive the elasticity of opponents' prices with respect to the firm's radial distance from the origin.[9] It is 0.71 in the duopoly and 0.41 in the three-firm oligopoly. Rivals' price cut response to a firm moving towards the origin in the first stage of the game will be weaker in the three-firm case than in the duopoly. This is because a price cut by one of the two firms that have not moved would not only affect the aggressive rival, but the other neighbor as well. This would provoke a response from a peaceful rival and would lead to lower profits made on consumers not affected by the aggressive firm. We have illustrated the incentive a firm in a three-firm oligopoly has when $R = 1$, when compared to the duopoly. It will gain relatively more demand directly and will be punished by relatively less severe a price cut by rivals in the second stage. If a firm residing at $R = 1$ in the duopoly had an incentive to move even farther away from the origin we expect this not to be the case with three firms anymore. Lemma 7 presents an interior radial distance from the origin for the three firms positioned equidistantly from each other, which does not make them want to relocate in radial direction.

Lemma 7: When $R^* = \dfrac{72 + 15\pi\sqrt{3} - 2\pi^2}{288 - 8\pi\sqrt{3}}$ (≈ 0.5476) and firms are located equidistantly, no firm finds it profitable to locally deviate from it.

Proof of Lemma 7 yields another result. Symmetric configuration with maximum distance between three firms, i.e. maximum differentiation, is never optimal.

Corollary 1: The three firms located on the perimeter of the disk, equidistantly from each other, is not a subgame perfect Nash equilibrium of the game.

Lemma 5 shows that locating at the perimeter of the disk will still yield the highest possible prices and profits in a symmetric configuration, but these are not sustainable since capturing the opponents' demand around the origin is too tempting; a classic prisoner dilemma on an oligopoly market.

Next, we show that, when firms are positioned equidistantly, none of them has an incentive to locally move along a polar direction.

Lemma 8: For any R, $\phi = \pi$ is locally optimal for Firm 3.

Results from Lemmas 5, 7, and 8 are summarized in Proposition 2.

[9] From part (c) in Lemmas 2 and 6.

Proposition 2: When in the first stage of the game the three firms locate their stores $R^* = \frac{72 + 15\pi\sqrt{3} - 2\pi^2}{288 - 8\pi\sqrt{3}}$ (≈ 0.5476) away from origin, equidistantly from each other, the equilibrium price they charge in the second stage is $\frac{\pi R^* \sqrt{3}}{3}$. Furthermore, every firm's location is a local best response to rivals' locations.

We lack analytical proof that every firm's location is in fact a global best reply to rivals' locations given the price competition in the second stage of the game. We therefore solve the system of first-order conditions for second-stage profit maximization with respect to prices (A.8-A.10) for an array of *Firm* 3's locations numerically to derive optimal profits. We find (see Figure 4) that locations we propose are globally optimal, and state our next result.

Result 2: Three firms positioning their stores at $R^* = \frac{72 + 15\pi\sqrt{3} - 2\pi^2}{288 - 8\pi\sqrt{3}}$ (≈ 0.5476), equidistantly from each other in the first stage of the game, and setting $p^* = \frac{\pi R^* \sqrt{3}}{3}$ in the second, is a subgame perfect Nash equilibrium of the game.

There are two novel perspectives on product differentiation to this result. First, in a setting that leads to differentiation in one characteristic only in duopoly, which is consistent with the existing literature, the three firms find it optimal to differentiate their products in two characteristics. Second, firms do not differentiate their products as much as they could have. In a setting that yields familiar a *min-max* differentiation result in the duopoly, a *medium-medium* type of product differentiation is observed. When all three firms are located on the perimeter of the disk, a radial deviation towards the center of the disk by a firm is followed by rivals' price cut that is less severe than the one observed in duopoly. Therefore, positive direct demand effect of such a move outweighs the negative strategic effect of rivals' price cuts, and firms move closer together.

We can speculate on whether the *min-min-…-min* part of the product differentiation result by Irmen and Thisse (1998) could be observed in a three-firm market with additional product characteristics. Firms may not want to differentiate their products in any additional characteristics, since they find *max-max* differentiation to be excessive in two dimensions, already. This suggests that firms may have no need to differentiate their products in another, third, dimension, since even the possibilities in two were not exhausted. We therefore predict that the *min-min-…-min* part of a product differentiation result is going to hold in our setting as well, which is what Feldin (2001) formally shows. This hypothesis rests on the market being uniformly populated by consumers. If there were some areas with higher population density there would be obviously more differentiation, since firms would try to tailor their products to meet the tastes of these different groups of customers.

