Fullerene patches I

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Abstract

Large carbon molecules, discovered at the end of the last century, are called fullerenes. The most famous of these, C_{60} has the structure of the soccer ball: the seams represent chemical bonds and the points where three seams come together represent the carbon atoms. We use the term *fullerene* to represent all mathematically possible structures for the chemical fullerenes: trivalent plane graphs with only hexagonal and pentagonal faces. A simple consequence of Euler's formula is that each fullerene has exactly 12 pentagonal faces; the only restriction on the number of hexagonal faces is that it not be 1. The chemical motivation for this paper is to answer the question: When is it possible to alter a fullerene by changing the structure inside a region of the fullerene bounded by a simple closed circuit? Such a region or patch is said to be ambiguous if alterations may be made to its interior without disturbing the structure of the fullerene outside of the region. In this paper, we show that, relative to the minimum distance between pentagonal faces, there are no small ambiguous patches.

Keywords: Fullerenes, fullerene patches, ambiguous patches, graphite patches.

Math. Subj. Class.: 05C10, 05C75, 92E10

1 Introduction

Let $\Gamma = (V, E, F)$ be a *fullerene*, a trivalent plane graph with only hexagonal and pentagonal faces. Let $B = (V(B), E(B))$ be an elementary circuit in Γ. The circuit B partitions the sphere into two open hemispheres. A *patch* Π of Γ is constructed by deleting all vertices, edges and faces interior to one of the hemispheres and replacing them by a single face bounded by B. Our notational convention $\Pi = (V_\Pi, E_\Pi, F_\Pi, B)$ should be interpreted as follows: (V_Π, E_Π) is the graph remaining after the deletion; F_Π is the collection of faces

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of Γ remaining after the deletion. Hence, $\hat{\Pi} = (V_\Pi, E_\Pi, \hat{F}_\Pi)$, where $\hat{F}_\Pi$ is the collection of faces F_Π along with the outside face, is a plane graph. The circuit B is called the *boundary* of the patch. *Graphite patches* are defined to be plane graphs with all hexagonal faces save one outside face and with all vertices on the boundary of the outside face having degree 2 or 3 and with all other (internal) vertices having degree 3. While investigating graphite patches, Guo, Hansen and Zheng [6] produced two nonisomorphic patches with the same boundary code, that is the same sequence of vertex degrees around the boundary. A fullerene or graphite patch that is not uniquely determined by its boundary is said to be *ambiguous*.

When the boundary curve of the Guo, Hansen and Zheng ambiguous patches is traced in the hexagonal tessellation of the plane or on a fullerene it intersects itself, that is, its boundary is not an elementary circuit. In fact, it was shown in [5] that an ambiguous graphite patch must triple cover some hexagon when projected onto the hexagonal tessellation of the plane. Our patches have elementary circuits for boundaries when drawn on the fullerene. However, the boundary curve of a fullerene patch with only hexagonal faces may have self intersections when projected onto the hexagonal tessellation of the plane. In spite of this, it seems to be the case that fullerene patches with only hexagonal faces or at most one pentagonal face cannot be ambiguous. But this has not been proved. In this paper, we show that ambiguous fullerene patches containing at most one pentagonal face, if they exist, must be relatively large - roughly as large or larger than the smallest patch of the fullerene containing two pentagonal faces. In the papers [1], by Brinkmann, Caporossi and Hansen, [2], by Brinkmann, Friedrichs and von Nathusius, and [3], by Brinkmann, Graver and Justus, it is shown that the number of faces in an ambiguous patch containing at most one pentagonal face is uniquely determined.

The smallest example of an ambiguous fullerene patch is due to Endo and Kroto [4] and is pictured in Figure 1. These two patches have the same bounding circuit and contain two pentagons. However, the relative positions of the pentagons are different in these two patches. In fact, most of the patches enclosing two or more pentagonal faces are ambiguous.
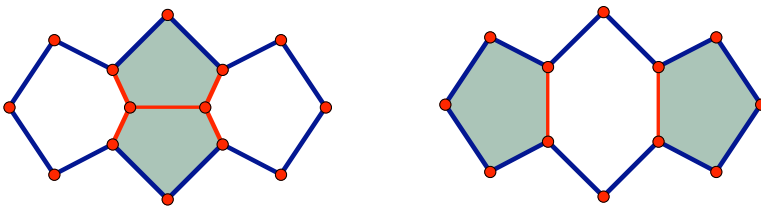

Figure 1: The Endo-Kroto ambiguous patches.

Given any patch consisting of two non-adjacent pentagons joined by a simple path of hexagons, we may construct a different patch with the same boundary. This generalization of the Endo-Kroto construction is illustrated in Figure 2.

