Locally spherical hypertopes from generalised cubes[*]

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Abstract

We show that every non-degenerate regular polytope can be used to construct a thin, residually-connected, chamber-transitive incidence geometry, i.e. a regular hypertope. These hypertopes are related to the semi-regular polyotopes with a tail-triangle Coxeter diagram constructed by Monson and Schulte. We discuss several interesting examples derived when this construction is applied to generalised cubes. In particular, we produce an example of a rank 5 finite locally spherical proper hypertope of hyperbolic type. No such examples were previously known.

Keywords: Regularity, thin geometries, hypermaps, hypertopes, abstract polytopes.

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1 Introduction

Hypertopes are a special type of incidence geometries that generalise the notions of abstract polytopes and of hypermaps. The concept was introduced in [9] with particular emphasis on regular hypertopes (that is, the ones with highest degree of symmetry). Although in [8, 10, 11] a number of interesting examples of regular hypertopes have been constructed, within the theory of abstract regular polytopes much more work has been done. Notably, [26] and [28] deal with universal constructions of polytopes, while in [5, 23, 24] some constructions with prescribed combinatorial conditions are explored. In another direction, in [3, 7, 14, 22] the questions of existence of polytopes with prescribed (interesting) groups

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are investigated. Much of the impetus to the development of the theory of abstract polytopes, as well as the inspiration with the choice of problems, was based on work of Branko Grünbaum [13] from the 1970s.

In this paper we generalise the halving operation on polyhedra (see 7B in [18]) to a certain class of regular abstract polytopes to construct regular hypertopes. More precisely, given a regular non-degenerate n-polytope $\mathcal{P}$, we construct a regular hypertope $\mathcal{H}(\mathcal{P})$ whose type-preserving automorphism group is a subgroup of $\mathrm{Aut}(\mathcal{P})$ of index at most 2. The hypertope $\mathcal{H}(\mathcal{P})$, as we shall see in Section 3, is closely related to the semi-regular polytopes with tail-triangle Coxeter diagram described by Monson and Schulte in [19, 20].

The paper is organised as follows. In Section 2 we review the basic theory of hypertopes (with particular focus on regular hypertopes) and revisit the notion of a regular polytope (first introduced in the early 1980s) within the theory of hypertopes. In Section 3 we explore the halving operation on an abstract polytope and show that the resulting incidence system is a regular hypertope. Finally, in Section 4 we give concrete examples arising from our construction. In particular, we focus on locally spherical hypertopes arising from Danzer's construction of generalised cubes. As a result we produce a number of new regular hypertopes including an example of a finite regular rank 5 proper hypertope which, because of the size of its automorphism group, could not previously be found (see [11, Section 6]).

2   Regular hypertopes

In this section we review the definition and basic properties of regular hypertopes. We introduce abstract polytopes as a special class of hypertopes. However, if the reader is interested in a classic and more detailed definition of abstract polytopes, we suggest [18, Section 2A].

The notion of a regular hypertope was introduced in [9] as a common generalisation of an abstract regular polytope and of a regular hypermap. In short, a *regular hypertope* is a *thin*, *residually-connected*, *chamber-transitive* geometry (the concepts are defined below). More details and an account of general theory can be found in [2].

An *incidence system* is a 4-tuple $\Gamma := (X, *, t, I)$ where X and I are sets, $t : X \to I$ is the *type function* and $*$ is a binary relation in X called *incidence*. The elements of X and I are called the *elements* and the *types* of Γ, respectively. The cardinality of I is the *rank* of Γ. An element x is said to be of type i, or an *i-element*, whenever $t(x) = i$, for $i \in I$. The relation $*$ is reflexive, symmetric and such that, for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$, then $x = y$.

A *flag* F is a subset of X in which every two elements are incident. An element x is incident to a flag F, denoted by $x * F$, when x is incident to all elements of F. For a flag F the set $t(F) := \{t(x) \,|\, x \in F\}$ is called *the type of F*. When $t(F) = I$, F is called a *chamber*.

An incidence system Γ is a *geometry* (or an *incidence geometry*) if every flag of Γ is contained in a chamber, that is, if all maximal flags of Γ are chambers.

The *residue* of a flag F of an incidence geometry Γ is the incidence geometry $\Gamma_F := (X_F, *_F, t_F, I_F)$ where $X_F := \{x \in X : x * F, x \notin F\}$, $I_F := I \setminus t(F)$, and where t_F and $*_F$ are restrictions of t and $*$ to X_F and I_F respectively.

An incidence system Γ is *thin* when every residue of rank one of Γ contains exactly two elements. If an incidence geometry is thin, then given a chamber C there exists exactly one chamber differing from C in its i-element. An incidence system Γ is *connected* if its

incidence graph is connected. Moreover, Γ is *residually connected* when Γ is connected and each residue of Γ of rank at least two is also connected. It is easy to see that this condition is equivalent to *strong connectivity* for polytopes (as defined in [18, pg. 23] and reviewed below) and the thinness is equivalent to the diamond condition for polytopes. A *hypertope* is a thin incidence geometry which is residually connected.