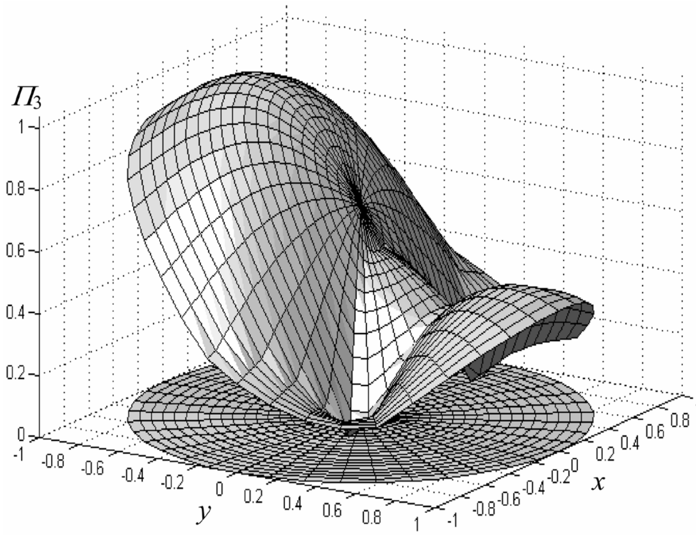

Figure 4: *Firm 3's profits with respect to location of its store and given equilibrium locations of opponents.*
Source: Own calculations.

4. WELFARE ANALYSIS

In this section we compare the extent of product differentiation in our competitive market to a social optimum for a duopoly and a three-firm oligopoly. A social planner who cares about well-being of all agents in the market would simply minimize the traveling costs borne by buyers. Each of n firms on the market will satisfy π/n of the market demand in a symmetric configuration. Every such piece of the pie can be split into two halves symmetrically over the line segment connecting the firm's location and the center of the disk. One such part of the disk is presented in Figure 5. To find the social optimum, we look at where a particular firm should have been in order to minimize the total traveling costs that consumers, from such a half of the disk, bear.

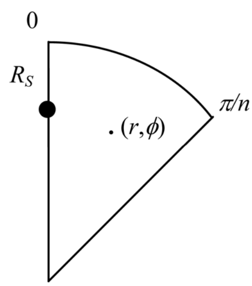

Figure 5: *Derivation of socially optimal product differentiation.*

The cost that a consumer residing at (ρ,ϕ) is faced with when buying from the firm located at R is:

$$c(\rho,\phi;R)=\rho^2\sin^2\phi+(R-\rho\cos\phi)^2=R^2-2\rho R\cos\phi+\rho^2.$$

The total cost borne by buyers from this portion of the disk is then:

$$C(R)=\int_0^{\frac{\pi}{n}}\int_0^1 c(\rho,\phi,R)\cdot\rho\,d\rho\,d\phi.$$

This integrates to: $C(R)=\dfrac{2R^2+1}{4}\cdot\dfrac{\pi}{n}-\dfrac{2}{3}R\sin\dfrac{\pi}{n}$.

Minimize this expression with respect to R to get the socially optimal distance from the center of the disk for the firms, R_S:

$$R_S=\frac{2n\sin\frac{\pi}{n}}{3\pi}.$$

We compare socially optimal locations with competitive equilibrium ones derived in Sections 3 and 4 in Table 1.

Table 1: *Social optimum vs. competitive equilibrium.*

n	R_S	R^*
2	*0.4244*	*1*
3	*0.5513*	*0.5476*

Source: Own calculations

This is analogous to what Brenner (2005) finds. Intense price competition in a duopoly drives firms to the edge of the market, and that is too far apart from a social standpoint. In a three-firm oligopoly the situation reverses. Product differentiation becomes too weak when compared to the social optimum. Price competition becomes less intense and firms move in on each other's demands. This gives rise to an interesting situation. It is in the social planner's interest to induce cooperation among firms. She might pass a regulation demanding the firms locate R_S away from the origin. This would help the firms to partly resolve their prisoner's dilemma. They would now be able to charge higher prices; therefore, they would be willing to accept such a regulation.

5. CONCLUDING REMARKS

We have presented some novel findings on the extent of product differentiation that firms in an oligopoly employ. Contrary to the literature on duopoly markets, we show that three firms will use two product characteristics to separate their products from rivals'. In a setting that usually yields *min-max* differentiation we find three firms to utilize a *medium-medium* type of differentiation. It remains to be seen how even a larger number of firms affects product differentiation in multi-characteristic space.

The demand side of the model is the sole driver of our results. If firms had to cover some R&D costs to introduce new varieties of products or improve some of the characteristics, which certainly is the case in reality, differentiation would be even weaker. We therefore note that firms deciding to introduce new characteristics to their products might want to be careful as not to engage in excessive differentiation that would not yield the maximum possible profits. Also, the fact that differentiation possibilities are not exhausted leads us to think that three firms will not want to use any new product characteristics to differentiate themselves from competitors. The first step towards such a *min-…-min-medium-medium* differentiation result is presented in Feldin (2001).