Let $\Pi = (V_\Pi, E_\Pi, F_\Pi, B)$ be a patch of a fullerene Γ. If f is a face of Π, we denote the boundary circuit of f by $B(f)$. The vertices of the boundary, B, of Π have degree 2 or 3 in Π and one easily sees that every patch of size greater than 1 admits at least two boundary vertices of degree 3. The paths in the circuit B joining two consecutive vertices of degree 3 are called the *segments* of B. The proof of the following lemma is straight forward.
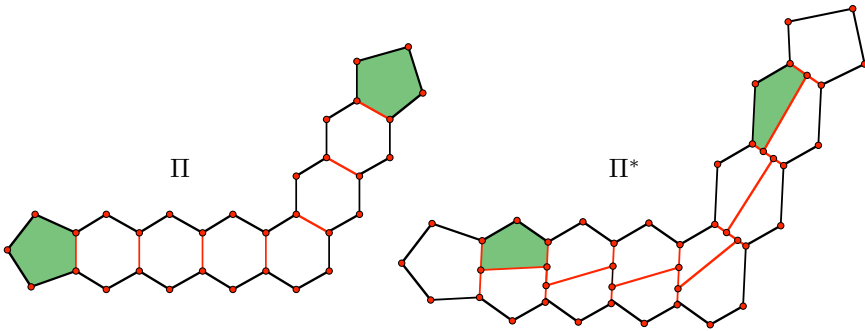
Figure 2: Two ambiguous patches using the generalized Endo-Kroto construction.

Lemma 1.1. *Let* $\Pi = (V_\Pi, E_\Pi, F_\Pi, B)$ *be a patch of size at least 2. The path* $S = x_0, e_0, x_1, \ldots, e_k, x_{k+1}$ *is a segment of* B *if and only if* S *is a connected component of* $B \cap B(f)$ *for some face* f *of* Π.

A face f is called an *i-face* if $B \cap B(f)$ consists of a single segment and that segment has length i. By the *distance* between two faces in a fullerene we mean the distance between the corresponding vertices in the dual graph. By the *spread* of a fullerene we mean the minimum distance between pentagonal faces taken over all pairs of pentagonal faces. By a *disk* of *radius* r, we mean a patch consisting of all faces of distance r or less from a central face (the *center* of the disk) as long as its boundary is an elementary circuit; see Figure 3. For any patch, its *diameter* is the maximum of the distances (measured in the entire fullerene) between pairs of faces in the patch. Finally, the *width* of a patch is the diameter of the disk of least diameter containing it.
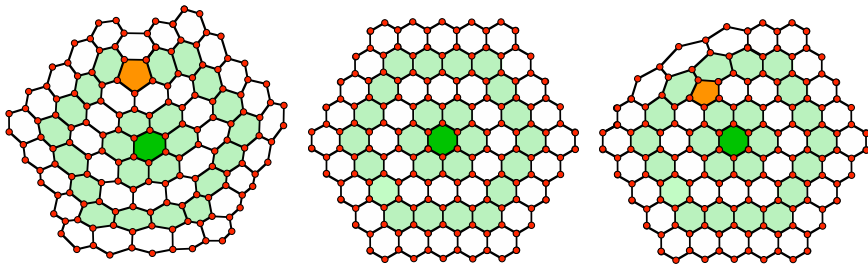

Figure 3: Three disks of radius 4.

The main result of this paper is:

Theorem 1.2. *Ambiguous disks contain at least two pentagonal faces. In particular, all disks with diameter less than the spread of the fullerene are unambiguous.*

As a simple corollary of the main result we have:

Corollary 1.3. *In a fullerene, any patch with width less than the spread of the fullerene is unambiguous.*

2   Proof of the main result

We start this section by proving several basic lemmas. All patches that we investigate will be considered as stand-alone patches. Hence they can be extended or altered even though the extension or alteration cannot be carried out in the fullerene from which they came. For the patch $\Pi = (V_\Pi, E_\Pi, F_\Pi, B)$, $p(\Pi)$ will denote the number of pentagonal faces in the patch and s_i, for $i = 1$ to 5, will denote the number of its i-faces.

Lemma 2.1. *The number of pentagonal faces in a patch Π, $p(\Pi)$, is given by $p(\Pi) = 6 + s_1 - s_3 - 2s_4 - 3s_5$.*

Proof. Let $\Pi = (V_\Pi, E_\Pi, F_\Pi, B)$ be a patch in the fullerene $\Gamma = (V, E, F)$. Denote the number of vertices on the boundary of Π that have degree j ($j = 2, 3$) in Π by d_j. Summing the vertex degrees, we have $3|V_\Pi| - d_2 = 2|E_\Pi|$ or $6|V_\Pi| - 4|E_\Pi| = 2d_2$. Summing the face degrees, we have $6h + 5p(\Pi) + |B| = 2|E_\Pi|$, where h denotes the number of hexagonal faces of Π. Combining these two equations gives:

$$6|V_\Pi| - 6|E_\Pi| = 2d_2 - (6h + 5p(\Pi) + |B|) = d_2 - d_3 - 6h - 5p(\Pi).$$