An *abstract polytope* of rank n is usually defined as a strongly-connected partially ordered set $(\mathcal{P}, \leqslant)$ that satisfies the *diamond condition* and in such a way that all maximal chains of $\mathcal{P}$ have the same length $(n + 2)$. In the language of incidence geometries, an abstract polytope is an incidence system $(\mathcal{P}, *_\leqslant, \mathrm{rk}, \{-1, \ldots, n\})$, where $*_\leqslant$ is the incidence relation defined by the order of $\mathcal{P}$ (i.e., $x *_\leqslant y$ if and only if $x \leqslant y$ or $y \leqslant x$) and rk is the *rank function*. We require that $\mathcal{P}$ has a unique (minimum) element of rank (type) -1 and a unique (maximum) element of rank n. Note that a flag (in the language of incidence systems) is what has been called a *chain* in the theory of abstract polytopes. Therefore, maximal chains of $\mathcal{P}$ are precisely the chambers of the corresponding incidence system. The fact that every maximal chain of $\mathcal{P}$ has $(n + 2)$ elements implies that $\mathcal{P}$ defines a geometry. It is well-known that for any two incident elements $F_i \leqslant F_j$ of $\mathcal{P}$, with $\mathrm{rk}(F_i) = i$ and $\mathrm{rk}(F_j) = j$, the *section* $F_j/F_i = \{x \in \mathcal{P} : F_i \leqslant x \leqslant F_j\}$ is a $(j - i - 1)$-polytope. We note that for polytopes, the residue of a chain F is a union of sections of $\mathcal{P}$ defined by the intervals of I_F.

Observe that the rank 2 hypertopes are precisely the abstract polygons and the rank 3 hypertopes are the non-degenerate hypermaps.

A *type-preserving automorphism* of an incidence system $\Gamma := (X, *, t, I)$ is a permutation α of X such that for every $x \in X$, $t(x) = t(x\alpha)$ and if $x, y \in X$, then $x * y$ if and only if $x\alpha * y\alpha$. The set of type-preserving automorphisms of Γ is denoted by $\mathrm{Aut}_I(\Gamma)$.

The group of type-preserving automorphisms of an incidence geometry Γ generalises the automorphism group of an abstract polytope. Some familiar symmetry properties of polytopes extend naturally to incidence geometries. For instance, $\mathrm{Aut}_I(\Gamma)$ acts faithfully on the set of chambers of Γ. Moreover, if Γ is a hypertope this action is semi-regular. In fact, if $\alpha \in \mathrm{Aut}_I(\Gamma)$ fixes a chamber C, it also fixes its i-adjacent chamber C^i. Since Γ is residually connected, α must be the identity.

We say that Γ is *chamber-transitive* if the action of $\mathrm{Aut}_I(\Gamma)$ on the chambers is transitive, and in that case the action of Γ on the set of chambers is regular. For that reason Γ is then called a *regular hypertope*. As expected, this generalises the concept of a regular polytope.

Observe that, when Γ is a geometry, chamber-transitivity is equivalent to *flag-transitivity* (meaning that for each $J \subseteq I$, there is a unique orbit on the flags of type J under the action of $\mathrm{Aut}_I(\Gamma)$; see for example Proposition 2.2 in [9]).

Let $\Gamma := (X, *, t, I)$ be a regular hypertope and let C be a fixed (base) chamber of Γ. For each $i \in I$ there exists exactly one automorphism ρ_i mapping C to C^i. If $F \subseteq C$ is a flag, then the automorphism group of the residue Γ_F is precisely stabiliser of F under the action of $\mathrm{Aut}_I(\Gamma)$. We denote this group by $\mathrm{Stab}_\Gamma(F)$. It is easy to see that

$$\mathrm{Stab}_\Gamma(F) = \langle \rho_i : i \in I_F \rangle.$$

If $I_F = \{i\}$, that is Γ_F is of rank $|I| - 1$, the thinness of Γ implies that

$$\rho_i^2 = 1. \tag{2.1}$$

If $I_F = \{i, j\}$, then there exists $p_{ij} \in \{2, \dots, \infty\}$ such that

$$(\rho_i \rho_j)^{p_{ij}} = 1; \tag{2.2}$$

in this situation the residue of F is an abstract p_{ij}-gon. Moreover, if J and K are arbitrary subsets of I and $F, G \subseteq C$ are flags such that $I_F = J$ and $I_G = K$, then

$$\mathrm{Stab}_\Gamma(F) \cap \mathrm{Stab}_\Gamma(G) = \mathrm{Stab}_\Gamma(F \cup G),$$

or equivalently

$$\langle \rho_j : j \in J \rangle \cap \langle \rho_k : k \in K \rangle = \langle \rho_i : i \in J \cap K \rangle. \tag{2.3}$$

We call the condition in (2.3) the *intersection property*. Following [9], a C-*group* is a group generated by involutions $\{\rho_i : i \in I\}$ that satisfies the intersection property. It follows that the type-preserving automorphism group of a regular hypertope is a C-group ([9, Theorem 4.1]).