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APPENDIX A: PROOFS OF LEMMAS AND PROPOSITIONS

Proof of Lemma 1: We first determine α and β that we need for the demand functions. They are given by the indifference conditions for the two consumers residing in points A and B: $p_1 + \sin^2 x + (R - \cos x)^2 = p_2 + (r\sin\phi - \sin x)^2 + (r\cos\phi - \cos x)^2$, where $x = \alpha, \beta$.

This equation simplifies to (A.1) and by using the fact that in symmetric equilibrium (if exists) $r = R$ and $\phi = \pi$, to (A.2):

$$2r\sin\phi \cdot \sin x + 2(r\cos\phi - R)\cdot\cos x = p_2 - p_1 + r^2 - R^2. \tag{A.1}$$

$$-4R\cdot\cos x = p_2 - p_1 \tag{A.2}$$

We state the necessary conditions for profit maximization with respect to own prices for *Firm* 1 and *Firm* 2 (from (2)):

$$\frac{1}{2}(z - \sin z) + \frac{p_1}{2}\cdot\left(z_{p_1} - \cos z\cdot z_{p_1}\right) = 0 \quad \text{and}$$

$$\frac{1}{2}(2\pi - z + \sin z) + \frac{p_2}{2}\cdot\left(-z_{p_2} + \cos z\cdot z_{p_2}\right) = 0. \tag{A.3}$$

Firms' behavior in prices will be symmetric, hence, it is enough to use only first equation in (A.3), the necessary condition for *Firm* 1. Since in equilibrium $z = \pi$, we get:

$$\pi + 2p_1 z_{p_1} = 0.$$

All derivatives we need in this section are collected in Appendix B. Using (B.1) completes the proof. *Q.E.D.*

Proof of Lemma 2: Since firms will charge symmetric prices: $\alpha = \pi/2$, $\beta = -\pi/2$, and $z = \pi$.

(a) In equilibrium line AB crosses the origin, therefore, any firm's infinitesimal radial move towards origin, $-dr$, yields new customers in the area $2\times dr/2$, hence, raising demand by dr.

(b) Using (1) and derivative (B.1) the partial derivative in question is:

$$\frac{\partial D_2}{\partial p_1} = \frac{1}{2}(-z_{p_1} + \cos z\cdot z_{p_1}) = -z_{p_1} = \frac{1}{2R}.$$

(c) Denote the total derivative of any variable x with respect to r by x' for the length of this appendix. The derivative in question is obtained from the system of first-order conditions (A.3) we used to derive equilibrium prices. Since this system must hold for any pair of locations, the total differentiation with respect to r yields p_1'. The total derivatives are (Note: $\sin z = 0$ and $\cos z = -1$ in equilibrium):

$$z' + p_1'\cdot z_{p_1} + p\cdot z_{p_1}' = 0 \text{ and } z' + p_2'\cdot z_{p_2} + p\cdot z_{p_2}' = 0.$$

The derivatives needed in above system are collected in (B.1)-(B.3). It becomes:

$-4p_1' + 2p_2' + 4R + \pi = 0$, and $2p_1' - 4p_2' - 4R + \pi = 0$,

with the unique solution $p_1' = \frac{\pi}{2} + \frac{2}{3}R$ and $p_2' = \frac{\pi}{2} - \frac{2}{3}R$. *Q.E.D.*

Proof of Lemma 3: Use the results from Lemma 2 and put them in parentheses in RHS of (3) to get the sign of the effect a radial move has on *Firm* 2's profits: $-\frac{2}{3} + \frac{\pi}{4R} > 0,\ \forall R \in (0.1]$. *Q.E.D.*

Proof of Lemma 4: The demand effect of any move in a polar direction is zero, since the new line of indifferent consumers (line *AB*) must pass through the origin (firms will charge identical prices). The demands firms face are unaffected by such a move, but, any such move that brings both firms closer together makes them more competitive and lowers the price they charge. The result follows. *Q.E.D.*

Proof of Lemma 5: First we characterize *x*, *y*, α, β, and γ, which define the firms' demands. Coordinates of point *D*, *x* and *y* solve the system:

$$p_1 + (R\sin(-\tfrac{\pi}{3}) - x)^2 + (R\cos(-\tfrac{\pi}{3}) - y)^2 = p_2 + (R\sin\tfrac{\pi}{3} - x)^2 + (R\cos\tfrac{\pi}{3} - y)^2,$$

$$p_3 + (r\sin\phi - x)^2 + (r\cos\phi - y)^2 = p_2 + (R\sin\tfrac{\pi}{3} - x)^2 + (R\cos\tfrac{\pi}{3} - y)^2,$$

