

Subgraphs of Kneser graphs with large girth and large chromatic number

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Abstract

It is well known that for any integers k and g , there is a graph with chromatic number at least k and girth at least g . In 1960s, Erdős and Hajnal conjectured that for any k and g , there exists a number $h(k, g)$, such that every graph with chromatic number at least $h(k, g)$ contains a subgraph with chromatic number at least k and girth at least g . The Erdős-Hajnal Conjecture has been recently proposed in the setting of the fractional chromatic number. In support of the original and the modified conjecture, we prove that every Kneser graph, whose (fractional) chromatic number is sufficiently large, contains a subgraph of large girth, whose (fractional) chromatic number is large.

Keywords: Kneser graph, graph coloring, chromatic number, fractional chromatic number.

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1 Introduction

A well known result, proved by Erdős [3] in 1950s, tells us that for every k and g , there exists a graph with chromatic number at least k and girth at least g . In 1960s, Erdős and Hajnal [4] proposed the following conjecture.

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Conjecture 1.1 (Erdős and Hajnal (1969)). *For any integers k and g , there exists a value $h(k, g)$ such that every graph G with $\chi(G) \geq h(k, g)$ contains a subgraph with chromatic number at least k and girth at least g .*

This conjecture has been resolved for $g = 4$ and arbitrary k by Rödl in 1977 [10], but it remains open for any larger girth. Recently, the authors extended Rödl's result to the setting of the fractional chromatic number χ_f by proving the following statement.

Theorem 1.2 ([9]). *For every real number $t \geq 1$, there exists a positive number $k(t)$ such that every graph G with $\chi_f(G) \geq k(t)$ contains a triangle-free subgraph H with $\chi_f(H) \geq t$.*

Let us recall that the *fractional chromatic number* $\chi_f(G)$ of a graph G is the minimum value of the following linear program:

$$\sum_{I \in \mathcal{I}(G)} y_I, \quad \text{subject to} \quad \sum_{I \in \mathcal{I}(G, v)} y_I \geq 1 \text{ for each } v \in V(G), \quad (1.1)$$

where $\mathcal{I}(G)$ denotes the family of all independent vertex-sets of G , $\mathcal{I}(G, v)$ is the family of all those independent sets which contain the vertex v , and y_I is a nonnegative real variable for each independent set I .

A version of the Erdős-Hajnal Conjecture for the fractional chromatic number has been proposed in [9].

Conjecture 1.3. *For all $t \geq 1$ and $g \geq 3$, there exists $k(t, g)$ such that every graph G with $\chi_f(G) \geq k(t, g)$ contains a subgraph H of girth at least g and with $\chi_f(H) \geq t$.*

Let us recall that the vertex-set of the Kneser graph $\text{KG}(n, r)$ consists of all r -sets of elements from $\{1, \dots, n\}$, and two such sets are adjacent if they are disjoint. It may be assumed that $r \leq n/2$. It is known [7] (see also [5]) that $\chi(\text{KG}(n, r)) = n - 2r + 2$ and that $\chi_f(\text{KG}(n, r)) = n/r$.

There is a tight relationship between Kneser graphs and the fractional chromatic number of general graphs. Namely, the following result holds (see, e.g. [5]).

Theorem 1.4. *For every graph G , we have the following:*

$$\chi_f(G) = \min\left\{\frac{n}{r} \mid G \text{ has a homomorphism into } \text{KG}(n, r)\right\}.$$

Theorem 1.4 shows that the first step towards a certification of Conjecture 1.3 should be its verification for the family of Kneser graphs. In this note we first observe the following.

Theorem 1.5. *For every k and g , every Kneser graph $\text{KG}(n, r)$ with n/r sufficiently large contains a subgraph whose fractional chromatic number is at least k and whose girth is at least g .*

This statement is an easy corollary of the fact that $\text{KG}(n, r)$ contains a complete subgraph of order $N = \lfloor n/r \rfloor$ and the fact proved in [3] that sufficiently large complete graph contains a subgraph of girth at least g and with fractional chromatic number at least k .¹

¹The proof of Erdős is actually for the usual chromatic number, but it is based on estimating the largest independent set, so it applies also for the fractional chromatic number.

The fact that the Erdős-Hajnal Conjecture is so resistant opens the question whether graphs with bounded fractional chromatic number (and large chromatic number) would still satisfy Conjecture 1.1. Therefore, it is natural to ask whether Conjecture 1.1 holds for Kneser graphs. The following theorem, which is the main result of this note, confirms that this is the case.