The number of face of the plane graph $\hat{\Pi}$ is $h + p(\Pi) + 1$ and we then have $6|\hat{F}_\Pi| = 6h + 6p(\Pi) + 6$. Adding this and the previous equation gives:

$$6|V_\Pi| - 6|E_\Pi| + 6|\hat{F}_\Pi| = d_2 - d_3 + p(\Pi) + 6.$$

By Euler's formula, the left-hand side equals 12; so, we have $p(\Pi) = 6 - d_2 + d_3$. Finally, one easily writes d_2 and d_3 in terms of $s_1, \ldots, s_5$: $d_3 = \sum_{i=1}^{5} s_i$ while $d_2 = \sum_{i=1}^{5}(i-1)s_i$ giving

$$p(\Pi) = 6 + \sum_{i=1}^{5} s_i - \sum_{i=1}^{5}(i-1)s_i = 6 - \sum_{i=1}^{5}(i-2)s_i$$

$\square$

The next lemma then follows at once.

Lemma 2.2. *If Π and Π^* are a pair of ambiguous patches then $p(\Pi) = p(\Pi^*)$.*

By a *side* of a patch Π we mean a set of boundary faces between two i-faces with $i \geq 3$. Note that $s_3 + s_4 + s_5$ is then the number of sides of Π. We define the *length* of a side in terms of the distance between the end faces in the dual. Specifically, if all of the faces of a side have bounding segments of length 2 and if there are $\ell - 1$ of them, we say the side has *length* ℓ or is an ℓ-*side*. We will adopt the convention that a bounding 4-face is a side of length 0; this will eliminate the need for special cases of the formulas that we will develop later. (5-faces will soon be eliminated from consideration, hence we adopt no special convention for them.) See Figure 4. A side is an $h|k$-*side* (h and k positive integers) if when rotating in the clockwise direction its first $h - 1$ faces and last $k - 1$ faces are 2-faces and they are separated by a single 1-face.
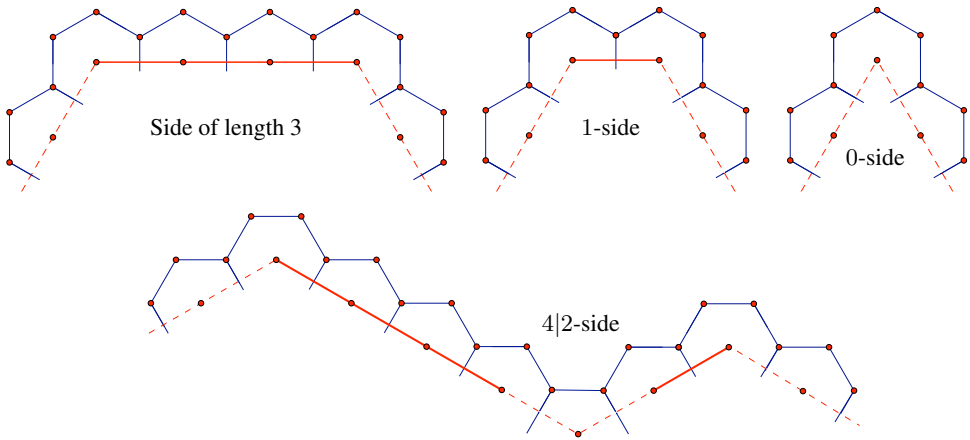
Figure 4: Side lengths.

A disk of radius r that consists solely of hexagons will be called an *H-disk*. A disk of radius r that contains exactly one pentagonal face is a *P-disk*. Notice that the only H-disk of radius 0 consists of a single hexagon, and the only P-disk of radius 0 is a single pentagon.

Proposition 2.3. *An H-disk with radius r greater than or equal to 1 has six r-sides. A P-disk with radius r greater than or equal to 1 has either*

   (i$_a$) *five r-sides,*
   (i$_b$) *four r-sides and one t-side with $r < t \leq 2r$ or*
   (ii) *five r-sides and one $h|k$-side with $h + k \leq r$.*
Furthermore, the boundary faces of an H- or P-disk of radius r are precisely the faces of distance r from the center face of the disk.

Proof. The only H- and P-disks of radius 1 are easily constructed and are pictured in Figure 5. It is clear that these disks all satisfy the conclusions of the proposition. (The simplest P-patch of type (ii) is pictured in Figure 7.)

