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The two-level Nambu–Jona-Lasinio model*

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Abstract. We show for a schematic quasispin model similar to the Nambu–Jona-Lasinio model that the Hartree-Fock and RPA approximations give accurate vacuum and pion properties in the limit of large number of quarks in the Dirac sea. This helps the understanding why the HF and RPA work so well in the full Nambu – Jona-Lasinio model, especially in the large N_c limit. We also show that the excitation spectrum in a box reveals rather accurately the pion scattering length.

1 The two-level Nambu–Jona-Lasinio model with one flavour

The Nambu–Jona-Lasinio model (NJL) has been successfully used in hadronic physics to describe the spontaneous chiral symmetry breaking, the formation of the massive constituent quark and the behaviour of pion and sigma meson as a chiral rotation and vibration. This model has not yet been solved exactly; so one does not know how accurate the approximate methods used with this model are. In order to gain some insights we simplify the NJL model from a field-theoretical model to an ordinary quantum-mechanical model with a fixed particle number N. This is achieved by

(i) a sharp 3-momentum cutoff $0 \leq |\mathbf{p}_i| \leq \Lambda$;
(ii) restricting the space to a box of volume $\mathcal{V}$ with periodic boundary conditions. This gives a finite number of discrete momentum states, $\mathcal{N} = N_c N_f \mathcal{V}\Lambda^3/3\pi^2$ in the Dirac sea and the same number available in the "Fermi sea" (positive energy states). In the ground state (vacuum) we assume also the same number of particles, $N = \mathcal{N}$, which, due to the interactions, are distributed between the Dirac and the Fermi sea.
For simplicity, we make two further approximations:
(iii) We restrict the system to one flavour, $N_f = 1$; many qualitative and even quantitative features will remain the same, but of course not all.
(iv) we assume all particles to have the same kinetic energy $\pm P$ instead of different individual values $\pm|\mathbf{p}_i|$: $|\mathbf{p}_i| \to P$. Furthermore, the following turns out to be a reasonable average: $P = \frac{3}{4}\Lambda$ and it is surprising how well it reproduces more detailed calculations.

* Talk delivered by M. Rosina.

Then the NJL Hamiltonian can be conveniently written in the first-quantized form [1]

$$H'_{NJL} = \sum_{i=1}^{N}\Big(\gamma_5(i)h(i)P + m_0\beta(i)\Big) + \\ - \frac{2G}{\mathcal{V}}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left(\beta(i)\beta(j) + \Big(i\beta(i)\gamma_5(i)\Big)\Big(i\beta(j)\gamma_5(j)\Big)\right)\mathcal{P}\,. \tag{1}$$

Here γ_5 is the chirality operator (handedness), $h = \boldsymbol{\sigma}\cdot\mathbf{p}/|\mathbf{p}|$ is the helicity and m_0 is the small bare quark mass which explicitly breaks the chiral symmetry. The interaction has two terms in order to be chirally symmetric. The projector

$$\mathcal{P} = \sum_{\mathbf{p}_i{}'}^{\Lambda}\sum_{\mathbf{p}_j{}'}^{\Lambda}\sum_{\mathbf{p}_i}^{\Lambda}\sum_{\mathbf{p}_j}^{\Lambda}\delta_{\mathbf{p}_i{}'+\mathbf{p}_j{}',\,\mathbf{p}_i+\mathbf{p}_j}\,|\,\mathbf{p}_i{}',\ \mathbf{p}_j{}'\rangle\langle\,\mathbf{p}_i\,,\ \mathbf{p}_j\,| \tag{2}$$

restricts momenta to a sharp cutoff Λ, but it allows any two quarks to scatter into any two other momentum states provided they conserve momentum (at infinite cutoff this would correspond to a contact interaction).

2 Relation to lattice calculations

The model assumption $0 \le |\mathbf{p_i}| \le \Lambda$ corresponds to the cell size (resolution) $a = 6^{1/3}\pi^{2/3}/\Lambda$. Here we assumed $N_c = 3$ colours, $N_f = 1$ flavours, and two helicities. The periodic boundary condition in $\mathcal{V}$ corresponds to the block size $L = \sqrt[3]{\frac{N}{6}} = \sqrt[3]{\mathcal{V}}$ with $N = \mathcal{V}\Lambda^3/\pi^2$.

In our present calculation with $N = 144$ ($\sqrt[3]{N/6} \approx 3$) and $\Lambda = 650\,\mathrm{MeV}$ this corresponds to the block size in three dimensions $L \approx 3a$ and $a \approx 1.2$ fm. It is surprising that such a poor resolution and block size yields excellent results. We shall discuss this point in the Discussion.