Theorem 1.6. *For every k and g , every Kneser graph $KG(n, r)$ with $n - 2r$ sufficiently large contains a subgraph whose chromatic number is at least k and whose girth is at least g .*

Proof of Theorem 1.6 occupies the rest of the paper.

2 Blow-ups and Kneser graphs

Our proof of Theorem 1.6 is motivated by the approach that was developed by Bollobás and Sauer [2] and generalized by Zhu [11]. They used their method to find uniquely k -colorable graphs of large girth and graphs of large girth that have unique homomorphism into any given core (a graph that does not have a homomorphism onto a proper subgraph of itself). The same approach was used in the setting of digraphs in [6].

The approach in [2, 11] used large blow-ups of graphs which have a homomorphism into some graph K but have no homomorphisms into some other fixed graph L .

However, the basic results in [2, 11] are not sufficiently strong to imply Theorem 1.6. We will thus develop the required strengthening which occupies the rest of this section.

Given a graph H , the *blow-up* of H with *power* m , denoted by $H^{(m)}$, is the graph obtained from H by replacing each vertex $a \in V(H)$ by an independent set V_a of cardinality m (called the *blow-up* of the vertex), and for each edge ab in H , adding a complete bipartite graph isomorphic to $K_{m,m}$ between V_a and V_b . The subgraph of $H^{(m)}$ replacing the edge ab will be referred to as the *blow-up* of that edge.

Let us recall that a graph L is said to be a *core* if every homomorphism $h: L \rightarrow L$ is an automorphism of L . A graph G is *L -colorable* if there exists a homomorphism $G \rightarrow L$. We have the following statement.

Theorem 2.1. *Let L be a graph of order $t = |L|$, and let G be a graph that is not L -colorable. Suppose that $g \geq 3$ and that $\Delta \geq \Delta(G)$ is an integer such that*

- (a) $(t\Delta)^{(g-2)/(2g)} > 4(1 + \log t)$ and
- (b) $\log(t\Delta) \geq 10s^{-1} + 8$.

If m is an integer that is larger than $t(t\Delta)^{2g-4}$, then $G^{(m)}$ contains a subgraph with girth more than g that is not L -colorable.

Let us observe that the proof of the main result in [11] contains a statement that is very close to the above theorem. However, the bound on m in [11] depends not only on the size of L and the maximum degree of G , but involves the number of vertices of G , which is not sufficient for our application.

To prove Theorem 2.1, we will need the following fact from [2, 11].

Lemma 2.2. *Let G be a graph that is not L -colorable, and let H be a subgraph of $G^{(m)}$. Suppose that for any edge $ab \in E(G)$ and for any subsets X, Y contained in the blow-ups V_a and V_b of a and b , respectively, with $|X| \geq \frac{m}{|L|}$, $|Y| \geq \frac{m}{|L|}$, there is an edge between X and Y in H . Then H is not L -colorable.*

In the proof of Theorem 2.1 we will take a random subgraph H of $G^{(m)}$, obtained by selecting each edge independently with probability $(\frac{m}{t})^{\frac{1}{4t}-1}$, and will prove that with positive probability thus obtained subgraph H has no short cycles, and for any edge $ab \in E(G)$ and for any pair X, Y contained in the respective blow-ups of a and b , and with $|X| \geq \frac{m}{t}$, $|Y| \geq \frac{m}{t}$, there is an edge between X and Y in H . To prove this, we will use the following asymmetric form of the Lovász Local Lemma (see [1]).

Theorem 2.3 (Lovász Local Lemma). *Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be a finite set of events in the probability space Ω . For $A \in \mathcal{A}$ let $\Gamma(A)$ denote a subset of \mathcal{A} such that A is independent from the collection of events $\mathcal{A} \setminus (\{A\} \cup \Gamma(A))$. If there exists an assignment of real numbers $y: \mathcal{A} \rightarrow (0, 1)$ to the events such that*

$$\forall A \in \mathcal{A} : \Pr(A) \leq y(A) \prod_{B \in \Gamma(A)} (1 - y(B))$$

then the probability of avoiding all events in \mathcal{A} is positive. In particular,

$$\Pr(\overline{A_1} \wedge \dots \wedge \overline{A_n}) \geq \prod_{A \in \mathcal{A}} (1 - y(A)).$$

Proof of Theorem 2.1. We may assume that L is connected. If $t = 1$, the statement is obvious. If $t = 2$, then $L = K_2$. In this case the proof is clear, since there is an odd cycle with length more than g in $G^{(m)}$. Thus we may assume henceforth that $t \geq 3$.