Every Coxeter group U is a C-group and in particular, it is the type-preserving auto-morphism of a regular hypertope [32, Section 3] called the *universal regular hypertope* associated with the Coxeter group U. Moreover, every C-group G is a quotient of a Coxeter group U. If $\mathcal{H}$ is a regular hypertope whose type-preserving automorphism group is G, the *universal cover* of $\mathcal{H}$ is the regular hypertope associated with U.

The *Coxeter diagram* of a C-group G is a graph with $|I|$ vertices corresponding to the generators of G and with an edge $\{i, j\}$ whenever the order p_{ij} of $\rho_i \rho_j$ is greater than 2. The edge is endowed with the label p_{ij} when $p_{ij} > 3$. The automorphism group of an abstract polytope is a *string C-group*, that is, a C-group having a linear Coxeter diagram. If $\mathcal{P}$ is a regular n-polytope, then we say that $\mathcal{P}$ is of (Schläfli) type $\{p_1, \dots, p_{n-1}\}$ whenever the Coxeter diagram of $\mathrm{Aut}(\mathcal{P})$ is

One of the most remarkable results in the theory of abstract regular polytopes is that the string C-groups are precisely the automorphism groups of the regular polytopes. In other words, given a string C-group G, there exists a regular polytope $\mathcal{P} = \mathcal{P}(G)$ such that $G = \mathrm{Aut}(\mathcal{P})$ (see [18, Section 2E]). This result was proved in [25, 27] for so-called *regular incidence complexes*, (combinatorial objects slightly more general than abstract polytopes). However, the results were essentially already known to Tits who constructed coset geometries from string Coxeter groups in [30], which preceded the introduction of the intersection property in a working paper from 1961 (see [32]). Nevertheless, Schulte was not aware of this. In [29] he gives a nice historical note on the development of the theory.

Analogously, it is also possible to construct, under certain conditions, a regular hypertope from a group, and particularly from a C-group, using the following proposition.

Proposition 2.1 (Tits Algorithm [32])**.** *Let n be a positive integer and $I := \{0, \dots, n-1\}$. Let G be a group together with a family of subgroups $(G_i)_{i \in I}$, X the set consisting of all cosets $G_i g$ with $g \in G$ and $i \in I$, and $t : X \rightarrow I$ defined by $t(G_i g) = i$. Define an incidence relation $*$ on $X \times X$ by:*

$$G_i g_1 * G_j g_2 \quad \text{if and only if} \quad G_i g_1 \cap G_j g_2 \neq \emptyset.$$

Then the 4-tuple $\Gamma := (X, *, t, I)$ is an incidence system having $\{G_i : i \in I\}$ as a chamber. Moreover, the group G acts by right multiplication as an automorphism group on Γ. Finally, the group G is transitive on the flags of rank less than 3.

The incidence system constructed using the proposition above will be denoted by $\Gamma(G; (G_i)_{i \in I})$ and called a *coset incidence system*.

Theorem 2.2 ([9, Theorem 4.6]). *Let $I = \{0, \ldots, n-1\}$, let $G = \langle \rho_i \,|\, i \in I \rangle$ be a C-group, and let $\Gamma := \Gamma(G; (G_i)_{i \in I})$ where $G_i := \langle \rho_j \,|\, j \neq i \rangle$ for all $i \in I$. If G is flag-transitive on Γ, then Γ is a regular hypertope.*

In other words, the coset incidence system $\Gamma = \Gamma(G, (G_i)_{i \in I})$ is a regular hypertope if and only if the group G is a C-group and Γ is flag-transitive. In order to prove that a given group G is a C-group, we can use the following result.

Proposition 2.3 ([7, Proposition 6.1]). *Let G be a group generated by n involutions $\rho_0, \ldots, \rho_{n-1}$. Suppose that G_i is a C-group for every $i \in \{0, \ldots, n-1\}$. Then G is a C-group if and only if $G_i \cap G_j = G_{i,j}$ for all $0 \leqslant i, j \leqslant n-1$.*

At the end of this section we introduce Lemma 2.4 whose proof is straightforward and will be used in Section 3 to prove our main results.

Lemma 2.4. *Let $G = \langle \rho_0, \ldots, \rho_{r-1} \rangle$ and $H = \langle \rho_r, \ldots, \rho_{r+s-1} \rangle$ be two C-groups. Then the group*

$$G \times H = \langle \rho_0, \ldots, \rho_{r-1}, \rho_r, \ldots, \rho_{r+s-1} \rangle$$

is a C-group.

3   Halving operation

In this section the halving operation is applied to the automorphism group of a non-degenerate regular polytope $\mathcal{P}$ producing $\mathrm{H}(\mathcal{P})$, which is a subgroup of $\mathrm{Aut}(\mathcal{P})$ of index at most 2. We prove that the group $\mathrm{H}(\mathcal{P})$ is a C-group and that the corresponding incidence system is flag-transitive. Therefore the group $\mathrm{H}(\mathcal{P})$ is the type-preserving automorphism group of a regular hypertope.

