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Locally s-arc-transitive graphs arising from product action*

Michael Giudici
Department of Mathematics and Statistics, The University of Western Australia, Perth WA 6009, Australia

Eric Swartz †
Department of Mathematics, William & Mary, P.O. Box 8795, Williamsburg, VA 23187-8795, USA

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Abstract

We study locally s-arc-transitive graphs arising from the quasiprimitive product action (PA). We prove that, for any locally $(G, 2)$-arc-transitive graph with G acting quasiprimitively with type PA on both G-orbits of vertices, the group G does not act primitively on either orbit. Moreover, we construct the first examples of locally s-arc-transitive graphs of PA type that are not standard double covers of s-arc-transitive graphs of PA type, answering the existence question for these graphs.

Keywords: Locally s-arc-transitive graph, quasiprimitive group, product action.

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1 Introduction

For an integer $s \geqslant 1$, an s-arc in a graph Γ is an $(s+1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices such that $\alpha_i \sim \alpha_{i+1}$ and $\alpha_i \neq \alpha_{i+2}$ for each i. We say that Γ is *s-arc-transitive* if Γ contains an s-arc and the automorphism group of Γ acts transitively on the set of all s-arcs.

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†Corresponding author.
E-mail addresses: michael.giudici@uwa.edu.au (Michael Giudici), easwartz@wm.edu (Eric Swartz)

If Γ is s-arc-transitive and each $(s-1)$-arc can be extended to an s-arc then any s-arc-transitive graph is also $(s-1)$-arc-transitive. The study of s-arc-transitive graphs goes back to the pioneering work of Tutte [32, 33], who showed that if Γ has valency three then $s \leqslant 5$. Weiss [35] later showed that if the valency restriction is relaxed to allow valency at least three then $s \leqslant 7$, with equality holding for the generalised hexagons arising from the groups $G_2(q)$ for $q = 3^f$.

Praeger [25] initiated a programme for the study of finite connected s-arc-transitive graphs by first showing that if $G \leqslant \mathrm{Aut}(\Gamma)$ acts transitively on the set of all s-arcs of Γ and $N \lhd G$ has at least three orbits on the set of vertices, then the quotient graph Γ_N whose vertices are the orbits of N is also s-arc-transitive. Moreover, Γ is a cover of Γ_N. This reduces the study of finite connected (G, s)-arc-transitive graphs to two basic types:

- those where G is *quasiprimitive* on the set of vertices, that is, where all nontrivial normal subgroups of G are transitive on vertices;
- those where G is *biquasiprimitive* on the set of vertices, that is, where all nontrivial normal subgroups of G have at most two orbits on vertices and there is a normal subgroup with two orbits.

Praeger showed that of the eight types of finite quasiprimitive groups, only four — HA (affine), TW (twisted wreath), AS (almost simple) and PA (product action) — can act 2-arc-transitively on a graph [25]. We use the types of quasiprimitive groups as given in [27] and define type PA, the main focus of this paper, in Section 2. These are slight variations on the types of primitive permutation groups given by the O'Nan–Scott Theorem. All graphs of type HA were classified by Praeger and Ivanov [18] while those of type TW were studied by Baddeley [1]. The 2-arc-transitive graphs for some families of almost simple groups have all been classified, for example the Suzuki groups [9], Ree groups [8] and $\mathrm{PSL}(2, q)$ [16]. The first examples of 2-arc-transitive graphs of PA type were given by Li and Seress [22] and studied further by Li, Seress, and Song [23]. Another family of quasiprimitive 2-arc-transitive graphs of PA type were constructed by Li, Ling, and Wu in [21].

In the biquasiprimitive case the graph is bipartite and such graphs were investigated in [26, 28]. An alternative way to study such graphs is via the notion of local s-arc-transitivity. We say that a graph Γ is *locally* (G, s)*-arc-transitive* for a group $G \leqslant \mathrm{Aut}(\Gamma)$ if for each vertex α, the vertex stabiliser G_α acts transitively on the set of all s-arcs starting at α. If G also acts transitively on the set of vertices then Γ is s-arc-transitive. If Γ is locally (G, s)-arc-transitive but G is intransitive on the set of vertices, then G has two orbits on vertices and Γ is bipartite. One way to construct locally s-arc-transitive graphs is to start with an s-arc-transitive graph Γ and take its standard double cover Σ, which has vertex set $V\Gamma \times \{1, 2\}$ and $(\alpha, i) \sim (\beta, j)$ precisely when $i \neq j$ and $\alpha \sim \beta$ in Γ. Then $\mathrm{Aut}(\Gamma)$ acts as automorphisms on Σ with two orbits on vertices and Σ is locally $(\mathrm{Aut}(\Gamma), s)$-arc-transitive [11].

If Γ is a bipartite graph and $G \leqslant \mathrm{Aut}(\Gamma)$ acts transitively on the set of vertices, then Γ is locally (G^+, s)-arc-transitive where G^+ is the index two subgroup that stabilises each part of the bipartition. Hence the study of locally s-arc-transitive graphs encompasses the study of all bipartite s-arc-transitive graphs and hence the biquasiprimitive case in Praeger's programme. It is also a wider class of graphs as the known generalised octagons are locally 9-arc-transitive but not vertex-transitive, and it has been shown by van Bon and Stellmacher [34] that this is best possible.

A programme for the study of finite connected locally s-arc-transitive graphs was mapped out by Giudici, Li and Praeger [11]. If Γ is locally (G,s)-arc-transitive with G having two orbits on vertices and $N \lhd G$ is intransitive on both G-orbits, then the quotient graph Γ_N is also locally s-arc-transitive. Moreover, Γ is a cover of Γ_N. This reduces the study of finite connected locally (G,s)-arc-transitive graphs for which G is vertex-intransitive into two basic types:

- those where G is quasiprimitive on each of its two orbits on vertices;
- those where G is quasiprimitive on only one of its two orbits on vertices.

In the second case, it was shown [11] that the quasiprimitive action must be of type HA, HS, AS, PA or TW. These were further studied in [12] where all examples where the quasiprimitive action has type PA preserving a product structure or type HS were classified. An infinite family of examples where the quasiprimitive action has type TW was given by Kaja and Morgan [19]. In the first case, either the two quasiprimitive actions have the same quasiprimitive type and are one of HA, AS, TW or PA, or they are different with one of type SD and one of type PA [11]. All 2-arc-transitive graphs of the latter type were classified in [13] and there are locally 5-arc-transitive examples in this case [14]. It was shown in [17, Lemma 3.2] that all locally 2-arc-transitive graphs where the quasiprimitive action is of type HA on both orbits are actually vertex-transitive but a complete classification has not been obtained – see [18, Section 2] for further discussion. All locally $(G,2)$-arc-transitive graphs have been classified in the cases where G is an almost simple group whose socle is a Ree group [7], Suzuki group [31], or $\mathrm{PSL}(2,q)$ [3], while the sporadic group case was studied in [20]. Examples also exist in the PA and TW cases as we can take standard double covers of s-arc-transitive graphs of type PA and TW respectively.

The aim of this paper is to study locally s-arc-transitive graphs of PA type. We prove that, for any locally $(G,2)$-arc-transitive graph with G acting quasiprimitively with type PA on both G-orbits of vertices, the group G does not act primitively on either orbit. Moreover, in the spirit of [22], we solve the existence problem for locally 2-arc-transitive graphs of PA type. In particular, we construct the first examples of locally s-arc-transitive graphs of PA type that are not standard double covers of s-arc-transitive graphs of PA type.

