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Classification of regular balanced Cayley maps of minimal non-abelian metacyclic groups*

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Abstract

In this paper, we classify the regular balanced Cayley maps of minimal non-abelian metacyclic groups. Besides the quaternion group Q_8, there are two infinite families of such groups which are denoted by $M_{p,q}(m,r)$ and $M_p(n,m)$, respectively. Firstly, we prove that there are regular balanced Cayley maps of $M_{p,q}(m,r)$ if and only if $q = 2$ and we list all of them up to isomorphism. Secondly, we prove that there are regular balanced Cayley maps of $M_p(n,m)$ if and only if $p = 2$ and $n = m$ or $n = m + 1$ and there is exactly one such map up to isomorphism in either case. Finally, as a corollary, we prove that any metacyclic p-group for odd prime number p does not have regular balanced Cayley maps.

Keywords: Regular balanced Cayley map, minimal non-abelian group, metacyclic group.

Math. Subj. Class.: 05C25, 05C30

1 Introduction

A *Cayley graph* $\Gamma = \mathrm{Cay}(G, X)$ is a graph based on a group G and a finite set $X = \{x_1, x_2, \dots, x_k\}$ of elements in G which does not contain 1_G, contains no repeated elements, is closed under the operation of taking inverses, and generates all of G. In this

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paper, we call X a *Cayley subset* of G. The vertices of the Cayley graph Γ are the elements of G, and two vertices g and h are adjacent if and only if $g = hx_i$ for some $x_i \in X$. The ordered pairs (h, hx) for $h \in G$ and $x \in X$ are called the darts of Γ. Let ρ be any cyclic permutation on X. Then the Cayley map $\mathcal{M} = \mathrm{CM}(G, X, \rho)$ is the 2-cell embedding of the Cayley graph $\mathrm{Cay}(G, X)$ in an orientable surface for which the orientation-induced local ordering of the darts emanating from any vertex $g \in G$ is always the same as the ordering of generators in X induced by ρ; that is, the neighbors of any vertex g are always spread counterclockwise around g in the order $(gx, g\rho(x), g\rho^2(x), \ldots, g\rho^{k-1}(x))$.

An (orientation preserving) *automorphism* of a Cayley map $\mathcal{M}$ is a permutation on the dart set of $\mathcal{M}$ which preserves the incidence relation of the vertices, edges, faces, and the orientation of the map. The full automorphism group of $\mathcal{M}$, denoted by $\mathrm{Aut}(\mathcal{M})$, is the group of all such automorphisms of $\mathcal{M}$ under the operation of composition. This group always acts semi-regularly on the set of darts of $\mathcal{M}$, that is, the stabilizer in $\mathrm{Aut}(\mathcal{M})$ of each dart of $\mathcal{M}$ is trivial. If this action is transitive, then we say that the Cayley map $\mathcal{M}$ is a regular Cayley map. As the left regular multiplication action of the underlying group G lifts naturally into the full automorphism group of any Cayley map $\mathrm{CM}(G, X, \rho)$, Cayley maps are proved to be a very good source of regular maps. There are many papers on the topic of regular Cayley maps, we refer the readers to [4, 10] and [11] and the references therein. Furthermore, A Cayley map $\mathrm{CM}(G, X, \rho)$ is called *balanced* if $\rho(x)^{-1} = \rho(x^{-1})$ for every $x \in X$. In [11], Škoviera and Širáň showed that a Cayley map $\mathrm{CM}(G, X, \rho)$ is regular and balanced if and only if there exists a group automorphism σ such that $\sigma|_X = \rho$, where $\sigma|_X$ denotes the restricted action of σ on X. Therefore, to determine all the regular balanced Cayley maps of a group is equivalent to determine all the orbits of its automorphisms that can be Cayley subsets.

In this paper, a non-abelian group G is called minimal if each of its proper subgroups H (that is $H < G$ but $H \neq G$) is abelian. In 1903, Miller and Moreno gave a full classification of minimal non-abelian groups, one may refer to [7] for detailed results. A group G is metacyclic if it has a cyclic normal subgroup N such that the factor group G/N is cyclic. As one can see in [7], there are three classes of minimal non-abelian metacyclic groups:

(1) the quaternion group Q_8;

(2) $M_{p,q}(m, r) = \langle a, b \mid a^p = 1, b^{q^m} = 1, b^{-1}ab = a^r \rangle$, where p and q are distinct prime numbers, m is a positive integer and $r \not\equiv 1 \pmod p$ but $r^q \equiv 1 \pmod p$;

(3) $M_p(n, m) = \langle a, b \mid a^{p^n} = b^{p^m} = 1, b^{-1}ab = a^{1+p^{n-1}}, n \geq 2, m \geq 1 \rangle$.

One can also cite [3, Theorem 2.1] for reference or [13, pp. 123] for details.

For regular balanced Cayley maps, it has been shown that all odd order abelian groups possess at least one regular balanced Cayley map [4]. Wang and Feng [12] classified all regular balanced Cayley maps for cyclic, dihedral and generalized quaternion groups. In [9], Oh proved the non-existence of regular balanced Cayley maps with semi-dihedral groups. In this paper, we pay our attentions to the regular balanced Cayley maps of minimal non-abelian metacyclic groups. Since the regular balanced Cayley maps of Q_8 have been classified in [12] (Q_8 has exactly one regular balanced Cayley map up to isomorphism), we only consider the groups $M_{p,q}(m, r)$ and $M_p(n, m)$. In Section 3, we show that $M_{p,q}(m, r)$ has regular balanced Cayley maps if and only if q is 2 and we list all of them up to isomorphism. In Section 4, we show that $M_p(n, m)$ has regular balanced Cayley maps if and only if $p = 2$ and $n = m$ or $n = m + 1$. In either case, it has exactly one regular balanced Cayley map up to isomorphism and the map has valency 4. Moreover, as a corollary any metacyclic p-group for odd prime p doesn't have regular balanced Cayley maps.