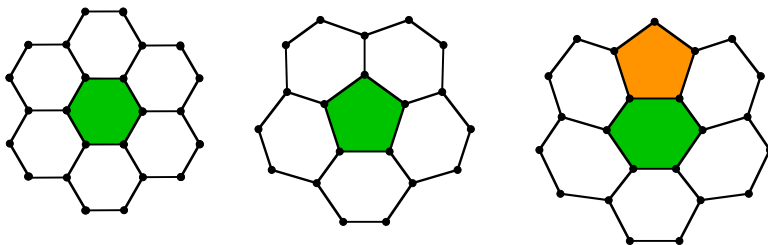Figure 5: One H-disk and two P-disks of radius one.

Let Π be an H- or P-disk of radius r. We first show that all of the boundary faces are a distance r from the center. Suppose that f is a boundary face of Π at a distance r' from

the center where $r' < r$. Since, f is a boundary face, there must be a face f' of Γ not in Π but adjacent to f. But then the distance of f' from the center is less than or equal to r and, by the definition of a disk, f' must be a face of the patch; contradiction.

We proceed by induction on the radius: assume that Π is an H- or P-disk of radius $r > 1$ and that all H-disks and P-disks of radius less than r are accurately described by the proposition. Delete all boundary faces of Π and any other faces that are a distance r from the center face c to get Π'. Since the faces of Π' are all the faces of Π at a distance of $r - 1$ or less from c and the faces of Π include all faces of Γ at a distance of r or less from c, the faces of Π' are all faces of Γ at a distance of $r - 1$ or less from c. Thus Π' is an H- or P-disk of radius $r - 1$ and, by induction, it has the boundary structure described above.

Now consider a face f of Γ that is a distance r from c. It must be a face of Π and it must share an edge with a boundary face of Π'. Hence, adding all of the faces of Γ not in Π' but adjacent to a boundary face of Π', reconstructs Π. In reconstructing Π, we observe that when adding only hexagons to the boundary of an ℓ-side produces an $\ell + 1$-side and adding only hexagons to the boundary of an $h|k$-side produces another $h|k$-side.

Thus, if Π' is an H-disk of radius $r - 1$, adding only hexagons creates another H-disk Π with six r-sides. If Π' is a P-disk, we may add only hexagons and we see immediately that: if Π' is of type (i$_a$) with five $r - 1$-sides, Π is a P-disk of type (i$_a$) with five r-sides; if Π' is of type (i$_b$) with four $r - 1$-sides and one t'-side ($r - 1 < t' < 2r - 2$), Π is a P-disk of type (i$_b$) with four r-sides and one $t = t' + 1$-side ($r < t < 2r - 1 < 2r$); finally if Π' is of type (ii) with five $r - 1$-sides and one $h|k$-side ($h + k \leq r - 1$), Π is a P-disk of type (ii) with five r-sides and one $h|k$-side ($h + k \leq r - 1 \leq r$).

The only remaining case is that of adding a boundary of hexagons and exactly one pentagon to an H-disk of radius $r - 1$. If the pentagon is adjacent to the central edge of a segment of length 3, the result is a P-disk of type (i$_b$) with four r-sides and one $2r$-side. If the pentagon is not adjacent to the central edge of a length 3 segment, then it will have just one edge on the boundary of the enlarged disk. Number the faces on the side of the enlarged disk that includes the pentagon $f_1, \ldots, f_{r+1}$, where f_1 and f_{r+1} have bounding segments of length 3. If the pentagon is face f_{h+1}, the P-disk resulting from adding the bounding faces will be of type (ii) with five r-sides and one $h|k$-side where $k + h = r$. $\qquad\square$

Corollary 2.4. *H-disks are unambiguous.*

Proof. When an H-disk is projected onto the hexagonal tessellation of the plane its boundary curve is an elementary circuit. Hence it is unambiguous. $\qquad\square$

To prove that P-disks are unambiguous, we are going to need to know just which boundary curves are possible for certain classes of hexagonal patches - patches with only hexagonal faces. By Lemma 2.1, when a hexagonal patch has $s_1 = s_4 = s_5 = 0$ it has exactly six sides. If the lengths of these sides are a, b, c, d, e, f in cyclic order (counterclockwise), we say the patch has *side parameters*, or simply *parameters*, (a, b, c, d, e, f). If a hexagonal patch has $s_1 = s_5 = 0$, $s_4 \geq 1$, then $s_3 = 6 - 2s_4$ and it will have $6 - s_4$ sides of positive length. By our convention, each 4-face is a side of length 0. Hence such a patch can also be described by six side parameters (a, b, c, d, e, f) where non-adjacent variables could take on the value 0; for example, the patch with parameters $(a, 0, c, d, e, f)$ where $a, c, d, e, f > 0$ is a 5-sided hexagonal patch with the a and c sides separated by a 4-face.

Lemma 2.5. *Let* Π *be a hexagonal patch with* $s_1 = s_5 = 0$ *and side parameters* (a, b, c, d, e, f). *Then there are no consecutive sides of length 0 and the following three equalities must hold:* $a + b = d + e$, $b + c = e + f$ *and* $c + d = f + a$.