3 The quasispin NJL-like model

In order to get a soluble model we simplify the interaction. In the NJL model the interaction conserves the sum of momenta of both quarks, but each quark changes its momentum in any direction (in the 2-body c.m. system) with equal probability. In the simplified interaction each quark conserves its momentum. The schematic Hamiltonian can then be written as

$$H = \sum_{k=1}^{N}\Big(\gamma_5(k)h(k)P + m_0\beta(k)\Big) + \\ - \frac{g}{2}\left(\sum_{k=1}^{N}\beta(k)\sum_{l=1}^{N}\beta(l) + \sum_{k=1}^{N}i\beta(k)\gamma_5(k)\sum_{l=1}^{N}i\beta(l)\gamma_5(l)\right)\quad . \tag{3}$$

Here $g = 4G/\mathcal{V}$.

The interaction part of Hamiltonian changes the chirality of each separate quark (since it does not commute with the Hamiltonian), but it conserves the helicity, color and momentum of each quark. That means that quarks have a unique label and can be treated as distinguishable.

In the interaction, the double sum has simplified into products of single sums which can be conveniently expressed with the following quasispin operators

$$j_x = \frac{1}{2}\,\beta\ , \quad j_y = \frac{1}{2}\,i\beta\gamma_5\ , \quad j_z = \frac{1}{2}\,\gamma_5\ ,$$

which obey (quasi)spin commutation relations and allow us to make full use of the angular momentum algebra.

The (quasi)spin commutation relations are also obeyed by separate sums over quarks with right and left helicity ($\alpha = x,\, y,\, z$)

$$R_\alpha = \sum_{k=1}^{N} \frac{1+h(k)}{2} j_\alpha(k)\ , \quad L_\alpha = \sum_{k=1}^{N} \frac{1-h(k)}{2} j_\alpha(k) \tag{4}$$

as well as by the total sum

$$J_\alpha = R_\alpha + L_\alpha = \sum_{k=1}^{N} j_\alpha(k)\,. \tag{5}$$

The model Hamiltonian can then be written as

$$H = 2P(R_z - L_z) + 2m_0 J_x - 2g(J_x^2 + J_y^2) \quad . \tag{6}$$

It commutes with R^2 and L^2 but not with R_z and L_z. Nevertheless, it is convenient to work in the basis $|\,R, L, R_z, L_z\,\rangle$.The Hamiltonian matrix elements can be easily calculated using the angular momentum algebra. By diagonalisation we then obtain the energy spectrum of the system.

A formally similar Hamiltonian has been studied already by Moszkowski [2] in the context of nuclear rotations and vibrations; instead of the NJL interaction, it is the quadrupole-quadrupole interactions that plays a similar role and leads to the spontaneous breaking of spherical symmetry. Also Civitarese et al. [3] used a two-level quark model to describe the low-lying mesonic spectra, but their interaction is not like NJL, they couple quarks to a one-level bosonic degree of freedom (representing gluon pairs or glueballs).

4 Model parameters and basic observables

Both the full NJL model as well as the quasispin model have three model parameters, Λ, G and m_0. We intend to adjust them to what we choose as the three basic observables $M = 335\,\text{MeV}$ (the dressed–constituent quark mass), $Q = \langle\, g\,|\,\bar{\psi}\psi\,|\,g\,\rangle = \frac{1}{\mathcal{V}}\langle\, g\,|\,\sum_i \beta(i)\,|\,g\,\rangle = \frac{2}{\mathcal{V}}\langle\, g\,|\,J_x\,|\,g\,\rangle = 250^3\ \text{MeV}^3$ (the chiral condensate) and the pion mass m_π.

We assume that the chiral condensate is related to the better defined observable, the pion decay constant, by the Gell-Mann Oakes Renner relation $f_\pi = -\sqrt{-2m_0 Q}/m_\pi = 93\,\text{MeV}$. We assume this relation also for the one-flavour case where f_π is not experimentally defined while Q has the same meaning as in the two-flavour case.

A detailed analysis of model parameters and quality of approximations for the ground state and the pionic excited state was performed by Oblak [4].