2 Preliminaries

Lemma 2.1. *Take an element $b^t a^s \in M_{p,q}(m, r)$, where $t \neq 0$, then the order of $b^t a^s$ is q^m if and only if $(t, q) = 1$.*

Proof. The group $M_{p,q}(m, r)$ is the union of one cyclic group of order p and p conjugate cyclic subgroups of order q^m. If $t \neq 0$, then $b^t a^s$ belongs to one of the cyclic subgroups of order q^m. Therefore, the order of $b^t a^s$ is q^m if and only if $(t, q) = 1$. □

Lemma 2.2. *The automorphism group of $M_{p,q}(m, r)$ is*

$$\mathrm{Aut}(M_{p,q}(m,r)) = \{\sigma \mid a^\sigma = a^i,\ b^\sigma = b^j a^k, 1 \leqslant i \leqslant p-1,\ 1 \leqslant j \leqslant q^m-1, q \mid (j-1)\}.$$

Proof. Assume $\sigma \in \mathrm{Aut}(M_{p,q}(m, r))$. According to Lemma 2.1, $a^\sigma = a^i, b^\sigma = b^j a^k$ for some $1 \leqslant i \leqslant p - 1, 1 \leqslant j \leqslant q^m - 1$ and $(j, q) = 1$. If $M_{p,q}(m, r) = \langle a^\sigma, b^\sigma\rangle$, then we can get the relation $q \mid (j - 1)$.

In fact, since $(a^r)^\sigma = (b^{-1}ab)^\sigma = (b^{-1})^\sigma a^\sigma b^\sigma = b^{-j} a^i b^j = a^{ir^j} = a^{ir}$, we have $a^{ir(r^{j-1}-1)} = 1$. Moreover, from $(ir, p) = 1$ and $a^p = 1$, we get $(r^{j-1} - 1) \equiv 0 \pmod p$, that is $r^{j-1} \equiv 1 \pmod p$. As $r^q \equiv 1 \pmod p$ and q is prime, we have $q \mid (j - 1)$. □

Lemma 2.3 ([5])**.** *The automorphism group of $M_p(n, m)$ is listed as follows:*

(i) If $n \leq m$, then $\mathrm{Aut}(M_p(n, m)) = \{\sigma \mid a^\sigma = b^j a^i, b^\sigma = b^s a^r, (i, p) = 1, 1 \leq i \leq p^n, j = kp^{m-n+1}, 0 \leq k < p^{n-1}, 1 \leq r \leq p^n, s \equiv 1 \pmod p, 1 \leq s \leq p^m\}$.

(ii) If p is odd and $n > m \geq 1$ or $p = 2$ and $n > m > 1$, then $\mathrm{Aut}(M_p(n, m)) = \{\sigma \mid a^\sigma = b^j a^i, b^\sigma = b^s a^r, (i, p) = 1, 1 \leq i \leq p^n, 1 \leq j \leq p^m, r = kp^{n-m}, 0 \leq k < p^m, s \equiv 1 \pmod p, 1 \leq s \leq p^m\}$.

The following Lemma 2.4 is a basic result in group theory and we omit the proof.

Lemma 2.4. *Let G be a finite group and N be a normal subgroup of G. Take $\alpha \in \mathrm{Aut}(G)$. If $N^\alpha = N$, then $\bar\alpha : Ng \mapsto Ng^\alpha$ is an automorphism of G/N which is called the induced automorphism of α.*

Lemma 2.5. *Let G be a finite group and N be a proper characteristic subgroup of G. Take $\alpha \in \mathrm{Aut}(G)$ and $g \in G$. If $X = g^{\langle\alpha\rangle}$ is a Cayley subset of G, then $\overline{X} = \overline{g^{\langle\alpha\rangle}} = \bar g^{\langle\bar\alpha\rangle}$ is a Cayley subset of $\overline{G} = G/N$. Moreover, if the order of α is a power of 2 and $\bar g$ is not an involution, then $|X| = |\overline{X}|$.*

Proof. By Lemma 2.4, $\bar\alpha$ is an automorphism of G/N induced by α. Set $X = g^{\langle\alpha\rangle}$, then $\overline{X} = \overline{g^{\langle\alpha\rangle}} = \bar g^{\langle\bar\alpha\rangle}$. If X is a Cayley subset of G, then the relations $\langle\overline{X}\rangle = \overline{G}$, $\overline{X} = \overline{X}^{-1}$ follow naturally. Since $N < G$, we have $\overline{X} \neq \bar 1^{\langle\bar\alpha\rangle}$ and then $\bar 1 \notin \overline{X}$. So, $\overline{X}$ is a Cayley subset of $\overline{G}$.

If the order of α is 2^s for some positive integer s, then the order of $\overline{\alpha}$ is 2^t for some integer $t \leq s$. From $g^{\alpha^{2^{s-1}}} = g^{-1}$, we have $\bar g^{\overline{\alpha}^{2^{s-1}}} = \bar g^{-1}$. While $\bar g^{\overline{\alpha}^{2^t}} = \bar g$, then $t > s - 1$. So, $s = t$ and $|X| = |\overline{X}|$. □

As a direct corollary of Lemmas 2.4 and 2.5, we give the following Corollary 2.6.

Corollary 2.6. *If a group G has regular balanced Cayley maps, then so does the quotient group G/N for any proper characteristic subgroup N of G.*

There are many ways to get proper characteristic subgroups. In the following, we give a method to get such subgroups. These results are exercises for students, so we omit the proof.