Let $s = \frac{m}{t}$, $\lambda = \frac{1}{4g}$ and $p = s^{\lambda-1}$. Let H be a random subgraph of $G^{(m)}$ obtained by picking each edge in $G^{(m)}$ independently at random with probability p .

If C is a cycle in $G^{(m)}$ of length at most g , let A_C be the event that all edges of C appear in H . Then $\Pr(A_C) = p^{|C|} = s^{(\lambda-1)|C|}$.

Let B be a copy of $K_{s,s}$ as a subgraph of a blow-up of an edge in G , let A_B be the event that H contains none of the edges of B . Then $\Pr(A_B) = (1 - p)^{s^2} \approx e^{-ps^2} = e^{-s^{1+\lambda}}$.

To prove that there exists a subgraph of $G^{(m)}$ with girth at least g that is not L -colorable, we will use Lemma 2.2. We just need to show

$$\Pr\left(\left(\bigwedge_{|C| \leq g} \overline{A_C}\right) \wedge \left(\bigwedge_{B \cong K_{s,s}} \overline{A_B}\right)\right) > 0.$$

This will be confirmed by applying the asymmetric form of the Lovász Local Lemma (Theorem 2.3) to the two types of events together.

Suppose that C is a cycle of length $|C| \leq g$. As the maximum degree in $G^{(m)}$ is at most $m\Delta = st\Delta$, there are at most $|C|(st\Delta)^{j-2}$ cycles of length j in $G^{(m)}$ that share edges with C , and there are at most $|C|\binom{m}{s}^2$ copies of $K_{s,s}$ that share edges with C .

Suppose that B is a copy of $K_{s,s}$. There are at most $s^2(st\Delta)^{j-2}$ cycles of length j in $G^{(m)}$ that share edges with B , and there are at most $\binom{m}{s}^2$ copies of $K_{s,s}$ that share edges with B .

For the Lovász Local Lemma, let $y_0 = y(A_B) = e^{-0.5s^{1+\lambda}} \approx \Pr(A_B)^{0.5}$, and for each cycle C of length $|C| \leq g$, let $y_{|C|} = y(A_C) = \Pr(A_C)^{1-\lambda} = s^{-(1-\lambda)^2|C|} < s^{(2\lambda-1)|C|}$. We just need to show:

$$(1) \Pr(A_C) \leq y_{|C|} \left(\prod_{3 \leq j \leq g} (1 - y_j)^{|C|(st\Delta)^{j-2}} \right) (1 - y_0)^{|C|\binom{m}{s}^2};$$

$$(2) \Pr(A_B) \leq y_0 \left(\prod_{3 \leq j \leq g} (1 - y_j)^{s^2(st\Delta)^{j-2}} \right) (1 - y_0)^{\binom{m}{s}}.$$

Let us take the logarithm on each side of the above inequalities, and use the fact that $0.9 \log(1 - z) > -z$ (when z is close to 0 as it appears to be in our case when $z = y_0$ or y_j). After simplifying, we see that it suffices to verify the following inequalities:

$$(3) 0.9\lambda(1 - \lambda) \log s \geq \sum_{3 \leq j \leq g} s^{(2\lambda-1)j} (st\Delta)^{j-2} + e^{-0.5s^{1+\lambda}} \binom{st}{s}^2.$$

$$(4) 0.4s^{1+\lambda} \geq \sum_{3 \leq j \leq g} s^{(2\lambda-1)j} s^2 (st\Delta)^{j-2} + e^{-0.5s^{1+\lambda}} \binom{st}{s}^2.$$

In order to prove (4), we first observe that:

$$\begin{aligned} \sum_{3 \leq j \leq g} s^{(2\lambda-1)j} s^2 (st\Delta)^{j-2} &= \sum_{3 \leq j \leq g} s^{2\lambda j} (t\Delta)^{j-2} \\ &< 1.1s^{2\lambda g} (t\Delta)^{g-2} \\ &= 1.1s^{0.5} (t\Delta)^{g-2}. \end{aligned}$$

By the assumption of the theorem, $s > (t\Delta)^{2g-4}$. Thus, $(t\Delta)^{g-2} < s^{0.5}$, and we have

$$\sum_{3 \leq j \leq g} s^{(2\lambda-1)j} s^2 (st\Delta)^{j-2} < 1.1s$$

Similarly, we also have $\sum_{3 \leq j \leq g} s^{(2\lambda-1)j} (st\Delta)^{j-2} < 1.1s^{-1}$, which will be used to prove (3).