In the two-level quasispin model the exact values of the observables are determined as

$$\begin{aligned} M &= \sqrt{\Big(E_g(N) - E_g(N-1)\Big)^2 - P^2} \\ Q &= \frac{2}{\mathcal{V}}\langle\, g\,|\,J_x\,|\,g\,\rangle \\ m_\pi &= E_1(N) - E_g(N)\,. \end{aligned} \tag{7}$$

As usual we have defined the constituent quark mass through the separation energy of the N-th quark and the pion mass as the energy difference between the first excited and ground state (note that pionic excitation conserves the momenta of all quarks and therefore carries no momentum).

We want to study the N-dependence of our results. Therefore it would be meaningful to adjust the model parameters for a particular N, for example for $N \to \infty$. Since we cannot calculate exactly for infinite N we rather choose as a reference the Hartree-Fock + RPA solution; anyway, also for the full NJL the model parameters have been adjusted in this way in the literature [5,6].

In the HF+RPA approximation, the relations (7) can be calculated explicitly

$$\begin{aligned} M &= \sqrt{\left(\alpha G\Lambda^3\right)^2 - P^2} \\ Q &= \frac{\Lambda^3}{\pi^2}\frac{M}{\sqrt{M^2+P^2}} \\ m_\pi &\approx \sqrt{\sqrt{\frac{M^2+P^2}{M^2}}\,G\Lambda^3 m_0}\,. \end{aligned} \tag{8}$$

Here $\alpha = (4/\pi^2)(1-1/N)(1-m_0/M)^{-1}$. It is then easy to determine the model parameters (we choose the limit $N \to \infty$):

$$\Lambda = 648\,\text{MeV}, \quad G = 40.6\,\text{MeV fm}, \quad m_0 = 4.58\,\text{MeV}.$$

These values compare favourably with those of full Nambu-Jona Lasinio

$$\begin{aligned} \text{Coimbra [5]}:\ \Lambda &= 631\,\text{MeV}, \quad G = 40\,\text{MeV fm}, \quad m_0 \approx 5\,\text{MeV}\,, \\ \text{Buballa [6]}:\ \Lambda &= 664\,\text{MeV}, \quad G = 37.8\,\text{MeV fm}, \quad m_0 = 5.0\,\text{MeV}\,. \end{aligned}$$

The agreement is not surprising for the following reasons.

(i) The Hartre-Fock solution in the quasispin model coincides with the Hartree-Fock solution in the two-level NJL model (apart from small Fock terms).

This can be seen from the potential energy contribution $\sum_{u \le v} V_{uvuv}$ which is the same in both cases. Hartree-Fock ignores the off-diagonal terms $V_{uvu'v'}$ which we have anyway thrown away in the quasispin model.

(ii) In order to make up for the contributions of the second flavour we have increased the coupling strength in (1) by a factor of two compared to the standard definition in the two-flavour NJL, $G \to 2G$

(iii) It seems that we have chosen a good average kinetic energy $P = \frac{3}{4}\Lambda$ when we replaced the individual values by an average.