2 PA type

Let G act quasiprimitively on a set Ω. We say that G has *type PA* if there exists a G-invariant partition $\mathcal{B}$ of Ω such that G acts faithfully on $\mathcal{B}$ and we can identify $\mathcal{B}$ with Δ^k for some set Δ and $k \geqslant 2$ such that $G \leqslant H \,\mathrm{wr}\, S_k$ acts in the usual product action of a wreath product on Δ^k, where $H \leqslant \mathrm{Sym}(\Delta)$ is an almost simple group acting quasiprimitively on Δ. Moreover, if $T = \mathrm{soc}(H)$ then G has a unique minimal normal subgroup $N = T^k$. Note that since G is quasiprimitive, N acts transitively on Ω and hence on $\mathcal{B}$. Thus $G = NG_\alpha = NG_B$, where $B \in \mathcal{B}$ is a block containing $\alpha \in \Omega$. As N is minimal normal in G we have that G transitively permutes the simple direct factors of N and hence so do both G_α and G_B. Thus given $B = (\delta,\ldots,\delta) \in \mathcal{B}$ we may assume that $N_B = T_\delta^k$ and for $\alpha \in B$ we have that N_α is a subdirect subgroup of N_B, that is, the projection of N_α onto each direct factor is isomorphic to T_δ.

Let $R = T_\delta$. Following the terminology of [22], if $N_\alpha \cong R$ then we call N_α a *diagonal subgroup* of $N_B = R^k$. Then there exists automorphisms $\varphi_2,\varphi_3,\ldots,\varphi_k$ of R such that

$$N_\alpha = \{(t, t^{\varphi_2},\ldots,t^{\varphi_k}) \mid t \in R\}.$$

If each of the φ_i is the trivial automorphism then we call N_α a *straight diagonal subgroup* while if some φ_i is nontrivial then we call N_α a *twisted diagonal subgroup*. Furthermore, if $N_\alpha \ncong R$ then we refer to N_α as being a *nondiagonal subgroup.* We refer to the quasiprimitive permutation group G of type PA as being of *straight diagonal*, *twisted diagonal*, or *nondiagonal* type according to the type of N_α.

Note that unlike for primitive groups of type PA, G does not necessarily preserve a product structure on Ω, only on some G-invariant partition $\mathcal{B}$. Indeed the following result shows that for locally 2-arc-transitive graphs this partition must be nontrivial on each of the bipartite halves.

Theorem 2.1. *Let Γ be a locally $(G,2)$-arc-transitive connected graph with G quasiprimitive of type PA on both orbits Ω_1 and Ω_2. Let $N = T^k = \mathrm{soc}(G)$ and for $i = 1, 2$, let $\mathcal{B}_i$ be a G-invariant partition of Ω_i such that G preserves a product structure Δ_i^k on each $\mathcal{B}_i$. Then $\mathcal{B}_i \neq \Omega_i$ for each i.*

Proof. Suppose that $\mathcal{B}_i = \Omega_i$ for some i. Without loss of generality suppose that $i = 1$. Also note that there is an almost simple group H with socle T such that $G \leqslant H \operatorname{wr} S_k$.

Let $\alpha = (\omega, \dots, \omega) \in \Omega_1$. Then $N_\alpha = T_\omega^k$ with $T_\omega \neq 1$ and $G_\alpha = G \cap (H_\omega \operatorname{wr} S_k)$. By [11, Lemma 3.2], $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive so either all neighbours of α lie in the same block of $\mathcal{B}_2$ or in distinct blocks. If they all lie in the same block then for each $\beta \in \Omega_1$ we have that the neighbours of β lie in the same block. However, this contradicts Γ being connected. Hence for each $\alpha \in \Omega_1$, the neighbours of α lie in distinct blocks. Hence G_α acts 2-transitively on the set X of blocks of $\mathcal{B}_2$ that contain neighbours of α. By [11, Lemma 6.2], $N_\alpha^{\Gamma(\alpha)}$ is a transitive subgroup of the 2-transitive group $G_\alpha^{\Gamma(\alpha)}$ and so N_α also acts transitively on X. Let $B = (\delta_1, \delta_2, \dots, \delta_k) \in \mathcal{B}_2$ be a block containing a neighbour γ of α. Then $X = (\delta_1, \delta_2, \dots, \delta_k)^{N_\alpha} = \delta_1^{T_\omega} \times \delta_2^{T_\omega} \times \cdots \times \delta_k^{T_\omega}$. By [29, Theorem 1.1(b)], the stabiliser G_1 in G of the first simple direct factor of N projects onto H in the first coordinate and so $(G_1)_\alpha$ projects onto H_ω in the first coordinate. Hence $\delta_1^{T_\omega} = \delta_1^{H_\omega}$. Since $G_\alpha \leqslant H_\omega \operatorname{wr} S_k$ and transitively permutes the k simple direct factors of N, it follows that $\delta_i^{T_\omega} = \delta_1^{T_\omega}$ for each i. In particular, $X = A^k$ for some set A and we could have chosen $B = (\delta, \dots, \delta)$ for some $\delta \in \Delta_2$. Thus $G_{\alpha\gamma} \leqslant G_{\alpha,B} \leqslant H_{\omega\delta} \operatorname{wr} S_k$. However, for $\delta' \in A \backslash \{\delta\}$ there is no element of $H_{\omega\delta} \operatorname{wr} S_k$ mapping $(\delta', \delta, \dots, \delta)$ to $(\delta', \delta', \delta, \dots, \delta)$, contradicting G_α acting 2-transitively on X. Thus $\mathcal{B}_1 \neq \Omega_1$. □

Corollary 2.2. *Let Γ be a locally $(G,2)$-arc-transitive connected graph with G quasiprimitive of type PA on both orbits. Then G is not primitive on either orbit.*

3 Constructions

Let G be a finite group with subgroups L and R. Let Δ_1 be the set $[G : L]$ of right cosets of L in G and Δ_2 be the set $[G : R]$ of right cosets of R in G. We define the *coset graph* $\Gamma = \mathrm{Cos}(G, L, R)$ to be the bipartite graph with vertex set the disjoint union $\Delta_1 \cup \Delta_2$ such that $\{Lx, Ry\}$ is an edge if and only if $Lx \cap Ry \neq \varnothing$, or equivalently $xy^{-1} \in LR$. Then G acts by right multiplication on both Δ_1 and Δ_2, and induces automorphisms of Γ. Note that the vertices in Δ_1 have valency $|L : L \cap R|$ while the vertices in Δ_2 have valency $|R : L \cap R|$. We say that Γ has *valency* $\{|L : L \cap R|, |R : L \cap R|\}$. Conversely, if Γ is a graph and $G \leqslant \mathrm{Aut}(\Gamma)$ acts transitively on the set of edges of Γ but not on the set of vertices then Γ can be constructed in this way [11, Lemma 3.7]. We refer to the triple $(L, R, L \cap R)$ as the associated *amalgam*.

We collect the following properties of coset graphs. We say that a subgroup H of a group G is *core-free* if $\cap_{g \in G} H^g = 1$.