We note that sin $\alpha = x$ and we proceed with β. It solves:

$$p_3 + (r\sin\phi - \sin\beta)^2 + (r\cos\phi - \cos\beta)^2 = p_2 + (R\sin\tfrac{\pi}{3} - \sin\beta)^2 + (R\cos\tfrac{\pi}{3} - \cos\beta)^2.$$

And, γ solves:

$$p_1 + (R\sin\text{-}\tfrac{\pi}{3} - \sin\gamma)^2 + (R\cos\text{-}\tfrac{\pi}{3} - \cos\gamma)^2 = p_3 + (r\sin\phi - \sin\gamma)^2 + (r\cos\phi - \cos\gamma)^2.$$

Last four equations simplify and lead to:

$$x = \sin\alpha = \frac{p_2 - p_1}{2R\sqrt{3}}, \tag{A.4}$$

$$y = \frac{R\sqrt{3}(2p_3 - p_1 - p_2 + 2r^2 - 2R^2) - 2r\sin\phi(p_2 - p_1)}{2R\sqrt{3}(2r\cos\phi - R)}, \tag{A.5}$$

$$(2r\sin\phi - R\sqrt{3})\sin\beta + (2r\cos\phi - R)\cos\beta = p_3 - p_2 + r^2 - R^2, \tag{A.6}$$

$$(2r\sin\phi + R\sqrt{3})\sin\gamma + (2r\cos\phi - R)\cos\gamma = p_3 - p_1 + r^2 - R^2. \tag{A.7}$$

The system of first-order conditions for profit maximization derived from (8), which we mainly need for future reference, is:

$$\begin{aligned}&\alpha-\gamma+x(\cos\alpha-\cos\gamma)-y(\sin\alpha-\sin\gamma)+p_1\big(\alpha_{p_1}-\gamma_{p_1}+x_{p_1}(\cos\alpha-\cos\gamma)\\&-x(\sin\alpha\cdot\alpha_{p_1}-\sin\gamma\cdot\gamma_{p_1})-y_{p_1}(\sin\alpha-\sin\gamma)-y(\cos\alpha\cdot\alpha_{p_1}-\cos\gamma\cdot\gamma_{p_1})\big)=0\text{'}\end{aligned}\qquad\text{(A.8)}$$

$$\begin{aligned}&\beta-\alpha+x(\cos\beta-\cos\alpha)-y(\sin\beta-\sin\alpha)+p_2\big(\beta_{p_2}-\alpha_{p_2}+x_{p_2}(\cos\beta-\cos\alpha)\\&-x(\sin\beta\cdot\beta_{p_2}-\sin\alpha\cdot\alpha_{p_2})-y_{p_2}(\sin\beta-\sin\alpha)-y(\cos\beta\cdot\beta_{p_2}-\cos\alpha\cdot\alpha_{p_2})\big)=0\text{'}\end{aligned}\qquad\text{(A.9)}$$

$$\begin{aligned}&2\pi-\beta+\gamma-x(\cos\beta-\cos\gamma)+y(\sin\beta-\sin\gamma)+p_3\big(-\beta_{p_3}+\gamma_{p_3}-x_{p_3}(\cos\beta-\cos\gamma)\\&+x(\sin\beta\cdot\beta_{p_3}-\sin\gamma\cdot\gamma_{p_3})+y_{p_3}(\sin\beta-\sin\gamma)+y(\cos\beta\cdot\beta_{p_3}-\cos\gamma\cdot\gamma_{p_3})\big)=0\end{aligned}\ .\qquad\text{(A.10)}$$

We use the assumed symmetry of equilibrium configuration to find the optimal prices. We first note that in equilibrium, $x = 0$, $y = 0$, $\alpha = 0$, $\beta = 2\pi/3$, and $\gamma = -2\pi/3$, which immediately simplifies (A.10):

$$\frac{2\pi}{3}+p_3\left(-\beta_{p_3}+\gamma_{p_3}+y_{p_3}\sqrt{3}\right)=0.$$

Next, symmetry yields $\gamma_{p_3}=-\beta_{p_3}$. Remaining two partial derivatives are in (B.8) and (B.14). We get:

$$\frac{2\pi}{3}+p_3\left(-\frac{2}{2R\sqrt{3}}-\frac{1}{3R}\sqrt{3}\right)=0,$$ which gives the result. *Q.E.D.*

Proof of Lemma 6: (a) Partially differentiate (7) with respect to r to get:[10]

$$D_r=\frac{1}{2}\left(-\beta_r+\gamma_r-x_r(\cos\beta-\cos\gamma)+y_r(\sin\beta-\sin\gamma)\right).$$