Lemma 2.7. *Let G be a finite group, $S \subseteq G$, $\sigma \in \mathrm{End}(G)$, K be a characteristic subgroup of G and n be a positive integer. Then,*

(i) $\langle S \rangle^{\sigma} = \langle S^{\sigma} \rangle$;

(ii) $H_1 = \langle x^n \mid x \in K \rangle$ *is a characteristic subgroup of G;*

(iii) $H_2 = \langle y \mid y \in G, y^n \in K \rangle$ *is a characteristic subgroup of G.*

As for isomorphism of regular maps, one may refer to [10] for the following Lemma 2.8.

Lemma 2.8. *Assume $M_1 = \mathrm{CM}(G, X_1, \rho_1)$ and $M_2 = \mathrm{CM}(G, X_2, \rho_2)$ are two regular balanced Cayley maps of the finite group G, where $X_1 = g^{\langle \sigma_1 \rangle}$ and $X_2 = h^{\langle \sigma_2 \rangle}$ are orbits of two group elements g and h under the action of two automorphisms σ_1 and σ_2 of G, respectively. Then M_1 and M_2 are isomorphic if and only if $|X_1| = |X_2| = k$ and there is some $\tau \in \mathrm{Aut}(G)$ such that $h^{\sigma_2^i} = g^{\sigma_1^i \tau}$, $1 \le i \le k$.*

As a special case and an application of Lemma 2.8, we have the following Lemma 2.9.

Lemma 2.9. *Let G be a finite group. Take $\alpha \in \mathrm{Aut}(G)$ and two elements $g, h \in G$. Assume $X = g^{\langle \alpha \rangle}$ is a Cayley subset of G. If there is some $\sigma \in \mathrm{Aut}(G)$ such that $g^{\sigma} = h$, then $Y = h^{\langle \sigma^{-1} \alpha \sigma \rangle}$ is also a Cayley subset of G and $Y = X^{\sigma}$. Under this situation, the two regular balanced Cayley maps $\mathrm{CM}(G, X, \alpha|_X)$ and $\mathrm{CM}(G, Y, \sigma^{-1} \alpha \sigma|_Y)$ are isomorphic.*

Proof. Because $Y = h^{\langle \sigma^{-1} \alpha \sigma \rangle} = g^{\sigma \langle \sigma^{-1} \alpha \sigma \rangle} = g^{\sigma \sigma^{-1} \langle \alpha \rangle \sigma} = g^{\langle \alpha \rangle \sigma} = X^{\sigma}$ and X is a Cayley subset, it follows that Y is also a Cayley subset. The result that $\mathrm{CM}(G, X, \alpha|_X)$ and $\mathrm{CM}(G, Y, \sigma^{-1} \alpha \sigma|_Y)$ are isomorphic follows from Lemma 2.8. □

3 Regular balanced Cayley maps of $M_{p,q}(m, r)$

As we mentioned in the introduction, to determine all the regular balanced Cayley maps of a group is equivalent to determine all the orbits of its automorphisms that can be Cayley subsets. In this section, we divide our discussion into two parts according to the parity of q.

Lemma 3.1. *The center $Z(M_{p,q}(m, r))$ of $M_{p,q}(m, r)$ is generated by b^q and the quotient group $M_{p,q}(m, r)/Z(M_{p,q}(m, r)) \cong M_{p,q}(1, r)$.*

Proof. From the defining relation of $M_{p,q}(m, r)$, we have $b^{-q} a b^q = a^{r^q} = a$. So, $b^q \in Z(M_{p,q}(m, r))$. Since $M_{p,q}(m, r)$ is not abelian and generated by a and b, we have $a, b \notin Z(M_{p,q}(m, r))$, hence $Z(M_{p,q}(m, r)) = \langle b^q \rangle$. The formula $M_{p,q}(m, r)/Z(M_{p,q}(m, r)) \cong M_{p,q}(1, r)$ follows directly from the definition of $M_{p,q}(m, r)$. □

Theorem 3.2. *If q is odd, then the group $M_{p,q}(1, r)$ does not have regular balanced Cayley maps.*

Proof. For brevity, set $H = M_{p,q}(1, r)$. Suppose there exists a $\sigma \in \mathrm{Aut}(H)$ and $b^v a^u \in H$ such that $X = (b^v a^u)^{\langle\sigma\rangle}$ is a Cayley subset of H. The derived subgroup of H is $H' = \langle a \rangle$ which is a characteristic subgroup. Let $\overline{H} = H/H'$ and $\overline{\sigma}$ be induced by σ. By Lemma 2.2, $b^\sigma = ba^k$ for some integer k and as a result $\overline{b}^{\overline{\sigma}} = \overline{b}$. So, $\overline{X} = \overline{b^v a^u}^{\langle\overline{\sigma}\rangle} = \overline{b^v}^{\langle\overline{\sigma}\rangle} = \{\overline{b^v}\}$. While $\overline{X} = \overline{X}^{-1}$, $o(b) = q$ and $o(\overline{b^v}) \mid o(b)$, we have $\overline{b^v} = \overline{1}$ and so $b^v \in H'$. It follows that $\langle X \rangle \leq H' < H$ contradicting to $H = \langle X \rangle$. □

As a corollary of Lemmas 3.1 and 2.5, we have the following Theorem 3.3.

Theorem 3.3. *If q is odd, then $M_{p,q}(m, r)$ does not have regular balanced Cayley maps.*

It is known that $\mathbb{Z}_{2^n}^* \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} = \langle \overline{-1} \rangle \times \langle \overline{5} \rangle$, where $\overline{-1}$ and $\overline{5}$ denote the class of integers equaling to -1 and 5 modular 2^n, respectively. In a p-group G, let $\mho_1(G) = \langle a^p \mid a \in G \rangle$. Then, $\mho_1(\mathbb{Z}_{2^n}^*) = \langle \overline{5}^2 \rangle$ which does not contain $\overline{-1}$.