As $\binom{st}{s}^2 < (et)^{2s}$, we have

$$e^{-0.5s^{1+\lambda}} \binom{st}{s}^2 < (e^{-0.5s^\lambda} (et)^2)^s = (e^{-0.5s^\lambda + 2 + 2\log t})^s.$$

Using the assumption that $s > (t\Delta)^{2g-4}$ and then using the assumption (a) from the statement of the theorem, we obtain that

$$s^\lambda > (t\Delta)^{\frac{2g-4}{4g}} > 4(1 + \log t).$$

This implies that, $e^{-0.5s^{1+\lambda}} \binom{st}{s}^2 < 1$.

Now, using the assumption (b), we see that

$$0.9\lambda(1 - \lambda) \log s \geq 0.9(1 - \lambda) \frac{2g-4}{4g} \log(t\Delta) > \frac{1}{8} \log(t\Delta) \geq 1.1s^{-1} + 1.$$

This implies that $0.4s^{1+\lambda} \geq 1.1s + 1$, and we conclude that both inequalities (3) and (4) are true.

In summary, if $m = st \geq t(t\Delta)^{2g-4}$, by the asymmetric form of the Lovász Local Lemma, the event that H has no cycles of length at most g and every $K_{s,s}$ as a subgraph of a blow-up of an edge has at least one edge, has positive probability. This means that there is a subgraph H of $G^{(m)}$ with these properties. Hence, by Lemma 2.2, H is not L -colorable. \square

The last ingredient needed in the forthcoming proofs is the following result from [8, Theorem 3.3] which shows that Kneser graphs contain blow-ups of smaller Kneser graphs with large power.

Theorem 2.4. *Let n, k, s , and t be nonnegative integers such that $0 < k < n$ and $t < ks$. The Kneser graph $\text{KG}(ns, ks - t)$ contains the blow-up of $\text{KG}(n, k)$ with power $\binom{k(s-1)}{t}$ as a subgraph. Furthermore, when $t < s$, it contains the blow-up of $\text{KG}(n, k)$ with power $\binom{ks}{t}$, and when $t = s$, it contains the blow-up of $\text{KG}(n, k)$ with power $\binom{ks}{t} - k$.*

3 Proof of Theorem 1.6

From Theorem 2.1, we know that graphs that are blow-ups of smaller graphs with sufficiently large power satisfy the Erdős-Hajnal Conjecture. In particular, Kneser graphs are such examples. This can be used to derive Theorem 1.6.

Let k and g be the parameters from the statement of the theorem. We may assume that k and g are both large. Let $\text{KG}(2n, n - 2x)$ be a Kneser graph with large chromatic number. Since $\chi(\text{KG}(2n, n - 2x)) = 4x + 2$, this just means that x is large in terms of k and g . Let $t = x/k$. By Theorem 2.4, $\text{KG}(2n, n - 2x)$ contains a blow-up of $\text{KG}(2n/t, (n - x)/t)$ with power $m = \binom{(n-x)(t-1)/t}{x}$. (Note that $\text{KG}(r, s)$ contains $\text{KG}(r - 1, s)$ as a subgraph, and thus, with some neglect of technicalities, we may assume that $x/k, 2n/t, (n - x)/t$, etc. are integers.) Each vertex of $\text{KG}(2n/t, (n - x)/t)$ has degree $\binom{(n+x)/t}{(n-x)/t} = \binom{(n+x)/t}{2x/t}$, which is at most

$$\Delta := ((n + x)/t)^{2x/t} = \left(\frac{nk}{x} + k\right)^{2k}.$$

In order to apply Theorem 2.1, we need power $m \geq k(k\Delta)^{2g-4}$ of a graph with chromatic number at least k . In the following we assume x is large in terms of g and k . Then we may assume that Δ is sufficiently large so that the conditions (a) and (b) of Theorem 2.1 are satisfied.