Again, symmetry, and partials (B.4), (B.9), and (B.15) give:

$$D_r=\frac{1}{2}\left(-2\frac{2R-1}{2R\sqrt{3}}-\frac{2}{3}\sqrt{3}\right)=\frac{1}{2}\left(-\frac{4R-2}{2R\sqrt{3}}-\frac{4R}{2R\sqrt{3}}\right)=-\frac{4R-1}{2R\sqrt{3}}.$$

(b) We proceed similarly as in part (a), to get:

$$D_{p_2}=\frac{1}{2}\left(-\beta_{p_2}+\gamma_{p_2}-x_{p_2}(\cos\beta-\cos\gamma)+y_{p_2}(\sin\beta-\sin\gamma)\right).$$

From (A.7) we note that $\gamma_{p_2}=0$ (the indifference line *CD* is unaffected by *Firm* 2's price change). With partials (B.4), (B.7) and (B.14) we obtain:

$$D_{p_2}=\frac{1}{2}\left(\frac{1}{2R\sqrt{3}}+\frac{1}{6R}\sqrt{3}\right)=\frac{1}{2R\sqrt{3}}.$$

[10] We denote partial derivatives with the variable with respect to which we differentiate in subscript and omit the index of the firm, since we always work with *Firm* 3.

(c) The reaction in first two firms' prices to a radial move by *Firm* 3 will be identical; hence, we can disregard *Firm* 1's behavior. Totally differentiate (A.9) and (A.10) with respect to r to get:

$$\begin{aligned}
&\beta' - \alpha' + x'(\cos\beta - \cos\alpha) - y'(\sin\beta - \sin\alpha) \\
&+ p_2'\big(\beta_{p_2} - \alpha_{p_2} + x_{p_2}(\cos\beta - \cos\alpha) - y_{p_2}(\sin\beta - \sin\alpha)\big) + p_2\big(\beta'_{p_2} - \alpha'_{p_2} \\
&+ x'_{p_2}(\cos\beta - \cos\alpha) - x_{p_2}(\sin\beta\cdot\beta' - \sin\alpha\cdot\alpha') - x'(\sin\beta\cdot\beta_{p_2} - \sin\alpha\cdot\alpha_{p_2}) \\
&- y'_{p_2}(\sin\beta - \sin\alpha) - y_{p_2}(\cos\beta\cdot\beta' - \cos\alpha\cdot\alpha') - y'(\cos\beta\cdot\beta_{p_2} - \cos\alpha\cdot\alpha_{p_2})\big) = 0
\end{aligned},$$

$$\begin{aligned}
&-\beta' + \gamma' - x'(\cos\beta - \cos\gamma) + y'(\sin\beta - \sin\gamma) \\
&+ p_3'\big(-\beta_{p_3} + \gamma_{p_3} - x_{p_3}(\cos\beta - \cos\gamma) + y_{p_3}(\sin\beta - \sin\gamma)\big) + p_3\big(-\beta'_{p_3} + \gamma'_{p_3} \\
&- x'_{p_3}(\cos\beta - \cos\gamma) + x_{p_3}(\sin\beta\cdot\beta' - \sin\gamma\cdot\gamma') + x'(\sin\beta\cdot\beta_{p_3} - \sin\gamma\cdot\gamma_{p_3}) \\
&+ y'_{p_3}(\sin\beta - \sin\gamma) + y_{p_3}(\cos\beta\cdot\beta' - \cos\gamma\cdot\gamma') + y'(\cos\beta\cdot\beta_{p_3} - \cos\gamma\cdot\gamma_{p_3})\big) = 0
\end{aligned}.$$

Next, we assume the first equality from part (c) of present Lemma and observe from (B.5) and (A.4) that $x' = 0$ and $\alpha' = 0$. We also use the second part of (B.5), which says that the total derivative with respect to r of partial derivative of x, and, hence by (A.5) also of α, with respect to any of the three prices is 0. Note again that $\alpha = 0$ in equilibrium, use (B.4) and symmetry of the problem to simplify the above system:

$$\begin{aligned}
&2\beta' - \sqrt{3}y' + p_2'\big(2\beta_{p_2} - 2\alpha_{p_2} - 3x_{p_2} - \sqrt{3}y_{p_2}\big) \\
&+ p\big(2\beta'_{p_2} - \sqrt{3}x_{p_2}\beta' - \sqrt{3}y'_{p_2} + y_{p_2}\beta' - y'(-\beta_{p_2} - 2\alpha_{p_2})\big) = 0
\end{aligned}$$

$$-2\beta' + \sqrt{3}y' + p_3'\big(-2\beta_{p_3} + \sqrt{3}y_{p_3}\big) + p\big(-2\beta'_{p_3} + \sqrt{3}y'_{p_3} - y_{p_3}\beta' - y'\beta_{p_3}\big) = 0\cdot$$