Lemma 3.4. *For a positive integer $n \geq 2$, the equation $x^k \equiv -1 \pmod{2^n}$ holds if and only if k is odd and $x \equiv -1 \pmod{2^n}$.*

Proof. It is obviously true when $n = 2$. So, we may assume $n \geq 3$. Let u be a solution of the equation $x^k \equiv -1 \pmod{2^n}$, then the integer u should be odd, so $\overline{u} \in \mathbb{Z}_{2^n}^* = \langle \overline{-1} \rangle \times \langle \overline{5} \rangle$. From the discussion preceding to the lemma, suppose k is even, then $\overline{-1} \equiv \overline{u}^k = (\overline{u}^{\frac{k}{2}})^2 \in \mho_1(\mathbb{Z}_{2^n}^*)$, a contradiction. So, k is odd.

Let $\overline{u} = \overline{a}\overline{b}$ for some $\overline{a} \in \langle \overline{-1} \rangle$ and $\overline{b} \in \langle \overline{5} \rangle$ such that $\overline{u^k} = \overline{-1}$. Then, $\overline{u^k} = \overline{a^k b^k} = \overline{-1}$. There are two choices of $\overline{a}$, that is $\overline{1}$ and $\overline{-1}$. But $\overline{a} \neq \overline{1}$, for otherwise $\overline{b^k} = \overline{-1}$, a contradiction. So, $\overline{b^k} = \overline{1}$ and as a result $\overline{b} = \overline{1}$ and $\overline{u} = \overline{-1}$. □

In a group G, for any element $g \in G$, we use $o(g)$ to denote the order of g. Now we look at the group $M_{p,2}(m, r)$. In the definition of $M_{p,2}(m, r)$, one can see that $r \equiv -1 \pmod{p}$. In particular, if $m = 1$, then $M_{p,2}(m, r)$ is a dihedral group of order $2p$. One may refer to [12] for the classification of the regular balanced Cayley maps of dihedral groups. For the sake of completeness, We restate the result in the following theorem.

Theorem 3.5 ([12, Theorem 3.3]). *The dihedral group D_{2p} of order $2p$ has $p - 1$ non-isomorphic regular balanced Cayley maps, where p is an odd prime number.*

When $m \geq 2$, we have the following Theorem 3.6.

Theorem 3.6. *Let $G = M_{p,2}(m, r)$, where $m \geq 2$, p is an odd prime and $r \equiv -1 \pmod{p}$. If $p - 1 = 2^e s$, where s is odd, then G has s non-isomorphic regular balanced Cayley maps. In particular, if p is a Fermat prime, then G has exactly one regular balanced Cayley map up to isomorphism.*

Proof. If the orbit of $b^v a^u$ under the action of $\sigma \in \mathrm{Aut}(G)$ is a Cayley subset of G, then the integer v must be odd. In fact, both the subgroups $\langle a \rangle$ and $Z(G) = \langle b^2 \rangle$ are characteristic in G, so $\langle (b^v a^u)^{\langle\sigma\rangle} \rangle$ is a proper subgroup of G if $(v, 2) \neq 1$. By Lemma 2.2, there is some $\alpha \in \mathrm{Aut}(G)$ such that $(b^v a^u)^\alpha = b$. According to Lemma 2.9, we only need to consider the orbit of b under the action of σ.

For brevity, we denote the automorphism $\sigma \in \mathrm{Aut}(G)$ satisfying $a^\sigma = a^i$ and $b^\sigma = b^j a^k$ by $\sigma_{i,j,k}$ and $X = b^{\langle\sigma_{i,j,k}\rangle}$ by $X_{i,j,k}$. Let $\rho_{i,j,k}$ be the arrangement of the elements in $X_{i,j,k}$ which respects the order of the elements in the orbit. Assume $X_{i,j,k}$ is a Cayley subset of G for some integer i coprime to p and odd integer j. Note that $k \not\equiv 0 \pmod p$ for otherwise contradicting to the Cayley subset assumption of $X_{i,j,k}$.

In the quotient group $\overline{G} = G/\langle a\rangle$, $\overline{X_{i,j,k}} = \overline{b}^{\langle \overline{\sigma_{i,j,k}}\rangle}$ should be a Cayley subset of $\overline{G}$. Therefore, there exists some integer t such that $\overline{b}^{\overline{\sigma_{i,j,k}}^t} = \overline{b}^{-1}$. Clearly, $\overline{b}^{\overline{\sigma_{i,j,k}}^t} = \overline{b}^{j^t} = \overline{b}^{-1}$, so $j^t \equiv -1 \pmod{2^m}$. From Lemma 3.4, t is odd and $j \equiv -1 \pmod{2^m}$. Moreover, as $X_{i,-1,1}^{\sigma_{k,1,0}} = X_{i,-1,k}$, we may assume $k = 1$. Under these conditions, we only need to pay attention to $X_{i,-1,1}$. By direct enumeration one can easily get

$$b^{\sigma_{i,-1,1}^{\ell}} = b^{(-1)^{\ell}} a^{i^{\ell-1}+i^{\ell-2}+\cdots+i+1},$$

for any positive integer ℓ. Since $X_{i,-1,1}$ is a Cayley subset, there exists some positive integer n such that $b^{\sigma_{i,-1,1}^{n}} = b^{-1}$. So, n is odd and

$$i^{n-1} + i^{n-2} + \cdots + i + 1 \equiv 0 \pmod p.$$

If $i \equiv 1 \pmod p$, then $b^{\sigma_{1,-1,1}^{p}} = b^{-1}$ and

$$X_{1,-1,1} = \{b, b^{-1}a, ba^2, \ldots, ba^{p-1}, b^{-1}, (b^{-1}a)^{-1}, \ldots, (ba^{p-1})^{-1}\}$$

is a Cayley subset of G of valency $2p$.