Proof. If we project Π onto the hexagonal tessellation, then the boundary of the patch can be represented in the dual (triangular) tessellation by the circuit joining consecutive bounding faces. In this representation, a side of length x is represented as straight lines of x dual edges; these straight lines make $120°$ angles at 3-faces and $60°$ angles at 4-faces. Since $s_1 = s_5 = 0$ and $s_3 + 2s_4 = 6$ (by Lemma 2.1), we see that Π has at most three 4-faces. If Π has three 4-faces, then Π is a "triangular" patch. It is clear that this triangle must be an equilateral triangle and that all sides of Π are the same length. Thus, the side parameters of Π are $(a, 0, a, 0, a, 0)$ which do satisfy the above equalities. (See Figure 6.)
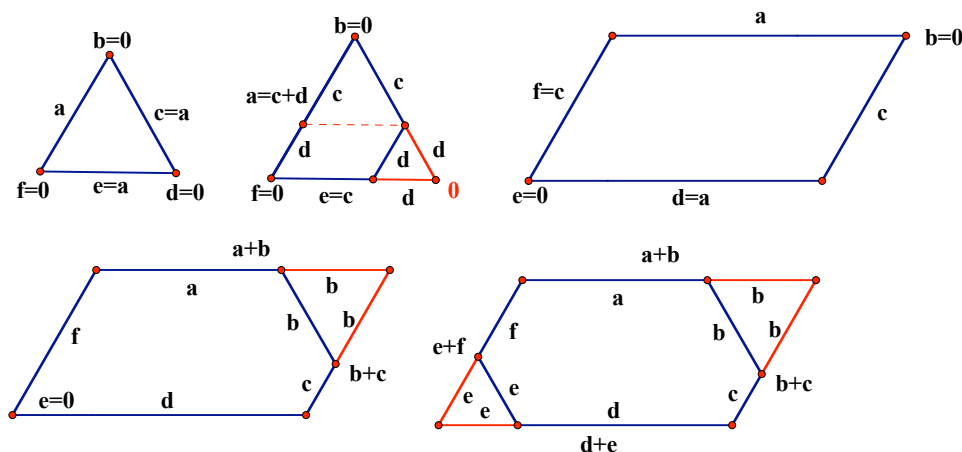

Figure 6: Hexagonal Patch Boundaries.

If Π has exactly two 4-faces, called f_1 and f_2, then it must have two 3-faces, g_1 and g_2. If f_1 and f_2 are separated by only 2-faces, then projecting the patch onto the hexagonal tessellation has as its diagram a trapezoid with sides of length $(a, 0, c, d, e, 0)$ where a is the side between f_1 and f_2. It is apparent that sides a and d are parallel and that $b + c = c = e = e + f$. We can enlarge the patch by extending sides c and e to sides of length $c + d$. This new patch is an equilateral triangle, and thus $a + b = a = e + d$ and $a + f = a = c + d$. On the other hand, if f_1 and f_2 are separated by 2-faces and one 3-face, then projecting Π onto the hexagonal tessellation yields a parallelogram with side parameters $(a, 0, c, d, 0, f)$ which must satisfy $a = d$ and $c = f$ giving $a + f = d + c$, $a + b = a = d = e + d$ and $b + c = c = f = e + f$.

If Π has exactly one 4-face, then projecting Π onto the hexagonal tessellation yields a patch with side parameters $(a, b, c, d, 0, f)$ where sides f and c are parallel and sides a and d are parallel. By extending sides a and c by b hexagons, we get a parallelogram which must satisfy $a + b = d$ and $b + c = f$. It follows that $a + b = d = e + d$ and $b + c = f = e + f$. Adding $a + b = d$ and $f = b + c$ gives $a + b + f = b + c + d$ and canceling the b gives $c + d = f + a$.

Finally, if Π has no 4-faces, then Π is a patch with side parameters (a, b, c, d, e, f)

where sides a and d are parallel, sides b and e are parallel, and sides f and c are parallel. By extending sides a and c by d hexagons, and extending sides f and d by e hexagons, we create a parallelogram (see Figure 6). This parallelogram gives $a+b = d+e$, $b+c = e+f$ and $a+b+e+f = b+c+d+e$ directly and $a+f = c+d$ by canceling the b and e in the last equation. $\square$