We next note that by (A.5) $\alpha_{p_2} = x_{p_2}/\cos\alpha = x_{p_2}$ in equilibrium. The remaining derivatives are collected in (B.7), (B.8), (B.11), (B.12), (B.13), (B.14), (B.17), and (B.18). Applying all these we get the final form of the system:

$$-54p_2' + 18p_3' + 36R - 9 + \pi\sqrt{3}\left(p_2' - p_3' - 2R + 3\right) = 0,$$

$$18p_2' - 36p_3' - 36R + 9 + \pi\sqrt{3}\left(-p_2' + p_3' + 2R + 3\right) = 0,$$

with a solution:

$$p_2' = \frac{-9 + 9\pi\sqrt{3} - \pi^2 + \left(36 - 2\pi\sqrt{3}\right)R}{90 - 3\pi\sqrt{3}}, \text{ and}$$

$$p_3' = \frac{18 + 12\pi\sqrt{3} - \pi^2 - \left(72 - 4\pi\sqrt{3}\right)R}{90 - 3\pi\sqrt{3}}.$$

Q.E.D.

Proof of Lemma 7: Collect results from Lemma 6 to calculate the expression in parentheses of (9) and equate it with zero:

$$-\frac{4R^*-1}{2R^*\sqrt{3}}+\frac{2}{2R^*\sqrt{3}}\cdot\frac{-9+9\pi\sqrt{3}-\pi^2+\left(36-2\pi\sqrt{3}\right)R^*}{90-3\pi\sqrt{3}}=0.$$

Simple algebra yields the result.

However, to establish that we have really found a firm's local best response to the rival locations we must check the second order condition also. The second order derivative of *Firm* 3's profit with respect to r is derived from (9):

$$\frac{d^2\Pi_3}{dr^2}=\frac{d}{dr}\left(\frac{\partial\Pi_3}{\partial p_3}\cdot\frac{dp_3}{dr}\right)+\frac{dp_3}{dr}\cdot\left(\frac{\partial D_3}{\partial r}+\frac{\partial D_3}{\partial p_1}\cdot\frac{dp_1}{dr}+\frac{\partial D_3}{\partial p_2}\cdot\frac{dp_2}{dr}\right)$$
$$+p_3\cdot\left(\left(\frac{\partial D_3}{\partial r}\right)'+\left(\frac{\partial D_3}{\partial p_1}\right)'\cdot\frac{dp_1}{dr}+\left(\frac{\partial D_3}{\partial p_2}\right)'\cdot\frac{dp_2}{dr}+\frac{\partial D_3}{\partial p_1}\cdot\frac{d^2p_1}{dr^2}+\frac{\partial D_3}{\partial p_2}\cdot\frac{d^2p_2}{dr^2}\right) \quad \text{(A.11)}$$

At $r = R^*$ both terms in the first row of (A.11) are zero. First one due to optimizing behavior in the second stage of the game, and the second one due to the expression in parentheses being zero at $r = R^*$ (F.O.C.). It remains to evaluate the sign of the big expression in parentheses in the second row of (A.11). We prove a series of claims.

Claim 1: $\left(\frac{\partial D_3}{\partial r}\right)'<0$.

We go back to the proof of Lemma 6(a) to obtain the relevant part of the partial derivative of interest. Using the symmetry of the problem and the fact that $x_r\equiv 0$ we get: $D_r=-\beta_r+y_r\sin\beta$. Hence,

$$\left(D_r\right)'=-\beta_r'+\frac{\sqrt{3}}{2}y_r'-\frac{1}{2}y_r\cdot\beta'.$$

The total derivatives are in (B.9), (B.10), (B.16) and (B.18). Putting them all together yields:

$$\left(D_r\right)'=-\frac{2R+1}{4R\sqrt{3}}-\frac{\sqrt{3}}{2}\frac{2}{9R}+\frac{1}{2}\frac{2}{3}\frac{p_3'-p_2'+2R-1}{2R\sqrt{3}}.$$

Evaluating this at $R = R^*$ and using result from Lemma 6(c) yields $\left(D_r\right)'\approx-1.3167$.

Claim 2: $\left(\frac{\partial D_3}{\partial p_2}\right)'<0$ and $\left(\frac{\partial D_3}{\partial p_1}\right)'<0$.