Lemma 3.1 ([11, Lemma 3.7]). *Let G be a group with proper subgroups L and R, and let $\Gamma = \mathrm{Cos}(G, L, R)$.*

(1) *Γ is connected if and only if $G = \langle L, R \rangle$.*

(2) *G acts faithfully on both $[G : L]$ and $[G : R]$ if and only if both L and R are core free in G.*

(3) *G acts transitively on the set of edges of Γ.*

(4) *Γ is locally $(G, 2)$-arc-transitive if and only if L acts 2-transitively on $[L : L \cap R]$ and R acts 2-transitively on $[R : L \cap R]$.*

We also need the following result, which essentially follows from the definition of a *completion* (and the *universal completion*) of an amalgam (see [15]) and results on covers of graphs (see, e.g., [2, Chapter 19]). The result is truly "folklore": while it seems to be taken for granted in the field, we also cannot find an explicit proof in the literature. We have included a proof here provided by Luke Morgan [24].

Lemma 3.2. *If Γ is a locally s-arc-transitive graph with amalgam $(L, R, L \cap R)$ and $s \geqslant 2$, then any other graph with amalgam $(L, R, L \cap R)$ is locally s-arc-transitive.*

Proof. Let $G := L *_{L \cap R} R$ be the universal completion of $(L, R, L \cap R)$ and let Γ^* denote the universal tree on which G acts edge-transitively. We identify L and R with their images in G, and label an edge $\{\alpha, \beta\}$ so that $G_\alpha = L$, $G_\beta = R$, and $G_{\alpha\beta} = L \cap R$. Since Γ is locally s-arc-transitive for $s \geqslant 2$, it is locally 2-arc-transitive and so the actions of L on the set of right cosets of $L \cap R$ in L, and of R on the set of right cosets of $L \cap R$ in R are 2-transitive [11, Lemma 3.2]. In particular, Γ^* is locally $(G, 2)$-arc-transitive.

Now let Σ be a graph with edge-transitive group of automorphisms H such that the amalgam $(H_\gamma, H_\delta, H_{\gamma\delta})$ is isomorphic to $(L, R, L \cap R)$, where $\{\gamma, \delta\}$ is an edge of Σ. By the universal property of G and of Γ^*, there is a map $\phi \colon G \to H$ such that the following diagrams commute:

$$\begin{CD} L @>>> G \\ @VVV @VVV \\ H_\gamma @>>> H \end{CD} \qquad \begin{CD} R @>>> G \\ @VVV @VVV \\ H_\delta @>>> H \end{CD} \qquad \begin{CD} L \cap R @>>> G \\ @VVV @VVV \\ H_{\gamma\delta} @>>> H \end{CD} .$$

Let N be the kernel of ϕ. Then, $\Sigma = \Gamma^*_N$, the quotient graph, and the kernel of the action of G on Σ is exactly N.

In particular, $\phi(G_\alpha) = H_\gamma$ and $\phi(G_\beta) = H_\delta$. Further, since $\phi(G_{\alpha\beta}) = H_{\gamma\delta}$, we have commutative diagrams of the following groups:

$$\begin{CD} G_\alpha @>>> G_\alpha^{\Gamma^*(\alpha)} \\ @VVV @VVV \\ H_\gamma @>>> H_\gamma^{\Sigma(\gamma)} \end{CD} \qquad \begin{CD} G_\beta @>>> G_\beta^{\Gamma^*(\beta)} \\ @VVV @VVV \\ H_\delta @>>> H_\delta^{\Sigma(\delta)} \end{CD} ,$$

where $G_\alpha^{\Gamma^*(\alpha)}$ denotes the induced action of G_α on $\Gamma^*(\alpha)$, etc.

We now claim that for $\varepsilon = \gamma, \delta$ and $\zeta \in \Gamma^*(\varepsilon)$, we have $\zeta^N \cap \Gamma^*(\varepsilon) = \{\zeta\}$. Indeed, this follows since $|G_\alpha : G_{\alpha\beta}| = |H_\gamma : H_{\gamma\delta}|$ and $|G_\beta : G_{\alpha\beta}| = |H_\delta : H_{\gamma\delta}|$.

Now suppose Γ^* is locally (G, r)-arc-transitive and Σ is locally (H, t)-arc-transitive. By [11, Lemma 5.1(3)], we have $t \geqslant r$.

Assume that $r < t$. We will show that Γ^* would be locally $(G, r+1)$-arc-transitive in this case, contradicting the maximality of r.

Suppose P and P' are $(r+1)$-paths in Γ^* with initial vertex α or β. Since $r \geqslant 1$, without loss of generality we may assume $P = (\alpha, \beta_1, \ldots, \beta_r, \beta_{r+1})$ and $P' = (\alpha, \beta_1, \ldots, \beta_r, \beta'_{r+1})$, where $\beta_1 = \beta$.

Consider the images of P^N and $(P')^N$ in Σ. Note that the images are two $(r+1)$-paths, since the equality $\beta_{i-1}^N = \beta_{i+i}^N$ would contradict our claim above. Hence, there is $h \in H_\gamma$ such that $(P^N)^h = (P')^N$. Since $\phi(G_\alpha) = H_\gamma$, we can take $h = \phi(g)$ for $g \in G_\alpha$, so g fixes α. Now, $(P^N)^h = (P')^N$ implies $(\beta^N)^g = \beta^N$. Thus, g fixes β^N, and, since g fixes α, g fixes the unique vertex in $\Gamma^*(\alpha) \cap \beta^N$, which is β; so, $g \in G_{\alpha\beta}$. Continuing in this way, we see that $g \in G_{\alpha\beta_1\ldots\beta_r}$. Now, $(\beta_{r+1}^N)^h = (\beta'_{r+1})^N$, and so β_{r+1}^g lies in the N-orbit of β'_{r+1}, and at the same time must be adjacent to β_r, since $g \in G_{\beta_r}$. Once more, the claim implies $\beta_{r+1}^g = \beta'_{r+1}$.

We have thus shown that G_α is transitive on $(r+1)$-arcs with initial vertex u. A similar argument establishes that same result for G_β, and hence Γ^* is locally $(G, r+1)$-arc-transitive. This contradicts the maximality of r, and, therefore, $r = t$, as desired. In particular, taking $\Sigma = \Gamma$ we see that $r = s$. Hence Γ^*, and so any graph with amalgam $(L, R, L \cap R)$, is locally s-arc-transitive. $\square$

Lemma 3.1 enables us to construct locally $(G, 2)$-arc-transitive graphs where G has two orbits Δ_1 and Δ_2 on vertices and acts quasiprimitively of type PA on each. Recall the three types straight diagonal, twisted diagonal and nondiagonal of quasiprimitive groups of type PA. Analogously to [22], we refer to a locally $(G, 2)$-arc-transitive graph Γ where G is quasiprimitive of type PA on each orbit by the type of the two PA actions. For example, if G is of straight diagonal type on Δ_1 and twisted diagonal type on Δ_2 then we refer to Γ as being of *straight-twisted type*.

3.1 Straight-twisted type

Construction 3.3. We begin with the following: let $(L, R, L \cap R)$ be an amalgam for a locally s-arc-transitive graph, and suppose further that $L = L_1 \rtimes K$ and $R = R_1 \rtimes K$ such that K acts trivially on R_1. Note that this implies $L \cap R = (L_1 \cap R_1)K$.

Let H be an almost simple group with socle T, and subgroups H_1 and H_2 such that

- $H_1 \cong L_1$, $H_2 \cong R_1$, $H_1 \cap H_2 \cong L_1 \cap R_1$, i.e., $\phi\colon H_1 \to L_1$, $\tau\colon H_2 \to R_1$ are isomorphisms with restrictions each sending $H_1 \cap H_2 \to L_1 \cap R_1$,
- $H = \langle H_1, H_2 \rangle$, and
- not all automorphisms of L_1 in K extend to automorphisms of T.