If a P-disk Π were ambiguous, there exists a patch Π^* with the same boundary. So we introduce a class of patches that will include all of these possible alternatives. By an *r-patch,* we mean a patch of one of the following two types. An r-patch of the first type is a patch with $s_1 = s_4 = s_5 = 0$, $s_3 = 5$ and with 4 sides of length r and one side of length t ($t \geq r$). That is, an r-patch of the first type has side parameters (r, r, r, r, t), where $t \geq r$. An r-patch of the second type is a patch with $s_4 = s_5 = 0$, $s_1 = 1$ and $s_3 = 6$, with five r-sides and one $h|k$-side. That is, an r-patch of the second type has side parameters $(r, r, r, r, r, h|k)$, where h and k are positive integers. It is an immediate corollary of Lemma 2.1 that an r-patch contains exactly one pentagon. One easily checks that there are only two 1-patches: the two P-disks of radius 1 with parameters $(1, 1, 1, 1, 1)$ and $(1, 1, 1, 1, 2)$. These are both r-patches of the first type and are pictured in Figure 5. The simplest r-patch of the second type has parameters $(2, 2, 2, 2, 2, 1|1)$ and is pictured in Figure 7; note that it is a P-disk of radius 2.
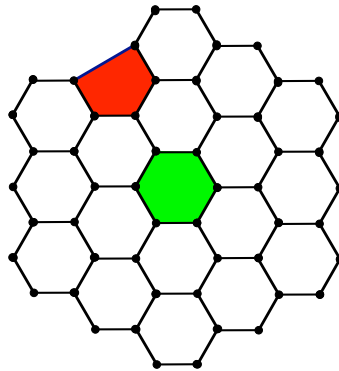

Figure 7: The r-patch of the second type with parameters $(2, 2, 2, 2, 2, 1|1)$.

The following lemma gives another property of r-patches — namely that we can delete all bounding faces of an r-patch ($r > 1$) and still get a legitimate patch. It will always be possible to remove all of the faces on a side unless one of those faces actually meets the boundary in two distinct segments — its removal would then leave a "disconnected" patch. It will always be possible to remove all of the faces on the boundary unless two faces on different sides share an edge. See Figure 8.

Lemma 2.6. *Let* $\Pi = (V_\Pi, E_\Pi, F_\Pi, B)$ *be an* r-patch $r > 1$.

   i. *Let* f *be a face of* Π. *Then,* $B(f) \cap B$ *consists of at most one path.*

   ii. *Let* f *and* g *be faces of* Π *that are not consecutive as one cyclically traverses the boundary. Then,* $B(f) \cap B(g) = \emptyset$.
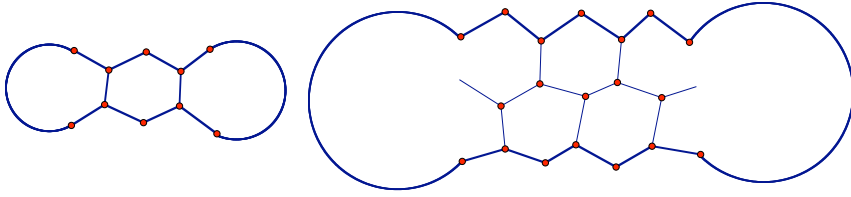
Figure 8: Impossible r-patch boundaries.

Proof. (i) Assume there exists a face f in Π consisting of 2 disjoint segments, a *cut-face*. Clearly both segments of f have length 2 or less. If both segments have length 2, then both faces adjacent to f are also cut-faces. Extending to the left, the string of cut-faces must terminate. Terminating with a single face is impossible since this is an r-patch and such a face would be a 4-face or 5-face. Hence the leftmost cut-face must meet two faces on its left and must have at least one bounding segment of length 1. If f meets two faces on its right, then it would have two bounding segments of length 1. If f meets only one face on its right then the right-most cut face in the string of cut-faces to the right must also have at least one bounding segment of length 1. Hence this patch must have at least two bounding segments of length 1 contradicting the assumption that it is an r-patch.

(ii) Consider any r-patch Π and suppose that it has a pentagonal 3-face f on its boundary. If the patch is of type (r, r, r, r, t), replacing f by a hexagon will result in a hexagonal patch with boundary lengths $(r, 0, r, r, r, t), (r, r, 0, r, r, t)$ or $(r, r, r, r, 0, t)$; all of which violate the conditions of Lemma 2.5. If the patch is of type $(r, r, r, r, h|k)$, replacing f by a hexagon will result in a hexagonal patch with boundary lengths $(r, 0, r, r, r, h|k)$, $(r, r, 0, r, r, r, h|k)$ or $(r, r, r, r, r, 0, h|k)$. Now we can eliminate the segment of length 1 in the $h|k$-side by attaching an $h \times k$ parallelogram of hexagons (see the left-hand diagram in Figure 9, also see Figure 10). The resulting six-sided hexagonal patches will then have boundary lengths $(h + r, 0, r, r, r, r + k), (h + r, r, r, 0, r, r + k)$ or $(h + r, r, r, r, r, k)$; all of which violate the conditions of Lemma 2.5. Hence Π has no pentagonal 3-face f on its boundary.