We take the partial derivative from Lemma 6(b) and proceed similarly as in previous claim: $D_{p_2} = -\beta_{p_2} + y_{p_2} \sin\beta$. So,

$$\left(D_{p_2}\right)' = -\beta'_{p_2} + \frac{\sqrt{3}}{2} y'_{p_2} - \frac{1}{2} y_{p_2} \cdot \beta'.$$

Derivatives needed are in (B.7), (B.11), (B.17), and (B.18). We get:

$$\left(D_{p_2}\right)' = -\frac{1}{4R^2\sqrt{3}} - \frac{\sqrt{3}}{2}\frac{1}{9R^2} - \frac{1}{2}\frac{1}{6R} \cdot \frac{p'_3 - p'_2 + 2R - 1}{2R\sqrt{3}}.$$

Evaluating this at $R = R^*$ and using result from Lemma 6(c) yields $\left(D_{p_2}\right)' \approx -0.8121$. Due to the symmetry it must also be $\left(D_{p_1}\right)' \approx -0.8121$

Claim 3: $\frac{d^2 p_2}{dr^2} < 0$ and $\frac{d^2 p_1}{dr^2} < 0$.

To prove this claim analytically we would have to solve the system (A.8)-(A.10) without asserting $r = R$ to get p'_1, p'_2 and p'_3 as functions of r. The dependence of relevant derivatives on r from Appendix B shows that such analytical solution is out of reach. We proceed as follows. We solve the system of the first-order conditions (A.8)-(A.10) for *Firms* 1 and 2 located R^* away from the origin, while varying *Firm* 3's location, r, around R^*, numerically. The results are:

$$p_1(r = R^* - 0.005) = p_2(r = R^* - 0.005) = 0.991583\,,$$

$$p_1(r = R^*) = p_2(r = R^*) = 0.993237\,, \text{ and}$$

$$p_1(r = R^* + 0.005) = p_2(r = R^* + 0.005) = 0.994882\,.$$

We are interested in two differences representing the first-order derivatives of p_2 with respect to r at R^*–0.0025 and R^*+0.0025:

$$p_2(r = R^*) - p_2(r = R^* - 0.005) = 1.654 \times 10^{-3}$$

$$p_2(r = R^* + 0.005) - p_2(r = R^*) = 1.645 \times 10^{-3}\,.$$

We see that in vicinity of R^* the first-order derivative of rivals' prices with respect to r is decreasing in r, hence, the second order derivative must be negative. This completes the proof of Claim 3.

One can verify that p'_1 and p'_2 presented in Lemma 6 (c) are positive when $r=R^*$, while Lemma 6(b) says that $\partial D_3/\partial p_1$ and $\partial D_3/\partial p_2$ are also positive. Using the three claims above, we see that every single summand in the big parentheses in the second row of (A.11) is negative. Hence, $\frac{d^2\Pi_3}{dr^2} < 0$ at $r = R^*$ at $r = R^*$. We have found a local maximum of *Firm* 3's profits with respect to r. *Q.E.D.*

Proof of Lemma 8: We are going to show that the expression in the parentheses in (10) is zero. First, it is obvious that $\partial D_3/\partial\phi = 0$, since the same number of customers won by a move along polar direction from one neighbor is lost to the other one.

Second, we claim $dp_1/d\phi = -dp_2/d\phi$. Clearly, the neighbor who becomes closer by such a move becomes more aggressive and reduces its price and *vice versa*, the neighbor that is now farther away becomes less aggressive and raises its price. The magnitude of the two derivatives is the same due to symmetry of the proposed configuration. The result follows. *Q.E.D.*

APPENDIX B: DERIVATIVES

We collect all the derivatives needed in preceding Appendix in subsections B.1 and B.2 for duopoly and three-firm oligopoly markets, respectively.

B.1 *Two firms*

In equilibrium $r = R$, $\phi = \pi$, and $\alpha = \pi/2$.

From (A.2): $\alpha_{p_1} = -\alpha_{p_2} = -\dfrac{1}{2(r+R)\cdot\sin\alpha} = -\dfrac{1}{4R}$ and $z_{p_1} = -z_{p_2} = -\dfrac{1}{2R}$ (B.1)

From (B.1): $\alpha'_{p_1} = -\alpha'_{p_2} = \dfrac{,2\sin\alpha}{4(r+R)^2\cdot\sin^2\alpha} = \dfrac{1}{2(r+R)^2} = \dfrac{1}{8R^2}$ and $z'_{p_1} = -z'_{p_2} = \dfrac{1}{4R^2}$ (B.2)

From (A.1): $\alpha' = \dfrac{p_2' - p_1' + 2r}{2(r+R)\cdot\sin\alpha} = \dfrac{p_2' - p_1' + 2R}{4R}$ and $z' = \dfrac{p_2' - p_1' + 2R}{2R}$ (B.3)