We will abuse notation slightly and assume $L_1, R_1 \leqslant H$. Let $k = |K|$ and let

$$F = \{f : K \to H\} \cong H^k.$$

For each $\ell \in L_1$ and $r \in R_1$, define $f_\ell, f_r \in F$ such that

$$f_\ell(\kappa) = \ell^\kappa,$$
$$f_r(\kappa) = r$$

for all $\kappa \in K$. Furthermore, we let

$$N_\alpha := \{f_\ell \mid \ell \in L_1\} \cong L_1,$$
$$N_\beta := \{f_r \mid r \in R_1\} \cong R_1.$$

Since K acts trivially on R_1, we have that

$$N_\alpha \cap N_\beta = \{f_r \mid r \in R_1 \cap L_1\} \cong L_1 \cap R_1.$$

Let $N := \langle N_\alpha, N_\beta \rangle$.

Now K acts on F via $f^\sigma(\kappa) = f(\sigma\kappa)$ for each $\sigma, \kappa \in K$. Then for $\ell \in L_1$ we have that

$$(f_\ell)^\sigma(\kappa) = f_\ell(\sigma\kappa) = \ell^{\sigma\kappa} = f_{\ell^\sigma}(\kappa).$$

Hence $(f_\ell)^\sigma = f_{\ell^\sigma}$ and so K normalises N_α. Similarly, $(f_r)^\sigma = f_r$ for all $r \in R_1$ so K normalises N_β and hence also N. Define

$$G_\alpha := N_\alpha \rtimes K,$$
$$G_\beta := N_\beta \rtimes K,$$
$$G := \langle G_\alpha, G_\beta \rangle.$$

Finally, we define $\Gamma := \mathrm{Cos}(G, G_\alpha, G_\beta)$.

Lemma 3.4. *Let* Γ *be a graph yielded by* Construction 3.3. *Then* Γ *is a connected locally* (G, s)*-arc-transitive graph such that* G *acts quasiprimitively with type* PA *on each orbit of vertices. Moreover, the action of* G *on* $[G : G_\beta]$ *is straight diagonal, and the action of* G *on* $[G : G_\alpha]$ *is twisted diagonal, that is,* Γ *is of straight-twisted type.*

Proof. Let $F_T = \{f \in F \mid f(\kappa) \in T \text{ for all } \kappa \in K\} \cong T^k$. For each $\kappa \in K$, let

$$\pi_\kappa : \begin{array}{ccc} F & \to & H \\ f & \mapsto & f(\kappa). \end{array}$$

Since $\langle R_1, L_1 \rangle = H$, we have that $\pi_\kappa(N) = H$ for all $\kappa \in K$ and so by [30, page 328, Lemma], $N \cap F_T$ is a direct product of diagonal subgroups, each isomorphic to T. Since there are elements $\kappa \in K$ that do not extend to an automorphism of T, it follows that $N \cap F_T$ is not itself a diagonal subgroup and so $N \cap F_T \cong T^j$ for some integer $2 \leqslant j \leqslant k$.

Since the action of K on N_α is isomorphic to the action of K on L_1 we see that $G_\alpha \cong L$ and similarly, $G_\beta \cong R$. Moreover, $G_\alpha \cap G_\beta \cong \langle L_1 \cap R_1, K \rangle = L \cap R$. Therefore $\Gamma := \mathrm{Cos}(G, G_\alpha, G_\beta)$ is a connected graph with amalgam $(L, R, L \cap R)$ and is thus a locally s-arc-transitive graph.

Finally, since K transitively permutes the simple direct factors of F_T it also transitively permutes the simple direct factors of $N \cap F_T$. Thus $\mathrm{soc}(G) \cong T^j$ and $G \lesssim H \,\mathrm{wr}\, S_j$ for some integer $j \geqslant 2$. Since $\pi_\kappa(N_\alpha) = L_1$ for all $\kappa \in K$ it follows that N_α is a subdirect subgroup of L_1^j and similarly, N_β is a subdirect subgroup of R_1^j. Therefore, G acts quasiprimitively with type PA on both $[G : G_\alpha]$ and $[G : G_\beta]$, and, by construction, the action of G on $[G : G_\beta]$ is straight diagonal, and the action of G on $[G : G_\alpha]$ is twisted diagonal. □

Example 3.5. This example is based on [22, Example 4.1]. First, $(\mathrm{AGL}(1,5) \times C_2, S_3 \times C_4, C_4 \times C_2)$ is an amalgam admitting a locally 2-arc-transitive connected graph of valency $\{3,5\}$: indeed, a GAP computation shows that in the group S_7 we can take $L = \langle (4,5,6,7), (3,4,5,7,6), (1,2) \rangle \cong \mathrm{AGL}(1,5) \times C_2$ and $R = \langle (1,2), (1,2,3), (4,5,6,7) \rangle \cong S_3 \times C_4$ such that $\langle L, R \rangle = S_7$, and $L \cap R \cong C_4 \times C_2$ [10].

Let $T = \mathrm{PSL}(2,p)$, where p is a prime and $p \equiv \pm 1 \pmod{60}$. Thus we may select $D < T$ such that $D \cong D_{60}$, with $D = \langle h, d \mid h^{30} = d^2 = 1, h^d = h^{-1} \rangle$. First, define $L_1 := \langle h^3 \rangle \cong C_{10} \cong C_5 \times C_2$. Noting that D has a subgroup $B := \langle h^{15}, d \rangle \cong C_2^2$, there exists an element x of T such that $B^x = B$ and $d^x = h^{15}$ [6]. Define $R_1 := \langle (h^{10})^x, d^x \rangle$ to be a subgroup of H^x isomorphic to S_3. Hence $\langle L_1, R_1 \rangle = T$ and $L_1 \cap R_1 = C_2$. Finally, the order four elements of $\mathrm{AGL}(1,5)$ cannot be extended to automorphisms of T since $\mathrm{Aut}(T) = \mathrm{PGL}(2,p)$ has no elements of order four normalising but not centralising a subgroup of order five. Thus we let $K = \langle k \rangle \cong C_4$ and $L = L_1 \rtimes K$. Note, as in [22, Example 4.1], that the action of k^2 on elements of T is the same as conjugation by d. Therefore, by Lemma 3.4, there is a locally 2-arc-transitive graph with amalgam $(\mathrm{AGL}(1,5) \times C_2, S_3 \times C_4, C_4 \times C_2)$ of straight-twisted type.

Theorem 3.6. *There is an infinite family of locally* 5*-arc-transitive graphs with valencies* $\{4,5\}$ *of straight-twisted type.*

Proof. By [20], there is an amalgam admitting a locally 5-arc-transitive connected graph of valency $\{4,5\}$ from the Mathieu group M_{24}, with $L = C_2^4 \rtimes (A_4 \times C_3)$, $R = A_5 \times A_4$, and $L \cap R = A_4 \times A_4$. Note that $L = L_1 \rtimes K$ and $R = R_1 \times K$ where $L_1 = C_2^4 \rtimes C_3$, $R_1 = A_5$ and $K = A_4$.