If $1 < i \le p-1$, then $i^{n-1} + i^{n-2} + \cdots + i + 1 \equiv 0 \pmod p$ if and only if $i^n \equiv 1 \pmod p$. Let $S = \{x \mid x \in \mathbb{Z}_p^*, o(x) \text{ is odd}\}$, then $|S| = s$. Since n is odd, any i satisfying $i^n \equiv 1 \pmod p$ corresponds to $\overline{i} \in S$. And for any $\overline{i} \in S \setminus \{1\}$, if $o(\overline{i}) = n$, then $b^{\sigma_{i,-1,1}^{n}} = b^{-1}$ and

$$X_{i,-1,1} = \{b, b^{-1}a, ba^{i+1}, b^{-1}a^{i^2+i+1}, \ldots, ba^{i^{n-2}+\cdots+i+1}, b^{-1}, \ldots, (ba^{i^{n-2}+\cdots+i+1})^{-1}\}$$

is a Cayley subset of G of valency $2n$. From all the above, when $i > 1$, $X_{i,-1,1}$ is a Cayley subset of G if and only if $\overline{i} \in S$ and $|X_{i,-1,1}|$ is twice of $o(\overline{i})$.

For any two distinct $\overline{i_1}$ and $\overline{i_2}$ in $S \setminus \{1\}$, Cayley maps $\mathrm{CM}(G, X_{i_1,-1,1}, \rho_{i_1,-1,1})$ and $\mathrm{CM}(G, X_{i_2,-1,1}, \rho_{i_2,-1,1})$ are not isomorphic. Otherwise, according to Lemma 2.8, there exists some $\beta \in \mathrm{Aut}(G)$ such that $b^{\beta} = b$ and for each $\ell \ge 1$,

$$(b^{(-1)^{\ell}} a^{i_1^{\ell-1}+i_1^{\ell-2}+\cdots+i_1+1})^{\beta} = b^{(-1)^{\ell}} a^{i_2^{\ell-1}+i_2^{\ell-2}+\cdots+i_2+1}.$$

In particular, $(b^{-1}a)^{\beta} = b^{-1}a$ and therefore β is the identical automorphism. Therefore, G has s non-isomorphic regular balanced Cayley maps. When p is a Fermat prime, then $p-1$ is a power of 2, so G has exactly one regular balanced Cayley map up to isomorphism. □

4 Regular balanced Cayley maps of $M_p(n, m)$

For minimal non-abelian p-group, one may refter to [1, 2] or [14] for the following Lemma 4.1.

Lemma 4.1 ([14, Theorem 2.3.6]). *Let G be a finite p-group, $d(G)$ be the number of elements in a minimal generating subset of G. Then, the followings are equivalent.*

(i) The group G is a minimal non-abelian group;

(ii) $d(G) = 2$ and $|G'| = p$;

(iii) $d(G) = 2$ *and* $Z(G) = \Phi(G)$, *where* $\Phi(G)$ *denotes the Frattini subgroup of* G.

Lemma 4.2. *Assume* G *is a finite* p*-group for some prime number* p *and* $d(G) = 2$. *Let* $\beta \in \mathrm{Aut}(G)$, $g \in G$ *and* $X = g^{\langle\beta\rangle}$. *If* $G = \langle X\rangle$, *then* $G = \langle g,\ g^{\beta}\rangle$.

Proof. Because $d(G) = 2$, it follows that $\overline{G} = G/\Phi(G) \cong Z_p \times Z_p$. Suppose $\langle g,\ g^{\beta}\rangle < G$, then in the quotient group the subgroup generated by g and g^{β} has order p, that is $|\overline{\langle g,\ g^{\beta}\rangle}| = p$. So, $g^{\beta} \in \langle g\Phi(G)\rangle$. As $\Phi(G)$ is a characteristic subgroup of G, for each $k > 1$ the element $g^{\beta^k} \in \langle g^{\beta^{k-1}}\Phi(G)\rangle$. Therefore, $X \subseteq \langle g\Phi(G)\rangle$ and then $\langle X\rangle \leq \langle g\Phi(G)\rangle < G$, a contradiction. So, $G = \langle g,\ g^{\beta}\rangle$. □

Remark Lemma 4.2 may not be true for a non-p-group. For example, the symmetry group S_n can be generated by two elements $(1\ 2)$ and $(1\ 2\ldots n)$. Take $g = (1\ 2) \in S_n$ and β the automorphism of S_n induced from the conjugation by the element $(2\ 3\ldots n)$, then $X = g^{\langle\beta\rangle} = \{(1\ 2), (1\ 3), \ldots, (1\ n)\}$ is a Cayley subset of S_n and $g^{\beta} = (1\ 3)$. But it is obvious that $S_n \neq \langle(1\ 2), (1\ 3)\rangle$ when $n \geq 4$.

Theorem 4.3. *Let* $G = M_p(n, n)$, *where* $n \geq 2$ *and* p *is an odd prime number. Then, the group* G *does not have regular balanced Cayley maps.*

Proof. Let $N = \langle x \in G \mid x^{p^{n-1}} \in G'\rangle$. According to Lemma 2.7, N is a characteristic subgroup of G. One can see from the defining relations of G that $G' = \langle a^{p^{n-1}}\rangle \cong \mathbb{Z}_p$ and $N = \langle a, b^p\rangle$. Take $\sigma \in \mathrm{Aut}(G)$ such that $a^{\sigma} = b^{kp}a^i$ and $b^{\sigma} = b^s a^r$, where the integers i, s, r satisfy the conditions in Lemma 2.3 and especially $s \equiv 1 \pmod p$. Suppose $X = (b^u a^v)^{\langle\sigma\rangle}$ is a Cayley subset of G. Then $b^u a^v \notin N$ and therefore $(u, p) = 1$. In the quotient group $\overline{G} = G/N$, $\overline{X} = \overline{(b^u a^v)^{\langle\sigma\rangle}} = \overline{b^u}^{\langle\bar{\sigma}\rangle}$ is a Cayley subset of $\overline{G}$. So, there exists some integer n such that $\overline{b^{-u}} = \overline{b^{s^n u}}$. As a result, one can get $s^n u \equiv -u \pmod p$. While $(u, p) = 1$, then $s^n \equiv -1 \pmod p$. But this result contradicts to $s \equiv 1 \pmod p$. □