Let us first consider the case when $n > x/(\frac{1}{2} - \frac{1}{2k})$. If we write $\frac{n}{x} = 2 + 2z$, this condition implies that $z \geq \frac{1}{k}$. If $x > k^2$, then

$$\frac{n-x}{x} \cdot \frac{t-1}{t} \geq 1 + z \quad (3.1)$$

and

$$\frac{n}{x} + 1 = 3 + 2z < (1 + z)^{x/2}. \quad (3.2)$$

Suppose, moreover, that $x \geq 10gk^2 \log k$. Then

$$k^{2g(2k+1)} < (1 + z)^{x/2}. \quad (3.3)$$

Using inequalities (3.1) – (3.3), we obtain:

$$\begin{aligned} k(k\Delta)^{2g-4} &= k^{2g-3} \left(\frac{nk}{x} + k\right)^{2k(2g-4)} \\ &< \left(\frac{(n-x)(t-1)/t}{x}\right)^x \\ &< \binom{(n-x)(t-1)/t}{x} = m. \end{aligned}$$

On the other hand, when x is large enough and $n \leq x/(\frac{1}{2} - \frac{1}{2k})$, then $n - 2x \leq n/k$ and $\text{KG}(2n, n - 2x)$ contains a blow-up of $\text{KG}(k, 1) = K_k$ with large power.

Thus, we can apply Theorem 2.1 and conclude that $\text{KG}(2n, n - 2x)$ contains a subgraph with girth more than g and chromatic number at least k . This completes the proof of Theorem 1.6.

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References

- [1] N. Alon and J. H. Spencer, *The Probabilistic Method*, Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 4th edition, 2016.
- [2] B. Bollobás and N. Sauer, Uniquely colourable graphs with large girth, *Canadian J. Math.* **28** (1976), 1340–1344, doi:10.4153/cjm-1976-133-5, <https://doi.org/10.4153/cjm-1976-133-5>.
- [3] P. Erdős, Graph theory and probability, *Canadian J. Math.* **11** (1959), 34–38, doi:10.4153/cjm-1959-003-9, <https://doi.org/10.4153/cjm-1959-003-9>.
- [4] P. Erdős, Problems and results in combinatorial analysis and graph theory, in: *Proof Techniques in Graph Theory*, Academic Press, New York, pp. 27–35, 1969.
- [5] C. Godsil and G. Royle, *Algebraic graph theory*, Springer-Verlag, New York, 2001, doi:10.1007/978-1-4613-0163-9, <https://doi.org/10.1007/978-1-4613-0163-9>.
- [6] A. Harutyunyan, P. M. Kayll, B. Mohar and L. Rafferty, Uniquely D -colourable digraphs with large girth, *Canad. J. Math.* **64** (2012), 1310–1328, doi:10.4153/cjm-2011-084-9, <https://doi.org/10.4153/cjm-2011-084-9>.
- [7] L. Lovász, Kneser’s conjecture, chromatic number, and homotopy, *J. Comb. Theory Ser. A* **25** (1978), 319–324, doi:10.1016/0097-3165(78)90022-5, [https://doi.org/10.1016/0097-3165\(78\)90022-5](https://doi.org/10.1016/0097-3165(78)90022-5).
- [8] B. Mohar and H. Wu, Dichromatic number and fractional chromatic number, *Forum Math. Sigma* **4** (2016), Paper No. e32, 14, doi:10.1017/fms.2016.28, <https://doi.org/10.1017/fms.2016.28>.
- [9] B. Mohar and H. Wu, Triangle-free subgraphs with large fractional chromatic number, *Combin. Probab. Comput.* **31** (2022), 136–143, doi:10.1017/s0963548321000250, <https://doi.org/10.1017/s0963548321000250>.
- [10] V. Rödl, On the chromatic number of subgraphs of a given graph, *Proc. Am. Math. Soc.* **64** (1977), 370–371, doi:10.2307/2041460, <https://doi.org/10.2307/2041460>.
- [11] X. Zhu, Uniquely H -colorable graphs with large girth, *J. Graph Theory* **23** (1996), 33–41, doi:10.1002/(SICI)1097-0118(199609)23:1<33::AID-JGT3>3.3.CO;2-P, [https://doi.org/10.1002/\(SICI\)1097-0118\(199609\)23:1<33::AID-JGT3>3.3.CO;2-P](https://doi.org/10.1002/(SICI)1097-0118(199609)23:1<33::AID-JGT3>3.3.CO;2-P).