Let $n \geqslant 2$ be an integer and $T = \mathrm{PSL}(2, 2^{2n})$. Then T contains a subgroup $R_1 \cong A_5 \cong \mathrm{PSL}(2,4)$ (see [6], for instance). Furthermore, T contains a subgroup Y isomorphic to $C_2^{2n} \rtimes C_{2^{2n}-1}$, and $2^{2n} - 1 \equiv 0 \pmod 3$. Let $Y = Y_2 \rtimes Y_1$, where $Y_2 \cong C_2^{2n}$ and $Y_1 = \langle y_1 \rangle \cong C_{2^{2n}-1}$. Thus Y_1 has a cyclic group of order three, which we will denote by $Y_3 = \langle y_1^{(2^{2n}-1)/3} \rangle$, acting semiregularly on the nonidentity elements of Y_2. Moreover, we may choose R_1 such that $Y_0 := R_1 \cap Y \cong A_4$ and $Y_3 \leqslant Y_0$. By [6, Theorem 260], we see that $N_T(Y_0) \leqslant Y$, and, noting that Y_1 acts regularly on the nonidentity elements of Y_2, we see that $N_T(Y_0) = Y_0$. By [6, Theorem 255], for each divisor m of $2n$, all subfield subgroups of T isomorphic to $\mathrm{PSL}(2, 2^m)$ are conjugate. This implies that Y_0 is contained in a unique subfield subgroup T_m isomorphic to $\mathrm{PSL}(2, 2^m)$ for each divisor m of $2n$, m even (if m is odd, then $2^2 - 1 = 3$ does not divide $2^m - 1$). Note also that this implies that the maximal subgroup of T_m isomorphic to $C_2^m \rtimes C_{2^m-1}$ is actually $T_m \cap Y$. We claim that no subfield subgroup T_m containing Y_0, for m a proper even divisor of $2n$, also contains $Y_0^{y_1}$. If some T_m contains $Y_0^{y_1}$, then, since the elements of order two in Y_0 and $Y_0^{y_1}$ commute and $Y_0 \cap Y_0^{y_1} = Y_3$, we have that $\langle Y_0, Y_0^{y_1} \rangle \leqslant T_m \cap Y \cong C_2^m \rtimes C_{2^m-1}$, where $T_m \cap Y_1$ acts regularly on the nonidentity elements of $T_m \cap Y_2$. However, Y_1 acts regularly on the nonidentity elements of Y_2, so y_1 is the unique element of Y_1 mapping, say, $y_2 \in Y_0 \cap Y_2$ to $y_2^{y_1} \in Y_0^{y_1} \cap Y_2$. On the other hand, $y_1 \notin T_m \cap Y_1 = \langle y_1^{(2^{2n}-1)/(2^m-1)} \rangle$, so we have a contradiction.

Let $L_1 := \langle Y_0, Y_0'^{y_1} \rangle$. Then $L_1 \cong 2^4{:}3$ (SmallGroup(48,50) in the GAP [10] small groups library) which is isomorphic to the subgroup L_1 in L, hence the abuse of notation. Moreover, $L_1 \cap R_1 \cong A_4$ and, since L_1 is not contained in any subfield subgroup, we have that $T = \langle L_1, R_1 \rangle$. Since $\mathrm{P\Gamma L}(2, 2^{2n})$ does not contain a subgroup isomorphic to L ([6, Theorem 260] and noting that the outer automorphism group of $\mathrm{PSL}(2, 2^{2n})$ is cyclic), it follows that not all automorphisms of L_1 in L extend to automorphisms of T. Hence by Lemma 3.4, Construction 3.3 yields a locally 5-arc-transitive graph of straight-twisted type. □

3.2 Twisted-twisted type

If G acts quasiprimitively with straight PA type on a set Ω, then there exists $\alpha \in \Omega$ such that $N_\alpha = \{(r, r, \ldots, r) \mid r \in R\}$, where $N = T^k$ is the unique minimal normal subgroup of G. If $g = (t_1, t_2, \ldots, t_k) \in R^k \leqslant N$ then $N_{\alpha^g} = (N_\alpha)^g = \{(r^{t_1}, r^{t_2}, \ldots, r^{t_k}) \mid r \in R\}$, which is a twisted diagonal subgroup if $t_i \notin C_T(R)$ for some i. Thus the examples given in the previous section can also be viewed as being of twisted-twisted type. However, if G acts quasiprimitively of type twisted PA on a set Ω then N_α is a twisted diagonal subgroup of R^k for some R but there may not be a $\beta \in \Omega$ such that N_β is a straight diagonal subgroup. Thus not all twisted-twisted type examples arise in this way. In this section we give an alternative construction.

Construction 3.7. Let $(L, R, L \cap R)$ be an amalgam for a locally s-arc-transitive graph, and suppose further that $L = L_1 \rtimes K$ and $R = R_1 \rtimes K$ such that $K = K_L \times K_R$ where $K_L \leqslant \mathrm{Aut}(L_1)$ such that $K_L \cap \mathrm{Inn}(L_1) = \{1\}$, K_L acts trivially on R_1, $K_R \leqslant \mathrm{Out}(R_1)$ and K_R acts trivially on L_1. Let H be an almost simple group with socle T, and subgroups H_1 and H_2 such that

- $H_1 \cong L_1$, $H_2 \cong R_1$, $H_1 \cap H_2 \cong L_1 \cap R_1$,
- $H = \langle H_1, H_2 \rangle$, and
- not all elements of K extend to automorphisms of T.

We will abuse notation slightly and assume $L_1, R_1 \leqslant H$. Let $k = |K|$ and let $F = \{f \colon K \to H\} \cong H^k$. For each $\ell \in L_1 \cup R_1$, define $f_\ell \in F$ such that $f_\ell(\kappa) = \ell^\kappa$ for all $\kappa \in K$. Furthermore, we let $N_\alpha := \{f_\ell \mid \ell \in L_1\} \cong L_1$ and $N_\beta = \{f_r \mid r \in R_1\} \cong R_1$. Moreover, $N_\alpha \cap N_\beta = \{f_r \mid r \in R_1 \cap L_1\} \cong L_1 \cap R_1$. Let $N := \langle N_\alpha, N_\beta \rangle$.

Now K acts on F via $f^\sigma(\kappa) = f(\sigma\kappa)$ for each $\sigma, \kappa \in K$. As in Construction 3.3, K normalises both N_α and N_β, and hence also N. Define $G_\alpha := N_\alpha \rtimes K$, $G_\beta := N_\beta \rtimes K$ and $G := \langle G_\alpha, G_\beta \rangle$. Let $\Gamma = \mathrm{Cos}(G, G_\alpha, G_\beta)$.

Lemma 3.8. *Let Γ be a graph yielded by* Construction 3.7. *Then Γ is a connected locally (G, s)-arc-transitive graph such that G acts quasiprimitively with type* PA *on each orbit on vertices. Moreover, the action of G on both $[G : G_\alpha]$ and $[G : G_\beta]$ is twisted diagonal, that is, Γ is of twisted-twisted type.*

Proof. The proof is analogous to that of Lemma 3.4. □

Example 3.9. First, $(C_{71}{:}C_{70} \times C_9, C_{19}{:}C_{18} \times C_{35}, C_{630})$ is an amalgam that admits a locally 2-arc-transitive graph; indeed, if $G = A_{89}$,

$$L := \langle (1,2,8,28,14,30,34,3,20,54,36,33,40,41,9,56,26,51,60,18,42,29,39,17,46,58, 47,10,15,70,62,13,32,59,57,31,66,22,24,67,48,27,35,50,45,12,23,11,52,4,64, 7,53,25,16,61,21,44,6,5,68,71,19,55,38,69,65,49,63,43,37), (2,3,4,\dots,71)(72,73,\dots,89)\rangle,$$

and

$$R := \langle (1,72,73,85,74,88,86,78,75,80,89,84,87,77,79,83,76,82,81), (2,3,4,\dots,71)(72,73,\dots,89)\rangle,$$

then, using GAP, we see that $L \cong C_{71}{:}C_{70} \times C_9$, $R \cong C_{19}{:}C_{18} \times C_{35}$, $L \cap R \cong C_{630}$, $\langle L, R\rangle = G$, and by Lemma 3.1, the coset graph $\mathrm{Cos}(G, L, R)$ is a connected locally $(G, 2)$-arc-transitive graph.