Let $n \geqslant 3$ and $\mathcal{P}$ be a regular, non-degenerate n-polytope of type $\{p_1, \ldots, p_{n-2}, p_{n-1}\}$ and automorphism group $\mathrm{Aut}(\mathcal{P}) = \langle \varrho_0, \ldots, \varrho_{n-1} \rangle$. The *halving operation* is the map

$$\eta : \langle \varrho_0, \ldots, \varrho_{n-1} \rangle \to \langle \rho_0, \ldots, \rho_{n-1} \rangle,$$

where

$$\rho_i = \begin{cases} \varrho_i, & \text{if } 0 \leqslant i \leqslant n-2, \\ \varrho_{n-1}\varrho_{n-2}\varrho_{n-1}, & \text{if } i = n-1, \end{cases} \tag{3.1}$$

The *halving group* of $\mathcal{P}$, denoted by $\mathrm{H}(\mathcal{P})$, is the image of $\mathrm{Aut}(\mathcal{P})$ under η.

Observe that the group $\mathrm{H}(\mathcal{P}) = \langle \rho_1, \ldots, \rho_{n-1} \rangle$ has the following diagram
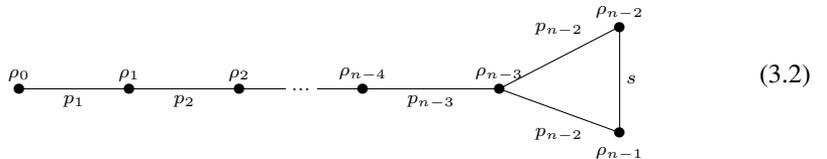
$$\tag{3.2}$$

where $s = p_{n-1}$ if p_{n-1} is odd, otherwise $s = \frac{p_{n-1}}{2}$. We denote by $\mathcal{H}(\mathcal{P})$ the coset incidence system $\Gamma\left(\mathrm{H}(\mathcal{P}), (H_i)_{i \in \{0,\ldots,n-1\}}\right)$, where H_i is the subgroup of $\mathrm{H}(\mathcal{P})$ generated by $\{\rho_j : j \neq i\}$. In Theorem 3.1 we show that the group $\mathrm{H}(\mathcal{P})$ satisfies the intersection property and in Proposition 3.2 we show that the corresponding incidence $\mathcal{H}(\mathcal{P})$ is flag-transitive. We conclude the section with Corollary 3.3 which states that $\mathcal{H}(\mathcal{P})$ is in fact a thin, chamber-transitive coset geometry, i.e. a regular hypertope.

The halving operation has been used before in the context of regular polyhedra of type $\{q, 4\}$ (see [17] and [18, Section 7B]) and the resulting incidence system is a regular polyhedron of type $\{q, q\}$.

The operation described above doubles the fundamental region of $\mathrm{Aut}(\mathcal{P})$ by gluing together the base flag Φ and the flag Φ^{n-1}.

As an example we explore the halving operation applied to the cubic tessellation $\{4, 3, 4\}$. The elements of type 0 and 1 of the resulting incidence system are the vertices and edges of the $\{4, 3, 4\}$, respectively. The elements of type 2 are half of the cubes and the elements of type 3 are the other half. This is the construction of the infinite hypertope described in [11, Example 2.5] and can also be seen as a semi-regular polytope (see [19, Section 3]).

It is easy to see that $\mathrm{H}(\mathcal{P})$ has index 2 in $\mathrm{Aut}(\mathcal{P})$ if and only if the set of facets of $\mathcal{P}$ is bipartite. This is only possible if p_{n-1} is even. If this is the case, then the elements of type i, for $i \in \{0, \ldots, n-3\}$ are the faces of rank i of $\mathcal{P}$. The elements of type $n-2$ are half of the facets of $\mathcal{P}$ (those belonging to the same partition as the base facet) and the elements of type $n-1$ are the other half of the facets, namely, those in the same partition as the facet of Φ^{n-1}.

In the remainder of the section we let $\mathcal{P}$ be a fixed regular n-polytope with a base flag Φ, the automorphism group $\mathrm{Aut}(\mathcal{P}) = \langle \varrho_0, \ldots \varrho_{n-1} \rangle$ and $H = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ the halving group of $\mathcal{P}$. For $i, j \in \{0, \ldots, n-1\}$ we let H_i and $H_{i,j}$ be the groups $\langle \rho_k : k \neq i \rangle$ and $\langle \rho_k : k \notin \{i, j\} \rangle$, respectively. Finally, by $\mathcal{H}(\mathcal{P})$ we denote the incidence system $\Gamma\left(H, (H_i)_{i \in \{0,\ldots,n-1\}}\right)$ and by Γ_i the residue of $\mathcal{H}(\mathcal{P})$ induced by H_i, that is $\Gamma_i = \Gamma\left(H_i, (H_{i,j})_{j \in \{0,\ldots,n-1\} \setminus \{i\}}\right)$.