Theorem 4.4. *Let* $G = M_p(n, m)$, *where* $n \geq 2, m \geq 1, m \neq n$ *and* p *is an odd prime number. Then, the group* G *does not have regular balanced Cayley maps.*

Proof. We firstly assume $m > n$. Set $N = \{x^{p^n} \mid x \in G\}$. By Lemma 2.7, $N = \langle b^{p^n}\rangle$ is a characteristic subgroup of G. The quotient group

$$\overline{G} = G/N = \langle \bar{a}, \bar{b} \mid \bar{a}^{p^n} = \bar{b}^{p^n} = 1, \bar{a}^{\bar{b}} = \bar{a}^{1+p^{n-1}}\rangle \cong M_p(n, n).$$

According to Theorem 4.3 and Lemma 2.5, G does not have regular balanced Cayley maps.

When $m < n$, suppose there exists some $\sigma \in \mathrm{Aut}(G)$ such that $X = (b^u a^v)^{\langle\sigma\rangle}$ is a Cayley subset of G. Because $Z(G) = \langle a^p,\ b^p\rangle$ is characteristic of G, one can assume $u = 0,\ v = 1$ from the results of Lemma 2.3 and Lemma 2.9. That is, $X = a^{\langle\sigma\rangle}$. Assume $a^{\sigma} = b^j a^i,\ o(\sigma) = 2k$ and $\tau = \sigma^k$, then $a^{\tau} = a^{-1},\ (b^j a^i)^{\tau} = a^{-i}b^{-j}$. Recall that $G' = \langle a^{p^{n-1}}\rangle \cong \mathbb{Z}_p$ and $[a, b^j] \in G' < \langle a\rangle$, so $[a, b^j]^{\tau} = [a, b^j]^{-1}$. While

$$[a, b^j]^{\tau} = ([a, a^i][a, b^j][a, b^j, a^i])^{\tau} = [a, b^j a^i]^{\tau} = [a^{\tau}, (b^j a^i)^{\tau}] = [a^{-1}, a^{-i}b^{-j}] = [a^{-1}, b^{-j}],$$

and $[a^{-1}, b^{-j}]$ belongs to the center, the result

$$[a, b^j]^{\tau} = [a^{-1}, b^{-j}] = b^{-j}a^{-1}[a^{-1}, b^{-j}]ab^j = [a, b^j]$$

follows. Therefore, $[a, b^j]^{-1} = [a, b^j]$, that is $[a, b^j]^2 = 1$. But the order of $[a, b^j]$ is a power of p which is coprime with 2, we get $[a, b^j] = 1$. And from Lemma 4.2, one can get $G = \langle a, a^\sigma \rangle = \langle a, b^j a^i \rangle$. So G is abelian, a contradiction. Thus in either case, G doesn't have regular balanced Cayley maps. □

Remark 4.5. In the paper of Newman and Xu ([8]), they claimed that for odd primes p every metacyclic p-group is isomorphic to one of the groups

$$G = \langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, b^{-1}ab = a^{1+p^r} \rangle, \tag{4.1}$$

where r, s, t, u are non-negative integers with r positive and $u \leq r$, and these groups are pairwise non-isomorphic. In the following Lemma 4.6, one will see that the metacyclic p-group has an 'intimate' connection with the minimal non-abelian metacyclic p-group.

Lemma 4.6. *Let G be a metacyclic p-group for some odd prime number p and $N < G'$ be a maximal subgroup of the derived subgroup G'. Then N is a characteristic subgroup of G and the quotient group $\overline{G} = G/N$ is a minimal non-abelian metacyclic p-group.*

Proof. Because G' is cyclic and G' is characteristic of G, it follows that N is also characteristic of G. While N is a proper subgroup of G', the quotient group $\overline{G} = G/N$ is non-abelian and metacyclic, generated by two elements because G is generated by two elements. As $\overline{G}' = \overline{G'} \cong \mathbb{Z}_p$ and so $|\overline{G}'| = p$. The quotient group $\overline{G}$ is mininal non-abelian follows from Lemma 4.1. □

From the results of Lemma 2.5 and Theorems 4.3 and 4.4, we get the following Corollary 4.7.

Corollary 4.7. *For any odd prime number p, the metacyclic p-group does not have regular balanced Cayley maps.*

Theorem 4.8. *Let $G = M_2(n, m)$, where m and n are positive integers and $m > n \geq 2$. Then G does not have regular balanced Cayley maps.*

Proof. According to Lemma 2.3, $\mathrm{Aut}(G) = \{\sigma \mid a^\sigma = b^j a^i,\ b^\sigma = b^s a^r\}$, where $(is, 2) = 1$, $1 \leq i \leq 2^n$, $1 \leq s \leq 2^m$, $j = 2^{m-n+1}k$, $0 \leq k < 2^{n-1}$, $1 \leq r \leq 2^n$. From the defining relations of G, one can see that both a^2 and b^2 belong to the center of G. Set $N = \langle a^2, b^4 \rangle = \{x \in Z(G) \mid x^{2^{m-2}} = 1\}$. By Lemma 2.7, N is a characteristic subgroup of $Z(G)$. Since $Z(G)$ is characteristic in G, N is characteristic in G. Suppose there is some $\sigma \in \mathrm{Aut}(G)$ and $b^u a^v \in G$ such that $X = (b^u a^v)^{\langle \sigma \rangle}$ is a Cayley subset of G. By Lemma 2.9, one may assume $u = 1$ and $v = 0$, that is, $X = b^{\langle \sigma \rangle}$.