Let $T = \mathbb{M}$, the Monster Group. By [4], T contains subgroups $L_1 \cong D_{142}$ and $R_1 \cong D_{38}$, and L_1 and R_1 may be selected such that $L_1 \cap R_1 \cong C_2$ (here, the element of order two is of type 2B). By [36] we see that $\mathbb{M}$ does not have a maximal subgroup of order divisible by 71 and 19. Thus $\langle L_1, R_1\rangle = T$. Let $K = C_{315} = C_{35} \times C_9$, and since T does not contain an element of order 315 [5], not all elements of K lift to an automorphism of T. Therefore, by Lemma 3.8, Construction 3.7 yields a locally 2-arc-transitive graph Γ with amalgam $(C_{71}{:}C_{70} \times C_9, C_{19}{:}C_{18} \times C_{35}, C_{630})$ of twisted-twisted type with valencies $\{71, 19\}$.

3.3 Straight-nondiagonal type

We first include an example of an *equidistant linear code* from [22], which proves useful in later constructions. A *linear (n,k)-code* C over $\mathrm{GF}(q)$ is a k-dimensional subspace of $\mathrm{GF}(q)^n$, a codeword has *weight* w if it has exactly w nonzero coordinates, and a code C is *equidistant* if all nonzero codewords have the same weight.

Example 3.10 ([22, Example 5.1]). Let $V = \mathrm{GF}(3)^4$, and let

$$C = \langle (1,1,1,0), (1,2,0,1)\rangle < V.$$

Then, C is a linear $(4, 2)$-code, and it contains eight nonzero code words:

$$(1,1,1,0), (1,2,0,1), (2,0,1,1), (0,2,1,2), (2,2,2,0), (2,1,0,2), (1,0,2,2), (0,1,2,1),$$

and hence C is equidistant of weight 3.

Let $\tau = (\sigma, 1, \sigma, \sigma)(1, 2, 3, 4) \in \mathrm{GL}(1, 3) \,\mathrm{wr}\, S_4 < \mathrm{GL}(V)$. Then, $\tau^4 = (\sigma, \sigma, \sigma, \sigma)$, $|\tau| = 8$, and τ permutes the eight nonzero words of C in the order given above.

Our next result constructs examples of straight-nondiagonal type.

Theorem 3.11. *For each integer $n \geqslant 3$, there exists a locally 2-arc-transitive graph of straight-nondiagonal type with valencies $\{n, 9\}$.*

Proof. We adapt the construction of [22, Lemma 5.2]. Let $H = S_{n+2}$. Then H contains subgroups $L \cong S_2 \times S_n$ and $R \cong S_3 \times S_{n-1}$ such that $\langle L, R\rangle = H$ and $L \cap R \cong S_2 \times S_{n-1}$

(this is realized by letting L be the stabilizer of $\{1,2\}$ and letting R be the stabilizer of $\{1,2,3\}$).

Based on the equidistant linear code defined in Example 3.10, we define $N_\alpha := \langle(\ell,\ell,\ell,\ell) \mid \ell \in L\rangle$. Moreover, if $R = R_1 \times R_2$, where $R_1 \cong S_3$, $R_2 \cong S_{n-1}$, and $R_1 = \langle h,\sigma | h^3 = \sigma^2 = hh^\sigma = 1\rangle$, we define $N_\beta := \langle(h,h,h,1),(h,h^{-1},1,h),(x,x,x,x)|x \in \langle\sigma\rangle \times R_2\rangle$. By choosing $\sigma \in L$ we have $N_\alpha \cap N_\beta \cong S_2 \times S_{n-1}$, and, as in [22, Lemma 5.2], $N_\beta \cong (C_3^2{:}C_2) \times S_{n-1} \not\cong R$. Let $N := \langle N_\alpha, N_\beta\rangle$. Since $\langle L,R\rangle \cong S_{n+2}$ it follows that N projects onto S_{n+2} in each of its four coordinates. Moreover, given any two of the four coordinates, N_β contains an element that is the identity in one coordinate and a nonidentity element of A_{n+2} in another. Thus $A_{n+2}^4 \lhd N$. Note that N is not necessarily all of S_{n+2}^4; indeed, the elements of N_β that do not have all entries equal have even permutations as their entries.

Define $\tau := (\sigma,1,\sigma,\sigma)(1,2,3,4)$. Then $\tau^4 = (\sigma,\sigma,\sigma,\sigma)$ and so $\tau^8 = 1$. Furthermore, τ centralizes N_α and normalises N_β. Let $G_\alpha := \langle N_\alpha,\tau\rangle$, $G_\beta := \langle N_\beta,\tau\rangle$, and $G := \langle G_\alpha, G_\beta\rangle$. By similar reasoning as in [22, Lemma 5.2], $A_{n+2}^4 \lesssim G$ and G induces C_4 on the 4 simple direct factors. Moreover, $G_\beta \cong \mathrm{AGL}(1,3^2) \times S_{n-1}$. We also see that $G_\alpha \cong C_8 \times S_n$, and $G_\alpha \cap G_\beta \cong C_8 \times S_{n-1}$.

Let $\Gamma := \mathrm{Cos}(G, G_\alpha, G_\beta)$. Since G_α acts on $[G_\alpha{:}G_\alpha \cap G_\beta]$ as S_n does on n points and G_β acts on $[G_\beta{:}G_\alpha \cap G_\beta]$ as $\mathrm{AGL}(1,3^2)$ does on $\mathrm{GF}(3^2)$, we see that Γ is a connected locally 2-arc-transitive graph with valencies $\{n,9\}$. Clearly, the action of G on $[G{:}G_\alpha]$ is straight diagonal, and the action of G on $[G{:}G_\beta]$ is nondiagonal (as in [22, Lemma 5.2]). Therefore, Γ is a locally 2-arc-transitive graph of straight-nondiagonal type with vertex valencies $\{n,9\}$. □

3.4 Twisted-nondiagonal type

As discussed at the start of Section 3.2, the straight-nondiagonal examples given by Theorem 3.11 can also be viewed as twisted-nondiagonal examples. We also have the following construction of a graph of twisted-nondiagonal type.

Example 3.12. Let $T = \mathrm{PSL}(2,61)$. By [6], T contains a maximal subgroup $M \cong D_{60}$. Now, M contains a subgroup X isomorphic to C_2^2, and $N_T(X) \cong A_4$. Now, $N_T(X)$ contains an element g of order three that is not in M. Thus we may select subgroups $L \leqslant M$ and $R \leqslant M^g$ such that $L \cong C_{10} \cong C_5 \times C_2$, $R \cong C_3{:}C_2$, $\langle L,R\rangle = T$ and $L \cap R = X \cong C_2$. Note that we may select presentations $L = \langle \ell, x | \ell^5 = x^2 = 1\rangle$ and $R = \langle r,x | r^3 = x^2 = rr^x = 1\rangle$.