Theorem 3.1. *Let $n \geqslant 3$ and $\mathcal{P}$ be a regular, non-degenerate n-polytope of type $\{p_1, \ldots, p_{n-1}\}$. Then the halving group $\mathrm{H}(\mathcal{P})$ is a C-group.*

Proof. The strategy of this proof is to use Proposition 2.3. To do so, we proceed by induction over n. Let $\Phi = \{F_{-1}, \ldots, F_n\}$ be the base flag of $\mathcal{P}$. Let F'_{n-1} be the facet of Φ^{n-1}.

If $n = 3$, we need to prove that the group $H_0 = \langle \rho_1, \rho_2 \rangle = \langle \varrho_1, \varrho_2 \varrho_1 \varrho_1 \rangle$ is a C-group. However, this group is a subgroup of the automorphism group of the polygonal section F_3/F_1 isomorphic to the dihedral group $\mathbb{D}_s$. The groups $H_1 = \langle \rho_0, \rho_2 \rangle$ and $H_2 = \langle \rho_0, \rho_1 \rangle$ are the automorphism groups of the polygonal sections F'_2/F_{-1} and F_2/F_{-1}, respectively. It follows that they are C-groups.

To finish our base case we only need to show

$$\langle \rho_0, \rho_1 \rangle \cap \langle \rho_0, \rho_2 \rangle = \langle \rho_0 \rangle, \tag{3.3}$$

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$$\langle \rho_0, \rho_2 \rangle \cap \langle \rho_1, \rho_2 \rangle = \langle \rho_2 \rangle. \tag{3.5}$$

To prove (3.3), just observe that $\langle \rho_0 \rangle = \operatorname{Stab}_{\mathcal{P}}(\{F_1, F_2\})$. Let $\gamma \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_0, \rho_2 \rangle$. Since $\gamma \in \langle \rho_0, \rho_1 \rangle$, γ fixes F_2. Similarly, since $\gamma \in \langle \rho_0, \rho_2 \rangle$, γ must fix F_2'. This implies that γ fixes F_1, since this is the only 1-face of $\mathcal{P}$ incident to both F_2 and F_2'. Therefore, $\gamma \in \operatorname{Stab}_{\mathcal{P}}(\{F_1, F_2\}) = \langle \rho_0 \rangle$. The other inclusion is obvious.

Similarly, we have that $\langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle \subseteq \operatorname{Stab}_{\mathcal{P}}(\{F_0, F_1\})$. This follows from the fact that the group $\langle \rho_0, \rho_1 \rangle$ fixes F_2 and the group $\langle \rho_1, \rho_2 \rangle$ fixes F_0. Then, $\langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle \subseteq \langle \rho_1 \rangle$. Again, the other inclusion is obvious. The proof of (3.4) follows from the same argument but now with respect to the flag $\Phi^2 = \{F_0, F_1, F_2'\}$ of $\mathcal{P}$. This completes the base case.

Assume that the halving group $\operatorname{H}(\mathcal{F})$ of every non-degenerate regular polytope $\mathcal{F}$ of rank r with $3 \leqslant r < n$ is a C-group.

Observe that the groups $H_{n-1} = \langle \rho_0, \ldots, \rho_{n-2} \rangle$ and $H_{n-2} = \langle \rho_0, \ldots, \rho_{n-3}, \rho_{n-1} \rangle$ are the automorphism groups of the sections F_{n-1}/F_{-1} and F_{n-1}'/F_{-1}, respectively. Hence, these groups are C-groups (see [18, Proposition 2B9]).

Observe that if $i \in \{0, 1, \ldots, n-3\}$ then $H_i = H_i^- \times H_i^+$ where $H_i^- = \langle \rho_j : j < i \rangle$ and $H_i^+ = \langle \rho_j : i < j \rangle$. Note that H_i^- is just the automorphism of the section F_i/F_{-1}, hence a C-group. If $i < n-3$ then H_i^+ is the halving group of the section F_n/F_i, which is a C-group by the inductive hypothesis. If $i = n-3$, then H_i^+ is isomorphic to a dihedral group $\mathbb{D}_s$. In any case, it follows from Lemma 2.4 that H_i is a C-group.

In order to use Proposition 2.3, we need to prove that for every pair $i, j \in \{0, \ldots, n-1\}$, with $i < j$, the equality

$$H_i \cap H_j = H_{i,j} \tag{3.6}$$

holds. We proceed in a similar way as in rank 3. If $\{i, j\} = \{n-1, n-2\}$, then observe that $H_i = H_{n-2}$ fixes F_{n-1} and $H_j = H_{n-1}$ fixes F_{n-1}'. This implies that an element $\gamma \in H_{n-2} \cap H_{n-1}$ must fix F_{n-2}. Thus $\gamma \in \operatorname{Stab}_{\mathcal{P}}(\{F_{n-2}, F_{n-1}\}) = \langle \rho_0, \ldots, \rho_{n-3} \rangle = H_{i,j}$. The other inclusion is obvious.