Assume $a^\sigma = b^j a^i$ and $b^\sigma = b^s a^r$, then $4 \mid j$, $(s, 2) = 1$ and so $s^2 \equiv 1 \pmod 4$. According to Lemma 4.2, $G = \langle b, b^s a^r \rangle = \langle b, a^r \rangle$ and so $(r, 2) = 1$. In the quotient group $\overline{G} = G/N$, $\overline{X} = \overline{b^{\langle \sigma \rangle}}$ should be a Cayley subset of $\overline{G}$. Noticing that $2 \mid (s + i)$, $4 \mid j$ and $G' \leq N$, we have $\overline{(b^s a^r)^\sigma} = \overline{(b^s a^r)^s (b^j a^i)^r} = \overline{b^{s^2} a^{rs} b^{jr} a^{ir}} = \overline{b^{s^2+jr} a^{r(s+i)}} = \overline{b^{s^2}}$. Since $o(\overline{b}) = 4$ and $s^2 \equiv 1 \pmod 4$, we have $\overline{b^{s^2}} = \overline{b}$. So, $\overline{X} = \{\overline{b}, \overline{b^s a^r}\}$. But $(r, 2) = 1$, $\overline{b}^{-1} \notin \overline{X}$. Then, $\overline{X}$ is not a Cayley subset, a contradiction. □

Theorem 4.9. *Let $G = M_2(n, m)$, where m and n are positive integers, $n > m + 1$ and $m \geq 2$. Then G does not have regular balanced Cayley maps.*

Proof. In this case, $\mathrm{Aut}(G) = \{\sigma \mid a^\sigma = b^j a^i,\ b^\sigma = b^s a^r\}$, where $(is, 2) = 1$, $1 \le i \le 2^n$, $1 \le s \le 2^m$, $1 \le j \le 2^m$, $r = k2^{n-m}$, $0 \le k < 2^m$. Let $N = \langle a^4, b^2 \rangle = \{x \in Z(G) \mid x^{2^{n-2}} = 1\}$. According to Lemma 2.7, N is characteristic in $Z(G)$. Since $Z(G)$ is characteristic in G, N is characteristic in G. Similar to the proof of Theorem 4.8, we only need to show that $X = a^{\langle\sigma\rangle}$ is not a Cayley subset of G for any $\sigma \in \mathrm{Aut}(G)$.

Assume $a^\sigma = b^j a^i$ and $b^\sigma = b^s a^r$. Then $(s, 2) = 1$, $4 \mid r$, $(i, 2) = 1$ and so $i^2 \equiv 1 \pmod 4$. And from Lemma 4.2, $G = \langle a, b^j a^i \rangle = \langle a, b^j \rangle$ and so $(j, 2) = 1$. If X is a Cayley subset, then $\overline{X} = \overline{a^{\langle\sigma\rangle}}$ is a Cayley subset of $\overline{G} = G/N$. While from $2 \mid (s+i)$, $4 \mid r$ and $G' \le N$, we have $\overline{(b^j a^i)^\sigma} = \overline{(b^s a^r)^j (b^j a^i)^i} = \overline{b^{sj} a^{rj} b^{ji} a^{i^2}} = \overline{b^{j(s+i)} a^{i^2+rj}} = \overline{a^{i^2}}$. And from $o(\overline{a}) = 4$, $i^2 \equiv 1 \pmod 4$, we have $\overline{a^{i^2}} = \overline{a}$. So, $\overline{X} = \{\overline{a}, \overline{b^j a^i}\}$. But $(j, 2) = 1$ implies $\overline{a}^{-1} \notin \overline{X}$. So, $\overline{X}$ is not a Cayley subset, a contradiction. □

In Theorem 4.9, if we allow $m = 1$ and so $n > 2$, then the group $M_2(n, 1)$ belongs to one of the p-groups with a cyclic maximal subgroup which had been considered by D. D. Hou, Y. Wang and H. P. Qu in [6]. We list the result in the following theorem.

Theorem 4.10 ([6, Theorem 3.3])**.** *For positive integers $n > 2$, $M_2(n, 1)$ does not have regular balanced Cayley maps.*

Now, there are still two cases about which we have not said anything, that is $M_2(n, n)$ for $n \ge 2$ and $M_2(n+1, n)$ for $n \ge 1$. One may look back at Lemma 2.3 and can easily see that the automorphism groups of both $M_2(n, n)$ and $M_2(n+1, n)$ are 2-groups.

Theorem 4.11. *Let $G = M_2(n, n)$, $n \ge 2$. Then G has exactly one regular balanced Cayley map of valency 4 in the sense of isomorphism.*

Proof. By Lemma 2.3, $\mathrm{Aut}(G) = \{\sigma \mid a^\sigma = b^{2k} a^i, b^\sigma = b^s a^r\}$, where $(si, 2) = 1$, $1 \le i, s, r \le 2^n$, $1 \le k \le 2^{n-1}$, and both a^2 and b^2 belong to $Z(G)$.

We firstly show that if for some $g \in G$ and $\sigma \in \mathrm{Aut}(G)$, $X = g^{\langle\sigma\rangle}$ is a Cayley subset of G, then $|X| = 4$. Set $N = \{x \in G \mid x^{2^{n-2}} \in G'\}$. According to Lemma 2.7, N is a characteristic subgroup of G and $N = \langle a^2, b^4 \rangle$. Without loss of generality, we assume $g = b$, then in the quotient group $\overline{G} = G/N \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, the order of $\overline{b}$ is 4. While there are exactly four order-4 elements in $\overline{G}$ and $\overline{X} = \overline{b}^{\langle\overline{\sigma}\rangle}$ is a Cayley subset of $\overline{G}$, $\overline{X}$ should contain all these four elements. Because the order of σ is a power of 2 and b is not involution, according to the results in Lemma 2.5, we have $|X| = |\overline{X}| = 4$.