Note that L has an isomorphism ϕ defined by $\phi\colon \ell \mapsto \ell^2, x \mapsto x$. We define $\overline{\ell} := (\ell,\ell^\phi,\ell^{\phi^2},\ell^{\phi^3}) = (\ell,\ell^2,\ell^4,\ell^3)$ and $\overline{x} := (x,x,x,x)$. Furthermore, we define $N_\alpha := \langle\overline{\ell},\overline{x}\rangle$, $N_\beta := \langle(r,r,r,1),(r,r^{-1},1,r),\overline{x}\rangle$, and $N := \langle N_\alpha,N_\beta\rangle$. As in [22, Lemma 5.2], none of the coordinates of N_β can be linked, so $N \cong T^4$. Moreover, $N_\alpha \cong L \cong C_5 \times C_2$, $N_\beta \cong C_3^2{:}C_2$ and $N_\alpha \cap N_\beta \cong C_2$.

Define $\tau := (x,1,x,x)(1,2,3,4)$. Then $\tau^4 = (x,x,x,x)$ and so $\tau^8 = 1$. Let $G_\alpha := \langle N_\alpha,\tau\rangle$, $G_\beta := \langle N_\beta,\tau\rangle$, and $G := \langle G_\alpha,G_\beta\rangle$. We note that τ centralizes $\overline{x}$, whereas $\overline{\ell}^\tau = (\ell^3,\ell,\ell^2,\ell^4) = \overline{\ell^3}$, and so $G_\alpha \cong C_2.\mathrm{AGL}(1,5)$. By similar reasoning as in [22, Lemma 5.2], we deduce that $G \cong \mathrm{PSL}(2,61) \,\mathrm{wr}\, C_4$ and $G_\beta \cong \mathrm{AGL}(1,3^2)$. We also see that $G_\alpha \cap G_\beta \cong C_8$.

Let $\Gamma := \mathrm{Cos}(G,G_\alpha,G_\beta)$. Since G_α acts on $[G_\alpha{:}G_\alpha \cap G_\beta]$ as $\mathrm{AGL}(1,5)$ does on $\mathrm{GF}(5)$ and G_β acts on $[G_\beta{:}G_\alpha \cap G_\beta]$ as $\mathrm{AGL}(1,3^2)$ does on $\mathrm{GF}(3^2)$, we see that Γ is a connected locally 2-arc-transitive graph with vertex valencies $\{5, 9\}$. Clearly, the action of G on $[G{:}G_\alpha]$ is twisted diagonal, and the action of G on $[G{:}G_\beta]$ is nondiagonal (as in [22, Lemma 5.2]). Therefore, Γ is a locally 2-arc-transitive graph of twisted-nondiagonal type with valencies $\{5, 9\}$.

3.5 Nondiagonal-nondiagonal type

Finally, in this subsection, we include a construction of a graph of nondiagonal-nondiagonal type.

Example 3.13. Let $T = J_2$, the second Janko group. By [5], T has two conjugacy classes of elements of order three, labelled 3A and 3B, and two conjugacy classes of involutions, labelled 2A and 2B. Moreover, the elements of type 3A are contained in a maximal subgroup isomorphic to $A_5 \times D_{10}$ which contains involutions from class 2B, and the elements of type 3B are contained in a maximal subgroup isomorphic to A_5 which also contains involutions of type 2B. Furthermore, within each of these maximal subgroups the elements of order three are normalized by an involution of type 2B. Using GAP, there are subgroups $L, R < T$, each isomorphic to S_3, such that $L \cap R \cong C_2$, L contains an element of order three of type 3A, R contains an element of order three of type 3B, and $\langle L, R\rangle = T$. Furthermore, by [5], the two conjugacy classes of order three are not fused by any outer automorphism of T. Let $L = \langle \ell, x | \ell^3 = x^2 = \ell\ell^x = 1\rangle$ and $R = \langle r, x | r^3 = x^2 = rr^x = 1\rangle$.

We again use the equidistant linear code as defined in Example 3.10. Define $N_\alpha := \langle(\ell, \ell, \ell, 1), (\ell, \ell^{-1}, 1, \ell), (x, x, x, x)\rangle$ and $N_\beta := \langle(r, r, r, 1), (r, r^{-1}, 1, r), (x, x, x, x)\rangle$. Note that $L \cap R \cong C_2$, and, reasoning as in [22, Lemma 5.2], we deduce that $N_\alpha \cong N_\beta \cong C_3^2{:}C_2 \not\cong L, R$. Also, given any two of the four coordinates, both N_α and N_β contain an element that is the identity in one coordinate and a nonidentity element in another, so $N := \langle N_\alpha, N_\beta\rangle \cong J_2^4$.

Define $\tau := (x, 1, x, x)(1, 2, 3, 4)$. Then $\tau^4 = (x, x, x, x)$ and so $\tau^8 = 1$. Let $G_\alpha := \langle N_\alpha, \tau\rangle$, $G_\beta := \langle N_\beta, \tau\rangle$, and $G := \langle G_\alpha, G_\beta\rangle$. By similar reasoning as in [22, Lemma 5.2], $G \cong J_2 \operatorname{wr} C_4$ and $G_\alpha \cong G_\beta \cong \mathrm{AGL}(1, 3^2)$. We also see that $G_\alpha \cap G_\beta \cong C_8$.

Let $\Gamma := \mathrm{Cos}(G, G_\alpha, G_\beta)$. Since G_α (respectively G_β) acts on $[G_\alpha{:}G_\alpha \cap G_\beta]$ (respectively $[G_\beta{:}G_\alpha \cap G_\beta]$) as $\mathrm{AGL}(1, 3^2)$ does on $\mathrm{GF}(3^2)$, we see that Γ is a connected locally $(G, 2)$-arc-transitive graph with valencies $\{9, 9\}$. Moreover, Γ cannot be a standard double cover of a $(G, 2)$-arc-transitive graph since L and R are not conjugate subgroups in $\mathrm{Aut}(J_2)$. Clearly, the action of G on both $[G{:}G_\alpha]$ and $[G{:}G_\beta]$ is nondiagonal (as in [22, Lemma 5.2]). Therefore, Γ is a locally $(G, 2)$-arc-transitive graph of nondiagonal-nondiagonal type that is regular of valency 9.

ORCID iDs

Michael Giudici https://orcid.org/0000-0001-5412-4656
Eric Swartz https://orcid.org/0000-0002-1590-1595

References

[1] R. W. Baddeley, Two-arc transitive graphs and twisted wreath products, *J. Algebr. Comb.* **2** (1993), 215–237, doi:10.1023/A:1022447514654, https://doi.org/10.1023/A:1022447514654.

[2] N. Biggs, *Algebraic Graph Theory*, Camb. Math. Libr., Cambridge University Press, Cambridge, 2nd edition, 1994, doi:10.1017/CBO9780511608704, https://doi.org/10.1017/CBO9780511608704.

[3] F. Buekenhout, J. De Saedeleer and D. Leemans, On the rank two geometries of the groups $\mathrm{PSL}(2, q)$: part II, *Ars Math. Contemp.* **6** (2013), 365–388, doi:10.26493/1855-3974.181.59e, https://doi.org/10.26493/1855-3974.181.59e.