If $j \in \{n-1, n-2\}$ and $i \leqslant n-3$ then (3.6) follows from the fact that H_j is a string C-group.

Assume that $0 \leqslant i < j \leqslant n-3$. Let $\mathcal{F}$ be the section F_j/F_{-1} of $\mathcal{P}$. Let $\gamma \in H_i \cap H_j$. Observe that $H_j = H_j^- \times H_j^+$ and that $\operatorname{Aut}(\mathcal{F}) = H_j^-$. Let $\alpha \in H_j^-$ and $\beta \in H_j^+$ be such that $\gamma = \alpha\beta$. Note that β fixes the face F_i of $\mathcal{P}$ and since $\gamma \in H_i$ it follows that α must fix F_i. Since H_j^- is a string C-group, it follows that $\alpha \in \langle \rho_0, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_{j-1} \rangle$. Then

$$\gamma \in \langle \rho_0, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_{j-1} \rangle \times \langle \rho_{j+1}, \ldots, \rho_{n-1} \rangle = H_{i,j}.$$

The other inclusion is obvious. $\qquad\square$

The halving group of a regular polytope $\mathcal{P}$ is a C-group with Coxeter diagram (3.2). The groups generated by involutions with this diagram are called *tail-triangle groups* (even when $s = 2$). In [19, 20] Monson and Schulte show that when a tail-triangle group is a C-group, it is the automorphism group of an alternating semi-regular polytope. We denote by $\mathcal{S}(\mathcal{P})$ the semi-regular polytope obtained by the halving operation on $\mathcal{P}$. The polytope $\mathcal{S}(\mathcal{P})$ has two orbits of isomorphic regular facets, namely the right cosets of $\langle \rho_0, \ldots, \rho_{n-3}, \rho_{n-2} \rangle$ and $\langle \rho_0, \ldots, \rho_{n-3}, \rho_{n-1} \rangle$. These *base facets* are incident with a regular polytope $\mathcal{R}$ of rank $n-2$. In fact, every flag of $\mathcal{R}$ can be extended to a flag of $\mathcal{S}(\mathcal{P})$ in two different ways. Moreover, any flag of $\mathcal{S}(\mathcal{P})$ belongs to the orbit of one of these two flags. The automorphisms of $\mathcal{S}(\mathcal{P})$ can now be used to show flag transitivity of $\mathcal{H}(\mathcal{P})$.

Proposition 3.2. *The incidence system $\mathcal{H}(\mathcal{P})$ associated with a non-degenerate regular polytope $\mathcal{P}$ is flag-transitive.*

Proof. Let $J \subseteq \{0, \ldots, n-1\}$ and let $F = \{H_i h : i \in J\}$ for some $h \in \mathrm{H}(\mathcal{P})$ be a flag of $\mathcal{H}(\mathcal{P})$ of type J. If $|J \cap \{n-2, n-1\}| \leqslant 1$, then F is a chain of $\mathcal{S}(\mathcal{P})$ of type J. By [19, Lemma 4.5b], the group $\mathrm{H}(\mathcal{P})$ is transitive on the chains of this type.

If $\{n-2, n-1\} \subseteq J$, then $F = \Upsilon \cup \Upsilon^{n-1}$, where Υ is the chain of $\mathcal{S}(\mathcal{P})$ of type $J' = J \setminus \{n-1\}$ whose faces are contained in F. Again, [19, Lemma 4.5b] implies that $\mathrm{H}(\mathcal{P})$ is transitive in chains of type J'. Finally, observe that if $\beta \in \mathrm{H}(\mathcal{P})$, then $(\Upsilon\beta)^{n-1} = (\Upsilon^{n-1})\beta$. It follows that $\mathrm{H}(\mathcal{P})$ is also transitive on flags of type J. $\qquad\square$

Corollary 3.3. *Let $\mathcal{P}$ be a non-degenerate, regular n-polytope and $I = \{0, \ldots, n-1\}$. Let $\mathrm{H}(\mathcal{P})$ be the halving group of $\mathcal{P}$. Then the incidence system $\mathcal{H}(\mathcal{P}) = \Gamma\left(\mathrm{H}(\mathcal{P}), (H_i)_{i \in I}\right)$ is a regular hypertope such that $\mathrm{Aut}_I\left(\mathcal{H}(\mathcal{P})\right) = \mathrm{H}(\mathcal{P})$.*

The assumption that $\mathcal{P}$ is non-degenerate is very important. When the halving operation is applied on the dual of the 4-hemicube the resulting incidence system is not a hypertope (see [11, Example 3.3]).

Theorem 2.5 in [20] implies that the semi-regular polytope $\mathcal{S}(\mathcal{P})$ is regular because the associated C-group admits a group automorphism interchanging the generators ρ_{n-1} and ρ_{n-2} (this automorphism is given by conjugation by ϱ_{n-1}). The polytope $\mathcal{S}(\mathcal{P})$ is in fact isomorphic to $\mathcal{P}$.