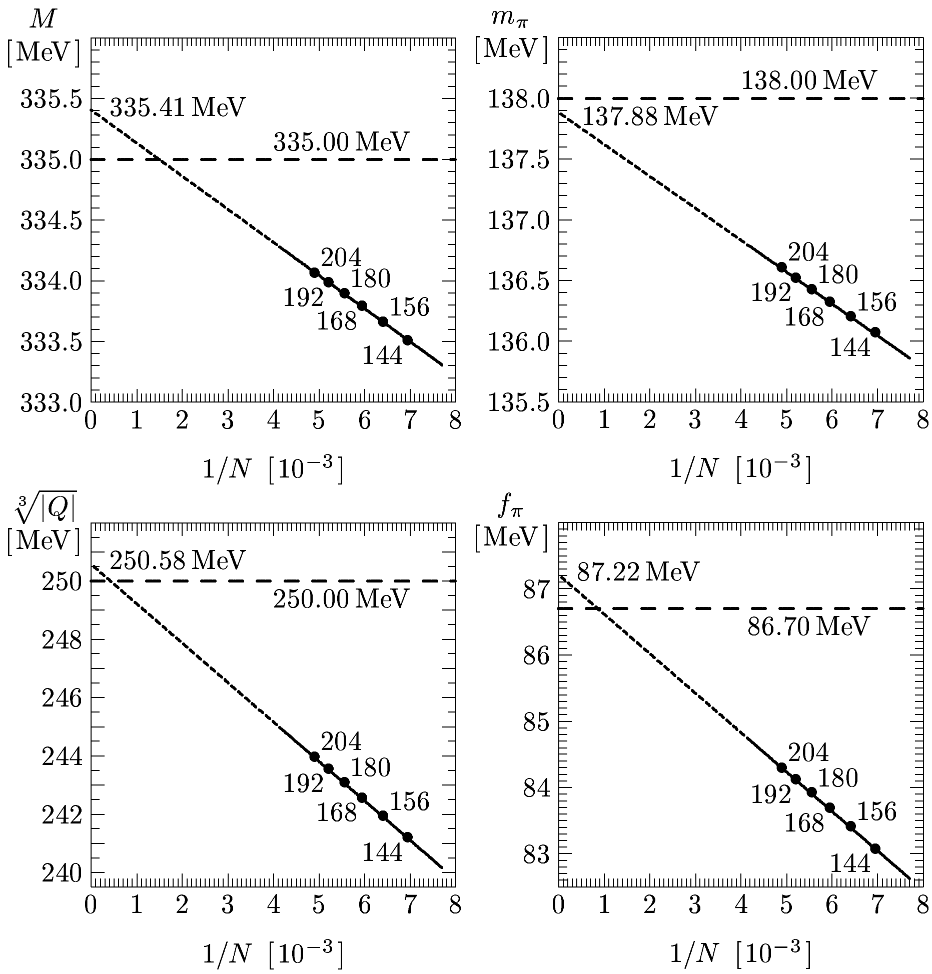

Fig. 1. Linear extrapolations of exact results of M, m_π, $\sqrt[3]{|Q|}$ and f_π for different values of N (144, 156, 168, 180, 192, 204) to the infinite N.

It is an interesting result that the exact values of the observables as a function of N approach the HF+RPA values in the limit $N \to \infty$. The linear extrapolation in Fig. 1 shows that HF+RPA is either exact or very accurate in this limit. Since full

NJL has the same HF+RPA solution, this fact gives a good credit to the HF+RPA approximation in full NJL (but does not yet prove the exactness).

5 Pion-pion scattering

An interesting application of the model is to calculate the pion-pion scattering length from the excitation spectrum in the box $\mathcal{V}$. Since we are working in a finite volume $\mathcal{V}$ with periodic boundary conditions we cannot impose scattering boundary conditions. Instead of a continuous spectrum of scattering states we obtain a discrete spectrum. However, we can interpret the ground state as vacuum and excited states as multi-pion states or sigma-meson excitations or superpositions of both. For his purpose we have to choose the spectrum with ground-state quantum numbers $R = L = N/4$. In this case pionic excitations conserve the momenta of all quarks and therefore carry no momentum (nor angular momentum). Such states correspond to n pions in s-state and allow the evaluation of the average effective pion-pion potential $\bar{V}$ and through it the pion-pion scattering length.

In Table 1 we present the spectrum for $N = 144$ and the model parameters listed in Section 4.

Table 1. The spectrum of the quasispin model with $N = 144$, quantum numbers $R+L = 36$ and model parameters listed in Section 4.

Parity	$E - E_0$ [MeV]	$\bar{V}$ [MeV]
+	932	-9.5
−	803	-11.7
+	771	-11.3
−	767	-8.8
+	646	-11.4
+	634	-12.2
−	580	-10.0
+	482	-10.5
−	378	-10.1
+	261	-10.3
−	136	
+	0	

For ideal n-pion states the energy should be

$$E_{n\pi} = n\, m_\pi + \frac{n(n-1)}{2}\bar{V}.$$

The quantity $\bar{V} = (E - E_0 - n\,m_\pi)/(\frac{1}{2}n(n-1))$ in Table1 is in fact rather constant throughout the spectrum, except around 600-700 MeV where two positive parity states appear in succession and signal the presence of a sigma-excitation causing the confusion.

Another test of the concept of average effective pion-pion potential is its N-dependence. Larger N means a larger normalization volume $\mathcal{V}$ and therefore more dilute pions leading to a proportionally smaller $\bar{V}$. In fact, for $N = 132$, the value $(132/144)\bar{V} = -10.6\,\mathrm{MeV}$, close to -10.3 MeV at $N = 144$. On the other hand, for $N = 108$, the value $(108/144)\bar{V} = -12.2\,\mathrm{MeV}$, rather far. This is an indication that above 132 we are already close enough to large-N limit, while 108 is still too small.

We calculate the s-state scattering length in the first-order Born approximation

$$a = \frac{m_\pi/2}{2\pi}\int V(\mathbf{r})\,d^3r = \frac{m_\pi}{4\pi}\bar{V}\mathcal{V}. \tag{9}$$

This formula was first quoted by M.Lüscher[7] in 1986 and 1991 and later by many authors. It was derived in a much more sophisticated way, but in our context it is just the first-order Born approximation.