Let f and g be adjacent, non-consecutative boundary faces - a *cut-pair*. Since they are non-consecutive, the face labeled y in the right-hand diagram of Figure 9 must belong to the patch. If neither of the faces labeled x and z belong to the patch, then y would be a hexagonal 4-face, a pentagonal 3-face or a cut face. None of these is possible in an r-patch. Hence we may assume that z belongs to the patch. We may also assume without loss of generality that f and g are the leftmost cut-pair and so x is also in the patch. By a similar argument v and one of u and w on the right must belong to the patch - say w. Hence g is a 1-face. If u is a face of the patch, then f is also a 1-face. If u is not a face of the patch, then v and w form a cut pair and the right-most cut pair in this chain of cut-pairs must also include a 1-face. Hence this patch must have at least two 1-faces contradicting the assumption that it is an r-patch. □

The *core* of a patch Π is the plane subgraph obtained by removing all the boundary faces of Π. By the construction of P-disks in Lemma 2.3, it is clear that a P-disk of radius r has either an H-disk or a P-disk of radius $r - 1$ as its core. By Lemma 2.6, all r-patches have a legitimate patch as their core. We use this fact in carrying out the next step to
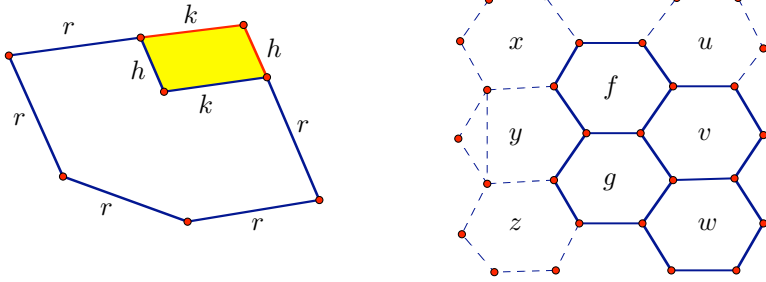
Figure 9: Diagrams for the proof of Lemma 5

eventually show that the only r-patches are the P-disks. The next step is to show that the parameters of any r-patches actually match the parameters of some P-disk.

Lemma 2.7. *Let Π be an r-patch of the first type with parameters (r, r, r, r, t) Then $t \leq 2r$ and*

   i. *if $t = 2r$, Π has a pentagon as the central face of the t-side;*

   ii. *if $t < 2r$, Π has only hexagons on the boundary.*

Proof. Let Π have parameters (r, r, r, r, t). In the proof of the previous lemma, we showed that a pentagon in the boundary of an r-patch cannot be a 3-face. Suppose that the ith face $(0 < i < r)$ in the first r-side is a pentagon. Replace this face by a hexagon. This new hexagonal patch has parameters $(i, r - i, r, r, r, t)$; which is impossible since $i + (r - i) \neq r + r$. We get the same kind of contradiction if we assume that any face on any of the r-sides are pentagons. Now assume that the ith face $(0 < i < t)$ in the t-side is a pentagon. Again, we replace the pentagon by a hexagon to get a hexagonal patch with parameters $(r, r, r, r, i, t - i)$. We must have $r + r = r + i$ and $r + r = i + (t - i)$. This is possible only if $i = r$ and $t = 2r$. Thus, the only way a pentagon can be on the boundary is if $t = 2r$ and the central face on the t-side is a pentagon.

To prove that $t < 2r$ when Π has only hexagonal boundary faces, we'll go by induction on r. This inequality holds for $r = 1$ since the only 1-patch with only hexagonal boundary faces has parameters $(1, 1, 1, 1, 1)$. Let Π be an r-patch with parameters (r, r, r, r, t) where $r > 1$ and with only hexagons on its boundary. We can then remove the boundary and consider its core. If a ℓ-side has only hexagons, removing them produces an $(\ell - 1)$-side. Thus, the core must be an $(r-1)$-patch with side parameters $(r-1, r-1, r-1, r-1, t-1)$. By induction, we have that $(t - 1) \leq 2(r - 1)$ or $t < 2r$.   □

Lemma 2.8. *Let Π be an r-patch of the second type with parameters $(r, r, r, r, r, h|k)$. Then $h + k \leq r$ and*

   i. *if $h + k = r$, then Π has a pentagon as the face with bounding segment of length 1;*

   ii. *if $h + k < r$, then Π has only hexagons on the boundary.*

Proof. Let Π have parameters $(r, r, r, r, r, h|k)$. Suppose that the ith face in the $h|k$-side, $i < h$, is a pentagon. We replace the pentagon by a hexagon to get a hexagonal patch with