4   Locally spherical hypertopes from generalised cubes.

A *spherical hypertope* is a universal regular hypertope whose Coxeter diagram is a union of diagrams of finite irreducible Coxeter groups. This definition is slightly different from that in [11]. A *locally spherical hypertope* is a regular hypertope whose all proper residues are spherical hypertopes. An irreducible regular hypertope is of *euclidean* (resp. *spherical*) *type* if the type-preserving automorphism group of its universal regular cover is an affine (resp. finite irreducible) Coxeter group. Similarly, an irreducible locally spherical hypertope is of *hyperbolic type* if the type-preserving automorphism group of its universal cover is a compact hyperbolic Coxeter group. It is well known that compact hyperbolic Coxeter groups exist only in ranks 3, 4 and 5.

In [11] the authors show that a locally spherical hypertope has to be of spherical, euclidean or hyperbolic type. The complete list of the Coxeter diagrams of these groups can be found in [11, Tables 1 and 2]. Whereas the first two classes are well understood, not much is known about the hyperbolic type. In particular, the authors were not successful in producing a finite example of rank 5 locally spherical proper hypertope of this type. In what follows we will use the halving operation on a certain class of polytopes first described by Danzer in [5] (see also [18, Theorem 8D2]) and in particular we produce an example of finite rank 5 proper regular hypertope of hyperbolic type.

We briefly review Danzer's construction of generalised cubes.

Let $\mathcal{K}$ be a regular finite non-degenerate n-polytope with vertex set $V = \{v_1, \ldots, v_m\}$. Consider the set

$$2^V = \prod_{j=1}^{m} \{0, 1\} = \{\bar{\mathrm{x}} = (x_1, \ldots, x_m) : x_j \in \{0, 1\}\}.$$

Given an i-face F of $\mathcal{K}$ and $\bar{\mathrm{x}} \in 2^V$ define the sets

$$F(\bar{\mathrm{x}}) = \left\{ \bar{y} = (y_1, \ldots, y_m) \in 2^V : y_j = x_j \text{ if } v_j \nleqslant F \right\}.$$

Then the polytope $2^{\mathcal{K}}$ is the set

$$\{\emptyset\} \cup 2^V \cup \left\{ F(\bar{\mathrm{x}}) : F \in \mathcal{K}, \bar{\mathrm{x}} \in 2^V \right\}$$

ordered by inclusion. The improper face of rank -1 of $2^{\mathcal{K}}$ is $\emptyset$ and if $i \geqslant 0$ the i-faces of $2^{\mathcal{K}}$ are the sets $F(\bar{\mathrm{x}})$ for F a certain $(i-1)$-face of $\mathcal{K}$ and some $\bar{\mathrm{x}} \in 2^V$. Note that $F_{-1}(\bar{\mathrm{x}}) = \{\bar{\mathrm{x}}\}$ for every $\bar{\mathrm{x}} \in 2^V$, hence $2^{\mathcal{K}}$ has $2^{|V|}$ vertices.

If $\mathcal{K}$ is a regular of type $\{p_1, \ldots, p_{n-2}\}$ then $2^{\mathcal{K}}$ is a regular polytope of type $\{4, p_1, \ldots, p_{n-2}\}$. In fact, all the vertex figures of $2^{\mathcal{K}}$ are isomorphic to $\mathcal{K}$. The polytope $2^{\mathcal{K}}$ is called a *generalised cube* since when $\mathcal{K}$ is the $(n-1)$-simplex, the polytope $2^{\mathcal{K}}$ is isomorphic to the n-cube.

For our purposes it is convenient to denote by $\hat{2}^{\mathcal{K}}$ the polytope $\left(2^{\mathcal{K}^*}\right)^*$, so that $\hat{2}^{\mathcal{K}}$ is a regular polytope of type $\{p_1, \ldots, p_{n-2}, 4\}$ whose facets are isomorphic to $\mathcal{K}$.

The automorphism group of $\hat{2}^{\mathcal{K}}$ is isomorphic to $\mathbb{Z}_2^m \rtimes \operatorname{Aut}(\mathcal{K})$, where m denotes the number of facets of $\mathcal{K}$ and the action of $\operatorname{Aut}(\mathcal{K})$ on $\mathbb{Z}_2^m$ is given by permuting coordinates in the natural way. In particular, the size of this group is $2^m \times |\operatorname{Aut}(\mathcal{K})|$ (see [18, Theorem 2C5] and [23]).

Remark 4.1. In [23] Pellicer generalises Danzer's construction of $2^{\mathcal{K}}$. Given a finite non-degenerate regular $(n-1)$-polytope $\mathcal{K}$ of type $\{p_1, \ldots, p_{n-2}\}$ and $s \in \mathbb{N}$, Pellicer's construction gives as a result an n-polytope $\mathcal{P}_s$ of type $\{p_1, \ldots p_{n-2}, 2s\}$. If $s = 2$, the polytope $\mathcal{P}_2$ is isomorphic to $\hat{2}^{\mathcal{K}}$. However, when our construction is applied to the polytopes $\mathcal{P}_s$ for $s \geqslant 3$, the resulting hypertopes are not locally spherical and therefore not included in this paper.