Take $\sigma_1 \in \mathrm{Aut}(G)$ such that $a^{\sigma_1} = b^2 a^{-1}$ and $b^{\sigma_1} = ba^{2^{n-1}-1}$. By a direct calculation, $X_1 = b^{\langle\sigma_1\rangle} = \{b, ba^{2^{n-1}-1}, b^{-1}, (ba^{2^{n-1}-1})^{-1}\}$ is clearly a Cayley subset of G.

For any $\sigma_2 \in \mathrm{Aut}(G)$ such that $a^{\sigma_2} = b^{2k} a^i$, $b^{\sigma_2} = b^s a^r$, where k, i, s, r satisfy the conditions listed in the first paragraph, and $X_2 = b^{\langle\sigma_2\rangle} = \{b, b^s a^r, b^{-1}, (b^s a^r)^{-1}\}$ is a Cayley subset of G, one may take $\tau \in \mathrm{Aut}(G)$ such that $a^\tau = b^{1-s} a^{-r(1+2^{n-1})}$ and $b^\tau = b$. It is easy to check that $(ba^{2^{n-1}-1})^\tau = b^s a^r$.

Therefore, by Lemma 2.8, the two regular balanced Cayley maps $\mathrm{CM}(G, X_1, \sigma_1|_{X_1})$ and $\mathrm{CM}(G, X_2, \sigma_2|_{X_2})$ are isomorphic. So, G has exactly one regular balanced Cayley map of valency 4 in the sense of isomorphism. □

Theorem 4.12. *Let $G = M_2(n+1, n)$, $n > 1$. Then G has exactly one regular balanced Cayley map up to isomorphism and this map is of valency 4.*

Proof. By Lemma 2.3, $\mathrm{Aut}(G) = \{\sigma \mid a^\sigma = b^j a^i, b^\sigma = b^s a^{2k}\}$, where $(si, 2) = 1$, $1 \le i \le 2^{n+1}$, $1 \le j, s, k \le 2^n$ and both a^2 and b^2 belong to $Z(G)$.

We firstly show that if $g \in G$, $\sigma \in \mathrm{Aut}(G)$ and $X = g^{\langle\sigma\rangle}$ is a Cayley subset of G, then $|X| = 4$. Set $N = \{x \in G \mid x^{2^{n-1}} = 1\}$. According to Lemma 2.7, N is a characteristic subgroup of G and $N = \langle a^4, b^2\rangle$. In the quotient group $\overline{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, the order of $\overline{a}$ is 4. There are exactly four order-4 elements in $\overline{G}$, similar to the proof of Theorem 4.11, $\overline{X} = \overline{a}^{\langle\overline{\sigma}\rangle}$ is a Cayley subset of $\overline{G}$ of order 4 and $|X| = |\overline{X}| = 4$.

Take $\sigma_1 \in \mathrm{Aut}(G)$ such that $a^{\sigma_1} = b^{-1}a$ and $b^{\sigma_1} = b^{-1}a^2$. Then, $Y_1 = a^{\langle\sigma_1\rangle} = \{a, b^{-1}a, a^{-1}, (b^{-1}a)^{-1}\}$ is a Cayley subset of G.

For any $\sigma_2 \in \mathrm{Aut}(G)$ such that $a^{\sigma_2} = b^j a^i$, $b^{\sigma_2} = b^s a^{2k}$, where j, i, s, k satisfy the conditions listed in the first paragraph, and $Y_2 = a^{\langle\sigma_2\rangle} = \{a, b^j a^i, a^{-1}, (b^j a^i)^{-1}\}$ is a Cayley subset of G, one may take $\tau \in \mathrm{Aut}(G)$ such that $a^\tau = a$ and $b^\tau = b^{-j}a^{1-i}$. It is easy to check that $(b^{-1}a)^\tau = b^j a^i$. Therefore, the two regular balanced Cayley maps $\mathrm{CM}(G, Y_1, \sigma_1|_{Y_1})$ and $\mathrm{CM}(G, Y_2, \sigma_2|_{Y_2})$ are isomorphic and so G has only one regular balanced Cayley map of valency 4 in the sense of isomorphism. □

To be more clear, we list the number of non-isomorphic regular balanced Cayley maps of minimal non-abelian metacyclic groups in Table 1. For brevity, we use $|G|$, N, RBCM and MNAMG to denote the order of group G, the number of regular balanced Cayley maps up to isomorphism, regular balanced Cayley maps and minimal non-abelian metacyclic groups, respectively.

Table 1: Number of RBCM of MNAMG.

	G	$\|G\|$	N
1	Q_8	8	1
2	$M_{p,2}(1,r) \cong D_{2p}$	$2p$	$p-1$
3	$M_{p,2}(m,r), m \ge 2, p-1 = 2^e s, (s,2) = 1$	$2^m p$	s
4	$M_{p,q}(m,r), q \ne 2$	pq^m	0
5	$M_2(2,1) \cong D_8$	8	2
6	$M_2(n,1), n > 2$	2^{n+1}	0
7	$M_2(n,n), n \ge 2$	2^{2n}	1
8	$M_2(n+1,n), n \ge 2$	2^{2n+1}	1
9	$M_2(n,m)$, $m \ne n$ and $m \ne n-1$	2^{n+m}	0
10	$M_p(n,m), p \ne 2$	p^{n+m}	0

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