[4] T. C. Burness, M. Giudici and R. A. Wilson, Prime order derangements in primitive permutation groups., *J. Algebra* **341** (2011), 158–178, doi:10.1016/j.jalgebra.2011.06.017, https://doi.org/10.1016/j.jalgebra.2011.06.017.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Oxford University Press, Oxford, 2005.

[6] L. E. Dickson, *Linear Groups: With an Exposition of the Galois Field Theory*, Dover Publications Inc., New York, 1958.

[7] X. G. Fang, C. H. Li and C. E. Praeger, The locally 2-arc transitive graphs admitting a Ree simple group, *J. Algebra* **282** (2004), 638–666, doi:10.1016/j.jalgebra.2004.08.013, https://doi.org/10.1016/j.jalgebra.2004.08.013.

[8] X. G. Fang and C. E. Praeger, Finite two-arc transitive graphs admitting a Ree simple group, *Commun. Algebra* **27** (1999), 3755–3769, doi:10.1080/00927879908826660, https://doi.org/10.1080/00927879908826660.

[9] X. G. Fang and C. E. Praeger, Finite two-arc transitive graphs admitting a Suzuki simple group, *Commun. Algebra* **27** (1999), 3727–3754, doi:10.1080/00927879908826659, https://doi.org/10.1080/00927879908826659.

[10] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008, https://www.gap-system.org.

[11] M. Giudici, C. H. Li and C. E. Praeger, Analysing finite locally s-arc transitive graphs, *Trans. Am. Math. Soc.* **356** (2004), 291–317, doi:10.1090/S0002-9947-03-03361-0, https://doi.org/10.1090/S0002-9947-03-03361-0.

[12] M. Giudici, C. H. Li and C. E. Praeger, Characterizing finite locally s-arc transitive graphs with a star normal quotient, *J. Group Theory* **9** (2006), 641–658, doi:10.1515/JGT.2006.041, https://doi.org/10.1515/JGT.2006.041.

[13] M. Giudici, C. H. Li and C. E. Praeger, Locally s-arc transitive graphs with two different quasiprimitive actions, *J. Algebra* **299** (2006), 863–890, doi:10.1016/j.jalgebra.2005.10.028, https://doi.org/10.1016/j.jalgebra.2005.10.028.

[14] M. Giudici, C. H. Li and C. E. Praeger, A new family of locally 5-arc transitive graphs, *Eur. J. Comb.* **28** (2007), 533–548, doi:10.1016/j.ejc.2004.11.005, https://doi.org/10.1016/j.ejc.2004.11.005.

[15] D. M. Goldschmidt, Automorphisms of trivalent graphs, *Ann. Math. (2)* **111** (1980), 377–406, doi:10.2307/1971203, https://doi.org/10.2307/1971203.

[16] A. Hassani, L. R. Nochefranca and C. E. Praeger, Two-arc transitive graphs admitting a two dimensional projective linear group, *J. Group Theory* **2** (1999), 335–353, doi:10.1515/jgth.1999.023, https://doi.org/10.1515/jgth.1999.023.

[17] M. E. Iofinova and A. A. Ivanov, Bi-primitive cubic graphs, in: *Investigations in Algebraic Theory of Combinatorial Objects*, Dordrecht: Kluwer Academic Publishers, pp. 459–472, 1993.

[18] A. A. Ivanov and C. E. Praeger, On finite affine 2-arc transitive graphs, *Eur. J. Comb.* **14** (1993), 421–444, doi:10.1006/eujc.1993.1047, https://doi.org/10.1006/eujc.1993.1047.

[19] E. Kaja and L. Morgan, A new infinite family of star normal quotient graphs of twisted wreath type, 2022, arXiv:2202.13396 [math.CO].

[20] D. Leemans, Locally s-arc-transitive graphs related to sporadic simple groups, *J. Algebra* **322** (2009), 882–892, doi:10.1016/j.jalgebra.2009.05.003, https://doi.org/10.1016/j.jalgebra.2009.05.003.

[21] C. H. Li, B. Ling and C. X. Wu, A new family of quasiprimitive 2-arc-transitive graphs of product action type, *J. Comb. Theory, Ser. A* **150** (2017), 152–161, doi:10.1016/j.jcta.2017.02.007, https://doi.org/10.1016/j.jcta.2017.02.007.

[22] C. H. Li and Á. Seress, Constructions of quasiprimitive two-arc transitive graphs of product action type, in: *Finite Geometries, Groups, and Computation*, Walter de Gruyter, Berlin, pp. 115–123, 2006.

[23] C. H. Li, Á. Seress and S. J. Song, s-arc-transitive graphs and normal subgroups, *J. Algebra* **421** (2015), 331–348, doi:10.1016/j.jalgebra.2014.08.032, https://doi.org/10.1016/j.jalgebra.2014.08.032.

[24] L. Morgan, Private communication, 2022.

[25] C. E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. Lond. Math. Soc., II. Ser.* **47** (1992), 227–239, doi:10.1112/jlms/s2-47.2.227, https://doi.org/10.1112/jlms/s2-47.2.227.

[26] C. E. Praeger, On a reduction theorem for finite, bipartite 2-arc-transitive graphs, *Australas. J. Comb.* **7** (1993), 21–36, https://ajc.maths.uq.edu.au/pdf/7/ocr-ajc-v7-p21.pdf.

[27] C. E. Praeger, Finite quasiprimitive graphs, in: *Surveys in combinatorics, Proceedings of the 16th British combinatorial conference*, Cambridge University Press, London, pp. 65–85, 1997.

[28] C. E. Praeger, Finite transitive permutation groups and bipartite vertex-transitive graphs, *Ill. J. Math.* **47** (2003), 461–475.

[29] C. E. Praeger and C. Schneider, Embedding permutation groups into wreath products in product action, *J. Aust. Math. Soc.* **92** (2012), 127–136, doi:10.1017/S1446788712000110, https://doi.org/10.1017/S1446788712000110.

[30] L. L. Scott, Representations in characteristic p, in: *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, Amer. Math. Soc., Providence, R.I.,, pp. 319–331, 1980.

[31] E. Swartz, The locally 2-arc transitive graphs admitting an almost simple group of Suzuki type, *J. Comb. Theory, Ser. A* **119** (2012), 949–976, doi:10.1016/j.jcta.2012.01.005, https://doi.org/10.1016/j.jcta.2012.01.005.

[32] W. T. Tutte, A family of cubical graphs, *Proc. Camb. Philos. Soc.* **43** (1947), 459–472.

[33] W. T. Tutte, On the symmetry of cubic graphs, *Can. J. Math.* **11** (1959), 621–624, doi:10.4153/CJM-1959-057-2, https://doi.org/10.4153/CJM-1959-057-2.

[34] J. van Bon and B. Stellmacher, Locally s-transitive graphs, *J. Algebra* **441** (2015), 243–293, doi:10.1016/j.jalgebra.2015.06.034, https://doi.org/10.1016/j.jalgebra.2015.06.034.

[35] R. Weiss, The nonexistence of 8-transitive graphs, *Combinatorica* **1** (1981), 309–311, doi:10.1007/BF02579337, https://doi.org/10.1007/BF02579337.

[36] R. A. Wilson, Maximal subgroups of sporadic groups, in: *Finite Simple Groups: Thirty Years of the Atlas and Beyond*, Amer. Math. Soc., Providence, RI, pp. 57–72, 2017, doi:10.1090/conm/694/13974, https://doi.org/10.1090/conm/694/13974.