B.2 *Three firms*

From (A.4): $x_{p_1} = -x_{p_2} = -\dfrac{1}{2R\sqrt{3}}, \quad x_{p_3} = 0, \quad x_r = 0,$ (B.4)

From (A.4) and (B.4): $x' = \dfrac{p_2' - p_1'}{2R\sqrt{3}}, \quad x'_{p_1} = x'_{p_2} = x'_{p_3} = 0,$ (B.5)

From (A.5): $y_{p_1} = \dfrac{-R\sqrt{3} + 2r\sin\phi}{2R\sqrt{3}(2r\cos\phi - R)} = \dfrac{1}{2(2r+R)} = \dfrac{1}{6R},$ (B.6)

From (A.5): $y_{p_2} = \dfrac{-R\sqrt{3} - 2r\sin\phi}{2R\sqrt{3}(2r\cos\phi - R)} = \dfrac{1}{2(2r+R)} = \dfrac{1}{6R},$ (B.7)

From (A.5): $y_{p_3} = \dfrac{2R\sqrt{3}}{2R\sqrt{3}(2r\cos\phi - R)} = -\dfrac{1}{2r+R} = -\dfrac{1}{3R},$ (B.8)

From (A.5): $y_r = \dfrac{R\sqrt{3}\cdot 4r}{2R\sqrt{3}(2r\cos\phi - R)} = -\dfrac{2r}{2r+R} = -\dfrac{2}{3},$ (B.9)

From (B.9): $y'_r = -\dfrac{2(2r+R) - 4r}{(2r+R)^2} = -\dfrac{2}{9R},$ (B.10)

From (B.6) and (B.7): $y'_{p_1} = y'_{p_2} = \dfrac{R\sqrt{3}\cdot 2\cos\phi}{2R\sqrt{3}(2r\cos\phi - R)^2} = \dfrac{-1}{(2r+R)^2} = -\dfrac{1}{9R^2},$ (B.11)

From (B.8): $y'_{p_3} = -\dfrac{2\cos\phi}{(2r\cos\phi - R)^2} = \dfrac{2}{(2r+R)^2} = \dfrac{2}{9R^2},$ (B.12)

From (A.5): $y' = \dfrac{R\sqrt{3}(2p_3' - p_1' - p_2' + 4r)}{2R\sqrt{3}(2r\cos\phi - R)} = \dfrac{2p_3' - p_1' - p_2' + 4r}{2(2r\cos\phi - R)} = -\dfrac{2p_3' - p_1' - p_2' + 4R}{6R}$, (B.13)

From (A.6):
$$\begin{aligned}\beta_{p_2} &= -\beta_{p_3} = -\frac{1}{(2r\sin\phi - R\sqrt{3})\cos\beta - (2r\cos\phi - R)\sin\beta} \\ &= -\frac{1}{R\frac{\sqrt{3}}{2} + (2r+R)\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}(r+R)} = -\frac{1}{2R\sqrt{3}}\end{aligned}, \qquad \text{(B.14)}$$

From (A.6): $\beta_r = \dfrac{2r - 2\cos\phi\cos\beta}{(2r\sin\phi - R\sqrt{3})\cos\beta - (2r\cos\phi - R)\sin\beta} = \dfrac{2r-1}{R\frac{\sqrt{3}}{2} + (2r+R)\frac{\sqrt{3}}{2}} = \dfrac{2R-1}{2R\sqrt{3}}$,
(B.15)

From (B.15): $\beta_r' = \left(\dfrac{2r-1}{\sqrt{3}(r+R)}\right)' = \dfrac{2\sqrt{3}\cdot 2R - (2R-1)\sqrt{3}}{12R^2} = \dfrac{2R+1}{4R\sqrt{3}}$, (B.16)

From (B.14):
$$\begin{aligned}\beta_{p_2}' &= \beta_{p_3}' = \frac{2\sin\phi\cos\beta + R\sqrt{3}\sin\beta\cdot\beta' - 2\cos\phi\sin\beta - (2r\cos\phi - R)\cos\beta\cdot\beta'}{\left((2r\sin\phi - R\sqrt{3})\cos\beta - (2r\cos\phi - R)\sin\beta\right)^2}, \\ &= \frac{\sqrt{3}}{3(r+R)^2} = \frac{1}{4R^2\sqrt{3}}\end{aligned}$$
(B.17)

From (A.6):

$$\beta' = \frac{p_3' - p_2' + 2r - 2\cos\phi\cos\beta}{(2r\sin\phi - R\sqrt{3})\cos\beta - (2r\cos\phi - R)\sin\beta} = \frac{p_3' - p_2' + 2r - 1}{\sqrt{3}(r+R)} = \frac{p_3' - p_2' + 2R - 1}{2R\sqrt{3}}.$$
(B.18)