In our example for $N = 144$ we have $\bar{V} = -10.3\,\mathrm{MeV}$ and $\mathcal{V} = \pi^2 N/\Lambda^3 = 40\,\mathrm{fm}^3$ This gives

$$a\,m_\pi = \frac{m_\pi^2}{4\pi}\bar{V}\mathcal{V} = -0.0836. \tag{10}$$

Of course, there are no experiments with one-flavour pions. It is, however, interesting to compare with the two-flavour value (I = 2). The chiral perturbation theory (soft pions) suggests in leading order $a_0^{I=2}\,m_\pi = -m_\pi^2/16\pi f_\pi^2 = -0.0445$. The old analysis of Gasser and Leutwyler gave -0.019 and the more recent analysis by Lesniak gave -0.034 ("non-uniform fit") or -0.044 ("uniform fit"). It is not yet clear to us why we get about twice larger value in our one-flavour model. Possibly this is due to the artifact that we made up for the second flavour by replacing $G \to 2G$ which might give too strong attraction between pions. We are still exploring this point.

6 Conclusion

From the quasispin model of the Nambu–Jona-Lasinio type one can learn several lessons:

(i) The Hartree-Fock solution is (almost) exact for a truncated Nambu–Jona-Lasinio model in which the off-diagonal interaction matrix elements (corresponding to scattering of two quarks into different final momenta) are neglected. Since the full NJL model has the same HF solution as the truncated one it is well approximated by HF provided the effect of the off-diagonal terms is suppressed.

(ii) The off-diagonal terms are important for pairing since scattering in all possible directions provides a large phase space for long range pairing correlations to develop. On the other hand, the interaction matrix elements which conserve each momentum and scatter the two quarks between the lower and the upper level (between the Dirac and Fermi sea) are responsible for the chiral deformation of the system (spontaneous chiral symmetry breaking). There is a competition between pairing and deformation. Here we draw a pictorial analogy with nuclear physics where it is also a competition between pairing and quadrupole deformation. The pairing energy is proportional to the number of valence nucleons and the deformation energy to the square of their number. Therefore near closed shells pairing prevails and nuclei are spherical, while far from closed shells (at large number of valence nucleons) the deformation prevails and nuclei are deformed. Similarly, due to a large number N of quarks and large chiral symmetry breaking we expect the pairing to be suppressed by order $1/N$. We have still to test this idea by studying the Hartree-Fock-Bogoliubov approximation of two-level NJL and verifying that the solution does not support pairing; if this is the case it would strongly support the idea that HF is an accurate approximation of NJL.

(iii) The picture that we have a chiral deformation of the mean field and of quark wavefunctions can be mapped into a picture in which we have quark-antiquark pairing. The interaction terms of the truncated NJL (and our quasispin model) scatter two quarks between the Dirac and Fermi sea but conserve their individual momenta; this leads to chiral deformation. We can, however, also call quark holes antiquarks, antiquarks carry opposite momenta as the missing quarks. Two quarks having whichever different momenta scatter back in the same momenta; in the other picture, quark and antiquark have opposite momenta and scatter in whichever pair of different opposite momenta. This is then just the condition stimulating pairing. The formal relation between the chiral deformation of Hartree-Fock quarks and the quark-antiquark pairing will be described elsewhere.

(iv) In the quasispin model it is very instructive that the number of colours N_c and the number of spatial states $\mathcal{V}\Lambda^3/6\pi^2$ appear on equal footing in the product $N = 2N_c\mathcal{V}\Lambda^3/6\pi^2$. The colour and the momentum quantum number together are just the house number of the particle since the interaction does not depend on them. Therefore it is the same limit $N \to \infty$ whether we take the large N_c limit or a large block $\mathcal{V}$. This explains why even with 3 colours the quasispin model behaves similarly as the theorems regarding large N_c limit suggest (good HF approximation, suppression of off-diagonal terms and their effects, etc.).

(v) The presented quasispin model is reminiscent of the schematic model of Lipkin, Agassi, Glick and Meshkov [8], popular in nuclear many-body problems. The purpose of the Lipkin model was to show essential features of approximations such as HF, perturbation theory, Projected HF, Time-dependent HF, RPA, Peierls-Yoccoz, Peierls-Thouless, Generator Coordinate Method, as well as to test their accuracy. Our schematic NJL-like model could be designed as "the Lipkin model of chiral symmetry".

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