parameters $(r, r, r, r, r, i, (h-i)|k)$. If we attach the hexagons of a $(h-i) \times k$ parallelogram (see Figure 10), we get a 6-sided hexagonal patch with parameters $(r, r, r, r, i+k, r+h-i)$. By Lemma 2.5, we have $(i+k) + (r+h-i) = 2r$; from which we conclude that we must have $h + k = r$. Considering another adjacent pair, we have $(r + h - i) + r = 2r$; from which we conclude that $i = h$. Thus the only case in which one of the faces on the $h|k$-side could be a pentagon is the case where $h + k = r$ and the pentagonal face is the face with boundary segment of length 1.
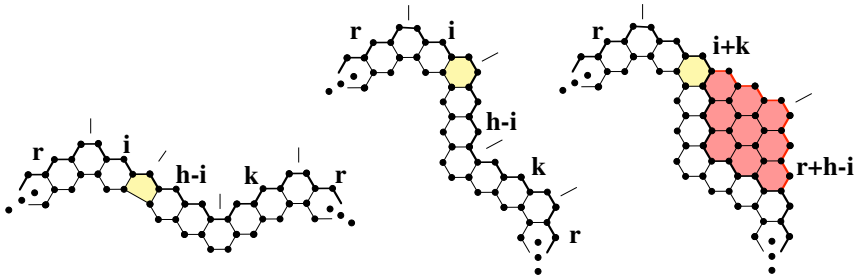

Figure 10: Diagrams for the proof of Lemma 2.8

Next suppose that the ith face in first r-side is a pentagon. Replace it in the patch by a hexagon. This results in a hexagonal patch with parameters $(i, r-i, r, r, r, r, h|k)$. Filling in to eliminate the length 1 segment then yields a 6-sided hexagonal patch with parameters $(i + h, r - i, r, r, r, r + k)$; which is impossible since $(r - i) + r = r + (r + k) > 2r$. We get the same kind of contradiction if we assume that any face on any of the r-sides are pentagons. Thus, if Π has a pentagon on the boundary, $h + k = r$ and the pentagon is the face with bounding segment of length 1.

To prove that $h + k < r$ when Π has only hexagonal boundary faces, we proceed by induction on r. One easily checks that there are no 1-patches of this second type and the only 2-patch of the second type is pictured in Figure 7 and has parameters $(2, 2, 2, 2, 2, 1|1)$. Hence, the lemma holds for all r-patches with $r \leq 2$. Let Π be an r-patch with parameters $(r, r, r, r, r, h|k)$ for $r > 2$ and with only hexagons on its boundary. We can remove the boundary and consider its core. If an r-side has only hexagons, removing them produces an $(r-1)$-side; if an $h|k$-side has only hexagons, removing them produces another $h|k$-side. Thus, the core must be an $(r-1)$-patch with sides $(r-1, r-1, r-1, r-1, r-1, h|k)$. By induction, we see that $h + k \leq r - 1$ or $h + k < r$. □

This final proposition concludes our proof of the main result.

Proposition 2.9. *P-disks are unambiguous.*

Proof. This proof will proceed by induction on r. It is easy to check that the only two *P*-disks of radius 1 are unambiguous. Hence we assume that Π is a *P*-disk of radius $r > 1$ and that *P*-disks of radius $r - 1$ are unambiguous. Let Π^* be another patch with the same boundary as Π. By Proposition 2.3, Π^* must be an r-patch with the same parameters as Π. We will show that Π^* is in fact the same patch as Π.

For the first case, we assume that Π has parameters (r, r, r, r, t) with $t < 2r$. By Lemma 2.7, we see that Π and Π^* have only hexagons on their boundaries. Stripping off

the hexagonal boundary faces of both Π and Π^* yields in each case a core that is an $(r-1)$-patch with parameters $(r-1, r-1, r-1, r-1, t-1)$. By Proposition 2.3, there is a P-disk with these parameters and by the induction hypothesis it must be the only $(r-1)$-patch with these parameters. Hence, Π and Π^* have the same P-disk as core. There is only one way to add hexagons around the boundary of a P-disk, so Π^* must be the same patch as Π.

Now, assume that Π has parameters $(r, r, r, r, 2r)$. By Lemma 2.7, both Π and Π^* have a pentagon as the central face of the $2r$-side. Thus the core of both Π and Π^* is the H-disk of radius $r-1$. Replacing the boundary faces, we see that $\Pi^* = \Pi$.

If Π has parameters $(r, r, r, r, r, h|k)$ with $h+k < r$, both Π and Π^* have only hexagons on the boundary. Thus, the core of Π is the P-disk with radius $r-1$ and parameters $(r-1, r-1, r-1, r-1, r-1, h|k)$. Since the core of Π^* is an r-patch with the same parameters, it is by induction the same P-disk as the core of Π. Again, replacing the boundary faces, we see that $\Pi^* = \Pi$.

Finally, assume Π has parameters $(r, r, r, r, r, h|k)$ with $h + k = r$. By Lemma 2.8, Π and Π^* must have a pentagon on the boundary and that pentagon is the boundary face with bounding segment of length 1. Thus, the core of both Π and Π^* is the H-disk of radius $r-1$. As above Π and Π^* have the same core and the same boundary faces and are therefore equal.   □

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