Now we discuss the locally spherical hypertopes resulting from applying the halving operation to the polytopes obtained from Danzer's construction. Since $\hat{2}^{\mathcal{K}}$ is of type $\{p_1, \ldots p_{n-2}, 4\}$, the hypertope $\mathcal{H}(\hat{2}^{\mathcal{K}})$ has the following Coxeter diagram:
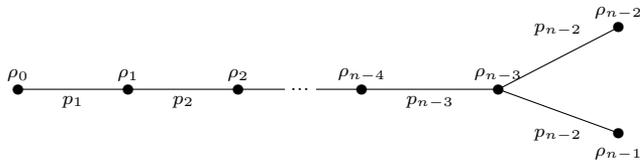
We naturally extend the Schläfli symbol and say that $\mathcal{H}(\hat{2}^{\mathcal{K}})$ is of type $\{p_{n-1}, \ldots, p_{n-3}, \overset{p_{n-2}}{p_{n-2}}\}$.

In rank 3 the polytope $\hat{2}^{\{p\}}$ is obtained by applying the construction on a regular polygon $\{p\}$ and the induced hypertope is in fact a self-dual polyhedron of type $\{p, p\}$. This polyhedron has 2^{p-1} vertices, $2^{p-2}p$ edges and 2^{p-1} faces and it is a map on a surface of genus $2^{p-3}(p-4) + 1$. For $p = 3$ the resulting hypertope is a spherical polyhedron $\{3, 3\}$, i.e. the tetrahedron. When $p = 4$ the polytope $\hat{2}^{\{4\}}$ is the toroid $\{4, 4\}_{(4,0)}$ and the induced hypertope is also of euclidean type, more precisely, it is the toroid $\{4, 4\}_{(2,2)}$.

To obtain locally spherical hypertopes in rank 4, $\mathcal{K}$ must be of type $\{p, 3\}$ with $p = 3, 4, 5$. The resulting hypertopes are of spherical, euclidean, and hyperbolic type, respectively. If $p = 3$, the hypertope $\mathcal{H}(\hat{2}^{\mathcal{K}})$ is the universal hypertope of Coxeter diagram D_4

and type $\left\{3, \frac{3}{3}\right\}$. When $p = 4$ the polytope $\hat{2}^{\mathcal{K}}$ is the toroid $\{4, 3, 4\}_{(4,0,0)}$ and $\mathcal{H}(\hat{2}^{\mathcal{K}})$ is a toroidal hypertope described by Ens in [6, Theorem 4.3], which we denote by $\left\{4, \frac{3}{3}\right\}_{(4,0,0)}$. The automorphism group of this hypertope has Coxeter diagram $\tilde{B}_3$. If $p = 5$, the resulting hypertope is of type $\left\{5, \frac{3}{3}\right\}$ with automorphism group of size $2^{11} \times 120 = 245,760$. This example is different from any of the examples listed in [11].

In rank 5 the polytope $\mathcal{K}$ must be of type $\{p, 3, 3\}$ with $p = 3, 4, 5$, the resulting hypertopes are of spherical, euclidean and hyperbolic type, respectively. If $p = 3$ then the hypertopes is the universal spherical hypertope of type $\left\{3, 3, \frac{3}{3}\right\}$, i.e. the universal hypertope of Coxeter diagram D_5. If $p = 4$ the polytope $\hat{2}^{\mathcal{K}}$ is the regular toroid $\{4, 3, 3, 4\}_{(4,0,0,0)}$. The induced hypertope is of euclidean type, hence a toroidal hypertope which we denote by $\left\{4, 3, \frac{3}{3}\right\}_{(4,0,0,0)}$. The Coxeter diagram of its automorphism group is $\tilde{B}_4$. For $p = 5$ the regular polytope $\hat{2}^{\mathcal{K}}$ is constructed from the 120-cell. The hypertope $\mathcal{H}(\hat{2}^{\mathcal{K}})$ is of type $\left\{5, 3, \frac{3}{3}\right\}$ and its automorphism group has size $2^{119} \times 14400$. It is not surprising that the authors of [11] could not find this example using a computational approach.

For rank $n \geqslant 6$ we can only obtain locally spherical hypertopes from our construction if $\mathcal{K}$ is the $(n-1)$-simplex $\{3^{n-2}\}$ or the $(n-1)$-cube $\{4, 3^{n-3}\}$. The polytope $\hat{2}^{\mathcal{K}}$ is the n-cross-polytope $\{3^{n-2}, 4\}$ or the toroid $\{4, 3^{n-3}, 4\}_{(4,0,\ldots 0)}$, respectively. In the former case the resulting hypertope is the universal spherical hypertope of type $\left\{3^{n-3}, \frac{3}{3}\right\}$ associated with the Coxeter diagram D_n while in the latter it is a toroidal hypertope associated with the Coxeter diagram $\tilde{B}_{n-1}$ which we denote by $\left\{4, 3^{n-4}, \frac{3}{3}\right\}_{(4,0,\ldots 0)}$.

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